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Convergence of solutions of kinetic variational inequalities in the rate-independent quasi-static limit

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Abstract

This paper discusses the convergence of kinetic variational inequalities to rate-independent quasi-static variational inequalities. Mathematical formulations as well as existence and uniqueness results for kinetic and rate-independent quasi-static problems are provided. Sharp a priori estimates for the kinetic problem are derived that imply that the kinetic solutions converge to the rate-independent ones, when the size of initial perturbations and the rate of application of the forces tend to 0. An application to three-dimensional elastic-plastic systems with hardening is given.

1 Introduction

Martins et al. [MS*04, MMP07] have discussed the connection between kinetic and quasi-static problems in mechanics, which is a problem of singular perturbations. They used the distinct time scales involved in kinetic and quasi-static problems, and performed a change of variables in the governing system of kinetic equations that consists of replacing the physical time t by a loading parameter $\tau = \varepsilon t$. This leads to a system of equations where the derivatives with respect to the loading parameter appear multiplied by ε. The quasi-static problem and its solutions are expected to be approached when ε tends to 0. In this paper the notions differ slightly from those in many engineering papers. On the one hand, often the term "quasi-static" is used for mechanical systems, where the kinetic term $M\ddot{q}$ is dropped, but various friction mechanisms (like viscous friction) may still be kept. This also includes the specific case of rate-independent friction, which is present in many plasticity models. Since we are interested in that case in the remainder of this paper we simply write "rate-independent system" to indicate "rate-independent quasi-static systems". On the other hand, we use the term "kinetic problem" for the mechanical problem with inclusion of the inertial term $M\ddot{q}$ (which is also often known as "dynamic problem").

We present here a generalization of the convergence result obtained in [MMP07] to general evolutionary variational inequalities including three-dimensional elastoplasticity with hardening. In contrast to [MMP07], where Yosida regularization and time differentiation were used, we rely on a difference quotient technique that is nicely adapted to nonsmooth variational inequalities and allows for relatively simple, explicit bounds. More precisely, we prove that the kinetic evolutions remain close to a rate-independent path when the load is applied sufficiently slowly and the kinetic evolutions start sufficiently close to that rate-independent path. In other words, we prove the *stability of the quasi-static path* in the sense of the definition in [MS*04, MMP07].

The paper is organized as follows. In Section 2, the mathematical formulations for kinetic and rate-independent problems as well as existence results are presented. We provide a priori estimates for the kinetic problem in Section 3. For example, using the slow time $\tau = \varepsilon t$ one of our results shows that the unique solution of the problem

$$\varepsilon^2 M \boldsymbol{q}''(\tau) + A\boldsymbol{q}(\tau) + \partial\mathcal{R}(\boldsymbol{q}'(\tau)) \ni \ell(\tau), \quad \boldsymbol{q}(0) = q_0, \quad \boldsymbol{q}'(0) = 0, \tag{1.1}$$

with $(\cdot)' = \frac{\mathrm{d}}{\mathrm{d}\tau}(\cdot)$ and $Aq_0 + \partial\mathcal{R}(0) \ni \ell(0)$, satisfies the a priori bound

$$\|\varepsilon M^{1/2}\boldsymbol{q}''(\tau)\|_H + \|A^{1/2}\boldsymbol{q}'(\tau)\|_H \le C\Big(\|\ell'(0)\|_* + \int_0^\tau \|\ell''(s)\|_* \,\mathrm{d}s\Big) \text{ for a.e } \tau \in [0, T_0],$$

where C is independent of ε, ℓ, and q_0. These estimates enable us to compare the kinetic solution to the rate-independent one in Section 4. If $\boldsymbol{q}_\varepsilon$ solves (1.1) and $\bar{q}$ solves (1.1) with $\varepsilon = 0$, we obtain

$$\begin{aligned}
&\big(\|\varepsilon M^{1/2}\boldsymbol{q}_\varepsilon'(\tau)\|_H^2 + \|A^{1/2}(\boldsymbol{q}_\varepsilon(\tau) - \bar{q}(\tau))\|_H^2\big)^{1/2} \\
&\le \big(\|\varepsilon M^{1/2}\boldsymbol{q}_\varepsilon'(0)\|_H^2 + \|A^{1/2}(\boldsymbol{q}_\varepsilon(0) - \bar{q}(0))\|_H^2\big)^{1/2} + C_\ell\sqrt{\varepsilon},
\end{aligned}$$

where C_ℓ is given explicitly in terms of $\ell \in \mathrm{W}^{2,1}([0, T_0]; \mathcal{D}(A^{-1/2}))$. In Section 5, we show that this convergence result can be applied for three-dimensional elastoplasticity with linear kinematic hardening.

2 Mathematical formulation

We start with a Hilbert space H with dual H^*, the dual pairing and the norm are respectively denoted by $\langle\,\cdot\,,\,\cdot\,\rangle : H \times H^* \to \mathbb{R}$ and $\|\cdot\|_H$. Let V be such that $V \subset H \subset V^*$ with dual V^*. We denote by $A : H \to H^*$ a symmetric, strictly positive operator with the domain of $A^{1/2}$ such that $\mathcal{D}(A^{1/2}) = V$. We use below the following norms: $\|\cdot\| \overset{\text{def}}{=} \sqrt{\langle A\,\cdot\,,\,\cdot\,\rangle}$, $\|\cdot\|_* \overset{\text{def}}{=} \sqrt{\langle\,\cdot\,, A^{-1}\,\cdot\,\rangle}$ and the following semi-norm: $|\cdot|_M \overset{\text{def}}{=} \|M^{1/2}\cdot\|_H$. We consider the variational inclusion

$$M\ddot{q}(t) + Aq(t) + \partial\mathcal{R}(\dot{q}(t)) \ni l(t), \tag{2.1}$$

where $(\dot{\ })$ denotes the time derivative $\frac{\mathrm{d}}{\mathrm{d}t}()$, M is a mass matrix operator, and l serves as input datum, also called external loading in mechanics. The dissipation functional $\mathcal{R} : H \to [0, \infty]$ is assumed to be convex, lower-semicontinuous, homogeneous of degree 1, i.e., $\mathcal{R}(\gamma q) = \gamma\mathcal{R}(q)$ for all $\gamma \ge 0$ and $q \in H$. Its subdifferential is given by $\partial\mathcal{R}(v) = \big\{\sigma \in H^* \,|\, \forall w \in H : \mathcal{R}(w) \ge \mathcal{R}(v) + \langle\sigma, w - v\rangle\big\}$. Using the definition of the subdifferential $\partial\mathcal{R}(\dot{q})$ leads to the *variational inequality*

$$\forall v \in H : \ \langle M\ddot{q} + Aq - l(t), v - \dot{q}\rangle + \mathcal{R}(v) - \mathcal{R}(\dot{q}) \ge 0. \tag{2.2}$$

