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Optimal Hölder index for density states of superprocesses with $(1+\beta)$-branching mechanism

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Abstract. For $0 < \alpha \leq 2$, a super-α-stable motion X in R^d with branching of index $1+\beta \in (1,2)$ is considered. If $d < \alpha/\beta$, a dichotomy for the density of states X_t at fixed times $t > 0$ holds: the density function is locally Hölder continuous if $d = 1$ and $\alpha > 1+\beta$, but locally unbounded otherwise. Moreover, in the case of continuity, we determine the optimal Hölder index.

Contents

1. Introduction and statement of results

1.1. **Background and purpose.** For $0 < \alpha \leq 2$, a super–α–stable motion $X = \{X_t : t \geq 0\}$ in R^d with branching of index $1+\beta \in (1,2]$ is a finite measure-valued process related to the log-Laplace equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u \;=\; \boldsymbol{\Delta}_\alpha u + au - bu^{1+\beta}, \tag{1.1}$$

where $a \in \mathsf{R}$ and $b > 0$ are any fixed constants. Its underlying motion is described by the fractional Laplacian $\boldsymbol{\Delta}_\alpha := -(-\boldsymbol{\Delta})^{\alpha/2}$ determining a symmetric α–stable motion in R^d of index $\alpha \in (0,2]$ (Brownian motion if $\alpha = 2$), whereas its continuous-state branching mechanism

$$v \;\mapsto\; -av + bv^{1+\beta} \;=:\; \Psi(v), \quad v \geq 0, \tag{1.2}$$

belongs to the domain of attraction of a stable law of index $1+\beta \in (1,2]$ (the branching is critical if $a = 0$).

As well-known, X has absolutely continuous states X_t at fixed times $t > 0$ in dimensions $d < \frac{\alpha}{\beta}$ (cf. Fleischmann [4] where $a = 0$, the non-critical case requires the obvious changes). By an abuse of notation, we sometimes denote a version of the density function of the measure $X_t = X_t(\mathrm{d}x)$ by the same symbol:

$X_t(\mathrm{d}x) = X_t(x)\,\mathrm{d}x$, that is $X_t = \{X_t(x) : x \in \mathsf{R}^d\}$. In the case of one-dimensional continuous super-Brownian motion ($\alpha = 2$, $\beta = 1$), even a joint-continuous density field $\{X_t(x) : t > 0, x \in \mathsf{R}\}$ exists, satisfying a stochastic equation (Konno and Shiga [10] as well as Reimers [14]).

From now on we assume that $d < \frac{\alpha}{\beta}$ and $\beta \in (0, 1)$. For the Brownian case $\alpha = 2$ and if $a = 0$ (critical branching) Mytnik [12] proved that a version of the density $\{X_t(x) : t > 0, x \in \mathsf{R}^d\}$ of the measure $X_t(\mathrm{d}x)\mathrm{d}t$ exists that satisfies in a weak sense the following stochastic partial deferential equation (SPDE)

$$\frac{\partial}{\partial t} X_t(x) = \boldsymbol{\Delta} X_t(x) + \bigl(bX_{t-}(x)\bigr)^{1/(1+\beta)} \dot{L}(t, x) \tag{1.3}$$

where $\dot{L}$ is a $(1+\beta)$-stable noise without negative jumps.

For the same model, in Mytnik and Perkins [13] regularity and irregularity properties of the density states at fixed times had been revealed. More precisely, these density states have continuous versions if $d = 1$, whereas they are locally unbounded on open sets of positive $X_t(\mathrm{d}x)$-measure in all higher dimensions $(d < \frac{2}{\beta})$.

The first *purpose* in the present paper is to allow also discontinuous underlying motions, that is to consider also all $\alpha \in (0, 2)$. Then the same type of *fixed time dichotomy* holds (recall that $d < \frac{\alpha}{\beta}$) : continuity of density states if $d = 1$ and $\alpha > 1 + \beta$, whereas local unboundedness is true if $d > 1$ or $\alpha \le 1 + \beta$.

However, the *main purpose* of the paper is to address the following question: What is the optimal Hölder index in the first case of existence of a continuous density? Here by optimality we mean that there is a critical index η_{c} such that there is a version of the density which is locally Hölder continuous of any index $\eta < \eta_{\mathrm{c}}$, whereas there is no locally Hölder continuous version with index $\eta \ge \eta_{\mathrm{c}}$.

In [13] continuity of the density at fixed times is proved by some moment methods, although moments of order larger than $1 + \beta$ are in general infinite in the $1 + \beta < 2$ case. A standard procedure to get local Hölder continuity is the Kolmogorov criterion by using "high" moments. This, for instance, can be done in the $\beta = 1$ case ($\alpha = 2$, $d = 1$) to show local Hölder continuity of any index smaller than $\frac{1}{2}$ (see the estimates in the proof of Corollary 3.4 in Walsh [17]).

Due, to the lack of "high" moments in our $\beta < 1$ case we cannot use moments to get the optimal Hölder index. Therefore we have to get deeply into the jump structure of the superprocess to obtain the needed estimates. As a result we are able to show the *local Hölder continuity* of all orders $\eta < \eta_{\mathrm{c}} := \frac{\alpha}{1+\beta} - 1$, provided that $d = 1$ and $\alpha > 1 + \beta$. We also verify that the bound η_{c} for the Hölder index is in fact *optimal* in the sense that there are points x_1, x_2 such that the density increments $\bigl|X_t(x_1) - X_t(x_2)\bigr|$ are of a larger order than $|x_1 - x_2|^\eta$ as $x_1 - x_2 \to 0$, for every $\eta \ge \eta_{\mathrm{c}}$. For precise formulations, see Theorem 1.1 below.

1.2. **Statement of results.** Write $\mathcal{M}_{\mathrm{f}}$ for the set of all finite measures μ defined on R^d and $|\mu|$ for its total mass $\mu(\mathsf{R}^d)$. Let $\|f\|_U$ denote the essential supremum (with respect to Lebesgue measure) of a function $f : \mathsf{R}^d \to \mathsf{R}_+ := [0, \infty)$ over a non-empty open set $U \subseteq \mathsf{R}^d$.

Let p^α denote the continuous α*–stable transition kernel* related to the fractional Laplacian $\boldsymbol{\Delta}_\alpha = -(-\boldsymbol{\Delta})^{\alpha/2}$, and S^α the related *semigroup.*

Recall that $0 < \alpha \le 2$, $1 + \beta \in (1, 2)$, and $d < \frac{\alpha}{\beta}$, and consider again the (α, d, β)–superprocess $X = \{X_t : t \ge 0\}$ in R^d related to (1.1). Recall also that for fixed $t > 0$, with probability one, the measure state X_t is absolutely continuous ([4]). Here is our *main result*:

Theorem 1.1 (Dichotomy for densities). *Fix $t > 0$ and $X_0 = \mu \in \mathcal{M}_{\mathrm{f}}$.*

(a) (Local Hölder continuity): *If $d = 1$ and $\alpha > 1+\beta$, then with probability one, there is a continuous version $\tilde{X}_t$ of the density function of the measure $X_t(\mathrm{d}x)$. Moreover, for each $\eta < \eta_{\mathrm{c}} := \frac{\alpha}{1+\beta} - 1$, this version $\tilde{X}_t$ is locally Hölder continuous of index η:*

$$\sup_{x_1,x_2\in K,\, x_1\neq x_2} \frac{\bigl|\tilde{X}_t(x_1) - \tilde{X}_t(x_2)\bigr|}{|x_1 - x_2|^\eta} < \infty, \quad \textit{compact } K \subset \mathsf{R}.$$

(b) (Optimal Hölder exponent): *Under conditions as in the beginning of part* (a)*, for every $\eta \geq \eta_{\mathrm{c}}$ with probability one, for any open $U \subseteq \mathsf{R}$,*

$$\sup_{x_1,x_2\in U,\, x_1\neq x_2} \frac{\bigl|\tilde{X}_t(x_1) - \tilde{X}_t(x_2)\bigr|}{|x_1 - x_2|^\eta} = \infty \quad \textit{whenever } X_t(U) > 0.$$

(c) (Local unboundedness): *If $d > 1$ or $\alpha \leq 1+\beta$, then with probability one, for all open $U \subseteq \mathsf{R}^d$,*

$$\|X_t\|_U = \infty \quad \textit{whenever } X_t(U) > 0.$$

Remark 1.2 (Any version). As in part (c), the statement in part (b) is valid also for any version X_t of the density function. $\diamond$

1.3. **Some discussion.** The result of Theorem 1.1(a,b) is a bit surprising. Let us recall again what is known about regularity properties of density states of (α, d, β)–superprocesses. The case of continuous super-Brownian motion ($\alpha = 2$, $\beta = 1$, $d = 1$) is very well studied. As already mentioned, densities exist at all times simultaneously, and they are locally Hölder continuous in the spatial variable for any index $\eta < \frac{1}{2}$. Moreover, it is known that $\frac{1}{2}$ is optimal in this case [1)]. Now let us consider our result in Theorem 1.1(a,b), specialized to $\alpha = 2$. Then we have $\eta_{\mathrm{c}} = \frac{2}{1+\beta} - 1 \downarrow 0$ as $\beta \uparrow 1$, where the limit 0 is different from the optimal Hölder index $\frac{1}{2}$ of continuous super-Brownian motion. This may confuse a reader and even raise a suspicion that something is wrong. However there is an intuitive explanation for this discontinuity as we would like to explain now.

Recall the notion of Hölder continuity at a point. A function f is Hölder continuous with index $\eta \in (0, 1)$ at a point x_0 if there is a neighborhood $U(x_0)$ such that

$$\bigl|f(x) - f(x_0)\bigr| \leq C\,|x - x_0|^\eta \ \text{ for all } x \in U(x_0). \tag{1.4}$$

The *optimal* Hölder index $H(x_0)$ of f at the point x_0 is defined as the supremum of all such η. Clearly, there are functions where $H(x_0)$ may vary with x_0, and the index of a local Hölder continuity in a domain cannot be larger than the smallest optimal Hölder index at the points of the domain. The density states of continuous super-Brownian motion are such that $H(x_0) = \frac{1}{2}$ for all x_0, whereas in our $\beta < 1$ case of discontinuous superprocesses the situation is quite different. The critical Hölder index $\eta_{\mathrm{c}} = \frac{\alpha}{1+\beta} - 1$ in our case is a result of the influence of relatively high

1) For instance, Walsh [16] showed optimality of $\frac{1}{2}$ in the case of the heat equation with noise, and together with the Konno-Shiga representation [10] and continuity this can be transferred to the present case of super-Brownian motion in the interior of the support at fixed times.

jumps of the superprocess that occur close to time t. So there are (random) points x_0 with $H(x_0) = \eta_c$. But these points are *exceptional* points, loosely speaking, there are not too many of them. We conjecture that at any *fixed* point x_0 the optimal Hölder index $H(x_0)$ equals $\bigl(\frac{1+\alpha}{1+\beta} - 1\bigr) \wedge 1 =: \bar{\eta}_c > \eta_c$. Now, if $\alpha = 2$, as $\beta \uparrow 1$ one gets the index $\frac{1}{2}$ corresponding to the case of continuous super-Brownian motion.

This observation raises in fact a number of very interesting *open problems*:

Conjecture 1.3 (**Multifractal spectrum**)**.** We conjecture that for any $\eta \in \bigl(\eta_c, \bar{\eta}_c\bigr)$ there are (random) points x_0 where the density state X_t is Hölder continuous with index η. What is the *Hausdorff dimension*, say $D(\eta)$, of the (random) set $\bigl\{x_0 : H(x_0) = \eta\bigr\}$? We conjecture that

$$\lim_{\eta \downarrow \eta_c} D(\eta) \,=\, 0 \quad \text{and} \quad \lim_{\eta \uparrow \bar{\eta}_c} D(\eta) = 1 \tag{1.5}$$

This function $\eta \mapsto D(\eta)$ reveals the so-called *multifractal* structure concerning the optimal Hölder index in points for the density states of superprocesses with branching of index $1+\beta < 2$ and is definitely worth studying. In this connection, we refer to Jaffard [8] where multifractal properties of one-dimensional Lévy processes are studied. ◇

Another interesting direction would be a generalization of our results to the case of SPDEs driven by Levy noises. In recent years there has been increasing interest to such SPDEs. Here we may mention the papers Bié [1], Mytnik [12], Mueller, Mytnik, and Stan [11], as well as Hausenblas [7]. Note that in these papers properties of solutions are described in some $\mathcal{L}^p$-sense. To the best of our knowledge not to many things are known about local Hölder continuity of solutions (in case of continuity). The only result we know in this direction is [13], where some local Hölder continuity of the fixed time density of super-Brownian motion ($\alpha = 2$, $\beta < 1$, $d < \frac{2}{\beta}$, $a = 0$) was established. However, the result there was far away from being optimal. With Theorem 1.1(a,b) we fill this gap. Our result also allows the following conjecture:

Conjecture 1.4 (**Regularity in case of SPDE with stable noise**)**.** Consider an SPDE of the kind

$$\frac{\partial}{\partial t} X_t(x) = \boldsymbol{\Delta}_\alpha X_t(x) + g\bigl(X_{t-}(x)\bigr)\dot{L}(t,x) \tag{1.6}$$

where $\dot{L}$ is a $(1+\beta)$-stable noise without negative jumps and g is such that solutions exist. Then there should exist versions of solutions such that at fixed times regularity holds just as described in Theorem 1.1(a,b), with the same parameter classification, in particular with the same η_c. ◇

1.4. **Martingale decomposition of X.** As in the $\alpha = 2$ case of [13], for the proof we need the martingale decomposition of X. For this purpose, we will work with the following *alternative description* of the continuous-state branching mechanism Ψ from (1.2):

$$\Psi(v) \,=\, -av \,+\, \varrho \int_0^\infty \mathrm{d}r\ r^{-2-\beta}\,\bigl(\mathrm{e}^{-vr} - 1 + vr\bigr), \quad v \geq 0, \tag{1.7}$$

where

$$\varrho \,:=\, b\,\frac{(1+\beta)\beta}{\Gamma(1-\beta)} \tag{1.8}$$

with Γ denoting the famous Gamma function. The martingale decomposition of X in the following lemma is basically proven in Dawson [2, Section 6.1].

Denote by $\mathcal{C}_{\mathrm{b}}$ the set of all bounded and continuous functions on R^d. We add the sign $+$ if the functions are additionally non-negative. $\mathcal{C}_{\mathrm{b}}^{(k),+}$ with $k \geq 1$ refers to the subset of functions which are k times differentiable and that all derivatives up to the order k belong to $\mathcal{C}_{\mathrm{b}}^{+}$, too.

Lemma 1.5 (**Martingale decomposition of** X). *Fix* $X_0 = \mu \in \mathcal{M}_{\mathrm{f}}$.

- **(a) (Discontinuities):** *All discontinuities of the process* X *are jumps upwards of the form* $r\delta_x$. *More precisely, there exists a random measure* $N\big(\mathrm{d}(s,x,r)\big)$ *on* $\mathsf{R}_+ \times \mathsf{R}^d \times \mathsf{R}_+$ *describing the jumps* $r\delta_x$ *of* X *at times* s *at sites* x *of size* r.
- **(b) (Jump intensities):** *The compensator* $\hat{N}$ *of* N *is given by*
$$\hat{N}\big(\mathrm{d}(s,x,r)\big) \;=\; \varrho\; \mathrm{d}s\, X_s(\mathrm{d}x)\, r^{-2-\beta}\mathrm{d}r,$$
that is, $\tilde{N} := N - \hat{N}$ *is a martingale measure on* $\mathsf{R}_+ \times \mathsf{R}^d \times \mathsf{R}_+$.
- **(c) (Martingale decomposition):** *For all* $\varphi \in \mathcal{C}_{\mathrm{b}}^{(2),+}$ *and* $t \geq 0$,
$$\langle X_t, \varphi\rangle \;=\; \langle \mu, \varphi\rangle + \int_0^t \mathrm{d}s\; \langle X_s, \boldsymbol{\Delta}_\alpha \varphi\rangle + M_t(\varphi) + a\, I_t(\varphi)$$
with discontinuous martingale
$$t \;\mapsto\; M_t(\varphi) \;:=\; \int_{(0,t]\times\mathsf{R}^d\times\mathsf{R}_+} \tilde{N}\big(\mathrm{d}(s,x,r)\big)\, r\, \varphi(x)$$
and increasing process
$$t \;\mapsto\; I_t(\varphi) \;:=\; \int_0^t \mathrm{d}s\; \langle X_s, \varphi\rangle.$$

From Lemma 1.5 we get the related *Green's function representation*

$$\begin{aligned}\langle X_t, \varphi\rangle \;=\; \langle \mu, S_t^\alpha \varphi\rangle \;+\; &\int_{(0,t]\times\mathsf{R}^d} M\big(\mathrm{d}(s,x)\big)\, S_{t-s}^\alpha \varphi\,(x)\\ +\; a &\int_{(0,t]\times\mathsf{R}^d} I\big(\mathrm{d}(s,x)\big)\, S_{t-s}^\alpha \varphi\,(x), \quad t \geq 0,\;\; \varphi \in \mathcal{C}_{\mathrm{b}}^{+},\end{aligned} \tag{1.9}$$

with M the martingale measure related to the martingale in part (c) and I the measure related to the increasing process there.

