How 1-dimensional hyperbolic attractors determine their basins

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Abstract. Two attractors Λ_i $(i = 1, 2)$ of diffeomorphisms $f_i\colon M_i \to M_i$ will be called intrinsically equivalent if there is a homeomorphism $h\colon \Lambda_1 \to \Lambda_2$ satisfying $f_2 h = h f_1$. If we can find a homeomorphism $g\colon W^s_{\Lambda_1} \to W^s_{\Lambda_2}$ of the basins $W^s_{\Lambda_i}$ of Λ_i such that $f_2 g = g f_1$, then we say that Λ_1, Λ_2 are basin equivalent. Let Λ_1, Λ_2 be transversely tame 1-dimensional hyperbolic attractors which are intrinsically equivalent. Then, if $W^s_{\Lambda_1}, W^s_{\Lambda_2}$ are orientable and $m = \dim M_1 = \dim M_2 \geq 4$, it is shown that Λ_1, Λ_2 are basin equivalent, provided these attractors are regarded, for some positive integer k, as attractors of f_1^k, f_2^k instead of f_1, f_2, respectively. This conclusion implies that $W^s_{\Lambda_1}, W^s_{\Lambda_2}$ are homeomorphic under a homeomorphism which maps Λ_1 to Λ_2 and the stable foliation $\mathfrak{W}^s_{\Lambda_1}$ of $W^s_{\Lambda_1}$ to the stable foliation $\mathfrak{W}^s_{\Lambda_2}$ of $W^s_{\Lambda_2}$. (To be transversely tame is a weak restriction; hence, roughly, speaking, these facts hold for "almost all" 1-dimensional hyperbolic attractors.) If transverse tameness and $m \geq 4$ is dropped from the assumption, then still the cartesian products $W^s_{\Lambda_i} \times \mathbb{R}$ are homeomorphic with a homeomorphism which maps $\Lambda_1 \times \{0\}$ to $\Lambda_2 \times \{0\}$ and $\mathfrak{W}^s_{\Lambda_1} \times \mathbb{R}$ to $\mathfrak{W}^s_{\Lambda_2} \times \mathbb{R}$.

1. Introduction

Let f be a diffeomorphism of an n-dimensional Riemannian C^∞ manifold M without boundary onto itself. (In this paper "differentiable" or "smooth" means "of class C^1".) By an attractor we mean a compact subset Λ of M which is invariant (i.e. $f(\Lambda) = \Lambda$) and attracts all points which are sufficiently close to Λ in the sense that there is a neighbourhood U of Λ in M such that

$$f(U) \subset U, \bigcap_{i \geq 0} f^i(U) = \Lambda.$$

For a description of an attractor Λ besides its intrinsic structure the structure of its basin

$$W^s_\Lambda = \{p \in M \mid \lim_{i \to \infty} \operatorname{dist}(f^i(p), \Lambda) = 0\},$$

i.e. the set of all points which are attracted by Λ, deserves our interest. We say that attractors Λ, Λ' of diffeomorphisms $f : M \to M, f' : M' \to M'$ are *intrinsically equivalent* if there is a homeomorphism $h : \Lambda \to \Lambda'$ such that $hf = f'h$ on Λ. The intrinsic type of Λ, i.e. the class of all attractors which are intrinsically equivalent to Λ, will be denoted by $\overline{\Lambda}$, and we say that Λ is a realization of $\overline{\Lambda}$. If there is a homeomorphism $g : W^s_\Lambda \to W^s_{\Lambda'}$, satisfying $gf = f'g$ on $W^s_{\Lambda'}$, then we say that Λ and Λ' have the same *basin type* or that they are *basin equivalent*. Obviously g maps Λ to Λ'. Therefore, if attractors Λ, Λ' have the same basin type then they are intrinsically equivalent, and the pairs (W^s_Λ, Λ) and $(W^s_{\Lambda'}, \Lambda')$ are homeomorphic in the sense that there is a homeomorphism which maps W^s_Λ to W^s_Λ, and Λ to Λ'.

In this paper we try to show to what extent the intrinsic type of an attractor Λ determines its basin type. Though facts similar to those proved below seem to hold in more general situations this aim motivates the restriction to a class of attractors whose intrinsic structure is well known. This class will consist of all 1-dimensional hyperbolic attractors i.e. attractors which are basic sets in the sense of S. Smale [9] and whose topological dimension is 1. Due to R.F. Williams [8], [12], [13] we have a satisfactory description of the intrinsic structure of these attractors.

Locally a 1-dimensional hyperbolic attractor Λ in an m-manifold $M(m \geq 2)$ can be described as follows. If $x \in \Lambda$ then there is a Cantor set C in the $(m-1)$-dimensional unit ball $\mathbb{D}^{m-1}$ and a homeomorphism h of $\mathbb{D}^1 \times \mathbb{D}^{m-1}$ onto a neighbourhood V of x in M such that $V \cap \Lambda = h(\mathbb{D}^1 \times C)$. (Here $\mathbb{D}^1$, the 1-dimensional unit ball, is the interval $[-1, 1]$.) This implies that the arc component W^u_x of x in Λ is the image of a C^0 immersion of $\mathbb{R}$ in M. It can be shown that W^u_x is a C^1 curve which is infinitely long in both directions. Indeed, W_x^u is the unstable manifold

$$W_x^u = \left\{ y \in M | \lim_{i\to-\infty} d(f^i(y), f^i(x)) = 0 \right\}$$

of x. By our definition of attractors Λ has finitely many components which are permuted by f, and since Λ is a basic set this permutation is cyclic. This justifies to assume, as we shall do, that Λ is connected (but not arcwise connected, of course). Then W_x^u is dense in Λ. For $y \in \Lambda$ the stable manifold

$$W_y^s = \left\{ z \in M | \lim_{i\to\infty} d(f^i(z), f^i(y)) = 0 \right\}$$

of y is a dense subset of W_Λ^s, and W_y^s can be obtained as the image of an injective C^1 immersion of $\mathbb{R}^{m-1}$ into W_Λ^s. The family of all these stable manifolds W_y^s $(y \in \Lambda)$ is a C^1 foliation of W_Λ^s which will be called the stable foliation of W_Λ^s and denoted by $\mathfrak{W}_\Lambda^s$.

By a transverse section S of Λ at a point $x \in \Lambda$ we mean an $(m-1)$-dimensional compact submanifold with boundary of W_x^s such that $x \in S\cap\Lambda = \operatorname{Int} S\cap\Lambda$. Then $S\cap\Lambda$ is a Cantor set. If $m-1 \geq 3$ then this Cantor set can be wildly imbedded in S, e.g., it can be similar to Antoine's necklace in $\mathbb{R}^3$ (see [1], [4]). This wild behaviour is excluded if we assume that $S\cap\Lambda$ can be covered by arbitrary small disjoint compact topological $(m-1)$-balls in S. Then $S\cap\Lambda$ is called a tame Cantor set in S, and we say that Λ is *transversely tame*, if for each transverse section S the set $S\cap\Lambda$ is tame in S. To prove that Λ is transversely tame it is sufficient that $S\cap\Lambda$ is tame for at least one S. For each at least 4-dimensional manifold M there are diffeomorphisms with transversely wild 1-dimensional hyperbolic attractors (see [4]), however it is not easy to find such examples (at least for $\dim M > 4$). Moreover a necessary (but by no means sufficient) condition for transverse wildness is $\dim_H \Lambda \geq \dim M - 2$, where $\dim_H$ denotes the Hausdorff dimension (see [11], [12]). Therefore some acquaintance with transversely wild attractors suggests the opinion that excluding transversely wild attractors, as we shall do in the main part of this paper, is a mild restriction. (For some facts concerning the basins of transversely wild 1-dimensional hyperbolic attractors see Corollary 1.3 below.)

There are four simple reasons why two 1-dimensional hyperbolic attractors Λ_i $(i = 1, 2)$ of diffeomorphisms $f_i \colon M_i \to M_i$, though intrinsically equivalent, can have topological different pairs $(W_{\Lambda_1}^s, \Lambda_1)$, $(W_{\Lambda_2}^s, \Lambda_2)$.

(a_0) The dimensions m_i of M_i can be different.

(b_0) The impact of Λ_i on $W_{\Lambda_i}^s$ is not strong enough to determine orientability or non orientability of $W_{\Lambda_i}^s$.

(c_0) It can happen that the strings W^u_x of Λ_i are tangled in $W^s_{\Lambda_i}$ in different ways.

(d_0) For transverse sections S_i of Λ_i the Cantor sets $S_i \cap \Lambda_i$ can have different structure.

This suggests the following assumptions.

(a) $\dim M_1 = \dim M_2 = m$.

(b) $W^s_{\Lambda_1}, W^s_{\Lambda_2}$ are orientable.

(c) $m \geq 4$. (Smooth curves can not be linked in at least 4-dimensional manifolds.)

(d) Λ is transversely tame.

Corollary 1.1 below shows that these conditions are in fact sufficient for the topological equivalence of the pairs $(W^s_{\Lambda_1}, \Lambda_1), (W^s_{\Lambda_2}, \Lambda_2)$. The following main result of this paper states a fact which is a little stronger than this equivalence.

Main Theorem. *Let Λ_i $(i = 1, 2)$ be transversely tame 1-dimensional hyperbolic attractors of diffeomorphismus $f_i\colon M_i \to M_i$, where $\dim M_1 = \dim M_2 \geq 4$ and the basins of Λ_1, Λ_2 are orientable. Then there is a positive integer k such that Λ_1, Λ_2 are basin equivalent, provided these attractors are regarded as attractors of f_1^k, f_2^k instead of f_1, f_2, respectively. (For a description of the common basin type of Λ_1, Λ_2 as attractors of f_1^k, f_2^k, respectively, see* Corollary 3.2 *in* Section 3.*)*

Remark. Examples (the attractors $\Lambda_{\underline{l}}$ in Section 3 e.g.) show that $k = 1$ is not always possible (f_i can twist $W^s_{\Lambda_i}$). If $M_1 = M_2 = \mathbb{R}^m$ then, presumably, the theorem holds with $k = 1$, but we can not prove this.