The energy associated with (2.1) is given by $\mathcal{E}(t,q,\dot q)=\frac12\langle M\dot q,\dot q\rangle+\frac12\langle Aq,q\rangle-\langle l(t),q\rangle$. The corresponding rate-independent system is obtained from (2.1) rescaling time via $\tau=\varepsilon t$, letting $\ell(\tau)=l(\tau/\varepsilon)$ and taking the limit $\varepsilon\to 0$:

$$A\bar q(\tau)+\partial\mathcal{R}(\bar q'(\tau))\ni\ell(\tau), \tag{2.3}$$

where $(\cdot)'=\frac{\mathrm{d}}{\mathrm{d}\tau}(\cdot)$ and the energy is given by $\bar{\mathcal{E}}(\tau,\bar q,\dot{\bar q})=\frac12\langle A\bar q,\bar q\rangle-\langle\ell(\tau),\bar q\rangle$. Analogously to the kinetic system, the variational inequality associated with (2.3) is

$$\forall\bar v\in H:\ \langle A\bar q-\ell(\tau),\bar v-\bar q'\rangle+\mathcal{R}(\bar v)-\mathcal{R}(\bar q')\geq 0. \tag{2.4}$$

Since we are interested in elastoplasticity, we want to be able to treat the case that M is degenerate. Thus, we assume that it has block structure and that q decomposes into two components correspondingly, i.e., $H=H_1\times H_2$, $V=V_1\times V_2$, $q\overset{\mathrm{def}}{=}(u,z)$ with $u\in H_1$ and $z\in H_2$ and

$$M\overset{\mathrm{def}}{=}\begin{pmatrix} m & 0\\ 0 & 0\end{pmatrix},\quad A\overset{\mathrm{def}}{=}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix},\quad \mathcal{R}(\dot q)\overset{\mathrm{def}}{=}\widetilde{\mathcal{R}}(\dot z),\quad l(t)\overset{\mathrm{def}}{=}\begin{pmatrix} f(t)\\ 0\end{pmatrix}. \tag{2.5}$$

For m we assume that it is invertible, more precisely

$$m=m^*\in\mathrm{Lin}(H_1,H_1)\quad\text{ and }\quad m^{-1}\in\mathrm{Lin}(H_1,H_1). \tag{2.6}$$

We denote by $\mathcal{R}^*(\cdot)$ the Legendre transform of $\widetilde{\mathcal{R}}(\cdot)$. The assumptions on $\mathcal{R}$ imply that $\mathcal{R}^*$ has the form $I_{\{0\}\times K}$, where $K\overset{\mathrm{def}}{=}\partial\widetilde{\mathcal{R}}(0)$ is a closed convex set in H_2^*. Then using the previous notations, the variational inclusion (2.1) can be rewritten in a form that may be studied using the theory of maximal monotone operators, namely the governing kinetic system

$$\begin{cases} m\ddot u+ \qquad a_{11}u+a_{12}z &= f(t),\\ \qquad \dot z+\partial\tilde{\mathcal{R}}^*(a_{21}u+a_{22}z) &\ni 0,\end{cases} \tag{2.7}$$

together with the initial conditions

$$(\dot u(0),q(0))=(\dot u(0),u(0),z(0))=(\dot u_0,u_0,z_0)=(\dot u_0,q_0). \tag{2.8}$$

The rate-independent system (2.3) can be rewritten as

$$\begin{cases} \qquad a_{11}\bar u+a_{12}\bar z &= \bar f(\tau),\\ \bar z'+\partial\tilde{\mathcal{R}}^*(a_{21}\bar u+a_{22}\bar z) &\ni 0,\end{cases} \tag{2.9}$$

where $\bar f(\tau)=f(\tau/\varepsilon)$, with initial conditions

$$\bar q(0)=(\bar u(0),\bar z(0))=(\bar u_0,\bar z_0)=\bar q_0. \tag{2.10}$$

Existence and uniqueness results for the kinetic and the rate-independent problem follow from [Gr78b] and [Gr78a, Joh78], respectively. In what concerns the kinetic problem, the theory of maximal monotone operators is used for this purpose. The reader can find this theory in many textbooks, see, e.g., [Bre73, Zei85]. In [Gr78b] it is assumed that $l\in\mathrm{W}^{1,2}([0,T];H)$ but it can be easily proved, using [ShS97, Thm. A], that the same result remains valid for $l\in\mathrm{W}^{1,1}([0,T];H)$.

Proposition 2.1 *Assume that* $(\dot u_0, q_0) \in H_1 \times V$ *such that* $0 \in a_{21}u_0{+}a_{22}z_0{+}\partial\widetilde{\mathcal{R}}(0)$ *is satisfied and* $l = (f,0)^\mathsf{T} \in \mathrm{W}^{1,1}([0,T]; H_1{\times}\{0\})$*. Then there exists a unique solution* $q \in \mathrm{W}^{1,\infty}([0,T]; V)$ *that solves* (2.7) *and* (2.8)*. This solution additionally satisfies* $Mq \in \mathrm{W}^{2,\infty}([0,T]; H)$*.*

The existence and uniqueness theory for the rate-independent case is classical, see e.g. [Gr78a, Joh78] or the surveys [Kre99, Thm. 3.6], [Mie05, Thm. 2.1].

