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Anisotropic growth of random surfaces in $2+1$ dimensions

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Abstract

We construct a family of stochastic growth models in $2+1$ dimensions, that belong to the anisotropic KPZ class. Appropriate projections of these models yield $1+1$ dimensional growth models in the KPZ class and random tiling models. We show that correlation functions associated to our models have determinantal structure, and we study large time asymptotics for one of the models.

The main asymptotic results are: (1) The growing surface has a limit shape that consists of facets interpolated by a curved piece. (2) The one-point fluctuations of the height function in the curved part are asymptotically normal with variance of order $\ln(t)$ for time $t \gg 1$. (3) There is a map of the $(2+1)$-dimensional space-time to the upper half-plane $\mathbb{H}$ such that on space-like submanifolds the multi-point fluctuations of the height function are asymptotically equal to those of the pullback of the Gaussian free (massless) field on $\mathbb{H}$.

Contents

1 Introduction

In recent years there has been a lot of progress in understanding large time fluctuations of driven interacting particle systems on the one-dimensional lattice, see e.g. [1, 2, 4, 7–9, 11, 22, 24, 25, 31, 38, 40–42]. Evolution of such systems is commonly interpreted as random growth of a one-dimensional interface, and if one views the time as an extra variable, the evolution produces a random surface (see e.g. Figure 4.5 in [36] for a nice illustration). In a different direction, substantial progress have also been achieved in studying the asymptotics of random surfaces arising from dimers on planar bipartite graphs, see the review [27] and references therein.

Although random surfaces of these two kinds were shown to share certain asymptotic properties (also common to random matrix models), no direct connection between them was known. One goal of this paper is to establish such a connection.

We construct a class of two-dimensional random growth models (that is, the principal object is a randomly growing surface, embedded in the four-dimensional space-time).

In two different projections these models yield random surfaces of the two kinds mentioned above (one reduces the spatial dimension by one, the second projection is fixing time). We partially compute the correlation functions of an associated (three-dimensional) random point process and show that they have determinantal form that is typical for determinantal point processes.

For one specific growth model we compute the correlation kernel explicitly, and use it to establish Gaussian fluctuations of the growing random surface. We then determine the covariance structure.

Let us describe our results in more detail.

1.1 A two-dimensional growth model

Consider a continuous time Markov chain on the state space of interlacing variables ($n = 1, 2, \dots$)

$$\mathcal{S}^{(n)} = \left\{ \{x_k^m\}_{\substack{k=1,\dots,m \\ m=1,\dots,n}} \subset \mathbb{Z}^{\frac{n(n+1)}{2}} \mid x_{k-1}^m < x_{k-1}^{m-1} \leq x_k^m \right\} \tag{1.1}$$

with the following evolution. Each of the particles x_k^m has an independent exponential clock of rate one, and when the x_k^m-clock rings the particle attempts to jump to the right by one. If at that moment $x_k^m = x_k^{m-1} - 1$ then the jump is blocked. If that is not the case, we find the largest $c \geq 1$ such that $x_k^m = x_{k+1}^{m+1} = \cdots = x_{k+c-1}^{m+c-1}$, and all c particles in this string jump to the right by one.

Informally speaking, the particles with smaller upper indices are heavier than those with larger upper indices, so that the heavier particles block and push the lighter ones in order for the interlacing conditions to be preserved. This anisotropy is essential, see more details in Section 1.4.

In this paper we consider only one initial condition for this Markov chain: at time moment $t = 0$ we have $x_k^m(0) = k - m - 1$ for all k, m. For any $t \geq 0$ denote by $\mathcal{M}^{(n)}(t)$ the resulting measure on $\mathcal{S}^{(n)}$ at time moment t.

Observe that $\mathcal{S}^{(n_1)} \subset \mathcal{S}^{(n_2)}$ for $n_1 \leq n_2$, and the definition of the evolution implies that $\mathcal{M}^{(n_1)}(t)$ is a marginal of $\mathcal{M}^{(n_2)}(t)$ for any $t \geq 0$. Thus, we can think of $\mathcal{M}^{(n)}$'s as marginals of the measure $\mathcal{M} = \varprojlim \mathcal{M}^{(n)}$ on $\mathcal{S} = \varprojlim \mathcal{S}^{(n)}$. In other words, $\mathcal{M}(t)$ are measures on the space $\mathcal{S}$ of infinite point configurations $\{x_k^m\}_{k=1,\dots,m,\, m\geq 1}$.

There are a number different viewpoints for these Markov chains and their parts.

1. The evolution of x_1^1 is the one-dimensional Poisson process of rate one.

2. The row $\{x_1^m\}_{m\geq 1}$ evolves as a Markov chain known as the *Totally Asymmetric Simple Exclusion Process* (TASEP), and the initial condition $x_1^m(0) = -m$ is commonly referred to as *step initial condition*.

3. The row $\{x_m^m\}_{m\geq 1}$ also evolves as a Markov chain that is sometimes called "long range TASEP"; it was also called PushASEP in [7].

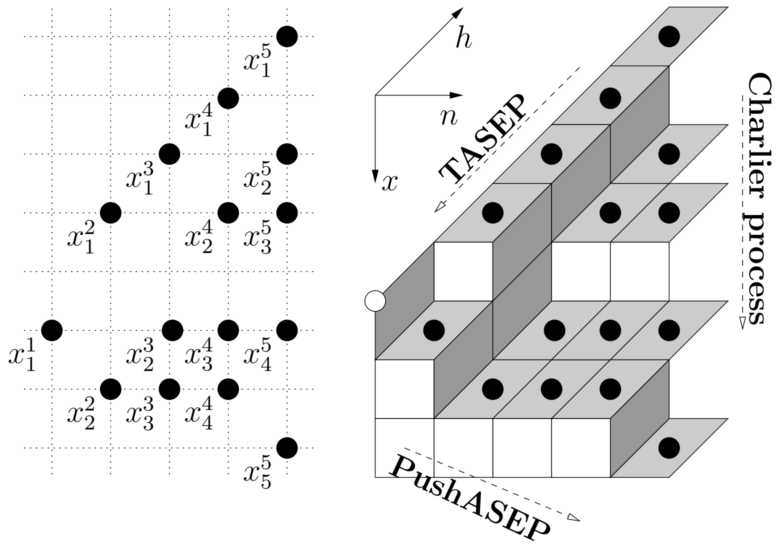

Figure 1.1: From particle configurations (left) to 3d visualization via lozenge tilings (right). The corner with the white circle has coordinates $(x, n, h) = (-1/2, 0, 0)$.

4. For our initial condition, the evolution of each row $\{x_k^m\}_{k=1,\dots,m}$, $m = 1, 2, \dots,$ is also a Markov chain. It was called *Charlier process* in [30] because of its relation to the classical orthogonal Charlier polynomials. It can be defined as h-Doob transform for m independent rate one Poisson processes with the harmonic function h equal to the Vandermonde determinant.

5. Infinite point configurations $\{x_k^m\} \in \mathcal{S}$ can be viewed as *Gelfand-Tsetlin schemes*. Then $\mathcal{M}(t)$ is the "Fourier transform" of a suitable irreducible character of the infinite-dimensional unitary group $U(\infty)$, see [13]. Interestingly enough, increasing t corresponds to a deterministic flow on the space of irreducible characters of $U(\infty)$.

6. Elements of $\mathcal{S}$ can also be viewed as lozenge tilings of a sector in the plane. To see that one surrounds each particle location by a rhombus of one type and draws edges through locations where there are no particles, see Figure 1.1.

7. Figure 1.1 has a clear three-dimensional connotation. Given the random configuration $\{x_k^n(t)\} \in \mathcal{S}$ at time moment t, define the random *height function*

$$h : (\mathbb{Z} + \tfrac{1}{2}) \times \mathbb{Z}_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}, \quad h(x, n, t) = \#\{k \in \{1, \dots, n\} \mid x_k^n(t) > x\}. \tag{1.2}$$

In terms of the tiling on Figure 1.1, the height function is defined at the vertices of rhombi, and it counts the number of particles down from a given vertex. (This definition is different by a simple linear function of (x, n) from the standard definition of the height function for lozenge tilings, see e.g. [27, 28].)

Thus, our Markov chain can be viewed as a random growth model of the surface given by the height function. In terms of the step surface of Figure 1.1, the evolution consists of removing all columns of (x,n,h)-dimensions $(1,*,1)$ that could be removed, independently with exponential waiting times of rate one.

One of the goals of this paper is to study asymptotic properties of the random height function as the time becomes large.

1.2 Determinantal formula, limit shape and one-point fluctuations

The first result about the Markov chain $\mathcal{M}(t)$ that we prove is the (partial) determinantal structure of the correlation functions. Introduce the notation

$$(n_1,t_1)\prec(n_2,t_2)\quad\text{iff}\quad n_1\le n_2, t_1\ge t_2,\ \text{and}\ (n_1,t_1)\neq(n_2,t_2). \tag{1.3}$$

Theorem 1.1. *For any $N=1,2,\dots,$ pick N triples*

$$\varkappa_j=(x_j,n_j,t_j)\in\mathbb{Z}\times\mathbb{Z}_{\ge 0}\times\mathbb{R}_{\ge 0}$$

such that

$$t_1\le t_2\le\dots\le t_N,\qquad n_1\ge n_2\ge\dots\ge n_N. \tag{1.4}$$

Then

$$\mathbb{P}\{\textit{For each } j=1,\dots,N \textit{ there exists a } k_j,\\ 1\le k_j\le n_j \textit{ such that } x_{k_j}^{n_j}(t_j)=x_j\}=\det\left[K(\varkappa_i,\varkappa_j)\right]_{i,j=1}^N, \tag{1.5}$$

where

$$\begin{aligned}K(x_1,n_1,t_1;x_2,n_2,t_2)=&-\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_0}\frac{\mathrm{d}w}{w^{x_2-x_1+1}}\frac{e^{(t_1-t_2)/w}}{(1-w)^{n_2-n_1}}\mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]}\\&+\frac{1}{(2\pi i)^2}\oint_{\Gamma_0}\mathrm{d}w\oint_{\Gamma_1}\mathrm{d}z\frac{e^{t_1/w}}{e^{t_2/z}}\frac{(1-w)^{n_1}}{(1-z)^{n_2}}\frac{w^{x_1}}{z^{x_2+1}}\frac{1}{w-z},\end{aligned} \tag{1.6}$$

the contours Γ_0, Γ_1 are simple closed paths, positively oriented, they include the poles 0 and 1, respectively, and no other poles.

The above kernel has in fact already appeared in [7] in connection with PushASEP. The determinantal structure makes it possible to study the asymptotics. Set

$$\mathcal{D}=\{(\nu,\eta,\tau)\in\mathbb{R}^3_{>0}\,|\,(\sqrt{\eta}-\sqrt{\tau})^2<\nu<(\sqrt{\nu}+\sqrt{\tau})^2\}. \tag{1.7}$$

It is exactly the set of triples $(\nu,\eta,\tau)\in\mathbb{R}^3_{>0}$ for which there exists a nondegenerate triangle with side lengths $(\sqrt{\nu},\sqrt{\eta},\sqrt{\tau})$. Denote by $(\pi_\nu,\pi_\eta,\pi_\tau)$ the angles of this triangle that are opposite to the corresponding sides (see Figure 3.1 too).

Theorem 1.2. *For any $(\nu,\eta,\tau)\in\mathcal{D}$ we have the moment convergence of random variables*

$$\lim_{L\to\infty}\frac{h([(\nu-\eta)L]+\frac{1}{2},[\eta L],\tau L)-\mathbb{E}\left(h([(\nu-\eta)L]+\frac{1}{2},[\eta L],\tau L)\right)}{\sqrt{\kappa\ln(L)}}=\xi\sim\mathcal{N}(0,1), \tag{1.8}$$

with $\kappa=(2\pi^2)^{-1}$. In particular,

$$\begin{aligned}\lim_{L\to\infty}\frac{\mathbb{E}\left(h([(\nu-\eta)L]+\frac{1}{2},[\eta L],\tau L)\right)}{L}&=:\bar{h}(\nu,\eta,\tau)\\ &=\frac{1}{\pi}\left(-\nu\pi_\eta+\eta(\pi-\pi_\eta)+\tau\frac{\sin\pi_\nu\sin\pi_\eta}{\sin\pi_\tau}\right).\end{aligned} \tag{1.9}$$

Theorem 1.2 describes the limit shape of our growing surface, and the domain $\mathcal{D}$ describes the points where this limit shape is *curved*. The remaining part consists of facets of the limit shape, that are just subsets of the facets in the initial condition. The logarithmic fluctuations is essentially a consequence of the asymptotic local behavior being governed by the sine kernel (this local behavior occurs also in tiling models [21, 26, 35]).

Using the connection with Charlier ensemble, see above, the formula (1.9) for the limit shape can be read off the formulas of [5].

Using Theorem 1.1 it is not hard to verify (although we do not do this in the paper) that near every point of the limit shape in the curved region, at any fixed time moment the random lozenge tiling approaches the unique translation invariant measure $M_{\pi_\nu,\pi_\eta,\pi_\tau}$ on lozenge tilings of the plane with prescribed slope (see [14,27,29] and references therein for discussions of these measures). The slope is exactly the slope of the tangent plane to the limit shape, given by

$$\frac{\partial\bar{h}}{\partial\nu}=-\frac{\pi_\eta}{\pi},\qquad\frac{\partial\bar{h}}{\partial\eta}=1-\frac{\pi_\nu}{\pi}. \tag{1.10}$$

This implies in particular, that $(\pi_\nu/\pi,\pi_\eta/\pi,\pi_\tau/\pi)$ are the asymptotic proportions of lozenges of three different types in the neighborhood of the point of the limit shape.

One also computes

$$\frac{\partial\bar{h}}{\partial\tau}=\frac{1}{\pi}\frac{\sin\pi_\nu\sin\pi_\eta}{\sin\pi_\tau}. \tag{1.11}$$

Since the right-hand side depends only on the slope of the tangent plane, this suggest that it should be possible to extend the definition of our surface evolution to the random surfaces distributed according to measures $M_{\pi_\nu,\pi_\eta,\pi_\tau}$; these measures have to remain invariant under evolution, and the speed of the height growth should be given by the right-hand side of (1.11). This is an interesting open problem that we do not address in this paper.

1.3 Complex structure and multipoint fluctuations

Set $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ and define a map $\Omega : \mathcal{D} \to \mathbb{H}$ by

$$|\Omega(\nu, \eta, \tau)| = \sqrt{\nu}/\sqrt{\tau}, \quad |1 - \Omega(\nu, \eta, \tau)| = \sqrt{\eta}/\sqrt{\tau}. \tag{1.12}$$

Observe that $\arg \Omega = \pi_\nu$ and $\arg(1 - \Omega) = -\pi_\eta$. The preimage of any $\Omega \in \mathbb{H}$ is a ray in $\mathcal{D}$ that consists of triples (ν, η, τ) with constant ratios $(\nu : \eta : \tau)$. Denote this ray by R_Ω. One sees that R_Ω's are also the level sets of the slope of the tangent plane to the limit shape. Since $\bar{h}(\alpha\nu, \alpha\eta, \alpha\tau) = \alpha\bar{h}(\nu, \eta, \tau)$ for any $\alpha > 0$, the height function grows linearly with time along each R_Ω.

Note also that the map Ω satisfies

$$(1 - \Omega)\frac{\partial \Omega}{\partial \nu} = \Omega \frac{\partial \Omega}{\partial \eta} = -\frac{\partial \Omega}{\partial \tau}, \tag{1.13}$$

and the first of these relations is the complex Burgers equation.

Denote by

$$\mathcal{G}(z, w) = -\frac{1}{2\pi} \ln \left| \frac{z - w}{z - \bar{w}} \right| \tag{1.14}$$

the Green function of the Laplace operator on $\mathbb{H}$ with Dirichlet boundary conditions.

Theorem 1.3. *For any $N = 1, 2, \dots$, let $\varkappa_j = (\nu_j, \eta_j, \tau_j) \in \mathcal{D}$ be any N triples such that*

$$\tau_1 < \tau_2 < \dots < \tau_N, \qquad \eta_1 \geq \eta_2 \geq \dots \geq \eta_N. \tag{1.15}$$

Denote

$$H_L(\nu, \eta, \tau) := \sqrt{\pi}\big[h([(\nu - \eta)L] + \tfrac{1}{2}, [\eta L], \tau L) - \mathbb{E}(h([(\nu - \eta)L] + \tfrac{1}{2}, [\eta L], \tau L))\big], \tag{1.16}$$

and $\Omega_j = \Omega(\nu_j, \eta_j, \tau_j)$. Then

$$\lim_{L\to\infty} \mathbb{E}\left(H_L(\varkappa_1) \cdots H_L(\varkappa_N)\right) = \begin{cases} \sum\limits_{\sigma \in \mathcal{F}_N} \prod\limits_{j=1}^{N/2} \mathcal{G}(\Omega_{\sigma(2j-1)}, \Omega_{\sigma(2j)}), & N \text{ is even}, \\ 0, & N \text{ is odd}, \end{cases} \tag{1.17}$$

where the summation is taken over all fixed point free involutions σ on $\{1, \dots, N\}$.

In addition to Theorem 1.3, a simple corollary of Theorem 1.2 gives

$$\mathbb{E}\left(H_L(\varkappa_1) \cdots H_L(\varkappa_N)\right) = O(L^\epsilon), \quad L \to \infty, \tag{1.18}$$

for any $\varkappa_j \in \mathcal{D}$ and any $\epsilon > 0$. This bounds the moments of $H_L(\varkappa_j)$ for infinitesimally close points $\varkappa_j$.

Conjecture 1.4. *The statement of Theorem 1.3 holds without the assumption (1.15).*

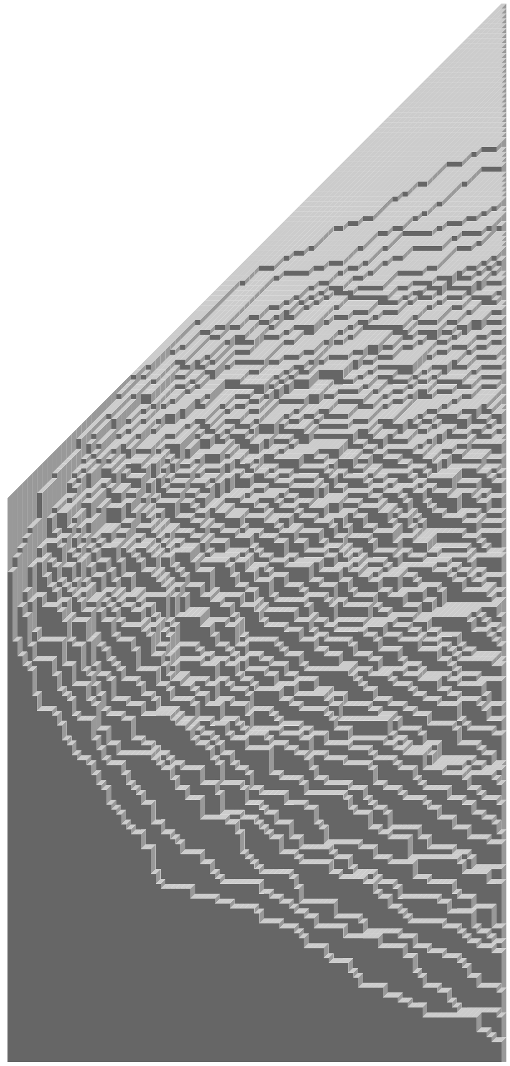

Figure 1.2: A configuration of the model analyzed with $N = 100$ particles at time $t = 25$, using the same representation as in Figure 1.1. In [20] there is a Java animation of the model.

Theorem 1.3 and Conjecture 1.4 indicate that the fluctuations of the height function along the rays R_Ω grow slower than in any other space-time direction. This statement can be rephrased more generally: The height function has smaller fluctuations along the curves where the slope of the limit shape remains constant. We have been able to find evidence for such a claim in one-dimensional random growth models as well.

Remark 1.5. In Figure 1.2 one can clearly see two facets and the curved part. This last region corresponds to the *bulk* for the particle system we analyze. Theorem 1.3 describes the fluctuations in the curved part of the surface. As shown in [7, 9, 25], the fluctuations of ledge bordering the surface are asymptotically described by the Airy_2 process [39], see also Section 1.6 below.

1.4 Universality class

In the terminology of physics literature, see e.g. [3], our Markov chain falls into the class of local growth models with relaxation and lateral growth, described by the Kardar-Parisi-Zhang (KPZ) equation

$$\partial_t h = \Delta h + Q(\partial_x h, \partial_y h) + \text{white noise}, \tag{1.19}$$

where Q is a quadratic form. Relations (1.10) and (1.11) imply that for our growth model the determinant of the Hessian of $\partial_t h$, viewed as a function of the slope, is strictly negative, which means that the form Q in our case has signature $(-1, 1)$. In such a situation the equation (1.19) is called *anisotropic* KPZ or AKPZ equation.

An example of such system is growth of vicinal surfaces, which are naturally anisotropic because the tilt direction of the surface is special. Using non-rigorous renormalization group analysis based on one-loop expansion, Wolf [45] predicted that large time fluctuations of the growth models described by AKPZ equation should be similar to those of linear models described by the Edwards-Wilkinson equation (heat equation with random term)

$$\partial_t h = \Delta h + \text{white noise}. \tag{1.20}$$

Our results can be viewed as the first rigorous analysis of a non-equilibrium growth model in the AKPZ class. (Some results, like logarithmic fluctuations, for an AKPZ model in a steady state were obtained in [37]. Another model with some numerical results is described in [23]). Indeed, Wolf's prediction correctly identifies the logarithmic behavior of height correlations and the appearance of the Gaussian free field. It does not predict, however, the complete structure (map Ω) of the fluctuations described in the previous section.

On the other hand, universality considerations imply that analogs of Theorems 1.2 and 1.3, as well as possibly Conjecture 1.4, should hold in any AKPZ growth model.

1.5 More general growth models

It turns out that the determinantal structure of the correlations functions stated in Theorem 1.1 holds for a much more general class of two-dimensional growth models. In the first half of the paper we develop an algebraic formalism needed to show that. At least three examples where this formalism applies, other than the Markov chain considered above, are worth mentioning.

1. In the Markov chain considered above one can make the particle jump rates depend on the upper index m in an arbitrary way. One can also allow the particles jump both right and left, with ratio of left and right jump rates possibly changing in time [7].

2. The shuffling algorithm for domino tilings of Aztec diamonds introduced in [19] also fits into our formalism. The corresponding discrete time Markov chain is described in Section 2 below, and its equivalence to domino shuffling is established in the recent paper [32].

3. A shuffling algorithm for lozenge tilings of the hexagon has been constructed in [12] using the formalism developed in this paper, see [12] for details.

Our original Markov chain is a suitable degeneration of each of these examples.

We expect our asymptotic methods to be applicable to many other two-dimensional growth models produced by the general formalism, and we plan to return to this discussion in a later publication.

1.6 Other connections

We have so far discussed the global asymptotic behavior of our growing surface, and its bulk properties (measures $M_{\pi_\nu,\pi_\eta,\pi_\tau}$), but have not discussed the edge asymptotics. As was mentioned above, rows $\{x_1^m\}_{m\geq 1}$ and $\{x_m^m\}_{m\geq 1}$ can be viewed as one-dimensional growth models on their own, and their asymptotic behavior was studied in [7] using essentially the same Theorem 1.1. This is exactly the edge behavior of our two-dimensional growth model.

Of course, the successive projections to $\{x_1^m\}_{m\geq 1}$ and then to a fixed (large) time commute. In the first ordering, this can be seen as the large time interface associated to the TASEP. In the second ordering, it corresponds to considering a tiling problem of a large region and focusing on the border of the facet.

Interestingly enough, an analog of Theorem 1.1 remains useful for the edge computations even in the cases when the measure on the space $\mathcal{S}$ is no longer positive (but its projection to $\{x_1^m\}_{m\geq 1}$ and $\{x_m^m\}_{m\geq 1}$ remains positive). These computations lead to the asymptotic results of [7–11, 42] for one-dimensional growth models with more general types of initial conditions.

Another natural asymptotic question that was not discussed is the limiting behavior of $\mathcal{M}^{(n)}(t)$ when $t\to\infty$ but n remains fixed. After proper normalization, in the limit one obtains the Markov chain investigated in [44].

Two of the four one-dimensional growth models constructed in [18] (namely, "Bernoulli with blocking" and "Bernoulli with pushing") are projections to $\{x_1^m\}_{m\geq 1}$ and $\{x_m^m\}_{m\geq 1}$ of one of our two-dimensional growth models, see Section 2 below. It remains unclear however, how to interpret the other two models of [18] in a similar fashion.

Finally, let us mention that our proof of Theorem 1.1 is based on the argument of [16] and [43], the proof of Theorem 1.3 uses several ideas from [28], and the algebraic formalism for two-dimensional growth models employs a crucial idea of constructing bivariate Markov chains out of commuting univariate ones from [17].

Outline. The rest of the paper is organized as follows. It has essentially two main parts. The first is Section 2. It contains the construction of the Markov chains, with the final result being the determinantal structure and the associated kernel (Theorem 2.27). Its continuous time analogue is Corollary 2.28, whose further specialization to particle-independent jump rate leads to Theorem 1.1. The second main part concerns the limit results for the continuous time model that we analyze. We start by collecting various geometric identities in Section 3. We also shortly discuss why our model is in the AKPZ class. In Section 4 we give a shifted version of the kernel, whose asymptotic analysis is the content of Section 7. These results allow then us to prove Theorem 1.2 in Section 5 and Theorem 1.3 in Section 6.

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2 Two dimensional dynamics

All the constructions below are based on the following basic idea. Consider two Markov operators P and P^* on state spaces $\mathcal{S}$ and $\mathcal{S}^*$, and a Markov link $\Lambda : \mathcal{S}^* \to \mathcal{S}$ that intertwines P and P^*, that is $\Lambda P = P^*\Lambda$. Then one can construct Markov chains on (subsets of) $\mathcal{S}^* \times \mathcal{S}$ that in some sense has both P and P^* as their projections. There are more than one way to realize this idea, and in this paper we employ two variants.

In one of them the image (y^*, y) of $(x^*, x) \in \mathcal{S}^* \times \mathcal{S}$ under the Markov operator is determined by *sequential update*: One first chooses y according to $P(x, y)$, and then one chooses y^* so that the needed projection properties are satisfied. A characteristic feature of the construction is that x and y^* are independent, given x^* and y. This bivariate Markov chain is denoted P_Λ; its construction is borrowed from [17].

In the second variant, the images y^* and y are independent, given (x, x^*), and we say that they are obtained by *parallel update*. The distribution of y is still $P(x, y)$, independently of what x^* is. This Markov chain is denoted P_Δ for the operator $\Delta = \Lambda P = P^*\Lambda$ that plays an important role.

By induction, one constructs multivariate Markov chains out of finitely many univariate ones and links that intertwine them. Again, we use two variants of the construction — with sequential and parallel updates.

The key property that makes these constructions useful is the following: If the chains P, P^*, and Λ, are h-Doob transforms of some (simpler) Markov chains, and the harmonic functions h used are consistent, then the transition probabilities of the bivariate Markov chains do not depend on h. Thus, participating univariate Markov chains may be fairly complex, while the transition probabilities of the univariate Markov chains remain simple.

Below we first explain the abstract construction of P_Λ, P_Δ, and their multivariate extensions. Then we exhibit a class of examples that are of interest to us. Finally, we show how the knowledge of certain averages (correlation functions) for the univariate Markov chains allows one to compute similar averages for the multivariate chains.

2.1 Bivariate Markov chains

Let $\mathcal{S}$ and $\mathcal{S}^*$ be discrete sets, and let P and P^* be stochastic matrices on these sets:

$$\sum_{y\in\mathcal{S}} P(x,y)=1,\quad x\in\mathcal{S};\qquad \sum_{y^*\in\mathcal{S}^*} P^*(x^*,y^*)=1,\quad x^*\in\mathcal{S}^*. \tag{2.1}$$

Assume that there exists a third stochastic matrix $\Lambda=\|\Lambda(x^*,x)\|_{x^*\in\mathcal{S}^*,\,x\in\mathcal{S}}$ such that for any $x^*\in\mathcal{S}^*$ and $y\in\mathcal{S}$

$$\sum_{x\in\mathcal{S}}\Lambda(x^*,x)P(x,y)=\sum_{y^*\in\mathcal{S}^*}P^*(x^*,y^*)\Lambda(y^*,y). \tag{2.2}$$

Let us denote the above quantity by $\Delta(x^*,y)$. In matrix notation

$$\Delta=\Lambda P=P^*\Lambda. \tag{2.3}$$

Set

$$\mathcal{S}_\Lambda=\{(x^*,x)\in\mathcal{S}^*\times\mathcal{S}\,|\,\Lambda(x^*,x)>0\},$$
$$\mathcal{S}_\Delta=\{(x^*,x)\in\mathcal{S}^*\times\mathcal{S}\,|\,\Delta(x^*,x)>0\}.$$

Define bivariate Markov chains on $\mathcal{S}_\Lambda$ and S_Δ by their corresponding transition probabilities

$$P_\Lambda((x^*,x),(y^*,y))=\begin{cases}\dfrac{P(x,y)P^*(x^*,y^*)\Lambda(y^*,y)}{\Delta(x^*,y)}, & \Delta(x^*,y)>0,\\ 0, & \text{otherwise},\end{cases} \tag{2.4}$$

$$P_\Delta((x^*,x),(y^*,y))=\frac{P(x,y)P^*(x^*,y^*)\Lambda(y^*,x)}{\Delta(x^*,x)}. \tag{2.5}$$

It is immediately verified that both matrices P_Λ and P_Δ are stochastic.

The chain P_Λ was introduced by Diaconis-Fill in [17], and we are using the notation of that paper.

One could think of P_Λ and P_Δ as follows.

For P_Λ, starting from (x^*,x) we first choose y according to the transition matrix $P(x,y)$, and then choose y^* using $\frac{P^*(x^*,y^*)\Lambda(y^*,y)}{\Delta(x^*,y)}$, which is the conditional distribution of the middle point in the successive application of P^* and Λ provided that we start at x^* and finish at y.

For P_Δ, starting from (x^*,x) we independently choose y according to $P(x,y)$ and y^* according to $\frac{P^*(x^*,y^*)\Lambda(y^*,x)}{\Delta(x^*,x)}$, which is the conditional distribution of the middle point in the successive application of P^* and Λ provided that we start at x^* and finish at x.

