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An inverse model problem in kinetic theory

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Abstract

The paper deals with an inverse model problem in linear kinetic theory: the identification of a density profile of a scattering medium in a slab geometry from measurement of the reflected portion of a particle flux entering the medium. We prove well-posedness of the problem and present a robust algorithm for the identification.

1 Introduction

1.1 Identification of scattering media

Inverse problems in kinetic theory have attracted considerable attention for a couple of years. One kind of question one poses is that of determining the optical thickness or - more ambitious - the spatial variation of some scattering medium by letting pass some test particle flow through the medium and measuring e.g. the transmitted or the reflected flow. In this context, different approaches lead to different kinds of mathematical problems. One of the most well-known examples is that of computer tomography. There the part of the flow is measured which passes the medium without interaction. The mathematical problem behind is that of the inversion of the Radon transform (see, e.g. [15]). This theory is applicable in situations of optically thin media, where in particular multiple collision effects are negligible.

The aim of this paper is to investigate a related inverse problem in a spatially one-dimensional situation, where multiple collisions cannot be neglected. This is the case e.g. if the thickness of the medium is in the order of one mean free path or more. Reviews in applied sciences where this problem is of interest have been provided in [10, 11, 12, 13, 14]. A large variety of methods has been proposed, mainly from a practical and experimental view point. However, as far as we know the question of well-posedness of these problems (in the sense of Hadamard [7]) in a strict mathematical sense seems to be open. This is - for a simplified model situation - the subject of this paper.

1.2 The mathematical framework

In linear kinetic theory, particle transport in some scattering medium Ω without absorption is described by an equation of the form

$$\left(\frac{\partial}{\partial t} + \Delta_x\right) f(t,x,v) = \int_{IR^3} c(x,v')f(t,x,v')k(v' \to v)dv' - c(x,v)f(t,x,v) \qquad (1)$$

for the density function in phase space $f = f(t,x,v)$, $(t,x,v) \in IR_+ \times \Omega \times IR^3$ (see, e.g. [6]). Here, $c(.,.)$ presents the inverse of the mean free path and thus corresponds to the density distribution of the medium; for each $v \in IR^3$, $k(v \to .)$ is a probability density which is related to the cross section of the scatterers. Solutions of an initial boundary value problem for this equation are usually well-defined if the initial condition

$f(0,.,.)$ in $\Omega \times IR^3$ and the inflow at the boundary, i.e. $f(t,x,v)$ for $x \in \partial\Omega$ and for $n(x)v < 0$, are known. ($n(x)$ is the outer normal on $\partial\Omega$ at x). Solutions for which *complete* knowledge at the boundary is available (e.g. by measuring the outgoing flow) present an overdetermined boundary value problem. The kind of question investigated here is that of deriving information about $c(.,.)$ from this additional knowledge. (For the complementary problem of determining $k(.,.)$ instead of $c(.)$ see [5].)

Let us restrict to the case of a slab $[0,a]$ with dependence on one space variable only (which for simplicity we again call x; we denote by v_x the velocity component in $x-$direction). Suppose given an impulse entering the slab at $x=0$ at time $t=0$. The inverse problem of determining c (which we assume not to depend on v) from measuring the reflected flow was studied in [3]. Keypoint was the construction of solutions of the initial boundary value problem by means of a related Markov process. The inverse problem is then that of finding density profiles of the scattering medium from the distribution of first exit times from the slab. Introducing a random time transformation it was shown that this problem is equivalent to that of determining an unknown function $l(.)$ (which is equal to $1/c(r(x))$ with a certain transformation $r(.)$) from the knowledge of some reflection operator $R_\lambda(0)$ which describes the Laplace transform of the first exit time distribution. $R_\lambda(x)$ is formally determined as the solution of a Riccati-type equation in some infinite-dimensional Banach algebra

$$\frac{d}{dx}R_\lambda = -B_\lambda^- - A_\lambda^- R_\lambda - R_\lambda A_\lambda^+ - R_\lambda B_\lambda^+ R_\lambda + \lambda l(.) C R_\lambda \tag{2}$$

with an additional condition $R_\lambda(a_0) = 0$. Since the right hand side is unbounded, a crucial point in the construction of an algorithm is an appropriate discretization of this equation. Such an approach was developed in [3] based on the strong Markov property of the underlying process. It turned out that this formalism may well be used to identify a few unknown parameters describing the function $l(.)$. However, for identification problems in higher dimensional - or even infinite dimensional - function spaces the use of the Laplace transform causes stability problems.

Using this as a starting point, we investigate here the inverse problem based on the first exit time distribution rather than on its Laplace transform. We restrict to a strongly simplified situation - a two-velocity model. This is mainly because such a model allows to recognize most clearly the mathematical structure behind the inverse problem. Much of the theory presented below should also be applicable in more complicated situations. However, the unboundedness of certain operators due to the presence of velocities with very small normal velocity components v_x leads to considerable technical problems. On the other hand, as we show at the end of the paper, there is some indication that these more complex models may be well-approximated by models without singularities.

2 The direct problem

2.1 A kinetic two-velocity model

The kinetic equation

$$\left(\frac{\partial}{\partial t} \pm \eta \frac{\partial}{\partial x}\right) u^{\pm} = (u^{\mp} - u^{\pm}) \tag{3}$$

describes the dynamics of a one-dimensional two-velocity particle system with admissible velocities $\eta, -\eta \in IR$ (we assume $\eta > 0$) with corresponding densities u^+, u^- ($u^{\pm} = u^{\pm}(x,t)$) in a *homogeneous* medium with density 1. An *inhomogeneous medium* is modelled by multiplying the right hand side with some density profile. Following [3], the inverse problem of determining such a profile in a slab $[0, a_0]$ with an inflow impulse at $x = 0$ at time $t = 0$ is related to the solution of the Riccati boundary value problem

$$\frac{d}{dx} r_\lambda(x) = \frac{1}{\eta}(-1 + 2(1 + \lambda l(x)) r(x) - r^2(x)) \tag{4}$$

with

$$r_\lambda(a) = 0 \tag{5}$$

and

$$r_\lambda(0) = \mathcal{L}u^-(0) = \int_0^\infty u^-(0,t) \exp(-\lambda t) dt \tag{6}$$

Performing the inverse Laplace transform yields the corresponding problem in the time domain

$$2l(.)f_t - \eta f_x = -2f + f * f \tag{7}$$

with the boundary conditions

$$f(0,t) = u^-(0,t) \tag{8}$$

and

$$f(a,t) = 0 \tag{9}$$

f satisfies in addition the compatibility condition $f(x,0) = 1/2l(x)$, a property which may be derived easily from analyzing the associated Markov process for small times (see [3]). (In fact, it is this property which we will exploit for the inverse problem.) $f * f$ is the convolution

$$f * f(t) = \int_0^t f(\tau) f(t-\tau) d\tau = \int_{IR} f(\tau) f(t-\tau) d\tau \tag{10}$$

(Note that we have extended the function $f(t)$ to the negative half-axis by assuming $f(t) = 0$ for $t < 0$, a convention which we shall follow in all of the paper.)