Proposition 2.2 *Assume that* $\bar q_0 \in V$ *and* $\ell \in \mathrm{W}^{1,\infty}([0,T_0]; V^*)$ *such that* $\ell(0) \in A\bar q_0 + \partial\mathcal{R}(0)$ *is satisfied. Then, the variational inequality* (2.4) *and hence also* (2.9) *have a unique solution* $\bar q \in \mathrm{W}^{1,\infty}([0,T_0]; V)$ *satisfying* (2.10)*.*

In particular, the solutions of (2.3) satisfy the relation

$$\langle A\bar q'(s), \bar q'(s)\rangle = \langle \ell'(s), \bar q'(s)\rangle \text{ for a.e. } s \in [0, T_0]. \tag{2.11}$$

Indeed, consider the variational inequality (2.4) with $\bar v = \lambda\bar q'(s)$, divide by λ and let $\lambda \to \infty$. We obtain $\langle A\bar q(\tau){-}\ell(\tau), \bar q'(s)\rangle + \mathcal{R}(\bar q'(s)) \geq 0$ for all $\tau \in [0,T_0]$ and a.a. $s \in [0,T_0]$. Moreover, for $s = \tau$ we have the opposite inequality by taking $\bar v = 0$ in (2.4). Differentiating with respect to τ we find (2.11), from which we easily obtain the a priori estimate

$$\|\bar q'(\tau)\| \leq \|\ell'(\tau)\|_* \text{ for a.a. } \tau \in [0, T_0]. \tag{2.12}$$

Remark 2.3 *In what concerns the rate-independent problem,* Johnson *[Joh78] formulates the plasticity problem as a variational inequality thereby extending the formulation of* Duvaut & Lions *[DuL76] to the case of a hardening material. Using Yosida regularization, the author has proved existence of a strong solution and, under some assumptions, he obtained a regularity result for the velocity field. Analogous results were obtained by* Gröger *[Gr78a], but remained largely unknown in the western world.*

3 A priori estimates for the kinetic problem

The aim of this section is to provide a priori estimates for the problem which allow us to control the term $M\ddot q$ in H instead of the usual estimates in V^*. The problem occurs through the fact that $\partial\mathcal{R}$ is nonsmooth and classical techniques for smooth problems do not suffice. One way to handle this is to use Yosida regularization leading to smooth systems and to show that a priori estimates stay uniform in the regularization parameter, see [MMP08]. Here we choose a different technique that is based on difference quotients.

To explain the methods we start with the basic energy estimate. We consider a solution and let $E(t) \overset{\text{def}}{=} \mathcal{E}(t, q(t), \dot q(t))$. Using $\langle \sigma, \dot q\rangle \geq 0$ for all $\sigma \in \partial\mathcal{R}(\dot q)$ we immediately find

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq -\langle \dot l(t), q(t)\rangle. \tag{3.1}$$

Our a priori estimates can be derived most easily by using the quadratic form $B : V \times H \times V^* \to \mathbb{R}$ defined via

$$B[q, \dot{q}, l] \stackrel{\text{def}}{=} |\dot{q}|_M^2 + \|q - A^{-1}l\|^2 + \|l\|_*^2. \tag{3.2}$$

The construction is such that for solutions we have

$$B[q(t), \dot{q}(t), l(t)] = 2\mathcal{E}(t, q(t), \dot{q}(t)) + 2\|l(t)\|_*^2 = 2E(t) + 2\|l(t)\|_*^2. \tag{3.3}$$

Moreover, one can notice that

$$\frac{1}{g^2}\big(\|q\|^2 + |\dot{q}|_M^2 + \|l\|_*^2\big) \leq B[q, \dot{q}, l] \leq g^2\big(\|q\|^2 + |\dot{q}|_M^2 + \|l\|_*^2\big), \tag{3.4}$$

with $g \stackrel{\text{def}}{=} \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio.

We now let $\beta(t) = B[q(t), \dot{q}(t), l(t)]$ and, combining (3.1), (3.3) and (3.4) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(t) \leq \|\dot{l}(t)\|_*\big(2\|q(t)\| + 4\|l(t)\|_*\big) \leq \|\dot{l}(t)\|_* 4g\sqrt{2}\sqrt{\beta(t)}.$$

Dividing by $2\sqrt{\beta}$ and integrating both sides we find the estimate

$$\sqrt{\beta(t)} \leq \sqrt{\beta(s)} + 2g\sqrt{2}\int_s^t \|\dot{l}(\tau)\|_* \,\mathrm{d}\tau \text{ for } 0 \leq s \leq t \leq T.$$

This provides a first, simple a priori bound for $(q, M^{1/2}\dot{q})$ in $V \times H$ in terms of the initial conditions, namely, using (3.4) we find

$$\begin{aligned}\big(\|q(t)\|^2 + |\dot{q}(t)|_M^2\big)^{1/2} &\leq g^2\big(\|q(0)\|^2 + |\dot{q}(0)|_M^2 + \|l(0)\|_*^2\big)^{1/2} \\ &\quad + 2g^2\sqrt{2}\int_0^t \|\dot{l}(\tau)\|_* \,\mathrm{d}\tau.\end{aligned} \tag{3.5}$$

Similarly, using (3.3) we obtain the a priori bound for the energy, namely

$$E(t) \leq \Big(\sqrt{E(0) + \|l(0)\|_*^2} + 2g\int_0^t \|\dot{l}(\tau)\|_* \,\mathrm{d}\tau\Big)^2 - \|l(t)\|_*^2.$$

The above estimates are just preliminary, but they already show the essential feature that the loading l appears on the right-hand side with a L^1 integral of $\dot{l}$ whereas the left-hand side provides an L^∞ estimate for $(q, M^{1/2}\dot{q})$ in $V \times H$. The crucial observation is now that the analogous estimate holds for the difference of two solutions, even if we treat different loadings l. These estimates are well known (see e.g., [Kre99, Mie05]) but we repeat it for the readers convenience and to have explicit constants.