Corollary 1.1. *Let M_i, f_i, Λ_i $(i = 1, 2)$ be as in the main theorem. Then there is a homeomorphism $h\colon W^s_{\Lambda_1} \to W^s_{\Lambda_2}$ which maps Λ_1 to Λ_2 and the stable foliation $\mathfrak{W}^s_{\Lambda_1}$ to $\mathfrak{W}^s_{\Lambda_2}$.*

Proof. $W^s_{\Lambda_i}$ and $\mathfrak{W}^s_{\Lambda_i}$ remain unchanged if Λ_i is regarded as an attractor of f_i^k instead of f_i. □

Corollary 1.2. *Let Λ be a transversely tame 1-dimensional hyperbolic attractor with orientable basin in an m-dimensional manifold. Then, if $m \geq 4, W^s_\Lambda$ is homeomorphic to an open subset of $\mathbb{R}^m$. (Even more, a result in [5] will imply that there is an open set G in $\mathbb{R}^3$ such that W^s_Λ is homeomorphic to $G \times \mathbb{R}^{m-3}$.)*

Proof. Since $m \geq 4$ it is easy to construct a diffeomorphism $f\colon \mathbb{R}^m \to \mathbb{R}^m$ with a transversely tame 1-dimensional hyperbolic attractor Λ' which is intrinsically equivalent to Λ. Then by Corollary 1.1 W^s_Λ is homeomorphic to $W^s_{\Lambda'}$. □

Corollary 1.3. *Let Λ_i $(i = 1, 2)$ be intrinsically equivalent 1-dimensional hyperbolic attractors (not necessary transversely tame) of diffeomorphisms $f_i\colon M_i \to M_i$, respectively, where M_1, M_2 are manifolds of the same dimension $m \geq 3$. Then, if $W^s_{\Lambda_1}, W^s_{\Lambda_2}$ are orientable, there is a homeomorphism $h\colon W^s_{\Lambda_1} \times \mathbb{R} \to W^s_{\Lambda_2} \times \mathbb{R}$ which maps $\Lambda_1 \times \{0\}$ to $\Lambda_2 \times \{0\}$ and $\mathfrak{W}^s_{\Lambda_1} \times \mathbb{R}$ to $\mathfrak{W}^s_{\Lambda_2} \times \mathbb{R}$. (This does not imply that $W^s_{\Lambda_1}, W^s_{\Lambda_2}$ are homeomorphic.)*

Proof. We consider the diffeomorphisms $\tilde{f}_i : M_i \times \mathbb{R} \to M_i \times \mathbb{R}$ which are defined by $\tilde{f}_i(x,t) = (f_i(x), \frac{1}{2}t)$. Then for $i = 1, 2$ we have the 1-dimensional hyperbolic attractor $\tilde{\Lambda}_i = \Lambda_i \times \{0\}$ of $\tilde{f}_i$, and this attractor is obviously intrinsically equivalent to Λ_i. Therefore $\tilde{\Lambda}_1, \tilde{\Lambda}_2$ are intrinsically equivalent too. Moreover, by [2] $\tilde{\Lambda}_1, \tilde{\Lambda}_2$ are transversely tame. Since $W^s_{\tilde{\Lambda}_i} = W^s_{\Lambda_i} \times \mathbb{R}$ the Corollary 1.3 follows from Corollary 1.1. □

In Section 3 we shall construct some manifolds $M_{\underline{l}}$ with diffeomorphisms $f_{\underline{l}}\colon M_{\underline{l}} \to M_{\underline{l}}$ each of which has a transversely tame 1-dimensional hyperbolic attractor $\Lambda_{\underline{l}}$ whose basin is the whole manifold $M_{\underline{l}}$. It will be shown that for each transversely tame 1-dimensional hyperbolic attractor Λ of a diffeomorphism $f\colon M \to M$, where $m = \dim M \geq 4$ and W^s_Λ is orientable, the pair (W^s_Λ, Λ) is homeomorphic to one of the pairs $(M_{\underline{l}}, \Lambda_{\underline{l}})$ (see Corollary 3.2 in Section 3).

Though the construction of $f_{\underline{l}} : M_{\underline{l}} \to M_{\underline{l}}$ is quite simple it does not illustrate the topological shape of the manifold $M_{\underline{l}}$ or, equivalently, the basin W^s_Λ. A way which leads to a satisfactory description of W^s_Λ proceeds as follows.

It can be proved that there is a diffeomorphism $f'\colon \mathbb{R}^3 \to \mathbb{R}^3$ with a 1-dimensional hyperbolic attractor Λ' which is intrinsically equivalent to Λ, provided Λ is regarded, for some $k \geq 1$, as attractor of f^k. Then we consider the diffeomorphism $f'' : \mathbb{R}^m \to \mathbb{R}^m$ which is defined by $f''(x_1, \ldots, x_m) = (f'(x_1, x_2, x_3), \frac{1}{2}x_4, \ldots, \frac{1}{2}x_m)$ and its attractor $\Lambda'' = \Lambda' \times \{(0, \ldots, 0)\}$. This attractor is transversely tame and by Corollary 1.1 the basins W^s_Λ and $W^s_{\Lambda''} = W^s_{\Lambda'} \times \mathbb{R}^{m-3}$ are homeomorphic. This shows that for each intrinsic type $\overline{\Lambda}$ of 1-dimensional hyperbolic attractors there is an open subset $G_{\overline{\Lambda}}$ of $\mathbb{R}^3$ such that for any transversely tame 1-dimensional hyperbolic attractors Λ in a manifold of dimension $m \geq 4$ the basin W^s_Λ, if orientable, is homeomorphic to $G_{\overline{\Lambda}} \times \mathbb{R}^{m-3}$, where $\overline{\Lambda}$ denotes the intrinsic type of Λ. The sets $G_{\overline{\Lambda}}$ have various topological structure. Since the number of intrinsic types of 1-dimensional hyperbolic attractors is countable the same holds for the number of sets $G_{\overline{\Lambda}}$. It is a natural task to describe these sets and to uncover the connection between $\overline{\Lambda}$ and $G_{\overline{\Lambda}}$. This will be done in [5].

Section 2 it devoted to some definitions and constructions which concern 1-dimensional hyperbolic attractors. Then in Section 3 we construct the attractors $\Lambda_{\underline{l}}$ which were already mentioned above and reduce the proof of the main theorem to three propositions. Each of these proposition is proved in one of the further sections.

We shall use the following notations: $\mathbb{R}^m$ denotes the m-dimensional real coordinate space, and $\mathbb{D}^m$ is the unit ball in $\mathbb{R}^m$ with boundary S^{m-1}, the $(m-1)$-dimensional unit sphere. So $\mathbb{D}^1 = [-1, 1]$, while I denotes the interval $[0, 1]$. The centre of $\mathbb{D}^m$ is denoted by o, and $\mathbb{D}^m(\rho)$ is the ball in $\mathbb{R}^m$ with centre o and radius ρ. If M is a manifold, then $\operatorname{Int} M$ and ∂M denote the interior and the boundary of M, respectively. The words "differentiable" or "smooth" mean "of class C^1".

2. W-representations

As pointed out in Section 1 all 1-dimensional hyperbolic attractors Λ are locally homeomorphic to the cartesian product of a Cantor set with an interval. Now to describe their global intrinsic structure we present a procedure by which each 1-dimensional hyperbolic attractor can be constructed up to intrinsic equivalence. As above $W^s_\Lambda, \mathfrak{W}^s_\Lambda$ will denote the basin of Λ and the stable foliation of W^s_Λ, respectively. In W^s_Λ we use a metric d^s, called *stable metric*, for which $d^s(x, y) = \infty$ if x, y lie in different leaves of $\mathfrak{W}^s_\Lambda$, and if x, y lie in the same leaf then $d^s(x, y)$ is their distance inside this leaf.

By a *branched* 1-*manifold* we mean a compact connected subset Σ of $\mathbb{R}^3$ which is the union of finitely many smooth arcs $A_1, \ldots, A_r$ with the following properties:

- (1_Σ) Two of the arcs A_i are disjoint or their intersection is a common end point.
- (2_Σ) No point belongs to more than three of the arcs A_i.
- (3_Σ) If τ is a common end point of A_i and A_j then the tangents $T_\tau A_i, T_\tau A_j$ of A_i and A_j at τ coincide.
- (4_Σ) Each point in Σ lies in the interior of a smooth subarc of Σ.

So Σ has a tangent at each of its points, and if there is a nowhere vanishing continuous vector field on Σ each of whose vectors is tangent to Σ then we say that Σ is *orientable*.

A point ϑ which belongs to three of the arcs A_i is called a *branch point* of Σ. Two of these arcs, say A_{i_1}, A_{i_2}, leave ϑ in the same and the remaining one, say A_{i_0}, in the opposite direction. A smooth arc in Σ one of whose end points is ϑ will be called a branch of ϑ if it leaves ϑ in the direction of one of the arcs A_{i_1}, A_{i_2} and does not contain further branch points. An arc which leaves ϑ in the opposite direction and does not contain further branch points is a stem of ϑ. The set of all branch points in Σ will be denoted by Θ.

Later we shall define standard tubular neighbourhoods N_Σ^m of branched 1-manifolds Σ. To make this definition unique we choose once for all in each branched 1-manifold Σ and for each branch point $\vartheta \in \Sigma$ an order in the set of the two branches of ϑ. Accordingly one branch will be called positive and the other negative.

In each branched 1-manifold Σ we choose a maximal finite subset Ξ of $\Sigma \setminus \Theta$ for which $\Sigma \setminus \Xi$ is connected. This set Ξ will be called the cutting set and its points the cutting points of Σ. Obviously $\Sigma \setminus \Xi$ does not contain closed curves, and $\Sigma \setminus \Xi$ is a tree. The number of points in Ξ can be regarded as the *number of handles* of Σ.

We assume that for each branched 1-manifold Σ a cutting set Ξ is fixed. So in each Σ we have the two uniquely defined finite subsets Θ and Ξ.

Let $\varphi : \Sigma \to \Sigma$ be a mapping. We call φ expanding if for any smooth arc A in Σ the restriction of φ to A is an expanding C^1 mapping in the sense that for each $\tau \in A$ and each tangent vector $v \in T_\tau A, v \neq 0$ we have $|d_\tau \varphi(v)| > |v|$, where $|v|$ denotes the length of vectors in $\mathbb{R}^3$. We shall use mappings $\varphi : \Sigma \to \Sigma$, called W-mappings, which have the following properties:

(1_φ) φ is expanding.
(2_φ) If $\vartheta \in \Sigma$ is a branch point, then for $k \geq 1$ the points $\varphi^k(\vartheta)$ are not branch points.
(3_φ) If A is any arc in Σ then there is a positive integer k such $\varphi^k(A) = \Sigma$.

Obviously, if $\varphi : \Sigma \to \Sigma$ is a W-mapping, so are the mappings $\varphi^k : \Sigma \to \Sigma$ $(k = 1.2, \dots)$.