2.2 Solutions of the direct problem

We choose $\eta := 1$ and start our investigations with the direct problem, the solution $f = f(x,t)$ on $[0,a] \times \mathbb{R}_+$ of the differential equation

$$2l(x)f_t - f_x = -2f + f * f \tag{11}$$

with the boundary conditions

$$f(.,0) = \frac{1}{2l(.)}\ , \ f(a,.) \equiv 0 \tag{12}$$

From now on, $a > 0$ is an arbitrary but fixed number, and l is a measurable function on $[0,a]$ satisfying $0 < l_{min} \leq l(.) \leq l_{max} < \infty$.

This problem will be solved by the method of characteristics. For this we need the mapping τ which is defined for $\alpha \in [0,a]$, $x \in [0,\alpha]$ by

$$\tau(x,\alpha) := 2\int_x^\alpha l(\xi)d\xi \tag{13}$$

and for $\alpha \geq a$, $x \in [0,a]$ by

$$\tau(x,\alpha) := \alpha - a + \int_x^a l(\xi)d\xi \tag{14}$$

It is easy to see that for each $(x,t) \in [0,a] \times \mathbb{R}_+$ there exists a unique $\alpha \geq x$ such that $t = \tau(x,\alpha)$. Given $(\tilde{x},\tilde{\alpha})$, denote

$$\tau_0 := \tau_0(\tilde{x},\tilde{\alpha}) := \begin{cases} \tilde{\alpha} - a & : \ \tilde{\alpha} \geq a \\ 0 & : \ else \end{cases}$$

and

$$x_0 := x_0(\tilde{x},\tilde{\alpha}) := \begin{cases} \tilde{\alpha} & : \ \tilde{\alpha} \leq a \\ a & : \ else \end{cases}$$

(x_0,τ_0) is the starting point of the unique characteristics $x \to (x,\tau(x,\alpha))$ through $(\tilde{x},\tau(\tilde{x},\tilde{\alpha}))$.

Following the standard terminology of kinetic theory, we define as *mild solutions* to the differential equation all functions satisfying - with $(x_0,\tau_0) = (x_0,\tau_0)(x,\alpha)$ -

$$f(x,\tau(x,\alpha)) = f(x_0,\tau_0) + \int_x^{x_0} [f * f - 2f]\,(\xi,\tau(\xi,\alpha))d\xi \text{ a.e.} \tag{15}$$

We now turn to properties of the solutions of the boundary value problem.

Properties of solutions 2.1:

1. *Existence and Uniqueness: For each l as described above, there exists a unique mild solution $f \in L^1$ of (1),(2).*

2. *Boundedness: This solution is non-negative, and the function*

$$\rho(x) := \int_0^\infty f(x,t)dt \tag{16}$$

is given by

$$\rho(x) = \frac{a-x}{1+(a-x)} \tag{17}$$

3. *Upper bound:* $f \in L^\infty$, *and* $\| f|_{[x,a]\times \mathbb{R}_+} \|_\infty = \| 1/2l(.)|_{[x,a]} \|_\infty$. *For* $t > 0$, $f(x,t) < 1/2l_{min}$.

4. *Domain of Dependency:* $f(x,t)$ *depends at most on the restriction of* $l(.)$ *to* $[x, \min\{a, x + t/2l_{min}\}]$. *The support of* $t \to f(x,t)$ *is contained in* $[0, 2l_{max}a]$.

Proof: *(i) Solution of a linear model problem:* We start by constructing a solution of the linear model equation

$$2lf_t - f_x = -2f + G \tag{18}$$

with boundary conditions

$$f(x,0) = \phi_1(x) \ , \ f(a,t) = 0 \tag{19}$$

and with given $G \in L^1_+([0,a]\times\mathbb{R}_+)$, $\phi_1 \in L^\infty_+([0,a])$. f is a mild solution, if $f(x,\tau(x,\alpha)) = \exp(2(x-a))g(x,\tau(x,\alpha))$, and g is a mild solution of $2lg_t - g_x = \hat{G}$ satisfying

$$g(x,0) = \phi_1(x)\exp(2(a-x)) \ , \ g(a,t) = 0 \tag{20}$$

and with $\hat{G}(x,\tau(x,\alpha)) = \exp(2(a-x)G(x,\tau(x,\alpha))$. This means that g is well defined by

$$g(x,\tau(x,\alpha)) := g(x_0,\tau_0) + \int_x^{x_0} \hat{G}(\xi,\tau(\xi,\alpha))d\xi \tag{21}$$

with $g(x_0,\tau_0)$ given by the boundary conditions. Nonnegativeness of g and f is obvious. Applying the transformation rule with

$$dt = \begin{cases} 2l(\alpha)d\alpha & : \alpha \le a \\ 1 & : \alpha > a \end{cases}$$

and $d\xi d\tau = dxdt$ we estimate

$$\rho_g(x) := \int_0^\infty g(x,t)dt \tag{22}$$

as follows

$$\rho_g(x) = \int_x^a 2\phi_1(\alpha)l(\alpha)d\alpha + \int_x^\infty \int_x^{x_0} \hat{G}(\xi,\tau(\xi,\alpha))d\xi\frac{d\tau}{d\alpha}d\alpha \tag{23}$$