Proposition 3.1 *Let $l_1, l_2 \in \mathrm{W}^{1,1}([0,T]; V^*)$ and q_1 and q_2 be solutions of* (2.1) *with right-hand sides l_1 and l_2 respectively, then $w = q_1 - q_2$ satisfies, for all $t \in [0,T]$, the estimate*

$$\begin{aligned}B[w(t), \dot{w}(t), l_1(t) - l_2(t)]^{1/2} &\leq B[w(0), \dot{w}(0), l_1(0) - l_2(0)]^{1/2} \\ &\quad + 2g\sqrt{2}\int_0^t \|\dot{l}_1(\tau) - \dot{l}_2(\tau)\|_* \,\mathrm{d}\tau.\end{aligned} \tag{3.6}$$

Proof. We use the variational inequalities (2.2) for q_1 and q_2 respectively and insert as test functions $v_1 = \dot q_2$ and $v_2 = \dot q_1$, respectively. Adding these two inequalities leads to a cancellation of all terms involving $\mathcal{R}$ and we find, with $l = l_1 - l_2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}|\dot w|_M^2 + \frac{1}{2}\|w\|^2 - \langle l, w\rangle\Big) \le -\langle \dot l, w\rangle.$$

For $\beta(t) = B[w(t), \dot w(t), l(t)]$ we find the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(t) \le \|\dot l(t)\|\big(2\|q(t)\| + 4\|l(t)\|\big) \le \|\dot l(t)\| 4g\sqrt{2}\sqrt{\beta(t)}.$$

Now (3.6) is obtained as above. □

We apply this result for deriving a priori estimates for the derivatives. Recall that Corollary 2.1 states $q \in \mathrm{W}^{1,\infty}([0,T];V)$ and $Mq \in \mathrm{W}^{2,\infty}([0,T];H)$. The idea is to consider difference quotients as a multiple of the difference between a solution and its time translation.

For arbitrary functions $y \in \mathrm{L}^\infty([0,T];Y)$, $h > 0$, and $t \in [0, T{-}h]$ we use the notation

$$\delta_h y(t) \stackrel{\text{def}}{=} \frac{1}{h}\big(y(t{+}h) - y(t)\big).$$

We use the fact that the norm of difference quotients can be bounded by the norm of the derivative. For all $p \in [1,\infty]$ and $y \in \mathrm{W}^{1,p}([0,T];Y)$ we have

$$\|\delta_h y\|_{\mathrm{L}^p([0,T-h];Y)} \le \|\dot y\|_{\mathrm{L}^p([0,T];Y)}. \tag{3.7}$$

For $p \in (1,\infty]$ the left-hand side even converges to the right-hand side for $h \to 0$.

Applying Proposition 3.1 with $q_1(t) = q(t{+}h)$ and $q_2(t) = q(t)$ and dividing by $h > 0$ we immediately find the a priori estimate

$$\begin{aligned} B[\delta_h q(t), \delta_h \dot q(t), \delta_h l(t)]^{1/2} &\le B[\delta_h q(s), \delta_h \dot q(s), \delta_h l(s)]^{1/2} \\ &\quad + 2g\sqrt{2}\int_s^t \|\delta_h \dot l(\tau)\|_* \,\mathrm{d}\tau, \end{aligned} \tag{3.8}$$

for all $0 \le s \le t \le T{-}h$. If it would be possible to pass to the limit $h \searrow 0$ on the right-hand side, then we would find the desired a priori bound for $(q, M^{1/2}\dot q)$ in $\mathrm{W}^{1,\infty}([0,T]; V \times H)$. However, in the general situation the initial conditions $q(0) = q_0 \in V$ and $\dot u(0) = \dot u_0 \in H_1$ do not guarantee the boundedness

$$\limsup_{h\searrow 0}\big(\|\delta_h q(0)\| + |\delta_h \dot q(0)|_M\big) < \infty.$$

Even the additional assumptions $\dot u_0 \in V_1$ and $l(0) \in \partial R(0) + Aq(0)$ do not help.

Here we have to make an additional assumption, which allows us to handle the nonsmoothness. For consistency we let $l_0 = l(0) = \lim_{h\searrow 0} l(h)$.

$$\begin{aligned} &\exists\,\rho > 0\ \exists\,\hat l \in \mathrm{W}^{2,1}([-\rho,0];V^*)\ \exists\, q = (u,z) \in \mathrm{W}^{1,\infty}([-\rho,0];V):\\ &(q(0),\dot u(0),\hat l(0)) = (q_0,\dot u_0,l_0),\ \ u \in \mathrm{W}^{2,\infty}([-\rho,0];H_1), \text{(2.1) is satisfied on } [-\rho,0]. \end{aligned} \tag{3.9}$$

Since l in (3.9) is defined on the t-interval $[-\rho, 0]$ the condition $l_0 = \hat{l}(0) = \lim_{s\nearrow 0} \hat{l}(s)$ is needed to guarantee that the concatenation of $\hat{l}$ and $l : [0, T] \to V^*$ is continuous, and we will denote this concatenation simply by $l : [-\rho, T] \to V^*$ in the sequel. This condition also implies that the stability condition $l(0) \in \partial R(0) + Aq_0$ holds and that the following limits for $h \searrow 0$ exist:

$$\delta_h q(-h) \to \dot{Q} \text{ in } V, \quad \delta_h \dot{u}(-h) \to \ddot{U} \text{ in } H_1, \quad \delta_h l(-h) \to \dot{L} \text{ in } V^*.$$

Remark 3.2 *There are two cases where this condition can be easily satisfied. The first one will be essential in the next section.*

(i) *If $\dot{u}_0 = 0$, then we may choose $q(t) = q_0$ for all $t \in [-\rho, 0]$ and let $\hat{l}(t) = l_0$. The limits then read $\dot{Q} = 0$, $\ddot{U} = 0$, and $\dot{L} = 0$.*

(ii) *If $\dot{u}_0 \in V_1$ and if the block structure of (2.5) is present, we may choose $q(t) = q_0 + t(\dot{u}_0, 0)^\mathsf{T}$ and let $\hat{l}(t) = l_0 + tA(\dot{u}_0, 0)^\mathsf{T}$. The limits here read $\dot{Q} = (\dot{u}_0, 0)^\mathsf{T}$, $\ddot{U} = 0$, and $\dot{L} = A(\dot{u}_0, 0)^\mathsf{T}$.*

Theorem 3.3 *Let $l \in \mathrm{W}^{2,1}([0, T]; V^*)$ and $(q_0, \dot{u}_0) \in V \times V_1$ be given such that condition (3.9) holds. Then, the unique solution q of (2.7) and (2.8) satisfies the a priori estimate*

$$\begin{aligned} B[\dot{q}(t), (\ddot{u}(t), 0)^\mathsf{T}, \dot{l}(t)]^{1/2} \leq{} & B[\dot{Q}, (\ddot{U}, 0)^\mathsf{T}, \dot{L}]^{1/2} \\ & + 2g\sqrt{2}\Big(\|\dot{L} - \dot{l}(0)\|_* + \int_0^t \|\ddot{l}(\tau)\|_* \,\mathrm{d}\tau \Big). \end{aligned} \tag{3.10}$$

Proof. The idea is to concatenate the artifical solution $q \in \mathrm{W}^{1,\infty}([-\rho, 0]; V)$ and the given solution $q \in \mathrm{W}^{1,\infty}([0, T]; V)$ as well as the loadings. The imposed conditions at $t = 0$ guarantee that we have a solution on all of $[-\rho, T]$ and estimate (3.8) holds for $-\rho \leq s \leq t \leq T-h$. In particular we may choose $s = -h$ and we see that the first term on the right-hand side of (3.8) converges to the first term on the right-hand side of (3.10).