Williams proved in [12] that for each 1-dimensional hyperbolic attractor Λ of a diffeomorphism $f : M \to M$ there is a W-mapping

$\varphi : \Sigma \to \Sigma$ of a 1-dimensional branched manifold Σ and a continuous projection $\pi_0 : \Lambda \to \Sigma$ with the following properties:

(1_W) The diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{f} & \Lambda \\ \pi \downarrow & & \downarrow \pi \\ \Sigma & \xrightarrow{\varphi} & \Sigma \end{array} \tag{W}$$

is commutative.

(2_W) For each $\tau \in \Sigma$ the preimage $\pi_0^{-1}(\tau)$ is a Cantor set in a stable manifold $W^s \in \mathfrak{W}^s_\Lambda$. The diameters of all sets $\pi^{-1}(\tau)$ $(\tau \in \Sigma)$ are bounded with respect to the stable metric d^s.

(3_W) If $x \in \Lambda$ then there is an arc in the unstable manifold W^u_x of x which contains x in its interior and which is mapped by π homeomorphically onto a smooth arc in Σ.

Under these conditions (W) will be called a *W-representation* for Λ.

If π is the projection in a W-representation of Λ, then for each stable manifold $W^s \in \mathfrak{W}^s_\Lambda$ the family of all preimages $\pi^{-1}(\tau)$ $(\tau \in \Sigma)$ which lie in W^s is locally finite with respect to the intrinsic topology of W^s.

Let Λ, Λ' be intrinsically equivalent 1-dimensional hyperbolic attractors of diffeomorphisms f, f', respectively, and let $h : \Lambda \to \Lambda'$ be a conjugating homeomorphism. Then a W-representations (W) of Λ defines, with $\pi' = \pi h^{-1}$, the W-representation

$$\begin{array}{ccc} \Lambda' & \xrightarrow{f'} & \Lambda' \\ \pi' \downarrow & & \downarrow \pi' \\ \Sigma & \xrightarrow{\varphi} & \Sigma \end{array} \tag{W$'$}$$

of Λ'. (Roughly speaking, intrinsically equivalent 1-dimensional hyperbolic attractors have the same W-representation.)

The importance of W representations results from the fact that their lower part, i.e., the W-mapping $\varphi : \Sigma \to \Sigma$, determines the intrinsic structure of Λ and that Λ, up to intrinsic equivalence, can be obtained from φ by simple constructions. For this paper the following construction using tubular neighbourhoods of branched 1-manifolds Σ is most convenient.

Let Σ be a branched 1-manifold with Θ the set of its branch points, and let $m \geq 2$ be an integer. Then by an *m-dimensional tubular neighbourhood* N^m_Σ of Σ we mean a compact connected toplogical manifold of dimension m which is defined by the following construction. If $\Theta = \emptyset$ then Σ is a smooth closed curve, and we define N^m_Σ to be the solid torus $\Sigma \times \mathbb{D}^{m-1}$. If $\Theta \neq \emptyset$ and Ξ is the cutting set of Σ then the components of $\Sigma \backslash (\Theta \cup \Xi)$ are open arcs $A_1, \ldots, A_s$ and we consider the closed arcs

$\overline{A}_1, \ldots, \overline{A}_s$ which are obtained by adding two end points to each A_i, where all these $2s$ end points are assumed to be different. Let $\overline{\Sigma}$ be the disjoint union of all $\overline{A}_i$. To each $\vartheta \in \Theta$ there correspond three end points $\vartheta_0, \vartheta_+, \vartheta_-$ in $\overline{\Sigma}$, where ϑ_0 belongs to the stem and ϑ_+, ϑ_- to the positive and the negative branch of ϑ, respectively. Now N^m_Σ is obtained from $\overline{\Sigma} \times \mathbb{D}^{m-1}$ by the following identifications. Firstly, if $\vartheta \in \Theta$ then for $i = \pm$ we identify $(\vartheta_i, x) \in \overline{\Sigma} \times \mathbb{D}^{m-1}$ with $(\vartheta_0, \delta_i(x))$, where the embeddings $\delta_i : \mathbb{D}^{m-1} \to \mathbb{D}^{m-1}$ are defined by

$$\delta_i(x) = \left(\tfrac{i1}{2}, 0, \ldots, 0\right) + \tfrac{1}{4}x \qquad (i = \pm).$$

Finally, if $\tau \in \Xi$ we take the two corresponding end points τ, τ' in $\overline{\Sigma}$ and identify all points $(\tau, x) \in \{\tau\} \times \mathbb{D}^{m-1}$ either with (τ', x) or with $(\tau', x')\,(x = (x_1, \ldots, x_{m-1}), x' = (-x_1, x_2, \ldots, x_{m-1}))$, where this identification is chosen so that N^m_Σ becomes orientable (see Figure 1).

There is a natural projection $\pi_\Sigma : N^m_\Sigma \to \Sigma$, and each disk $\pi_\Sigma^{-1}(\tau)$ has a well defined linear euclidean structure; even more, in $\pi_\Sigma^{-1}(\tau)$ a centre $o(\tau)$ and the directions of $m-1$ coordinate axes are defined (the first possibly without orientation). If Σ is orientable then for all $\tau \in \Xi$ the identification $(\tau, x) = (\tau', x)$ or $(\tau, x) = (\tau', x')$ are of the first kind. In this case for each $\tau \in \Sigma$ we have a well defined diffeomorphism $\nu_\tau : \pi_\Sigma^{-1}(\tau) \to \mathbb{D}^{m-1}$ and therefore standard coordinates in $\pi_\Sigma^{-1}(\tau)$. Since for $\Theta \neq \emptyset$ the boundary of N^m_Σ has corners N^m_Σ is not a C^1 manifold in the ordinary sense, but we can define C^1 mappings of N^m_Σ in the obvious way. (Warning: N^m_Σ is called neighbourhood though there is no natural embedding $\Sigma \to N^m_\Sigma$.)

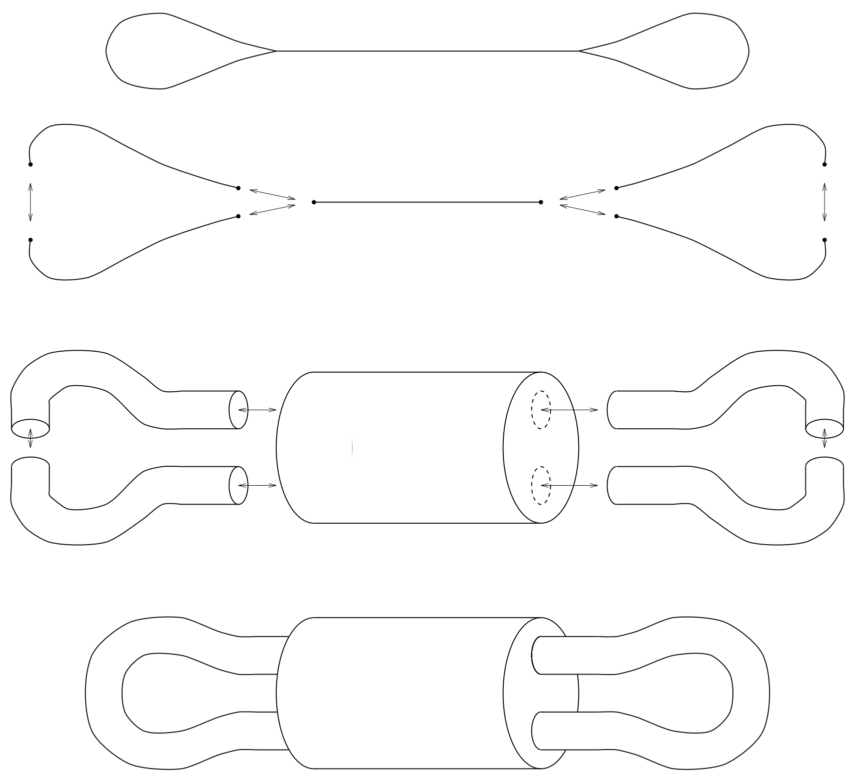


Figure 1

As pointed out above the number $\#\Xi$ of points in a cutting set Ξ for Σ can be regarded as the number of handles of Σ. This becomes more concrete if we consider a tubular neighbourhood N^m_Σ of Σ. Indeed, N^m_Σ is a handlebody with exactly $\#\Xi$ handles.

Now, as announced above, we show how a tubular neighbourhood N^m_Σ of the branched 1-manifold Σ in (W) can be used to describe the intrinsic type of Λ. Let $m \geq 3$. Then there is an orientation preserving C^1 embedding $f_\varphi : N^m_\Sigma \to \operatorname{Int} N^m_\Sigma$ such that for each $\tau \in \Sigma$ the restriction of f_φ to $\pi_\Sigma^{-1}(\tau)$ is a contracting similarity mapping into $\pi_\Sigma^{-1}(\varphi(\tau))$. We assume that f_φ does not twist N^m_Σ in the sense that each coordinate direction in $\pi_\Sigma^{-1}(\tau)$ is mapped to the same coordinate direction in $\pi_\Sigma^{-1}(\varphi(\tau))$, where the first direction is regarded without orientation (see Figure 2). We assume that for any W-mapping $\varphi : \Sigma \to \Sigma$ and for any $m \geq 3$ the embedding $f_\varphi \colon N^m_\Sigma \to N^m_\Sigma$ is fixed once and for all so that the following constructions are uniquely determined by φ and m.