Lemma 2.1. *For any $(x^*,x)\in\mathcal{S}_\Lambda,\ y\in\mathcal{S}$, we have*

$$\begin{aligned}\sum_{y^*\in\mathcal{S}^*:(y^*,y)\in\mathcal{S}_\Lambda} P_\Lambda((x^*,x),(y^*,y)) &= P(x,y),\\ \sum_{y^*\in\mathcal{S}^*:(y^*,y)\in\mathcal{S}_\Delta} P_\Delta((x^*,x),(y^*,y)) &= P(x,y),\end{aligned}\tag{2.6}$$

and for any $x^*\in\mathcal{S}^*,\ (y^*,y)\in\mathcal{S}_\Lambda$,

$$\begin{aligned}\sum_{x\in\mathcal{S}:(x^*,x)\in\mathcal{S}_\Lambda} \Lambda(x^*,x)P_\Lambda((x^*,x),(y^*,y)) &= P^*(x^*,y^*)\Lambda(y^*,y),\\ \sum_{x\in\mathcal{S}:(x^*,x)\in\mathcal{S}_\Delta} \Delta(x^*,x)P_\Delta((x^*,x),(y^*,y)) &= P^*(x^*,y^*)\Delta(y^*,y).\end{aligned}\tag{2.7}$$

Proof of Lemma 2.1. Straightforward computation using the relation $\Delta=\Lambda P=P^*\Lambda$. $\square$

Proposition 2.2. *Let $m^*(x^*)$ be a probability measure on $\mathcal{S}^*$. Consider the evolution of the measure $m(x^*)\Lambda(x^*,x)$ on $\mathcal{S}_\Lambda$ under the Markov chain P_Λ and denote by $(x^*(j),x(j))$ the result after $j=0,1,2,\dots$ steps. Then for any $k,l=0,1,\dots$ the joint distribution of*

$$(x^*(0),x^*(1),\dots,x^*(k),x(k),x(k+1),\dots,x(k+l))\tag{2.8}$$

coincides with the stochastic evolution of m^* under transition matrices

$$(\underbrace{P^*,\dots,P^*}_{k},\Lambda,\underbrace{P,\dots,P}_{l}).\tag{2.9}$$

Exactly the same statement holds for the Markov chain P_Δ and the initial condition $m^*(x^*)\Delta(x^*,x)$ with Λ replaced by Δ in the above sequence of matrices.

Proof of Proposition 2.2. Successive application of the first relations of Lemma 2.1 to evaluate the sums over $x^*(k+l),\dots,x^*(k+1)$, and of the second relations to evaluate the sums over $x(1),\dots,x(k-1)$. $\square$

Note that Proposition 2.2 also implies that the joint distribution of $x^*(k)$ and $x(k)$ has the form $m_k^*(x^*(k))\Lambda(x^*(k),x(k))$, where m_k^* is the result of k-fold application of P^* to m^*.

The above constructions can be generalized to the nonautonomous situation.

Assume that we have a time variable $t \in \mathbb{Z}$, and our state spaces as well as transition matrices depend on t, which we will indicate as follows:

$$\mathcal{S}(t), \quad \mathcal{S}^*(t), \quad P(x,y\,|\,t), \quad P^*(x^*,y^*\,|\,t), \quad \Lambda(x^*,x\,|\,t), \quad P(t), \quad P^*(t), \quad \Lambda(t). \tag{2.10}$$

The commutation relation (1.3) is replaced by $\Lambda(t)P(t) = P^*(t)\Lambda(t+1)$ or

$$\Delta(x^*,y\,|\,t) := \sum_{x\in\mathcal{S}(t)} \Lambda(x^*,x\,|\,t)P(x,y\,|\,t) = \sum_{y^*\in\mathcal{S}^*(t+1)} P^*(x^*,y^*\,|\,t)\,\Lambda(y^*,y\,|\,t+1). \tag{2.11}$$

Further, we set

$$\begin{aligned} \mathcal{S}_\Lambda(t) &= \{(x^*,x)\in\mathcal{S}^*(t)\times\mathcal{S}(t)\,|\,\Lambda(x^*,x\,|\,t)>0\}, \\ \mathcal{S}_\Delta(t) &= \{(x^*,x)\in\mathcal{S}^*(t)\times\mathcal{S}(t+1)\,|\,\Delta(x^*,x\,|\,t)>0\}, \end{aligned} \tag{2.12}$$

and

$$P_\Lambda((x^*,x),(y^*,y)\,|\,t) = \begin{cases} \dfrac{P(x,y\,|\,t)P^*(x^*,y^*\,|\,t)\Lambda(y^*,y\,|\,t+1)}{\Delta(x^*,y\,|\,t)}, & \Delta(x^*,y\,|\,t)>0, \\ 0, & \text{otherwise}, \end{cases} \tag{2.13}$$

$$P_\Delta((x^*,x),(y^*,y)\,|\,t) = \frac{P(x,y\,|\,t+1)P^*(x^*,y^*\,|\,t)\Lambda(y^*,x\,|\,t+1)}{\Delta(x^*,x\,|\,t)}. \tag{2.14}$$

The nonautonomous generalization of Proposition 2.2 is proved in exactly the same way as Proposition 2.2. Let us state it.

Proposition 2.3. *Fix $t_0\in\mathbb{Z}$, and let $m^*(x^*)$ be a probability measure on $\mathcal{S}^*(t_0)$. Consider the evolution of the measure $m(x^*)\Lambda(x^*,x\,|\,t_0)$ on $\mathcal{S}_\Lambda(t_0)$ under the Markov chain $P_\Lambda(t)$, and denote by $(x^*(t_0+j),x(t_0+j))\in\mathcal{S}_\Lambda(t_0+j)$ the result after $j=0,1,2,\dots$ steps. Then for any $k,l=0,1,\dots$ the joint distribution of*

$$(x^*(t_0),x^*(t_0+1),\dots,x^*(t_0+k),x(t_0+k),x(t_0+k+1),\dots,x(t_0+k+l)) \tag{2.15}$$

coincides with the stochastic evolution of m^* under transition matrices

$$P^*(t_0),\dots,P^*(t_0+k-1),\Lambda(t_0+k),P(t_0+k),\dots,P(t_0+k+l-1) \tag{2.16}$$

(for $k=l=0$ only $\Lambda(t_0)$ remains in this string).

A similar statement holds for the Markov chain $P_\Delta(t)$ and the initial condition $m^*(x^*)\Delta(x^*,x\,|\,t_0)$: For any $k,l=0,1,\dots$ the joint distribution of

$$(x^*(t_0),x^*(t_0{+}1),\dots,x^*(t_0{+}k),x(t_0{+}k{+}1),x(t_0{+}k{+}2),\dots,x(t_0{+}k{+}l{+}1)) \tag{2.17}$$

coincides with the stochastic evolution of m^* under transition matrices

$$P^*(t_0),\dots,P^*(t_0+k-1),\Delta(t_0+k),P(t_0+k+1),\dots,P(t_0+k+l). \tag{2.18}$$

Remark 2.4. Observe that there is a difference in the sequences of times used in (2.8) and (2.17). The reason is that for nonautonomous P_Δ, the state space at time t is a subset of $\mathcal{S}^*(t)\times\mathcal{S}(t+1)$, and we denote its elements as $(x^*(t),x(t+1))$. In the autonomous case, an element of the state space $\mathcal{S}_\Delta$ at time t was denoted as $(x^*(t),x(t))$.

2.2 Multivariate Markov chains

We now aim at generalizing the constructions of Section 2.1 to more than two state spaces.

Let $\mathcal{S}_1, \dots, \mathcal{S}_n$ be discrete sets, $P_1, \dots, P_n$ be stochastic matrices defining Markov chains on them, and let $\Lambda_1^2, \dots, \Lambda_{n-1}^n$ be stochastic links between these sets:

$$\begin{aligned} P_k : \mathcal{S}_k \times \mathcal{S}_k \to [0,1], \quad & \sum_{y \in \mathcal{S}_k} P_k(x,y) = 1, \quad x \in \mathcal{S}_k, \quad k = 1, \dots, n; \\ \Lambda_{k-1}^k : \mathcal{S}_k \times \mathcal{S}_{k-1} \to [0,1], \quad & \sum_{y \in \mathcal{S}_{k-1}} \Lambda_{k-1}^k(x,y) = 1, \quad x \in \mathcal{S}_k, \quad k = 2, \dots, n. \end{aligned} \tag{2.19}$$

Assume that these matrices satisfy the commutation relations

$$\Delta_{k-1}^k := \Lambda_{k-1}^k P_{k-1} = P_k \Lambda_{k-1}^k, \qquad k = 2, \dots, n. \tag{2.20}$$

The state spaces for our multivariate Markov chains are defined as follows

$$\begin{aligned} \mathcal{S}_\Lambda^{(n)} &= \Big\{ (x_1, \dots, x_n) \in \mathcal{S}_1 \times \dots \times \mathcal{S}_n \mid \prod_{k=2}^n \Lambda_{k-1}^k(x_k, x_{k-1}) \neq 0 \Big\}, \\ \mathcal{S}_\Delta^{(n)} &= \Big\{ (x_1, \dots, x_n) \in \mathcal{S}_1 \times \dots \times \mathcal{S}_n \mid \prod_{k=2}^n \Delta_{k-1}^k(x_k, x_{k-1}) \neq 0 \Big\}. \end{aligned} \tag{2.21}$$

The transition probabilities for the Markov chains $P_\Lambda^{(n)}$ and $P_\Delta^{(n)}$ are defined as (we use the notation $X_n = (x_1, \dots, x_n)$, $Y_n = (y_1, \dots, y_n)$)

$$P_\Lambda^{(n)}(X_n, Y_n) = \begin{cases} P_1(x_1, y_1) \prod_{k=2}^n \dfrac{P_k(x_k, y_k) \Lambda_{k-1}^k(y_k, y_{k-1})}{\Delta_{k-1}^k(x_k, y_{k-1})}, & \prod_{k=2}^n \Delta_{k-1}^k(x_k, y_{k-1}) > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{2.22}$$

$$P_\Delta^{(n)}(X_n, Y_n) = P(x_1, y_1) \prod_{k=2}^n \frac{P_k(x_k, y_k) \Lambda_{k-1}^k(y_k, x_{k-1})}{\Delta_{k-1}^k(x_k, x_{k-1})}. \tag{2.23}$$

One way to think of $P_\Lambda^{(n)}$ and $P_\Delta^{(n)}$ is as follows.

For $P_\Lambda^{(n)}$, starting from $X = (x_1, \dots, x_n)$, we first choose y_1 according to the transition matrix $P(x_1, y_1)$, then choose y_2 using $\frac{P_2(x_2,y_2)\Lambda_1^2(y_2,y_1)}{\Delta_1^2(x_2,y_1)}$, which is the conditional distribution of the middle point in the successive application of P_2 and Λ_1^2 provided that we start at x_2 and finish at y_1, after that we choose y_3 using the conditional distribution of the middle point in the successive application of P_3 and Λ_2^3 provided that we start at x_3 and finish at y_2, and so on. Thus, one could say that Y is obtained by the *sequential update*.

For $P_\Delta^{(n)}$, starting from $X=(x_1,\dots,x_n)$ we *independently* choose $y_1,\dots,y_n$ according to $P_1(x_1,y_1)$ for y_1 and $\frac{P_k(x_k,y_k)\Lambda_{k-1}^k(y_k,x_{k-1})}{\Delta_{k-1}^k(x_k,x_{k-1})}$, for y_k, $k=2,\dots,n$. The latter formula is the conditional distribution of the middle point in the successive application of P_k and Λ_{k-1}^k provided that we start at x_k and finish at x_{k-1}. Thus, it is natural to say that this Markov chains corresponds to the *parallel update*.

We aim at proving the following generalization of Proposition 2.2.

Proposition 2.5. *Let $m_n(x_n)$ be a probability measure on $\mathcal{S}_n$. Consider the evolution of the measure*

$$m_n(x_n)\Lambda_{n-1}^n(x_n,x_{n-1})\cdots\Lambda_1^2(x_2,x_1) \tag{2.24}$$

on $\mathcal{S}_\Lambda^{(n)}$ under the Markov chain $P_\Lambda^{(n)}$, and denote by $(x_1(j),\dots,x_n(j))$ the result after $j=0,1,2,\dots$ steps. Then for any $k_1\ge k_2\ge\cdots\ge k_n\ge 0$ the joint distribution of

$$\begin{aligned}(x_n(0),\dots,x_n(k_n),x_{n-1}(k_n),x_{n-1}(k_n+1),\dots,x_{n-1}(k_{n-1}),\\ x_{n-2}(k_{n-1}),\dots,x_2(k_2),x_1(k_2),\dots,x_1(k_1))\end{aligned}$$

coincides with the stochastic evolution of m_n under transition matrices

$$(\underbrace{P_n,\dots,P_n}_{k_n},\Lambda_{n-1}^n,\underbrace{P_{n-1},\dots,P_{n-1}}_{k_{n-1}-k_n},\Lambda_{n-2}^{n-1},\dots,\Lambda_1^2,\underbrace{P_1,\dots,P_1}_{k_1-k_2}). \tag{2.25}$$

Exactly the same statement holds for the Markov chain $P_\Delta^{(n)}$ and the initial condition

$$m(x_n)\Delta_{n-1}^n(x_n,x_{n-1})\cdots\Delta_1^2(x_2,x_1) \tag{2.26}$$

with Λ's replaced by Δ's in the above sequence of matrices.

The following lemma is useful.

Lemma 2.6. *Consider the matrix $\Lambda:\mathcal{S}_n\times\mathcal{S}_\Lambda^{(n-1)}\to[0,1]$ given by*

$$\Lambda(x_n,(x_1,\dots,x_{n-1})):=\Lambda_{n-1}^n(x_n,x_{n-1})\cdots\Lambda_1^2(x_2,x_1). \tag{2.27}$$

Then $\Lambda P_\Lambda^{(n-1)}=P_n\Lambda$. If we denote this matrix by Δ then

$$P_\Lambda^{(n)}(X_n,Y_n)=\begin{cases}\dfrac{P_\Lambda^{(n-1)}(X_{n-1},Y_{n-1})P_n(x_n,y_n)\Lambda(y_n,Y_{n-1})}{\Delta(x_n,Y_{n-1})}, & \Delta(x_n,Y_{n-1})>0,\\ 0, & \textit{otherwise.}\end{cases} \tag{2.28}$$

Also, using the same notation,

$$P_\Delta^{(n)}(X_n,Y_n)=\frac{P_\Delta^{(n-1)}(X_{n-1},Y_{n-1})P_n(x_n,y_n)\Lambda(y_n,X_{n-1})}{\Delta(x_n,X_{n-1})}. \tag{2.29}$$

Proof of Lemma 2.6. Let us check the commutation relation $\Lambda P_\Lambda^{(n-1)} = P_n\Lambda$. We have

$$\Lambda P_\Lambda^{(n-1)}(x_n, Y_{n-1}) = \sum_{x_1,\dots,x_{n-1}} \Lambda_{n-1}^n(x_n, x_{n-1}) \cdots \Lambda_1^2(x_2, x_1) \times P_1(x_1, y_1) \prod_{k=2}^{n-1} \frac{P_k(x_k, y_k)\Lambda_{k-1}^k(y_k, y_{k-1})}{\Delta_{k-1}^k(x_k, y_{k-1})}, \quad (2.30)$$

where the sum is taken over all $x_1, \dots, x_{n-1}$ such that $\prod_{k=2}^{n-1} \Delta_{k-1}^k(x_k, y_{k-1}) > 0$. Computing the sum over x_1 and using the relation $\Lambda_1^2 P_1 = \Delta_1^2$ we obtain

$$\Lambda P_\Lambda^{(n-1)}(x_n, Y_{n-1}) = \sum_{x_2,\dots,x_{n-1}} \Lambda_{n-1}^n(x_n, x_{n-1}) \cdots \Lambda_2^3(x_3, x_2) \times P_2(x_2, y_2)\Lambda_1^2(y_2, y_1) \prod_{k=3}^{n-1} \frac{P_k(x_k, y_k)\Lambda_{k-1}^k(y_k, y_{k-1})}{\Delta_{k-1}^k(x_k, y_{k-1})}. \quad (2.31)$$

Now we need to compute the sum over x_2. If $\Delta_1^2(x_2, y_1) = 0$ then $P_2(x_2, y_2) = 0$ because otherwise the relation $\Delta_1^2 = P_2\Lambda_1^2$ implies that $\Lambda_1^2(y_2, y_1) = 0$, which contradicts to the hypothesis that $Y_{n-1} \in \mathcal{S}_\Lambda^{(n-1)}$. Thus, we can extend the sum to all $x_2 \in \mathcal{S}_2$, and the relation $\Lambda_2^3 P_2 = \Delta_2^3$ gives

$$\Lambda P_\Lambda^{(n-1)}(x_n, Y_{n-1}) = \sum_{x_3,\dots,x_{n-1}} \Lambda_{n-1}^n(x_n, x_{n-1}) \cdots \Lambda_3^4(x_4, x_3) \times P_3(x_3, y_3)\Lambda_2^3(y_3, y_2)\Lambda_1^2(y_2, y_1) \prod_{k=4}^{n-1} \frac{P_k(x_k, y_k)\Lambda_{k-1}^k(y_k, y_{k-1})}{\Delta_{k-1}^k(x_k, y_{k-1})}. \quad (2.32)$$

Continuing like that we end up with

$$\Lambda_{n-2}^{n-1}(y_{n-1}, y_{n-2}) \cdots \Lambda_1^2(y_2, y_1) \sum_{x_{n-1}} \Lambda_{n-1}^n(x_n, x_{n-1}) P_{n-1}(x_{n-1}, y_{n-1}), \quad (2.33)$$

which, by $\Lambda_{n-1}^n P_{n-1} = P_n \Lambda_{n-1}^n$ is exactly $P_n\Lambda(x_n, Y_{n-1})$. Let us also note that

$$\Delta(x_n, Y_{n-1}) = \Delta_{n-1}^n(x_n, y_{n-1})\Lambda_{n-2}^{n-1}(y_{n-1}, y_{n-2}) \cdots \Lambda_1^2(y_2, y_1). \quad (2.34)$$

The needed formulas for $P_\Lambda^{(n)}$ and $P_\Delta^{(n)}$ are now verified by straightforward substitution. □

Proof of Proposition 2.5. Let us give the argument for $P_\Lambda^{(n)}$; for $P_\Delta^{(n)}$ the proof is literally the same. By virtue of Lemma 2.6, we can apply Proposition 2.2 by taking

$$\mathcal{S}^* = \mathcal{S}_n, \quad \mathcal{S} = \mathcal{S}_\Lambda^{(n-1)}, \quad P^* = P_n, \quad P = P_\Lambda^{(n-1)}, \quad k = k_n, \quad l = k_1 - k_n, \quad (2.35)$$

and $\Lambda(x_n, X_{n-1})$ as in Lemma 2.6. Proposition 2.2 says that the joint distribution

$$(x_n(0), x_n(1), \dots, x_n(k_n), X_{n-1}(k_n), X_{n-1}(k_n+1), \dots, X_{n-1}(k_1)) \quad (2.36)$$

is the evolution of m_n under

$$(\underbrace{P_n,\dots,P_n}_{k_n},\Lambda,\underbrace{P_\Lambda^{(n-1)},\dots,P_\Lambda^{(n-1)}}_{k_1-k_n}). \tag{2.37}$$

Induction on n completes the proof. □

As in the previous section, Proposition 2.5 can be also proved in the nonautonomous situation. Let us give the necessary definitions.

We now have a time variable $t\in\mathbb{Z}$, and our state spaces as well as transition matrices depend on t:

$$\mathcal{S}_k(t),\quad P_k(x,y\,|\,t),\quad k=1,\dots,n,\qquad \Lambda^k_{k-1}(x_k,x_{k-1}\,|\,t),\quad k=2,\dots,n. \tag{2.38}$$

The commutation relations are

$$\Delta^k_{k-1}(t):=\Lambda^k_{k-1}(t)P_{k-1}(t)=P_k(t)\Lambda^k_{k-1}(t+1),\qquad k=2,\dots,n. \tag{2.39}$$

The multivariate state spaces are defined as

$$\begin{aligned}
\mathcal{S}_\Lambda^{(n)} &= \Big\{(x_1,\dots,x_n)\in\mathcal{S}_1(t)\times\dots\times\mathcal{S}_n(t)\,|\,\prod_{k=2}^n\Lambda^k_{k-1}(x_k,x_{k-1}\,|\,t)\neq 0\Big\},\\
\mathcal{S}_\Delta^{(n)} &= \Big\{(x_1,\dots,x_n)\in\mathcal{S}_1(t+n-1)\times\dots\times\mathcal{S}_n(t)\,|\,\prod_{k=2}^n\Delta^k_{k-1}(x_k,x_{k-1}\,|\,t+n-k)\neq 0\Big\}.
\end{aligned}$$

Then the transition matrices for $P_\Lambda^{(n)}$ and $P_\Delta^{(n)}$ are defined as

$$P_\Lambda^{(n)}(X_n,Y_n\,|\,t)=P_1(x_1,y_1\,|\,t)\prod_{k=2}^n\frac{P_k(x_k,y_k\,|\,t)\Lambda^k_{k-1}(y_k,y_{k-1}\,|\,t+1)}{\Delta^k_{k-1}(x_k,y_{k-1}\,|\,t)} \tag{2.40}$$

if $\prod_{k=2}^n\Delta^k_{k-1}(x_k,y_{k-1}\,|\,t)>0$ and 0 otherwise; and

$$\begin{aligned}
P_\Delta^{(n)}(X_n,Y_n) &= P(x_1,y_1\,|\,t+n-1)\\
&\qquad\times\prod_{k=2}^n\frac{P_k(x_k,y_k\,|\,t+n-k)\Lambda^k_{k-1}(y_k,x_{k-1}\,|\,t+n-k+1)}{\Delta^k_{k-1}(x_k,x_{k-1}\,|\,t+n-k)}.
\end{aligned} \tag{2.41}$$

Proposition 2.7. *Fix $t_0\in\mathbb{Z}$, and let $m_n(x_n)$ be a probability measure on $\mathcal{S}_n(t_0)$. Consider the evolution of the measure*

$$m_n(x_n)\Lambda^n_{n-1}(x_n,x_{n-1}\,|\,t_0)\cdots\Lambda^2_1(x_2,x_1\,|\,t_0) \tag{2.42}$$

on $\mathcal{S}_\Lambda^{(n)}(t_0)$ under $P_\Lambda^{(n)}(t)$. Denote by $(x_1(t_0+j),\dots,x_n(t_0+j))$ the result after $j=0,1,2,\dots$ steps. Then for any $k_1\geq k_2\geq\dots\geq k_n\geq t_0$ the joint distribution of

$$\begin{aligned}
&(x_n(t_0),\dots,x_n(k_n),x_{n-1}(k_n),x_{n-1}(k_n+1),\dots,x_{n-1}(k_{n-1}),\\
&\qquad\qquad x_{n-2}(k_{n-1}),\dots,x_2(k_2),x_1(k_2),\dots,x_1(k_1))
\end{aligned}$$

coincides with the stochastic evolution of m_n under transition matrices

$$P_n(t_0), \ldots, P_n(k_n-1), \Lambda^n_{n-1}(k_n), P_{n-1}(k_n), \ldots, P_{n-1}(k_{n-1}-1),\\ \Lambda^{n-1}_{n-2}(k_{n-1}), \ldots, \Lambda^2_1(k_2), P_1(k_2), \ldots, P_1(k_1-1).$$

A similar statement holds for the Markov chain $P^{(n)}_\Delta(t)$ and the initial condition

$$m(x_n)\Delta^n_{n-1}(x_n, x_{n-1} \mid t_0) \cdots \Delta^2_1(x_2, x_1 \mid t_0+n-2). \tag{2.43}$$

For any $k_1 > k_2 > \cdots > k_n \geq t_0$ the joint distribution of

$$(x_n(t_0), \ldots, x_n(k_n), x_{n-1}(k_n+1), x_{n-1}(k_n+2), \ldots, x_{n-1}(k_{n-1}),\\ x_{n-2}(k_{n-1}+1), \ldots, x_2(k_2), x_1(k_2+1), \ldots, x_1(k_1))$$

coincides with the stochastic evolution of m_n under transition matrices

$$P_n(t_0), \ldots, P_n(k_n-1), \Delta^n_{n-1}(k_n), P_{n-1}(k_n+1), \ldots, P_{n-1}(k_{n-1}-1),\\ \Delta^{n-1}_{n-2}(k_{n-1}), \ldots, \Delta^2_1(k_2), P_1(k_2+1), \ldots, P_1(k_1-1).$$

The proof is very similar to that of Proposition 2.5.

2.3 Toeplitz-like transition probabilities

The goal of this section is to provide some general recipe on how to construct commuting stochastic matrices.

Proposition 2.8. *Let $\alpha_1, \ldots, \alpha_n$ be nonzero complex numbers, and let $F(x)$ be an analytic function in an annulus A centered at the origin that contains all α_j^{-1}'s. Assume that $F(\alpha_1^{-1}) \cdots F(\alpha_n^{-1}) \neq 0$. Then*

$$\frac{1}{F(\alpha_1^{-1}) \cdots F(\alpha_n^{-1})} \sum_{y_1 < \cdots < y_n \in \mathbb{Z}} \det\left[\alpha_i^{y_j}\right]_{i,j=1}^n \det\left[f(x_j - y_i)\right]_{i,j=1}^n = \det\left[\alpha_i^{x_j}\right]_{i,j=1}^n \tag{2.44}$$

where

$$f(m) = \frac{1}{2\pi \mathrm{i}} \oint \frac{F(z)\mathrm{d}z}{z^{m+1}}, \tag{2.45}$$

and the integral is taken over any positively oriented simple loop in A.

Proof of Proposition 2.8. Since the left-hand side is symmetric with respect to permutations of y_j's and it vanishes when two y_j's are equal, we can extend the sum to $\mathbb{Z}^n$ and divide the result by $n!$. We obtain

$$\sum_{y_1, \ldots, y_n \in \mathbb{Z}} \det\left[\alpha_i^{y_j}\right]_{i,j=1}^n \det\left[f(x_j - y_i)\right]_{i,j=1}^n = n! \det\Big[\sum_{y=-\infty}^{+\infty} \alpha_k^y f(x_j - y)\Big]_{k,j=1}^n. \tag{2.46}$$

Further,

$$\sum_{y=-\infty}^{+\infty}\alpha_k^y f(x_j-y)=\sum_{y=-\infty}^{+\infty}\frac{1}{2\pi\mathrm{i}}\oint\frac{\alpha_k^y F(z)\mathrm{d}z}{z^{x_j-y+1}}$$

$$=\frac{1}{2\pi\mathrm{i}}\oint_{|z|=c_1<|\alpha_k|^{-1}}F(z)\mathrm{d}z\sum_{y=x_j+1}^{+\infty}\frac{\alpha_k^y}{z^{x_j-y+1}}+\frac{1}{2\pi\mathrm{i}}\oint_{|z|=c_2>|\alpha_k|^{-1}}F(z)\mathrm{d}z\sum_{y=-\infty}^{x_j}\frac{\alpha_k^y}{z^{x_j-y+1}}$$

$$=\frac{1}{2\pi\mathrm{i}}\oint_{|z|=c_1<|\alpha_k|^{-1}}\frac{\alpha_k^{x_j+1}F(z)}{1-\alpha_k z}-\frac{1}{2\pi\mathrm{i}}\oint_{|z|=c_2>|\alpha_k|^{-1}}\frac{\alpha_k^{x_j+1}F(z)}{1-\alpha_k z}=\alpha_k^{x_j}F(\alpha_k^{-1}).$$

□

Proposition 2.9. *In the notation of Proposition 2.8, assume that the variable y_n is virtual, $y_n=\mathrm{virt}$, and set $f(x_k-\mathrm{virt})=\alpha_n^{x_k}$ for any $k=1,\dots,n$. Then*

$$\frac{1}{F(\alpha_1^{-1})\cdots F(\alpha_{n-1}^{-1})}\sum_{y_1<\cdots<y_{n-1}\in\mathbb{Z}}\det\left[\alpha_i^{y_j}\right]_{i,j=1}^{n-1}\det\left[f(x_j-y_i)\right]_{i,j=1}^{n}=\det\left[\alpha_i^{x_j}\right]_{i,j=1}^{n}. \tag{2.47}$$

Proof of Proposition 2.9. Expansion of $\det\left[f(x_j-y_i)\right]_{i,j=1}^n$ along the last row gives

$$\det\left[f(x_j-y_i)\right]_{i,j=1}^n=\sum_{k=1}^n(-1)^{n-k}\alpha_n^{x_k}\cdot\det\left[f(x_j-y_i)\right]_{\substack{i=1,\dots,n-1\\ j=1,\dots,k-1,k+1,\dots,n}}. \tag{2.48}$$

The application of Proposition 2.8 to each of the resulting summands in the left-hand side of the desired equality produces the expansion of $\det\left[\alpha_i^{x_j}\right]_{i,j=1}^n$ along the last row. □

For $n=1,2,\dots,$ denote

$$\mathfrak{X}_n=\{(x_1,\dots,x_n)\in\mathbb{Z}^n\mid x_1<\cdots<x_n\}. \tag{2.49}$$

In what follows we assume that the (nonzero) complex parameters $\alpha_1,\alpha_2,\dots$ are such that the ratios $\det[\alpha_i^{x_j}]_{i,j=1}^n/\det\left[\alpha_i^{j-1}\right]_{i,j=1}^n$ are nonzero for all $n=1,2,\dots$ and all $(x_1\dots,x_n)$ in $\mathfrak{X}^n$. This holds, for example, when all α_j's are positive. The Vandermonde determinant in the denominator is needed to make sense of $\det[\alpha_i^{x_j}]_{i,j=1}^n$ when some of the α_j's are equal.

Under this assumption, define the matrices $\mathfrak{X}_n\times\mathfrak{X}_n$ and $\mathfrak{X}_n\times\mathfrak{X}_{n-1}$ by

$$\begin{aligned}
T_n(\alpha_1,\dots,\alpha_n;F)(X,Y) &= \frac{\det\left[\alpha_i^{y_j}\right]_{i,j=1}^n}{\det\left[\alpha_i^{x_j}\right]_{i,j=1}^n}\,\frac{\det\left[f(x_i-y_j)\right]_{i,j=1}^n}{\prod_{j=1}^n F(\alpha_j^{-1})},\quad X,Y\in\mathfrak{X}_n,\\
T_{n-1}^n(\alpha_1,\dots,\alpha_n;F)(X,Y) &= \frac{\det\left[\alpha_i^{y_j}\right]_{i,j=1}^{n-1}}{\det\left[\alpha_i^{x_j}\right]_{i,j=1}^n}\,\frac{\det\left[f(x_i-y_j)\right]_{i,j=1}^n}{\prod_{j=1}^{n-1} F(\alpha_j^{-1})},\quad X\in\mathfrak{X}_n,\ Y\in\mathfrak{X}_{n-1},
\end{aligned}$$

where in the second formula $y_n = \text{virt}$. By Propositions 2.8 and 2.9, the sums of entries of these matrices along rows are equal to 1. We will often omit the parameters α_j from the notation so that the above matrices will be denoted as $T_n(F)$ and $T^n_{n-1}(F)$.

We are interested in these matrices because they have nice commutation relations, as the following proposition shows.