The second term on the right-hand side of (3.8) can be estimated explicitly by taking care of the fact, that $l : [-h, T] \to V^*$ is defined piecewise. With

$$\int_{-h}^t \|\delta_h \dot{l}(\tau)\|_* \,\mathrm{d}\tau = \frac{1}{h} \int_{-h}^0 \|\dot{l}(\tau+h) - \dot{l}(\tau)\|_* \,\mathrm{d}\tau + \int_0^t \|\delta_h \dot{l}(\tau)\|_* \,\mathrm{d}\tau,$$

we see that the first term converges to $\|\dot{L} - \dot{l}(0)\|_*$, since on the one hand $\dot{l}(\tau) \to \dot{L}$ as $-h < \tau < 0$ and $\dot{l} \in \mathrm{C}^0([-\rho, 0]; V^*)$ and on the other hand $\dot{l}(\tau+h) \to \dot{l}(0)$ for the analogous reasons. Finally the second term can be estimated by (3.7) with $p = 1$ applied to $y = \dot{l}$, and the result is established. $\square$

4 Rate-independent limit $\varepsilon \to 0$

To consider systems with very slow loading rates we introduce the slow process time $\tau = \varepsilon t$ and assume that the loading l used in (2.1) and Section 3 is given in the form $l(t) = \ell(\varepsilon t)$, where now $\ell : [0, T_0] \to H^*$ is fixed, and the loading rate $\varepsilon > 0$ eventually tends to 0. We introduce

$$\boldsymbol{q}_\varepsilon(\tau) = (\boldsymbol{u}_\varepsilon(\tau), \boldsymbol{z}_\varepsilon(\tau)) \stackrel{\text{def}}{=} (u(\tau/\varepsilon), z(\tau/\varepsilon)) = q(\tau/\varepsilon)$$

for the solution as a function of the slow process time.

Applying this transformation to system (2.1) and using that the rate-independent friction term remains unchanged, as $\partial\mathcal{R}(\cdot)$ is homogeneous of degree 0, we arrive at the rescaled problems

$$\varepsilon^2 M \boldsymbol{q}_\varepsilon''(\tau) + A\boldsymbol{q}_\varepsilon(\tau) + \partial\mathcal{R}(\boldsymbol{q}_\varepsilon'(\tau)) \ni \ell(\tau), \quad \boldsymbol{q}_\varepsilon(0) = q_0, \quad \boldsymbol{u}_\varepsilon'(0) = u_1, \tag{4.1}$$

and

$$A\bar{q}(\tau) + \partial\mathcal{R}(\bar{q}'(\tau)) \ni \ell(\tau), \quad \bar{q}(0) = \bar{q}_0. \tag{4.2}$$

The whole theory in Section 3 remains valid when M is replaced by $\varepsilon^2 M$ and $\dot{(\cdot)}$ by $(\cdot)'$ with now $|\boldsymbol{q}_\varepsilon'|_{\varepsilon^2 M} = \varepsilon|\boldsymbol{q}_\varepsilon'|_M = \varepsilon\|M^{1/2}\boldsymbol{q}_\varepsilon'\|_H$.

As in [MMP08] we have the following estimate between the kinetic solution $\boldsymbol{q}_\varepsilon$ and the rate-independent quasistatic limit $\bar{q}$.

Proposition 4.1 *Assume that $\ell \in \mathrm{W}^{1,1}([0, T_0]; V^*)$, $(M(u_1, 0)^\mathsf{T}, q_0) \in H \times V$ and $\bar{q}_0 \in V$. Then, for all $\tau \in [0, T_0]$ we have*

$$\begin{aligned}\varepsilon^2|\boldsymbol{q}_\varepsilon'(\tau)|_M^2 + \|\boldsymbol{q}_\varepsilon(\tau) - \bar{q}(\tau)\|^2 \le{}& \varepsilon^2|(u_1, 0)^\mathsf{T}|_M^2 + \|q_0 - \bar{q}_0\|^2 \\ &+ 2\,\varepsilon^2 \Big(\operatorname*{ess\,sup}_{s\in[0,T_0]} |\boldsymbol{q}_\varepsilon''(s)|_M\Big) \int_0^\tau |\bar{q}'(s)|_M \,\mathrm{d}s.\end{aligned}$$

Proof. Theorem 3.3 guarantees that all quantities on the right-hand side are finite. To obtain the estimate we use the standard trick of adding the corresponding variational inequalities, cf. (2.2) and (2.4), but now in the slow process time. Choosing $v = \bar{q}'$ and $\bar{v} = \boldsymbol{q}_\varepsilon'$ all terms involving $\mathcal{R}$ cancel and we obtain $\langle \varepsilon^2 M\boldsymbol{q}_\varepsilon'', \boldsymbol{q}_\varepsilon' - \bar{q}'\rangle + \langle A(\boldsymbol{q}_\varepsilon - \bar{q}), \boldsymbol{q}_\varepsilon' - \bar{q}'\rangle \le 0$. Integrating over $[0, \tau]$ yields

$$\varepsilon^2|\boldsymbol{q}_\varepsilon'(\tau)|_M^2 + \|\boldsymbol{q}_\varepsilon(\tau) - \bar{q}(\tau)\|^2 \le \varepsilon^2|(u_1, 0)^\mathsf{T}|_M^2 + \|q_0 - \bar{q}_0\|^2 + 2\int_0^\tau \langle \varepsilon^2 M\boldsymbol{q}_\varepsilon''(s), \bar{q}'(s)\rangle \,\mathrm{d}s.$$

The Cauchy-Schwarz inequality and taking out the essential supremum provides the desired result. ☐

The final result provides an estimate between $\boldsymbol{q}_\varepsilon$ and $\bar{q}$ that is explicitly given in terms of the data. For this we need the a priori estimates on the solutions $\bar{q}$ and