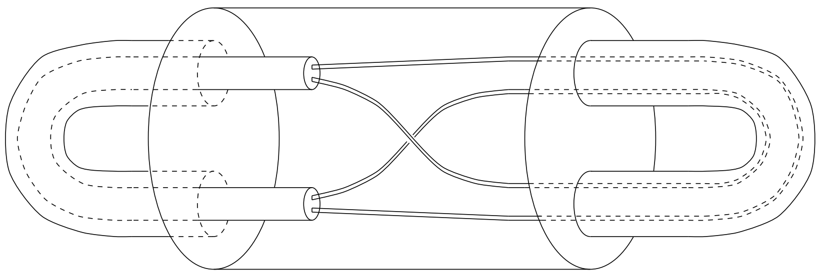
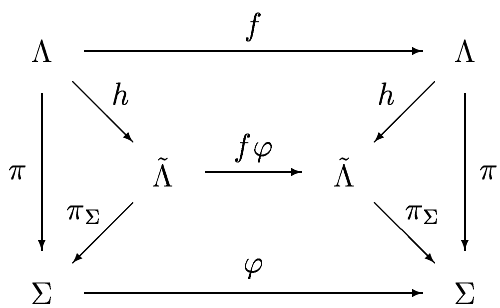


Figure 2

The intersection $\tilde{\Lambda} = \bigcap_{i=0}^{\infty} f_{\varphi}^{i}(N_{\Sigma}^{m})$ is invariant under f_{φ}, and if for $x \in \Lambda$ we define $h(x)$ to be the single point in $\bigcap_{i=0}^{\infty} f_{\varphi}^{i}\pi_{\Sigma}^{-1}\pi f^{-i}(x)$ then it is not hard to see that h is a homeomorphism $h\colon \Lambda \to \tilde{\Lambda}$ for which



is commutative. This shows that $f : \Lambda \to \Lambda$ and $f_{\varphi} : \tilde{\Lambda} \to \tilde{\Lambda}$ are topologically conjugate, and so $\tilde{\Lambda}$ describes the intrinsic type of Λ.

In Proposition 3.2 of Section 3 we shall show that for certain 1-dimensional hyperbolic attractors Λ with a W-representation (W) a tubular neighbourhood N_{Σ}^{m} of Σ (m the dimension of the manifold containing Λ) can be embedded in W_{Λ}^{s} so that we get a compact neighbourhood of Λ. These neighbourhoods, called *tubular neighbourhoods* of Λ, will be the main tool for our investigation of attractor basins. More exactly, if Λ is a 1-dimensional hyperbolic attractor in an m-dimensional manifold and (W) is a W-representation of Λ with the W-mapping $\varphi\colon \Sigma \to \Sigma$, then by a tubular neighbourhood N of Λ belonging to (W) we mean a compact neighbourhood of Λ in W_{Λ}^{s} which is the image of a C^0 embedding $h\colon N_{\Sigma}^{m} \to M$ with the following property. *If $x \in \Lambda, \tau = \pi(x) \in \Sigma$, then h maps the disk $\pi_{\Sigma}^{-1}(\tau)$ diffeomorphically onto a disk $N(\tau)$ in the stable manifold W_{x}^{s}, and $N(\tau) \cap \Lambda = \operatorname{Int} N(\tau) \cap \Lambda = \pi^{-1}(\tau)$, where $\pi\colon \Lambda \to \Sigma$ is the projection in (W) and $\pi_{\Sigma}\colon N_{\Sigma}^{m} \to \Sigma$ is the projection considered above.* The projection π can be extended to

$$\pi_N = \pi_\Sigma h^{-1} : N \to \Sigma.$$

If in addition to $(1_N), (2_N), (3_N)$ we have $f(N) \subset \operatorname{Int} N$, then we get the following commutative diagram which will be called an *extended W-representation* or an *extension* of (W)

$$\begin{array}{ccc} N & \xrightarrow{f} & N \\ \pi_N \downarrow & & \downarrow \pi_N \\ \Sigma & \xrightarrow{\varphi} & \Sigma \end{array} \qquad (W_N)$$

3. The attractors $\Lambda_{\underline{l}}$ and the plan for the proof of the main theorem

In this section after having defind the attractors $\Lambda_{\underline{l}}$ we state three propositions and show how they imply the main theorem. The propositions will be proved in the following sections.

Let $\varphi \colon \Sigma \to \Sigma$ be a W-mapping. In the preceding section we have defined for each $m \geq 3$ a tubular neighbourhood N^m_Σ of Σ with a projection $\pi_\Sigma : N^m_\Sigma \to \Sigma$ and a C^1 embedding $f_\varphi : N^m_\Sigma \to N^m_\Sigma$. Here for $m \geq 4$ we generalize this construction by adding twists to f_φ. If $\Xi = \{\tau_1, \ldots, \tau_r\}$ is the cutting set of Σ we choose arcs $A_1, \ldots, A_r$ in Σ, where A_i and τ_i lie in the same component of $\Sigma \setminus \Theta$, but $\tau_i \notin A_i$. Then $\pi_\Sigma^{-1}(A_i) = A_i \times \mathbb{D}^{m-1}$. For $i = 1, \ldots, r$ we choose a monotone C^1 function $\chi_i : A_i \to [0, 2\pi]$ which is 0 near one end point of A_i and 1 near the other end point and define $\vartheta_i : N^m_\Sigma \to N^m_\Sigma$ to be the diffeomorphism which is the identity outside $\pi_\Sigma^{-1}(A_i)$ and which on $\pi_\Sigma^{-1}(A_i) = A_i \times \mathbb{D}^{m-1}$ is given by

$$\begin{aligned} &\vartheta_i(\tau, t_1, ..., t_{m-1}) \\ &= (\tau, t_1 \cos\alpha(\tau) - t_2 \sin\alpha(\tau), t_1 \sin\alpha(\tau) + t_2 \cos\alpha(\tau), t_3, ..., t_{m-1}). \end{aligned}$$

The mapping ϑ_i twists the handle of N^m_Σ which corresponds to τ_i. If $\underline{l} = (l_1, \ldots, l_r) \in \{0,1\}^r$ is a sequence of r elements each of which is 0 or 1 we define $\vartheta_{\underline{l}} = \vartheta_{l_1} \ldots \vartheta_{l_r}$, i.e. $\vartheta_{\underline{l}}$ twists exactly those handles which corresponds to cutting points τ_i for which $l_i = 1$.

In the next step of our construction we define m-manifolds $W_{\underline{l}}$ ($\underline{l} \in \{0,1\}^r$) and diffeomorphisms $f_{\underline{l}} \colon W_{\underline{l}} \to W_{\underline{l}}$ such that each $W_{\underline{l}}$ contains N^m_Σ and $f_{\underline{l}}$ is an extension of the embedding $f_\varphi \vartheta_{\underline{l}} : N^m_\Sigma \to N^m_\Sigma$. To this aim we consider disjoint copies $R_1, R_2, \ldots$ of $R_0 = N^m_\Sigma \setminus \operatorname{Int} f_\varphi(N^m_\Sigma)$. Then $W_{\underline{l}}$ is obtained from the disjoint union $N^m_\Sigma \cup R_1 \cup R_2 \cup \ldots$ by identifying each point $x \in R_i$ which corresponds to a point x' in R_0 lying on ∂N^m_Σ with the point in R_{i+1} which corresponds to $f_\varphi \vartheta_{\underline{l}}(x')$ in

R_0 $(i = 0, 1, 2, \dots)$. The extension $f_{\underline{l}} : W_{\underline{l}} \to W_{\underline{l}}$ of $f_\varphi \vartheta_{\underline{l}} : N^m_\Sigma \to N^m_\Sigma$ maps each point $x \in R_i$ $(i \geq 1)$ to the point in R_{i-1} which is the copy of x in R_{i-1}. It is a simple task to equip $W_{\underline{l}}$ with a C^1 structure so that each $f_{\underline{l}}$ becomes a diffeomorphism. Obviously

$$\Lambda_{\underline{l}} = \bigcap_{j=0}^{\infty} f^j_{\underline{l}}(N^m_\Sigma)$$

is a transversely tame 1-dimensional hyperbolic attractor of $f_{\underline{l}}$ with tubular neighbourhood N^m_Σ, and the basin of $\Lambda_{\underline{l}}$ is the whole manifold $W_{\underline{l}}$. Moreover all $\Lambda_{\underline{l}}$ $(\underline{l} \in \{0,1\}^r)$ are intrinsically equivalent.

Proposition 3.1. *Let $\varphi : \Sigma \to \Sigma$ be a W-mapping, where Σ has r handles, and let $\underline{l}, \underline{l}'$ be sequences in $\{0,1\}^r$ with $\underline{l}' = (0, \dots, 0)$. Then there is an integer $k \geq 1$ such that the diffeomorphisms $f^k_{\underline{l}} : W_{\underline{l}} \to W_{\underline{l}}$ and $f^k_{\underline{l}'} : W_{\underline{l}'} \to W_{\underline{l}'}$ are topologically conjugate.*

Proposition 3.2. *If (W) is a W-representation of a transversely tame 1-dimensional hyperbolic attractor Λ whose basin is orientable, then, if the manifold containing Λ is at least 4-dimensional, Λ has a tubular neighbourhood belonging to (W).*

Corollary 3.1. *If Λ, W are as in the proposition then there is an integer $k_0 \geq 1$ such that for any $k \geq k_0$ the diagram*

$$\begin{array}{ccc} \Lambda & \xrightarrow{f^k} & \Lambda \\ \pi \downarrow & & \downarrow \pi \\ \Sigma & \xrightarrow{\varphi^k} & \Sigma \end{array}$$

(Σ, π, φ as in (W)), when regarded as W-representation for Λ as attractor f^k, has an extension.

Proof. Since each tubular neighbourhood N of Λ is a compact subset of W^s_Λ which contains Λ in its interior there is an integer $k_0 \geq 1$ such that $f^k(N) \subset \operatorname{Int} N$ for all $k \geq k_0$. □

Remark 3.1. Proposition 3.2 does not hold for 1-dimensional hyperbolic attractors Λ in 3-manifolds. In this case each Λ has a W-representation (W) with a tubular neighbourhood belonging to (W), but there are examples where (W) can not be chosen arbitrarily (see [4], where the 3-dimensional case is considered).

Proposition 3.3. *Let Λ be a 1-dimensional hyperbolic attractor in an m-dimensional manifold, where $m \geq 4$. It is assumed that Λ has an extended W-representation (W_N). Then, if $\varphi : \Sigma \to \Sigma$ is the W-mapping in (W_N) and r is the number of handles of Σ, our attractor Λ is basin equivalent to one of the attractors $\Lambda_{\underline{l}}$ $(\underline{l} \in \{0,1\}^r)$ which are constructed above for $\varphi\colon \Sigma \to \Sigma$ and m.*

Corollary 3.2. *Let Λ be a transversely tame 1-dimensional hyperbolic attractor of a diffeomorphism $f\colon M \to M$, where $m = \dim M \geq 4$ and W^s_Λ is orientable. Then there is an integer $k \geq 1$ such that the attractor Λ, when regarded as attractor of f^k, is basin equivalent to one of the attractors $\Lambda_{\underline{l}}$.*

Proof. The corollary is immediately implied by Corollary 3.1 and Proposition 3.3. □

Proof of the main theorem. Let $f_i : M_i \to M_i, \Lambda_i$ $(i = 1, 1)$ and $m = \dim M_1 = \dim M_2$ be as in the main theorem. Since Λ_1, Λ_2 are intrinsically equivalent we can choose W-representations for these attractors with the same expanding mapping $\varphi : \Sigma \to \Sigma$. Then by Proposition 3.2 and Proposition 3.3 there is an integer $k' \geq 1$ with the following property, where Λ_i' denotes Λ_i regarded as attractor of $f_i^{k'}$, and $\overline{\Lambda}'$ is the common intrinsic type of Λ_1', Λ_2'.