Proposition 2.10. *Let F_1 and F_2 be two functions holomorphic in an annulus containing α_j^{-1}'s, that are also nonzero at these points. Then*

$$\begin{aligned} T_n(F_1)T_n(F_2) = T_n(F_2)T_n(F_1) = T_n(F_1F_2), \\ T_n(F_1)T^n_{n-1}(F_2) = T^n_{n-1}(F_1)T_{n-1}(F_2) = T^n_{n-1}(F_1F_2). \end{aligned} \tag{2.50}$$

Proof of Proposition 2.10. The first line and the relation $T^n_{n-1}(F_1)T_{n-1}(F_2) = T^n_{n-1}(F_1F_2)$ are proved by straightforward computations using the fact the Fourier transform of F_1F_2 is the convolution of those of F_1 and F_2. The only additional ingredient in the proof of the relation $T_n(F_1)T^n_{n-1}(F_2) = T^n_{n-1}(F_1F_2)$ is

$$\sum_{y\in\mathbb{Z}} f_1(x-y)f_2(y-\text{virt}) = \sum_{y\in\mathbb{Z}} f_1(x-y)\alpha_n^y = F_1(\alpha_n^{-1})\alpha_n^x. \tag{2.51}$$

□

Remark 2.11. In the same way one proves the commutation relation

$$T^n_{n-1}(F_1)T^{n-1}_{n-2}(F_2) = T^n_{n-1}(F_2)T^{n-1}_{n-2}(F_1) \tag{2.52}$$

but we will not need it later.

2.4 Minors of some simple Toeplitz matrices

The goal of the section is to derive explicit formulas for $T_n(F)$ and $T^n_{n-1}(F)$ from the previous section for some simple functions F.

Lemma 2.12. *Consider $F(z) = 1 + pz$, that is*

$$f(m) = \begin{cases} p, & m = 1, \\ 1, & m = 0, \\ 0, & \textit{otherwise.} \end{cases} \tag{2.53}$$

Then for integers $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$

$$\det\left[f(x_i - y_j)\right]_{i,j=1}^n = \begin{cases} p^{\sum_{i=1}^n (x_i - y_i)}, & \textit{if } y_i - x_i \in \{-1, 0\} \textit{ for all } 1 \le i \le n, \\ 0, & \textit{otherwise.} \end{cases} \tag{2.54}$$

Proof of Lemma 2.12. If $x_i < y_i$ for some i then $x_k < y_l$ for $k \le i$ and $l \ge i$, which implies that $f(x_k - y_l) = 0$ for such k, l, and thus the determinant in question vanishes. If $x_i > y_i + 1$ then $x_k > y_l + 1$ for $k \ge i$ and $l \le i$, which means $f(x_k - y_l) = 0$, and the determinant vanishes again. Hence, it remains to consider the case when $x_i - y_i \in \{0, 1\}$ for all $1 \le i \le n$.

Split $\{x_i\}_{i=1}^n$ into blocks of neighboring integers with distance between blocks being at least 2. Then it is easy to see that $\det[f(x_i - y_j)]$ splits into the product of determinants corresponding to blocks. Let $(x_k, \dots, x_{l-1})$ be such a block. Then there exists m, $k \le m < l$, such that $x_i = y_i + 1$ for $l \le i < m$, and $x_i = y_i$ for $m \le i < l$. The determinant corresponding to this block is the product of determinants of two triangular matrices, one has size $m - k$ and diagonal entries equal to p, while the other one has size $l - m$ and diagonal entries equal to 1. Thus, the determinant corresponding to this block is equal to p^{m-k}, and collecting these factors over all blocks yields the result. □

Lemma 2.13. *Consider $F(z) = (1 - qz)^{-1}$, that is*

$$f(m) = \begin{cases} q^m, & m \ge 0, \\ 0, & \textit{otherwise}. \end{cases} \tag{2.55}$$

(i) For integers $x_1 < \dots < x_n$ and $y_1 < \dots < y_n$

$$\det[f(x_i - y_j)]_{i,j=1}^n = \begin{cases} q^{\sum_{i=1}^n (x_i - y_i)}, & x_{i-1} < y_i \le x_i,\ 1 \le i \le n, \\ 0, & \textit{otherwise}. \end{cases} \tag{2.56}$$

(The condition $x_0 < y_1$ above is empty.)

(ii) For integers $x_1 < \dots < x_n$ and $y_1 < \dots < y_{n-1}$, and with virtual variable $y_n = \mathrm{virt}$ such that $f(x - \mathrm{virt}) = q^x$,

$$\det[f(x_i - y_j)]_{i,j=1}^n = \begin{cases} (-1)^{n-1} q^{\sum_{i=1}^n x_i - \sum_{i=1}^{n-1} y_i}, & x_i < y_i \le x_{i+1},\ 1 \le i \le n-1, \\ 0, & \textit{otherwise}. \end{cases} \tag{2.57}$$

Proof of Lemma 2.13. (i) Let us first show that the needed inequalities are satisfied. Indeed, if $x_i < y_i$ for some i then $\det[f(x_i - y_j)] = 0$ by the same reasoning as in the previous lemma. On the other hand, if $x_{i-1} \ge y_i$ then $x_k \ge y_l$ for $k \ge i - 1$, $l \le i$. Let i be the smallest number such that $x_{i-1} \ge y_i$. Then columns i and $i + 1$ have the form

$$\begin{bmatrix} 0 & \dots & 0 & q^{x_{i-1} - y_{i-1}} & q^{x_i - y_{i-1}} & * & * & \dots \\ 0 & \dots & 0 & q^{x_{i-1} - y_i} & q^{x_i - y_i} & * & * & \dots \end{bmatrix}^T, \tag{2.58}$$

where the 2×2 block with powers of q is on the main diagonal. This again implies that the determinant vanishes. On the other hand, if the interlacing inequalities are satisfied then the matrix $[f(x_i - y_j)]$ is triangular, and computing the product of its diagonal entries yields the result.

(ii) The statement follows from (i). Indeed, we just need to multiply both sides of (i) by q^{y_1}, denote $y_1(\le x_1)$ by virt, and then cyclically permute y_j's. $\square$

Lemma 2.14. *Consider $F(z) = p + qz(1-qz)^{-1}$, that is*

$$f(m) = \begin{cases} p, & m = 0, \\ q^m, & m \ge 1, \\ 0, & \textit{otherwise.} \end{cases} \tag{2.59}$$

(i) For integral $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$

$$\det\left[f(x_i - y_j)\right]_{i,j=1}^n = q^{\sum_{i=1}^n (x_i - y_i)} p^{\#\{i\,|\,x_i = y_i\}} (1-p)^{\#\{i\,|\,x_{i-1} = y_i\}} \tag{2.60}$$

if $x_{i-1} \le y_i \le x_i$ for all $1 \le i \le n$, and 0 otherwise.

(ii) For integral $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_{n-1}$, and with virtual variable $y_n = \mathrm{virt}$ such that $f(x - \mathrm{virt}) = q^x$,

$$\det\left[f(x_i - y_j)\right]_{i,j=1}^n = (-1)^{n-1} q^{\sum_{i=1}^n x_i - \sum_{i=1}^{n-1} y_i} p^{\#\{i\,|\,x_{i+1} = y_i\}} (1-p)^{\#\{i\,|\,x_i = y_i\}} \tag{2.61}$$

if $x_i \le y_i \le x_{i+1}$ for all $1 \le i \le n-1$, and 0 otherwise.

Proof of Lemma 2.14. (i) The interlacing conditions are verified by the same argument as in the proof of Lemma 2.13(i) (although the conditions themselves are slightly different). Assuming that they are satisfied, we observe that the matrix elements of $[f(x_i - y_j)]$ are zero for $j \ge i+2$ because $x_i \le y_{i+1} < y_{i+2}$ and $f(m) = 0$ for $m < 0$. Further, the $(i, i+1)$-element is equal to p if $x_i = y_{i+1}$ or 0 if $x_i < y_{i+1}$. Thus, the matrix is block-diagonal, with blocks being either of size 1 with entry $f(x_i - y_i)$, or of large size having the form

$$\begin{pmatrix} q^{x_k - y_k} & p & 0 & \dots & 0 \\ q^{x_{k+1} - y_k} & q^{x_{k+1} - y_{k+1}} & p & \dots & 0 \\ q^{x_{k+2} - y_k} & q^{x_{k+2} - y_{k+1}} & q^{x_{k+2} - y_{k+2}} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ q^{x_l - y_k} & q^{x_l - y_{k+1}} & q^{x_l - y_{k+2}} & \dots & q^{x_l - y_l} \end{pmatrix} \tag{2.62}$$

with $x_k = y_{k+1}, \dots, x_{l-1} = y_l$, and $x_{k-1} < y_k$, $x_l < y_{l+1}$. The determinant of (2.62) is computable via Lemma 1.2 of [6], and it is equal to

$$q^{x_l - y_k} (1-p)^{l-k} = q^{x_k + \cdots + x_l - (y_k + \cdots + y_l)} (1-p)^{l-k}. \tag{2.63}$$

Collecting all the factors yields the desired formula.

The proof of (ii) is very similar to that of Lemma 2.13(ii). $\square$

Although the next statement is not used in the sequel, it is still interesting that the minors can also be explicitly evaluated in slightly more complicated cases.

Lemma 2.15. *Consider*

$$F(z)=\frac{(a-b)z}{(1-az)(1-bz)}, \quad \text{that is} \quad f(m)=\begin{cases} a^m-b^m, & m\geq 1,\\ 0, & \text{otherwise.}\end{cases} \tag{2.64}$$

(i) For integral $x_1<\cdots<x_n$ and $y_1<\cdots<y_n$

$$\det\left[f(x_i-y_j)\right]_{i,j=1}^n=\prod_{k=1}^{n-1}(ab)^{\max(x_k-y_{k+1},0)}\prod_{k=1}^{n}f\big(\min(x_k,y_{k+1})-\max(x_{k-1},y_k)\big). \tag{2.65}$$

(ii) For integral $x_1<\cdots<x_n$ and $y_1<\cdots<y_{n-1}$, and with virtual variable $y_n=\mathrm{virt}$ such that $f(x-\mathrm{virt})=a^x$,

$$\begin{aligned}&\det\left[f(x_i-y_j)\right]_{i,j=1}^n\\ &\quad=(-1)^{n-1}a^{\min(x_1,y_1)}\prod_{k=1}^{n-1}(ab)^{\max(x_k-y_k,0)}\prod_{k=1}^{n}f\big(\min(x_{k+1},y_{k+1})-\max(x_k,y_k)\big).\end{aligned}$$

Remark 2.16. Because of the definition of f, the right-hand side of (i) is nonzero iff $x_k>y_k>x_{k-2}$ for all k such that the inequalities makes sense. Similarly, the right-hand side of (ii) is nonzero iff $x_{k+1}>y_k>x_{k-1}$ for all k.

Proof of Lemma 2.15. We use the Cauchy-Binet formula

$$\frac{\det\left[f(x_i-y_j)\right]_{i,j=1}^n}{(a-b)^n}=\sum_{z_1<\cdots<z_n}\det\left[f_a(x_i-z_j)\right]_{i,j=1}^n\det\left[f_b(z_i-y_j-1)\right]_{i,j=1}^n \tag{2.66}$$

with $f_a(m)=a^m$, $f_b(m)=b^m$ for $m\geq 0$ and $f_a(m)=f_b(m)=0$ for $m<0$. Here the denominator $(a-b)^n$ is responsible for the factor $(a-b)$ in the formula for $F(z)$, and the shift by 1 in the last determinant comes from the factor z in $F(z)$. The determinants with f_a and f_b have been computed in Lemma 2.13(i). Possible locations of z_k are determined from two groups of inequalities coming from Lemma 2.13(i):

$$x_{k-1}<z_k\leq x_k, \qquad y_k<z_k\leq y_{k+1}. \tag{2.67}$$

Denote by $z_k^{\min}$ and $z_k^{\max}$ the minimal and maximal z_k satisfying these inequalities:

$$z_k^{\min}=\max(x_{k-1},y_k)+1, \qquad z_k^{\max}=\min(x_k,y_{k+1}). \tag{2.68}$$

Clearly, the segments $[z_k^{\min},z_k^{\max}]$ do not intersect, and the condition that none of them is empty coincides with the conditions of nonvanishing from the remark above. We can now write

$$\begin{aligned}\frac{\det\left[f(x_i-y_j)\right]_{i,j=1}^n}{(a-b)^n}&=\sum_{\substack{z_k^{\min}\leq z_k\leq z_k^{\max}\\ k=1,\ldots,n}}a^{\sum_{i=1}^n(x_i-z_i)}b^{\sum_{i=1}^n(z_i-y_i-1)}\\ &=a^{\sum_{i=1}^n(x_i-z_i^{\max})}b^{\sum_{i=1}^n(z_i^{\min}-y_i-1)}\prod_{k=1}^n\frac{a^{z_k^{\max}-z_k^{\min}+1}-b^{z_k^{\max}-z_k^{\min}+1}}{a-b},\end{aligned}$$

which concludes the proof of (i).

(ii) In the formula of (i) let us take y_1 very negative. Observe that for any $m \in \mathbb{Z}$

$$\frac{f(m-y_1)}{a^{-y_1}} = \frac{a^{m-y_1} - b^{m-y_1}}{a^{-y_1}} \to a^m \tag{2.69}$$

as $y_1 \to -\infty$ provided that $|a| < |b|$. Thus, multiplying both sides of (i) by a^{y_1}, sending $y_1 \to -\infty$, and cyclically permuting y_j's we arrive at (ii). The restriction $|a| < |b|$ is irrelevant as the statement is a polynomial identity in a and b. $\square$

2.5 Examples of bivariate Markov chains

We now use the formulas from the previous two sections to make the constructions of the first two sections more explicit.

Let us start with bivariate Markov chains. Set $\mathcal{S}^* = \mathfrak{X}_n$ and $\mathcal{S} = \mathfrak{X}_{n-1}$, where the sets $\mathfrak{X}_m$, $m = 1, 2, \dots$, were introduced in Section 2.3. We will also take

$$\Lambda = T^n_{n-1}(\alpha_1, \dots, \alpha_n; (1-\alpha_n z)^{-1}) \tag{2.70}$$

for some fixed $\alpha_1, \dots, \alpha_n > 0$.

The first case we consider is

$$P = T_{n-1}(\alpha_1, \dots, \alpha_{n-1}; 1+\beta z), \quad P^* = T_n(\alpha_1, \dots, \alpha_n; 1+\beta z), \qquad \beta > 0. \tag{2.71}$$

Then Proposition 2.10 implies that

$$\Delta = \Lambda P = P^* \Lambda = T^n_{n-1}(\alpha_1, \dots, \alpha_n; (1+\beta z)/(1-\alpha_n z)). \tag{2.72}$$

According to (2.22), (2.23), we have to compute expressions of the form

$$\frac{P^*(x^*, y^*)\Lambda(y^*, y)}{\Delta(x^*, y)}, \qquad \frac{P^*(x^*, y^*)\Lambda(y^*, x)}{\Delta(x^*, x)}$$

for the sequential and parallel updates, respectively.

We start with the condition probability needed for the Markov chain P_Λ.

Proposition 2.17. *Assume that $x^* \in \mathcal{S}^*$ and $y \in \mathcal{S}$ are such that $\Delta(x^*, y) > 0$, that is, $x^*_k \le y_k \le x^*_{k+1}$ for all $1 \le k \le n-1$. Then the probability distribution*

$$\frac{P^*(x^*, y^*)\Lambda(y^*, y)}{\Delta(x^*, y)}, \qquad y^* \in \mathcal{S}^*, \tag{2.73}$$

has nonzero weights iff

$$y^*_k - x^*_k \in \{-1, 0\}, \qquad y_{k-1} \le y^*_k < y_k, \qquad k = 1, \dots, n, \tag{2.74}$$

(equivalently, $\max(x_k^* - 1, y_{k-1}) \le y_k^* \le \min(x_k^*, y_k - 1)$ *for all* k*), and these weights are equal to*

$$\prod_{\substack{\max(x_k^*-1, y_{k-1}) < \min(x_k^*, y_k - 1)\\ k=1,\dots,n}} \left(\frac{\beta}{\alpha_n + \beta}\right)^{x_k^* - y_k^*} \left(\frac{\alpha_n}{\alpha_n + \beta}\right)^{1 - x_k^* + y_k^*} \tag{2.75}$$

with empty product equal to 1.

Remark 2.18. One way to think about the distribution of $y^* \in \mathcal{S}^*$ is as follows. For each k there are two possibilities for y_k^*: Either $\max(x_k^* - 1, y_{k-1}) = \min(x_k^*, y_k - 1)$, in which case y_k^* is forced to be equal to this number, or $\max(x_k^* - 1, y_{k-1}) = x_k^* - 1$ and $\min(x_k^*, y_k - 1) = x_k^*$, in which case y_k^* is allowed to take one of the two values x_k^* or $x_k^* - 1$. Then in the latter case, $x_k^* - y_k^*$ are i. i. d. Bernoulli random variables with the probability of the value 0 equal to $\alpha_n/(\alpha_n + \beta)$.

Proof of Proposition 2.17. The conditions for non-vanishing of the weights follow from those of Lemmas 2.12 and 2.13, namely from (2.54) and (2.57). Using these formulas we extract the factors of $P^*(x^*, y^*)\Lambda(y^*, y)$ that depend on y^*. This yields $(\alpha_n/\beta)^{\sum_{i=1}^n y_i^*}$. Normalizing these weights so that they provide a probability distribution leads to the desired formula. □

Let us now look at the conditional distribution involved in the definition of the Markov chain P_Δ. The following statement is a direct consequence of Proposition 2.17.

Corollary 2.19. *Assume that* $x^* \in \mathcal{S}^*$ *and* $x \in \mathcal{S}$ *are such that* $\Delta(x^*, x) > 0$*, that is,* $x_k^* \le x_k \le x_{k+1}^*$ *for all* $1 \le k \le n-1$*. Then the probability distribution*

$$\frac{P^*(x^*, y^*)\Lambda(y^*, x)}{\Delta(x^*, x)}, \qquad y^* \in \mathcal{S}^*, \tag{2.76}$$

has nonzero weights iff $\max(x_k^* - 1, x_{k-1}) \le y_k^* \le \min(x_k^*, x_k - 1)$*, and these weights are equal to*

$$\prod_{\substack{\max(x_k^*-1, x_{k-1}) < \min(x_k^*, x_k - 1)\\ k=1,\dots,n}} \left(\frac{\beta}{\alpha_n + \beta}\right)^{x_k^* - y_k^*} \left(\frac{\alpha_n}{\alpha_n + \beta}\right)^{1 - x_k^* + y_k^*}. \tag{2.77}$$

Let us now proceed to the case

$$P = T_{n-1}(\alpha_1, \dots, \alpha_{n-1}; (1 - \gamma z)^{-1}), \quad P^* = T_n(\alpha_1, \dots, \alpha_n; (1 - \gamma z)^{-1}). \tag{2.78}$$

We assume that $0 < \gamma < \min\{\alpha_1, \dots, \alpha_n\}$.

By Proposition 2.10

$$\Delta = \Lambda P = P^* \Lambda = T_{n-1}^n\big(\alpha_1, \dots, \alpha_n; 1/((1 - \alpha_n z)(1 - \gamma z))\big). \tag{2.79}$$

Again, let us start with P_Λ.

Proposition 2.20. *Assume that $x^* \in \mathcal{S}^*$ and $y \in \mathcal{S}$ are such that $\Delta(x^*, y) > 0$, that is, $x^*_{k-1} < y_k - 1 < x^*_{k+1}$ for all k. Then the probability distribution*

$$\frac{P^*(x^*, y^*)\Lambda(y^*, y)}{\Delta(x^*, y)}, \qquad y^* \in \mathcal{S}^*, \tag{2.80}$$

has nonzero weights iff

$$x^*_{k-1} < y^*_k \le x^*_k, \qquad y_{k-1} \le y^*_k < y_k, \qquad k = 1, \dots, n-1, \tag{2.81}$$

(equivalently, $\max(x^*_{k-1}+1, y_{k-1}) \le y^*_k \le \min(x^*_k, y_k - 1)$ *for all k), and these weights are equal to*

$$\prod_{k=1}^{n} \frac{(\alpha_n/\gamma)^{y^*_k}}{\sum\limits_{l=\max(x^*_{k-1}+1, y_{k-1})}^{\min(x^*_k, y_k-1)} (\alpha_n/\gamma)^l}. \tag{2.82}$$

Here $\max(x^*_0 + 1, y_0)$ *is assumed to denote $-\infty$.*

Remark 2.21. Less formally, these formulas state the following: Each y^*_k has to belong to the segment $[\max(x^*_{k-1}+1, y_{k-1}), \min(x^*_k, y_k - 1)]$, and the restriction that $\Delta(x^*, y) > 0$ guarantees that these segments are nonempty. Then the claim is that y^*_k's are independent, and the distribution of y^*_k in the corresponding segment is proportional to the weights $(\alpha_n/\gamma)^{y^*_k}$. In other words, this is the geometric distribution with ratio α_n/γ conditioned to live in the prescribed segment.

Proof of Proposition 2.20. Similarly to the proof of Proposition 2.17, we use Lemmas 2.13 to derive the needed inequalities and to single out the part of the ratio $P^*(x^*, y^*)\Lambda(y^*, y)/\Delta(x^*, y)$ that depends on y^*. One readily sees that it is equal to $(\alpha_n/\gamma)^{\sum_{k=1}^{n} y^*_k}$, and this concludes the proof. $\square$

Let us state what this computation means in terms of the conditional distribution used in the construction of P_Δ.

Corollary 2.22. *Assume that $x^* \in \mathcal{S}^*$ and $x \in \mathcal{S}$ are such that $\Delta(x^*, x) > 0$, that is, $x^*_{k-1} < x_k - 1 < x^*_{k+1}$ for all k. Then the probability distribution*

$$\frac{P^*(x^*, y^*)\Lambda(y^*, x)}{\Delta(x^*, x)}, \qquad y^* \in \mathcal{S}^*, \tag{2.83}$$

has nonzero weights iff $\max(x^*_{k-1} + 1, x_{k-1}) \le y^*_k \le \min(x^*_k, x_k - 1)$ *for all k, and these weights are equal to*

$$\prod_{k=1}^{n} \frac{(\alpha_n/\gamma)^{y^*_k}}{\sum\limits_{l=\max(x^*_{k-1}+1, x_{k-1})}^{\min(x^*_k, x_k-1)} (\alpha_n/\gamma)^l}. \tag{2.84}$$

In the four statements above we computed the ingredients needed for the constructions of the bivariate Markov chains for the simplest possible Toeplitz-like transition matrices. In these examples we always had $x_k^* \geq y_k^*$, or, informally speaking, "particles jump to the left". Because of the previous works on the subject, it is more convenient to deal with the case when particles "jump to the right". The arguments are very similar, so let us just state the results.

Consider

$$P = T_{n-1}(\alpha_1, \ldots, \alpha_{n-1}; 1 + \beta z^{-1}), \quad P^* = T_n(\alpha_1, \ldots, \alpha_n; 1 + \beta z^{-1}), \qquad \beta > 0. \tag{2.85}$$

- For P_Λ, we have $\max(x_k^*, y_{k-1}) \leq y_k^* \leq \min(x_k^* + 1, y_k - 1)$. This segment consists of either 1 or 2 points, in the latter case $y_k^* - x_k^*$ are i. i. d. Bernoulli random variables with the probability of 0 equal to $(1 + \alpha_n \beta)^{-1}$.
- For P_Δ, we have $\max(x_k^*, x_{k-1}) \leq y_k^* \leq \min(x_k^* + 1, x_k - 1)$, and the rest is the same as for P_Λ.

Now consider

$$P = T_{n-1}(\alpha_1, \ldots, \alpha_{n-1}; (1 - \gamma z^{-1})^{-1}), \quad P^* = T_n(\alpha_1, \ldots, \alpha_n; (1 - \gamma z^{-1})^{-1}), \tag{2.86}$$

for $0 < \gamma < \min\{\alpha_1^{-1}, \ldots, \alpha_n^{-1}\}$.

- For P_Λ, we have $\max(x_k^*, y_{k-1}) \leq y_k^* \leq \min(x_{k+1}^*, y_k) - 1$, and y_k^* are independent geometrically distributed with ratio $(\alpha_n \gamma)$ random variables conditioned to stay in these segments.
- For P_Δ, we have $\max(x_k^*, x_{k-1}) \leq y_k^* \leq \min(x_{k+1}^*, x_k) - 1$, and the rest is the same as for P_Λ.

Thus, we have so far considered eight bivariate Markov chains. It is natural to denote them as

$$P_\Lambda(1 + \beta z^{\pm 1}), \quad P_\Delta(1 + \beta z^{\pm 1}), \quad P_\Lambda((1 - \gamma z^{\pm 1})^{-1}), \quad P_\Delta((1 - \gamma z^{\pm 1})^{-1}). \tag{2.87}$$

Observe that although all four chains of type P_Λ live on one and the same state space, all four chains of type P_Δ live on different state spaces. For the sake of completeness, let us list those state spaces:

$$\begin{aligned}
S_\Lambda &= \{(x^*, x) \in \mathfrak{X}_n \times \mathfrak{X}_{n-1} \,|\, x_k^* + 1 \leq x_k \leq x_{k+1}^* \text{ for all } k\} \\
S_\Delta(1 + \beta z) &= \{(x^*, x) \in \mathfrak{X}_n \times \mathfrak{X}_{n-1} \,|\, x_k^* \leq x_k \leq x_{k+1}^* \text{ for all } k\} \\
S_\Delta(1 + \beta z^{-1}) &= \{(x^*, x) \in \mathfrak{X}_n \times \mathfrak{X}_{n-1} \,|\, x_k^* + 1 \leq x_k \leq x_{k+1}^* + 1 \text{ for all } k\} \\
S_\Delta((1 - \gamma z)^{-1}) &= \{(x^*, x) \in \mathfrak{X}_n \times \mathfrak{X}_{n-1} \,|\, x_{k-1}^* + 2 \leq x_k \leq x_{k+1}^* \text{ for all } k\} \\
S_\Delta((1 - \gamma z^{-1})^{-1}) &= \{(x^*, x) \in \mathfrak{X}_n \times \mathfrak{X}_{n-1} \,|\, x_k^* + 1 \leq x_k \leq x_{k+2}^* - 1 \text{ for all } k\}
\end{aligned}$$

In the above formulas we always use the convention that if an inequality involves a nonexistent variable (like x_0 or x_{n+1}^*), it is omitted.

2.6 Examples of multivariate Markov chains

Let us now use some of the examples of the bivariate Markov chains from the previous section to construct explicit examples of multivariate (not necessarily autonomous) Markov chains following the recipe of Section 2.1.

For any $m \geq 0$ we set $\mathcal{S}_m = \mathfrak{X}_m$, which is the set of strictly increasing m-tuples of integers. In this section we will denote these integers by $x_1^m < \cdots < x_m^m$.

Fix an integer $n \geq 1$, and choose n positive real numbers $\alpha_1, \ldots, \alpha_n$. We take the maps Λ_{k-1}^k to be

$$\Lambda_{k-1}^k = T_{k-1}^k(\alpha_1, \ldots, \alpha_k; (1-\alpha_k z)^{-1}), \qquad k = 2, \ldots, n. \tag{2.88}$$

We consider the Markov chain $S_\Lambda^{(n)}$, i.e. the sequential update, first. Its state space has the form

$$\begin{aligned} S_\Lambda^{(n)} &= \Big\{(x^1, \ldots, x^n) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \mid \prod_{m=2}^{n} \Lambda_{m-1}^m(x^m, x^{m-1}) > 0\Big\} \\ &= \Big\{\{x_k^m\}_{\substack{m=1,\ldots,n \\ k=1,\ldots,m}} \subset \mathbb{Z}^{\frac{n(n+1)}{2}} \mid x_k^{m+1} < x_k^m \leq x_{k+1}^{m+1} \text{ for all } k, m\Big\}. \end{aligned} \tag{2.89}$$

In other words, this is the space of n interlacing integer sequences of length $1, \ldots, n$.

Let t be an integer time variable. We now need to choose the transition probabilities $P_m(t)$, $m = 1, \ldots, n$.

Let $\{F_t(z)\}_{t \geq t_0}$ be a sequence of functions each of which has one of the four possibilities:

$$F_t(z) = (1 + \beta_t^+ z) \quad \text{or} \quad (1 + \beta_t^-/z) \quad \text{or} \quad (1 - \gamma_t^+ z)^{-1} \quad \text{or} \quad (1 - \gamma_t^-/z)^{-1}. \tag{2.90}$$

Here we assume that

$$\beta_t^\pm, \gamma_t^\pm > 0, \qquad \gamma_t^+ < \min\{\alpha_1, \ldots, \alpha_n\}, \quad \gamma_t^- < \min\{\alpha_1^{-1}, \ldots, \alpha_n^{-1}\}. \tag{2.91}$$

We set

$$P_m(t) = T_m(\alpha_1, \ldots, \alpha_m; F_t(z)), \qquad m = 1, \ldots, n. \tag{2.92}$$

Then all needed commutation relations are satisfied, thanks to Proposition 2.10.

The results of Section 2.5 enable us to describe the resulting Markov chain on $\mathcal{S}_\Lambda^{(n)}$ as follows.

At time moment t we observe a (random) point $\{x_k^m(t)\} \in S_\Lambda^{(n)}$. In order to obtain $\{x_k^m(t+1)\}$, we perform the sequential update from level 1 to level n. When we are at level m, $1 \leq m \leq n$, the new positions of the particles $x_1^m < \cdots < x_m^m$ are decided independently.

(1) For $F_t(z) = 1 + \beta_t^+ z$, the particle x_k^m is either forced to stay where it is if $x_{k-1}^{m-1}(t+1) = x_k^m(t)$, or it is forced to jump to the left by 1 if $x_k^{m-1}(t+1) = x_k^m(t)$, or it chooses between staying put or jumping to the left by 1 with probability of staying equal to $1/(1+\beta_t^+\alpha_m^{-1})$. This follows from Proposition 2.17.

(2) For $F_t(z) = 1+\beta_t^-/z$, the particle x_k^m is either forced to stay where it is if $x_k^{m-1}(t+1) = x_k^m(t)+1$, or it is forced to jump to the right by 1 if $x_{k-1}^{m-1}(t+1) = x_k^m(t)+1$, or it chooses between staying put or jumping to the right by 1 with probability of staying equal to $1/(1+\beta_t^-\alpha_m)$.

(3) For $F_t(z) = (1-\gamma_t^+ z)^{-1}$, the particle x_k^m chooses its new position according to a geometric random variable with ratio α_m/γ_t^+ conditioned to stay in the segment

$$[\max(x_{k-1}^m(t)+1, x_{k-1}^{m-1}(t+1)), \min(x_k^m(t), x_k^{m-1}(t+1)-1)]. \tag{2.93}$$

In other words, it tries to jump to the left using the geometric distribution of jump length, but it is conditioned not to overcome $x_{k-1}^m(t)+1$ (in order not to "interact" with the jump of x_{k-1}^m), and it is also conditioned to obey the interlacing inequalities with the updated particles on level $m-1$. This follows from Proposition 2.20.