We add also the following lemma which can be proved as Lemma 3.1 in Le Gall and Mytnik [6]. For $p \geq 1$, let $\mathcal{L}_{\mathrm{loc}}^p(\mu) = \mathcal{L}_{\mathrm{loc}}^p\big(\mathsf{R}_+ \times \mathsf{R}^d,\, S_s^\alpha \mu\,(x)\,\mathrm{d}s\mathrm{d}x\big)$ denote the space of equivalence classes of measurable functions ψ such that

$$\int_0^T \mathrm{d}s \int_{\mathsf{R}^d} \mathrm{d}x\; S_s^\alpha \mu\,(x)\, \big|\psi(s,x)\big|^p \;<\; \infty, \quad T > 0. \tag{1.10}$$

Lemma 1.6 (L^p**-space with martingale measure**). *Let* $X_0 = \mu \in \mathcal{M}_{\mathrm{f}}$ *and* $\psi \in \mathcal{L}_{\mathrm{loc}}^p(\mu)$ *for some* $p \in (1+\beta, 2)$. *Then the martingale*

$$t \mapsto \int_{(0,t]\times\mathsf{R}^d} M\big(\mathrm{d}(s,x)\big)\, \psi(s,x) \tag{1.11}$$

is well-defined.

Fix $t>0,\ \mu\in\mathcal{M}_{\mathrm{f}}$. Suppose $d<\frac{\alpha}{\beta}$. Then the random measure X_t is a.s. absolutely continuous. From (1.9) we get the following representation of a version of its *density function* (cf. [13, 6]:

$$\begin{aligned} X_t(x) \;&=\; \mu{*}p_t^{\alpha}(x) \;+\; \int_{(0,t]\times\mathsf{R}^d} M\big(\mathrm{d}(s,y)\big)\,p_{t-s}^{\alpha}(x-y) \\ &\quad+\; a\int_{(0,t]\times\mathsf{R}^d} I\big(\mathrm{d}(s,y)\big)\,p_{t-s}^{\alpha}(x-y) \\ &=:\; Z_t^1(x)+Z_t^2(x)+Z_t^3(x), \quad x\in\mathsf{R}^d, \end{aligned} \tag{1.12}$$

with notation in the obvious correspondence.

This representation is the starting point for the proof of the local Hölder continuity as claimed in Theorem 1.1(a). Main work has to be done to deal with Z_t^2.

1.5. **Organization of the paper.** In Section 2 we develop some tools that will be used in the following sections for the proof of Theorem 1.1. Also on the way, in Subsection 2.3 we are able to verify partially Theorem 1.1(a) for some range of parameters α,β using simple moment estimates. The proof of Theorem 1.1(a) is completed in Section 3 using a more delicate analysis of the jump structure of the process. Section 4 is devoted to the proof of part (c) of Theorem 1.1. In Section 5, which is the most technically involved section, we verify Theorem 1.1(b).

2. Auxiliary tools

In this section we always assume that $d=1$.

2.1. **On the transition kernel of α-stable motion.** The symbol C will always denote a generic positive constant, which might change from place to place. On the other hand, $c_{(\#)}$ denotes a constant appearing in formula line (#).

We start with two estimates concerning the α-stable transition kernel p^{α}.

Lemma 2.1 (**α-stable density increment**). *For every* $\delta\in[0,1]$,

$$\big|p_t^{\alpha}(x)-p_t^{\alpha}(y)\big| \;\le\; C\,\frac{|x-y|^{\delta}}{t^{\delta/\alpha}}\,\big(p_t^{\alpha}(x/2)+p_t^{\alpha}(y/2)\big), \quad t>0,\ x,y\in\mathsf{R}. \tag{2.1}$$

Proof. For the case $\alpha=2$, see e.g. Rosen [15, (2.4e)]. Suppose $\alpha<2$. Then we use the well-known subordination formula

$$p_t^{\alpha}(z) \;=\; \int_0^{\infty}\mathrm{d}s\;q_t^{\alpha/2}(s)\,p_s^{(2)}(z), \quad t>0,\ z\in\mathsf{R}, \tag{2.2}$$

where $q^{\alpha/2}$ denotes the continuous transition kernel of a stable process on R_+ of index $\alpha/2$, and by an abuse of notation, $p^{(2)}$ refers to p^{α} in case $\alpha=2$. Consequently,

$$\big|p_t^{\alpha}(x)-p_t^{\alpha}(y)\big| \;\le\; \int_0^{\infty}\mathrm{d}s\;q_t^{\alpha/2}(s)\,\big|p_s^{(2)}(x)-p_s^{(2)}(y)\big|. \tag{2.3}$$

Hence, from the $\alpha=2$ case,

$$\big|p_t^{\alpha}(x)-p_t^{\alpha}(y)\big| \;\le\; C\,|x-y|^{\delta}\int_0^{\infty}\mathrm{d}s\;q_t^{\alpha/2}(s)\,s^{-\delta/2}\big(p_s^{(2)}(x/2)+p_s^{(2)}(y/2)\big). \tag{2.4}$$

The lemma will be proved if we show that

$$\int_0^\infty \mathrm{d}s\; q_t^{\alpha/2}(s)\, s^{-\delta/2}\, p_s^{(2)}(x/2) \;\le\; C\, t^{-\delta/\alpha}\, p_t^\alpha(x/2), \quad t>0,\;\; x\in\mathsf{R}. \tag{2.5}$$

First, in view of (2.2),

$$\begin{aligned}\int_{t^{2/\alpha}}^\infty \mathrm{d}s\; q_t^{\alpha/2}(s)\, s^{-\delta/2}\, p_s^{(2)}(x/2) \;&\le\; t^{-\delta/\alpha}\int_{t^{2/\alpha}}^\infty \mathrm{d}s\; q_t^{\alpha/2}(s)\, p_s^{(2)}(x/2)\\ &\le\; t^{-\delta/\alpha}\, p_t^\alpha(x/2).\end{aligned} \tag{2.6}$$

Second, by scaling $q_t^{\alpha/2}(s) = t^{-2/\alpha}\, q_1^{\alpha/2}(t^{-2/\alpha}s)$ and substitution $t^{-2/\alpha}s = u$,

$$\int_0^{t^{2/\alpha}} \mathrm{d}s\; q_t^{\alpha/2}(s)\, s^{-\delta/2}\, p_s^{(2)}(x/2) \;=\; t^{-\delta/\alpha}\int_0^1 \mathrm{d}u\; q_1^{\alpha/2}(u)\, u^{-\delta/2}\, p_{t^{2/\alpha}u}^{(2)}(x/2). \tag{2.7}$$

By Brownian scaling, (2.7) can be continued with

$$\begin{aligned}&=\; t^{-(\delta+1)/\alpha}\int_0^1 \mathrm{d}u\; q_1^{\alpha/2}(u)\, u^{-(\delta+1)/2}\, p_1^{(2)}\Big(\frac{x/2}{t^{1/\alpha}u^{1/2}}\Big)\\ &\le\; t^{-(\delta+1)/\alpha}\, p_1^{(2)}(x/2t^{1/\alpha})\int_0^1 \mathrm{d}u\; q_1^{\alpha/2}(u)\, u^{-(\delta+1)/2}\\ &\le\; C\, t^{-(\delta+1)/\alpha}\, p_1^{(2)}(x/2t^{1/\alpha}),\end{aligned} \tag{2.8}$$

where in the last step we have used the fact that $q_1^{\alpha/2}(u)$ decreases, as $u\downarrow 0$, exponentially fast (cf. [3, Theorem 13.6.1]). Since $p_1^{(2)}(x) = \mathrm{o}\big(p_1^\alpha(x)\big)$ as $x\uparrow\infty$, we have $p_1^{(2)}(x)\le C\, p_1^\alpha(x)$, $x\in\mathsf{R}$. Hence,

$$\int_0^{t^{2/\alpha}} \mathrm{d}s\; q_t^{\alpha/2}(s)\, s^{-\delta/2}\, p_s^{(2)}(x) \le C\, t^{-(\delta+1)/\alpha}\, p_1^\alpha(x/2t^{1/\alpha}) =\; C\, t^{-\delta/\alpha}\, p_t^\alpha(x/2). \tag{2.9}$$

Combining (2.6) and (2.9) gives (2.5), completing the proof. □

Lemma 2.2 (**Integrals of α-stable density increment**). *If* $\theta\in[1,1+\alpha)$ *and* $\delta\in[0,1]$ *satisfy* $\delta<(1+\alpha-\theta)/\theta$, *then*

$$\begin{aligned}&\int_0^t \mathrm{d}s\int_{\mathsf{R}} \mathrm{d}y\; p_s^\alpha(y)\,\big|p_{t-s}^\alpha(x_1-y) - p_{t-s}^\alpha(x_2-y)\big|^\theta\\ &\le\; C\,(1+t)\,|x_1-x_2|^{\delta\theta}\big(p_t^\alpha(x_1/2)+p_t^\alpha(x_2/2)\big), \quad t>0,\;\; x_1,x_2\in\mathsf{R}.\end{aligned} \tag{2.10}$$

Proof. By Lemma 2.1, for every $\delta\in[0,1]$,

$$\begin{aligned}&\big|p_{t-s}^\alpha(x_1-y) - p_{t-s}^\alpha(x_2-y)\big|^\theta\\ &\le\; C\,\frac{|x_1-x_2|^{\delta\theta}}{(t-s)^{\delta\theta/\alpha}}\Big(p_{t-s}^\alpha\big((x_1-y)/2\big)+p_{t-s}^\alpha\big((x_2-y)/2\big)\Big)^\theta,\end{aligned} \tag{2.11}$$

$t>s\ge 0$, $x_1,x_2,y\in\mathsf{R}$. Noting that $p_{t-s}^\alpha(\cdot)\le C\,(t-s)^{-1/\alpha}$, we obtain

$$\begin{aligned}&\big|p_{t-s}^\alpha(x_1-y) - p_{t-s}^\alpha(x_2-y)\big|^\theta\\ &\le\; C\,\frac{|x_1-x_2|^{\delta\theta}}{(t-s)^{(\delta\theta+\theta-1)/\alpha}}\Big(p_{t-s}^\alpha\big((x_1-y)/2\big)+p_{t-s}^\alpha\big((x_2-y)/2\big)\Big),\end{aligned} \tag{2.12}$$

$t>s\geq 0,\ x_1,x_2,y\in\mathsf{R}$. Therefore,

$$\int_0^t \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}y\; p_s^\alpha(y)\,\big|p_{t-s}^\alpha(x_1-y)-p_{t-s}^\alpha(x_2-y)\big|^\theta \;\leq\; C\,|x_1-x_2|^{\delta\theta}\times$$
$$\times\int_0^t \mathrm{d}s\;(t-s)^{-(\delta\theta+\theta-1)/\alpha}\int_{\mathsf{R}}\mathrm{d}y\; p_s^\alpha(y)\Big(p_{t-s}^\alpha\big((x_1-y)/2\big)+p_{t-s}^\alpha\big((x_2-y)/2\big)\Big).$$

By scaling of p^α,

$$\int_{\mathsf{R}}\mathrm{d}y\; p_s^\alpha(y)\,p_{t-s}^\alpha\big((x-y)/2\big) \;=\; \frac{1}{2}\int_{\mathsf{R}}\mathrm{d}y\; p_{2^{-\alpha}s}^\alpha(y/2)\,p_{t-s}^\alpha\big((x_2-y)/2\big)$$
$$=\;\frac{1}{2}\,p_{2^{-\alpha}s+t-s}^\alpha(x/2)\;=\;\frac{1}{2}\big(2^{-\alpha}s+t-s\big)^{-1/\alpha}\,p_1^\alpha\big((2^{-\alpha}s+t-s)^{-1/\alpha}x/2\big)$$
$$\leq\; t^{-1/\alpha}\,p_1^\alpha(t^{-1/\alpha}x/2)\;=\;p_t^\alpha(x/2), \tag{2.13}$$

since $2^{-\alpha}t\leq 2^{-\alpha}s+t-s\leq t$. As a result we have the inequality

$$\int_0^t \mathrm{d}s\int_{\mathsf{R}}\mathrm{d}y\; p_s^\alpha(y)\,\big|p_{t-s}^\alpha(x_1-y)-p_{t-s}^\alpha(x_2-y)\big|^\theta$$
$$\leq\; C\,|x_1-x_2|^{\delta\theta}\big(p_t^\alpha(x_1/2)+p_t^\alpha(x_2/2)\big)\int_0^t\mathrm{d}s\; s^{-(\delta\theta+\theta-1)/\alpha}. \tag{2.14}$$

Noting that the latter integral is bounded by $C\,(1+t)$, since $(\delta\theta+\theta-1)/\alpha<1$, we get the desired inequality. ☐

2.2. **An upper bound for a spectrally positive stable process.** Let $L=\{L_t: t\geq 0\}$ denote a spectrally positive stable process of index $\kappa\in(1,2)$. That is, L is an R-valued time-homogeneous process with independent increments and with Laplace transform given by

$$\mathbf{E}\,\mathrm{e}^{-\lambda L_t}\;=\;\mathrm{e}^{t\lambda^\kappa},\quad \lambda,t\geq 0. \tag{2.15}$$

Note that L is the unique solution to the following martingale problem:

$$t\mapsto \mathrm{e}^{-\lambda L_t}-\int_0^t\mathrm{d}s\;\mathrm{e}^{-\lambda L_s}\lambda^\kappa \text{ is a martingale for any } \lambda>0. \tag{2.16}$$

Let $\Delta L_s:=L_s-L_{s-}$ denote the jumps of L.

Lemma 2.3 (Big values of the process in case of bounded jumps). *We have*

$$\mathbf{P}\Big(\sup_{0\leq u\leq t} L_u 1\big\{\sup_{0\leq v\leq u}\Delta L_v\leq y\big\}\geq x\Big)\;\leq\;\Big(\frac{C\,t}{xy^{\kappa-1}}\Big)^{x/y},\quad t>0,\ x,y>0. \tag{2.17}$$

Proof. Since for $\tau>0$ fixed, $\{L_{\tau t}: t\geq 0\}$ equals in law to $\tau^{1/\kappa}L$, for the proof we may assume that $t=1$. Let $\{\xi_i: i\geq 1\}$ denote a family of independent copies of L_1. Set

$$W_{ns}\;:=\;\sum_{1\leq k\leq ns}\xi_k,\quad L_s^{(n)}\;:=\;n^{-1/\kappa}W_{ns},\quad 0\leq s\leq 1,\ n\geq 1. \tag{2.18}$$

Denote by $D_{[0,1]}$ the Skorohod space of càdlàg functions $f:[0,1]\to\mathsf{R}$. For fixed $y>0$, let $H: D_{[0,1]}\mapsto\mathsf{R}$ be defined by

$$H(f)\;=\;\sup_{0\leq u\leq 1} f(u)\,1\Big\{\sup_{0\leq v\leq u}\Delta f(v)\leq y\Big\},\quad f\in D_{[0,1]}. \tag{2.19}$$

It is easy to verify that H is continuous on the set $D_{[0,1]} \setminus J_y$, where $J_y := \{f \in D_{[0,1]} : \Delta f(v) = y \text{ for some } v \in [0,1]\}$. Since $\mathbf{P}(L \in J_y) = 0$, from the invariance principle for $L^{(n)}$ we conclude that

$$\mathbf{P}\big(H(L) \geq x\big) \;=\; \lim_{n\uparrow\infty} \mathbf{P}\big(H(L^{(n)}) \geq x\big), \quad x > 0. \tag{2.20}$$

Consequently, the lemma will be proved if we show that

$$\begin{aligned} &\mathbf{P}\Big(\sup_{0\leq u\leq 1} W_{nu} 1\big\{\max_{1\leq k\leq nu} \xi_k \leq y n^{1/\kappa}\big\} \geq x n^{1/\kappa}\Big) \\ &\quad\leq \Big(\frac{C}{x y^{\kappa-1}}\Big)^{x/y}, \quad x, y > 0,\ n \geq 1. \end{aligned} \tag{2.21}$$

To this end, for fixed $y', h \geq 0$, we consider the sequence

$$\Lambda_0 := 1, \quad \Lambda_n \;:=\; \mathrm{e}^{hW_n} 1\big\{\max_{1\leq k\leq n} \xi_k \leq y'\big\}, \quad n \geq 1. \tag{2.22}$$

It is easy to see that

$$\mathbf{E}\{\Lambda_{n+1} \,|\, \Lambda_n = \mathrm{e}^{hu}\} \;=\; \mathrm{e}^{hu}\, \mathbf{E}\{\mathrm{e}^{hL_1};\, L_1 \leq y'\} \quad \text{for all } u \in \mathsf{R} \tag{2.23}$$

and that

$$\mathbf{E}\{\Lambda_{n+1} \,|\, \Lambda_n = 0\} \;=\; 0. \tag{2.24}$$

In other words,

$$\mathbf{E}\{\Lambda_{n+1} \,|\, \Lambda_n\} \;=\; \Lambda_n\, \mathbf{E}\{\mathrm{e}^{hL_1};\, L_1 \leq y'\}. \tag{2.25}$$