If r is the number of handles of Σ then there are sequences $\underline{l}_1, \underline{l}_2 \in \{0,1\}^r$ such that Λ_i' is basin equivalent to $\Lambda_{\underline{l}_i}$ $(i = 1, 2)$ where these attractors are constructed for the W-mapping $\varphi^k\colon \Sigma \to \Sigma$. By Proposition 3.1 there is an integer $k'' \geq 1$ such that $\Lambda_{\underline{l}_1}, \Lambda_{\underline{l}_2}$ as attractors of $f_{\underline{l}_1}^{k''}, f_{\underline{l}_2}^{k''}$, respectively, are basin equivalent. This shows that Λ_1, Λ_2 as attractors of $f_1^{k'+k''}, f_2^{k'+k''}$, respectively, are basin equivalent too. □

4. Proof of Proposition 3.3

We start with a W-mapping $\varphi\colon \Sigma \to \Sigma$ and a tubular neighbourhood N^m_Σ of Σ with the projection $\pi_\Sigma\colon N^m_\Sigma \to \Sigma$, where $m \geq 4$. This neighbourhood will be denoted by N and the disks $\pi_\Sigma^{-1}(\tau)$ $(\tau \in \Sigma)$ by $N(\tau)$. By our construction of N each disk $N(\tau)$ has a well defined euclidean structure and is isometric with $\mathbb{D}^{m-1}$. If r is the number of handles of Σ we have the mappings $f_\varphi : N \to \operatorname{Int} N, \vartheta_{\underline{l}} : N \to N, f_{\underline{l}} = f_\varphi \vartheta_{\underline{l}} : N \to \operatorname{Int} N$ $(\underline{l} \in \{0,1\}^r)$ as defined in Section 2 and Section 3. The common image of the embeddings $f_\varphi, f_{\underline{l}}$ will be denoted by N' and the disks $f_\varphi(N(\tau)) = f_{\underline{l}}(N(\tau))$ by $N'(\tau)$. By an *embedding* $f\colon N \to \operatorname{Int} N$

over φ we mean an orientation preserving C^1 embedding such that $f(N(\tau)) \subset \operatorname{Int} N(\varphi(\tau))$ and

$$\lim_{i\to\infty} \max_{\tau\in\Sigma} \operatorname{diam} f^i(N(\tau)) = 0.$$

Then

$$\Lambda_f = \bigcap_{i=0}^{\infty} f^i(N)$$

is the attractor of f. An embedding $\varphi\colon N \to \operatorname{Int} N$ over φ will be called *fibrewise linear* on *fibrewise similar* if for each $\tau \in \Sigma$ the restriction of f to $N(\tau)$ is a linear mapping or a similarity mapping onto a disk in $N(\varphi(\tau))$. An orientation preserving homeophism or diffeomorphism $h\colon N \to N$ will be called *fibre preserving* if $h(N(\tau)) = N(\tau)$ $(\tau \in \Sigma)$ and in addition $h = id$ in a neighbourhood of ∂N. We say that two embeddings $f, f' : N \to \operatorname{Int} N$ over φ are C^1 conjugated and write $f \sim_1 f'$ if there is a fibre preserving homeomorphism $h\colon N \to N$ such that $hf = f'h$ and the restriction of h to $N \setminus \Lambda_f$ is a diffeomorphism onto $N \setminus \Lambda_{f'}$.

The way in which the following lemma implies Proposition 3.3 is so short and straightforward (N in the lemma can, up to topological equivalence, be identified with the neighbourhood N of (W_N) in the proposition) that its description can be omitted and we merely have to prove the lemma.

Lemma 4.1. *For each embedding $f : N \to \operatorname{Int} N$ over φ there is a sequence $\underline{l} \in \{0,1\}^r$ such that $f \sim_1 f_{\underline{l}}$.*

The proof of Lemma 4.1 will be divided in three steps (the following lemmas). Lemma 4.4 is more general than needed here. Indeed, to prove Lemma 4.1 we merely have to use the case where $f' = f_\varphi$ and therefore $f'\vartheta_{\underline{l}} = f_{\underline{l}}$. The reason for the general form of Lemma 4.4 is our later application of this lemma in the proof of Proposition 3.1.

Lemma 4.2. *If $f, f' : N \to \operatorname{Int} N$ are embeddings over φ and if $h_0 : N \to N$ is a fibre preserving diffeomorphism such that $h_0 f = f'$, then $f \sim_1 f'$.*

Lemma 4.3. *Let $f, f'\colon N \to \operatorname{Int} N$ be embeddings over φ, where f' is fibrewise similar. Then there is a fibrewise similar embedding $f''\colon N \to \operatorname{Int} N$ over φ and a fibre preserving diffeomorphism $h_0 : N \to N$ such that $f''(N(\tau)) = f'(N(\tau))\,(\tau \in \Sigma)$ and $h_0 f = f''$.*

Lemma 4.4. *If $f, f' : N \to \operatorname{Int} N$ are fibrewise similar embeddings over φ such that $f(N(\tau)) = f'(N(\tau))\,(\tau \in \Sigma)$, then there is a sequence $\underline{l} \in \{0,1\}^r$ and a fibrewise preserving diffeomorphism $h_0 \colon N \to N$ such that $h_0 f = f' \vartheta_{\underline{l}}$.*

Proof of Lemma 4.2. We have to find a fibre preserving homeomorphism $h \colon N \to N$ satisfying $hf = f'h$ such that the restriction of h to $N \setminus \Lambda_f$ is a diffeomorphism to $N \setminus \Lambda_{f'}$.

Since $h_0 = id$ on ∂N we have $f' h_0 f^{-1} = f' f^{-1} = h_0$ on $\partial f(N)$, and the restriction of h_0 to $N \setminus \operatorname{Int} f(N)$ together with the restriction of $f' h_0 f^{-1}$ to $f(N)$ define a fibre preserving homeomorphism $N \to N$. It is not hard to modify h_0 near ∂N so that this homeomorphism becomes a diffeomorphism. We define h on

$$N \setminus \Lambda_f = \bigcup_{i=0}^{\infty} f^i(N \setminus \operatorname{Int} f(N))$$

by

$$h = f'^i h_0 f^{-i} \text{ on } f^i(N) \setminus \operatorname{Int} f^{i+1}(N).$$

Then h is a diffeomorphism satisfying $hf = f'h$ which maps $N \setminus \Lambda_f$ to $N \setminus \Lambda_{f'}$. To extend h over Λ_f we consider a point $x \in \Lambda_f$ and the points $\tau_i \in \Sigma\,(i = 0, 1, \dots)$ for which $f^{-i}(x) \in N(\tau_i)$. Then $\varphi(\tau_{i+1}) = \tau_i$ and therefore $f'^{i+1}(N(\tau_{i+1})) \subset f'^i(N(\tau_i))$. Hence $\bigcap_{i=0}^{\infty} f'^i(N(\tau_i))$ contains exactly one point y, and we define $h(x) = y$. □

Proof of Lemma 4.3. Using the techniques of tubular neighbourhood theory (see [7], Chapter 4.5) it is a standard procedure to define first a fibre preserving diffeomorphism $h_1 : N \to N$ such that $h_1 f$ is fibrewise linear and then a fibre preserving diffeomorphism $h_2 \colon N \to N$ such that $h_2 h_1 f$ is fibrewise similar. Therefore it is sufficient to prove the lemma under the assumption that f is fibrewise similar.

With this assumption it is easy to construct a fibre preserving diffeomorphism $h_3 : N \to N$ such that $h_3 f$ is still fibrewise similar and $h_3 f(o(\tau))$ coincides with $f'(o(\tau))$ for all τ in a neighbourhood of the branch point set Θ of Σ, where $o(\tau)$ denotes the centre of $N(\tau)$. In the next step of our construction we define a fibre preserving diffeomorphism $h_4 : N \to N$ such that $h_4 h_3 f$ is still fibrewise similar and $h_4 h_3 f(o(\tau)) = f'(o(\tau))$ holds for all $\tau \in \Sigma$. To get h_4 we merely have to apply the following general fact (which depends on $m \geq 4$).

Let $S_1, \dots, S_q; S'_1, \dots, S'_q$ be smooth arcs in $\mathbb{D}^1 \times \mathbb{D}^{m-1}$ each of which is transverse to the disks $\{t\} \times \mathbb{D}^{m-1}\,(t \in \mathbb{D}^1)$ and has one end point on $\{-1\} \times \mathbb{D}^{m-1}$ and one on $\{1\} \times \mathbb{D}^{m-1}$. Moreover, we assume that

S_i and S'_i coincide on a neighbourhood of their end points. Then there is a diffeomorphism $g\colon \mathbb{D}^1 \times \mathbb{D}^{m-1} \to \mathbb{D}^1 \times \mathbb{D}^{m-1}$ which is the identity near $\partial(\mathbb{D}^1 \times \mathbb{D}^{m-1})$ and satisfies $g(\{t\} \times \mathbb{D}^{m-1}) = \{t\} \times \mathbb{D}^{m-1}, g(S_i) = S'_i\, (t \in \mathbb{D}^1, 1 \leq i \leq q)$.

The step from $h_4 h_3$ to a fibre preserving $h_0 : N \to N$ with the properties required in the lemma is so easy that it can be left to the reader. □

Proof of Lemma 4.4. We shall prove the lemma under the additional assumption that Σ is orientable (see Sect. 2). This restricted case avoids some technical considerations, but it presents all ideas for a general proof. The gain of our restriction is that for each $\tau \in \Sigma$ we have a fixed diffeomorphism $\nu_\tau\colon N(\tau) \to \mathbb{D}^{m-1}$, i.e. $N(\tau)$ has standard coordinates. Therefore for each euclidean ball D in a disk $N(\tau)$ the group $SO(m-1)$ of all orthogonal matrices with determinant $+1$ operators on D, and each orientation preserving isometric map $g\colon D \to D$ determines an element g^* of $SO(m-1)$. Moreover, each mapping $\vartheta_{\underline{l}}\colon N \to N$ ($\underline{l} \in \{0,1\}^r$) by $\vartheta^*_{\underline{l}}(\tau) = (\vartheta_{\underline{l}}|_{N(\tau)})^*$ defines a C^1 mapping $\vartheta^*_{\underline{l}}\colon \Sigma \to SO(m-1)$, and for each pair $f_1, f_2\colon N \to \operatorname{Int} N$ of fibrewise similar embedding over φ satisfying $f_1(N(\tau)) = f_2(N(\tau))$ we get the C^1 mappings $(f_2 f_1^{-1})^*\colon \Sigma \to SO(m-1), (f_1^{-1} f_2)^*\colon \Sigma \to SO(m-1)$ which are defined by $(f_2 f_1^{-1})^*(\tau) = (f_2 f_1^{-1}|_{f_1(N(\tau))})^*, (f_1^{-1} f_2)^*(\tau) = (f_1^{-1} f_2|_{N(\tau)})^*$, respectively.