(4) For $F_t(z) = (1-\gamma_t^-/z)^{-1}$, the particle x_k^m chooses its new position according to a geometric random variable with ratio $\alpha_m\gamma_t^-$ conditioned to stay in the segment

$$[\max(x_k^m(t), x_{k-1}^{m-1}(t+1)), \min(x_{k+1}^m(t), x_k^{m-1}(t+1))-1]. \tag{2.94}$$

In other words, it tries to jump to the right using the geometric distribution of jump length, but it is conditioned not to overcome $x_{k+1}^m(t)-1$ (so that it does not interact with jumps of x_{k+1}^m), and it is also conditioned to obey the interlacing inequalities with the updated particles on level $m-1$.

Projection to $\{x_1^m\}_{m\geq 1}$. A remarkable property of the Markov chain $P_\Lambda^{(n)}$ with steps of the first three types is that its projection onto the n-dimensional subspace $\{x_1^1 > x_1^2 > \cdots > x_1^n\}$ (the smallest coordinates on each level) is also a Markov chain. Moreover, since these are the leftmost particles on each level, they have no interlacing condition on their left to be satisfied, which makes the evolution simpler. Let us describe these Markov chains.

At time moment t we observe $\{x_1^1(t) > x_1^2(t) > \cdots > x_1^n(t)\}$. In order to obtain $\{x_1^m(t+1)\}_{m=1}^n$, we perform the sequential update from x_1^1 to x_1^n.

(1) For $F_t(z) = 1+\beta_t^+ z$, the particle x_1^m is either forced to jump (it is being *pushed*) to the left by 1 if $x_1^{m-1}(t+1) = x_1^m(t)$, or it chooses between not moving at all or jumping to the left by 1 with probability of not moving equal to $1/(1+\beta_t^+\alpha_m^{-1})$.

(2) For $F_t(z) = 1+\beta_t^-/z$, the particle x_1^m is either forced to stay where it is if $x_1^{m-1}(t+1) = x_1^m(t)+1$, or it chooses between staying put or jumping to the right by 1 with probability of staying equal to $1/(1+\beta_t^-\alpha_m)$.

(3) For $F_t(z) = (1-\gamma_t^+ z)^{-1}$, the particle x_1^m chooses its new position according to a geometrically distributed with ratio γ_t^+/α_n jump to the left from the point $\min(x_1^m(t), x_1^{m-1}(t+1)-1)$. That is, if $x_1^m(t) < x_1^{m-1}(t+1)$ then x_1^m simply jumps to the left with the geometric distribution of the jump, while if $x_1^m(t) \geq x_1^{m-1}(t+1)$

then x_1^m is first being pushed to the position $x_1^{m-1}(t+1)-1$ and then it jumps to the left using the geometric distribution.

(4) For the transition probability with $F_t(z) = (1-\gamma_t^-/z)^{-1}$, the particle x_1^m is conditioned to stay below $\min(x_2^m(t), x_1^{m-1}(t+1)) - 1$, which involves x_2^m, thus the projection is *not Markovian*.

The Markov chains on $\{x_1^1 > \cdots > x_1^n\}$ corresponding to $1+\beta_t^+ z$ and $1+\beta_t^-/z$ are the "Bernoulli jumps with pushing" and "Bernoulli jumps with blocking" chains discussed in [18].

Projection to $\{x_m^m\}_{m\ge1}$. Similarly, the projection of the "big" Markov chain to $\{x_1^1 \le x_2^2 \le \cdots \le x_n^n\}$ is Markovian for the steps of types one, two, and four, but it is not Markovian for the step of the third type $F_t(z) = (1-\gamma_t^+ z)^{-1}$.

Let us now consider the parallel update Markov chain $P_\Delta^{(n)}$, or rather one of them.

Choose a sequence of functions $G_t(z) = 1+\beta_t z^{-1}$ with $\beta_t \ge 0$, and set

$$P_m(t) = T_m(\alpha_1,\dots,\alpha_m; G_t(z)), \qquad m=1,\dots,n. \tag{2.95}$$

In case $\beta_t = 0$, $P_m(t)$ is the identity matrix. As before, the needed commutation relations are satisfied by Proposition 2.10.

The (time-dependent) state space of our Markov chain is

$$\begin{aligned}\mathcal{S}_\Delta^{(n)}(t) &= \Big\{(x^1,\dots,x^n) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \mid \prod_{m=2}^n \Delta_{m-1}^m(x^m, x^{m-1} \mid t+n-m) > 0\Big\} \\ &= \Big\{\{x_k^m\}_{\substack{m=1,\dots,n\\k=1,\dots,m}} \subset \mathbb{Z}^{\frac{n(n+1)}{2}} \mid x_k^m < x_k^{m-1} \le x_{k+1}^m \ \text{ if } \quad \beta_{t+n-m} = 0, \\ &\qquad\qquad x_k^m < x_k^{m-1} \le x_{k+1}^m + 1 \ \text{ if } \quad \beta_{t+n-m} > 0\Big\}. \end{aligned} \tag{2.96}$$

The update rule follows from the analog of Corollary 2.19 for $(1+\beta_t z^{-1})$. Namely, assume we have $\{x_k^m(t)\} \in S_\Delta^{(n)}(t)$. Then we choose $\{x_k^m(t)\}$ *independently of each other* as follows. We have

$$\max(x_k^m(t), x_{k-1}^{m-1}(t)) \le x_k^m(t+1) \le \min(x_k^m(t)+1, x_k^{m-1}(t)-1). \tag{2.97}$$

This segment consists of either 1 or 2 points, and in the latter case $x_k^{m+1}(t+1)$ has probability of not moving equal to $(1+\alpha_m\beta_{t-n+m})^{-1}$, and it jumps to the right by 1 with remaining probability. In particular, if $\beta_{t-n+m} = 0$ then $x_k^m(t+1) = x_k^m(t)$ for all $k=1,\dots,m$.

Less formally, each particle x_k^m either stays put or moves to the right by 1. It is forced to stay put if $x_k^m(t) = x_k^{m-1}(t)-1$, and it is forced to move by 1 if $x_k^m(t) = x_{k-1}^{m-1}(t)-1$. Otherwise, it jumps with probability $1-(1+\alpha_n\beta_{t-n+m})^{-1}$.

Projection to $\{x_1^m\}_{m\ge1}$. Once again, the projection of this Markov chain to $\{x_1^1 > \cdots > x_1^n\}$ is also a Markov chain, and its transition probabilities are as follows: Each particle x_1^m at time moment t is either forced to stay if $x_1^m(t) = x_1^{m-1}(t)-1$ or it stays with probability $(1+\alpha_n\beta_{t-n+m})^{-1}$ and jumps to the right by 1 with complementary probability. This Markov chain has no pushing because x_1^m's do not have neighbors on the left. This is the "TASEP with parallel update", see e.g. [10].

Projection to $\{x_m^m\}_{m\geq 1}$. We can also restrict our "big" Markov chain to the particles $\{x_1^1, x_2^2, \ldots, x_n^n\}$. Then at time moment t they satisfy the inequalities

$$x_{m-1}^{m-1}(t) \leq x_m^m(t) \quad \text{if} \quad \beta_{t+n-m} = 0, \qquad x_{m-1}^{m-1}(t) \leq x_m^m(t) + 1 \quad \text{if} \quad \beta_{t+n-m} > 0, \tag{2.98}$$

and the update rule is as follows. If $x_{m-1}^{m-1}(t) = x_m^m(t) + 1$ then x_m^m moves to the right by 1: $x_m^m(t+1) = x_m^m(t)$. However, if $x_{m-1}^{m-1}(t) \leq x_m^m(t)$ then x_m^m stays put with probability $(1+\alpha_n\beta_{t-n+m})^{-1}$, and it jumps to the right by 1 with the complementary probability.

In the special case when all $\alpha_j = 1$,

$$\beta_k = \begin{cases} \beta, & k \geq n-1, \\ 0, & k < n-1, \end{cases} \tag{2.99}$$

and with the densely packed initial condition $x_k^m(n-m) = k-m-1$, the Markov chain $P_\Delta^{(n)}$ discussed above is equivalent to the so-called shuffling algorithm on domino tilings of the Aztec diamonds that at time n produces a random domino tiling of the diamond of size n distributed according to the measure that assigns to a tiling the weight proportional to β raised to the number of vertical tiles, see [32].

2.7 Continuous time multivariate Markov chain

Many of the (discrete time) Markov chains considered above admit degenerations to continuous time Markov chains. Let us work out one of the simplest examples.

As in the previous sections, we fix an integer $n \geq 1$ and n positive real numbers $\alpha_1, \ldots, \alpha_n$, and take

$$\Lambda_{k-1}^k = T_{k-1}^k(\alpha_1, \ldots, \alpha_k; (1-\alpha_k z)^{-1}), \qquad k = 2, \ldots, n. \tag{2.100}$$

We will consider a limit of the Markov chain $S_\Lambda^{(n)}$, so our state space is

$$S_\Lambda^{(n)} = \Big\{ \{x_k^m\}_{\substack{m=1,\ldots,n \\ k=1,\ldots,m}} \subset \mathbb{Z}^{\frac{n(n+1)}{2}} \mid x_k^{m+1} < x_k^m \leq x_{k+1}^{m+1} \text{ for all } k, m \Big\}. \tag{2.101}$$

In the notation of the previous section, let us take $F_t(z) = 1 + \beta^-/z$ for a fixed $\beta_- > 0$ and $t = 1, 2, \ldots$. Thus, we obtain an autonomous Markov chain on $\mathcal{S}_\Lambda^{(n)}$, whose transition probabilities are determined by the following recipe.

In order to obtain $\{x_k^m(t+1)\}$ from $\{x_k^m(t)\}$, we perform the sequential update from level 1 to level n. When we are at level m, $1 \leq m \leq n$, for each $k = 1, \ldots, m$ the particle x_k^m is either forced to stay if $x_k^{m-1}(t+1) = x_k^m(t)+1$, or it is forced to jump to the right by 1 if $x_{k-1}^{m-1}(t+1) = x_k^m(t)+1$, or it chooses between staying put or jumping to the right by 1 with probability of staying equal to $(1+\beta^-\alpha_m)^{-1}$. Note that, since particles can only move to the right, it is easy to order the elements of the state space so that the matrix of transition probabilities is triangular.

We are now interested in taking the limit $\beta^- \to 0$.

Lemma 2.23. *Let $A(\epsilon)$ be a (possibly infinite) triangular matrix, whose matrix elements are polynomials in an indeterminate $\epsilon > 0$:*

$$A(\epsilon) = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \dots, \tag{2.102}$$

and assume that $A_0 = \mathbb{1}$. Then for any $\tau \in \mathbb{R}$,

$$\lim_{\epsilon\to 0}(A(\epsilon))^{[\tau/\epsilon]} = \exp(\tau A_1). \tag{2.103}$$

Proof of Lemma 2.23. For the finite size matrix the claim is standard, and the triangularity assumption reduces the computation of any fixed matrix element of $(A(\epsilon))^{[\tau/\epsilon]}$ to the finite matrix case. □

This lemma immediately implies that the transition probabilities of the Markov chain described above converge, in the limit $\beta^- \to 0$ and time rescaling by β^-, to those of the continuous time Markov chain on $S_\Lambda^{(n)}$, whose generator is the linear in β^- term of the generator of the discrete time Markov chain. Let denote this linear term by $L^{(n)}$. Its off-diagonal entries are easy to compute:

$$L^{(n)}\left(\{x_k^m\}_{\substack{m=1,\dots,n,\\ k=1,\dots,m}}, \{y_k^m\}_{\substack{m=1,\dots,n\\ k=1,\dots,m}}\right) = 1 \tag{2.104}$$

if there exists $1 \le a \le b$, $1 \le b \le n$, $0 \le c \le n-b$ such that

$$\begin{gathered} x_a^b = x_{a+1}^{b+1} = \dots = x_{a+c}^{b+c} = x, \\ y_a^b = y_{a+1}^{b+1} = \dots = y_{a+c}^{b+c} = x+1, \end{gathered}$$

and $x_k^m = y_k^m$ for all other values of (k,m), and

$$L^{(n)}\left(\{x_k^m\}_{\substack{m=1,\dots,n,\\ k=1,\dots,m}}, \{y_k^m\}_{\substack{m=1,\dots,n\\ k=1,\dots,m}}\right) = 0 \tag{2.105}$$

in all other cases.

Less formally, this continuous time Markov chain can be described as follows. Each of the particles x_k^m has its own exponential clock, all clocks are independent. When x_a^b-clock rings, the particle checks if its jump by one to the right would violate the interlacing condition. If no violation happens, that is, if

$$x_a^b < x_a^{b-1} - 1 \quad \text{and} \quad x_a^b < x_{a+1}^{b+1}, \tag{2.106}$$

then this jump takes place. If $x_a^b = x_a^{b-1} - 1$ then the jump is blocked. On the other hand, if $x_a^b = x_{a+1}^{b+1}$ then we find the longest string $x_a^b = x_{a+1}^{b+1} = \cdots = x_{a+c}^{b+c}$ and move all the particles in this string to the right by one. One could think that the particle x_a^b has pushed the whole string.

We denote this continuous time Markov chain by $\mathcal{P}^{(n)}$.

Similarly to $P_\Lambda^{(n)}$, each of the Markov chains P_m on $\mathcal{S}_m$ also has a continuous limit as $\beta^- \to 0$. Indeed, the transition probabilities of the Markov chain generated by $T_m(\alpha_1, \dots, \alpha_m; 1 + \beta^-/z)$ converge to $(x^m, y^m \in \mathcal{S}_m)$

$$\left(\lim_{\beta^- \to 0} \left(T_m(\alpha_1, \dots, \alpha_m; 1 + \beta^-/z)\right)^{[\tau/\beta^-]}\right)(x^m, y^m) \\ = \frac{\det\left[\alpha_i^{y_j^m}\right]_{i,j=1}^m}{\det\left[\alpha_i^{x_j^m}\right]_{i,j=1}^m} \frac{\det\left[\exp(\tau(y_i^m - x_j^m))\right]_{i,j=1}^m}{\exp(m\tau)}. \quad (2.107)$$

Thus, the limit of P_m is the Doob h-transform of m independent Poisson processes by the harmonic function $h(x_1, \dots, x_m) = \det\left[\alpha_i^{x_j}\right]_{i,j=1}^m$, cf. [33]. Let us denote this continuous time Markov chain by $\mathcal{P}_m$, and the above matrix of its transition probabilities over time τ by $\mathcal{P}_m(\tau)$.

Taking the same limit β^- in Proposition 2.5 leads to the following statement.

Proposition 2.24. *Let $m_n(x^n)$ be a probability measure on $\mathcal{S}_n$. Consider the evolution of the measure*

$$m_n(x^n)\Lambda_{n-1}^n(x^n, x^{n-1}) \cdots \Lambda_1^2(x^2, x^1) \quad (2.108)$$

on $\mathcal{S}_\Lambda^{(n)}$ under the Markov chain $\mathcal{P}^{(n)}$, and denote by $(x^1(t), \dots, x^n(t))$ the result after time $t \geq 0$. Then for any

$$0 = t_n^0 \leq \cdots \leq t_n^{c(n)} = t_{n-1}^0 \leq \cdots \leq t_{c(n-1)}^{n-1} = t_{n-2}^0 \leq \cdots \leq t_2^{c(2)} = t_1^0 \leq \cdots \leq t_1^{c(1)} \quad (2.109)$$

(here $c(1), \dots, c(n)$ are arbitrary nonnegative integers) the joint distribution of

$$x^n(t_n^0), \dots, x^n(t_n^{c(n)}), x^{n-1}(t_{n-1}^0), x^{n-1}(t_{n-1}^1), \dots, x^{n-1}(t_{n-1}^{c(n-1)}), \\ x^{n-2}(t_{n-2}^0), \dots, x^2(t_2^{c(2)}), x^1(t_1^0), \dots, x^1(t_1^{c(1)})$$

coincides with the stochastic evolution of m_n under transition matrices

$$\mathcal{P}_n(t_n^1 - t_n^0), \dots, \mathcal{P}_n\left(t_n^{c(n)} - t_n^{c(n)-1}\right), \Lambda_{n-1}^n, \\ \mathcal{P}_{n-1}(t_{n-1}^1 - t_{n-1}^0), \dots, \mathcal{P}_{n-1}\left(t_{n-1}^{c(n-1)} - t_{n-1}^{c(n-1)-1}\right), \Lambda_{n-2}^{n-1}, \dots, \\ \dots, \Lambda_1^2, \mathcal{P}_1(t_1^1 - t_1^0), \dots, \mathcal{P}_1\left(t_1^{c(1)} - t_1^{c(1)-1}\right).$$

Remark 2.25. It is not hard to see that if in the construction of $P_\Lambda^{(n)}$ we used $F_t(z) = (1 - \gamma^-/z)^{-1}$ and took the limit $\gamma^- \to 0$ then the resulting continuous

Markov chains would have been exactly the same. On the other hand, if we used $F_t(z) = (1+\beta^+ z)$ or $F_t(z) = (1-\gamma^+ z)^{-1}$ then the limiting continuous Markov chain would have been similar to $\mathcal{P}^{(n)}$, but with particles jumping to the left.

It is slightly technically harder to establish the convergence of Markov chains with alternating steps, for example,

$$F_{2s}(z) = 1 + \beta^+(s)z, \qquad F_{2s+1} = 1 + \beta^-(s)/z, \tag{2.110}$$

because the transition matrix is no longer triangular (particles jump in both directions). It is possible to prove, however, the following fact:

For any two continuous functions $a(\tau)$ and $b(\tau)$ on $\mathbb{R}_+$ with $a(0) = b(0) = 0$, consider the limit as $\epsilon \to 0$ of the Markov chain $P_\Lambda^{(n)}$ with alternating F_t's as above,

$$\beta^-(s) = \epsilon a(\epsilon s), \qquad \beta^+(s) = \epsilon b(\epsilon s), \tag{2.111}$$

and the time rescaled by ϵ. Then this Markov chain converges to a continuous time Markov chain, whose generator at time τ is equal to $a(\tau)$ times the generator of $\mathcal{P}^{(n)}$ plus $b(\tau)$ times the generator of the Markov chain similar to $\mathcal{P}^{(n)}$ but with particles jumping to the left.

The statement of Proposition 2.24 also remains true, but in the definition of the Markov chains $\mathcal{P}_m$ one needs to replace the Poisson process by the one-dimensional process whose generator is $a(\tau)$ times the generator of the Poisson process plus $b(\tau)$ times the generator of the Poisson process jumping to the left.

2.8 Determinantal structure of the correlation functions

The goal of this section is to compute certain averages often called correlation functions for the Markov chains $P_\Lambda^{(n)}$ and $P_\Delta^{(n)}$ with $F_t(z) = (1+\beta_t^\pm z^{\pm 1})$ or $(1-\gamma_t^\pm z^{\pm 1})^{-1}$, and their continuous time counterpart $\mathcal{P}^{(n)}$, starting from certain specific initial conditions.

As usual, we begin with $P_\Lambda^{(n)}$. The initial condition that we will use is natural to call *densely packed initial condition*. It is defined by

$$x_k^m(0) = k - m - 1, \qquad k = 1, \dots, m, \; m = 1, \dots, n. \tag{2.112}$$

Definition 2.26. *For any $M \geq 1$, pick M points*

$$\varkappa_j = (y_j, m_j, t_j) \in \mathbb{Z} \times \{1, \dots, n\} \times \mathbb{Z}_{\geq 0} \quad \text{or} \quad \mathbb{Z} \times \{1, \dots, n\} \times \mathbb{R}_{\geq 0}, \tag{2.113}$$

$j = 1, \dots, M$. The value of the Mth correlation function ρ_M of $P_\Lambda^{(n)}$ (or $P_\Delta^{(n)}$) at $(\varkappa_1, \dots, \varkappa_M)$ is defined as

$$\rho_M(\varkappa_1, \dots, \varkappa_M) = \mathrm{Prob}\{\text{For each } j = 1, \dots, M \text{ there exists a } k_j,\ 1 \leq k_j \leq m_j, \text{ such that } x_{k_j}^{m_j}(t_j) = y_j\}. \tag{2.114}$$

The goal of this section is to partially evaluate the correlation functions corresponding to the densely packed initial condition.

Introduce a partial order on pairs $(m,t) \in \{1,\dots,n\} \times \mathbb{Z}_{\geq 0}$ or $\{1,\dots,n\} \times \mathbb{R}_{\geq 0}$ via

$$(m_1,t_1) \prec (m_2,t_2) \quad \text{iff} \quad m_1 \leq m_2,\ t_1 \geq t_2 \text{ and } (m_1,t_1) \neq (m_2,t_2). \tag{2.115}$$

In what follows we use positive numbers $\alpha_1,\dots,\alpha_n$ that specify the links Λ^k_{k-1} as in Section 2.6, and as before we assume that

$$\beta_t^{\pm}, \gamma_t^{\pm} > 0, \qquad \gamma_t^+ < \min\{\alpha_1,\dots,\alpha_n\}, \quad \gamma_t^- < \min\{\alpha_1^{-1},\dots,\alpha_n^{-1}\}. \tag{2.116}$$

Theorem 2.27. *Consider the Markov chain $P_\Lambda^{(n)}$ with the densely packed initial condition and $F_t(z) = (1+\beta_t^{\pm} z^{\pm 1})$ or $(1-\gamma_t^{\pm} z^{\pm 1})^{-1}$. Assume that triplets $\varkappa_j = (y_j, m_j, t_j)$, $j=1,\dots,M$, are such that any two distinct pairs (m_j,t_j), $(m_{j'},t_{j'})$ are comparable with respect to $\prec$. Then*

$$\rho_M(\varkappa_1,\dots,\varkappa_M) = \det\left[K(\varkappa_i,\varkappa_j)\right]_{i,j=1}^M, \tag{2.117}$$

where

$$\begin{aligned} K(y_1,m_1,t_1;y_2,m_2,t_2) = &-\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}w}{w^{y_2-y_1+1}} \frac{\prod_{t=t_2}^{t_1-1} F_t(w)}{\prod_{l=m_1+1}^{m_2}(1-\alpha_l w)} \mathbb{1}_{[(m_1,t_1)\prec(m_2,t_2)]} \\ &+ \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}w \oint_{\Gamma_{\alpha^{-1}}} \mathrm{d}z \frac{\prod_{t=0}^{t_1-1} F_t(w)}{\prod_{t=0}^{t_2-1} F_t(z)} \frac{\prod_{l=1}^{m_1}(1-\alpha_l w)}{\prod_{l=1}^{m_2}(1-\alpha_l z)} \frac{w^{y_1}}{z^{y_2+1}} \frac{1}{w-z}, \end{aligned}$$

the contours Γ_0, $\Gamma_{\alpha^{-1}}$ are closed and positively oriented, and they include the poles 0 and $\{\alpha_1^{-1},\dots,\alpha_n^{-1}\}$, respectively, and no other poles.

This statement obviously implies

Corollary 2.28. *For the Markov chain $\mathcal{P}^{(n)}$, with the notation of Theorem 2.27 and densely packed initial condition, the correlation functions are given by the same determinantal formula with the kernel*

$$\begin{aligned} K(y_1,m_1,\tau_1;y_2,m_2,\tau_2) = &-\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}w}{w^{y_2-y_1+1}} \frac{e^{(t_1-t_2)/w}}{\prod_{l=m_1+1}^{m_2}(1-\alpha_l w)} \mathbb{1}_{[(m_1,t_1)\prec(m_2,t_2)]} \\ &+ \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}w \oint_{\Gamma_{\alpha^{-1}}} \mathrm{d}z \frac{e^{t_1/w}}{e^{t_2/z}} \frac{\prod_{l=1}^{m_1}(1-\alpha_l w)}{\prod_{l=1}^{m_2}(1-\alpha_l z)} \frac{w^{y_1}}{z^{y_2+1}} \frac{1}{w-z}. \end{aligned}$$

Remark 2.29. For the more general continuous time Markov chain described in Remark 2.25 a similar to Corollary 2.28 result holds true, where one needs to replace the function $e^{t/w}$ by $e^{a(t)/w+b(t)w}$.

Proof of Theorem 2.27. The starting point is Proposition 2.7. The densely packed initial condition is a measure on $\mathcal{S}_\Lambda^{(n)}$ of the form $m_n(x^n)\Lambda^n_{n-1}(x^n,x^{n-1})\cdots\Lambda^2_1(x^2,x^1)$ with m_n being the delta-measure at the point $(-n,-n+1,\dots,-1) \in \mathcal{S}_n$.

This delta-measure can be rewritten (up to a constant) as $\det[\Psi^n_{n-l}(x^n_k)]_{k,l=1,\dots,n}$ with

$$\Psi^n_{n-l}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \prod_{j=l+1}^{n} (1-\alpha_j w) w^{x+l} \frac{\mathrm{d}w}{w}, \qquad l = 1, \dots, n. \tag{2.118}$$

Indeed, $\mathrm{Span}(\Psi^n_{n-l} \,|\, l = 1, \dots, n)$ is exactly the space of all functions on $\mathbb{Z}$ supported by $\{-1, \dots, -n\}$.

We are then in a position to apply Theorem 4.2 of [7]. (In fact, the change of notation that facilitates the application was already used in Proposition 2.24 above.) The computation of the matrix M^{-1} of that theorem follows verbatim the computation in the proof of Theorem 3.2 of [13], where θ_j of [13] have to be replaced by α_j^{-1} for all $j = 1, \dots, n$. Arguing exactly as in that proof we arrive at the desired integral representation for the correlation kernel. □

Finally, one can also derive similar formulas for the Markov chain $P^{(n)}_\Delta$. As the state space $\mathcal{S}^{(n)}_\Delta$ is now

$$\mathcal{S}^{(n)}_\Delta(t) = \{(x^n(t), x^{n-1}(t+1), \dots, x^1(t+n-1)\}, \tag{2.119}$$

we need to define the densely packed initial condition differently, cf. the end of Section 2.6. We set

$$x^m_k(n-m) = k-m-1, \qquad k = 1, \dots, m, \; m = 1, \dots, n, \tag{2.120}$$

and assume that $F_t(z) \equiv 1$ for $t = 0, \dots, n-2$. This means that

$$\Delta^m_{m-1}(x^m, x^{m-1} \,|\, n-m) = \Lambda^m_{m-1}(x^m, x^{m-1}), \qquad m = 2, \dots, n, \tag{2.121}$$

and our initial condition is of the form (2.43).

Corollary 2.30. *For the Markov chain $P^{(n)}_\Delta$, with the above assumptions, notation of Theorem 2.27, and densely packed initial condition, under the additional assumption that for any two pairs $(m_j, t_j) \prec (m_{j'}, t_{j'})$ we have*

$$t_j - t_{j'} \geq m_{j'} - m_j,$$

the correlation functions are given by the same determinantal formula as in Theorem 2.27.

Proof of Corollary 2.30. Comparing the formulas for the joint distributions for $P^{(n)}_\Lambda$ and $P^{(n)}_\Delta$ in Proposition 2.7 we see that with the densely packed initial conditions they simply coincide. Hence, the correlation functions are the same. □

Note that according to the remark at the end of Section 2.6, the correlation functions for the shuffling algorithm of domino tilings of Aztec diamonds can be obtained from Theorem 2.27 and Corollary 2.30.

3 Geometry

3.1 Macroscopic behavior, limit shape

It is more convenient for us to slightly modify the definition of the height function (1.2) by assuming that its first argument varies over $\mathbb{Z}$, and

$$h(x,n,t) = |\{k | x_k^n(t) > x\}| . \tag{3.1}$$

Clearly, this modification has no effect on asymptotic statements.

We are interested in large time behavior of the interface. The macroscopic choice of variable is

$$x = [(\nu - \eta)L], \quad n = [\eta L], \quad t = \tau L, \tag{3.2}$$

where $(\nu, \eta, \tau) \in \mathbb{R}_+^3$ and $L \gg 1$ is a large parameter setting the macroscopic scale. For fixed η and τ, $h(x,n,t) = n$ for ν small enough (e.g., $\nu = 0$) and $h(x,n,t) = 0$ for ν large enough. Define the x-density of our system as the average number of particles in the x-direction. Then, for large L, one expects that $-L^{-1}\partial h/\partial\nu \simeq x$-density. Thus, our model has facets when the x-density is constant (equal to 0 or 1 in our situation), which are interpolated by curved pieces of the surface, see Figure 1.2.

Claim 3.1. The domain $\mathcal{D} \subset \mathbb{R}_+^3$, where the x-density of our system is asymptotically strictly between 0 and 1 is given by

$$|\sqrt{\tau} - \sqrt{\eta}| < \sqrt{\nu} < \sqrt{\tau} + \sqrt{\eta}. \tag{3.3}$$

Equivalently, x-density $\in (0,1)$ iff there exists a (non-degenerate) triangle with sides $\sqrt{\nu}$, $\sqrt{\eta}$, $\sqrt{\tau}$. Denote by π_ν, π_η and π_τ the angles of this triangle as indicated in Figure 3.1.

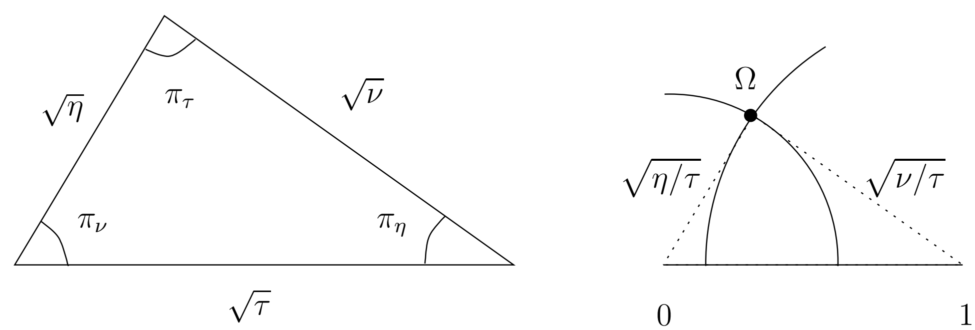

Figure 3.1: The triangle of (3.3) on the left and its scaled version defined by intersection of circles on the right.

The condition (3.3) is also equivalent to saying that the circle centered at 0 of radius $\sqrt{\eta/\tau}$ has two disjoints intersections with the circle centered at 1 of radius $\sqrt{\nu/\tau}$.