This means that $\{\Lambda_n : n \geq 1\}$ is a supermartingale (submartingale) if h satisfies $\mathbf{E}\{\mathrm{e}^{hL_1};\, L_1 \leq y'\} \leq 1$ (respectively $\mathbf{E}\{\mathrm{e}^{hL_1}; L_1 \leq y'\} \geq 1$). If Λ_n is a submartingale, then by Doob's inequality,

$$\mathbf{P}\big(\max_{1\leq k\leq n} \Lambda_k \geq \mathrm{e}^{hx'}\big) \;\leq\; \mathrm{e}^{-hx'}\, \mathbf{E}\Lambda_n\,, \quad x' > 0. \tag{2.26}$$

But if Λ_n is a supermartingale, then

$$\mathbf{P}\big(\max_{1\leq k\leq n} \Lambda_k \geq \mathrm{e}^{hx'}\big) \;\leq\; \mathrm{e}^{-hx'}\, \mathbf{E}\Lambda_0 \;=\; \mathrm{e}^{-hx'}, \quad x' > 0. \tag{2.27}$$

From these inequalities and (2.25) we get

$$\mathbf{P}\big(\max_{1\leq k\leq n} \Lambda_k \geq \mathrm{e}^{hx'}\big) \;\leq\; \mathrm{e}^{-hx'} \max\Big\{1, \big(\mathbf{E}\{\mathrm{e}^{hL_1};\, L_1 \leq y'\}\big)^n\Big\}. \tag{2.28}$$

It was proved by Fuk and Nagaev [5] that

$$\mathbf{E}\{\mathrm{e}^{hL_1};\, L_1 \leq y'\} \;\leq\; 1 + h\mathbf{E}\{L_1\,;\, L_1 \leq y'\} + \frac{\mathrm{e}^{hy'} - 1 - hy'}{(y')^2}\, V(y'), \quad h, y' > 0,$$

where $V(y') := \int_{-\infty}^{y'} \mathbf{P}(L_1 \in \mathrm{d}u)\, u^2 > 0$. Noting that the assumption $\mathbf{E}L_1 = 0$ yields that $\mathbf{E}\{L_1\,;\, L_1 \leq y'\} \leq 0$, we obtain

$$\mathbf{E}\{\mathrm{e}^{hL_1};\, L_1 \leq y'\} \;\leq\; 1 + \frac{\mathrm{e}^{hy'} - 1 - hy'}{(y')^2}\, V(y'), \quad h, y' > 0. \tag{2.29}$$

Now note that

$$\begin{aligned} \Big\{\max_{1\leq k\leq n} W_k 1\big\{\max_{1\leq i\leq k} \xi_i \leq y'\big\} \geq x'\Big\} \;&=\; \Big\{\max_{1\leq k\leq n} \mathrm{e}^{hW_k}\, 1\big\{\max_{1\leq i\leq k} \xi_i \leq y'\big\} \geq \mathrm{e}^{hx'}\Big\} \\ &=\; \big\{\max_{1\leq k\leq n} \Lambda_k \geq \mathrm{e}^{hx'}\big\}. \end{aligned} \tag{2.30}$$

Thus, combining (2.30), (2.29), and (2.28), we get

$$\mathbf{P}\Big(\max_{1\le k\le n} W_k \mathbf{1}\{\max_{1\le i\le k}\xi_i \le y'\} \ge x'\Big) \le \mathbf{P}\big(\max_{1\le k\le n}\Lambda_k \ge \mathrm{e}^{hx'}\big)$$
$$\le \exp\Big\{-hx' + \frac{\mathrm{e}^{hy'}-1-hy'}{(y')^2}\, n\, V(y')\Big\}. \tag{2.31}$$

Choosing $h := (y')^{-1}\log\big(1 + x'y'/n\,V(y')\big)$, we arrive, after some elementary calculations, at the bound

$$\mathbf{P}\Big(\max_{1\le k\le n} W_k \mathbf{1}\{\max_{1\le i\le k}\xi_i \le y'\} \ge x'\Big) \le \Big(\frac{\mathrm{e}\, n\, V(y')}{x'y'}\Big)^{x'/y'}, \quad x', y' > 0. \tag{2.32}$$

Since $\mathbf{P}(L_1 > u) \sim C\, u^{-\kappa}$ as $u \uparrow \infty$, we have $V(y') \le C\,(y')^{2-\kappa}$ for all $y' > 0$. Therefore,

$$\mathbf{P}\Big(\max_{1\le k\le n} W_k \mathbf{1}\{\max_{1\le i\le k}\xi_i \le y'\} \ge x'\Big) \le \Big(\frac{Cn}{x'(y')^{\kappa-1}}\Big)^{x'/y'}, \quad x', y' > 0. \tag{2.33}$$

Choosing finally $x' = xn^{1/\kappa}$, $y' = yn^{1/\kappa}$, we get (2.21) from (2.33). Thus, the proof of the lemma is complete. □

Lemma 2.4 (**Small process values**). *There is a constant* c_κ *such that*

$$\mathbf{P}\Big(\inf_{u\le t} L_u < -x\Big) \le \exp\Big\{-c_\kappa\,\frac{x^{\kappa/(\kappa-1)}}{t^{1/(\kappa-1)}}\Big\}, \quad x, t > 0. \tag{2.34}$$

Proof. It is easy to see that for all $h > 0$,

$$\mathbf{P}\Big(\inf_{u\le t} L_u < -x\Big) = \mathbf{P}\Big(\sup_{s\le t}\mathrm{e}^{-hL_u} > \mathrm{e}^{hx}\Big). \tag{2.35}$$

Applying Doob's inequality to the submartingale $t \mapsto \mathrm{e}^{-hL_t}$, we obtain

$$\mathbf{P}\Big(\inf_{u\le t} L_u < -x\Big) \le \mathrm{e}^{-hx}\,\mathbf{E}\,\mathrm{e}^{-hL_t}. \tag{2.36}$$

Taking into account definition (2.15), we have

$$\mathbf{P}\Big(\inf_{u\le t} L_u < -x\Big) \le \exp\Big\{-hx + th^\kappa\Big\}. \tag{2.37}$$

Minimizing the function $h \mapsto -hx + th^\kappa$, we get the inequality in the lemma with $c_\kappa = (\kappa-1)/(\kappa)^{\kappa/(\kappa-1)}$. □

2.3. **Local Hölder continuity with some index.** In this subsection we prove Theorem 1.1(a) for parameters $\beta \ge \frac{\alpha-1}{2}$ (see Remark 2.10), whereas for parameters $\beta < \frac{\alpha-1}{2}$ we obtain local Hölder continuity only with non-optimal bound on indexes. We use the Kolmogorov criterion for local Hölder continuity to get these results. The proof of Theorem 1.1(a) for parameters $\beta < \frac{\alpha-1}{2}$ will be finished in Section 3.

Fix $t > 0$, $\mu \in \mathcal{M}_\mathrm{f}$, and suppose $\alpha > 1+\beta$. Since our theorem is trivially valid for $\mu = 0$, from now on we everywhere suppose that $\mu \ne 0$. Since we are dealing with the case $d = 1$, the random measure X_t is a.s. absolutely continuous. Recall decomposition (1.12).

Clearly, the deterministic function Z_t^1 is Lipschitz continuous, by Lemma 2.1. Next we turn to the random function Z_t^3.

Lemma 2.5 (**Hölder continuity of** Z_t^3). *With probability one,* Z_t^3 *is Hölder continuous of each index* $\eta < \alpha - 1$.

Proof. From Lemma 2.1 we get for fixed $\delta \in (0, \alpha - 1)$,

$$\left|p^{\alpha}_{t-s}(x_1 - y) - p^{\alpha}_{t-s}(x_2 - y)\right| \leq C\, \frac{|x_1 - x_2|^{\delta}}{(t-s)^{(\delta+1)/\alpha}}\,, \quad t > s > 0,\ \ x_1, x_2, y \in \mathsf{R}.$$

Therefore,

$$\begin{aligned}
\left|Z^3_t(x_1) - Z^3_t(x_2)\right| &\leq |a| \int_0^t \mathrm{d}s \int_{\mathsf{R}} X_s(\mathrm{d}x)\ \left|p^{\alpha}_{t-s}(x_1 - y) - p^{\alpha}_{t-s}(x_2 - y)\right| \\
&\leq C\left(\sup_{s\leq t} X_s(\mathsf{R})\right) |x_1 - x_2|^{\delta} \int_0^t \mathrm{d}s\ (t-s)^{-(\delta+1)/\alpha} \\
&\leq C\, \frac{\alpha}{\alpha - 1 - \delta} \left(\sup_{s\leq t} X_s(\mathsf{R})\right) |x_1 - x_2|^{\delta}, \quad x_1, x_2 \in \mathsf{R}. \qquad (2.38)
\end{aligned}$$

Consequently,

$$\sup_{x_1 \neq x_2} \frac{\left|Z^3_t(x_1) - Z^3_t(x_2)\right|}{|x_1 - x_2|^{\delta}} < \infty \ \text{ a.s.,} \qquad (2.39)$$

and the proof is complete. □

Our main work concerns Z^2_t.

Lemma 2.6 (q**-norm**). *For each* $\theta \in (1+\beta, 2)$ *and* $q \in (1, 1+\beta)$,

$$\begin{aligned}
&\mathbf{E}\left|Z^2_t(x_1) - Z^2_t(x_2)\right|^q \\
&\leq C\left[\left(\int_0^t \mathrm{d}s \int_{\mathsf{R}} S^{\alpha}_s \mu(\mathrm{d}y) \left|p^{\alpha}_{t-s}(x_1 - y) - p^{\alpha}_{t-s}(x_2 - y)\right|^{\theta}\right)^{q/\theta}\right. \\
&\qquad \left. + \int_0^t \mathrm{d}s \int_{\mathsf{R}} S^{\alpha}_s \mu(\mathrm{d}y) \left|p^{\alpha}_{t-s}(x_1 - y) - p^{\alpha}_{t-s}(x_2 - y)\right|^{q}\right], \quad x_1, x_2 \in \mathsf{R}. \qquad (2.40)
\end{aligned}$$

The proof can be done similarly to the proof of inequality (3.1) in [6].

Corollary 2.7 (q**-norm**). *For each* $\theta \in (1+\beta, 2)$, $q \in (1, 1+\beta)$ *and* $\delta > 0$ *satisfying* $\delta < \min\{1, (1+\alpha-\theta)/\theta, (1+\alpha-q)/q\}$,

$$\mathbf{E}\left|Z^2_t(x_1) - Z^2_t(x_2)\right|^q \leq C\,|x_1 - x_2|^{\delta q}, \quad x_1, x_2 \in \mathsf{R}. \qquad (2.41)$$

Proof. For every $\varepsilon \in (1, 1+\alpha)$,

$$\begin{aligned}
&\int_0^t \mathrm{d}s \int_{\mathsf{R}} S^{\alpha}_s \mu(\mathrm{d}y) \left|p^{\alpha}_{t-s}(x_1 - y) - p^{\alpha}_{t-s}(x_2 - y)\right|^{\varepsilon} \\
&= \int_{\mathsf{R}} \mu(\mathrm{d}z) \int_0^t \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}y\ p^{\alpha}_s(y - z) \left|p^{\alpha}_{t-s}(x_1 - z) - p^{\alpha}_{t-s}(x_2 - z)\right|^{\varepsilon} \\
&= \int_{\mathsf{R}} \mu(\mathrm{d}z) \int_0^t \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}y\ p^{\alpha}_s(y) \left|p^{\alpha}_{t-s}(x_1 - z - y) - p^{\alpha}_{t-s}(x_2 - z - y)\right|^{\varepsilon}. \qquad (2.42)
\end{aligned}$$

Using Lemma 2.2, we get for every positive $\delta < \min\{1, (1+\alpha-\varepsilon)/\varepsilon\}$,

$$\begin{aligned}
&\int_0^t \mathrm{d}s \int_{\mathsf{R}} S^{\alpha}_s \mu(\mathrm{d}y) \left|p^{\alpha}_{t-s}(x_1 - y) - p^{\alpha}_{t-s}(x_2 - y)\right|^{\varepsilon} \\
&\leq C\,|x_1 - x_2|^{\delta\varepsilon} \int_{\mathsf{R}} \mu(\mathrm{d}z) \left(p^{\alpha}_t\big((x_1 - z)/2\big) + p^{\alpha}_t\big((x_2 - z)/2\big)\right) \leq C\,|x_1 - x_2|^{\delta\varepsilon},
\end{aligned}$$

since μ, t are fixed. Applying this bound to both summands at the right hand side of (2.40) finishes the proof of the lemma. □

Using our Lemma 2.5, Corollary 2.7, and Corollary 1.2 of Walsh [17], one can easily get

Corollary 2.8 (**Local boundedness of** X_t)**.** *If* $K \subset \mathsf{R}$ *is a compact, then*

$$\mathbf{E} \sup_{x \in K} X_t(x) < \infty. \tag{2.43}$$

Furthermore, Corollary 2.7 allows us to prove the following result:

Proposition 2.9 (**Local Hölder continuity of** Z_t^2)**.** *With probability one,* Z_t^2 *has a version which is locally Hölder continuous of all orders* $\eta > 0$ *satisfying*

$$\eta \;<\; \eta_{\mathrm{c}}' \;:=\; \begin{cases} \frac{\alpha}{1+\beta} - 1, & \text{if } \beta \geq (\alpha-1)/2, \\ \frac{\beta}{1+\beta}, & \text{if } \beta \leq (\alpha-1)/2. \end{cases} \tag{2.44}$$

Proof. Let θ, q and δ satisfy the conditions in Corollary 2.7. Then almost surely Z_t^2 has a version which is locally Hölder continuous of all orders smaller than $\delta - 1/q$, cf. [17, Corollary 1.2].

Let $\varepsilon > 0$ satisfy $\varepsilon < 1-\beta$ and $\varepsilon < \beta$. Then $\theta = \theta_\varepsilon := 1+\beta+\varepsilon$ and $q = q_\varepsilon := 1+\beta-\varepsilon$ are in the range of parameters we are just considering. Moreover, the condition $\delta < \min\bigl\{1, (1+\alpha-\theta)/\theta, (1+\alpha-q)/q\bigr\}$ reads as

$$\delta \;<\; \min\Bigl\{1,\, \frac{\alpha-\beta-\varepsilon}{1+\beta+\varepsilon}\,,\, \frac{\alpha-\beta+\varepsilon}{1+\beta-\varepsilon}\Bigr\} \;=:\; f(\varepsilon). \tag{2.45}$$

Hence, for all sufficiently small $\varepsilon > 0$ we can choose $\delta = \delta_\varepsilon := f(\varepsilon) - \varepsilon$. Thus, Z_t^2 has a version which is locally Hölder continuous of all orders smaller than $\delta_\varepsilon - 1/q_\varepsilon$ for this choice of $\theta_\varepsilon, q_\varepsilon, \delta_\varepsilon$. Now

$$\delta_\varepsilon - \frac{1}{q_\varepsilon} \underset{\varepsilon \downarrow 0}{\longrightarrow} \min\Bigl\{1,\, \frac{\alpha-\beta}{1+\beta}\,,\, \frac{\alpha-\beta}{1+\beta}\Bigr\} - \frac{1}{1+\beta} \;=\; \min\Bigl\{1,\, \frac{\beta}{1+\beta}\,,\, \frac{\alpha-\beta-1}{1+\beta}\Bigr\}.$$

where this limit coincides with the claimed value of η_{c}', finishing the proof. □

Remark 2.10 (**Proof of Theorem 1.1(a) for** $\beta \geq \frac{\alpha-1}{2}$)**.** By Lemma 2.5 and Proposition 2.9, the proof of Theorem 1.1(a) is finished for $\beta \geq \frac{\alpha-1}{2}$. ◇

2.4. **Further estimates.** We continue to fix $t > 0$, $\mu \in \mathcal{M}_{\mathrm{f}} \backslash \{0\}$, and to suppose $\alpha > 1+\beta$.