Since $m-1 \geq 3$ the fundamental group of $SO(m-1)$ is of order two (see [7] p. 439), i.e. there is a loop $\lambda_0\colon [0,1] \to SO(m-1)\, (\lambda_0(0) = \lambda_0(1) = \mathbb{1})$ such that each loop in $SO(m-1)$ is either contractible or homotopic to λ_0. If $\gamma : \Sigma \to SO(m-1)$ is continuous then each handle of Σ determines a loop in $SO(m-1)$, and the homotopy class of γ is determined by these loops. If $\underline{l} = (l_1, \ldots, l_r)$ is a sequence in $\{0,1\}^r$ then for a handle of Σ the corresponding loop of $\vartheta^*_{\underline{l}} : \Sigma \to SO(m-1)$ is not contractible if and only if $\vartheta_{\underline{l}}$ twists this handle, i.e. if the corresponding l_i in $\underline{l}$ is 1. Therefore it is not hard to see that for the embeddings f, f' in the lemma there is a twist mapping $\vartheta_{\underline{l}}$ such that $(f^{-1} f' \vartheta_{\underline{l}})^* : \Sigma \to SO(m-1)$ is contractible. Then $(f' \vartheta_{\underline{l}} f^{-1})^* : \Sigma \to SO(m-1)$ is also contractible, and we get a family of mappings (contraction) $g_t : \Sigma \to SO(m-1)$ such that $g_t = (f' \vartheta_{\underline{l}} f^{-1})^*$ for all t near 0 and $g_t \equiv \mathbb{1}$ for all t near 1. Since $(f' \vartheta_{\underline{l}} f^{-1})^*$ is C^1 we can choose g_t to be differentible too.

For our construction of h_0 some preliminary definitions are necessary; especially we introduce two "thickenings" N^* and N^{**} of N' in N. If $\tau \in \Sigma$ and $\rho > 0$ we define $N'(\tau, \rho)$ to be the ball in $N(\varphi(\tau))$ which is concentric with $N'(\tau)$ and whose radius is ρ times the radius of

$N'(\tau)$. Here we assume that ρ is small enough, i.e. that $N'(\tau,\rho)$ lies in $N(\varphi(\rho))$. We choose $\varepsilon > 0$ so small that for each $\tau \in \Sigma$ the ball $N'(\tau, 1+\varepsilon)$ is defined, that $\tau \neq \tau'$ implies $N'(\tau,1+\varepsilon) \cap N'(\tau',1+\varepsilon) = \emptyset$ and that for each branch point ϑ and each branch A of ϑ the limit of the disks $N'(\tau, 1+\varepsilon)$ for $\tau \in A \setminus \{\vartheta\}$, $\tau \to \vartheta$ lies in $\operatorname{Int} N'(\vartheta)$. The ball $N'(\tau, 1+\varepsilon)$ will be denoted by $N^*(\tau)$. Then

$$N^* = \bigcup_{\tau\in\Sigma} N^*(\tau)$$

can be regarded as a thickening of N'.

Now let ϑ be a branch point of Σ. We choose two branches $A_{\vartheta-}, A_{\vartheta+}$ of ϑ such that $\varphi(A_{\vartheta-}) = \varphi(A_{\vartheta+}) = A_\vartheta$ is a smooth arc in Σ and consider a C^1 embedding

$$\xi_\vartheta : A_\vartheta \times \mathbb{D}^{m-1}(1+\varepsilon) \to \operatorname{Int} N$$

with the properties (1) - (3) below, where we use the following notations

$$\begin{aligned} Z_\vartheta &= \xi_\vartheta(A \times \mathbb{D}^{m-1}(1+\varepsilon)),\\ Z_\vartheta(\tau,\rho) &= \xi_\vartheta(\{\tau\} \times \mathbb{D}^{m-1}(\rho)) \quad (\tau \in \Sigma, 0 < \rho \leq 1+\varepsilon),\\ Z_\vartheta(\tau) &= Z_\vartheta(\tau, 1+\varepsilon). \end{aligned}$$

(1) $Z_\vartheta(\tau) \subset \operatorname{Int} N(\tau)$ and the restriction of ξ_ϑ to $\{\tau\} \times \mathbb{D}^{m-1}(1+\varepsilon)$ is a similarity mapping into $N(\tau)$, so that $Z_\vartheta(\tau)$ is a euclidean ball.

(2)
$$\xi_\vartheta(\varphi(\vartheta), x) = f(\nu_\vartheta^{-1}(x)) \qquad (x \in \mathbb{D}^{m-1}),$$
where $\nu_\tau : N(\tau) \to \mathbb{D}^{m-1}$ is the coordinate mapping mentioned above. Even more, if B_ϑ is a stem of ϑ such that $\overline{A}_\vartheta = A_\vartheta \cup \varphi(B_\vartheta)$ is a smooth arc, then the mapping $\overline{A}_\vartheta \times \mathbb{D}^{m-1} \to N$ given on $A_\vartheta \times \mathbb{D}^{m-1}$ by ξ_ϑ and on $\varphi(B_\vartheta) \times \mathbb{D}^{m-1}$ by $(\varphi(\tau), x) \mapsto f(\nu_\tau^{-1}(x))$ is a C^1 embedding. (This means that the coordinates in $Z_\vartheta(\tau)$ ($\tau \in A_\vartheta$) given by $\xi_\vartheta(\tau, x) \mapsto x$ define a C^1 extension of the coordinates in $N'(\tau')$ ($\tau' \in B_\vartheta$) given by $x \mapsto \nu_{\tau'} f^{-1}(x)$.)

(3) If $\tau_+ \in A_{\vartheta_+}, \tau_- \in A_{\vartheta_-}, \varphi(\tau_+) = \varphi(\tau_-) = \tau \neq \varphi(\vartheta)$ then
$$Z_\vartheta(\tau) \cap N^* = N^*(\tau_+) \cup N^*(\tau_-) \subset \operatorname{Int} Z_\vartheta(\tau, 1).$$
Therefore $Z_\vartheta \cap N^*$ is the union of two curved cylinders and the disk $N^*(\vartheta)$.

We assume that the "cylinders" Z_ϑ for different branch points ϑ are disjoint and define N^{**} to be the union of N^* with all these cylinders Z_ϑ ($\vartheta \in \Theta$). Obviously N^{**} is a manifold which has corners if $\Theta \neq \emptyset$. Moreover N^{**} is a neighbourhood of N' in N, and we shall define h_0 so that it is the identity outside N^{**}, i.e. we shall find a sequence $\underline{l} \in \{0,1\}^r$ and construct a diffeomorphism $h_0 \colon N^{**} \to N^{**}$ which is the identity near ∂N^{**}, which maps each component of $N^{**} \cap N(\tau)$ $(\tau \in \Sigma)$ onto itself and which coincides with $f' \vartheta_{\underline{l}} f^{-1}$ on N'.

We start the definition of the diffeomorphism $h_0 \colon N \to N$ by fixing h_0 on the set

$$\begin{aligned} N_1^* &= Cl(N^* \setminus \bigcup_{\vartheta \in \Theta} Z_\vartheta) \\ &= Cl(N^{**} \setminus \bigcup_{\vartheta \in \Theta} Z_\vartheta) \\ &= \bigcup_{\tau \in \Sigma_1} N^*(\tau), \\ \text{where } \Sigma_1 &= \Sigma \setminus \bigcup_{\vartheta \in \Theta} (\operatorname{Int} A_{\vartheta_+} \cup \operatorname{Int} A_{\vartheta_-}). \end{aligned}$$

If $\tau \in \Sigma_1$ then with the contraction $g_t : \Sigma \to SO(m-1)$ constructed above h_0 on $N^*(\tau)$ is defined by

$$\begin{aligned} h_0 &= f' \vartheta_{\underline{l}} f^{-1} = (f' \vartheta_{\underline{l}} f^{-1})^*(\tau) = g_0(\tau) \text{ on } N'(\tau) = N'(\tau, 1), \\ h_0 &= g_t(\tau) \qquad\qquad\qquad\qquad \text{on } \partial N^*(\tau, 1+\varepsilon) \ (0 \le t \le 1). \end{aligned}$$

Obviously $h_0 \colon N_1^* \to N_1^*$ is a diffeomorphism such that $h_0 = f' \vartheta_{\underline{l}} f^{-1}$ on $N' \cap N_1^*, h_0(N^*(\tau)) = N^*(\tau)$ for $N^*(\tau) \subset N_1^*$ and $h_0 = id$ near $\partial N^* \cap N_1^*$.

Now we consider the cylinder Z_ϑ of a branch point ϑ and define h_0 on Z_ϑ. This will be done in three steps. First we choose a diffeomorphism $h_1 \colon Z_\vartheta \to Z_\vartheta$ which is fibre preserving in the sense that $h_1(Z_\vartheta(\tau)) = Z_\vartheta(\tau)$ $(\tau \in A_\vartheta)$ and which satisfies the following conditions.

(1) On $N^*(\vartheta) = Z_\vartheta(\varphi(\vartheta))$ (where h_0 is already defined) h_1 coincides with h_0 and the combination of h_0 and h_1 at $N^*(\vartheta)$ is differentiable.
(2) For each $\tau \in A_\vartheta$ the restriction of h_1 to $Z_\vartheta(\tau, 1)$ is a rotation in $SO(m-1)$.
(3) $h_1 = id$ near $\bigcup_{\tau \in A_\vartheta} \partial Z_\vartheta(\tau) \cup Z_\vartheta(\tau_1)$, where τ_1 is the second end point of A_ϑ.

These conditions imply the following properties of h_1.