In that case, the two intersections are complex conjugate. Denote by $\Omega(\nu,\eta,\tau)$ the intersection in

$$\mathbb{H}=\{z\in\mathbb{C}\,|\,\mathrm{Im}(z)>0\}. \tag{3.4}$$

Then, we have the following properties

$$|\Omega|^2=\frac{\eta}{\tau},\quad |1-\Omega|^2=\frac{\nu}{\tau},\quad \arg(\Omega)=\pi_\nu,\quad \arg(1-\Omega)=-\pi_\eta. \tag{3.5}$$

The cosine rule gives the angles π_*'s in $(0,\pi)$ by

$$\begin{aligned}\pi_\nu &= \arccos\left(\frac{\tau+\eta-\nu}{2\sqrt{\tau\eta}}\right),\\ \pi_\eta &= \arccos\left(\frac{\tau+\nu-\eta}{2\sqrt{\tau\nu}}\right),\\ \pi_\tau &= \arccos\left(\frac{\eta+\nu-\eta}{2\sqrt{\nu\eta}}\right).\end{aligned} \tag{3.6}$$

Proposition 3.2 (Bulk scaling limit)**.** *For any $k=1,2,\ldots,$ consider*

$$\varkappa_j(L)=(x_j(L),n_j(L),t_j(L)),\qquad j=1,\ldots,k,$$

such that for any $i\neq j$ and any $L>0$ either $\varkappa_i(L)\prec\varkappa_j(L)$ or $\varkappa_j(L)\prec\varkappa_i(L)$. Assume that

$$\lim_{L\to\infty}\frac{x_j}{L}=\nu,\qquad \lim_{L\to\infty}\frac{n_j}{L}=\eta,\qquad \lim_{L\to\infty}\frac{t_j}{L}=\tau,\qquad j=1,\ldots,k;$$

we have $(\nu,\eta,\tau)\in\mathcal{D}$; and also all the differences x_i-x_j, n_i-n_j, t_i-t_j do not depend on the large parameter L. Then the k-point correlation function $\rho^{(k)}(\varkappa_1,\ldots,\varkappa_k)$ converges to the determinant $\det[K_{ij}^{\mathrm{bulk}}]_{1\le i,j\le k}$*, where*

$$K_{ij}^{\mathrm{bulk}}=\frac{1}{2\pi\mathrm{i}}\int_{1-\Omega(\nu,\eta,\tau)}^{1-\overline{\Omega}(\nu,\eta,\tau)}\mathrm{d}w\frac{(1-w)^{n_i-n_j}e^{(t_j-t_i)w}}{w^{x_i-x_j+1}}, \tag{3.7}$$

where for $(n_i,t_i)\not\prec(n_j,t_j)$ the integration contour crosses $\mathbb{R}_-$, while for $(n_i,t_i)\prec(n_j,t_j)$ the contour crosses $\mathbb{R}_+$.

Proof of Proposition 3.2. One follows exactly the same steps as in Section 3.2 of [34], replacing the double integral (35) in there by (4.1). The deformed paths are then like in Figure 7.3 but with $z_c=w_c$. $\square$

Corollary 3.3. *Let ρ denote the asymptotic x-density. Then, in $\mathcal{D}$, it is given by*

$$\rho(\nu,\eta,\tau)=\lim_{L\to\infty}\rho^{(1)}([\nu L],[\eta L],\tau L)=\pi_\eta/\pi\in[0,1]. \tag{3.8}$$

Consequently,

$$\lim_{L\to\infty}\frac{\mathbb{E}\left(h([(\nu-\eta)L],[\eta L],\tau L)\right)}{L}=\bar{h}(\nu,\eta,\tau):=\frac{1}{\pi}\int_\nu^{(\sqrt{\tau}+\sqrt{\eta})^2}\pi_\eta(\nu',\eta,\tau)\mathrm{d}\nu'. \tag{3.9}$$

Below we perform the integral in (3.9) to get an explicit expression for the limit shape $\bar{h}$. Along the way we derive some interesting geometric relations. First of all, $\bar{h}$ is homogeneous of degree one (since it is the scaling limit under same scaling in all directions).

Lemma 3.4. *For any $\alpha > 0$,*

$$\bar{h}(\alpha\nu, \alpha\eta, \alpha\tau) = \alpha\bar{h}(\nu, \eta, \tau), \tag{3.10}$$

from which it follows

$$\left(\nu\frac{\partial}{\partial\nu} + \eta\frac{\partial}{\partial\eta} + \tau\frac{\partial}{\partial\tau}\right)\bar{h}(\nu, \eta, \tau) = \bar{h}(\nu, \eta, \tau). \tag{3.11}$$

Proof of Lemma 3.4. It follows directly from the geometric property $\pi_\eta(\alpha\nu, \alpha\eta, \alpha\tau) = \pi_\eta(\nu, \eta, \tau)$. □

Therefore, we need just to compute the partial derivatives, then the limit shape $\bar{h}$ will be determined by the l.h.s. of (3.11).

Proposition 3.5. *The partial derivatives of the limit shape $\bar{h}$ are given by*

$$\frac{\partial\bar{h}}{\partial\nu} = -\frac{\pi_\eta}{\pi}, \quad \frac{\partial\bar{h}}{\partial\eta} = 1 - \frac{\pi_\nu}{\pi}, \quad \frac{\partial\bar{h}}{\partial\tau} = \frac{\sin(\pi_\nu)\sin(\pi_\eta)}{\pi\sin(\pi_\tau)}. \tag{3.12}$$

As a corollary of Lemma 3.4 and Proposition 3.5, the limit shape is given as follows.

Corollary 3.6. *For $(\nu, \eta, \tau) \in \mathcal{D}$, we have*

$$\bar{h}(\nu, \eta, \tau) = \frac{1}{\pi}\left(-\nu\pi_\eta + \eta(\pi - \pi_\nu) + \tau\frac{\sin(\pi_\nu)\sin(\pi_\eta)}{\sin(\pi_\tau)}\right). \tag{3.13}$$

Proof of Proposition 3.5. From (3.9) we immediately have the first relation: $\partial\bar{h}/\partial\nu = -\pi_\eta/\pi$. In the derivative of $\bar{h}$ with respect to τ and η we have one term coming from the boundary term and one from the internal derivative. The boundary terms will actually be zero, since the density at the upper edge is zero. We need to compute

$$\frac{\partial\pi_\eta}{\partial\eta} = \frac{1}{\sqrt{4\eta\tau - (\nu-\eta-\tau)^2}}, \quad \frac{\partial\pi_\eta}{\partial\tau} = \frac{\nu-\eta-\tau}{2\tau\sqrt{4\eta\tau - (\nu-\eta-\tau)^2}}. \tag{3.14}$$

Then, we apply the indefinite integrals

$$\int\frac{\mathrm{d}x}{a^2 - x^2} = \arcsin(x/|a|) + C, \quad \int\frac{x\mathrm{d}x}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2} + C. \tag{3.15}$$

For the derivative with respect to η,

$$\begin{aligned}\pi\frac{\partial\bar{h}}{\partial\eta} &= \int_\nu^{(\sqrt{\eta}+\sqrt{\tau})^2}\frac{\partial\pi_\eta}{\partial\eta}\mathrm{d}\nu' + (1 + \sqrt{\tau/\eta})\pi_\eta\left((\sqrt{\eta}+\sqrt{\tau})^2, \eta, \tau\right) \\ &= \pi/2 + \arcsin\left(\frac{\eta+\tau-\nu}{2\sqrt{\eta\tau}}\right) = \pi - \arccos\left(\frac{\eta+\tau-\nu}{2\sqrt{\eta\tau}}\right),\end{aligned} \tag{3.16}$$

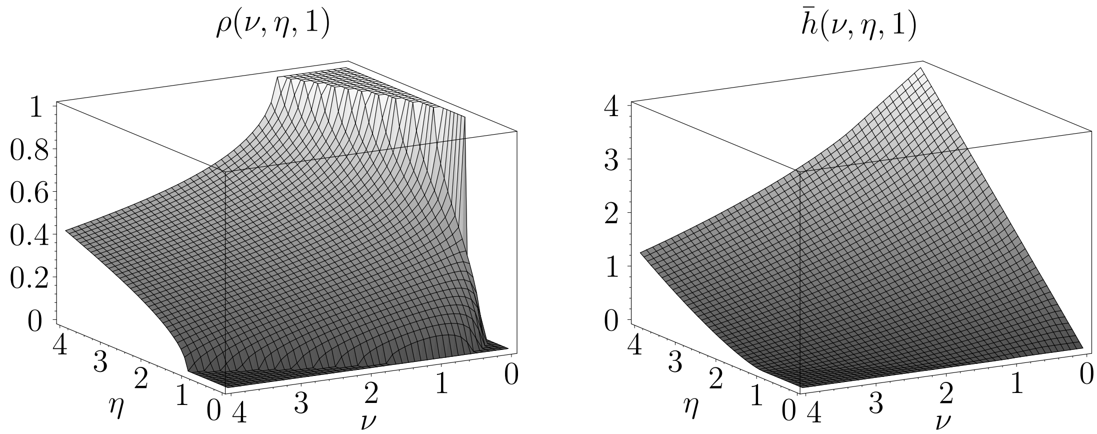

Figure 3.2: (a) Limiting density of the particles with $\tau = 1$. (b) The associated limiting height function. Two facets are visible.

the latter being π_ν. Finally,

$$\begin{aligned}
\pi \frac{\partial \bar{h}}{\partial \tau} &= \int_\nu^{(\sqrt{\eta}+\sqrt{\tau})^2} \frac{\partial \pi_\eta}{\partial \tau} \mathrm{d}\nu' + (1+\sqrt{\eta/\tau})\pi_\eta\left((\sqrt{\eta}+\sqrt{\tau})^2, \eta, \tau\right) \\
&= \frac{\sqrt{4\eta\tau - (\nu-\eta-\tau)^2}}{2\tau} = \sqrt{\eta/\tau}\sin(\pi_\nu), \qquad (3.17)
\end{aligned}$$

and by the sinus theorem to the triangle of Figure 3.1 we have $\sqrt{\eta}/\sqrt{\tau} = \sin(\pi_\eta)/\sin(\pi_\tau)$. □

3.2 Growth model in the anisotropic KPZ class

For fixed τ, the macroscopic slopes of the interface in the x- and n-directions are given by $u_x = \partial_\nu \bar{h}$ and $u_n = \partial_\eta \bar{h}$. The speed of growth of the surface, $\partial_\tau \bar{h}$, depends only on these two slopes. Indeed, by (3.12), we can rewrite

$$v = \frac{\partial \bar{h}}{\partial \tau} = -\frac{1}{\pi}\frac{\sin(\pi u_x)\sin(\pi u_n)}{\sin(\pi(u_x+u_n))}. \qquad (3.18)$$

Remark that the speed of growth is monotonically decreasing with the slope

$$\frac{\partial v(u_x,u_n)}{\partial u_x} < 0, \quad \frac{\partial v(u_x,u_n)}{\partial u_n} < 0 \qquad (3.19)$$

for $u_x, u_n, u_x+u_n \in (0,1)$.

To see in which universality class our model belongs to, we need to compute the determinant of the Hessian of $v = v(u_x,u_n)$. Explicit computations give

$$\begin{vmatrix} \partial_{u_x}\partial_{u_x} v & \partial_{u_x}\partial_{u_n} v \\ \partial_{u_n}\partial_{u_x} v & \partial_{u_n}\partial_{u_n} v \end{vmatrix} = -4\pi^2 \frac{\sin(\pi u_x)^2 \sin(\pi u_n)^2}{\sin(\pi(u_x+u_n))^4} < 0 \qquad (3.20)$$

for $u_x, u_n, u_x + u_n \in (0,1)$, i.e., for $(\nu, \eta, \tau) \in \mathcal{D}$. Thus, our model belongs to the *anisotropic* KPZ universality class of growth models in $2+1$ dimensions.

Let us give a more intuitive explanation. Consider Figure 1.1 and focus on the random surface seen from the (n,h) plane; call $y(n,h,t)$ the function whose graph is the surface. Along the direction of fixed n, an increase of the density corresponds to a larger slope $\partial y/\partial h$. Along this direction, the particles evolve according to TASEP dynamics. Therefore, a larger slope corresponds to a smaller speed of growth for $\partial y/\partial t$. Secondly, consider the direction of fixed h. Also in this case, a larger density implies a larger slope $\partial y/\partial n$. But along this direction, particles evolve as PushASEP dynamics, i.e., a larger slope corresponds to a larger speed of growth. Therefore, the speed of growth will be monotonically increasing with $\partial y/\partial h$ and $-\partial y/\partial n$.

3.3 A few other geometric properties

During the asymptotic analysis we will use a few more geometric quantities, which we collect in this section. The key function to be analyzed is

$$G(w) \equiv G(w|\nu,\eta,\tau) = \tau w + \nu \ln(1-w) - \eta \ln(w), \quad w \in \mathbb{C}. \tag{3.21}$$

The critical points of G coincide with Ω as stated below.

Proposition 3.7. *Away from $\{0,1\}$, the function G has two critical points(counted with multiplicities). These two points are distinct and complex conjugate if and only if $(\nu,\eta,\tau) \in \mathcal{D}$, in which case the critical points are $\{\Omega, \overline{\Omega}\}$.*

Proof of Proposition 3.7. The derivative of G gives

$$G'(w) = \frac{\tau}{w(w-1)}\left(\left(w - \frac{\eta+\tau-\nu}{2\tau}\right)^2 + \frac{4\eta\tau - (\eta+\tau-\nu)^2}{4\tau^2}\right), \tag{3.22}$$

and we have two distinct complex conjugate solutions iff $4\eta\tau - (\eta+\tau-\nu)^2 > 0$, i.e., iff $(\nu,\eta,\tau) \in \mathcal{D}$. Also, from (3.5) and (3.6) we get

$$\mathrm{Re}(\Omega) = \frac{\eta+\tau-\nu}{2\tau}, \quad \mathrm{Im}(\Omega) = \frac{\sqrt{4\eta\tau - (\eta+\tau-\nu)^2}}{2\tau}. \tag{3.23}$$

Thus, Ω and $\overline{\Omega}$ are the two solutions of $G'(w) = 0$, i.e., the two critical points. $\square$

The main formulas needed later are the partial derivatives of Ω as well as $G''(\Omega)$.

Proposition 3.8. *Denote $\kappa = 2\tau \mathrm{Im}(\Omega) = \sqrt{4\eta\tau - (\eta+\tau-\nu)^2}$. Then we have*

$$G''(\Omega) = \frac{-\mathrm{i}\kappa}{\Omega(1-\Omega)}, \tag{3.24}$$

which implies

$$|G''(\Omega)| = \frac{\kappa}{|\Omega(1-\Omega)|}, \quad \arg(G''(\Omega)) = -\frac{\pi}{2} - \pi_\nu + \pi_\eta. \tag{3.25}$$

Moreover,

$$\frac{\partial\Omega}{\partial\nu} = \frac{\mathrm{i}\Omega}{\kappa}, \quad \frac{\partial\Omega}{\partial\eta} = \frac{\mathrm{i}(1-\Omega)}{\kappa}, \quad \frac{\partial\Omega}{\partial\tau} = \frac{-\mathrm{i}\Omega(1-\Omega)}{\kappa}. \tag{3.26}$$

Proof of Proposition 3.8. From (3.22) we get

$$G''(\Omega) = \frac{2\tau}{\Omega(\Omega-1)}(\Omega - \mathrm{Re}(\Omega)) = \frac{2\mathrm{i}\tau\mathrm{Im}(\Omega)}{\Omega(\Omega-1)}. \tag{3.27}$$

The modulus is immediate, while the argument is obtained using (3.5).

Since Ω is the intersection point of the circles $|z| = \sqrt{\eta/\tau}$ and $|1-z| = \sqrt{\nu/\tau}$, the direction of $\partial_\nu\Omega$ is orthogonal to the vector Ω and $\partial_\eta\Omega$ is orthogonal to $1-\Omega$. Therefore, for some $c_1, c_2 \in \mathbb{R}$,

$$\frac{\partial\Omega}{\partial\nu} = c_1\Omega\mathrm{i}, \quad \frac{\partial\Omega}{\partial\eta} = c_2(1-\Omega)\mathrm{i}. \tag{3.28}$$

Looking at the real part of these equations, we get $\partial_\nu\mathrm{Re}(\Omega) = -c_1\mathrm{Im}(\Omega)$, and $\partial_\nu\mathrm{Re}(\Omega) = c_2\mathrm{Im}(\Omega)$. On the other hand,

$$\mathrm{Re}(\Omega) = \frac{\eta+\tau-\nu}{2\tau} \quad \Rightarrow \quad \partial_\nu\mathrm{Re}(\Omega) = -\frac{1}{2\tau}, \quad \partial_\eta\mathrm{Re}(\Omega) = \frac{1}{2\tau}. \tag{3.29}$$

From this we conclude that

$$\partial_\nu\Omega = \frac{\mathrm{i}\Omega}{2\tau\mathrm{Im}(\Omega)}, \quad \partial_\eta\Omega = \frac{\mathrm{i}(1-\Omega)}{2\tau\mathrm{Im}(\Omega)}. \tag{3.30}$$

To get $\partial_\tau\Omega$, we can use the following property: $\Omega(a\nu, a\eta, a\tau) = \Omega(\nu,\eta,\tau)$ for any $a > 0$, which implies

$$(\nu\partial_\nu + \eta\partial_\eta + \tau\partial_\tau)\,\Omega = 0. \tag{3.31}$$

This equation leads to

$$\partial_\tau\Omega = -\frac{\mathrm{i}}{2\tau\mathrm{Im}(\Omega)}\left(\frac{\nu}{\tau}\Omega + \frac{\eta}{\tau}(1-\Omega)\right) = -\frac{\mathrm{i}\Omega(1-\Omega)}{2\tau\mathrm{Im}(\Omega)}, \tag{3.32}$$

using $|\Omega|^2 = \eta/\tau$ and $|1-\Omega|^2 = \nu/\tau$, see (3.5). □

Another important function appearing in the asymptotics of the kernel is the imaginary part of $G(\Omega)$ (and their derivatives).

Proposition 3.9. *We have*

$$\gamma(\nu,\eta,\tau) := \mathrm{Im}(G(\Omega)) = \tau\mathrm{Im}(\Omega) - \nu\pi_\eta - \eta\pi_\nu, \tag{3.33}$$

Its derivatives are

$$\frac{\partial\mathrm{Im}(G(\Omega))}{\partial\nu} = -\pi_\eta, \quad \frac{\partial\mathrm{Im}(G(\Omega))}{\partial\eta} = -\pi_\nu, \tag{3.34}$$

and

$$\frac{\partial^2\mathrm{Im}(G(\Omega))}{\partial\nu\partial\eta} = -\frac{1}{\kappa}, \quad \kappa = 2\tau\mathrm{Im}(\Omega). \tag{3.35}$$

Proof of Proposition 3.9. The relation (3.33) is a direct consequence of (3.5). The rest are just simple computations. □

4 Kernel representations

For the analysis of the variance we will use a representation in terms of Charlier polynomials. These polynomials are defined on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, while our particles at level n live on $\{-n, -n+1, \ldots\}$. Thus, it is convenient to shift the position at level n by $-n$, i.e., the positions of particles at level n will be denoted by $-n+x$, $x \geq 0$. With this shift, the kernel (1.6) is equivalent to the following one.

$$
\begin{aligned}
&K(x_1, n_1, t_1; x_2, n_2, t_2) \\
&= \begin{cases} \frac{e^{t_1-t_2}}{(2\pi \mathrm{i})^2} \oint_{\Gamma_1} \mathrm{d}z \oint_{\Gamma_0} \mathrm{d}w \frac{z^{n_1}}{e^{t_1 z}(1-z)^{x_1+1}} \frac{e^{t_2 w}(1-w)^{x_2}}{w^{n_2}} \frac{1}{w-z}, & (n_1, t_1) \not\prec (n_2, t_2) \\ \frac{e^{t_1-t_2}}{(2\pi \mathrm{i})^2} \oint_{\Gamma_1} \mathrm{d}z \oint_{\Gamma_{0,z}} \mathrm{d}w \frac{z^{n_1}}{e^{t_1 z}(1-z)^{x_1+1}} \frac{e^{t_2 w}(1-w)^{x_2}}{w^{n_2}} \frac{1}{w-z}, & (n_1, t_1) \prec (n_2, t_2) \end{cases}
\end{aligned} \tag{4.1}
$$

To get it, we just have to do the change of variables $z \to 1/(1-w)$ and $w \to 1/(1-z)$, followed by a conjugation. We can get this kernel also from (3.11) in [7] (and change of variable: $z \to 1-w, w \to 1-z$).

We can write the kernel also using Charlier polynomials $C_n(x, t)$.

Proposition 4.1 (Charlier extended kernel)**.** *The extended kernel is given by*

$$
K(x_1, n_1, t_1; x_2, n_2, t_2) = \begin{cases} \sum_{k=1}^{n_2} \Psi_{n_1-k}^{n_1,t_1}(x_1) \Phi_{n_2-k}^{n_2,t_2}(x_2), & (n_1, t_1) \not\prec (n_2, t_2) \\ -\sum_{l=0}^{\infty} \Psi_{n_1+l}^{n_1,t_1}(x_1) \Phi_{n_2+l}^{n_2,t_2}(x_2), & (n_1, t_1) \prec (n_2, t_2) \end{cases} \tag{4.2}
$$

where

$$
\Psi_k^{n,t}(x) = \frac{t^{x/2} e^{t/2}}{\sqrt{x!}} \frac{\sqrt{k!}}{t^{k/2}} q_k(x, t), \quad \Phi_k^{n,t}(x) = \left(\frac{t^{x/2} e^{t/2}}{\sqrt{x!}} \frac{\sqrt{k!}}{t^{k/2}} \right)^{-1} q_k(x, t), \tag{4.3}
$$

where

$$
q_n(x, t) = w_t(x)^{1/2} \frac{t^{n/2}}{\sqrt{n!}} C_n(x, t), \tag{4.4}
$$

$$
w_t(x) = \frac{e^{-t} t^x}{x!}, \quad C_n(x, t) = \frac{n!}{t^n} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \frac{(1-w)^x e^{wt}}{w^{n+1}}.
$$

Remark 4.2. For later use, we rewrite q_n as

$$
q_n(x, t) = B_{n,t}(x) I_{n,t}(x), \quad B_{n,t}(x) = \frac{e^{-t/2} t^{x/2}}{\sqrt{x!}} \frac{\sqrt{n!}}{t^{n/2}}, \tag{4.5}
$$

and

$$
I_{n,t}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \frac{(1-w)^x e^{wt}}{w^{n+1}}. \tag{4.6}
$$

Then, we will have to do the asymptotics of the integral $I_{n,t}$.

Remark 4.3. In the proof of Theorem 1.3 we will use the following property

$$-\partial_{s'} K(m,n,s;m',n',s') = K(m,n,s;m'+1,n',s'), \tag{4.7}$$

which holds for both (4.1) and (4.2). We could have conjugated out the prefactor $e^{t_1-t_2}$ in (4.1) to obtain an equivalent kernel. But then (4.7) would not hold (there would be a shift in n' instead) and it might have been confusing.

Proof of Proposition 4.1. From [10], Proposition 3.1, Lemma 3.2 and Lemma 3.3 (just need to shift the positions at level n by $-n$, so that $x \geq 0$), we have the formula for the starting extended kernel, namely,

$$K(x_1,n_1,t_1;x_1,n_2,t_2) = -\phi^{((n_1,t_1),(n_2,t_2))}(x_1,x_2) + \sum_{k=1}^{n_2} \Psi^{n_1,t_1}_{n_1-k}(x_1)\Phi^{n_2,t_2}_{n_2-k}(x_2) \tag{4.8}$$

with

$$\begin{aligned} \Psi^{n,t}_k(x) &= \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{0,1}} \mathrm{d}w \frac{e^{tw}(1-w)^k}{w^{x+1}}, \\ \Phi^{n,t}_k(x) &= \frac{-1}{2\pi\mathrm{i}}\oint_{\Gamma_1} \mathrm{d}z \frac{z^x e^{-tz}}{(1-z)^{k+1}}, \\ \phi^{((n_1,t_1),(n_2,t_2))}(x_1,x_2) &= \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{0,1}} \mathrm{d}w \frac{e^{w(t_1-t_2)}}{w^{x_1-x_2+1}(w-1)^{n_2-n_1}} \mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]}. \end{aligned} \tag{4.9}$$

Using the integral representations, it is not difficult to check that

$$\sum_{k\geq 0} \Psi^{n_1,t_1}_k(x)\Phi^{n_2,t_2}_k(y) = \phi^{((n_1,t_1),(n_2,t_2))}(x,y). \tag{4.10}$$

Thus (4.8) becomes (4.2). This new expression is good because in (4.9) we never have the case when the pole at 0 in $\Psi^{n,t}_k$ survives. Finally, using the integral representation (4.4) of the Charlier polynomials and the relation $C_n(x,t) = C_x(n,t)$, we can express the Ψ and the Φ in terms of them.

The same formula can be obtained starting from (4.2). For example, when $(n_1,t_1) \not\prec (n_2,t_2)$, one uses the geometric series $z/(z-w) = \sum_{k\geq 0}(w/z)^k$ to go back to (4.8). ☐

For the computation of the variance, we will need only the kernel at fixed (n,t). Thus, we denote $K_{n,t}(x,y) \equiv K(x,n,t;y,n,t)$.

Corollary 4.4. *The kernel $K_{n,t}$ is given by*

$$K_{n,t}(x,y) = \sqrt{nt}\frac{q_{n-1}(x,t)q_n(y,t) - q_n(x,t)q_{n-1}(y,t)}{x-y}. \tag{4.11}$$

Proof of Corollary 4.4. The proof is just a specialization of Proposition 4.1 to the case $n_1 = n_2 = n$, $t_1 = t_2 = t$, and then use Christoffel-Darboux formula. ☐

5 Gaussian fluctuations

In this section we look only at the height function at given time. Therefore, it is convenient to set $\lambda = \nu/\tau$ and $c = \eta/\tau$ so that we have $n = [\eta L] = [ct]$ and $x = [\nu L] = [\lambda t]$. In these variables, the equation for the bulk region given by (3.3) rewrites as

$$(1-\sqrt{c})^2 < \lambda < (1+\sqrt{c})^2. \tag{5.1}$$

First we compute the variance of the height.

Proposition 5.1. *For any* $\lambda \in ((1-\sqrt{c})^2, (\sqrt{c}+1)^2)$,

$$\lim_{t\to\infty} \frac{\mathrm{Var}(h([(\lambda-c)t],[ct],t))}{\ln(t)} = \frac{1}{2\pi^2}. \tag{5.2}$$

With this we can prove Theorem 1.2.

Proof of Theorem 1.2. It is a consequence of Proposition 5.1 and [43]. More precisely, in Section 2 of [43] the convergence in distribution (a generalization of the result for the sine kernel of [16]) is stated. However, following the proof of the theorem, one realizes that it is done by controlling the cumulants, i.e., also the moments converge. □

Proof of Proposition 5.1. The variance can be written in terms of the one and two point correlation functions $\rho^{(1)}$ and $\rho^{(2)}$. Namely,

$$\mathrm{Var}(h([(\lambda-c)t],[ct],t)) = \sum_{x,y>[\lambda t]} \rho^{(2)}(x,y) + \sum_{x>[\lambda t]} \rho^{(1)}(x) - \Big(\sum_{x>[\lambda t]} \rho^{(1)}(x)\Big)^2, \tag{5.3}$$

where $\rho^{(2)}(x,y) = K_{n,t}(x,x)K_{n,t}(y,y) - K_{n,t}(x,y)K_{n,t}(y,x)$ and $\rho^{(1)}(x) = K_{n,t}(x,x)$. Using $K_{n,t}^2 = K_{n,t}$ on $\ell_2(\mathbb{Z}_+)$, we have

$$\begin{aligned}
\mathrm{Var}(h([(\lambda-c)t],[ct],t)) &= \sum_{x>[\lambda t]} K_{n,t}(x,x) - \sum_{x,y>[\lambda t]} K_{n,t}(x,y)K_{n,t}(y,x) \\
&= \sum_{x>[\lambda t]} \sum_{y=0}^{\infty} K_{n,t}(x,y)K_{n,t}(y,x) - \sum_{x,y>[\lambda t]} K_{n,t}(x,y)K_{n,t}(y,x) \\
&= \sum_{x>[\lambda t]} \sum_{y\leq[\lambda t]} (K_{n,t}(x,y))^2, \quad n=[ct].
\end{aligned} \tag{5.4}$$

We use the expression (4.11) for the kernel $K_{n,t}$. We decompose the sum in (5.4) into the following three sets:

$$\begin{aligned}
M &= \{x,y \in \mathbb{Z}_+^2 | x > [\lambda t], y \leq [\lambda t], y-x \leq \varepsilon_1 t\}, \\
R_1 &= \{x,y \in \mathbb{Z}_+^2 | x > [\lambda t], y \leq [\lambda t], \varepsilon_1 t < y-x < \varepsilon_2 t\}, \\
R_2 &= \{x,y \in \mathbb{Z}_+^2 | x > [\lambda t], y \leq [\lambda t], \varepsilon_2 t \leq y-x\},
\end{aligned} \tag{5.5}$$

where the parameter $\varepsilon_2 = \frac{1}{2}\min\{(1+\sqrt{c})^2 - \lambda, \lambda - (1-\sqrt{c})^2\}$ is chosen so that R_1 is a subset of the bulk. Thus

$$\mathrm{Var}(h([(\lambda - c)t], [ct], t)) = M_t + R_{t,1} + R_{t,2}, \tag{5.6}$$

with

$$M_t = \sum_{x,y\in M} |K_{n,t}(x,y)|^2, \quad R_{t,k} = \sum_{x,y\in R_k} |K_{n,t}(x,y)|^2. \tag{5.7}$$

Remark: The parameter ε_1, small, will be chosen t-dependent in the end.

(1) Bound on $R_{t,2}$. For $x,y \in R_2$, we use $y - x \geq \varepsilon_2 t$, and and extend the sum to infinities

$$\begin{aligned} R_{t,2} \quad &\leq \quad \frac{1}{\varepsilon_2^2}\sum_{x\geq\lambda t}\sum_{y\leq\lambda t}\big(|q_{[ct]}(x,t)|^2|q_{[ct]-1}(y,t)|^2 + |q_{[ct]-1}(x,t)|^2|q_{[ct]}(y,t)|^2| \\ &\quad +2|q_{[ct]-1}(x,t)q_{[ct]}(x,t)||q_{[ct]-1}(y,t)q_{[ct]}(y,t)|\big) \leq \frac{4}{\varepsilon_2^2}. \end{aligned} \tag{5.8}$$

The last inequality follows from Cauchy-Schwarz and the property

$$\sum_{x\geq 0}|q_k(x,t)|^2 = \langle \Psi_k^{n,t}, \Phi_k^{n,t}\rangle = 1, \quad \text{for all } k. \tag{5.9}$$

(2) Bound on $R_{t,1}$. Since this time $x,y \in R_1$ are always in the bulk, we just use the bound of Lemma 7.10 and get

$$\begin{aligned} R_{t,1} \quad &\leq \quad \text{const} \sum_{x,y\in R_1}\frac{1}{(x-y)^2} = \text{const}\sum_{z=[\varepsilon_1 t]}^{[\varepsilon_2 t]}\frac{1}{z} \\ &= \quad \Psi([\varepsilon_2 t]+1) - \Psi([\varepsilon_1 t]), \end{aligned} \tag{5.10}$$

where $\Psi(x)$ is the *digamma* function, which has the series expansion at infinity given by

$$\Psi(x) = \ln(x) - 1/(2x) + \mathcal{O}(1/x^2). \tag{5.11}$$

Thus

$$R_{t,1} \leq \text{const } \ln(1/\varepsilon_1), \tag{5.12}$$

with const t-independent.