Lemma 2.11 (**Local boundedness of uniformly smeared out density**)**.** *Fix a non-empty compact* $K \subset \mathsf{R}$ *and a constant* $c \geq 1$. *Then*

$$V \;:=\; V_t^c(K) \;:=\; \sup_{0 \leq s \leq t,\, x \in K} S^\alpha_{c(t-s)} X_s\,(x) \;<\; \infty \quad \textit{almost surely.} \tag{2.46}$$

Proof. Assume that the statement of the lemma does not hold, i.e. there exists an event A of positive probability such that $\sup_{0 \leq s \leq t,\, x \in K} S^\alpha_{c\,(t-s)} X_s\,(x) = \infty$ for every $\omega \in A$. Let $n \geq 1$. Put

$$\tau_n \;:=\; \begin{cases} \inf\Bigl\{s < t: \text{ there exists } x \in K \text{ such that } S^\alpha_{c\,(t-s)} X_s\,(x) > n\Bigr\}, & \omega \in A, \\ t, \quad \omega \in A^{\mathrm{c}}. \end{cases}$$

If $\omega \in A$, choose $x_n = x_n(\omega) \in K$ such that $S^{\alpha}_{c\,(t-\tau_n)} X_{\tau_n}(x_n) > n$, whereas if $\omega \in A^{\mathrm{c}}$, take any $x_n = x_n(\omega) \in K$. Using the strong Markov property gives

$$\begin{aligned}(2.47)\qquad \mathbf{E} S^{\alpha}_{(c-1)(t-\tau_n)} X_t\,(x_n) \;&=\; \mathbf{E}\mathbf{E}\big[S^{\alpha}_{(c-1)(t-\tau_n)} X_t\,(x_n)\,\big|\,\mathcal{F}_{\tau_n}\big] \\ &=\; \mathbf{E}\,\mathrm{e}^{a(t-\tau_n)} S^{\alpha}_{(c-1)(t-\tau_n)} S^{\alpha}_{(t-\tau_n)} X_{\tau_n}(x_n) \;\geq\; \mathrm{e}^{-|a|t}\, \mathbf{E} S^{\alpha}_{c\,(t-\tau_n)} X_{\tau_n}(x_n).\end{aligned}$$

From the definition of (τ_n, x_n) we get

$$(2.48)\qquad \mathbf{E} S^{\alpha}_{c\,(t-\tau_n)} X_{\tau_n}(x_n) \;\geq\; n\, \mathbf{P}(A) \to \infty \ \text{ as } \ n \uparrow \infty.$$

In order to get a contradiction, we want to prove boundedness in n of the expectation in (2.47). If $c = 1$, then

$$(2.49)\qquad \mathbf{E} X_t(x_n) \;\leq\; \mathbf{E} \sup_{x\in K} X_t(x) \;<\; \infty,$$

the last step by Corollary 2.8. Now suppose $c > 1$. Choosing a compact $K_1 \supset K$ satisfying $\operatorname{dist}\big(K, (K_1)^{\mathrm{c}}\big) \geq 1$, we have

$$\begin{aligned}&\mathbf{E} S^{\alpha}_{(c-1)(t-\tau_n)} X_t\,(x_n) \\ &= \mathbf{E} \int_{K_1} \mathrm{d}y\ X_t(y)\, p^{\alpha}_{(c-1)(t-\tau_n)}(x_n - y) + \mathbf{E} \int_{(K_1)^{\mathrm{c}}} \mathrm{d}y\ X_t(y)\, p^{\alpha}_{(c-1)(t-\tau_n)}(x_n - y) \\ &\leq \mathbf{E} \sup_{y\in K_1} X_t(y) + \mathbf{E} X_t(\mathsf{R}) \sup_{y\in (K_1)^{\mathrm{c}},\, x\in K,\, 0\leq s\leq t} p^{\alpha}_{(c-1)s}(x - y).\end{aligned}$$

By our choice of K_1 we obtain the bound

$$(2.50)\qquad \mathbf{E} S^{\alpha}_{(c-1)(t-\tau_n)} X_t\,(x_n) \;\leq\; \mathbf{E} \sup_{y\in K_1} X_t(y) + C \;=\; C,$$

the last step by Corollary 2.8. Altogether, (2.47) is bounded in n, and the proof is finished. □

Lemma 2.12 (Randomly weighted kernel increments). *Fix* $\theta \in [1, 1+\alpha)$, $\delta \in [0, 1]$ *with* $\delta < (1 + \alpha - \theta)/\theta$, *and a non-empty compact* $K \subset \mathsf{R}$. *Then*

$$\begin{aligned}&\int_0^t \mathrm{d}s \int_{\mathsf{R}} X_s(\mathrm{d}y)\, \big|p^{\alpha}_{t-s}(x_1 - y) - p^{\alpha}_{t-s}(x_2 - y)\big|^{\theta} \\ (2.51)\qquad &\leq\; C\, V\, |x_1 - x_2|^{\delta\theta}, \quad x_1, x_2 \in K, \ \textit{a.s.},\end{aligned}$$

with $V = V_t^{2^{\alpha}}(K)$ *from Lemma* 2.11.

Proof. Using (2.12) gives

$$\begin{aligned}&\int_0^t \mathrm{d}s \int_{\mathsf{R}} X_s(\mathrm{d}y)\, \big|p^{\alpha}_{t-s}(x_1 - y) - p^{\alpha}_{t-s}(x_2 - y)\big|^{\theta} \;\leq\; C\, |x_1 - x_2|^{\delta\theta} \times \\ &\times \int_0^t \mathrm{d}s\ (t-s)^{-(\delta\theta+\theta-1)/\alpha} \int_{\mathsf{R}} X_s(\mathrm{d}y)\, \Big(p^{\alpha}_{t-s}\big((x_1 - y)/2\big) + p^{\alpha}_{t-s}\big((x_2 - y)/2\big)\Big),\end{aligned}$$

uniformly in $x_1, x_2 \in \mathsf{R}$. Recalling the scaling property of p^{α}, we get

$$\begin{aligned}&\int_0^t \mathrm{d}s \int_{\mathsf{R}} X_s(\mathrm{d}y)\, \big|p^{\alpha}_{t-s}(x_1 - y) - p^{\alpha}_{t-s}(x_2 - y)\big|^{\theta} \\ &\leq\; C\, |x_1 - x_2|^{\delta\theta} \int_0^t \mathrm{d}s\ (t-s)^{-(\delta\theta+\theta-1)/\alpha} \Big(S^{\alpha}_{2^{\alpha}(t-s)} X_s(x_1) + S^{\alpha}_{2^{\alpha}(t-s)} X_s(x_2)\Big).\end{aligned}$$

We complete the proof by applying Lemma 2.11. □

Remark 2.13 (**Lipschitz continuity of Z_t^3**)**.** Using Lemma 2.12 with $\theta = 1 = \delta$, we see that Z_t^3 is in fact Lipschitz continuous. ◇

Let $\Delta X_s := X_s - X_{s-}$ denote the jumps of the measure-valued process X.

Lemma 2.14 (**Total jump mass**)**.** *Let $\varepsilon > 0$ and $\gamma \in \bigl(0,(1+\beta)^{-1}\bigr)$. There exists a constant $c_{(2.52)} = c_{(2.52)}(\varepsilon,\gamma)$ such that*

$$\mathbf{P}\Bigl(|\Delta X_s| > c_{(2.52)}\,(t-s)^{(1+\beta)^{-1}-\gamma} \text{ for some } s<t\Bigr) \le \varepsilon. \tag{2.52}$$

Proof. Recall the random measure N from Lemma 1.5(a). For any $c>0$, set

$$Y_0 := N\Bigl([0,2^{-1}t)\times\mathsf{R}\times\bigl(c\,2^{-\lambda}t^{\lambda},\infty\bigr)\Bigr), \tag{2.53}$$

$$Y_n := N\Bigl(\bigl[(1-2^{-n})t,(1-2^{-n-1})t\bigr)\times\mathsf{R}\times\bigl(c\,2^{-\lambda(n+1)}t^{\lambda},\infty\bigr)\Bigr),\quad n\ge 1, \tag{2.54}$$

where $\lambda := (1+\beta)^{-1}-\gamma$. It is easy to see that

$$\mathbf{P}\Bigl(|\Delta X_s| > c\,(t-s)^{\lambda} \text{ for some } s<t\Bigr) \le \mathbf{P}\Bigl(\sum_{n=0}^{\infty} Y_n \ge 1\Bigr) \le \sum_{n=0}^{\infty}\mathbf{E}Y_n\,, \tag{2.55}$$

where in the last step we have used the classical Markov inequality. From the formula for the compensator $\hat{N}$ of N in Lemma 1.5(b),

$$\mathbf{E}Y_n = \varrho\int_{(1-2^{-n})t}^{(1-2^{-n-1})t}\mathrm{d}s\;\mathbf{E}X_s(\mathsf{R})\int_{c\,2^{-\lambda(n+1)}t^{\lambda}}^{\infty}\mathrm{d}r\;r^{-2-\beta},\quad n\ge 1. \tag{2.56}$$

Now

$$\mathbf{E}X_s(\mathsf{R}) = X_0(\mathsf{R})\,\mathrm{e}^{as} \le |\mu|\,\mathrm{e}^{|a|t} =: c_{(2.57)}. \tag{2.57}$$

Consequently,

$$\mathbf{E}Y_n \le \frac{\varrho}{1+\beta}\,c_{(2.57)}c^{-1-\beta}\,2^{-(n+1)\gamma(1+\beta)}\,t^{\gamma(1+\beta)}. \tag{2.58}$$

Analogous calculations show that (2.58) remains valid also in the case $n=0$. Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbf{E}Y_n &\le \frac{\varrho}{1+\beta}\,c_{(2.57)}c^{-1-\beta}\,t^{\gamma(1+\beta)}\sum_{n=0}^{\infty}2^{-(n+1)\gamma(1+\beta)}\\ &= \frac{\varrho}{1+\beta}\,c_{(2.57)}c^{-1-\beta}\,t^{\gamma(1+\beta)}\,\frac{2^{-\gamma(1+\beta)}}{1-2^{-\gamma(1+\beta)}}\,. \end{aligned}\tag{2.59}$$

Choosing $c = c_{(2.52)}$ such that the expression in (2.59) equals ε, and combining with (2.55), the proof is complete. □

2.5. **Representation as time-changed stable process.** We return to general $t>0$. Recall the martingale measure M related to the martingale in Lemma 1.5(c) and Lemma 1.6.

Lemma 2.15 (**Representation as time-changed stable process**)**.** *Suppose $p\in(1+\beta,2)$ and let $\psi\in\mathcal{L}^p_{\mathrm{loc}}(\mu)$ with $\psi\ge 0$. Then there exists a spectrally positive $(1+\beta)$-stable process $\{L_t:\,t\ge 0\}$ such that*

$$Z_t(\psi) := \int_{(0,t]\times\mathsf{R}} M\bigl(\mathrm{d}(s,y)\bigr)\,\psi(s,y) = L_{T(t)}\,,\quad t\ge 0, \tag{2.60}$$

where $T(t) := \int_0^t \mathrm{d}s \int_{\mathsf{R}} X_s(\mathrm{d}y)\,\big(\psi(s,y)\big)^{1+\beta}$.

Proof. Let us write Itô's formula for $\mathrm{e}^{-Z_t(\psi)}$:

$$
\begin{aligned}
\mathrm{e}^{-Z_t(\psi)} - 1 \;&=\; \text{local martingale} \\
&+ \varrho \int_0^t \mathrm{d}s\; \mathrm{e}^{-Z_s(\psi)} \int_{\mathsf{R}} X_s(\mathrm{d}y) \int_0^\infty \mathrm{d}r\, \big(\mathrm{e}^{-r\psi(s,y)} - 1 + r\,\psi(s,y)\big)\, r^{-2-\beta}.
\end{aligned} \tag{2.61}
$$

Define $\tau(t) := T^{-1}(t)$ and put $t^* := \inf\{t : \tau(t) = \infty\}$. Then it is easy to get for every $v > 0$,

$$
\begin{aligned}
\mathrm{e}^{-vZ_{\tau(t)}(\psi)} \;&=\; 1 + \int_0^t \mathrm{d}s\; \mathrm{e}^{-vZ_{\tau(s)}(\psi)}\, \frac{X_{\tau(s)}\big(v^{1+\beta}\psi^{1+\beta}(s,\cdot)\big)}{X_{\tau(s)}\big(\psi^{1+\beta}(s,\cdot)\big)} + \text{loc. mart.} \\
&=\; 1 + \int_0^t \mathrm{d}s\; \mathrm{e}^{-vZ_{\tau(s)}(\psi)}\, v^{1+\beta} + \text{loc. mart.}, \quad t \le t^*.
\end{aligned} \tag{2.62}
$$

Since the local martingale is bounded, it is in fact a martingale. Let $\tilde{L}$ denote a spectrally positive process of index $1+\beta$, independent of X. Define

$$
L_t \;:=\; \begin{cases} Z_{\tau(t)}(\psi), & t \le t^*, \\ Z_{\tau(t^*)}(\psi) + \tilde{L}_{t-t^*}\,, & t > t^* \text{ (if } t^* < \infty). \end{cases} \tag{2.63}
$$

Then we can easily get that L satisfies the martingale problem (2.16) with κ replaced by $1+\beta$. Now by time change back we obtain

$$
Z_t(\psi) = \tilde{L}_{T(t)} = L_{T(t)}\,, \tag{2.64}
$$

finishing the proof. □

3. Local Hölder continuity: proof of Theorem 1.1(a)

We continue to assume that $d = 1$, and that $t > 0$ and $\mu \in \mathcal{M}_{\mathrm{f}} \backslash \{0\}$ are fixed. For $\beta \ge (\alpha-1)/2$ the desired existence of a locally Hölder continuous version of Z_t^2 of required orders is already proved in Proposition 2.9. Therefore, in what follows we shall consider the complementary case $\beta < (\alpha-1)/2$. Fix any compact K and $x_1 < x_2$ belonging to it. By definition (1.12) of Z_t^2,

$$
\begin{aligned}
Z_t^2(x_1) - Z_t^2(x_2) \;&=\; \int_{(0,t]\times\mathsf{R}} M\big(\mathrm{d}(s,y)\big)\, \big(p^\alpha_{t-s}(x_1-y) - p^\alpha_{t-s}(x_2-y)\big) \\
&=\; \int_{(0,t]\times\mathsf{R}} M\big(\mathrm{d}(s,y)\big)\, \varphi_+(s,y) - \int_{(0,t]\times\mathsf{R}} M\big(\mathrm{d}(s,y)\big)\, \varphi_-(s,y),
\end{aligned} \tag{3.1}
$$

where $\varphi_+(s,y)$ and $\varphi_-(s,y)$ are the positive and negative parts of $p^\alpha_{t-s}(x_1-y) - p^\alpha_{t-s}(x_2-y)$. It is easy to check that φ_+ and φ_- satisfy the assumptions in Lemma 2.15. Thus, there exist stable processes L^1 and L^2 such that

$$
Z_t^2(x_1) - Z_t^2(x_2) \;=\; L^1_{T_+} - L^2_{T_-}\,, \tag{3.2}
$$

where $T_\pm := \int_0^t \mathrm{d}s \int_{\mathsf{R}} X_s(\mathrm{d}y)\,\big(\varphi_\pm(s,y)\big)^{1+\beta}$.