(4) If $\tau \in A_{\vartheta_+} \cup A_{\vartheta_-}$ then the restriction of h_1 to $N^*(\tau)$ is an orientation preserving similarity mapping.
(5) If Z_+, Z_- are the closures of the two components of $(Z_\vartheta \setminus N^*(\vartheta)) \cap N^*$ then the disks $Z_+ \cap N^*(\vartheta), Z_- \cap N^*(\vartheta)$ are invariant under h_1 (see Figure 3).

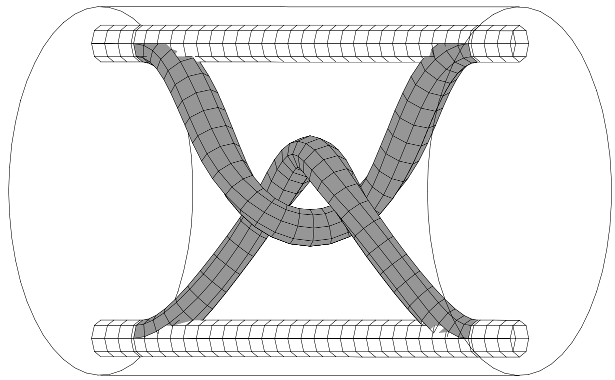


Figure 3

Using that $m \geq 4$ in the second step we can find a fibre preserving diffeomorphism $h_2 \colon Z_\vartheta \to Z_\vartheta$ which satisfies the following conditions.

(1) If $\tau \in A_{\vartheta_+} \cup A_{\vartheta_-}$ then the restriction of h_2 to $h_1(N^*(\tau))$ is an orientation preserving similarity mapping to $N^*(\tau)$.

(2) $h_2 = id$ on $N^*(\vartheta)$, near $\bigcup_{\tau \in A_\vartheta} \partial Z_\vartheta(\tau)$ and near $Z_\vartheta(\tau_1)$ where τ_1 is the second end point of A_ϑ. Moreover, for $x \in N^*(\vartheta)$ the differential of h_2 at x is the identity.

Then $h_2 h_1(N^* \cap Z_\vartheta) = N^* \cap Z_\vartheta$ and $h_2 h_1(N' \cap Z_\vartheta) = N' \cap Z_\vartheta$ but we can not yet be sure that $h_2 h_1 = f' \vartheta_{\underline{l}} f^{-1}$ on $N' \cap Z_\vartheta$. To obtain this equality we apply the method by which h_0 on N_1^* was constructed above to the closure of each of the two components of $(Z_\vartheta \cap N^*) \setminus N^*(\varphi(\vartheta))$. So we get a fibre preserving diffeomorphism $h_3 \colon Z_\vartheta \to Z_\vartheta$ with the following properties.

(1) $h_3 = id$ on $Z_\vartheta \setminus N^*$ and on $N^*(\vartheta)$, and for $x \in N^*(\vartheta)$ the differential of h_3 at x is the identity.

(2) $h_3 = h_0$ on the two disks at which $N_1^* \setminus N^*(\vartheta)$ intersects Z_ϑ.

(3) $h_3 h_2 h_1 = f' \vartheta_{\underline{l}} f^{-1}$ on $N' \cap Z_\vartheta$.

The diffeomorphism $h_3 h_2 h_1$ defines an extension of $h_0 \colon N_1^* \to N_1^*$ to a diffeomorphism $h_0 : N_1^* \cup Z_\vartheta \to N_1^* \cup Z_\vartheta$. To obtain h_0 on N^{**} we extend $h_0 : N_1^* \to N_1^*$ in this way over all cylinders Z_ϑ ($\vartheta \in \Theta$). This construction implies $h_0 = id$ on $N \setminus N^{**}$ we get a diffeomorphism h_0 on the whole manifold N which by the remark above proves the lemma. □

5. Proof of Proposition 3.1

Let $\underline{l}, \underline{l}' \in \{0,1\}^r$ be fixed. To prove the proposition we have to find an integer $k \geq 1$ and a homeomorphism $h \colon W_{\underline{l}} \to W_{\underline{l}'}$ such that

$hf_{\underline{l}}^k = f_{\underline{l}'}^k h'$ and the restriction of h to the complement of the attractor $\Lambda_{f_{\underline{l}}}$ of $f_{\underline{l}}$ is a diffeomorphism. We shall use the following notations.

$$N = N_\Sigma^m, \qquad N(\tau) = \pi_\Sigma^{-1}(\tau) \quad (\tau \in \Sigma),$$
$$N_i = f_{\underline{l}}^i(N) \qquad N_i' = f_{\underline{l}'}^i(N) \qquad (i \in \mathbb{Z}).$$

The first step in our construction of h is the following lemma.

Lemma 5.1. *There is a fibre preserving homeomorphism $h_0 \colon N \to N$ such that for certain sequences $l_1^*, l_2^*, \dots$ in $\{0,1\}^r$ we have for $i = 1, 2, \dots$*

$$h_0(N_i) = N_i'$$
$$h_0 = f_{\underline{l}'}^i, \vartheta_{\underline{l}_i^*} f_{\underline{l}}^{-i} \text{ on } \partial N_i.$$

The restriction of h_0 to $N \setminus \Lambda_{f_{\underline{l}}}$ is a diffeomorphism.

Proof. The homeomorphism h_0 will be the limit of a sequence $h_0^* = id, h_1^*, h_2^*, \dots$ of fibre preserving diffeomorphisms $h_i^* : N \to N$ which together with certain sequences $\underline{l}_0^* = (0, \dots, 0), \underline{l}_1^*, \underline{l}_2^*, \dots$ in $\{0,1\}^r$ have the following properties

$$h_i^* = f_{\underline{l}'}^i \vartheta_{\underline{l}_i^*} f_{\underline{l}}^{-i} \text{ on } N_i,$$
$$h_j^* = h_i^* \text{ on } N \setminus N_i \text{ if } 0 \le i < j.$$

We construct the $\underline{l}_i^*, h_i^*$ inductively, i.e. we assume that for some $i \ge 0$ both $\underline{l}_i^*$ and h_i^* are already fixed and define $\underline{l}_{i+1}^*, h_{i+1}^*$. To this aim we shall show that there is a sequence $\underline{l}_{i+1}^* \in \{0,1\}^r$ and a fibre preserving diffeomorphism $h' \colon N \to N$ such that

$$h' f_{\underline{l}} = \vartheta_{\underline{l}_i^*}^{-1} f_{\underline{l}'} \vartheta_{\underline{l}_{i+1}^*}.$$

Then the lemma is proved, for by

$$h_{i+1}^* = \begin{cases} h_i^* f_{\underline{l}}^i h' f_{\underline{l}}^{-i} & \text{on } N_i \\ h_i^* & \text{on } N \setminus N_i \end{cases}$$

we get a fibre preserving diffeomorphism which on N_{i+1} satisfies

$$\begin{aligned} h_{i+1}^* &= h_i^* f_{\underline{l}}^i h' f_{\underline{l}}^{-i} \\ &= f_{\underline{l}'}^i \vartheta_{\underline{l}_i^*} f_{\underline{l}_i}^{-i} f_{\underline{l}}^i \vartheta_{\underline{l}_i^*}^{-1} f_{\underline{l}'} \vartheta_{\underline{l}_{i+1}^*} f_{\underline{l}}^{-1} f_{\underline{l}}^{-i} \\ &= f_{\underline{l}'}^{i+1} \vartheta_{\underline{l}_{i+1}^*} f_{\underline{l}}^{-(i+1)}. \end{aligned}$$

The construction of h' is done in two steps. In the first we apply Lemma 4.3 with $f = f_{\underline{l}}, f' = \vartheta_{\underline{l}_i^*}^{-1} f_{\underline{l}'}$ and get a fibrewise similar embedding f'':

$N \to \operatorname{Int} N$ over φ and a fibre preserving diffeomorphism $h_1'\colon N \to N$ such that

$$h_1' f_{\underline{l}} = f'', \quad f''(N(\tau)) = \vartheta_{\underline{l}_i^*}^{-1} f_{\underline{l}'}(N(\tau)).$$

In the second step, applying Lemma 4.4, we get a sequence $\underline{l}_{i+1}^*$ and a fibre preserving diffeomorphism $h_2'\colon N \to N$ such that

$$h_2' f'' = \vartheta_{\underline{l}_i^*}^{-1} f_{\underline{l}'} \vartheta_{\underline{l}_{i+1}^*}$$

and therefore with $h' = h_2' h_1'$

$$h' f_{\underline{l}} = h_2' f'' = \vartheta_{\underline{l}_i^*}^{-1} f_{\underline{l}'} \vartheta_{\underline{l}_{i+1}^*}.$$

□

Now we continue our construction of $k \geq 0$ and $h\colon W_{\underline{l}} \to W_{\underline{l}'}$. Since $\{0,1\}^r$ is finite, there are integers $0 \leq i < j$ such that the sequences $\underline{l}_i^*, \underline{l}_j^*$ coincide. Then for the mapping h_0 in Lemma 5.1 we have

$$h_0 = f_{\underline{l}'}^i \vartheta_{\underline{l}_i^*} f_{\underline{l}}^{-i} \quad \text{on } \partial N_i,$$
$$h_0 = f_{\underline{l}'}^j \vartheta_{\underline{l}_i^*} f_{\underline{l}}^{-j} \quad \text{on } \partial N_j,$$

and this implies

$$h_0 = f_{\underline{l}'}^{j-i} h_0 f_{\underline{l}}^{i-j} \quad \text{on } \partial N_j.$$

We define $k = j - i$. Therefore

$$h_0 = f_{\underline{l}'}^k h_0 f_{\underline{l}}^{-k} \quad \text{on } \partial N_j$$

and we modify h_0 near ∂N_i so that in addition the differentials of h_0 and $f_{\underline{l}'}^k h_0 f_{\underline{l}}^{-k}$ coincide on N_j. Then by

$$h = f_{\underline{l}'}^{jk} h_0 f_{\underline{l}}^{-jk} \text{ on } f^{jk}(N_i \setminus N_j) \quad (j \in \mathbb{Z})$$

we get a diffeomorphism

$$h : \bigcup_{j \in \mathbb{Z}} f_{\underline{l}}^{jk}(N_i \setminus N_j) = W_{\underline{l}} \setminus \Lambda_{f_{\underline{l}}} \to \bigcup_{j \in \mathbb{Z}} f_{\underline{l}'}^{jk}(N_i' \setminus N_j') = W_{\underline{l}'} \setminus \Lambda_{f_{\underline{l}'}},$$

and this diffeomorphism satisfies on each set $f^{jk}(N_i \setminus N_j)$ and therefore on $W_{\underline{l}} \setminus \Lambda_{f_{\underline{l}}}$

$$h = f_{\underline{l}'}^{jk} h_0 f_{\underline{l}}^{-jk} = f_{\underline{l}'}^k f_{\underline{l}'}^{(j-1)k} h_0 f_{\underline{l}}^{-(j-1)k} f_{\underline{l}}^{-k} = f_{\underline{l}'}^k h f_{\underline{l}}^{-k}.$$

The extension of h over $\Lambda_{f_{\underline{l}}}$ to a diffeomorphism $h\colon W_{\underline{l}} \to W_{\underline{l}'}$ can be done as in the proof of Lemma 4.2.