$\boldsymbol{q}_\varepsilon$ derived in Section 2 and 3, respectively. Moreover, following [MMP08] we will estimate the distance between $\boldsymbol{q}_\varepsilon$ and $\bar q$ by introducing a special intermediate solution $\hat{\boldsymbol{q}}_\varepsilon$ for which Theorem 3.3 is applicable. In our case, this special kinetic solution $\hat{\boldsymbol{q}}_\varepsilon = (\hat{\boldsymbol{u}}_\varepsilon, \hat{\boldsymbol{z}}_\varepsilon)$ satisfies (4.1) together with initial conditions $(\hat{\boldsymbol{u}}_\varepsilon'(0), \hat{\boldsymbol{u}}_\varepsilon(0), \hat{\boldsymbol{z}}_\varepsilon(0)) = (0, \bar u_0, \bar z_0)$. In particular, we impose that the initial velocity $\hat{\boldsymbol{u}}_\varepsilon'(0) = 0$ whereas in [MMP08] the initial velocity $\tilde{\boldsymbol{u}}_\varepsilon'(0) = \varepsilon \bar u_0'$ was used, which lead to the additional assumption $\bar u_0' \in V_1$, which is not needed any more. Nevertheless, our final estimate (4.3) is the same as the one obtained in [MMP08].

Theorem 4.2 *Let the above assumptions on M, A and $\mathcal{R}$ hold and assume $\ell = (\bar f, 0)^\mathsf{T} \in \mathrm{W}^{2,1}([0,T_0], V^*)$. For $\bar q_0 \in V$ with $\ell(0) \in \partial\mathcal{R}(0) + A\bar q_0 \subset \{0\} \times H_2$ let $\bar q$ be the unique solution of (4.2). For arbitrary $q_0 = (u_0, z_0) \in V$ and $u_1 \in H_1$ satisfying $0 \in a_{21}u_0 + a_{22}z_0 + \partial\widetilde{\mathcal{R}}(0)$ let $\boldsymbol{q}_\varepsilon$ be the unique solution of (4.1). Then the difference between $\boldsymbol{q}_\varepsilon$ and $\bar q$ can be estimated via*

$$\big(|\varepsilon \boldsymbol{q}_\varepsilon'(\tau)|_M^2 + \|\boldsymbol{q}_\varepsilon(\tau) - \bar q(\tau)\|^2\big)^{1/2} \le \big(|(\varepsilon u_1, 0)^\mathsf{T}|_M^2 + \|q_0 - \bar q_0\|^2\big)^{1/2} + \sqrt{\varepsilon C[\ell](\tau)}, \quad (4.3)$$

where

$$C[\ell](\tau) \overset{\text{def}}{=} 2g^2\sqrt{2}\,\mu \int_0^\tau \|\ell'(s)\|_* \,\mathrm{d}s \Big(\|\ell'(0)\|_* + \int_0^\tau \|\ell''(s)\|_* \,\mathrm{d}s\Big) \quad \textit{and} \quad \mu \overset{\text{def}}{=} \sup_{\|v\|=1} |v|_M.$$

Proof. Proposition 2.1 provides the existence of the special kinetic solution $\hat{\boldsymbol{q}}_\varepsilon$ solving (4.1) with $\hat{\boldsymbol{q}}_\varepsilon(0) = \bar q_0$ and $\hat{\boldsymbol{u}}'(0) = 0$. This choice allows us to satisfy condition (3.9) via Part (i) in Remark 3.2. Using $Q' = U'' = L' = 0$ estimate (3.10) provides the a priori bound

$$\varepsilon|\hat{\boldsymbol{q}}_\varepsilon''(\tau)|_M \le gB[\hat{\boldsymbol{q}}_\varepsilon'(\tau), \hat{\boldsymbol{q}}_\varepsilon''(\tau), \ell'(\tau)]^{1/2} \le gB[0,0,0]^{1/2} + C_1[\ell](\tau),$$

with $C_1[\ell](\tau) \overset{\text{def}}{=} 2g^2\sqrt{2}\Big(\|\ell'(0)\|_* + \int_0^\tau \|\ell''(s)\|_* \,\mathrm{d}s\Big)$, and the right-hand side is independent of ε. Now Proposition 4.1 can be used to obtain

$$\varepsilon^2|\hat{\boldsymbol{q}}_\varepsilon'(\tau)|_M^2 + \|\hat{\boldsymbol{q}}_\varepsilon(\tau) - \bar q(\tau)\|^2 \le 2C_1[\ell](\tau) \int_0^\tau \varepsilon|\bar q'(s)|_M \,\mathrm{d}s \le \varepsilon C[\ell](\tau), \qquad (4.4)$$

where we used $|\bar q'(s)|_M \le \mu\|\bar q'(s)\| \le \mu\|\ell'(s)\|_*$ with the last estimate following from (2.12). For the difference between the given solution $\boldsymbol{q}_\varepsilon$ and the special solution $\hat{\boldsymbol{q}}_\varepsilon$ we use Proposition 3.1 and obtain, because of $\ell_1 = \ell_2 = \ell$, the simple estimate

$$\begin{aligned} &\varepsilon^2|\boldsymbol{q}_\varepsilon'(\tau) - \hat{\boldsymbol{q}}_\varepsilon'(\tau)|_M^2 + \|\boldsymbol{q}_\varepsilon(\tau) - \hat{\boldsymbol{q}}_\varepsilon(\tau)\|^2 \\ &\le \varepsilon^2|(u_1, 0)^\mathsf{T}|_M^2 + \|q_0 - \bar q_0\|^2 \text{ for all } \tau \in [0, T_0]. \end{aligned} \qquad (4.5)$$

Taking the square roots of (4.4) and (4.5) and using the triangle inequality gives the desired result. ☐

In [DuL76, Ch. V.3.5] the limit $\varepsilon \to 0$ is used to prove existence for the quasistatic case. However, the viscoplastic case is treated there, i.e., the viscosity parameter

$\mu > 0$ (see [DuL76, p.234]) and the necessary ε-independent a priori estimates corresponding to our estimate (3.10) are simply obtained by differentiating in time. The convergence stated in [DuL76] is weak $*$ only, whereas our result provides quantitative error estimates.

5 Elastic-plastic systems with hardening

We relate now the result obtained in the Theorem 4.2 to an elastic-plastic model with linear kinematic hardening which leads to a generalization of the convergence result obtained by Martins et al. in [MMP08].