Fix any $\varepsilon \in (0,1)$. According to Lemma 2.11, there exists a constant c_ε such that $\mathbf{P}(V \le c_\varepsilon) \ge 1-\varepsilon$, where $V = V_t^{2^\alpha}(K)$. Consider again $\gamma \in \big(0, (1+\beta)^{-1}\big)$ and set

$$
A^\varepsilon \;:=\; \Big\{|\Delta X_s| \le c_{(2.52)}\,(t-s)^{(1+\beta)^{-1}-\gamma} \text{ for all } s < t\Big\} \cap \{V \le c_\varepsilon\}. \tag{3.3}
$$

Evidently,

$$\mathbf{P}(A^{\varepsilon}) \geq 1-2\varepsilon. \tag{3.4}$$

Define $Z_t^{2,\varepsilon}(x) := Z_t^2(x)1(A^{\varepsilon})$. We first show that $Z_t^{2,\varepsilon}$ has a version which is locally Hölder continuous of all orders η smaller than η_{c}. It follows from (3.2) that

$$\begin{aligned} &\mathbf{P}\Big(\big|Z_t^{2,\varepsilon}(x_1)-Z_t^{2,\varepsilon}(x_2)\big| \geq 2\,|x_1-x_2|^{\eta}\Big) \\ &\leq\ \mathbf{P}\big(L^1_{T_+} \geq |x_1-x_2|^{\eta},\, A^{\varepsilon}\big)+\mathbf{P}\big(L^2_{T_-} \geq |x_1-x_2|^{\eta},\, A^{\varepsilon}\big). \end{aligned} \tag{3.5}$$

If the jumps of $M\big(\mathrm{d}(s,y)\big)$ do not exceed $c_{(2.52)}\,(t-s)^{(1+\beta)^{-1}-\gamma}$, then the jumps of the process $u\mapsto\int_{(0,u]\times\mathsf{R}}M\big(\mathrm{d}(s,y)\big)\,\varphi_{\pm}(s,y)$ are bounded by

$$c_{(2.52)}\,\sup_{s<t}(t-s)^{(1+\beta)^{-1}-\gamma}\,\sup_{y\in\mathsf{R}}\varphi_{\pm}(s,y). \tag{3.6}$$

Obviously,

$$\sup_{y\in\mathsf{R}}\varphi_{\pm}(s,y)\ \leq\ \sup_{y\in\mathsf{R}}\big|p^{\alpha}_{t-s}(x_1-y)-p^{\alpha}_{t-s}(x_2-y)\big|. \tag{3.7}$$

Assume additionally that $\gamma<\eta_{\mathrm{c}}/\alpha$. Using Lemma 2.1 with $\delta=\eta_{\mathrm{c}}-\alpha\gamma$ gives

$$\begin{aligned} &\sup_{y\in\mathsf{R}}\big|p^{\alpha}_{t-s}(x_1-y)-p^{\alpha}_{t-s}(x_2-y)\big| \\ &\leq\ C\,|x_1-x_2|^{\eta_{\mathrm{c}}-\alpha\gamma}\,(t-s)^{-\eta_{\mathrm{c}}/\alpha+\gamma}\,\sup_{z\in\mathsf{R}}p^{\alpha}_{t-s}(z) \\ &\leq\ C\,|x_1-x_2|^{\eta_{\mathrm{c}}-\alpha\gamma}\,(t-s)^{-\eta_{\mathrm{c}}/\alpha+\gamma}\,(t-s)^{-1/\alpha} \\ &=\ C\,|x_1-x_2|^{\eta_{\mathrm{c}}-\alpha\gamma}\,(t-s)^{-\frac{1}{1+\beta}+\gamma}. \end{aligned} \tag{3.8}$$

Combining (3.6) – (3.8), we see that all jumps of $u\mapsto\int_{(0,u]\times\mathsf{R}}M\big(\mathrm{d}(s,y)\big)\,\varphi_{\pm}(s,y)$ on the set A^{ε} are bounded by $c_{\varepsilon}\,|x_1-x_2|^{\eta_{\mathrm{c}}-\alpha\gamma}$ for some constant c_{ε}. Therefore,

$$\begin{aligned} &\mathbf{P}\big(L_{T_{\pm}} \geq |x_1-x_2|^{\eta},\, A^{\varepsilon}\big) \\ &=\ \mathbf{P}\Big(L_{T_{\pm}} \geq |x_1-x_2|^{\eta},\ \sup_{u<T_{\pm}}\Delta L_u \leq c_{\varepsilon}\,|x_1-x_2|^{\eta_{\mathrm{c}}-\alpha\gamma},\, A^{\varepsilon}\Big) \\ &=\ \mathbf{P}\bigg(\sup_{v\leq T_{\pm}}L_v 1\Big\{\sup_{u<v}\Delta L_u \leq c_{\varepsilon}\,|x_1-x_2|^{\eta_{\mathrm{c}}-\alpha\gamma}\Big\} \geq |x_1-x_2|^{\eta},\, A^{\varepsilon}\bigg). \end{aligned} \tag{3.9}$$

Since

$$T_{\pm}\ \leq\ \int_0^t \mathrm{d}s\int_{\mathsf{R}}X_s(\mathrm{d}y)\,\big|p^{\alpha}_{t-s}(x_1-y)-p^{\alpha}_{t-s}(x_2-y)\big|^{1+\beta}, \tag{3.10}$$

applying Lemma 2.12 with $\theta=1+\beta$ and $\delta=1$ (since $\beta<(\alpha-1)/2$), we get the bound

$$T_{\pm}\ \leq\ c_{\varepsilon}\,|x_1-x_2|^{1+\beta}\quad\text{on }\{V\leq c_{\varepsilon}\}. \tag{3.11}$$

Consequently,

$$\begin{aligned} &\mathbf{P}\big(L_{T_{\pm}} \geq |x_1-x_2|^{\eta},\, A^{\varepsilon}\big) \\ &\leq\ \mathbf{P}\Big(\sup_{v\leq c_{\varepsilon}|x_1-x_2|^{1+\beta}}L_v\,1\big\{\sup_{u<v}\Delta L_u \leq c_{\varepsilon}\,|x_1-x_2|^{\eta_{\mathrm{c}}-\alpha\gamma}\big\} \geq |x_1-x_2|^{\eta}\Big). \end{aligned} \tag{3.12}$$

Using Lemma 2.3 with $\kappa = 1+\beta$, $t = c_\varepsilon|x_1-x_2|^{1+\beta}$, $x = |x_1-x_2|^\eta$, and $y = c_\varepsilon\,|x_1-x_2|^{\eta_{\rm c}-\alpha\gamma}$, and noting that

$$(3.13)\qquad 1+\beta-\eta-\beta(\eta_{\rm c}-\alpha\gamma) \;=\; 2+2\beta-\alpha+(\eta_{\rm c}-\eta)+\beta\alpha\gamma \;>\; 2+2\beta-\alpha,$$

we obtain

$$(3.14)\qquad \mathbf{P}\big(L_{T_\pm}\geq |x_1-x_2|^\eta,\ A^\varepsilon\big) \;\leq\; \big(c_\varepsilon\,|x_1-x_2|^{(2\beta+2-\alpha)}\big)^{\left(c_\varepsilon^{-1}|x_1-x_2|^{\eta-\eta_{\rm c}+\alpha\gamma}\right)}.$$

Applying this bound with $\gamma=(\eta_{\rm c}-\eta)/2\alpha$ to the summands at the right hand side in (3.5), and noting that $2\beta+2-\alpha$ is also constant here, we have

$$(3.15)\ \mathbf{P}\Big(\big|Z_t^{2,\varepsilon}(x_1)-Z_t^{2,\varepsilon}(x_2)\big|\geq 2|x_1-x_2|^\eta\Big) \;\leq\; 2\big(c_\varepsilon\,|x_1-x_2|\big)^{\left(c_\varepsilon^{-1}|x_1-x_2|^{(\eta-\eta_{\rm c})/2}\right)}.$$

This inequality yields

$$(3.16)\qquad \mathbf{P}\Big(\big|Z_t^{2,\varepsilon}(x_1)-Z_t^{2,\varepsilon}(x_2)\big|\geq 2\,|x_1-x_2|^\eta\Big) \;\leq\; c_\varepsilon\,|x_1-x_2|^2.$$

Using standard arguments, we conclude that almost surely $Z_t^{2,\varepsilon}$ has a version which is locally Hölder continuous of all orders $\eta<\eta_{\rm c}$.

By an abuse of notation, from now on the symbol $Z_t^{2,\varepsilon}$ always refers to this continuous version. Consequently,

$$(3.17)\qquad \lim_{k\uparrow\infty}\mathbf{P}\bigg(\sup_{x_1,x_2\in K,\,x_1\neq x_2}\frac{\big|Z_t^{2,\varepsilon}(x_1)-Z_t^{2,\varepsilon}(x_2)\big|}{|x_1-x_2|^\eta}>k\bigg) \;=\; 0.$$

Combining this with the bound

$$(3.18)\qquad \mathbf{P}\bigg(\sup_{x_1,x_2\in K,\,x_1\neq x_2}\frac{\big|Z_t^{2}(x_1)-Z_t^{2}(x_2)\big|}{|x_1-x_2|^\eta}>k\bigg)$$
$$\leq \mathbf{P}\bigg(\sup_{x_1,x_2\in K,\,x_1\neq x_2}\frac{\big|Z_t^{2,\varepsilon}(x_1)-Z_t^{2,\varepsilon}(x_2)\big|}{|x_1-x_2|^\eta}>k,\ \ A^\varepsilon\bigg)+\mathbf{P}(A^{\varepsilon,{\rm c}})$$

(with $A^{\varepsilon,{\rm c}}$ denoting the complement of A^ε) gives

$$(3.19)\qquad \limsup_{k\uparrow\infty}\mathbf{P}\bigg(\sup_{x_1,x_2\in K,\,x_1\neq x_2}\frac{\big|Z_t^{2}(x_1)-Z_t^{2}(x_2)\big|}{|x_1-x_2|^\eta}>k\bigg) \;\leq\; 2\varepsilon.$$

Since ε may be arbitrarily small, this immediately implies

$$(3.20)\qquad \sup_{x_1,x_2\in K,\,x_1\neq x_2}\frac{\big|Z_t^{2}(x_1)-Z_t^{2}(x_2)\big|}{|x_1-x_2|^\eta}<\infty\quad\text{almost surely.}$$

This is the desired local Hölder continuity of Z_t^2, for all $\eta<\eta_{\rm c}$. Because $\eta_{\rm c}<\alpha-1$, together with Lemma 2.5 the proof of Theorem 1.1(a) is complete. □

4. Local unboundedness: proof of Theorem 1.1(c)

The proof goes along the lines of the proof of Theorem 1.1(b) of [13]. In this section, suppose $d>1$ or $\alpha\leq 1+\beta$. Recall that $t>0$ and $X_0=\mu\in\mathcal{M}_{\rm f}\backslash\{0\}$ are fixed. We want to verify that for each version of the density function X_t the property

$$(4.1)\qquad \|X_t\|_B=\infty\ \ \mathbf{P}\text{-a.s. on the event }\ \big\{X_t(B)>0\big\}$$

holds whenever B is a fixed open ball in R^d. Then the claim of Theorem 1.1(c) follows as in the proof of Theorem 1.2 in [13]. We thus fix such B.

Lemma 4.1 (**Path continuity at fixed times**). *For the fixed* $t>0$,

$$\lim_{s\to t} X_s(B) \;=\; X_t(B) \;\; a.s. \tag{4.2}$$

Proof. Since t is fixed, X is continuous at t with probability 1. The statement of the lemma is then an easy consequence of the fact that $X_t(\mathrm{d}x)$ is absolutely continuous and that the ball B is a Lebesgue-continuity set. □

Lemma 4.2 (**Explosion**). *Let* $f:(0,t)\to(0,\infty)$ *be measurable such that*

$$\int_{t-\delta}^{t} \mathrm{d}s\; f(t-s) = \infty \;\; \textit{for all sufficiently small } \delta\in(0,t). \tag{4.3}$$

Then for these δ,

$$\int_{t-\delta}^{t} \mathrm{d}s\; X_s(B) f(t-s) \;=\; \infty \quad \mathbf{P}\textit{-a.s. on the event } \big\{X_t(B)>0\big\}. \tag{4.4}$$

Proof. Fix δ as in the lemma. Fix also ω such that $X_t(B)>0$ and $X_s(B)\to X_t(B)$ as $s\uparrow t$. For this ω, there is an $\varepsilon\in(0,\delta)$ such that $X_s(B)>\varepsilon$ for all $s\in(t-\varepsilon,t)$. Hence

$$\int_{t-\delta}^{t} \mathrm{d}s\; X_s(B) f(t-s) \;\geq\; \varepsilon \int_{t-\varepsilon}^{t} \mathrm{d}s\; f(t-s) \;=\; \infty, \tag{4.5}$$

and we are done. □

Set

$$\vartheta \;:=\; \frac{1}{1+\beta}\,, \tag{4.6}$$

and for $\varepsilon\in(0,t)$ let $\tau_\varepsilon(B)$ denote the infimum over the set

$$\Big\{ s\in(t-\varepsilon,t):\; |\Delta X_s|(B) > (t-s)^{\vartheta}\, \log^{\vartheta}\Big(\frac{1}{t-s}\Big)\Big\} \tag{4.7}$$

if this set is non-empty, and put it ∞ otherwise.

Lemma 4.3 (**Existence of big jumps**). *For* $\varepsilon\in(0,t)$ *and the open ball* B,

$$\mathbf{P}(\tau_\varepsilon(B)=\infty) \;\leq\; \mathbf{P}\big(X_t(B)\;=\;0\big). \tag{4.8}$$

Proof. For simplicity, through the proof we write τ for $\tau_\varepsilon(B)$. It suffices to show that

$$\mathbf{P}\big\{\tau=\infty,\; X_t(B)>0\big\} \;=\; 0. \tag{4.9}$$

To verify (4.9) we will mainly follow the lines of the proof of Theorem 1.2(b) of [6]. For $u\in(0,\varepsilon]$, define

$$Z_u := N\bigg((s,x,r):\; s\in(t-\varepsilon,\,t-\varepsilon+u),\; x\in B,\; r>(t-s)^{\vartheta}\log^{\vartheta}\Big(\frac{1}{t-s}\Big)\bigg),$$

with the random measure N introduced in Lemma 1.5(a). Then

$$\{\tau=\infty\} \;=\; \{Z_\varepsilon=0\}. \tag{4.10}$$

Recall the formula for the compensator $\hat{N}$ of N in Lemma 1.5(b). From a classical time change result for counting processes (see e.g. Theorem 10.33 in [9]), we get that there exists a standard Poisson process $A = \{A(v) : v \geq 0\}$ such that

$$Z_u = A\Big(\varrho \int_{t-\varepsilon}^{t-\varepsilon+u} \mathrm{d}s\; X_s(B) \int_{(t-s)^{\vartheta}\log^{\vartheta}\left(\frac{1}{t-s}\right)}^{\infty} \mathrm{d}r\; r^{-2-\beta}\Big)$$

$$= A\Big(\frac{\varrho}{1+\beta} \int_{t-\varepsilon}^{t-\varepsilon+u} \mathrm{d}s\; X_s(B)\, \frac{1}{(t-s)\log\left(\frac{1}{t-s}\right)}\Big), \tag{4.11}$$

where we used notation (4.6). Then

$$\mathbf{P}\big(Z_\varepsilon = 0,\, X_t(B) > 0\big)$$

$$\leq \mathbf{P}\Big(\int_{t-\varepsilon}^{t} \mathrm{d}s\; X_s(B)\, \frac{1}{(t-s)\log\left(\frac{1}{t-s}\right)} < \infty,\; X_t(B) > 0\Big). \tag{4.12}$$

It is easy to check that

$$\int_{t-\delta}^{t} \mathrm{d}s\; \frac{1}{(t-s)\log\left(\frac{1}{t-s}\right)} = \infty \quad \text{for all } \delta \in (0,\varepsilon). \tag{4.13}$$

Therefore, by Lemma 4.2,

$$\int_{t-\varepsilon}^{t} \mathrm{d}s\; X_s(B)\, \frac{1}{(t-s)\log\left(\frac{1}{t-s}\right)} = \infty \quad \text{on } \big\{X_t(B) > 0\big\}. \tag{4.14}$$

Thus, the probability in (4.12) disappears. Hence, together with (4.10) claim (4.9) follows. □

Set $\varepsilon_n := 2^{-n}$, $n \geq 1$. Then we choose open balls $B_n \uparrow B$ such that

$$\overline{B_n} \subset B_{n+1} \subset B \quad \text{and} \quad \sup_{y\in B^c,\, x\in B_n,\, 0<s\leq\varepsilon_n} p_s^{\alpha}(x-y) \underset{n\uparrow\infty}{\longrightarrow} 0. \tag{4.15}$$

Fix $n \geq 1$ such that $\varepsilon_n < t$. Define $\tau_n := \tau_{\varepsilon_n}(B_n)$.

In order to get a lower bound for $\|X_t\|_B$ we use the following inequality

$$\|X_t\|_B \geq \int_B \mathrm{d}y\; X_t(y)\, p_r^{\alpha}(y-x), \quad x \in B,\; r > 0. \tag{4.16}$$

On the event $\{\tau_n < t\}$, denote by ζ_n the spatial location in B_n of the jump at time τ_n, and by r_n the size of the jump, meaning that $\Delta X_{\tau_n} = r_n \delta_{\zeta_n}$. Then specializing (4.16),

$$\|X_t\|_B \geq \int_B \mathrm{d}y\; X_t(y)\, p_{t-\tau_n}^{\alpha}(y-\zeta_n) \text{ on the event } \{\tau_n < t\}. \tag{4.17}$$

From the strong Markov property at time τ_n, together with the branching property of superprocesses, we know that conditionally on $\{\tau_n < t\}$, the process $\{X_{\tau_n+u} : u \geq 0\}$ is bounded below in distribution by $\{\widetilde{X}_u^n : u \geq 0\}$, where $\widetilde{X}^n$ is a super-Brownian motion with initial value $r_n\delta_{\zeta_n}$. Hence, from (4.17) we get

$$\mathbf{E}\exp\big\{-\|X_t\|_B\big\} \tag{4.18}$$

$$\leq \mathbf{E}\,\mathbf{1}_{\{\tau_n<t\}} \exp\Big\{-\int_B \mathrm{d}y\; X_t(y)\, p_{t-\tau_n}^{\alpha}(y-\zeta_n)\Big\} + \mathbf{P}(\tau_n = \infty)$$

$$\leq \mathbf{E}\,\mathbf{1}_{\{\tau_n<t\}} \mathbf{E}_{r_n\delta_{\zeta_n}} \exp\Big\{-\int_B \mathrm{d}y\; X_{t-\tau_n}(y)\, p_{t-\tau_n}^{\alpha}(y-\zeta_n)\Big\} + \mathbf{P}(\tau_n = \infty).$$

Note that on the event $\{\tau_n < t\}$, we have

$$r_n \geq (t-\tau_n)^{\vartheta} \log^{\vartheta}\Big(\frac{1}{t-\tau_n}\Big) \;=:\; h_\beta(t-\tau_n). \tag{4.19}$$

We now claim that

$$\lim_{n\uparrow\infty} \sup_{0<s<\varepsilon_n,\, x\in B_n,\, r\geq h_\beta(s)} \mathbf{E}_{r\delta_x} \exp\Big\{-\int_B \mathrm{d}y\ X_s(y)\, p_s^\alpha(y-x)\Big\} \;=\; 0. \tag{4.20}$$

To verify (4.20), let $s\in(0,\varepsilon_n)$, $x\in B_n$ and $r\geq h_\beta(s)$. Then, using the Laplace transition functional of the superprocess we get

$$\begin{aligned}\mathbf{E}_{r\delta_x} \exp\Big\{-\int_B \mathrm{d}y\ X_s(y)\, p_s^\alpha(y-x)\Big\} \;&=\; \exp\big\{-r\, v_{s,x}^n(s,x)\big\}\\ &\leq\; \exp\big\{-h_\beta(s)\, v_{s,x}^n(s,x)\big\}\end{aligned} \tag{4.21}$$

where the non-negative function $v_{s,x}^n = \big\{v_{s,x}^n(s',x') : s'>0,\ x'\in\mathsf{R}^d\big\}$ solves the log-Laplace integral equation

$$\begin{aligned} v_{s,x}^n(s',x') \;=&\; \int_{\mathsf{R}^d} \mathrm{d}y\ p_{s'}^\alpha(y-x')\, 1_B(y)\, p_s^\alpha(y-x)\\ &+ \int_0^{s'} \mathrm{d}r' \int_{\mathsf{R}^d} \mathrm{d}y\ p_{s'-r'}^\alpha(y-x') \Big[a v_{s,x}^n(r',y) - b\big(v_{s,x}^n(r',y)\big)^{1+\beta}\Big]\end{aligned} \tag{4.22}$$

related to (1.1).