Using

$$\int_x^\infty \int_x^{x_0} \hat{G}(\xi, \tau(\xi,\alpha))d\xi \frac{d\tau}{d\alpha} d\alpha = \int_x^a \int_0^\infty \hat{G}(x,t)dtdx =: \int_x^a \rho_{\hat{G}}(x)dx \tag{24}$$

follows $\rho_g(a) = 0$ and

$$d\rho_g(x) = -(2\phi_1(x)l(x)\exp(2(a-x)) + \rho_{\hat{G}}(x))dx \tag{25}$$

which tells us in particular that $g \in L^1([0,a] \times \mathbb{R}_+)$. Finally, from the construction follows that

$$\sup_{[x,a]\times\mathbb{R}_+} g(.,.) \leq \sup_{[x,a]}[\exp(2(a-.))\phi_1(.)] + \int_x^a \sup_{[\xi,a]\times\mathbb{R}_+} \hat{G}d\xi \tag{26}$$

For

$$\rho_f(x) := \int_0^\infty f(x,t)dt \tag{27}$$

we find $\rho_f(a) = 0$ and

$$d\rho_f(x) = (2\rho_f(x) - 2\phi_1(x)l(x) - \rho_G(x))dx \tag{28}$$

The supremum of f is bounded by the upper bound for g given above. Further, if $\phi_1^{(1)} \geq \phi_1^{(2)}$, and $G^{(1)} \geq G^{(2)}$, and if $f^{(i)}, g^{(i)}, i = 1,2$ are the solutions of the corresponding linear problems as constructed above, then $f^{(1)} \geq f^{(2)}$.
(ii) Solution of the nonlinear problem: The nonlinear problem can be solved via iteration: Start with $f^{(0)} \equiv 0$ and define $f^{(n+1)}$ as solution of the linear problem

$$2lf_t - f_x = -2f + f^{(n)} * f^{(n)} \tag{29}$$

as shown in *(i)*, with the correct boundary conditions. Then $f^{(n+1)} \geq f^{(n)}$ (since obviously $g^{(n+1)} \geq g^{(n)}$), and

$$\rho^{(n+1)} := \int_0^\infty f^{(n+1)}dt \tag{30}$$

is bounded by the unique solution ρ of $\rho(a) = 0$, $\rho_x = -1 + 2\rho - \rho^2$, which is given by

$$\rho(x) = \frac{a-x}{1+(a-x)} \tag{31}$$

From Lebesgue's monotone convergence theorem follows that $f^{(n)}$ converges to a non-negative solution of the nonlinear problem, and that $\rho_f = \rho$. From the estimate for the supremum of the linear problem and from

$$f^{(n)} * f^{(n)}(x,.) \leq \sup_{\mathbb{R}_+} f^{(n)}(x,.)\rho^{(n)}(x) \leq \sup_{\mathbb{R}_+} f^{(n)}(x,.) \tag{32}$$

we find by induction

$$\sup_{[x,a]\times \mathbb{R}_+} f^{(n)}(.,.) \leq \frac{1}{2l_{min}} \sum_0^n \frac{1}{k!}(a-x)^k \tag{33}$$

and in particular that the supremum of f is bounded.
(iii) Uniqueness follows from standard arguments using the Gronwall lemma.
(iv) Upper bound: Define $f_\infty := \sup_{[0,a]\times \mathbb{R}_+} f(x,t)$ (which we know is finite), and $G(.,.) \equiv f_\infty$. The monotonicity criterion of *(i)* tells us that f is bounded by the solution of the linear problem $2lg_t - g_x = -2g + G$, $g(x,0) = 1/2l_{min}$, $g(a,t) = 1/2l_{min}$, which is given by

$$g(x, \tau(x,\alpha)) = \frac{1}{2l_{min}} \exp(2(x-x_0)) + \frac{f_\infty}{2}(1 - \exp(2(x-x_0))) \tag{34}$$

If $1/2l_{min}$ were less or equal to $f_\infty/2$ then

$$f_\infty \leq \sup_{[0,a]\times \mathbb{R}_+} g(.,.) \leq \frac{f_\infty}{2} \tag{35}$$

which is a contradiction. Therefore, $1/2l_{min} > f_\infty/2$, and $f_\infty \leq 1/2l_{min}$. For $t > 0$ (i.e. for $x_0 > x$), $f(x,t) < 1/2l_{min}$.
(v) Domain of Dependency: follows from the construction in (i). ◇

3 The inverse problem

3.1 The problem

The inverse problem to be solved reads: Given an integrable function ϕ_0, find (in appropriate function spaces) functions $f(x,t)$ and $l(x)$ satisfying

$$f(0,.) = \phi_0 \tag{36}$$

$$f_x = 2lf_t + 2f - f * f \tag{37}$$

$$f(x,0) = \frac{1}{2l(x)} \tag{38}$$

We transform this problem into its integral form. Define

$$L(x) := \int_0^x l(\xi)d\xi \tag{39}$$

Then the differential equation transforms into the integral equation

$$f(x,t) = \phi_0(t + 2L(x))\exp(2x) - \int_0^x f * f(\xi, t + 2(L(x) - L(\xi)))\exp(2(x-\xi))d\xi \tag{40}$$

From this representation, the condition $f(x,0) = 1/2l(x)$ can be reformulated by integrating over x and using the transformation rule. It then reads

$$\int_0^{2L(x)} \phi_0(\tau)d\tau \quad - \quad \int_0^x \exp(-2\xi) \int_0^{2(L(x)-L(\xi))} f * f(\xi,\tau)d\tau d\xi$$
$$= \frac{1}{2}(1 - \exp(-2x)) \tag{41}$$

If $\phi_o > 0$ a.e., then $\Phi_0(x) := \int_0^x \phi_0(\xi)d\xi$ is invertible, and we obtain the final form

$$2L(x) = \Phi_0^{-1}\left(\frac{1}{2}(1 - \exp(-2x)) + \int_0^x \exp(-2\xi) \int_0^{2(L(x)-L(\xi))} f * f(\xi,\tau)d\tau d\xi\right) \tag{42}$$

It is obvious, that properties like global $L^\infty-$boundedness and nonnegativity (and as a consequence, global L^1-boundedness) which we derived for solutions of the direct problem, do not hold in general for solutions of the inverse problem. For these, we obtain no more than local bounds.