(3) Limit value for M_t. This time we need more than just a bound. Remind that $n = ct$ and set $x = [\lambda t] + \xi_1$, $y = [\lambda t] - \xi_2$. We have $1 \leq \xi_1 + \xi_2 \leq \varepsilon_1 t$. Lemma 7.7 gives

$$\begin{aligned} q_{[ct]-\ell}(\lambda t + \xi, t) = \frac{1}{\sqrt{\pi}}\frac{t^{-1/2}}{\sqrt[4]{c - \frac{(1+c-\lambda)^2}{4}}}\Bigg[&\mathcal{O}(t^{-1/2}) + \mathcal{O}(\varepsilon_1) \\ &+ \cos\Big[t\alpha(c,\lambda+\xi/t) + \beta(c,\lambda) - \ell\partial_c\alpha(c,\lambda)\Big]\Bigg]. \end{aligned} \tag{5.13}$$

We use it with $\ell = 0, 1$, together with the trigonometric identity

$$\cos(b_1 + \delta)\cos(b_2) - \cos(b_1)\cos(b_2 + \delta) = \sin(\delta)\sin(b_2 - b_1), \tag{5.14}$$

with $\delta = -\partial_c \alpha(c, \lambda)$, $b_1 = t\alpha(c, \lambda + \xi_1/t) + \beta(c, \lambda)$, $b_2 = t\alpha(c, \lambda - \xi_1/t) + \beta(c, \lambda)$. The factor $\sin^2(\delta)$ cancels the $\sqrt[4]{\cdots}$ term exactly. We obtain

$$\begin{aligned} M_t = \sum_{\xi_1=1}^{[\varepsilon_1 t]} \sum_{\xi_2=0}^{\xi_1 - 1} \frac{1}{\pi^2} \frac{1}{(\xi_1 + \xi_2)^2} \bigg[& \mathcal{O}(t^{-1/2}) + \mathcal{O}(\varepsilon_1) \\ & + \sin^2 \Big[t(\alpha(c, \lambda - \xi_2/t) - \alpha(c, \lambda + \xi_1/t)) \Big] \bigg] \end{aligned} \tag{5.15}$$

The contribution of the error terms can be bounded by $\ln(\varepsilon_1 t)\mathcal{O}(t^{-1/2}, \varepsilon_1)$ and the remainder is

$$\sum_{\xi_1=1}^{[\varepsilon_1 t]} \sum_{\xi_2=0}^{\xi_1 - 1} \frac{1}{\pi^2} \frac{1}{(\xi_1 + \xi_2)^2} \sin^2 \Big[t(\alpha(c, \lambda - \xi_2/t) - \alpha(c, \lambda + \xi_1/t)) \Big]. \tag{5.16}$$

Let $b(\lambda) = -\alpha(c, \lambda)$, then

$$b'(\lambda) = \arccos\left(\frac{1 + \lambda - c}{2\sqrt{\lambda}}\right) \in (0, \pi), \quad \text{for } (1 - \sqrt{c})^2 < \lambda < (1 + \sqrt{c})^2. \tag{5.17}$$

By Lemma 5.2 below, for large t the leading term in the sum is identical to the one where $\sin^2(\cdots)$ is replaced by its mean, i.e., $1/2$. Thus

$$\begin{aligned} (5.16) &= (1 + \mathcal{O}(\varepsilon_1, (\varepsilon_1^2 \sqrt{t})^{-1}) \sum_{\xi_1=1}^{[\varepsilon_1 t]} \sum_{\xi_2=0}^{\xi_1 - 1} \frac{1}{2\pi^2} \frac{1}{(\xi_1 + \xi_2)^2} \\ &= \frac{1}{2\pi^2} \ln(\varepsilon_1 t)(1 + \mathcal{O}(\varepsilon_1, (\varepsilon_1^2 \sqrt{t})^{-1})). \end{aligned} \tag{5.18}$$

Thus,

$$M_t = \ln(\varepsilon_1 t) \left(\frac{1}{2\pi^2} + \mathcal{O}(t^{-1/2}, \varepsilon_1, (\varepsilon_1^2 \sqrt{t})^{-1}) \right). \tag{5.19}$$

Now we choose $\varepsilon_1 = 1/\ln(t)$. Then,

$$\mathrm{Var}(h([(\lambda - c)t], [ct], t)) = \frac{1}{2\pi^2} \ln(t) + \mathcal{O}\big(1, \ln(\ln(t)), (\ln(t))^3/\sqrt{t}\big), \tag{5.20}$$

from which it follows (5.2). $\square$

Lemma 5.2. *Let $b(x)$ be a smooth function (C^2 is enough) with $b'(0) \in (0, \pi)$. Then*

$$\sum_{\xi_1=1}^{[\varepsilon t]} \sum_{\xi_2=0}^{\xi_1 - 1} \frac{\sin^2 [tb(\xi_1/t) - tb(-\xi_2/t)]}{(\xi_1 + \xi_2)^2} = \sum_{\xi_1=1}^{[\varepsilon t]} \sum_{\xi_2=0}^{[\varepsilon t] - 1} \frac{1}{2(\xi_1 + \xi_2)^2} \left(1 + \mathcal{O}\Big(\varepsilon; \frac{1}{\varepsilon^2 \sqrt{t}}\Big) \right). \tag{5.21}$$

uniformly for $\varepsilon > 0$ small enough.

Proof of Lemma 5.2. We divide the sum into two regions.

$$\begin{aligned} I_1 &= \{\xi_1 \geq 1, \xi_2 \geq 0 | 1 \leq \xi_1 + \xi_2 \leq \varepsilon\sqrt{t}\}, \\ I_2 &= \{\xi_1 \geq 1, \xi_2 \geq 0 | \varepsilon\sqrt{t} < \xi_1 + \xi_2 \leq \varepsilon t\}. \end{aligned} \tag{5.22}$$

Let us evaluate the contribution to (5.21) of $(\xi_1, \xi_2) \in I_1$. We set $z = \xi_1 + \xi_2$ and get

$$\sum_{z=1}^{[\varepsilon\sqrt{t}]} \sum_{\xi_1=1}^{z} \frac{1}{z^2} \sin^2 \left[tb(\xi_1/t) - tb((\xi_1 - z)/t)\right]. \tag{5.23}$$

Series expansion around zero leads to

$$tb(\xi_1/t) - tb((\xi_1 - z)/t) = zb'(0) + \mathcal{O}(\varepsilon^2). \tag{5.24}$$

Thus

$$(5.23) = \sum_{z}^{[\varepsilon\sqrt{t}]} \frac{1}{z} \left(\sin^2 [zb'(0)] + \mathcal{O}(\varepsilon^2)\right). \tag{5.25}$$

The sum with the sinus square can be explicitly obtained,

$$\sum_{z=1}^{M} \frac{\sin^2(\sigma z)}{z} = \frac{1}{2} \ln(M) + \mathcal{O}(1), \qquad \text{as } M \to \infty \tag{5.26}$$

provided $0 < \sigma < \pi$. Since $\sum_{z=1}^{M} 1/z = \ln(M)/2 + \mathcal{O}(1/M)$, we have

$$\sum_{z=1}^{M} \frac{\sin^2(\sigma z)}{z} = \sum_{z=1}^{M} \frac{1}{2z} \left(1 + \mathcal{O}(1/M)\right). \tag{5.27}$$

Using $M = [\varepsilon\sqrt{t}]$ and going back to the original variables (ξ_1, ξ_2) we have

$$\sum_{(\xi_1,\xi_2)\in I_1} \frac{\sin^2 [tb(\xi_1/t) - tb(-\xi_2/t)]}{(\xi_1 + \xi_2)^2} = \sum_{(\xi_1,\xi_2)\in I_1} \frac{1}{2(\xi_1 + \xi_2)^2} \left(1 + \mathcal{O}\Big(\frac{1}{\varepsilon\sqrt{t}}, \varepsilon^2\Big)\right). \tag{5.28}$$

Now we evaluate the contribution to (5.21) of $(\xi_1, \xi_2) \in I_2$. Let $(X, Y) \in I_2$, then we have $X + Y \geq \varepsilon\sqrt{t}$. We consider a neighborhood of size $M = [\varepsilon^2\sqrt{t}]$ around (X, Y), namely the contribution

$$\sum_{x,y=0}^{M} \frac{1}{(X + Y + x + y)^2} \sin^2 \left[tb((X + x)/t) - tb(-(Y + y)/t)\right]. \tag{5.29}$$

Since $\sin^2(\cdots) \geq 0$ and $\frac{1}{(X+Y)^2} - \frac{1}{(X+Y+x+y)^2} \geq 0$, if we replace $\frac{1}{(X+Y+x+y)^2}$ by $\frac{1}{(X+Y)^2}$ in (5.29) the error made is bounded by

$$\begin{aligned} &\sum_{x,y=0}^{M} \left(\frac{1}{(X + Y)^2} - \frac{1}{(X + Y + x + y)^2}\right) \qquad (5.30) \\ = &\sum_{x,y=0}^{M} \frac{1}{(X + Y)^2} \left(1 - \frac{1}{(1 + \mathcal{O}(\varepsilon))^2}\right) = \sum_{x,y=0}^{M} \frac{1}{(X + Y)^2} \mathcal{O}(\varepsilon). \end{aligned}$$

because $(x+y)/(X+Y) \leq 2\varepsilon$. This relation can be inverted and we also get

$$\sum_{x,y=0}^{M} \frac{1}{(X+Y)^2} = \sum_{x,y=0}^{M} \frac{1}{(X+Y+x+y)^2} \left(1+\mathcal{O}(\varepsilon)\right). \tag{5.31}$$

Therefore we have

$$(5.29) = \sum_{x,y=0}^{M} \frac{\mathcal{O}(\varepsilon)}{(X+Y+x+y)^2} + \sum_{x,y=0}^{M} \frac{\sin^2\left[tb((X+x)/t) - tb(-(Y+y)/t)\right]}{(X+Y)^2}. \tag{5.32}$$

Now we apply series expansion to the argument in the sinus square. Denote by $\kappa_1 = tb(X/t) - tb(-Y/t)$, $\theta_1 = b'(X/t)$ and $\theta_2 = b'(-Y/t)$. Then the argument in the $\sin^2(\cdots)$ is $\kappa_1 + \theta_1 x + \theta_2 y + \mathcal{O}(\varepsilon^2)$. The ε^2 error term is smaller than the $\mathcal{O}(\varepsilon)$ in (5.32), thus

$$(5.32) = \sum_{x,y=0}^{M} \frac{\mathcal{O}(\varepsilon)}{(X+Y+x+y)^2} + \sum_{x,y=0}^{M} \frac{\sin^2\left[\kappa_1 + \theta_1 x + \theta_2 y\right]}{(X+Y)^2}. \tag{5.33}$$

Since b is smooth and $b'(0) \in (0,\pi)$, also in a neighborhood of 0, $b' \in (0,\pi)$. Thus, for ε small enough, $0 < \theta_1, \theta_2 < \pi$ uniformly in t, because $|Y|/t \leq \varepsilon$ and $|X|/t \leq \varepsilon$. Also this sum can be carried out explicitly. For $0 < \theta_1, \theta_2 < \pi$ we have the identity

$$\begin{aligned} &\sum_{x,y=0}^{M} \sin^2\left[\kappa_1 + \theta_1 x + \theta_2 y\right] \\ =\ & \frac{(M+1)^2}{2} - \frac{\cos(2\kappa_1 + \theta_1 M + \theta_2 M)\sin(\theta_1(M+1))\sin(\theta_2(M+1))}{2\sin(\theta_1)\sin(\theta_2)} \\ =\ & \sum_{x,y=0}^{M} \frac{1}{2}(1+\mathcal{O}(1/M^2)). \end{aligned} \tag{5.34}$$

We replace (5.34) into (5.33) and finally obtain

$$\begin{aligned} &\sum_{x,y=0}^{M} \frac{\sin^2\left[tb((X+x)/t) - tb(-(Y+y)/t)\right]}{(X+Y+x+y)^2} \\ =\ & \sum_{x,y=0}^{M} \frac{1}{2(X+Y+x+y)^2}\left(1+\mathcal{O}(\varepsilon,(\varepsilon^4 t)^{-1})\right). \end{aligned} \tag{5.35}$$

This estimate holds for all the region I_2, thus

$$\sum_{(\xi_1,\xi_2)\in I_2} \frac{\sin^2\left[tb(\xi_1/t) - tb(-\xi_2/t)\right]}{(\xi_1+\xi_2)^2} = \sum_{(\xi_1,\xi_2)\in I_2} \frac{1}{2(\xi_1+\xi_2)^2}\left(1+\mathcal{O}(\varepsilon,(\varepsilon^4 t)^{-1})\right). \tag{5.36}$$

The estimates of (5.28) and (5.36) imply the statement of the Lemma. $\square$

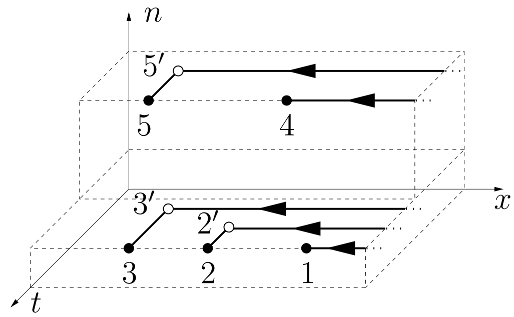

Figure 6.1: Example of how the integration over time is actually used.

6 Correlations along space-like paths

6.1 Height differences as time integration of fluxes

In this section we will prove Theorem 1.2. To determine the height function at a position (m, n) at a given time t, one can act in several ways. Let $\eta(x, n, t)$ be the point process at (n, t) and site x given by the occupancy of particles. Then,

$$h(m, n, t) = \sum_{x>m} \eta(x, n, t). \tag{6.1}$$

This formula is however not very practical when dealing with joint distributions of height functions at different points $(m_1, n, t), \ldots, (m_K, n, t)$, because then the height functions are linear functions but not of disjoint regions of the point process η. The consequence are tedious formulas. However, the height function can be also computed as follows:

$$h(m, n, t) = h(m, n, t') + J_{t',t}(m, n), \tag{6.2}$$

where $J_{t',t}(m, n)$ is the number of particles which jumped from site (m, n) to site $(m + 1, n)$ from time t' to t. Therefore, the expression

$$\mathbb{E}\bigg(\prod_{k=1}^{N} [h(m_k, n_k, t_k) - \mathbb{E}(h(m_k, n_k, t_k))] \bigg) \tag{6.3}$$

can be expressed as a sum of terms of the form

$$\mathbb{E}\bigg(\prod_{k=1}^{M} [h(m_k, n_k, t_k) - \mathbb{E}(h(m_k, n_k, t_k))] \prod_{j=M+1}^{N} \Big[J_{t'_j,t_j}(m_j, n_j) - \mathbb{E}(J_{t'_j,t_j}(m_j, n_j)) \Big] \bigg). \tag{6.4}$$

In the example of Figure 6.1, the coordinates of the points are:

$$\begin{array}{lll} 1 \equiv (m_1, n_1, t_1), & 2 \equiv (m_2, n_2 = n_1, t_2 = t_1), & 2' \equiv (m_2, n_2, t'_2), \\ 3 \equiv (m_3, n_3 = n_1, t_3 = t_1), & 3' \equiv (m_3, n_3, t'_3), & \\ 4 \equiv (m_4, n_4, t_4), & 5 \equiv (m_5, n_5 = n_4, t_5 = t_4), & 5' \equiv (m_5, n_5, t'_5). \end{array}$$

We now derive a formula for (6.4).

Lemma 6.1. *Denote the paths* $\pi_k = \{(x,n_k,t_k)|\, x>m_k\}$ *and* $\tilde\pi_j = \{(m_j,n_j,t)|\, t\in [t_j',t_j]\}$. *Assume that these paths do not intersect. Also, assume that* $\{(n_k,t_k), k=1,\ldots,M\}$ *together with* $\{(n_j,t_j),(n_j,t_j'), j=M+1,\ldots,N\}$ *are space-like. Then,*

$$\begin{aligned}
&\mathbb{E}\bigg(\prod_{k=1}^{M}[h(m_k,n_k,t_k)-\mathbb{E}(h(m_k,n_k,t_k))]\\
&\quad\times\prod_{j=M+1}^{N}\Big[J_{t_j',t_j}(m_j,n_j)-\mathbb{E}(J_{t_j',t_j}(m_j,n_j))\Big]\bigg) \qquad (6.5)\\
&=\sum_{x_1>m_1}\cdots\sum_{x_M>m_M}\int_{t'_{M+1}}^{t_{M+1}}\mathrm{d}s_{M+1}\cdots\int_{t'_N}^{t_N}\mathrm{d}s_N\det\begin{bmatrix}A_{1,1} & A_{1,2}\\ A_{2,1} & A_{2,2}\end{bmatrix},
\end{aligned}$$

with the matrix blocks $A_{i,j}$ *as follows:*

$$\begin{aligned}
A_{1,1}&=[(1-\delta_{i,j})K(x_i,n_i,t_i;x_j,n_j,t_j)]_{1\le i,j\le M},\\
A_{1,2}&=\big[-\partial_{s_j}K(x_i,n_i,t_i;m_j,n_j,s_j)\big]_{1\le i\le M,\,M+1\le j\le N},\\
A_{2,1}&=[K(m_i,n_i,s_i;x_j,n_j,t_j)]_{M+1\le i\le N,\,1\le j\le N},\\
A_{2,2}&=\big[-(1-\delta_{i,j})\partial_{s_j}K(m_i,n_i,s_i;m_j,n_j,s_j)\big]_{M+1\le i,j\le N}.
\end{aligned}\qquad (6.6)$$

Proof of Lemma 6.1. Below we prove that

$$\mathbb{E}\bigg(\prod_{k=1}^{M}h(m_k,n_k,t_k)\prod_{j=M+1}^{N}J_{t_j',t_j}(m_j,n_j)\bigg) \qquad (6.7)$$

is equal to (6.5) but without the $1-\delta_{i,j}$ terms. The fact that the subtraction of the averages is given by putting zeros on the diagonal is a simple but important property, which was already noticed for example in [28] (see proof of Theorem 7.2).

For $M=N$, (6.7) is true because for space-like paths η is a determinantal point process. The flux of particle can be written as

$$J_{t',t}(m,n)=\lim_{D\to\infty}\sum_{\ell=1}^{D}\eta(m,n,\tau_{i-1})(1-\eta(m,n,\tau_i)) \qquad (6.8)$$

with $\tau_i=t'+i\Delta\tau$, $i=0,\ldots,D$, $\Delta\tau=(t-t')/D$. The quantity $\eta(m,n,\tau_{i-1})(1-\eta(m,n,\tau_i))$ is one only if site (m,n) was occupied at time τ_{i-1} and empty at time τ_i. Each particle try to jump independently with an exponentially waiting time. Every time a particle try to move, if succeeds it can also push other particles. Anyway, since there is a finite number of particles, the probability that a particle has more than one jump during $\Delta\tau$ is of order $\Delta\tau^2$. Thus, the limit $\Delta\tau\to 0$ is straightforward.

To obtain (6.7) we have to determine expression at first order in $\Delta\tau$ of

$$\mathbb{E}\left(\eta(m,n,\tau_{i-1})(1-\eta(m,n,\tau_i))\prod_{j=1}^{Q}\eta(m_j,n_j,t_j)\right). \qquad (6.9)$$

Then, in the $\Delta\tau \to 0$ limit we will get an integral from t' to t.

Denote by $\overline{K}_{x,n}(t_1;t_2) = \sum_{k=0}^{n-1}\Psi_k^{n,t_1}(x)\Phi_k^{n,t_2}(x)$. Remark that in (4.8), $\phi^{((n,\tau_i),(n,\tau_{i-1}))}(x,x)=1$. Then, since $\tau_i > \tau_{i-1}$, from (4.8) we obtain

$$(6.9) = \det\begin{bmatrix} \overline{K}_{m,n}(\tau_{i-1};\tau_{i-1}) & \overline{K}_{m,n}(\tau_{i-1};\tau_i) & K(m,n,\tau_{i-1};q) \\ 1-\overline{K}_{m,n}(\tau_i;\tau_{i-1}) & 1-\overline{K}_{m,n}(\tau_i;\tau_i) & -K(m,n,\tau_i,m;q) \\ K(q;m,n,\tau_{i-1}) & K(q;m,n,\tau_i) & K(q,q) \end{bmatrix} \tag{6.10}$$

where with q we denoted the triples (m_j,n_j,t_j), for $j\in\{1,\ldots,Q\}$. The second line is just one in the diagonal minus the entries of the kernel. Written in terms of $\overline{K}$ it becomes as above, since the $(2,1)$ entry has a 1 coming from ϕ. Next we do two operations keeping the determinant invariant:

$$\begin{aligned} \text{Second row} &\to \text{Second row} + \text{First row} \\ \text{Second column} &\to \text{Second column} - \text{First column}. \end{aligned}$$

We get that (6.10) is equal to

$$\begin{aligned} &\det\begin{bmatrix} \overline{K}_{m,n}(\tau_{i-1};\tau_{i-1}) & \Delta\tau\partial_2\overline{K}_{m,n}(\tau_{i-1};\tau_{i-1}) & K(m,n,\tau_{i-1};q) \\ 1-\mathcal{O}(\Delta\tau) & \mathcal{O}(\Delta\tau^2) & \mathcal{O}(\Delta\tau) \\ K(q;m,n,\tau_{i-1}) & \Delta\tau\partial_2 K(q;m,n,\tau_{i-1}) & K(q,q) \end{bmatrix} \\ = \; &-\Delta\tau\det\begin{bmatrix} \partial_2\overline{K}_{m,n}(\tau_{i-1};\tau_{i-1}) & K(m,n,\tau_{i-1};q) \\ \partial_2 K(q;m,n,\tau_{i-1}) & K(q,q) \end{bmatrix} + \mathcal{O}(\Delta\tau^2) \end{aligned} \tag{6.11}$$

where ∂_2 means the derivative with respect to τ_{i-1} in the second entry of the kernel. This formula and (6.8) imply

$$\begin{aligned} &\mathbb{E}\left(J_{t',t}(m,n)\prod_{j=1}^{Q}\eta(m_j,n_j,t_j)\right) \qquad (6.12)\\ = &\int_{t'}^{t}\mathrm{d}s\det\begin{bmatrix} -\partial_2 K(m,n,s;m,n,s) & K(m,n,s;m_j,n_j,t_j) \\ -\partial_2 K(m_i,n_i,t_i;m,n,s) & K(m_i,n_i,t_i;m_j,n_j,t_j) \end{bmatrix}_{1\le i,j\le Q}. \end{aligned}$$

The case of several factors J is obtained by induction. $\square$

6.2 Proof of Theorem 1.3

In the proof of the theorem, we take advantage of the above trick. We also use the asymptotics of the kernel, which is contained in Section 7.

Proof of Theorem 1.3. First we want to prove that we have a Gaussian process and then we will compute the covariance. Consider

$$\mathbb{E}\Big(\prod_{k=1}^{N}[h(m_k,n_k,t_k)-\mathbb{E}(h(m_k,n_k,t_k))]\Big) \tag{6.13}$$

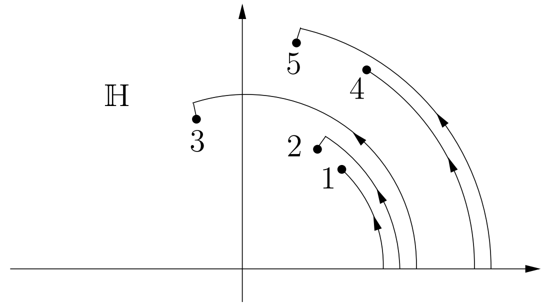

Figure 6.2: Complex plane mapping of the paths of Figure 6.1.

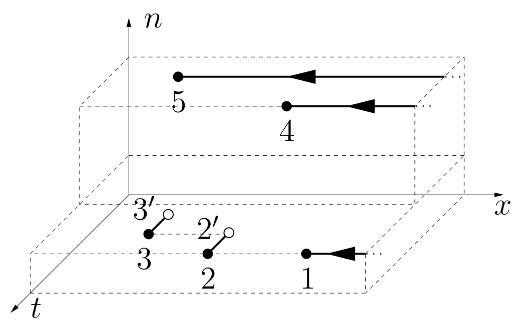

Figure 6.3: A typical configuration of paths which comes from the decomposition of the paths in Figure 6.1.

and determine its limit as $L \to \infty$ under the macroscopic scaling: $t_k = [\tau_k L]$, $n_k = [\eta_k L]$, $m_k = [(\nu_k - \eta_k)L]$, with $\nu_k \in ((\sqrt{\eta_k} - \sqrt{\tau_k})^2, (\sqrt{\eta_k} + \sqrt{\tau_k})^2)$. (6.13) is given as linear combinations of expressions in Lemma 6.1. The summations are of the type of Figure 6.3.

Let $\mathcal{S}_N$ be the permutation group of $\{1, \dots, N\}$. In the case $M = N$, the determinant in (6.5) writes

$$\sum_{\sigma \in \mathcal{S}_N} (-1)^{|\sigma|} (1 - \delta_{i,\sigma_i}) \prod_{i=1}^{N} K(x_i, n_i, t_i; x_{\sigma_i}, n_{\sigma_i}, t_{\sigma_i}). \tag{6.14}$$

Similarly, for $M < N$, but where some of the K replaced by $\partial_{s_{\sigma_j}} K$.

The contribution of all permutations with fixed points is zero (because the diagonal matrix elements are zeroes). All other permutations can be written as unions of several cycles of length $\ell \geq 2$. The contributions of the permutations only with cycles of length 2 is the final result, i.e., to prove the Theorem we need to show that the sum of the contributions of cycles of length $\ell \geq 3$ is zero in the $L \to \infty$ limit.

Consider all cycles of length $\ell \geq 3$ and use the indices $1, \dots, \ell$ to the corresponding points t_i, n_i, m_i. Let us order them so that

$$\eta_1 \geq \eta_2 \geq \dots \geq \eta_\ell, \quad \tau_1 \leq \tau_2 \leq \dots \leq \tau_\ell, \quad \text{no double points,} \tag{6.15}$$

i.e., $(\eta_j,\tau_j)\prec(\eta_{j-1},\tau_{j-1})$ (as in Figure 6.3).

The signature is constant for cycles of same length. So, for a ℓ-cycle of $\mathcal{S}_\ell$ we need to take the product of the kernels (or their time derivatives depending on the case), and do the summation over $y_1\leq\nu_1 L$ or the integration over $[\tau_i' L,\tau_i L]$ depending on whether in (6.5) we have an integral or a sum. We will need to take in account the signature just when computing the covariance (a 2-cycle).

We first collect all the factors related with the point with index i. There are two possible cases:

(a) The point i is related with a summation variable. Then we have to analyze:

$$\sum_{x>[\nu_i L]} K(x,\eta_i L,\tau_i L;x_{\sigma_i},n_{\sigma_i},t_{\sigma_i})K(x_{\sigma_i^{-1}},n_{\sigma_i^{-1}},t_{\sigma_i^{-1}};x,\eta_i L,\tau_i L); \tag{6.16}$$

(b) The point i is related with an integrated variable. We have in this case

$$\int_{\tau_i' L}^{\tau_i L} \mathrm{d}t K(\nu_i L,\eta_i L,t;x_{\sigma_i},n_{\sigma_i},t_{\sigma_i})K(x_{\sigma_i^{-1}},n_{\sigma_i^{-1}},t_{\sigma_i^{-1}};\nu_i L+1,\eta_i L,t). \tag{6.17}$$

We analyze these two expression in the $L\to\infty$ limit using results of Section 7.3.

First of all, since w_c-z_c remains bounded away from zero all along the integrals/sums, the bounds of Section 7.3 imply that the sum of all contributions of the error term $\mathcal{O}(L^{-5/12})$ in (7.42) is of the same order, namely $\mathcal{O}(L^{-5/12})$. Therefore we can get rid of it immediately. We first consider case (a) and divide the sum in three parts for which we use Propositions 7.11-7.15. Let

$$\begin{aligned} I_1 &= \{x\in\mathbb{N}, x\geq(\sqrt{\tau_i}+\sqrt{\eta_i})^2-\ell L^{1/3}\},\\ I_2 &= \{x\in\mathbb{N},(\sqrt{\tau_i}+\sqrt{\eta_i})^2-L^{2/3}<x<(\sqrt{\tau_i}+\sqrt{\eta_i})^2-\ell L^{1/3}\},\\ I_3 &= \{x\in\mathbb{N},[\nu_i L]<x\leq(\sqrt{\tau_i}+\sqrt{\eta_i})^2-L^{2/3}\}. \end{aligned} \tag{6.18}$$

Then, by Propositions 7.14-7.15,

$$\begin{aligned} &\left|\sum_{x\in I_1} K(x,\eta_i L,\tau_i L;x_{\sigma_i},n_{\sigma_i},t_{\sigma_i})K(x_{\sigma_i^{-1}},n_{\sigma_i^{-1}},t_{\sigma_i^{-1}};x,\eta_i L,\tau_i L)\right|\\ \leq&\sum_{x\in I_1}\frac{\mathrm{const}}{L^{2/3}}\exp\left(-2\frac{x-(\sqrt{\tau_i}+\sqrt{\eta_i})^2L}{(\tau_i L)^{1/3}}\right)\times\text{terms in }\sigma_i,\sigma_i^{-1}\\ \leq&\frac{\mathrm{const}}{L^{1/3}}\times\text{terms in }\sigma_i,\sigma_i^{-1}. \end{aligned} \tag{6.19}$$

Therefore, as $L\to\infty$, the contribution of this part of the facet/edge goes to zero. By Proposition 7.13,

$$\begin{aligned} &\left|\sum_{x\in I_2} K(x,\eta_i L,\tau_i L;x_{\sigma_i},n_{\sigma_i},t_{\sigma_i})K(x_{\sigma_i^{-1}},n_{\sigma_i^{-1}},t_{\sigma_i^{-1}};x,\eta_i L,\tau_i L)\right|\\ \leq&\sum_{x\in I_2}\frac{\mathrm{const}}{L\sqrt{\eta_i\tau_i-\frac14(\tau_i+\eta_i-x/L)^2}}\times\text{terms in }\sigma_i,\sigma_i^{-1}\\ \leq&\frac{\mathrm{const}}{L^{1/6}}\times\text{terms in }\sigma_i,\sigma_i^{-1}. \end{aligned} \tag{6.20}$$

Therefore, as $L \to \infty$, this contribution also vanishes.