Lemma 4.4 (Another explosion). *Under the conditions $d>1$ or $\alpha\leq 1+\beta$, we have*

$$\lim_{n\uparrow\infty}\Big(\inf_{0<s<\varepsilon_n,\, x\in B_n} h_\beta(s)\, v_{s,x}^n(s,x)\Big) \;=\; +\infty\,. \tag{4.23}$$

Let us postpone the proof of Lemma 4.4.

Completion of Proof of Theorem 1.1(c). Our claim (4.20) readily follows from estimate (4.21) and (4.23). Moreover, according to (4.20), by passing to the limit $n\uparrow\infty$ in the right hand side of (4.18), and then using Lemma 4.3, we arrive at

$$\mathbf{E}\exp\big\{-\|X_t\|_B\big\} \;\leq\; \limsup_{n\uparrow\infty} \mathbf{P}\,(\tau_n=\infty) \;\leq\; \limsup_{n\uparrow\infty} \mathbf{P}\big(X_t(B_n)=0\big). \tag{4.24}$$

Since the event $\big\{X_t(B)=0\big\}$ is the non-increasing limit as $n\uparrow\infty$ of the events $\big\{X_t(B_n)=0\big\}$ we get

$$\mathbf{E}\exp\big\{-\|X_t\|_B\big\} \;\leq\; \mathbf{P}\big(X_t(B)\,=\,0\big). \tag{4.25}$$

Since obviously $\|X_t\|_B = 0$ if and only if $X_t(B)=0$, we see that (4.1) follows from this last bound. The proof of Theorem 1(c) is finished for $U=B$. □

Proof of Lemma 4.4. We start with a determination of the asymptotics of the first term at the right hand side of the log-Laplace equation (4.22) at $(s',x')=(s,x)$. Note that

$$\begin{aligned}&\int_{\mathsf{R}^d} \mathrm{d}y\ p_s^\alpha(y-x)\, 1_B(y)\, p_s^\alpha(y-x)\\ &= \int_{\mathsf{R}^d} \mathrm{d}y\ p_s^\alpha(y-x)\, p_s^\alpha(y-x) - \int_{B^c} \mathrm{d}y\ p_s^\alpha(y-x)\, p_s^\alpha(y-x).\end{aligned} \tag{4.26}$$

In the latter formula line, the first term equals $p^\alpha_{2s}(0) = Cs^{-d/\alpha}$, whereas the second one is bounded from above by

$$\sup_{0<s<\varepsilon_n,\, x\in B_n,\ y\in B^{\mathrm{c}}} p^\alpha_s(y-x) \underset{n\uparrow\infty}{\longrightarrow} 0, \tag{4.27}$$

where the last convergence follows by assumption (4.15) on B_n. Hence from (4.26) and (4.27) we obtain

$$\int_{\mathsf{R}^d} \mathrm{d}y\ p^\alpha_s(y-x)\, 1_B(y)\, p^\alpha_s(y-x) \;=\; C\, s^{-d/\alpha} + \mathrm{o}(1) \ \text{ as } \ n\uparrow\infty, \tag{4.28}$$

uniformly in $s\in(0,\varepsilon_n)$ and $x\in B_n$.

To simplify notation, we write $v^n := v^n_{s,x}$. Next, from (4.22) we can easily get the upper bound

$$v^n(s',x') \;\le\; \mathrm{e}^{|a|s'} \int_{\mathsf{R}^d} \mathrm{d}y\ p^\alpha_{s'}(y-x')\, p^\alpha_s(y-x) \;=\; \mathrm{e}^{|a|s'}\, p^\alpha_{s'+s}(x-x'). \tag{4.29}$$

Then we have

$$\begin{aligned} &\int_0^s \mathrm{d}r' \int_{\mathsf{R}^d} \mathrm{d}y\ p^\alpha_{s-r'}(y-x)\, \big(v^n(r',y)\big)^{1+\beta} \\ &\le\; \mathrm{e}^{|a|(1+\beta)s} \int_0^s \mathrm{d}r' \int_{\mathsf{R}^d} \mathrm{d}y\ p^\alpha_{s-r'}(y-x)\, \big(p^\alpha_{r'+s}(x-y)\big)^{1+\beta} \\ &\le\; \mathrm{e}^{|a|(1+\beta)s} \big(p^\alpha_s(0)\big)^\beta \int_0^s \mathrm{d}r' \int_{\mathsf{R}^d} \mathrm{d}y\ p^\alpha_{s-r'}(y-x)\, p^\alpha_{r'+s}(x-y) \\ &=\; \mathrm{e}^{|a|(1+\beta)s} \big(p^\alpha_s(0)\big)^\beta \int_0^s \mathrm{d}r'\ p^\alpha_{2s}(0) \;=\; C\, \mathrm{e}^{|a|(1+\beta)s} s^{1-d(1+\beta)/\alpha} \end{aligned} \tag{4.30}$$

and similarly

$$\int_0^s \mathrm{d}r' \int_{\mathsf{R}^d} \mathrm{d}y\ p^\alpha_{s-r'}(y-x)\, a v^n(r',y) \;\ge\; -C\,|a|\, \mathrm{e}^{|a|s}\, s^{1-d/\alpha}. \tag{4.31}$$

Summarizing, by (4.22), (4.28), (4.30), and (4.31),

$$v^n(s,x) \ge C\, s^{-d/\alpha} + \mathrm{o}(1) - C\, \mathrm{e}^{|a|(1+\beta)s}\, s^{1-d(1+\beta)/\alpha} - C\,|a|\, \mathrm{e}^{|a|s}\, s^{1-d/\alpha} \tag{4.32}$$

uniformly in $s\in(0,\varepsilon_n)$ and $x\in B_n$. According to our general assumption $d<\alpha/\beta$, we conclude that the right hand side of (4.32) behaves like $Cs^{-d/\alpha}$ as $s\downarrow 0$, uniformly in $s\in(0,\varepsilon_n)$. Now recalling definitions (4.19) and (4.6) as well as our assumption that $d>1$ or $\alpha\le 1+\beta$, we immediately get

$$\lim_{n\uparrow\infty}\ \inf_{0<s<\varepsilon_n} h_\beta(s)\, s^{-d/\alpha} \;=\; +\infty\,. \tag{4.33}$$

By (4.32), this implies (4.23), and the proof of the lemma is finished. □

5. Optimal Hölder index: proof of Theorem 1.1(b)

We return to $d=1$ and continue to assume that $t>0$ and $\mu\in\mathcal{M}_{\mathrm{f}}\backslash\{0\}$ are fixed. In the proof of Theorem 1.1(b) we implement the following idea. We show that there exists a sequence of "big" jumps of X that occur close to time t and these jumps in fact destroy the local Hölder continuity of any index greater or equal than η_{c}.

As in the proof of Theorem 1.1(c) in the previous section, we may work with a fixed open interval U. For simplicity we consider $U = (0,1)$. Put

$$I_k^{(n)} := \Big[\tfrac{k}{2^n}, \tfrac{k+1}{2^n}\Big), \quad n \geq 1,\ 0 \leq k \leq 2^n - 1. \tag{5.1}$$

Choose n_0 such that $2^{-\alpha n_0} < t$. For $n \geq n_0$ and $2 \leq k \leq 2^n + 1$, denote by $A_{n,k}$ the following event

$$\left\{ \Delta X_s(I_{k-2}^{(n)}) \geq \frac{(2\alpha)^{\frac{1}{1+\beta}}}{2^{\frac{\alpha}{1+\beta}n}}\, n^{\frac{1}{1+\beta}} \ \text{ for some } s \in \big[t - 2^{-\alpha n},\, t - 2^{-\alpha(n+1)}\big) \right\}, \tag{5.2}$$

and for $N \geq n_0$ write

$$\widetilde{A}_N := \bigcup_{k=N}^{\infty} \bigcup_{k=2}^{2^n+1} A_{n,k}\,. \tag{5.3}$$

Lemma 5.1 (Again existence of big jumps). *For any $N \geq n_0$,*

$$\mathbf{P}\big\{\widetilde{A}_N \,\big|\, X_t(U) > 0\big\} = 1. \tag{5.4}$$

Proof. For $s \in \big[t - 2^{-\alpha n},\, t - 2^{-\alpha(n+1)}\big)$ we have

$$\Big((t-s)\log\big(\tfrac{1}{t-s}\big)\Big)^{\frac{1}{1+\beta}} \leq \Big(2^{-\alpha n}\log 2^{\alpha(n+1)}\Big)^{\frac{1}{1+\beta}} \leq (2\alpha)^{\frac{1}{1+\beta}}\, 2^{-\frac{\alpha}{1+\beta}n}\, n^{\frac{1}{1+\beta}}, \tag{5.5}$$

since $2^{\alpha(n+1)} \leq \mathrm{e}^{\alpha 2n}$. Therefore,

$$\bigcup_{k=2}^{2^n+1} A_{n,k} \supseteq \left\{\Delta X_s(U) \geq \Big((t-s)\log\big(\tfrac{1}{t-s}\big)\Big)^{\frac{1}{1+\beta}} \text{ for some } s \in \big[t-2^{-\alpha n},\, t-2^{-\alpha(n+1)}\big)\right\}$$

and, consequently,

$$\begin{aligned} \widetilde{A}_N &= \bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} A_{n,k} \\ &\supseteq \left\{\Delta X_s(U) \geq \Big((t-s)\log\big(\tfrac{1}{t-s}\big)\Big)^{\frac{1}{1+\beta}} \ \text{ for some } s \geq t - 2^{-N}\right\}, \end{aligned} \tag{5.6}$$

and we are done by Lemma 4.3. □

Now we are going to define increments of Z_t^2 on the dyadic sets $\big\{\frac{k}{2^n} : k = 0,\dots,2^n\big\}$. By definition (1.12),

$$\begin{aligned} &Z_t^2\left(\tfrac{k}{2^n}\right) - Z_t^2\left(\tfrac{k+1}{2^n}\right) \\ &= \int_{(0,t]\times\mathsf{R}} M\big(\mathrm{d}(s,y)\big)\Big(p_{t-s}^{\alpha}\big(\tfrac{k}{2^n} - y\big) - p_{t-s}^{\alpha}\big(\tfrac{k+1}{2^n} - y\big)\Big) \\ &= \int_{(0,t]\times\mathsf{R}} M\big(\mathrm{d}(s,y)\big)\Big(p_{t-s}^{\alpha}\big(\tfrac{k}{2^n} - y\big) - p_{t-s}^{\alpha}\big(\tfrac{k+1}{2^n} - y\big)\Big)_+ \\ &\quad + \int_{(0,t]\times\mathsf{R}} M\big(\mathrm{d}(s,y)\big)\Big(p_{t-s}^{\alpha}\big(\tfrac{k}{2^n} - y\big) - p_{t-s}^{\alpha}\big(\tfrac{k+1}{2^n} - y\big)\Big)_- . \end{aligned} \tag{5.7}$$

Then according to Lemma 2.15 there exist spectrally positive stable processes $L_{n,k}^+$ and $L_{n,k}^-$ of index $1+\beta$ such that

$$Z_t^2\left(\tfrac{k}{2^n}\right) - Z_t^2\left(\tfrac{k+1}{2^n}\right) = L_{n,k}^+(T_+) - L_{n,k}^-(T_-) \tag{5.8}$$

where

$$T_\pm := \int_0^t \mathrm{d}s \int_{\mathsf{R}} X_s(\mathrm{d}y)\, \Big(p^\alpha_{t-s}\big(\tfrac{k}{2^n}-y\big) - p^\alpha_{t-s}\big(\tfrac{k+1}{2^n}-y\big)\Big)^{1+\beta}_\pm . \tag{5.9}$$

Fix $\varepsilon \in \big(0, \frac{1}{1+\beta}\big)$ for a while. Let us define the following events

$$\begin{aligned} B_{n,k} &:= \Big\{L^+_{n,k}(T_+) \geq 2^{-\eta_c n} n^{\frac{1}{1+\beta}-\varepsilon}\Big\} \cap \Big\{L^-_{n,k}(T_-) \leq 2^{-\eta_c n - \varepsilon n}\Big\} \\ &=: B^+_{n,k} \cap B^-_{n,k} \end{aligned} \tag{5.10}$$

(with notation in the obvious correspondence). Define the following event

$$\begin{aligned} D_N &:= \bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} (A_{n,k} \cap B_{n,k}) \\ &\supseteq \bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} A_{n,k} \setminus \bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} \big(A_{n,k} \cap B^{\mathrm{c}}_{n,k}\big). \end{aligned} \tag{5.11}$$

An estimation of the probability of D_N is crucial for the proof of Theorem 1.1(b). In fact we are going to show that conditionally on $\{X_t(U) > 0\}$, the event D_N happens with probability one for any N. This in turn implies that for any N one can find $n \geq N$ sufficiently large such that there exists an interval $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ on which the increment $Z^2_t(\frac{k}{2^n}) - Z^2_t(\frac{k+1}{2^n})$ is of order $L^+_{n,k}(T_+) \geq 2^{-\eta_c n} n^{\frac{1}{1+\beta}-\varepsilon}$ (since the other term $L^-_{n,k}(T_-)$ is much smaller on that interval). This implies the statement of Theorem 1.1(b). Detailed arguments follow.

By Lemma 5.1 we get

$$\mathbf{P}\big\{D_N \,\big|\, X_t(U) > 0\big\} \;\geq\; 1 - \mathbf{P}\bigg\{ \bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} \big(A_{n,k} \cap B^{\mathrm{c}}_{n,k}\big) \,\bigg|\, X_t(U) > 0 \bigg\}. \tag{5.12}$$

Recall A^ε defined in (3.3). Note that

$$\begin{aligned} &\mathbf{P}\bigg(\bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} \big(A_{n,k} \cap B^{\mathrm{c}}_{n,k}\big) \bigg) \\ &\quad\leq\; \mathbf{P}(A^{\varepsilon,\mathrm{c}}) + \mathbf{P}\bigg(\bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} \big(A^\varepsilon \cap A_{n,k} \cap B^{\mathrm{c}}_{n,k}\big) \bigg) \\ &\quad\leq\; 2\varepsilon + \mathbf{P}\bigg(\bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} \big(A^\varepsilon \cap A_{n,k} \cap B^{\mathrm{c}}_{n,k}\big) \bigg). \end{aligned} \tag{5.13}$$

Lemma 5.2 (**Probability of small increments**). *For all $\varepsilon > 0$ sufficiently small,*

$$\lim_{N\uparrow\infty} \mathbf{P}\bigg(\bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} \big(A^\varepsilon \cap A_{n,k} \cap B^{\mathrm{c}}_{n,k}\big) \bigg) \;=\; 0. \tag{5.14}$$

We postpone the proof of this lemma to the end of this section. Instead we will show now, how it implies Theorem 1.1(b).