Lemma 3.1: *Suppose that $\|\phi_0\|_{L^1} \leq 1$ and that $f \in L^\infty([0,\bar{x}], L^1(IR_+))$ satisfies the integral equation (.) a.e. Denote*

$$\rho_f(x) := \int_0^\infty |f(x,t)|dt \tag{43}$$

and

$$f_\infty(x) := \|f(x,.)\|_\infty \tag{44}$$

1. *ρ_f is bounded by the solution*

$$\rho^{(1)}(x) := \frac{0.5 + x}{0.5 - x} \tag{45}$$

of the initial value problem $\rho_x = 1 + 2\rho + \rho^2$, $\rho(0) = 1$.

2. *f_∞ is bounded by the solution*

$$\rho^\infty(x) := \|\phi_0\|_\infty \exp\left(\int_0^x (2 + \rho^{(1)}(\xi))d\xi\right) \tag{46}$$

of the initial value problem $\rho_x = \rho(2 + \rho^{(1)})$, $\rho(0) = \|\phi_0\|_\infty$

The proof of this follows from straightforward calculations and is omitted here. Notice that under the hypothesis of nonnegativity of f, we obtain global bounds of the form $c_0 \exp(2x)$ for both ρ_f and f_∞.

3.2 An auxiliary problem

In all what follows, we will make the following assumptions on ϕ_0.

Assumptions 3.2:

1. $\phi_0 \in L^1(R_+) \cap L^\infty$, $\|\phi_0\|_{L^1} \leq 1$ and $\|\phi_0\|_\infty < 1$.
(The condition on the sup-norm can always be achieved for bounded ϕ_0 by introducing an appropriate time scale: replace $f(x,t)$ by $\lambda f(x, \lambda t)$ (and in the same way ϕ_0), and l by l/λ. This leads to an equivalent boundary value problem.)

2. *$\bar{x}$ and $\bar{t}$ are finite positive numbers (to be fixed later) which are related by*

$$\bar{t} = \frac{1}{2}(1 - \exp(-2\bar{x})) + \int_0^{\bar{x}} (\rho^{(1)}(\xi))^2 \exp(-2\xi) d\xi \tag{47}$$

3. *There exists a constant $\phi_{min} > 0$ such that $\inf_{[0,\bar{t}/\phi_{min}]} \phi_0(.) < \phi_{min}$. (From this follows immediately that $\Phi_0^{-1}|_{[0,\bar{t}]}$ is Lipschitz continuous with Lipschitz constant bounded by $1/\phi_{min}$.)*

We extend Φ_0^{-1} - which is defined on $[0, \bar{t}]$ - to the interval $[-\bar{t}, \bar{t}]$ by setting $\Phi_0^{-1}(-t) := -\Phi_0^{-1}(t)$.

Denote by $\mathcal{S}$ the set of functions $f \in L^\infty([0,\bar{x}], L^1(IR_+))$ satisfying $\|f(x,.)\|_\infty \leq \rho^\infty(x)$ a.e., and $\|f(x,.)\|_{L^1} \leq \rho^{(1)}(x)$ a.e. Further, denote by $\mathcal{LC}$ the set of Lipschitz continuous functions L from $[0,\bar{x}]$ to IR with $L(0) = 0$, and by $\mathcal{LC}[\Lambda]$ the subset of $\mathcal{LC}$ of functions with Lipschitz constant bounded by Λ. $\mathcal{LC}[\Lambda]$ is closed under the sup-norm. For simplicity, we extend functions $f \in \mathcal{S}$ to $[0,\bar{x}] \times IR$ by defining $f(x,t) := 0$ for $t < 0$.

Given $(p, R) \in \mathcal{S} \times \mathcal{LC}$, define $f = T_S(p, R)$ by

$$f(x,t) := \tag{48}$$
$$\phi_0(t + 2R(x)) \exp(2x) - \int_0^x p * p(\xi, t + 2(R(x) - R(\xi))) \exp(2(x - \xi)) d\xi$$

for $t \geq 0$, and $L = T_{LC}(p, R)$ by

$$2L(x) := \tag{49}$$
$$\Phi_0^{-1}\left(\frac{1}{2}(1 - \exp(-2x)) + \int_0^x \exp(-2\xi) \int_0^{2(R(x)-R(\xi))} p * p(\xi, \tau) d\tau d\xi\right)$$

Lemma 3.3:

1. *$T := T_S \times T_{LC}$ maps $\mathcal{S} \times \mathcal{LC}$ into $\mathcal{S} \times \mathcal{LC}$.*

2. *For $\bar{x}$ small enough, there exists $\Lambda > 0$ such that T maps $\mathcal{S} \times \mathcal{LC}[\Lambda]$ into $\mathcal{S} \times \mathcal{LC}[\Lambda]$.*

Proof: Due to the construction of $\rho^{(1)}$ and ρ^∞, $f = T_S(p, R)$ is bounded by these functions, whenever p is.

$L = T_{LC}(p, R)$ is Lipschitz continuous on $[0, \bar{x}]$ since Φ_0^{-1} is Lipschitz continuous on $[-\bar{t}, \bar{t}]$, and since for $x \leq \bar{x}$

$$\frac{1}{2}(1 - \exp(-2x)) + \int_0^x \exp(-2\xi) \int_0^{2(R(x)-R(\xi))} p * p(\xi, \tau) d\tau d\xi \leq \bar{t} \tag{50}$$

To find an estimate for the Lipschitz constant, we observe

$$|(1 - \exp(-2x)) - (1 - \exp(-2y))| \leq 2|x - y| \tag{51}$$

and (for $x \leq y$)

$$\left|\int_0^x \exp(-2\xi)\int_0^{2(R(x)-R(\xi))} p*p(\xi,\tau)d\tau d\xi - \int_0^y \exp(-2\xi)\int_0^{2(R(y)-R(\xi))} p*p(\xi,\tau)d\tau d\xi\right|$$
$$\leq \int_x^y \int_0^\infty |p*p|d\xi d\tau + \int_0^{\bar{x}} \left|\int_{2(R(x)-R(\xi))}^{2(R(y)-R(\xi))} p*p(\xi,\tau)d\tau\right| d\xi$$