Finally, we need to compute the sum over I_3. Define the function

$$A(\nu, \eta, \tau) = \frac{1}{2\pi B(\nu, \eta, \tau)^2 \sqrt{\nu/\tau}} \tag{6.21}$$

with B given in (7.44). Then, by Proposition 7.11 we have

$$\begin{aligned}
&\sum_{x\in I_3} K(x, \eta_i L, \tau_i L; x_{\sigma_i}, n_{\sigma_i}, t_{\sigma_i}) K(x_{\sigma_i^{-1}}, n_{\sigma_i^{-1}}, t_{\sigma_i^{-1}}; x, \eta_i L, \tau_i L) \\
= \quad &\sum_{x\in I_3} \frac{A(x/L, \eta_i, \tau_i)}{L} \bigg[\frac{e^{-\mathrm{i}\beta_1(i)}}{\omega(\sigma_i) - \omega(i)} \frac{e^{\mathrm{i}\beta_2(i)}}{\omega(i) - \omega(\sigma_i^{-1})} \\
&+ \frac{e^{-\mathrm{i}\beta_1(i)}}{\omega(\sigma_i) - \omega(i)} \frac{e^{-\mathrm{i}\beta_2(i)}}{\bar\omega(i) - \omega(\sigma_i^{-1})} e^{-2\mathrm{i}F(x/L, \eta_i, \tau_i)} \\
&+ \frac{e^{\mathrm{i}\beta_1(i)}}{\omega(\sigma_i) - \bar\omega(i)} \frac{e^{\mathrm{i}\beta_2(i)}}{\omega(i) - \omega(\sigma_i^{-1})} e^{2\mathrm{i}F(x/L, \eta_i, \tau_i)} \\
&+ \frac{e^{\mathrm{i}\beta_1(i)}}{\omega(\sigma_i) - \bar\omega(i)} \frac{e^{-\mathrm{i}\beta_2(i)}}{\bar\omega(i) - \omega(\sigma_i^{-1})} + \circlearrowleft \bigg] \times \text{terms in } \sigma_i, \sigma_i^{-1}
\end{aligned} \tag{6.22}$$

where we used the notation $\omega(i) = \Omega(\nu_i, \eta_i, \tau_i)$ and $\circlearrowleft$ means the other 12 terms obtained by replacing $\omega(\sigma_i)$ by $\bar\omega(\sigma_i)$ and/or $\omega(\sigma_i^{-1})$ by $\bar\omega(\sigma_i^{-1})$.

First we want to show that the terms with F in the exponential are irrelevant in the $L \to \infty$ limit. For that, we sum over $N = L^{1/3}$ positions around any νL in the bulk. Then, for $0 \leq x \leq L^{1/3}$ it holds

$$F(\nu + x/L, \eta, \tau) = L\gamma(\nu, \eta, \tau) + x\partial_\nu \gamma(\nu, \eta, \tau) + \mathcal{O}(L^{-1/3}). \tag{6.23}$$

All the other functions (A, β_1, β_2, and $\omega(i)$) are smooth functions in ν_i, i.e., over an interval $L^{1/3}$ vary only by $\sim L^{-2/3}$. Then, for $0 < b < \pi$, we use

$$\frac{1}{N} \sum_{x=0}^{N-1} e^{\mathrm{i}bx} = \frac{e^{\mathrm{i}bN} - 1}{N(e^{\mathrm{i}b} - 1)}. \tag{6.24}$$

In our case, b is strictly between 0 and π as soon as we are away from the facet. When we reach the lower facet, $b \to 0$. However, in the sum over I_3 we are at least at a distance $L^{2/3}$ from the facet, i.e., $b \geq \text{const}\, L^{-1/6}$. Therefore

$$|(6.24)| \leq \text{const}\,/(bN) \leq L^{-1/6}. \tag{6.25}$$

Since this holds uniformly in the domain I_3, we have shown that the contribution of the terms where the $\exp(\pm 2\mathrm{i}F)$ is present is at worst of order $L^{-1/6}$. Therefore the only non-vanishing terms in (6.22) are

$$\begin{aligned}
&\sum_{x\in I_3} \frac{A(x/L, \eta_i, \tau_i)}{L} \bigg[\frac{e^{-\mathrm{i}\beta_1(i)}}{\omega(\sigma_i) - \omega(i)} \frac{e^{\mathrm{i}\beta_2(i)}}{\omega(i) - \omega(\sigma_i^{-1})} \\
&+ \frac{e^{\mathrm{i}\beta_1(i)}}{\omega(\sigma_i) - \bar\omega(i)} \frac{e^{-\mathrm{i}\beta_2(i)}}{\bar\omega(i) - \omega(\sigma_i^{-1})} + \circlearrowleft \bigg] \times \text{terms in } \sigma_i, \sigma_i^{-1}.
\end{aligned} \tag{6.26}$$

The last step is a change of variable. All the functions appearing now are smooth and changing over distances $x \sim L$. Thus, defining $x = \nu L$, the sum becomes, up to an error of order $\mathcal{O}(L^{-1/3})$, the integral

$$\begin{aligned}&\int_{\nu_i}^{(\sqrt{\tau_i}+\sqrt{\eta_i})^2} \mathrm{d}\nu A(\nu,\eta_i,\tau_i)\bigg[\frac{e^{-\mathrm{i}\beta_1(i)}}{\omega(\sigma_i)-\omega(i)}\frac{e^{\mathrm{i}\beta_2(i)}}{\omega(i)-\omega(\sigma_i^{-1})}\\ &+\frac{e^{\mathrm{i}\beta_1(i)}}{\omega(\sigma_i)-\bar{\omega}(i)}\frac{e^{-\mathrm{i}\beta_2(i)}}{\bar{\omega}(i)-\omega(\sigma_i^{-1})}+\circlearrowleft\bigg]\times \text{terms in } \sigma_i,\sigma_i^{-1}.\end{aligned} \tag{6.27}$$

The final step is a change of variable. For the term with $\omega(i)$, we set $z_i^+ = \omega(i) = \Omega(\nu,\eta_i,\tau_i)$. Denote the new integration path by $\Gamma_+^i = \{\Omega(\nu,\eta_i,\tau_i), \nu : (\sqrt{\tau_i}+\sqrt{\eta_i})^2 \to \nu_i\}$. The Jacobian was computed in Proposition 3.8, namely

$$\frac{\partial\omega(i)}{\partial\nu} = \frac{\mathrm{i}\omega(i)}{\kappa} = 2\pi\mathrm{i}A\, e^{\mathrm{i}\beta_2(i)}e^{-\mathrm{i}\beta_1(i)}. \tag{6.28}$$

For the term with $\bar{\omega}(i)$ we set $z_i^- = \bar{\omega}(i) = \overline{\Omega}(\nu,\eta_i,\tau_i)$ and $\Gamma_-^i = \bar{\Gamma}_+^i$. Then (6.27) becomes

$$\frac{-1}{2\pi\mathrm{i}}\sum_{\varepsilon_i=\pm}\varepsilon_i\int_{\Gamma_{\varepsilon_i}^i}\mathrm{d}z_{\varepsilon_i}^i\bigg[\frac{1}{z_{\varepsilon_i}^i-\omega(\sigma_i)}\frac{1}{\omega(\sigma_i^{-1})-z_{\varepsilon_i}^i}+\circlearrowleft\bigg]\times \text{terms in } \sigma_i,\sigma_i^{-1}. \tag{6.29}$$

The factor -1 comes from the orientation of $\Gamma_{\varepsilon_i}^i$, see Figure 6.2.

The second case to be considered is (b), namely when we do an integration over a time interval. This time we do not have to deal with the edges, since, by assumption, we remain in the bulk of the system. We need to compute

$$\begin{aligned}&\int_{\tau_i' L}^{\tau_i L}\mathrm{d}t K(\nu_i L,\eta_i L,t;x_{\sigma_i},n_{\sigma_i},t_{\sigma_i})K(x_{\sigma_i^{-1}},n_{\sigma_i^{-1}},t_{\sigma_i^{-1}};\nu_i L+1,\eta_i L,t)\\ =\;&\int_{\tau_i'}^{\tau_i}\mathrm{d}\tau A(\nu_i,\eta_i,\tau)\bigg[\frac{e^{-\mathrm{i}\beta_1(i)}}{\omega(\sigma_i)-\omega(i)}\frac{e^{\mathrm{i}\beta_2(i)}}{\omega(i)-\omega(\sigma_i^{-1})}(1-\omega(i))\\ &+\frac{e^{-\mathrm{i}\beta_1(i)}}{\omega(\sigma_i)-\omega(i)}\frac{e^{-\mathrm{i}\beta_2(i)}}{\bar{\omega}(i)-\omega(\sigma_i^{-1})}(1-\bar{\omega}(i))e^{-2\mathrm{i}F(\nu_i,\eta_i,\tau)}\\ &+\frac{e^{\mathrm{i}\beta_1(i)}}{\omega(\sigma_i)-\bar{\omega}(i)}\frac{e^{\mathrm{i}\beta_2(i)}}{\omega(i)-\omega(\sigma_i^{-1})}(1-\omega(i))e^{2\mathrm{i}F(\nu_i,\eta_i,\tau)}\\ &+\frac{e^{\mathrm{i}\beta_1(i)}}{\omega(\sigma_i)-\bar{\omega}(i)}\frac{e^{-\mathrm{i}\beta_2(i)}}{\bar{\omega}(i)-\omega(\sigma_i^{-1})}(1-\bar{\omega}(i))+\circlearrowleft\bigg]\times \text{terms in } \sigma_i,\sigma_i^{-1}.\end{aligned} \tag{6.30}$$

The only rapidly changing function is F, which, as for the sum, makes the contributions of the term with it vanishing small as $L\to\infty$. We do the same change of variable as above, i.e., $z_i^+ = \omega(i) = \Omega(\nu_i,\eta_i,\tau)$. Denote the new integration path by $\Gamma_+^i = \{\Omega(\nu_i,\eta_i,\tau), \tau \in [\tau_i',\tau_i]\}$. The Jacobian is computed in Proposition 3.8, namely

$$\frac{\partial\omega(i)}{\partial\tau} = \frac{-\mathrm{i}\omega(i)(1-\omega(i))}{\kappa} = -2\pi\mathrm{i}Ae^{\mathrm{i}\beta_2(i)}e^{-\mathrm{i}\beta_1(i)}(1-\omega(i)). \tag{6.31}$$

Thus, we obtain again (6.29).

So, after summing / integrating all the ℓ variables, we get the contribution of the ℓ-cycles, namely

$$
\begin{aligned}
& \frac{(-1)^\ell}{(2\pi \mathrm{i})^\ell} \sum_{\varepsilon_1,\dots,\varepsilon_\ell=\pm} \prod_{i=1}^{\ell} \varepsilon_i \int_{\Gamma^1_{\varepsilon_1}} \mathrm{d}z_1^{\varepsilon_1} \cdots \int_{\Gamma^\ell_{\varepsilon_\ell}} \mathrm{d}z_\ell^{\varepsilon_\ell} \prod_{i=1}^{\ell} \frac{1}{z_i^{\varepsilon_i} - z_{\sigma_i}^{\varepsilon_{\sigma_i}}}, \\
= \; & \frac{(-1)^\ell}{(2\pi \mathrm{i})^\ell} \sum_{\varepsilon_1,\dots,\varepsilon_\ell=\pm} \prod_{i=1}^{\ell} \varepsilon_i \int_{\Gamma^1_{\varepsilon_1}} \mathrm{d}z_1^{\varepsilon_1} \cdots \int_{\Gamma^\ell_{\varepsilon_\ell}} \mathrm{d}z_\ell^{\varepsilon_\ell} \prod_{i=1}^{\ell} \frac{1}{z_{\sigma_i}^{\varepsilon_{\sigma_i}} - z_{\sigma_{i-1}}^{\varepsilon_{\sigma_{i-1}}}},
\end{aligned} \tag{6.32}
$$

where we set $\sigma_0 := \sigma_\ell$. By Lemma 7.3 in [28], which refers back to [15],

$$
\sum_{\ell-\text{cycle in } \mathcal{S}_\ell} \prod_{i=1}^{\ell} \frac{1}{Y_{\sigma_i} - Y_{\sigma_{i-1}}} = 0, \qquad \text{for } \ell \geq 3. \tag{6.33}
$$

Therefore, the sum of (6.32) over the ℓ-cycles gives zero for $\ell \geq 3$.

We have shown that we have a Gaussian type formula (sum over all couplings) for points macroscopically away. We still need to compute explicitly the covariance for such points. The covariance is obtained by (6.32) for $\ell = 2$. We need now to consider the signature, which for a 2-cycle is -1. It is a sum of 4 terms which can be put together into

$$
\begin{aligned}
& \frac{1}{(2\pi \mathrm{i})^2} \int_{\bar{\Omega}(\nu_1,\eta_1,\tau_1)}^{\Omega(\nu_1,\eta_1,\tau_1)} \mathrm{d}z_1 \int_{\bar{\Omega}(\nu_2,\eta_2,\tau_2)}^{\Omega(\nu_2,\eta_2,\tau_2)} \mathrm{d}z_2 \frac{1}{(z_1 - z_2)^2} \qquad (6.34) \\
= \; & \frac{-1}{4\pi^2} \ln \left(\frac{(\Omega(\nu_1,\eta_1,\tau_1) - \Omega(\nu_2,\eta_2,\tau_2))(\overline{\Omega}(\nu_1,\eta_1,\tau_1) - \overline{\Omega}(\nu_2,\eta_2,\tau_2))}{(\overline{\Omega}(\nu_1,\eta_1,\tau_1) - \Omega(\nu_2,\eta_2,\tau_2))(\Omega(\nu_1,\eta_1,\tau_1) - \overline{\Omega}(\nu_2,\eta_2,\tau_2))} \right).
\end{aligned}
$$

□

We finally prove the short distance bound (1.18).

Lemma 6.2. *For any $\varkappa_j \in \mathcal{D}$ and any $\varepsilon > 0$, we have*

$$
\mathbb{E}\left(H_L(\varkappa_1) \cdots H_L(\varkappa_N)\right) = \mathcal{O}(L^\epsilon), \quad L \to \infty. \tag{6.35}
$$

Proof of Lemma 6.2. Theorem 1.2 implies, for any integer $m \geq 1$,

$$
\mathbb{E}(H_L(\varkappa_j)^{2m}) = \mathcal{O}(\ln(L)^m). \tag{6.36}
$$

By Chebyshev inequality,

$$
\mathbb{P}(|H_L(\varkappa_j)| \geq X \ln(L)) = \mathcal{O}(1/X^{2m}), \quad \mathbb{P}(|H_L(\varkappa_j)| \geq Y) = \mathcal{O}(\ln(L)^m / Y^{2m}). \tag{6.37}
$$

The final ingredient is that $|H_L(\varkappa_j)| \leq \mathcal{O}(L)$, since we have $\mathcal{O}(L)$ points. Therefore, for any Y, we can bound

$$
\begin{aligned}
|\mathbb{E}(H_L(\varkappa_1) \cdots H_L(\varkappa_N))| &\leq \mathbb{P}(|H_L(\varkappa_1)| \leq Y, \dots, |H_L(\varkappa_N)| \leq Y) Y^N \\
&+ \mathbb{P}(\exists j \text{ s.t. } |H_L(\varkappa_j)| > Y) \mathcal{O}(L)^N \\
&\leq \mathcal{O}(Y^N) + \mathcal{O}(L^N \ln(L)^m / Y^{2m}).
\end{aligned} \tag{6.38}
$$

Taking $Y = L^{\varepsilon/2}$ and $m \gg 1$ large enough, we obtain

$$|\mathbb{E}(H_L(\varkappa_1)\cdots H_L(\varkappa_N))| \leq \mathcal{O}(L^\varepsilon), \quad \text{for any given } \varepsilon > 0. \tag{6.39}$$

□

7 Asymptotics analysis

7.1 Asymptotics at the edge

Now we first determine the edge asymptotics of $I_{n,t}$. For the asymptotic analysis of $I_{n,t}$ at the edges we apply exactly the same strategy as in previous papers. Here we write down only the difference, namely the functions, their series expansion around the critical point and we also determine the steep descent paths used in the proof. The rest of the argument of the proofs of Lemma 7.1 and 7.3 below, is identical to the one of Propositions 15 and 17 in [10].

In the following, we will determine first the asymptotics for the upper edge, $x \simeq (1+\sqrt{c})^2$. Then, for completeness, we state the analogue result for the lower edge, $x \simeq (1-\sqrt{c})^2$ (although it will not be used).

Lemma 7.1. *Let $n = ct$ and $x = (1+\sqrt{c})^2 t + st^{1/3}$, for any $c > 0$. Then,*

$$\lim_{t\to\infty} t^{1/3} I_{n,t}(x) \frac{(-\sqrt{c})^n}{e^{-\sqrt{c}t}(1+\sqrt{c})^x} = \tilde{\kappa}_2 \mathrm{Ai}(\kappa_2 s), \tag{7.1}$$

uniformly for s in bounded sets, with $\kappa_2 = c^{1/6}(1+\sqrt{c})^{-2/3}$, and $\tilde{\kappa}_2 = (1+\sqrt{c})^{1/3} c^{-1/3}$.

Proof of Lemma 7.1. (7.1) is (up to a $-$ sign) equal to (5.15) in [10], with the following replacements. The critical point is $z_c = -\sqrt{c}$, and the functions f_0 to f_3 are

$$\begin{aligned} f_0(z) &= g(z) - g(z_c), \quad g(z) = z + (1+\sqrt{c})^2 \ln(1-z) - c\ln(z), \\ f_1(z) &= 0, \\ f_2(z) &= s\ln(1-z), \\ f_3(z) &= -\ln(z). \end{aligned} \tag{7.2}$$

Their series expansions around the critical point are

$$\begin{aligned} f_0(z) &= \tfrac{1}{3}\kappa_0 (z+\sqrt{c})^3 + \mathcal{O}((z+\sqrt{c})^4), \quad \kappa_0 = 1/(\sqrt{c}(1+\sqrt{c})), \\ f_2(z) &= f_2(-\sqrt{c}) - \frac{s}{1+\sqrt{c}}(z+\sqrt{c}) + \mathcal{O}((z+\sqrt{c})^2), \\ f_3(z) &= -\ln(-\sqrt{c}) + \mathcal{O}(z+\sqrt{c}). \end{aligned} \tag{7.3}$$

The steep descent path used in the analysis is made up pieces of the two following paths, γ_ρ and $\gamma_{\rm loc}$. $\gamma_\rho = \{-\rho e^{\mathrm{i}\phi}, \phi \in (-\pi,\pi]\}$. For $\rho \in (0,\sqrt{c}]$, γ_ρ is steep descent path for f_0. In fact, we get

$$\frac{\mathrm{dRe}(f_0(z=\rho e^{\mathrm{i}\phi}))}{\mathrm{d}\phi} = -\frac{\rho\sin\phi}{|1-z|^2}(c-\rho^2+2\sqrt{c}-2\rho\cos\phi). \tag{7.4}$$

The last term is minimal for $\phi = 0$, where his value becomes

$$(\sqrt{c}-\rho)(\rho+\sqrt{c}+2) \geq 0, \tag{7.5}$$

for $\rho \in (0,\sqrt{c}]$. γ_ρ is a steep descent path for f_0 because the value zero is attained only for $\rho = \sqrt{c}$ and, in that case, only at one point, $\phi = 0$. However, close to the critical point it is not optimal. Let us consider $\gamma_{\rm loc} = \{-\sqrt{c} + e^{-\pi\mathrm{i}/3\,\mathrm{sgn}(x)}|x|, x \in [0,\sqrt{c}/2]\}$. By symmetry, consider just $x \geq 0$, then we compute

$$\frac{\mathrm{dRe}(f_0(z=-\sqrt{c}+e^{-\pi\mathrm{i}/3}x))}{\mathrm{d}x} = -\frac{x^2 Q(x)}{|z|^2|1-z|^2}, \tag{7.6}$$

with $Q(x) = \sqrt{c}(1+\sqrt{c}) - x(1+x)/2 - \sqrt{c}x$. $Q(0) > 0$, and the computation of the (at most) two zeros of $Q(x)$ shows that none are in the interval $[0,\sqrt{c}/2]$. Thus $\gamma_{\rm loc}$ is also a steep descent path for f_0. Since this is the steepest descent path for f_0 around the critical point, we choose as path Γ_0 in $I_{n,t}(x)$ the one formed by $\gamma_{\rm loc}$ close to the critical point, until it intersect $\gamma_{\rho=\sqrt{3c/4}}$ and then we follow $\gamma_{\sqrt{3c/4}}$. The result of Lemma 7.1 follows then from the same argument as the proof of Proposition 15 in [10]. □

Lemma 7.2. *Let $n = ct$ with $0 < c < 1$ and $x = (1-\sqrt{c})^2 t - st^{1/3}$. Then,*

$$\lim_{t\to\infty} t^{1/3} I_{n,t}(x) \frac{c^{n/2}}{e^{\sqrt{c}t}(1-\sqrt{c})^x} = \tilde{\kappa}_1 \mathrm{Ai}(\kappa_1 s), \tag{7.7}$$

uniformly for s in bounded sets, with $\kappa_1 = c^{1/6}(1-\sqrt{c})^{-2/3}$, and $\tilde{\kappa}_1 = (1-\sqrt{c})^{1/3}c^{-1/3}$.

Proof of Lemma 7.2. The proof is analogue of Lemma 7.1. The only relevant difference is that instead of (7.5) we have

$$(\sqrt{c}-\rho)(2-\rho-\sqrt{c}) \geq 0 \tag{7.8}$$

for $\rho \in (0,\sqrt{c}]$, provided that $0 < c < 1$. This is not just a technical restriction, because for $c > 1$ the lower edge has a density going to one instead of zero, and the asymptotics is different. □

Lemma 7.3. *Fix an $\ell > 0$ and consider the scaling of Lemma 7.1. Then*

$$\left| t^{1/3} I_{n,t}(x) \frac{(-\sqrt{c})^n}{e^{-\sqrt{c}t}(1+\sqrt{c})^x} \right| \leq \mathrm{const}\, e^{-s}, \tag{7.9}$$

uniformly for $s \geq -\ell$ and where const *is a constant independent of t.*

Proof of Lemma 7.3. For $s \in [-\ell, 2\ell]$ it is a consequence of Lemma 7.1. For $s \geq 2\ell$, the proof is analogue the one of Proposition 17 in [10]. Here we just indicate the differences. In Lemma 7.1 we already showed that γ_ρ is steep descent path for f_0, for any $\rho \in (0, \sqrt{c}]$. Let $\tilde{s} = (s + 2\ell)t^{-2/3} > 0$ and $\tilde{f}_0(z) = f_0(z) + \tilde{s}\ln(1-z)$. Since $\tilde{s} > 0$, γ_ρ is also steep descent for $\tilde{s}\ln(1-z)$. We set then

$$\rho = \begin{cases} -\sqrt{c} + (\tilde{s}/\kappa_0)^{1/2}, & \text{if } 0 \leq \tilde{s} \leq \varepsilon, \\ -\sqrt{c} + (\varepsilon/\kappa_0)^{1/2}, & \text{if } \tilde{s} \geq \varepsilon. \end{cases} \tag{7.10}$$

with κ_0 in (7.3). $\square$

Lemma 7.4. *Fix an $\ell > 0$ and consider the scaling of Lemma 7.2. Then*

$$\left| t^{1/3} I_{n,t}(x) \frac{\sqrt{c}^n}{e^{\sqrt{c}t}(1-\sqrt{c})^x} \right| \leq \mathrm{const}\, e^{-s}, \tag{7.11}$$

uniformly for $s \geq -\ell$ and where const *is a constant independent of t.*

To get the needed bound on q_n around the edge, we use the bound of Lemma 7.4 on $I_{n,t}$ which has still to be multiplied by $B_{n,t}(x)$.

Lemma 7.5. *Let $n = ct$ and $x = (1+\sqrt{c})^2 t + st^{1/3}$. Fix an $\ell > 0$, then*

$$|q_n(x,t)| \leq \mathrm{const}\, t^{-1/3} e^{-s}, \tag{7.12}$$

for any $s \geq -\ell$, and const *is a t-independent constant.*

Proof of Lemma 7.5. This result follows from Lemma 7.3 if

$$\tilde{B}_{n,t}(x) = \left| B_{n,t}(x) \frac{e^{-\sqrt{c}t}(1+\sqrt{c})^x}{(-\sqrt{c})^n} \right| \leq \mathrm{const}\,. \tag{7.13}$$

For the factorials we use Stirling formula, namely

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{f_n}, \quad \frac{1}{1+12n} \leq f_n \leq \frac{1}{12n}. \tag{7.14}$$

We obtain

$$\tilde{B}_{ct,t}((1+\sqrt{c})^2 t) = \sqrt[4]{c/(1+\sqrt{c})^2}(1 + \mathcal{O}(1/t)). \tag{7.15}$$

For $x = \xi t$, $\xi \in [(1-\sqrt{c})^2, \infty)$, we compute

$$\frac{\tilde{B}_{ct,t}(\xi t)}{\tilde{B}_{ct,t}((1+\sqrt{c})^2 t)} = \sqrt[4]{(1+\sqrt{c})^2/\xi}(1+\mathcal{O}(1/t))e^{th(\xi)}, \tag{7.16}$$

with $h(\xi) = \frac{1}{2}\xi(1 - \ln(\xi) + 2\ln(1+\sqrt{c})) - \frac{1}{2}(1+\sqrt{c})^2$. It satisfies $\frac{\mathrm{d}h(\xi)}{\mathrm{d}\xi} = \frac{1}{2}\ln((1+\sqrt{c})^2/\xi) \leq 0$ for $\xi \geq (1+\sqrt{c})^2$. $\square$

Lemma 7.6. *Let $n = ct$ with $0 < c < 1$ and $x = (1-\sqrt{c})^2 t - st^{1/3}$. Fix an $\ell > 0$, then*

$$|q_n(x,t)| \leq \mathrm{const}\, t^{-1/3} e^{-s}, \tag{7.17}$$

for any $s \geq -\ell$, and const *is a t-independent constant.*

7.2 Asymptotics in the bulk

In this section we derive a precise expansion for $x = \lambda t$, $\lambda \in ((1-\sqrt{c})^2, (1+\sqrt{c})^2)$. Let $n = ct$ and $x = \lambda t$ for any fixed $c > 0$. Then

$$I_{n,t}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}w}{w} e^{tg(w)}, \quad g(w) = G(w|\lambda, c, 1). \tag{7.18}$$

Recall a few results from Section 3. For $\lambda \in ((1-\sqrt{c})^2, (1+\sqrt{c})^2)$, g has two complex conjugate critical points, w_c and $\bar{w}_c$, with $w_c = \Omega(\lambda, c, 1)$. In particular, $|w_c| = \sqrt{c}$, $|1-w_c| = \sqrt{\lambda}$, and $|g''(w_c)| = \frac{1}{\sqrt{\lambda c}}\sqrt{4c-(1+c-\lambda)^2}$ Denote by π_c the angle π_η with $\eta = c, \tau = 1$, and by π_λ the angle π_ν with $\nu = \lambda, \tau = 1$. Then

$$\begin{aligned} \mathrm{Re}(g(w_c)) &= \frac{1+c-\lambda}{2} - \frac{c}{2}\ln(c) + \frac{\lambda}{2}\ln(\lambda), \\ \mathrm{Im}(g(w_c)) &= \mathrm{Im}(w_c) - \lambda \pi_c - c\pi_\lambda. \end{aligned} \tag{7.19}$$

Lemma 7.7. *Let us set $\alpha = \mathrm{Im}(g(w_c))$ and $\beta = -\frac{1}{2}(\pi_c + \pi_\lambda - \pi/2)$. Then,*

$$\begin{aligned} I_{ct,t}(\lambda t) \quad = \quad & \frac{e^{t\mathrm{Re}(g(w_c))}}{\sqrt{|g''(w_c)|t}} \left[\sqrt{\frac{2}{\pi |w_c|^2}} \cos(t\alpha + \beta) \right. \\ & \left. + \mathcal{O}(t^{-1/2}) + \mathcal{O}\Big(\sqrt{|g''(w_c)|t}\, e^{-\mathrm{const}\, |g''(w_c)|\delta^2 t} \Big) \right], \end{aligned} \tag{7.20}$$

for some $0 < \delta \ll |g''(w_c)|$. The errors are uniform for λ in a compact *subset of $((1-\sqrt{c})^2, (1+\sqrt{c})^2)$.*

Proof of Lemma 7.7. The critical points of g, the points such that $g'(w) = 0$, are w_c and its complex conjugate. Close to the critical point the series expansion has a first relevant term which is quadratic,

$$g(w) = g(w_c) + \tfrac{1}{2} g''(w_c)(w-w_c)^2 + \mathcal{O}((w-w_c)^3). \tag{7.21}$$

Now we construct the steep descent path used in the asymptotics. By symmetry we consider only $\mathrm{Im}(w) \geq 0$, the path for $\mathrm{Im}(w) \leq 0$ will be the complex conjugate image of the first one. Let $\gamma_\rho = \{w = \rho e^{\mathrm{i}\phi}, \phi \in [0, \pi]\}$, then

$$\frac{\mathrm{d}}{\mathrm{d}\phi}(\mathrm{Re}(g(w = \rho e^{\mathrm{i}\phi}))) = \rho \sin(\phi) \left[\frac{\lambda}{|1-w|^2} - 1 \right]. \tag{7.22}$$

This is positive if we $|1-w| < \sqrt{\lambda}$, and negative otherwise.

Locally, consider the path $\gamma_{\mathrm{loc}} = \{w = w_c + \hat{\theta}x, x \in [-\delta, \delta]\}$. Then

$$g(w) = g(w_c) + \tfrac{1}{2} g''(w_c)\hat{\theta}^2 x^2 + \mathcal{O}(x^3), \tag{7.23}$$

where we choose

$$\hat{\theta} = \exp\left(\frac{\mathrm{i}\pi}{2} - \frac{\mathrm{i}}{2}\arg(g''(w_c)) \right). \tag{7.24}$$

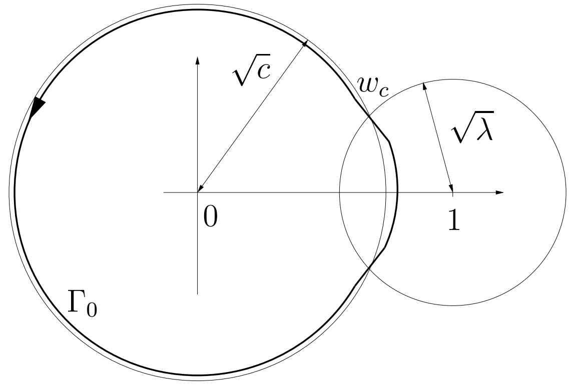

Figure 7.1: Illustration of the steep descent path.

This corresponds to, for $-\delta < x < 0$, our path being closer to 1 than $\sqrt{\lambda}$, and for $0 < x < \delta$ our path is farther from 1 than from $\sqrt{\lambda}$. This is possible since our θ must have an angle between $\pi/4$ and $3\pi/4$ to the tangent to the circle $|1-w| = \sqrt{\lambda}$ (otherwise it would be in contradiction with (7.22)).