6. Existence of tubular neighbourhoods (Proof of Proposition 3.2)

In this section we prove that for each transversely tame 1-dimensional hyperbolic attractor Λ with an orientable basin belonging to a C^1 diffeomorphism $f : M \to M$ of an m-dimensional manifold M $(m \geq 4)$ and for each W-representation (W) of Λ (see Section 2) there is a corresponding tubular neighbourhood N of Λ.

Let A be a smooth arc in Σ for which $A \cap \Theta$ (Θ the set of branch points) is either empty or an end point of A. Then by a cylinder over A we mean a C^1 embedding $\xi\colon A \times \mathbb{D}^n \to W^s_\Lambda \quad (n = m-1)$ such that for each $\tau \in \Sigma$ the disk $\xi(\{\tau\} \times \mathbb{D}^n)$ lies in the stable manifold which contains $\pi^{-1}(\tau)$, where π is the projection in (W). The image $\xi(A \times \mathbb{D}^n)$ will be denoted by $|\xi|$, and for $\tau \in A$ we write $|\xi|_\tau = \xi(\{\tau\} \times \mathbb{D}^n)$. A cylinder ξ over A will be called adapted if for $\tau \in A\backslash\Theta$

$$|\xi|_\tau \cap \Lambda = \mathrm{Int}\, |\xi|_\tau \cap \Lambda = \pi^{-1}(\tau)$$

and for $\tau \in A \cap \Theta$

$$|\xi|_\tau \cap \Lambda = \mathrm{Int}\, |\xi|_\tau \cap \Lambda = Cl(\xi((A\backslash\{\tau\}) \times \mathbb{D}^n) \cap \Lambda) \backslash (\xi((A\backslash\{\tau\}) \times \mathbb{D}^n) \cap \Lambda).$$

We start the construction of N by choosing for each $\tau \in \Sigma$ an n-dimensional compact manifold Q_τ in the stable manifold which contains $\pi^{-1}(\tau)$ such that $\pi^{-1}(\tau)$ lies in $\mathrm{Int}\, Q_\tau$ and that these manifolds Q_τ for different points τ of Σ are disjoint. Since the sets $\pi^{-1}(\tau)$ are Cantor sets we can bore holes into the manifolds Q_τ (if necessary) so that none of them disconnects the corresponding stable manifold. Then after connecting different components of Q_τ by thin tubes we may assume that each Q_τ is connected. Now we use our assumption that Λ is transversely tame (i.e. that $\pi^{-1}(\tau)$ can be covered by arbitrary small disjoint balls in Q_τ) to define in each Q_τ an n-ball D_τ which contains $\pi^{-1}(\tau)$ in its interior. Obviously, these balls D_τ $(\tau \in \Sigma)$ are disjoint.

Now for each $\tau \in \Sigma$ by thickening D_τ we can find a smooth arc A in Σ containing τ and an adapted cylinder ξ over A such that $|\xi|_\tau = D_\tau$. (Here we use the well known fact that the stable foliation of a 1-dimensional hyperbolic attractor is of class C^1.) Then, since Σ is compact, it is not hard to construct a decomposition of Σ, consisting of smooth arcs $A_1, \ldots, A_q$ each pair of which has at most one end point in common and adapted cylinders ξ_i over A_i $(i = 1, \ldots, q)$ such that $A_i \cap A_j \neq \emptyset$, $i \neq j$ implies that A_i, A_j have a common end point τ_0 and $|\xi_i| \cap |\xi_j| = |\xi_i|_{\tau_0} \cap |\xi_j|_{\tau_0}$.

The next task in our construction is a modification of the cylinders ξ_i after which th union $|\xi_1|\cup\cdots\cup|\xi_q|$ becomes a tubular neighbourhood of Λ.

First we consider the common end point τ_0 of exactly two arcs A_i, A_j. We shall define a new adapted cylinder ξ_j^* over A_j such that ξ_i, ξ_j^* together define an adapted cylinder ξ' over $A' = A_i \cup A_j$ in the sense that $|\xi'|_\tau = |\xi_i|_\tau$ for $\tau \in A_i$ and $|\xi'|_\tau = |\xi_j^*|_\tau$ for $\tau \in A_j$. For $1 \le k \le q$, $i \ne k \ne j$ this cylinder ξ' will satisfy $|\xi'| \cap |\xi_k| = (|\xi_i| \cup |\xi_j|) \cap |\xi_k|$. Moreover there will be a proper subarc A_j^* of A_j with one end point τ_0 such that $|\xi_j^*|_\tau = |\xi_j|_\tau$ for $\tau \in A_j \setminus A_j^*$.

To find A_j^* and ξ_j^* we choose an n-disk D in the stable manifold $W^s_{\tau_0}$ containing $|\xi_i|_{\tau_0}$ and $|\xi_j|_{\tau_0}$ which contains $|\xi_i|_{\tau_0} \cup |\xi_j|_{\tau_0}$ in its interior and whose boundary does not intersect any $|\xi_k|$ $(1 \le k \le q)$. (This is possible since $W^s_{\tau_0} \cap (|\xi_1|\cup\cdots\cup|\xi_q|)$ is the union of n-disks which, with a finite number of exceptions corresponding to end points of $A_1, \ldots, A_q$, are disjoint.) Then it is easy to find a proper subarc A_j^* of A_j one of whose end point is τ_0 and over which there is an adapted cylinder ξ with the following properties.

(1) $|\xi|_{\tau_0} = D$.
(2) $\bigcup_{\tau \in A_j^*} \partial|\xi|_\tau \cap (|\xi_1| \cup \cdots \cup |\xi_q|) = \emptyset$.
(3) For $1 \le k \le q$ the intersection $|\xi| \cap |\xi_k|$ is either a disk in $\mathrm{Int}\,|\xi|_{\tau_0}$ or there is a subarc A_k^* of A_k such that $|\xi| \cap |\xi_k| = \xi_k(A_k^* \times \mathbb{D}^n)$.

Using this cylinder ξ the construction of ξ_j^* is a straightforeward application of the following lemma which states a version of the well known fact that in dimensions higher than three all braids with 1-dimensional strings are trivial. Indeed, we have to apply this lemma, where $Z = |\xi|$, $D_0 = |\xi_i|_{\tau_0}$, $D_1 = |\xi_j|_{\tau_1}$ (τ_1 is the second end point of A_j^*), $h_{i'}(Z)\,(k_0 < i' \le k)$ are the closures of the components of $\xi((A_j^* \setminus \{\tau_0\}) \times \mathbb{D}^n) \cap \bigcup_{j' \ne j} |\xi_{j'}|$, and the tubes $h_{i'}(Z)\,(1 \le i' \le k_0)$ must be choosen so that they cover $\Lambda \cap \xi_j(A_j^* \times \mathbb{D}^n)$. [At this point of the construction we need $m \ge 4$!]

Lemma 6.1. *Let $Z = I \times \mathbb{D}^n$ $(n \ge 3)$ be the $(n+1)$-dimensional standard cylinder, and let $h_i : Z \to Z$ $(i = 1, \ldots, k)$ be C^1 emdeddings with disjoint images $Z_1, \ldots, Z_k$ such that*

$$h_i(\{t\} \times \mathbb{D}^n) \subset \mathrm{Int}(\{t\} \times \mathbb{D}^n) \qquad (i = 1, \ldots, k;\ t \in I).$$

Moreover, let $g_j : \mathbb{D}^n \to \{j\} \times \mathrm{Int}\,\mathbb{D}^n$ $(j = 0, 1)$ be two orientation preserving C^1 embedding with images $D_j = g_j(\mathbb{D}^n)$ $(j = 0, 1)$. It is *assumed that for some* $k_0 \le k$ *we have*

$$h_i(\{j\} \times \mathbb{D}^n) \subset \operatorname{Int} D_j \qquad (1 \le i \le k_0;\ j = 0,1)$$
$$h_i(\{j\} \times \mathbb{D}^n) \cap D_j = \emptyset \qquad (k_0 < i \le k;\ j = 0,1).$$

Then there is a C^1 *embedding* $h : Z \to Z$ *such that*

$$h(\{t\} \times \mathbb{D}^n) \subset \operatorname{Int}(\{t\} \times \mathbb{D}^n) \qquad (t \in I),$$
$$h(j,x) = g_j(x) \qquad (j = 0,1;\ x \in \mathbb{D}^n),$$
$$Z_1 \cup \cdots \cup Z_{k_0} \subset h(I \times \operatorname{Int} \mathbb{D}^n),$$
$$h(Z) \cap (Z_{k_0+1} \cup \cdots \cup Z_{k_0}) = \emptyset.$$

□

After this construction is done for all end points τ_0 of exactly two of the arcs $A_1, \ldots, A_q$, the union N_0 of the modified cylinders $|\xi_1|, \ldots, |\xi_q|$ has the following property. If $\tau \in \Sigma$ is not a branch point then there is an adapted cylinder ξ over an arc A which contains τ in its interior such that $|\xi| \subset N_0$, $|\xi| \cap Cl(N_0 \setminus |\xi|) = |\xi|_{\tau_0} \cup |\xi|_{\tau_1}$ (τ_0, τ_1 the end points of A) and $|\xi|_{\tau'} \cap \Lambda = \operatorname{Int} |\xi|_{\tau'} \cap A = \pi^{-1}(\tau')$ for each $\tau' \in A$.

To get a tubular neighbourhood N of λ belonging to (W) it remains to modify N_0 at those parts which lie over neighbourhoods of branch points of Σ. This can be done (using Lemma 6.1) by the same methods by which the end points τ_0 of exactly two of the arcs $A_1, \ldots, A_q$ were treated above. Therefore we can omit a detailed description of this final step, and the construction of a tubular neighbourhood N of Λ is finished.

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