Therefore $L \in \mathcal{LC}[\Lambda]$, if R has Lipschitz constant bounded by Λ, and if

$$\Lambda \geq \left(\rho^{(1)}(\bar{x})(\rho^{(1)}(\bar{x}) + 2\Lambda\bar{x}\rho^\infty(\bar{x})) + 1\right)/(2\phi_{min}), \tag{52}$$

a condition which is satisfied for $\bar{x}$ small and Λ large enough. ♢

Our main aim now is to find some contraction property of T. To this end we define distances in $\mathcal{S}$ and in $\mathcal{S} \times \mathcal{LC}$ by

$$D_\mathcal{S}(f,g) := \sup_{[0,\bar{x}]}\left\{\sup_I |\int_I f(x,t) - g(x,t)dt|\right\} \tag{53}$$

where $\sup_I$ is the supremum over all intervals $I \subset R$, and

$$D((f,K),(g,L)) := \max\{D_\mathcal{S}(f,g), \|K-L\|_\infty\} \tag{54}$$

Lemma 3.4: *There exists a continuous function κ_1 with $\kappa_1(0) < 1$ such that $D(T(q,R),T(q,S)) \leq \kappa_1(\bar{x})D((q,R),(q,S))$ for all $q \in \mathcal{S}$, $R, S \in \mathcal{LC}$.*

Proof: Define $(e,K) := T(q,R)$ and $(g,L) := T(q,S)$.
Since $|q*q(x,t)| \leq \rho^{(1)}(x)\rho^\infty(x)$, we find for arbitrary $a, b \in R_+$ and intervals I

$$\begin{aligned}&\left|\int_I [q*q(x,t+a) - q*q(x,t+b)]dt\right| \\ \leq\ & \lambda((I+a)\Delta(I+b))\rho^{(1)}(x)\rho^\infty(x) \\ \leq\ & 2\rho^{(1)}(x)\rho^\infty(x)|b-a|\end{aligned} \tag{55}$$

(λ is the Lebesgue measure on R and $A\Delta B = A\cup B - A\cap B$ denotes the symmetric difference between A and B) and hence

$$\begin{aligned}&|\int_I\int_0^x [q*q(\xi,t+2(R(x)-R(\xi))) - q*q(\xi,t+2(S(x)-S(\xi)))] \\ &\exp(2(x-\xi))d\xi dt| \leq 4\|R-S\|_\infty \int_0^x \rho^{(1)}(\xi)\rho^\infty(\xi)\exp(2(x-\xi))d\xi\end{aligned} \tag{56}$$

from which follows

$$D_\mathcal{S}(e,g) \leq 4\|R-S\|_\infty \int_0^{\bar{x}} \rho^{(1)}(\xi)\rho^\infty(\xi)\exp(2(x-\xi))d\xi \tag{57}$$

Since

$$\left|\int_I \phi_0(t+2R(x))dt - \int_I \phi_0(t+2S(x))dt\right| \leq 4\|R-S\|_\infty\|\phi_0\|_\infty \tag{58}$$

it follows that

$$D_S(e,g) \leq 4\|R-S\|_\infty \exp(2\bar{x}) \left(\|\phi_0\|_\infty + \int_0^{\bar{x}} \rho^{(1)}(\xi)\rho^\infty(\xi)\exp(-2\xi)d\xi\right) \quad (59)$$

and from the Lipschitz continuity of Φ_0 follows

$$\|K-L\|_\infty \leq \frac{2\|R-S\|_\infty}{\phi_{min}} \int_0^{\bar{x}} \exp(-2\xi)\rho^{(1)}(\xi)\rho^\infty(\xi)d\xi \quad (60)$$

$\diamondsuit$

Lemma 3.5: *There exists a continuous function κ_2 with $\kappa_2(0)=0$ such that $D(T(p,R),T(q,R)) \leq \kappa_2(\bar{x})D((p,R),(q,R))$ for all $p,q \in \mathcal{S}$, $R \in \mathcal{LC}$.*

Proof: Denote $(f,K) := T(p,R)$ and $(g,L) := T(q,R)$. From

$$\left|\int_I (p*p-q*q)(x,t)dt\right| \leq \int_0^\infty (|p(\tau)|+|q(\tau)|)D_S(p,q)d\tau \leq 2\rho^{(1)}D_S(p,q) \quad (61)$$

follows

$$D_S(f,g) \leq 2\int_0^{\bar{x}} \rho^{(1)}(\xi)\exp(2\xi)d\xi D_S(p,q) \quad (62)$$

Furthermore,

$$\|K-L\|_\infty \leq \frac{2}{\phi_{min}} \int_0^{\bar{x}} \rho^{(1)}(\xi)\exp(-2\xi)d\xi D_S(p,q) \quad (63)$$

$\diamondsuit$

An immediate consequence of the two lemmas is

Corollary 3.6: *For $\bar{x} > 0$ small enough, there exists a constant $\kappa < 1$ such that for all $p,q \in \mathcal{S}$, $R,S \in \mathcal{LC}$*

$$D(T(p,R),T(q,S)) \leq \kappa D((p,R),(q,S)) \quad (64)$$

We need one further result on the smoothness of $T_S(q,R)$. For this, we define the modulus of continuity of a function $f(.)$ by

$$\omega_f(h) := \int_{R_+} |f(t+h)-f(t)|dt \quad (65)$$

for $h \geq 0$, and similarly $\omega_f(x,h) := \omega_{f(x,.)}$ for a function $f(.,.)$.

Lemma 3.7: *For $(q,R) \in \mathcal{S} \times \mathcal{LC}$ and $f := T_S(q,R)$*

$$\omega_f(x,h) \leq \exp(2x)\left(\omega_{\phi_0}(h) + \int_0^x \rho^{(1)}(\xi)\omega_p(\xi,h)\exp(-2\xi)d\xi\right) \quad (66)$$

In particular, if $f = T_S(f,R)$, then $\omega_f(.,h)$ is bounded by the solution y_h of $y_h' = y_h(2+\rho^{(1)})$, $y_h(0) = \omega_{\phi_0}(h)$.