We consider a material with a reference configuration $\Omega \subset \mathbb{R}^d$ with $d \in \{2,3\}$. We assume that Ω is an open bounded set with a 1-regular smooth boundary (see [RaT83]) and $|\Omega| < \infty$. This body may undergo displacements $u(\tau,\cdot) : \Omega \to \mathbb{R}^d$. The plastic strain will be characterized by $z = e_{\rm pl} : \Omega \to \mathbb{S}^d_0$ where $\mathbb{S}^d_0$ is the space of symmetric $d \times d$ tensors with vanishing trace. Further, we will denote by $\mathbb{S}^d$ the space of symmetric $d \times d$ tensors endowed with the scalar product $v{:}w \stackrel{\text{def}}{=} \operatorname{tr}(v^\mathsf{T} w)$ and the corresponding norm is given by $|v|^2 \stackrel{\text{def}}{=} v{:}v$ for all $v, w \in \mathbb{S}^d$. Here $\operatorname{tr}(\cdot)$ denotes the trace of the matrix $(\cdot)$.

The set of admissible displacements $\mathcal{F}$ is chosen as a suitable subset of $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^d)$ by prescribing Dirichlet data on the subset $\Gamma_{\rm Dir}$ of $\partial\Omega$, i.e.,

$$\mathcal{F} \stackrel{\text{def}}{=} \big\{ u \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^d) \,|\, u_{|\Gamma_{\rm Dir}} = 0 \big\}.$$

The plastic variables $e_{\rm pl}$ belongs to $\mathcal{Z} \stackrel{\text{def}}{=} \mathrm{L}^2(\Omega;\mathbb{S}^d_0)$ and the linearized strain tensor $e = \boldsymbol{e}(u)$ is given by $\boldsymbol{e}(u) \stackrel{\text{def}}{=} \frac{1}{2}(\nabla u + \nabla u^\mathsf{T}) \in \mathbb{S}^d$. We assume that $\partial\Omega$ is smooth enough and that $\operatorname{mes}(\Gamma_{\rm Dir}) > 0$ such that the Korn's inequality holds, i.e. there exists $c_{\rm Korn} > 0$ with

$$\forall u \in \mathcal{F} : \int_\Omega |\boldsymbol{e}(u)|^2 \,\mathrm{d}x \geq c_{\rm Korn} \|u\|^2_{\mathrm{W}^{1,2}}. \tag{5.1}$$

For more details on Korn's inequality and its consequences, we refer to [DuL76] or [KoO88].

We consider now the following kinetic equation

$$\varepsilon^2 \rho u'' - \operatorname{div}\big(\mathbb{E}(\boldsymbol{e}(u) - e_{\rm pl})\big) = \ell_{\rm ext}(\tau), \quad x \in \Omega, \quad \tau \in [0, T_0], \tag{5.2}$$

where $\rho > 0$, $\ell_{\rm ext}$ are the density and the applied mechanical loading respectively; $\mathbb{E}$ is a symmetric, uniformly positive definite elasticity tensor. The behavior of plastic element is characterized by the plastic flow rule in the form

$$-\mathbb{E}(\boldsymbol{e}(u) - e_{\rm pl}) + \mathbb{H} e_{\rm pl} + \partial R(e'_{\rm pl}) \ni 0, \quad x \in \Omega, \quad \tau \in [0, T_0], \tag{5.3}$$

where $\mathbb{H}$ is a symmetric, uniformly positive definite hardening tensor. The dissipation potential is given by

$$\widetilde{\mathcal{R}}(e'_{\rm pl}) \stackrel{\text{def}}{=} \int_\Omega R(x, e'_{\rm pl}(x)) \,\mathrm{d}x,$$

with $R \in \mathrm{L}^\infty_{\mathrm{loc}}(\bar\Omega \times \mathbb{S}^d_0)$ such that there exist r_1, r_2 with $0 < r_1 < r_2$ with

$$\forall\,(x,v) \in \bar\Omega \times \mathbb{S}^d_0 :\ r_1|v| \le R(x,v) \le r_2|v|.$$

We assume also that $R(x,\cdot) : \mathbb{S}^d_0 \to [0,\infty)$ is 1-homogeneous and convex. Notice that (5.3) is equivalent to

$$e'_{\mathrm{pl}} \in \partial R^*\big(\mathbb{E}(\boldsymbol{e}(u){-}e_{\mathrm{pl}}) - \mathbb{H}e_{\mathrm{pl}}\big), \quad x \in \Omega, \quad \tau \in [0,T_0], \tag{5.4}$$

where R^* is the Legendre transform of $R(\cdot)$. From (5.2) and (5.4), we finally obtain the governing system

$$\left.\begin{array}{rcl} \varepsilon^2\rho u'' &=& \mathrm{div}\big(\mathbb{E}(\boldsymbol{e}(u){-}e_{\mathrm{pl}})\big) - \ell_{\mathrm{ext}}(\tau), \\ e'_{\mathrm{pl}} &\in& \partial R^*\big(\mathbb{E}(\boldsymbol{e}(u){-}e_{\mathrm{pl}}) - \mathbb{H}e_{\mathrm{pl}}\big), \end{array}\right\} \quad x \in \Omega, \quad \tau \in [0,T_0], \tag{5.5}$$

together with Dirichlet boundary conditions

$$u = 0 \ \text{ on } \ \Gamma_{\mathrm{Dir}} \times [0,T_0], \tag{5.6}$$

and initial conditions

$$(u(0), u'(0), e_{\mathrm{pl}}(0)) = (u_0, u_1, e^0_{\mathrm{pl}}). \tag{5.7}$$

The corresponding rate-independent system is then

$$\left.\begin{array}{rcl} 0 &=& \mathrm{div}\big(\mathbb{E}(\boldsymbol{e}(u){-}e_{\mathrm{pl}})\big) - \ell_{\mathrm{ext}}(\tau), \\ e'_{\mathrm{pl}} &\in& \partial R^*\big(\mathbb{E}(\boldsymbol{e}(u){-}e_{\mathrm{pl}}) - \mathbb{H}e_{\mathrm{pl}}\big), \end{array}\right\} \quad x \in \Omega, \quad \tau \in [0,T_0], \tag{5.8}$$

with Dirichlet boundary conditions $\bar u = 0$ on $\Gamma_{\mathrm{Dir}} \times [0,T_0]$, and initial conditions

$$(\bar u(0), \bar e_{\mathrm{pl}}(0)) = (\bar u_0, \bar e^0_{\mathrm{pl}}). \tag{5.9}$$