So, the steep descent path used is the following: we extend $\gamma_{\rm loc}$ by adding two circular arcs of type γ_ρ, for adequate ρ, which connect to the real axis; finally we add the complex conjugate image, see Figure 7.1 too.

In this way, we have a steep descent path. Thus,

$$I_{n,t}(x) = e^{t\mathrm{Re}(g(w_c))}\mathcal{O}(e^{-\mu t}) + 2\mathrm{Re}\left(\frac{1}{2\pi \mathrm{i}}\int_{\gamma_{\rm loc}} \frac{\mathrm{d}w}{w} e^{tg(w)}\right) \tag{7.25}$$

with $\mu \sim |g''(w_c)|\delta^2$, as soon as $|g''(w_c)| > 0$, i.e., as soon as the second order expansion dominates all higher order terms in the series expansion.

The term in the real part of the last term in (7.25) is given by

$$\begin{aligned}\frac{1}{2\pi \mathrm{i}}\int_{\gamma_{\rm loc}} \frac{\mathrm{d}w}{w} e^{tg(w)} &= \frac{1}{2\pi \mathrm{i}}\int_{-\delta}^{\delta} \mathrm{d}x \frac{\hat{\theta}}{w_c} e^{tg(w_c)} e^{-\frac{1}{2}t|g''(w_c)|x^2} e^{\mathcal{O}(tx^3)}\mathcal{O}(x) \\ &= \frac{1}{2\pi \mathrm{i}}\frac{\hat{\theta}}{w_c}\int_{-\delta}^{\delta} \mathrm{d}x e^{tg(w_c)} e^{-\frac{1}{2}t|g''(w_c)|x^2} + E_1 \end{aligned} \tag{7.26}$$

where

$$E_1 = \frac{1}{2\pi \mathrm{i}}\frac{\hat{\theta}}{w_c}\int_{-\delta}^{\delta} \mathrm{d}x e^{tg(w_c)} e^{-\frac{1}{2}t|g''(w_c)|x^2} e^{\mathcal{O}(tx^3)}\mathcal{O}(tx^3, x). \tag{7.27}$$

Here we used $|e^x - 1| \leq |x|e^{|x|}$. By change of variable $y = x\sqrt{t}$, we get that

$$\begin{aligned}|E_1| &\leq \mathrm{const}\, e^{t\mathrm{Re}(g(w_c))}\frac{1}{t}\int_{-\delta\sqrt{t}}^{\delta\sqrt{t}} \mathrm{d}y e^{-|g''(w_c)|y^2/2}\mathcal{O}(y)e^{\mathcal{O}(y^3/\sqrt{t})} \\ &\leq \mathrm{const}\, \frac{e^{t\mathrm{Re}(g(w_c))}}{t\sqrt{|g''(w_c)|}} \end{aligned} \tag{7.28}$$

for δ small enough, i.e., for $0 < \delta \ll |g''(w_c)|$. In this small neighborhood, the quadratic term controls the higher order ones. The final step is to extend the integral on the rest of r.h.s. of (7.26) to $\pm\infty$ instead of $\pm\delta$. This can be made up to an error $e^{t\mathrm{Re}(g(w_c))}\mathcal{O}(e^{-\mu t})$ as above.

Resuming we have

$$\begin{aligned} I_{n,t} &= e^{t\mathrm{Re}(g(w_c))}\left[\mathcal{O}(e^{-\mu t}) + \mathcal{O}(1/(t\sqrt{|g''(w_c)|})\right] \\ &\quad +2\mathrm{Re}\left(\frac{1}{2\pi\mathrm{i}}\frac{\hat{\theta}}{w_c}\int_{\mathbb{R}} \mathrm{d}x e^{tg(w_c)} e^{-\frac{1}{2}t|g''(w_c)|x^2}\right). \end{aligned} \tag{7.29}$$

The error terms are the one indicated in (7.20), and the Gaussian integral for the last term gives

$$\frac{2e^{t\mathrm{Re}(g(w_c))}}{\sqrt{2\pi t\,|w_c|^2\,|g''(w_c)|}}\mathrm{Re}\left(-\mathrm{i}\hat{\theta}\frac{|w_c|}{w_c}e^{\mathrm{i}t\mathrm{Im}(g(w_c))}\right). \tag{7.30}$$

We then define $\beta = \arg(-\mathrm{i}\hat{\theta}/w_c)$, which is computed by using (3.5) and (3.25). For λ in a compact subset of $((1-\sqrt{c})^2, (1+\sqrt{c})^2)$, $|g''(w_c)|$ is uniformly bounded away from zero and infinity. Thus the Lemma is proven. $\square$

The consequence for q is the following asymptotics.

Lemma 7.8. *With the notations of Lemma 7.7,*

$$q_{ct}(\lambda t, t) = \frac{1}{\sqrt{\pi}}\frac{t^{-1/2}}{\sqrt[4]{c - \frac{(1+c-\lambda)^2}{4}}}\left[\cos(t\alpha+\beta) + \mathcal{O}(t^{-1/2})\right]. \tag{7.31}$$

The errors are uniform for λ in a compact *subset of* $((1-\sqrt{c})^2, (1+\sqrt{c})^2)$.

Proof of Lemma 7.8. We just have to compute the prefactor $B_{ct,t}(\lambda t)e^{t\mathrm{Re}(g(w_c))}$. We have (7.19) and applying Stirling formula for the factorials in $B_{ct,t}(\lambda t)$ we get that

$$B_{ct,t}(\lambda t)e^{t\mathrm{Re}(g(w_c))} = \sqrt[4]{\lambda/c}(1+\mathcal{O}(1/t)). \tag{7.32}$$

$\square$

Remark 7.9. Actually, it is not difficult to see that the expression (7.31) holds also until $\lambda = (1\pm\sqrt{c})^2 \mp t^{-1/3}$ but with the error term $\mathcal{O}(t^{-1/2})$ changed into $\mathcal{O}(t^{-5/12})$. Indeed, in this limit regime, $|g''(w_c)| \sim t^{-1/6}$ and the previous analysis in unchanged if we set $\delta = t^{-1/4}$.

The result of Lemma 7.8 is needed since we will have to know the density at one point. Now we need to fill the gap between the bulk and the edge. In this region we do not need a precise asymptotics, just a bound. Approaching the edge, $g''(w_c)$ goes to zero, so also δ has to be taken to zero. This is not a real problem as $|g'''(w_c)| \neq 0$ at the edges.

Lemma 7.10. *For $\varepsilon_0 > 0$ fixed but small enough, and $\ell > 0$ large, fixed, let either*

$$\lambda \in [(1+\sqrt{c})^2 - \varepsilon_0, (1+\sqrt{c})^2 - \ell t^{-2/3}], \quad \text{for any } c \geq 0, \tag{7.33}$$

or

$$\lambda \in [(1-\sqrt{c})^2 + \ell t^{-2/3}, (1-\sqrt{c})^2 + \varepsilon_0], \quad \text{for any } c \in (0,1). \tag{7.34}$$

Then we have the uniform bound

$$|q_{ct}(\lambda t, t)| \leq \text{const}\, \frac{t^{-1/2}}{\sqrt[4]{c - \frac{(1+c-\lambda)^2}{4}}}. \tag{7.35}$$

Proof of Lemma 7.10. Close to the edges, say for $\varepsilon_0 > 0$ small enough, we can compute explicitly the direction $\hat{\theta}$. It is a continuous function and, as $\lambda \uparrow (1+\sqrt{c})^2$, $\hat{\theta} \uparrow e^{\mathrm{i}5\pi/4}$, while as $\lambda \downarrow (1-\sqrt{c})^2$, then $\hat{\theta} \downarrow e^{\mathrm{i}3\pi/4}$ (with $0 < c < 1$). Since we need just a bound, we choose $\hat{\theta} = e^{\mathrm{i}5\pi/4}$ or $\hat{\theta} = e^{\mathrm{i}3\pi/4}$ respectively. The second derivative of g vanishes at the boundary, but the third derivative not: $g'''(w_c) \to -2/\sqrt{c}(1-\sqrt{c})$ as $\lambda \to (1-\sqrt{c})^2$, see (7.3), and $g'''(w_c) \to -2/\sqrt{c}(1+\sqrt{c})$ as $\lambda \to (1+\sqrt{c})^2$. Once more we see the restriction $0 < c < 1$ at the lower edge is essential.

Therefore, in this case we can replace the error term $\mathcal{O}(e^{-\text{const}\,|g''(w_c)|\delta^2 t})$ in Lemma 7.7, by an error term $\mathcal{O}(e^{-\text{const}\,|g'''(w_c)|\delta^3 t})$ and keep $0 < \delta \ll 1$ not vanishing (the condition $\delta \ll |g''(w_c)|$ is not required anymore). The rest of the estimates of Lemma 7.7 carry over here too. The only difference is that in (7.27) higher order are controlled by the third term expansion. □

7.3 Asymptotic of the kernel

It is convenient to conjugate the kernel. For the lower edge, we will use (compare with Lemma 7.4 and Remark 4.3)

$$W_{i,l} = \exp\left(\sqrt{n_i t_i} + x_i \ln(1 - \sqrt{n_i/t_i}) - n_i \ln(\sqrt{n_i/t_i}) - t_i\right), \tag{7.36}$$

for the upper edge (compare with Lemma 7.3 and Remark 4.3)

$$W_{i,u} = \exp\left(-\sqrt{n_i t_i} + x_i \ln(1 + \sqrt{n_i/t_i}) - n_i \ln(-\sqrt{n_i/t_i}) - t_i\right), \tag{7.37}$$

and in the bulk (see Lemma 7.7 and Remark 4.3)

$$W_{i,b} = \exp\left(\tfrac{1}{2}(t_i + n_i - x_i) - \tfrac{1}{2} n_i \ln(n_i/t_i) + \tfrac{1}{2} x_i \ln(x_i/t_i) - t_i\right). \tag{7.38}$$

Then, define the conjugation as

$$W_i = \begin{cases} W_{i,l}, & \text{for } x_i \leq (\sqrt{t_i} - \sqrt{n_i})^2, \\ W_{i,b}, & \text{for } (\sqrt{t_i} - \sqrt{n_i})^2 \leq x_i \leq (\sqrt{t_i} + \sqrt{n_i})^2, \\ W_{i,u}, & \text{for } x_i \geq (\sqrt{t_i} + \sqrt{n_i})^2. \end{cases} \tag{7.39}$$

Remark that W_i is continuous. Moreover, $|W_{i,l} - W_{i,b}| \leq \mathcal{O}(L^{-1/3})$ for $x_i \leq (\sqrt{t_i} - \sqrt{n_i})^2 - \ell L^{-1/3}$ for ℓ bounded. Similarly for the upper edge. Therefore in such a neighborhood it is actually irrelevant which formula to use.

Proposition 7.11. *Let us consider*

$$x_i = [\nu_i L], \quad n_i = [\eta_i L], \quad t_i = \tau_i L. \tag{7.40}$$

Define the function $G_i(w) = G(w|\nu_i,\eta_i,\tau_i)$, $w_c = \Omega(\nu_2,\eta_2,\tau_2)$ *and* $z_c = \Omega(\nu_1,\eta_1,\tau_1)$ *where* Ω *is given in (3.5). Assume* $z_c \neq w_c$. *Then, for*

$$(\sqrt{\tau_i}-\sqrt{\eta_i})^2 + L^{-1/3} \leq \nu_i \leq (\sqrt{\tau_i}+\sqrt{\eta_i})^2 - L^{-1/3}, \tag{7.41}$$

the asymptotic expansion

$$\begin{aligned} K(x_1,n_1,t_1;x_2,n_2,t_2) &= \frac{W_2/W_1}{2\pi L\sqrt{B(\nu_2,\eta_2,\tau_2)B(\nu_1,\eta_1,\tau_1)\nu_1/\tau_1}}\Bigg[\mathcal{O}(L^{-5/12}) \\ &+\frac{1}{w_c-z_c}\frac{e^{\mathrm{i}F(\nu_2,\eta_2,\tau_2)+\mathrm{i}\beta_2}}{e^{\mathrm{i}F(\nu_1,\eta_1,\tau_1)+\mathrm{i}\beta_1}} + \frac{1}{w_c-\bar z_c}\frac{e^{\mathrm{i}F(\nu_2,\eta_2,\tau_2)+\mathrm{i}\beta_2}}{e^{-\mathrm{i}F(\nu_1,\eta_1,\tau_1)-\mathrm{i}\beta_1}} \\ &+\frac{1}{\bar w_c-z_c}\frac{e^{-\mathrm{i}F(\nu_2,\eta_2,\tau_2)-\mathrm{i}\beta_2}}{e^{\mathrm{i}F(\nu_1,\eta_1,\tau_1)+\mathrm{i}\beta_1}} + \frac{1}{\bar w_c-\bar z_c}\frac{e^{-\mathrm{i}F(\nu_2,\eta_2,\tau_2)-\mathrm{i}\beta_2}}{e^{-\mathrm{i}F(\nu_1,\eta_1,\tau_1)-\mathrm{i}\beta_1}}\Bigg]. \end{aligned} \tag{7.42}$$

holds, with the error uniform in L *for* $L \geq L_0 \gg 1$. *The phases* β_1 *and* β_2 *are defined by*

$$\beta_1 = -\frac{3\pi}{4} - \frac{\pi_{\nu_1}}{2} - \frac{\pi_{\eta_1}}{2}, \quad \beta_2 = \frac{5\pi}{4} + \frac{\pi_{\nu_2}}{2} - \frac{\pi_{\eta_2}}{2}. \tag{7.43}$$

The function F *and* B *are given by*

$$F(\nu,\eta,\tau) = L\mathrm{Im}(G(\Omega(\nu,\eta,\tau))), \quad B(\nu,\eta,\tau) = \frac{2\tau}{\sqrt{\eta\nu}}\sqrt{\eta\tau - \tfrac14(\tau+\eta-\nu)^2}. \tag{7.44}$$

Proof of Proposition 7.11. The analysis is made on the double integral representation (4.1) of the kernel. The analysis for the cases $(n_1,t_1) \not\prec (n_2,t_2)$ and $(n_1,t_1) \prec (n_2,t_2)$ are very similar. Let us explain the first case. The asymptotics are very close to the one of Lemma 7.7. The first case corresponds to $\eta_1 > \eta_2$ and $\tau_1 < \tau_2$, or $(\eta_1,\tau_1) = (\eta_2,\tau_2)$. Since the asymptotic analysis is very close to the of of Lemma 7.7, we introduce the notations

$$c_i = \eta_i/\tau_i \Rightarrow n_i = c_i t_i, \quad \lambda_i = \nu_i/\tau_i \Rightarrow x_i = \lambda_i t_i. \tag{7.45}$$

The conjugation factor $e^{t_1-t_2}$ will not appear in the following computations, since it comes trivially in the factors W_1/W_2. We need to analyze, see (4.1),

$$\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}w\oint_{\Gamma_1}\mathrm{d}z e^{t_2 g_2(w)-t_1 g_1(z)}\frac{1}{(1-z)(w-z)} \tag{7.46}$$

with $g_i(w) = w + \lambda_i\ln(1-w) - c_i\ln(w) \equiv G(w|\lambda_i,c_i,1)$, $i=1,2$.

For a moment, ignore the fact that the paths Γ_0 and Γ_1 do not intersect, i.e., do not care about the factor $1/(w-z)$. The critical points of $g_2(w)$ and $g_1(z)$ are given by

$$w_c = \Omega(\lambda_2,c_2,1) = \Omega(\nu_2,\eta_2,\tau_2), \quad z_c = \Omega(\lambda_1,c_1,1) = \Omega(\nu_1,\eta_1,\tau_1). \tag{7.47}$$

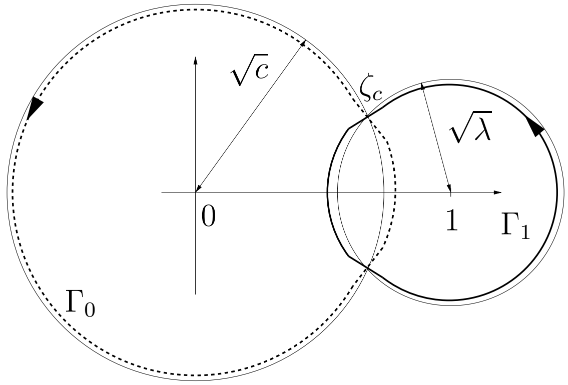

Figure 7.2: Illustration of the steep descent paths.

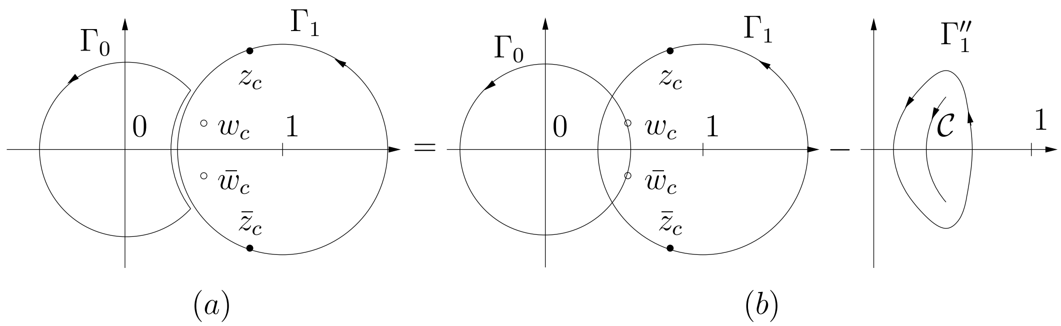

Figure 7.3: The subdivision of the integration (7.46). The first integral in (b) is in the Principal Value sense. We have $|z_c| \geq |w_c|$ and when $|z_c| = |w_c|$, they are not at the same position.

The integrals over w is, up to the factor $w/(z-w)$, as in Lemma 7.7. Therefore, the steep descent path Γ_0 is chosen as in Lemma 7.7 and the steep descent path Γ_1 in a similar way. We illustrate these paths in the case if the critical point are ζ_c, see Figure 7.2. In particular, $|w_c| = \sqrt{\eta_2/\tau_2}$ and $|z_c| = \sqrt{\eta_1/\tau_2}$. In our case, we have $|w_c| \leq |z_c|$ and $|w_c - z_c| > 0$. The steep descent paths described above actually intersect. Therefore, we have to correct (7.46) by subtracting the residue at $z = w$, as indicated in Figure 7.3. The integral with the paths Γ_0 and Γ_1 crossing, is intended as the principal value integral.

Both integrals can be divided as the part on $\mathbb{H}$ and their complex conjugate. Therefore, in the final expression we get the sum of four terms. Now, we restrict our attention to the integral over Γ_0 and Γ_1 restricted to $\mathbb{H}$. The analysis of the integral over Γ_0 is the same as in Lemma 7.7 except for the missing $1/w_c$ factor and that instead of $2\mathrm{Re}(\cdots)$ we just have $(\cdots)$ in (7.29). The integral over Γ_1 is on the same line. Moreover, in the δ-neighborhood of the critical points w_c and z_c, we have $|w-z| > 0$ uniformly for δ small enough. Therefore, the first part of the Principal

Value integral is given by

$$
\begin{aligned}
&\left[e^{t_2\mathrm{Re}(g_2(w_c))}\mathcal{O}\left(e^{-\mu_2 t_2}, 1/(t_2\sqrt{|g_2''(w_c)|})\right) + \frac{e^{t_2 g_2(w_c)}\hat{\theta}_2(w_c)}{\sqrt{2\pi t_2|g_2''(w_c)|}}\right] \\
\times\;&\left[e^{-t_1\mathrm{Re}(g_1(z_c))}\mathcal{O}\left(e^{-\mu_1 t_1}, 1/(t_1\sqrt{|g_1''(z_c)|})\right) + \frac{e^{-t_1 g_1(z_c)}\hat{\theta}_1(z_c)}{\sqrt{2\pi t_1|g_1''(z_c)|}}\frac{e^{\mathrm{i}\pi\eta_1}}{|1-z_c|}\right] \\
\times\;&\frac{1}{w_c - z_c}
\end{aligned}
\tag{7.48}
$$

with $\mu_1 \sim |g_1''(z_c)|\delta_1^2$, $\mu_2 \sim |g_2''(w_c)|\delta_2^2$, and $0 < \delta_1 \ll |g_1''(z_c)|$, $0 < \delta_2 \ll |g_2''(w_c)|$. The term $e^{\mathrm{i}\pi\eta_1}$ is the phase of $1/(1-z_c)$, while $\hat{\theta}_i$ are the directions of the steepest descent path at the critical point. Explicitly,

$$
\hat{\theta}_1(z_c) = \exp(\mathrm{i}(\pi - \tfrac{1}{2}\arg(g''(z_c)))), \quad \hat{\theta}_2(w_c) = \exp(\mathrm{i}(\tfrac{\pi}{2} - \tfrac{1}{2}\arg(g''(w_c)))). \tag{7.49}
$$

We can rewrite it as

$$
\begin{aligned}
&\frac{e^{t_2\mathrm{Re}(g_2(w_c)) - t_1\mathrm{Re}(g_1(z_c))}}{2\pi\sqrt{t_1 t_2|1-z_c|^2|g_2''(w_c)||g_1''(z_c)|}}\frac{\hat{\theta}_2(w_c)\hat{\theta}_1(z_c)e^{\mathrm{i}\pi\eta_1}}{w_c - z_c} \\
\times\;&\left[\mathcal{O}\left(\sqrt{|g_2''(w_c)|t_2}e^{-\mathrm{const}\,|g_2''(w_c)|\delta_2^2 t_2}, 1/\sqrt{t_2}\right) + e^{\mathrm{i}t_2\mathrm{Im}(g_2(w_c))}\right] \\
\times\;&\left[\mathcal{O}\left(\sqrt{|g_1''(z_c)|t_1}e^{-\mathrm{const}\,|g_1''(z_c)|\delta_1^2 t_1}, 1/\sqrt{t_1}\right) + e^{-\mathrm{i}t_1\mathrm{Im}(g_1(z_c))}\right].
\end{aligned}
\tag{7.50}
$$

For λ_1 in a compact subset of $((1-\sqrt{c_1})^2, (1+\sqrt{c_1})^2)$, the error terms are $\mathcal{O}(t_1^{-1/2})$ since $|g_1''(z_c)|$ is uniformly away form zero. However, we can do more. The estimate (7.50) holds even closer to the facets. Namely, we can extend it without problems until $\lambda_1 = (1\pm\sqrt{c_1})^2 \mp t_1^{-1/3}$. Indeed, in that case, $|g_1''(z_c)| \sim t_1^{-1/6}$, therefore we can choose $\delta = t_1^{-1/4}$ and get a bound on the error term $\mathcal{O}(t_1^{-5/12})$. Thus, for

$$
(1-\sqrt{c_i})^2 + t_i^{-1/3} \leq \lambda_i \leq (1+\sqrt{c_i})^2 - t_i^{-1/3}, \quad i = 1, 2, \tag{7.51}
$$

the complete contribution of the Principal Value integral is given by

$$
\begin{aligned}
&\frac{e^{t_2\mathrm{Re}(g_2(w_c)) - t_1\mathrm{Re}(g_1(z_c))}}{2\pi\sqrt{t_1 t_2|1-z_c|^2|g_2''(w_c)||g_1''(z_c)|}}\Bigg[\mathcal{O}(L^{-5/12}) \\
+\;&\frac{1}{w_c - z_c}\frac{e^{\mathrm{i}t_2\mathrm{Im}(g_2(w_c))+\mathrm{i}\beta_2}}{e^{\mathrm{i}t_1\mathrm{Im}(g_1(z_c))+\mathrm{i}\beta_1}} + \frac{1}{w_c - \bar{z}_c}\frac{e^{\mathrm{i}t_2\mathrm{Im}(g_2(w_c))+\mathrm{i}\beta_2}}{e^{-\mathrm{i}t_1\mathrm{Im}(g_1(z_c))-\mathrm{i}\beta_1}} \\
+\;&\frac{1}{\bar{w}_c - z_c}\frac{e^{-\mathrm{i}t_2\mathrm{Im}(g_2(w_c))-\mathrm{i}\beta_2}}{e^{\mathrm{i}t_1\mathrm{Im}(g_1(z_c))+\mathrm{i}\beta_1}} + \frac{1}{\bar{w}_c - \bar{z}_c}\frac{e^{-\mathrm{i}t_2\mathrm{Im}(g_2(w_c))-\mathrm{i}\beta_2}}{e^{-\mathrm{i}t_1\mathrm{Im}(g_1(z_c))-\mathrm{i}\beta_1}}\Bigg].
\end{aligned}
\tag{7.52}
$$

Finally, we replace $G_i(w)L = g_i(w)t_i$ and $e^{t_2\mathrm{Re}(g_2(w_c)) - t_1\mathrm{Re}(g_1(z_c))} = W_2/W_1$ to get (7.42).

We still have to estimate the contribution of the residue (the last case of Figure 7.3). This term is given by

$$
\frac{1}{2\pi\mathrm{i}}\int_{\zeta}^{\bar{\zeta}} \mathrm{d}z \frac{e^{(\tau_2-\tau_1)Lz}e^{(\eta_1-\eta_2)L\ln(z)}}{(1-z)^{(\nu_1-\nu_2)L+1}}, \tag{7.53}
$$

where ζ and $\bar{\zeta}$ are the two intersection points of the steep descent path Γ_0 and Γ_1 in the Principal Value integral. Since $\tau_2 - \tau_1 \geq 0$, $\eta_1 - \eta_2 \geq 0$, and $|1-z| = \mathrm{const}$ along the piece of Γ_1 inside Γ_0, we have $\mathrm{Re}(z) \leq \mathrm{Re}(\zeta)$ and $\mathrm{Re}(\ln(z)) \leq \mathrm{Re}(\ln(\zeta))$. Therefore,

$$|(7.53)| \leq e^{t_2 \mathrm{Re}(g_2(\zeta)) - t_1 \mathrm{Re}(g_1(\zeta))} \leq e^{t_2 \mathrm{Re}(g_2(w_c)) - t_1 \mathrm{Re}(g_1(z_c))} \mathcal{O}(e^{-\mu_1 t_1} e^{-\mu_2 t_2}), \tag{7.54}$$

for some $\mu_1 \geq 0$ and $\mu_2 \geq 0$, but not simultaneously equal to zero. This follows from the fact that either one (or both) critical points are away of order one from ζ, and ζ lies on the steep descent paths of $g_2(w)$ and $-g_1(z)$. □

We will also need the following corollary when we do time integration.

Corollary 7.12. *In the same setting of Proposition 7.11, the formula for $K(x_1, n_1, t_1; x_2 + 1, n_2, t_2)$ is the same as (7.42) but with an extra factor $(1 - w_c)$ (resp. $(1 - \bar{w}_c)$) to the terms with $e^{i\beta_2}$ (resp. $e^{-i\beta_2}$).*

Proof of Corollary 7.12. It follows by the same analysis by noticing that in (7.46) we have an extra term $(1 - w)$. □

Proposition 7.13. *Consider the setting of Proposition 7.11, but with one or both of the ν_i close to the edge. More precisely, with*

$$\begin{aligned} & (\sqrt{\tau_i} - \sqrt{\eta_i})^2 + \ell L^{-2/3} \leq \nu_i \leq (\sqrt{\tau_i} - \sqrt{\eta_i})^2 + L^{-1/3} \\ \text{or} \quad & (\sqrt{\tau_i} + \sqrt{\eta_i})^2 - L^{-1/3} \leq \nu_i \leq (\sqrt{\tau_i} + \sqrt{\eta_i})^2 - \ell L^{-2/3}. \end{aligned} \tag{7.55}$$

Then, there exists a ℓ large enough, such that

$$|K(x_1, n_1, t_1; x_2, n_2, t_2)| \leq \mathrm{const} \frac{W_2/W_1}{L \prod_{i=1}^{2} \sqrt[4]{\eta_i \tau_i - \frac{1}{4}(\tau_i + \eta_i - \nu_i)^2}} \tag{7.56}$$

uniformly in L for $L \geq L_0 \gg 1$.

Proof of Proposition 7.13. The proof follows the same argument as Lemma 7.10 for the variables which are close to the edge. For the one which is away from the edges, it is a consequence of the analysis Proposition 7.11. □

When one or both positions are at the edge, we need a different bound. We state it first in the case when ν_1 is at the lower edge and ν_2 in the bulk. Similar bounds are obtained in the same way when ν_1 is at the upper edge.

Proposition 7.14. *Consider the setting of Proposition 7.11, but now with ν_1 at the edge or in the facet. More precisely, with*

$$\nu_1 \geq (\sqrt{\tau_1} + \sqrt{\eta_1})^2 - \ell L^{-2/3} \tag{7.57}$$

for any fixed ℓ. Then,

$$\begin{aligned} |K(x_1,n_1,t_1;x_2,n_2,t_2)| \;\leq\; & \frac{\mathrm{const}\, W_2/W_1}{\sqrt{L}\sqrt[4]{\eta_2\tau_2-\frac{1}{4}(\tau_2+\eta_2-\nu_2)^2}} \\ & \times\; \frac{1}{L^{1/3}}\exp\left(-\frac{x_1-(\sqrt{\tau_1}+\sqrt{\eta_1})^2L}{(\tau_1 L)^{1/3}}\right), \end{aligned} \tag{7.58}$$

uniformly in L for $L \geq L_0 \gg 1$.

Proof of Proposition 7.14. The proof is obtained along the same lines as Lemmas 7.1 and 7.3 for the upper edge, and Lemmas 7.2 and 7.4 for the lower edge. With respect to those cases, the integral has however an extra factor $1/(w-z)$. Since we need just a bound, it can simply be replaced by $1/(w_c - z_c)$ as follows. In Lemma 7.1 w_c is replaced by $-\sqrt{c}$, while in Lemma 7.3, we need to replace w_c by ρ in (7.10). Notice that we can take $|w_c+\sqrt{c}|$ as small as desired. In particular, we take it small enough so that, even when two integrals will be at the lower edge, they this pole will not collide. □

In the case when both ν_1 and ν_2 are at the lower edge we have the following statement.

Proposition 7.15. *Consider the setting of Proposition 7.11, but now with ν_1 at the edge or in the facet. More precisely, with*

$$\nu_i \geq (\sqrt{\tau_i}+\sqrt{\eta_i})^2 - \ell L^{-2/3}, \quad i=1,2, \tag{7.59}$$

for any fixed ℓ. Then,

$$\begin{aligned} &|K(x_1,n_1,t_1;x_2,n_2,t_2)| \leq \mathrm{const}\, W_2/W_1 \\ &\frac{1}{L^{2/3}}\exp\left(-\frac{x_2-(\sqrt{\tau_2}+\sqrt{\eta_2})^2L}{(\tau_2 L)^{1/3}}\right)\exp\left(-\frac{x_1-(\sqrt{\tau_1}+\sqrt{\eta_1})^2L}{(\tau_1 L)^{1/3}}\right), \end{aligned} \tag{7.60}$$

uniformly in L for $L \geq L_0 \gg 1$.

Proof of Proposition 7.15. The proof is like Proposition 7.14. □

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