The **proof** follows from

$$\int_{R_+} |\phi_0(t+s+h) - \phi_0(t+s)|dt \leq \omega_{\phi_0}(h) \tag{67}$$

and

$$\begin{aligned}
&\int_{R_+} |p * p(\xi, t+s+h) - p * p(\xi, t+s)|dt \\
\leq\ & \int_0^\infty |p(\xi,\tau)| \int_0^\infty |p(\xi, t+s+h-\tau) - p(\xi, t+s-\tau)|dt d\tau \\
\leq\ & \omega_p(\xi, h) \int_0^\infty |p(\xi, t)dt
\end{aligned} \tag{68}$$

◇

3.3 Well-posedness of the inverse problem

With the terminology of the previous section, the inverse problem reads: Find $(f, L) \in \mathcal{S} \times \mathcal{LC}$ satisfying $(f, L) = T(f, L)$. We are now able to state the main result of this section, the (local) well-posedness (in the sense of Hadamard) of the inverse problem.

Theorem 3.8: *For $\bar{x}$ as in the corollary, there exists a unique solution of the inverse problem in $[0, \bar{x}]$. This solution depends continuously on the initial condition ϕ_0 in the following sense: There exists a constant $c < \infty$ such that for the solutions $(f^{(1)}, L^{(1)})$, $(f^{(2)}, L^{(2)})$ of the inverse problems with initial conditions ϕ_0 and ψ_0*

$$D((f^{(1)}, L^{(1)}), (f^{(2)}, L^{(2)})) \leq cD_{\mathcal{S}}(\phi_0, \psi_0) \tag{69}$$

Proof: Step 1: Existence. A solution may be obtained by the following iteration. Choose an arbitrary element $(f^{(0)}, L^{(0)}) \in \mathcal{S} \times \mathcal{LC}$ with $\omega_{f^{(0)}}(., h) \leq y_h(.)$. ($y_h$ was defined in Lemma 3.7.) Define by induction

$$(f^{(n+1)}, L^{(n+1)}) := T(f^{(n)}, L^{(n)}) \tag{70}$$

This generates a bounded sequence $f^{(n)}$ in $L^1([0, \bar{x}] \times I\!R_+) \cap L^\infty([0, \bar{x}] \times I\!R_+)$. In particular, this sequence has a subsequence converging *weak** to some f^∞ in $L^\infty([0, \bar{x}] \times I\!R_+)$. Take any bounded interval J in $[0, \bar{x}] \times I\!R_+$. Without restriction, we may assume that $f^{(n)}$ is a sequence of nonnegative functions. (If they are not, we replace them with $f^{(n)} + \rho^\infty(\bar{x})$ and obtain a nonnegative sequence which is bounded in $L^1(J)$.) Since with $\kappa < 1$,

$$D((f^{(n+1)}, L^{(n+1)}), (f^{(n)}, L^{(n)})) \leq \kappa^n D((f^{(1)}, L^{(1)}), (f^{(0)}, L^{(0)})) \tag{71}$$

we find that

$$\int_J f^{(n)}(x,t) dt dx \to \int_J f^\infty(x,t) dx dt \tag{72}$$

The set of bounded intervals is a convergence-determining class (see, e.g. [4]). Therefore the restriction of $f^{(n)}$ to any bounded interval converges *weak*$*$ to f^{∞}.

By induction follows that $\omega_{f^{(n)}}(.,h)$ is for all n bounded by the function y_h defined in Lemma 3.7. Therefore, for each $x \in [0,\bar{x}]$ the set $f^{(n)}(x,.)$ is precompact in $L^1(IR_+)$ and has a subsequence converging in L^1 to some function $f(x,.)$. In order to show that even $f^{(n)}(x,.)$ converges, assume that another subsequence converges to some function $\tilde{f}(x,.)$. From the contraction property follows for arbitrary intervals J

$$\int_J f(x,t)dt = \int_J \tilde{f}(x,t)dt \tag{73}$$

Now define for $T>0$ and for $t \in [2^{-n}kT, 2^{-n}(k+1)T)$

$$g_n(t) := \frac{2^n}{T}\int_{2^{-n}kT}^{2^{-n}(k+1)T} f(x,t)dt = \frac{2^n}{T}\int_{2^{-n}kT}^{2^{-n}(k+1)T} \tilde{f}(x,t)dt \tag{74}$$

It is well-known (see, e.g. Thm 0.5.3 in [9]) that g_n converges in $L^1([0,T])$ to $f(x,.)$ and to $\tilde{f}(x,.)$. Therefore $f(x,.) = \tilde{f}(x,.)$ a.e. So $f^{(n)}(x,.)$ is a Cauchy sequence in $L^1(IR_+)$. Denote $r_N(x) := \sup_{n,m\geq N}\int_{IR_+}|f^{(n)}(x,t)-f^{(m)}(x,t)|dt$; then $r_N(x)$ converges monotonically to 0 for all x, and the theorem on monotone convergence shows that $f^{(n)}$ is a Cauchy sequence in $L^1([0,\bar{x}]\times IR_+)$. From this and the *weak*$*$ convergence it follows that $f^{(n)}$ converges in $L^1([0,\bar{x}]\times IR_+)$ to f^{∞}.

We have to show that (f,L) is a solution to the inverse problem. This, however, follows immediately from the following observations:

1. If $f^{(n)}$ is a bounded sequence in $L^1(IR_+)\cap L^{\infty}(IR_+)$ converging in $L^1(IR_+)$ to f, then $f^{(n)} * f^{(n)}$ converges in $L^1(IR_+)$ to $f*f$.