Completion of proof of Theorem 1.1(b). From Lemma 5.2 and (5.13) it follows that

$$\limsup_{N\uparrow\infty} \mathbf{P}\bigg\{ \bigcup_{n=N}^{\infty} \bigcup_{k=2}^{2^n+1} \big(A_{n,k} \cap B^{\mathrm{c}}_{n,k}\big) \,\bigg|\, X_t(U) > 0 \bigg\} \;\leq\; \frac{2\varepsilon}{\mathbf{P}\big(X_t(U) > 0\big)} . \tag{5.15}$$

Since ε can be arbitrarily small, the latter lim sup expression disappears. Combining this with estimate (5.12), we get

$$\lim_{N\uparrow\infty} \mathbf{P}\big\{D_N \,\big|\, X_t(U)>0\big\} \;=\; 1. \tag{5.16}$$

Since $D_N \downarrow \bigcap_{N=n_0}^{\infty} D_N =: D_\infty$ as $N\uparrow\infty$, we conclude that

$$\mathbf{P}\big\{D_\infty \,\big|\, X_t(U)>0\big\} \;=\; 1. \tag{5.17}$$

This means that, almost surely on $\big\{X_t(U)>0\big\}$, there is a sequence (n_j,k_j) such that

$$Z_t^2\big(\tfrac{k_j}{2^{n_j}}\big) - Z_t^2\big(\tfrac{k_j+1}{2^{n_j}}\big) \;\geq\; 2^{-\eta_{\mathrm{c}}\, n_j}\, n_j^{\frac{1}{1+\beta}-\varepsilon}. \tag{5.18}$$

This inequality implies the claim in Theorem 1.1(b). □

We now prepare for the proof of Lemma 5.2. Actually by using (5.10), we represent the probability in (5.14) as a sum of two probabilities:

$$\begin{aligned}
&\mathbf{P}\Bigg(\bigcup_{n=N}^{\infty}\bigcup_{k=2}^{2^n+1}\big(A^\varepsilon\cap A_{n,k}\cap B_{n,k}^{\mathrm{c}}\big)\Bigg)\\
&= \mathbf{P}\Bigg(\bigcup_{n=N}^{\infty}\bigcup_{k=2}^{2^n+1}\Big(A^\varepsilon\cap A_{n,k}\cap B_{n,k}^{+,\mathrm{c}}\Big)\Bigg) + \mathbf{P}\Bigg(\bigcup_{n=N}^{\infty}\bigcup_{k=2}^{2^n+1}\Big(A^\varepsilon\cap A_{n,k}\cap B_{n,k}^{-,\mathrm{c}}\Big)\Bigg).
\end{aligned} \tag{5.19}$$

Now we will handle each term on the right hand side of (5.19) separately.

Lemma 5.3 (First term in (5.19)). *For* $\varepsilon\in\big(0,\frac{1}{1+\beta}\big)$,

$$\lim_{N\uparrow\infty}\mathbf{P}\Bigg(\bigcup_{n=N}^{\infty}\bigcup_{k=2}^{2^n+1}\Big(A^\varepsilon\cap A_{n,k}\cap B_{n,k}^{+,\mathrm{c}}\Big)\Bigg) \;=\; 0. \tag{5.20}$$

Proof. Consider the process $L_{n,k}^+(s),\ s\leq T_+$. On $A_{n,k}$ there exists a jump of the martingale measure M of the form $r^*\delta_{s^*,y^*}$ for some

$$r^*\geq(2\alpha)^{\frac{1}{1+\beta}}\,2^{-\frac{\alpha}{1+\beta}n}n^{\frac{1}{1+\beta}},\quad s^*\in\big[t-2^{-\alpha n},\,t-2^{-\alpha(n+1)}\big],\quad y^*\in I_{k-2}^{(n)}. \tag{5.21}$$

Hence

$$\begin{aligned}
\Delta L_{n,k}^+(s^*)\geq \inf_{y\in I_{k-2}^{(n)},\ s\in[2^{-\alpha(n+1)},2^{-\alpha n}]} &\Big(p_s^\alpha\big(\tfrac{k}{2^n}-y\big)-p_s^\alpha\big(\tfrac{k+1}{2^n}-y\big)\Big)_+\\
&\times(2\alpha)^{\frac{1}{1+\beta}}\,2^{-\frac{\alpha}{1+\beta}n}\,n^{\frac{1}{1+\beta}}.
\end{aligned} \tag{5.22}$$

It is easy to get

$$\begin{aligned}
&\inf_{y\in I^{(n)}_{k-2},\ s\in[2^{-\alpha(n+1)},2^{-\alpha n}]} \Big(p^\alpha_s\big(\tfrac{k}{2^n}-y\big)-p^\alpha_s\big(\tfrac{k+1}{2^n}-y\big)\Big)_+\\
&= \inf_{\substack{2^{-n}\le z\le 2^{-n+1},\\ s\in[2^{-\alpha(n+1)},2^{-\alpha n}]}} \Big(p^\alpha_s(z)-p^\alpha_s\big(z+2^{-n}\big)\Big)_+\\
&= \inf_{\substack{2^{-n}\le z\le 2^{-n+1},\\ s\in[2^{-\alpha(n+1)},2^{-\alpha n}]}} s^{-1/\alpha}\Big(p^\alpha_1\big(zs^{-1/\alpha}\big)-p^\alpha_1\big((z+2^{-n})s^{-1/\alpha}\big)\Big)_+\\
&\ge\ 2^n \inf_{\substack{2^{-n}\le z\le 3\cdot 2^{-n},\\ s\in[2^{-\alpha(n+1)},2^{-\alpha n}]}} \Big|(p^\alpha_1)'\big(zs^{-1/\alpha}\big)\Big|\,2^{-n}s^{-1/\alpha}
\end{aligned}$$

$$\ge\ 2^n \inf_{1\le x\le 6}\big|(p^\alpha_1)'\,(x)\big|\ =:\ c_{(5.23)}2^n, \tag{5.23}$$

where $c_{(5.23)} > 0$. In fact, from (2.2),

$$\frac{\mathrm{d}}{\mathrm{d}z}p^\alpha_1(z)\ =\ -\int_0^\infty \mathrm{d}s\ q^{\alpha/2}_1(s)\,\frac{z}{2s}\,p^{(2)}_s(z)\ \ne\ 0,\quad z\ne 0, \tag{5.24}$$

and $(p^{(2)}_\alpha)'(x)\ne 0$ for any $x\ne 0$. Apply (5.23) in (5.22) to arrive at

$$\Delta L^+_{n,k}(s^*)\ \ge\ c_{(5.25)}2^{(1-\frac{\alpha}{1+\beta})n}\,n^{\frac{1}{1+\beta}}\ =\ c_{(5.25)}2^{-\eta_c n}\,n^{\frac{1}{1+\beta}}. \tag{5.25}$$

Using Lemma 2.12 with $\theta = 1+\beta$ and

$$\delta\ =\ (1+\beta)1_{\{2\beta<\alpha-1\}}+(\alpha-\beta-\varepsilon)1_{\{2\beta\ge\alpha-1\}}\,, \tag{5.26}$$

we get, with c_ε appearing in definition (3.3) of A^ε,

$$T_\pm\ \le\ c_\varepsilon\Big(2^{-n(1+\beta)}1_{\{2\beta<\alpha-1\}}+2^{-n(\alpha-\beta-\varepsilon)}1_{\{2\beta\ge\alpha-1\}}\Big)\ =:\ t_n\ \text{ on }\ A^\varepsilon. \tag{5.27}$$

Hence for all n sufficiently large we obtain

$$\begin{aligned}
&\mathbf{P}\Big(L^+_{n,k}(T_+)<2^{-\eta_c n}\,n^{\frac{1}{1+\beta}-\varepsilon},\ A^\varepsilon\cap A_{n,k}\Big)\\
&\le \mathbf{P}\Big(L^+_{n,k}(T_+)<2^{-\eta_c n}\,n^{\frac{1}{1+\beta}-\varepsilon},\ \Delta L^+_{n,k}(s^*)\ge c_{(5.25)}2^{-\eta_c n}\,n^{\frac{1}{1+\beta}},\ A^\varepsilon\Big)\\
&\le \mathbf{P}\Big(\inf_{s\le T^+}L^+_{n,k}(s)<-\frac12\,c_{(5.25)}2^{-\eta_c n}\,n^{\frac{1}{1+\beta}},\ A^\varepsilon\Big)\\
&\le \mathbf{P}\Big(\inf_{s\le t_n}L^+_{n,k}(s)<-\frac12\,c_{(5.25)}2^{-\eta_c n}\,n^{\frac{1}{1+\beta}}\Big)
\end{aligned}$$

$$\le\ \exp\Big\{-c_\beta(t_n)^{-1/\beta}(c_{(5.25)}2^{-\eta_c n}\,n^{\frac{1}{1+\beta}})^{(1+\beta)/\beta}\Big\} \tag{5.28}$$

$$\le\ \exp\Big\{-c_\varepsilon n^{1/\beta}(t_n^{-1}2^{-\eta_c(1+\beta)n})^{1/\beta}\Big\}\ \le\ \exp\big\{-c_\varepsilon n^{1/\beta}2^{(1-\varepsilon)n}\big\}$$

where (5.28) follows by Lemma 2.4 and the rest is simple algebra. From this we get that for N sufficiently large

$$\mathbf{P}\left(\bigcup_{n=N}^{\infty}\bigcup_{k=2}^{2^n+1}\left(A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{+,\mathrm{c}}\right)\right)\le\sum_{n=N}^{\infty}\sum_{k=2}^{2^n+1}\mathbf{P}\left(A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{+,\mathrm{c}}\right)$$
$$\le\sum_{n=N}^{\infty}\sum_{k=2}^{2^n+1}\exp\bigl\{-c_{\varepsilon}n^{1/\beta}\,2^{(1-\varepsilon)n}\bigr\}\;=\;\sum_{n=N}^{\infty}2^n\exp\bigl\{-c_{\varepsilon}n^{1/\beta}\,2^{(1-\varepsilon)n}\bigr\} \tag{5.29}$$

which converges to 0 as $N\uparrow\infty$, and we are done with the proof of Lemma 5.3. □

Lemma 5.4 (**Second term in (5.19)**). *For all $\varepsilon>0$ sufficiently small,*

$$\lim_{N\uparrow\infty}\mathbf{P}\left(\bigcup_{n=N}^{\infty}\bigcup_{k=2}^{2^n+1}\left(A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{-,\mathrm{c}}\right)\right)\;=\;0. \tag{5.30}$$

The proof of this lemma will be postponed almost to the end of the section. For its preparation, fix $\rho\in(0,\frac{1}{2})$. Define

$$A_n^{\rho}\;:=\;\Bigl\{\omega:\ \text{there exists } I_k^{(n)}\ \text{with}\sup_{s\in[t-2^{-\alpha(1-\rho)n},\,t)}X_s(I_k^{(n)})\ge 2^{-n(1-2\rho)}\Bigr\}.$$

Note that

$$\mathbf{P}\left(\bigcup_{k=2}^{2^n+1}\left(A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{-,\mathrm{c}}\right)\right)$$
$$\le\;\mathbf{P}(A_n^{\rho})+\mathbf{P}\left(\bigcup_{k=2}^{2^n+1}\left(A_n^{\rho,\mathrm{c}}\cap A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{-,\mathrm{c}}\right)\right)$$
$$\le\;\mathbf{P}(A_n^{\rho})+\sum_{k=2}^{2^n+1}\mathbf{P}\left(A_n^{\rho,\mathrm{c}}\cap A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{-,\mathrm{c}}\right). \tag{5.31}$$

Now let us introduce the notation

$$B_{n,k}^{-,1}\;:=\;\Bigl\{\sup_{s\le T_-}\Delta L_{n,k}^{-}(s)\le 2^{-\eta_{\mathrm{c}}n-\varepsilon n}\Bigr\}. \tag{5.32}$$

Then we have

$$\mathbf{P}\left(\bigcup_{k=2}^{2^n+1}\left(A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{-,\mathrm{c}}\right)\right)$$
$$\le\;\mathbf{P}(A_n^{\rho})+\sum_{k=2}^{2^n+1}\mathbf{P}\left(A_n^{\rho,\mathrm{c}}\cap A^{\varepsilon}\cap A_{n,k}\cap B_{n,k}^{-,\mathrm{c}}\right)$$
$$\le\;\mathbf{P}(A_n^{\rho})+\sum_{k=2}^{2^n+1}\mathbf{P}\left(A^{\varepsilon}\cap B_{n,k}^{-,\mathrm{c}}\cap B_{n,k}^{-,1}\right)$$
$$+\sum_{k=2}^{2^n+1}\mathbf{P}\left(A^{\varepsilon}\cap A_n^{\rho,\mathrm{c}}\cap A_{n,k}\cap B_{n,k}^{-,1,\mathrm{c}}\right)$$
$$=:\;\mathbf{P}(A_n^{\rho})+\sum_{k=2}^{2^n+1}P_{n,k}^{\varepsilon}+\sum_{k=2}^{2^n+1}P_{n,k}^{\varepsilon,\varrho}. \tag{5.33}$$

In the following lemmas we consider the three terms in (5.33) separately.

Lemma 5.5 (**First term in (5.33)**). *There exists a constant* $c_{(5.34)}$ *independent of* $\rho \in (0, \frac{1}{2})$ *such that*

$$\mathbf{P}(A_n^\rho) \le c_{(5.34)}\, 2^{-\rho n}, \quad n \ge n_0\,. \tag{5.34}$$

Proof. Fix $n \ge n_0$. Define the stopping time $\tau_n = \tau_n(\rho)$ as

$$\inf\Big\{ s \in [t - 2^{-\alpha(1-\rho)n}, t) : X_s(I_k^{(n)}) \ge 2^{-n(1-2\rho)} \text{ for some } I_k^{(n)} \Big\} \tag{5.35}$$

if $\omega \in A_n^\rho$, and as t if $\omega \in A_n^{\rho,\mathrm{c}}$. Fix any $\omega \in A_n^\rho$. By definition of τ_n there exists a sequence $\big\{(s_j, I_{k_j}^{(n)}) : j \ge 1\big\}$ such that

$$s_j \downarrow \tau_n \text{ as } j \uparrow \infty \quad \text{and} \quad X_{s_j}(I_{k_j}^{(n)}) \ge 2^{-n(1-2\rho)}, \ j \ge 1. \tag{5.36}$$

There exists a subsequence $\{j_r : r \ge 1\}$ such that $I_{k_{j_r}}^{(n)} = I_{\tilde k}^{(n)}$ for some $\tilde k \in \mathsf{Z}$. Hence, for the fixed $\omega \in A_n^\rho$,

$$X_{\tau_n}(I_{\tilde k}^{(n)}) = \lim_{r\to\infty} X_{s_{j_r}}(I_{\tilde k}^{(n)}) \ge 2^{-n(1-2\rho)}. \tag{5.37}$$

Put $\tilde B := \big[\tilde k 2^{-n} - 2^{-n(1-\rho)}, (\tilde k + 1)2^{-n} + 2^{-n(1-\rho)}\big]$. Then there is a constant $c_{(5.38)}$ independent of ρ such that

$$\int_{\tilde B} \mathrm{d}y\; p_{t-s}^\alpha(y-z) \ge c_{(5.38)} \quad \text{for all } z \in I_{\tilde k}^{(n)} \text{ and } s \in [t - 2^{-\alpha(1-\rho)n}, t). \tag{5.38}$$

Now, by the strong Markov property,

$$\mathbf{E} X_t(\tilde B) = \mathbf{E}\, \mathrm{e}^{a(t-\tau_n)} S_{t-\tau_n}^\alpha X_{\tau_n}(\tilde B) \ge \mathrm{e}^{-|a|t}\, \mathbf{E}\Big\{ \int_{\tilde B} \mathrm{d}y \int_{\mathsf{R}} X_{\tau_n}(\mathrm{d}z) p_{t-\tau_n}^\alpha(y-z); A_n^\rho \Big\}$$

$$\ge \mathrm{e}^{-|a|t}\, \mathbf{E}\Big\{ \int_{I_{\tilde k}^{(n)}} X_{\tau_n}(\mathrm{d}z) \int_{\tilde B} \mathrm{d}y\; p_{t-\tau_n}^\alpha(y-z);\ A_n^\rho \Big\} \ge c_{(5.38)}\, \mathbf{E} X_{\tau_n}(I_{\tilde k}^{(n)}). \tag{5.39}$$

Taking into account (5.37) and (5.38) then gives

$$\mathbf{E} X_t(\tilde B) \ge c_{(5.38)}\, 2^{-n(1-2\rho)} \mathbf{P}(A_n^\rho). \tag{5.40}$$

On the other hand, in view of Corollary 2.8,

$$\begin{aligned} \mathbf{E} X_t(\tilde B) &\le |\tilde B|\, \mathbf{E} \sup_{0\le x\le 1} X_t(x) \\ &\le 2\big(2^{-n} + 2^{-n(1-\rho)}\big)\, \mathbf{E} \sup_{0\le x\le 1} X_t(x) \le C\, 2^{-n(1-\rho)}, \end{aligned} \tag{5.41}$$

where we wrote $|\tilde B|$ for the length of the interval $\tilde B$. Combining (5.40) and (5.41) completes the proof. □

Lemma 5.6 (**Second term in (5.33)**). *For fixed* $\varepsilon \in \big(0, \frac{1}{1+\beta}\big)$ *and all* n *large enough,*

$$P_{n,k}^\varepsilon \le 2^{-3n/2}, \quad 2 \le k \le 2^n + 1. \tag{5.42}$$

Proof. Since $T_- \le t_n$ on A^ε (recall notation (5.27)),

$$P_{n,k}^\varepsilon \le \mathbf{P}\bigg(\sup_{v \le t_n} L_v 1\Big\{ \sup_{u\le v} \Delta L_u \le 2^{-n(\eta_\mathrm{c}+\varepsilon)} \Big\} \ge 2^{-n\eta_\mathrm{c}} \bigg). \tag{5.43}$$