Further, the energy associated with (5.5) is given by

$$\mathcal{E}(\tau,u,e_{\mathrm{pl}},u') = \frac12 \int_\Omega \big(\rho|\varepsilon u'|^2 + (\boldsymbol{e}(u){-}e_{\mathrm{pl}}){:}\mathbb{E}(\boldsymbol{e}(u){-}e_{\mathrm{pl}}) + e_{\mathrm{pl}}{:}\mathbb{H}e_{\mathrm{pl}}\big)\,\mathrm{d}x - \langle \ell_{\mathrm{ext}}(\tau), u\rangle.$$

For a given external loading ℓ_{ext}, a given elasticity tensor $\mathbb{E}$ and a given hardening tensor $\mathbb{H}$ with

$$\ell_{\mathrm{ext}} \in \mathrm{C}^1([0,T_0]; \mathrm{W}^{1,2}(\Omega;\mathbb{R}^d)^*), \tag{5.10a}$$

$$\mathbb{E} \in \mathrm{L}^\infty(\Omega; \mathrm{Lin}(\mathbb{S}^d,\mathbb{S}^d)) \text{ with } \mathbb{E}(x) \ge \eta\mathbf{1} \text{ a.e.}, \tag{5.10b}$$

$$\mathbb{H} \in \mathrm{L}^\infty(\Omega; \mathrm{Lin}(\mathbb{S}^d_0,\mathbb{S}^d_0)) \text{ with } \mathbb{H}(x) \ge \eta\mathbf{1} \text{ a.e.}, \tag{5.10c}$$

where $\eta > 0$, we recall existence and uniqueness result for kinetic and rate-independent problems. First, one can identify $H = H_1 \times H_2 \overset{\mathrm{def}}{=} \mathrm{L}^2(\Omega;\mathbb{R}^d) \times \mathrm{L}^2(\Omega;\mathbb{S}^d_0)$ and $V = V_1 \times V_2 \overset{\mathrm{def}}{=} \mathcal{F} \times H_2$. Second, (5.5) and (5.8) can be rewritten in the form of (2.7) and (2.9), respectively. More precisely, one has to choose $a_{11} = -\mathrm{div}(\mathbb{E}\boldsymbol{e}(\cdot))$, $a_{12} = \mathrm{div}(\mathbb{E}(\cdot))$, $a_{21} = -\mathbb{E}\boldsymbol{e}(\cdot)$, $a_{22} = \mathbb{E}(\cdot){+}\mathbb{H}(\cdot)$, $f(\tau) = \ell_{\mathrm{ext}}(\tau)$ and $m = \varepsilon^2\rho$. Then, Proposition 2.1 gives the following result.

Proposition 5.1 *Assume that* (5.10) *holds and that* $(u_0, u_0', e_{\mathrm{pl}}^0) \in \mathcal{F} \times \mathcal{F} \times \mathcal{Z}$ *such that* $0 \in \mathbb{E}(e_{\mathrm{pl}^0} - \boldsymbol{e}(u_0)) + \mathbb{H}e_{\mathrm{pl}}^0 + \partial R(0) \subset \mathcal{Z}$ *is satisfied. Then there exists a unique solution* $(u, e_{\mathrm{pl}}) \in \mathrm{W}^{1,\infty}([0,T_0];V)$ *that solves* (5.6) *and* (5.7).

The existence and uniqueness theory for the rate-independent elastoplasticity problem is standard, see [Gr78a, Joh78].

Proposition 5.2 *Assume that* (5.10) *holds and that* $(\bar{u}_0, \bar{e}_{\mathrm{pl}}^0) \in \mathcal{F} \times \mathcal{Z}$ *such that* $(\ell_{\mathrm{ext}}(0), 0)^\mathsf{T} \in A(\bar{u}_0, \bar{e}_{\mathrm{pl}}^0)^\mathsf{T} + \{0\} \times \partial R(0) \subset \mathcal{F} \times \mathcal{Z}$ *is satisfied. Then, there exists a unique solution* $(\bar{u}, \bar{e}_{\mathrm{pl}}) \in \mathrm{W}^{1,\infty}([0,T_0];V)$ *that solves* (5.8) *and* (5.9).

Applying Theorem 4.2 and using (5.1), we deduce the following result.

Corollary 5.3 *Assume that* (5.10) *holds and* $(u_0, u_0', e_{\mathrm{pl}}^0) \in \mathcal{F} \times \mathcal{F} \times \mathcal{Z}$ *and* $(\bar{u}_0, \bar{e}_{\mathrm{pl}}^0) \in \mathcal{F} \times \mathcal{Z}$ *satisfy* $0 \in \mathbb{E}(e_{\mathrm{pl}}^0 - \boldsymbol{e}(u_0)) + \mathbb{H}e_{\mathrm{pl}}^0 + \partial R(0)$ *and* $(\ell_{\mathrm{ext}}(0), 0)^\mathsf{T} \in A(\bar{u}_0, \bar{e}_{\mathrm{pl}}^0)^\mathsf{T} + \{0\} \times \partial R(0)$, *respectively. Then there exist* $c, C > 0$ *such that for all* $\varepsilon > 0$, *we have*

$$\begin{aligned}&\big(\|\varepsilon\rho^{1/2}u'(\tau)\|_{\mathrm{L}^2}^2 + \|u(\tau) - \bar{u}(\tau)\|_{\mathrm{W}^{1,2}}^2 + \|e_{\mathrm{pl}}(\tau) - \bar{e}_{\mathrm{pl}}(\tau)\|_{\mathrm{L}^2}^2\big)^{1/2}\\ &\leq c\big(\|\varepsilon\rho^{1/2}u_0'\|_{\mathrm{L}^2}^2 + \|u_0 - \bar{u}_0\|_{\mathrm{W}^{1,2}}^2 + \|e_{\mathrm{pl}}^0 - \bar{e}_{\mathrm{pl}}^0\|_{\mathrm{L}^2}^2\big)^{1/2} + C\sqrt{\varepsilon},\end{aligned}$$

for all $\tau \in [0, T_0]$.

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