2. If $f^{(n)} \to f$ in $L^1(IR)$ and $r^n \to r$ then $f^{(n)}(.+r^n) \to f(.+r)$ in $L^1(IR)$.

<u>Step 2:</u> Uniqueness. This is a direct consequence of the contraction property of T.
<u>Step 3:</u> Continuous dependence. Denote by (f,K) resp. (g,L) the solution of the inverse problem corresponding to the initial condition ϕ_0 resp. ψ_0. Further, define $(h,M) := T(g,L)$. (As before, T is defined with initial condition ϕ_0.) Then

$$\begin{aligned} D((f,K),(g,L)) &\leq D((f,K),(h,M)) + D((h,M),(g,L)) \\ &\leq \kappa D((f,K),(g,L)) + D((h,M),(g,L)) \end{aligned} \tag{75}$$

and therefore

$$D((f,K),(g,L)) \leq \frac{1}{1-\kappa} D((h,M),(g,L)) \tag{76}$$

From the definition of (g,L) and (h,M) follows

$$h(x,t) - g(x,t) = [\phi_0(t+2L(x)) - \psi_0(t+2L(x))]\exp(2x) \tag{77}$$

and

$$\Psi_0(2L(x)) = \Phi_0(2M(x)) \tag{78}$$

This yields

$$D_S(g,h) \leq D_S(\phi_0,\psi_0)\exp(2\bar{x}) \tag{79}$$

and

$$\int_0^{2L(x)} \psi_0(\tau)d\tau - \int_0^{2L(x)} \phi_0(\tau)d\tau = \int_{2L(x)}^{2M(x)} \phi_0(\tau)d\tau \tag{80}$$

From the latter follows

$$2|L(x) - M(x)|\phi_{min} \leq \left|\int_0^{2L(x)} (\phi_0(\tau) - \psi_0(\tau))d\tau\right| \leq D_S(\phi_0,\psi_0) \tag{81}$$

$\diamondsuit$

We want to stress that the domain of well-posedness $[0,\bar{x}]$ can be extended iteratively, as long as $f(\bar{x},.)$ satisfies the assumptions on ϕ_0. However, the arguments break down in particular when $f(\bar{x},.)$ is not bounded away from 0 in a neighborhood of $t = 0$.

4 Numerical examples

4.1 The algorithm

Our algorithm for the numerical solution of the inverse problem includes the following elements.

1. A numerical scheme for the discretization of $d_x f - 2ld_t f = g$;
2. An integration scheme for the calculation of the convolution;
3. An estimate for $l(.)$ in the next discretization step;

For the discretization of the PDE we use a classical upwind difference scheme (see, e.g. [8]):

$$\begin{aligned} f((h+1)\Delta x, k\Delta t) \ &:= \ (1-c_{CFL})f(h\Delta x, k\Delta t) + c_{CFL}f(h\Delta x,(k+1)\Delta t) \\ &+ \ \Delta x\left((1-\frac{c_{CFL}}{2})g(h\Delta x,k\Delta t) + \frac{c_{CFL}}{2}g(h\Delta x,(k+1)\Delta t)\right) \end{aligned} \tag{82}$$

with the constant

$$c_{CFL} = 2l(h\Delta x)\frac{\Delta x}{\Delta t} \tag{83}$$

which for reasons of stability has to satisfy the Courant-Friedrichs-Lewy condition

$$c_{CFL} \leq 1 \tag{84}$$

The convolution $h(t) := f * f(t)$ is discretized by

$$h(k\Delta t) := \sum_{l=0}^{k-1} f(l\Delta t) f((k-l)\Delta t)\Delta t \tag{85}$$

Denote $l_0 := l(h\Delta x)$ and $l_1 := l((h+1)\Delta x)$. An estimate for $\Delta l := l_1 - l_0$ comes from the requirement

$$f((h+1)\Delta x, 0) = \frac{1}{2l_1} = \frac{1}{2l_0}\left(1 - \frac{\Delta l}{l_0}\right) \tag{86}$$

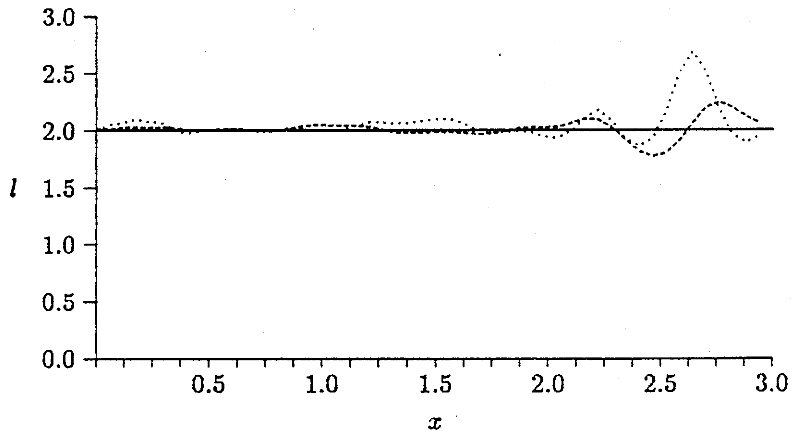
together with the estimate for $f((h+1)\Delta x, 0)$

$$\begin{aligned} f((h+1)\Delta x, 0) &= f(h\Delta x, 2l_{eff}\Delta x)\exp(2\Delta x) \qquad (87)\\ &= \left(f(h\Delta x, 0) + 2l_{eff}\Delta x\frac{f(h\Delta x, \Delta t) - f(h\Delta x, 0)}{\Delta t}\right)(1 + 2\Delta x) \\ &= f(h\Delta x, 0) + 2\Delta x\left(f(h\Delta x, 0) + l_{eff}\frac{f(h\Delta x, \Delta t) - f(h\Delta x, 0)}{\Delta t}\right) \end{aligned}$$

with $l_{eff} := l_0 + \Delta l/2$. (Note that the convolution can be neglected for t small.) It follows the estimate $\bar{\Delta l}$ for Δl

$$\bar{\Delta l} = -4l_0^2\Delta x\left(f(h\Delta x, 0) + l_0\frac{f(h\Delta x, \Delta t) - f(h\Delta x, 0)}{\Delta t}\right) \tag{88}$$

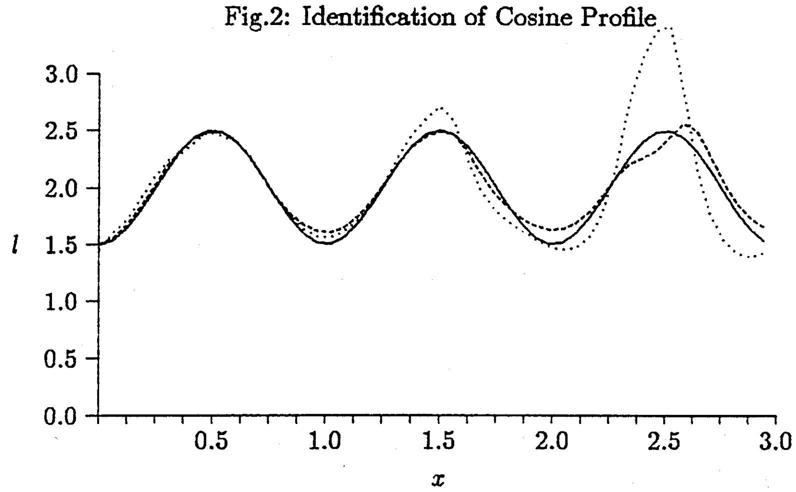
Fig.1: Identification of Constant Profile