Applying now Lemma 2.3, with notation of t_n (5.27) we obtain

$$P^{\varepsilon}_{n,k} \leq \Big(c_\varepsilon 2^{\varepsilon\beta n-(1-\eta_{\rm c})(1+\beta)n} + c_\varepsilon 2^{\eta_{\rm c}(1+\beta)n+\varepsilon\beta n-(\alpha-\beta-\varepsilon)n}\Big)^{(2^{n\varepsilon})}. \tag{5.44}$$

Inserting the definition of $\eta_{\rm c}$ and making n sufficiently large, the estimate in the lemma follows. □

In order to deal with the third term $P^{\varepsilon,\varrho}_{n,k}$, we need to define additional events

$$A^{\varepsilon,\rho,1}_{n,k} := \Big\{\text{There exists a jump of } M \text{ of the form } r^*\delta_{(s^*,y^*)} \text{ for some } (r^*,s^*,y^*) \text{ such that } r^* \geq (t-s)^{\frac{1}{1+\beta}+2\varepsilon/\alpha}, \ \big|\tfrac{k+1}{2^n}-y^*\big| \leq (t-s)^{1/\alpha-2\varepsilon}, \ \ s^* \geq t-2^{-\alpha(1+\rho)n}\Big\} \tag{5.45}$$

and

$$A^{\varepsilon,\rho,2}_{n,k} := A^{\rho,{\rm c}}_n \cap A_{n,k} \cap \Big\{\text{There exists a jump of } M \text{ of the form } r^*\delta_{(s^*,y^*)} \text{ for some } (r^*,s^*,y^*) \text{ such that } r^* \geq (t-s)^{\frac{1}{1+\beta}+2\varepsilon/\alpha}, \ y^* \in \big[\tfrac{k+1/2}{2^n}, \tfrac{k+1+2^{\rho n+\alpha 2\varepsilon(1-\rho)n}}{2^n}\big], \ \ s^* \in \big[t-2^{-\alpha(1-\rho)n}, t-2^{-\alpha(1+\rho)n}\big]\Big\}.$$

So far we assumed that $\varepsilon \in \big(0,\frac{1}{1+\beta}\big)$ and $\rho \in (0,\frac{1}{2})$. Suppose additionally that

$$\frac{\alpha(\alpha+1)2\varepsilon}{1-\eta_{\rm c}+2\varepsilon(\alpha^2+\alpha-1)} \leq \rho. \tag{5.46}$$

Lemma 5.7 (**Splitting of the third term in (5.33)**). *For $\rho,\varepsilon>0$ sufficiently small and satisfying* (5.46) *we have*

$$P^{\varepsilon,\varrho}_{n,k} \leq \mathbf{P}(A^{\varepsilon,\rho,1}_{n,k}) + \mathbf{P}(A^{\varepsilon,\rho,2}_{n,k}) \tag{5.47}$$

for all $0 \leq k \leq 2^n-1$ *and* $n \geq n_\varepsilon$.

Proof. First let us describe the strategy of the proof. We are going to show that whenever the jump of $L^-_{n,k}(s),\ s \leq T_-$, occurs which is greater than $2^{-n(\eta_{\rm c}+\varepsilon)}$, then it may happen only in the points indicated in the definition of $A^{\varepsilon,\rho,1}_{n,k}$ and $A^{\varepsilon,\rho,2}_{n,k}$. To show this we will in fact show that outsides the sets mentioned in $A^{\varepsilon,\rho,1}_{n,k}$ and $A^{\varepsilon,\rho,2}_{n,k}$ the jumps of $L^-_{n,k}(s),\ s \leq T_-$, are less than $2^{-n(\eta_{\rm c}+\varepsilon)}$.

To implement this strategy, first let us recall that all the jumps of $L^-_{n,k}(s),\ s \leq T_-$, equal to

$$\Delta X_{s*}(y^*)\Big(p^\alpha_{t-s}\big(\tfrac{k+1}{2^n}-y^*\big) - p^\alpha_{t-s}\big(\tfrac{k}{2^n}-y^*\big)\Big)_+ \tag{5.48}$$

for some $(s^*,y^*) \in [0,t) \times \mathsf{R}$.

Recall that by definition (3.3), on the event A^ε,

$$|\Delta X_s| \leq c_{(2.52)}(t-s)^{(1+\beta)^{-1}-\gamma} \tag{5.49}$$

with $\gamma \in \big(0,(1+\beta)^{-1}\big)$. On the other hand using Lemma 2.1 with $\delta=1$ we obtain

$$p^\alpha_{t-s}\big(\tfrac{k+1}{2^n}-y\big) - p^\alpha_{t-s}\big(\tfrac{k}{2^n}-y\big) \leq C2^{-n}(t-s)^{-2/\alpha}. \tag{5.50}$$

From (5.49) and (5.50) we infer

$$\sup_{s\le t-2^{-\alpha(1-\rho)n}} \Delta X_s \sup_{y\in\mathsf{R}} \Big(p^\alpha_{t-s}\big(\tfrac{k+1}{2^n}-y\big) - p^\alpha_{t-s}\big(\tfrac{k}{2^n}-y\big)\Big) \le C c_{(2.52)} 2^{-n} \big(2^{-\alpha(1-\rho)n}\big)^{\frac{1}{1+\beta}-\gamma-2/\alpha} = C\,2^{-n\big(\eta_{\mathrm{c}}-\alpha\gamma+\rho(1-\eta_{\mathrm{c}}+\alpha\gamma)\big)}. \tag{5.51}$$

Furthermore if the jump ΔX_s occurs at the point y^* with

$$\big|y^*-\tfrac{k+1}{2^n}\big| \ge (t-s)^{1/\alpha-2\varepsilon}, \tag{5.52}$$

then again by Lemma 2.1, for any $\delta\in[0,1]$,

$$p^\alpha_{t-s}\big(\tfrac{k+1}{2^n}-y\big) - p^\alpha_{t-s}\big(\tfrac{k}{2^n}-y\big) \le C\,2^{-n\delta}(t-s)^{-\delta/\alpha}\; p^\alpha_{t-s}\big((t-s)^{1/\alpha-2\varepsilon}\big). \tag{5.53}$$

Since

$$p^\alpha_1(x) \le C\,x^{-1-\alpha}, \quad x\in\mathsf{R}, \tag{5.54}$$

we get the bound

$$p^\alpha_{t-s}\big(\tfrac{k+1}{2^n}-y\big) - p^\alpha_{t-s}\big(\tfrac{k}{2^n}-y\big) \le C\,2^{-n\delta}(t-s)^{-\frac{\delta+1}{\alpha}+2\varepsilon(\alpha+1)}. \tag{5.55}$$

Hence,

$$\sup_{s<t}\ \sup_{y:\ |y-\frac{k+1}{2^n}|\ge(t-s)^{1/\alpha-2\varepsilon}} \Delta X_s(y)\Big(p^\alpha_{t-s}\big(\tfrac{k+1}{2^n}-y\big) - p^\alpha_{t-s}\big(\tfrac{k}{2^n}-y\big)\Big) \le C c_{(2.52)}\,2^{-n\delta}(t-s)^{-\frac{\delta+1}{\alpha}+2\varepsilon(\alpha+1)+\frac{1}{\beta+1}-\gamma}. \tag{5.56}$$

Set

$$\delta := \eta_{\mathrm{c}} + \alpha\big(2\varepsilon(\alpha+1)-\gamma\big). \tag{5.57}$$

Note that for all ε and γ sufficiently small, we have $\delta\in[0,1]$, and we can apply the previous estimates. Thus we obtain

$$\sup_{s<t}\ \sup_{y:\ |y-\frac{k+1}{2^n}|\ge(t-s)^{1/\alpha-2\varepsilon}} \Delta X_s(y)\Big(p^\alpha_{t-s}\big(\tfrac{k+1}{2^n}-y\big) - p^\alpha_{t-s}\big(\tfrac{k}{2^n}-y\big)\Big) \le C c_{(2.52)}\,2^{-n\big(\eta_{\mathrm{c}}+\alpha(2\varepsilon(\alpha+1)-\gamma)\big)}. \tag{5.58}$$

Now if we take $\gamma = 2\varepsilon\,(\alpha+1-1/\alpha)$, which belongs to these admissible γ, and ρ as in (5.46) we conclude that the right hand side of (5.51) and (5.58) is bounded by

$$c_\varepsilon\,2^{-n(\eta_{\mathrm{c}}+2\varepsilon)} \tag{5.59}$$

for some $c_\varepsilon<\infty$.

For any jump $r^*\delta_{(s^*,y^*)}$ of M such that $r^*\le (t-s)^{\frac{1}{1+\beta}+2\varepsilon/\alpha}$ and $s^*<t$ we my apply Lemma 2.1 with $\delta=\eta_{\mathrm{c}}+2\varepsilon$ to get that

$$\Delta X_{s*}(y^*)\Big(p^\alpha_{t-s}\big(\tfrac{k+1}{2^n}-y^*\big) - p^\alpha_{t-s}\big(\tfrac{k}{2^n}-y^*\big)\Big) \le C\,2^{-n(\eta_{\mathrm{c}}+2\varepsilon)}. \tag{5.60}$$

Now recall (5.48). Hence combining (5.51), (5.58), (5.59), and (5.60), the conclusion of Lemma 5.7 follows. □

In the next two lemmas we will bound the two probabilities on the right hand side of (5.47).

Lemma 5.8 (**First term in (5.47)**). *For all $\rho, \varepsilon > 0$ sufficiently small and satisfying*

$$6\,\varepsilon(\alpha+1+\beta) \,\le\, \rho \tag{5.61}$$

we have

$$\mathbf{P}(A^{\varepsilon,\rho,1}_{n,k}) \,\le\, 2^{-n-n\rho/2} \tag{5.62}$$

for all k, n considered.

Proof. It is easy to see that

$$\begin{aligned} A^{\varepsilon,\rho,1}_{n,k} \,\subseteq\, \bigcup_{l=(1+\rho)n}^{\infty} \Big\{ & \text{There exists a jump of } M \text{ of the form } r^*\delta_{(s^*,y^*)} \\ & \text{for some } (r^*,s^*,y^*) \text{ such that } r^* \ge 2^{-l\left(\frac{\alpha}{1+\beta}+2\varepsilon\right)}, \\ & \big|\tfrac{k+1}{2^n} - y^*\big| \le 2^{-l(1-2\varepsilon\alpha)},\ s^* \in \big[t-2^{-\alpha l},\ t-2^{-\alpha(l+1)}\big) \Big\} \\ =: \bigcup_{l=(1+\rho)n}^{\infty} & A^{\varepsilon,\rho,1}_{n,k,l}\,. \end{aligned}$$

Recall the random measure N describing the jumps of X. Write $Y_{n,k,l}$ for the N-measure of

$$\big[t(1-2^{-\alpha l}), t(1-2^{-\alpha(l+1)})\big] \times \big[\tfrac{k+1}{2^n} - 2^{-l(1-2\alpha\varepsilon)}, \tfrac{k+1}{2^n} + 2^{-l(1-2\alpha\varepsilon)}\big] \times \big[2^{-l(\frac{\alpha}{1+\beta}+2\varepsilon)}, \infty\big).$$

Then, by Markov's inequality,

$$\mathbf{P}(A^{\varepsilon,\rho,1}_{n,k,l}) \,=\, \mathbf{P}(Y_{n,k,l} \ge 1) \,\le\, \mathbf{E}Y_{n,k,l}\,. \tag{5.63}$$

Therefore,

$$\mathbf{P}(A^{\varepsilon,\rho,1}_{n,k}) \,\le\, \sum_{k \ge (1+\rho)n} \mathbf{P}(A^{\varepsilon,\rho,1}_{n,k,l}) \,\le\, \sum_{k \ge (1+\rho)n} \mathbf{E}Y_{n,k,l}\,. \tag{5.64}$$

From the formula for the compensator of N we get

$$\begin{aligned} \mathbf{E}Y_{n,k,l} \,&=\, \varrho \int_{t(1-2^{-\alpha l}}^{t(1-2^{-\alpha(l+1)}} \mathrm{d}s\ \mathbf{E}X_s\Big(\Big[\tfrac{k+1}{2^n} - 2^{-l(1-2\alpha\varepsilon)}, \tfrac{k+1}{2^n} + 2^{-l(1-2\alpha\varepsilon)}\Big]\Big) \\ &\qquad \times \int_{2^{-l(\frac{\alpha}{1+\beta}+2\varepsilon)}}^{\infty} \mathrm{d}r\ r^{-2-\beta} \\ &\le\, C\,2^{-\alpha l}\,2^{-l(1-2\alpha\varepsilon)}\,2^{l\left(\alpha+2\varepsilon(1+\beta)\right)}. \end{aligned} \tag{5.65}$$

Consequently,

$$\mathbf{P}(A^{\varepsilon,\rho,1}_{n,k,l}) \,\le\, C \sum_{k \ge (1+\rho)n} 2^{-l+2\varepsilon(\alpha+1+\beta)l} \,\le\, C\,2^{-(1+\rho)n+2\varepsilon(\alpha+1+\beta)(1+\rho)n}. \tag{5.66}$$

Noting that $2\varepsilon(\alpha+1+\beta)(1+\rho) \le \rho/2$ under the conditions in the lemma, we complete the proof. □

Lemma 5.9 (**Second term in (5.47)**). *For all $\varepsilon, \rho > 0$ sufficiently small,*

$$\mathbf{P}(A^{\varepsilon,\rho,2}_{n,k}) \,\le\, 2^{-3n/2} \tag{5.67}$$

for all k, n considered.

Proof. It is easy to see by construction that

$$A_{n,k}^{\varepsilon,\rho,2} \subseteq A_n^{\rho,\mathrm{c}} \cap \Big\{ \text{There exist at least two jumps of } M \text{ of the form } r^*\delta_{(s^*,y^*)} \text{ such that}$$

$$r^* \geq 2^{-n\left(\frac{\alpha(1+\rho)}{1+\beta}+2\varepsilon(1+\rho)\right)}, \tag{5.68a}$$

$$y^* \in \left[\tfrac{k-2}{2^n},\ \tfrac{k+1+2^{\rho n+2\alpha\varepsilon(1-\rho)n}}{2^n}\right], \tag{5.68b}$$

$$s^* \in \left[t-2^{-\alpha(1-\rho)n},\ t-2^{-\alpha(1+\rho)n}\right]\Big\}. \tag{5.68c}$$

On the event $A_n^{\rho,\mathrm{c}}$, for the intensity of jumps satisfying (5.68a)-(5.68c) we have

$$\int_{t-2^{-\alpha(1-\rho)n}}^{t-2^{-\alpha(1+\rho)n}} \mathrm{d}s\ X_s\left(\left[\tfrac{k-2}{2^n},\ \tfrac{k+1+2^{\rho n+2\alpha\varepsilon(1-\rho)n}}{2^n}\right]\right)\int_{2^{-n\left(\frac{\alpha(1+\rho)}{1+\beta}+2\varepsilon(1+\rho)\right)}}^{\infty} \mathrm{d}r\ r^{-2-\beta}$$

$$\leq\ 2^{-\alpha(1-\rho)n}\,2^{-n(1-2\rho)}\,2^{\rho n+2\alpha\varepsilon(1-\rho)n+2}\,2^{n\left(\alpha(1+\rho)+2\varepsilon(1+\rho)(1+\beta)\right)}$$

$$\leq\ 2^{-n}\,2^{10(\rho+2\varepsilon)n}\ \leq\ 2^{-\frac{3}{4}n}$$

for all ε and ρ sufficiently small. Since the number of such jumps can be represented by means of a time-changed standard Poisson process, the probability to have at least two jumps is bounded by the square of the above bound and we are done. □

Lemma 5.10 (Third term in (5.33)). *For all $\rho, \varepsilon > 0$ sufficiently small, satisfying* (5.46) *and* (5.61)*, we have*

$$P_{n,k}^{\varepsilon,\varrho}\ \leq\ 2^{-3n/2}+C2^{-n-\rho n/2}, \quad 2\leq k\leq 2^n+1,\ n\geq n_\varepsilon. \tag{5.69}$$

Proof. Immediate by Lemmas 5.7, 5.8, and 5.9. □

Proof of Lemma 5.4. Applying Lemmas 5.5, 5.6, and 5.10 to (5.33) we obtain

$$\mathbf{P}\left(\bigcup_{k=2}^{2^n+1}\left(A^\varepsilon\cap A_{n,k}\cap B_{n,k}^{-,\mathrm{c}}\right)\right)\ \leq\ c_{(5.34)}\,2^{-\rho n}+2^{-n/2}+C2^{-\rho n/2}+2^{-n/2} \tag{5.70}$$

for all $\rho, \varepsilon > 0$ sufficiently small satisfying (5.46) and (5.61) as well as all $n \geq n_\varepsilon$. Since these terms are summable in n, the claim of the lemma follows. □

Proof of Lemma 5.2. Immediate by (5.10) and Lemmas 5.3 and 5.4. □

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