4.2 Numerical experiments

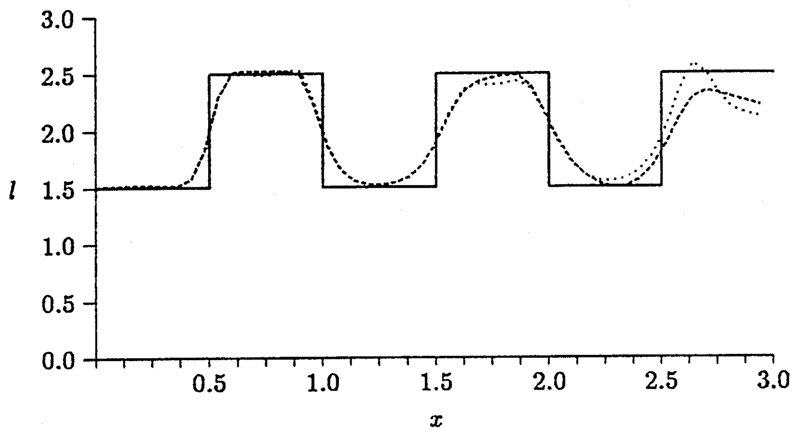
In our implementation of the above scheme for the inverse problem we have used as space discretization $\Delta x = 0.005$ (with one mean free path as the reference length) and $\Delta t = 0.040$. As the profiles $l(.)$ to be identified we have chosen the constant profile $l(.) \equiv 2$, the continuous profile $l(x) = 2 + 0.5\cos(2\pi x)$ and a discontinuous, piecewise constant profile oscillating around the value 2.

Fig.2: Identification of Cosine Profile



In order to test the algorithm for the inverse problem, we have to generate appropriate input data, i.e. an approximation of the function ϕ_0. Of course, we could use a scheme similar to that described above for the direct problem. However, an independent scheme for the direct problem seems us to be more appropriate. One way is to construct samples of an associated Markov process ([2]). Varying the number of samples and thus using more or less randomly perturbed input data gives us then an intuition on the robustness of the inverse algorithm.

Fig.3: Identification of Piecewise Constant Profile



The identified profiles are shown in Figs. 1 to 3. There the solid lines represent the profiles to be identified; the dotted lines show the identified data based on 100000 (Figs. 1,2) resp. 500000 samples (Fig. 3) of the Markov process; the dashed lines result from less perturbed input data generated with 500000 resp. 1000000 samples. All obtained results show a good agreement with the lines to be reconstructed up to a depth of two mean free paths. For larger values of

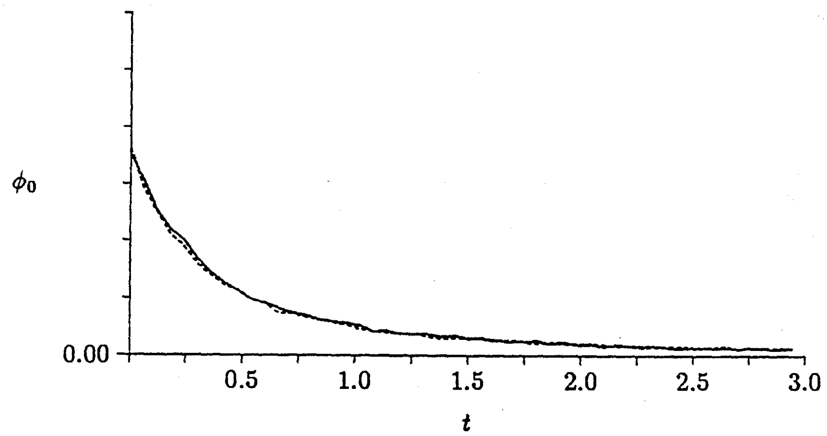
4.3 Modelling aspects

One possible extension of the results presented above would be to try to apply theory and numerics as developed here to more complicated kinetic model equations. This would certainly be possible to some extent in a straightforward manner. Let us consider as an example a semi-discrete kinetic model describing a two-dimensional particle system with the set $\{(\cos\theta, \sin\theta), \theta \in [0, 2\pi]\}$ as the range of admissible velocities. The corresponding one-dimensional stationary equation for a homogeneous medium reads

$$\cos\theta \frac{\partial}{\partial x} f(x,\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(x,\theta')d\theta' - f(x,\theta) \tag{89}$$

and has a singularity at $\cos\theta = 0$. Of course we again find an associated Riccati equation and a corresponding time-dependent version for the inverse problem (see [3]). However, the singularity causes a couple of technical problems with immediate consequences on the

Fig.4: Input data for two different models



quality of numerical solutions of the inverse problem. Therefore an alternative might be to find out whether this model can be reasonably well approximated by a simpler ersatz model without singularity. Therefore we finish our investigation with a comparison of the input data profiles ϕ_0 for the inverse problem produced by the semi-discrete model and a two-velocity model as defined in section 2. We have one free parameter for this simpler model, which is the modulus η of the velocity, and which has to be adjusted. As a good approximation we find $\eta = 0.7$. Fig. 4 shows the input profiles ϕ_0 for the semi-discrete model (solid line) and for the ersatz model (dashed line) which exhibit an excellent agreement. This gives rise to the hope that inverse problems for models with continuum range of velocities may be well approximated by models with finite (and small) number of velocities. If none of the x-components of the finite number of velocities vanishes then the resulting kinetic equation has no singularity, and the inverse problem may be expected to be well-posed, since all of the arguments described above seem to apply in this case. It is interesting to note that even in the case of zero velocity components, the Riccati equation is well-defined. (This is an immediate consequence of the Markov theory approach developed in [3].) In the corresponding time-domain equation, zero components introduce an extra convolution which make a numerical scheme a little bit more complicated, but still well tractable.

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