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G.B. Di Masi[1,2], E. Platen[3,4], W.J. Runggaldier[1]

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[1] Università di Padova
Dipartimento di Matematica
Pura ed Applicata
I – 35100 Padova
Italy

[2] CNR-LADSEB
I – 35100 Padova
Italy

[3] Australian National University
GPO Box 4
Canberra ACT, 2601
Australia

[4] Institut für Angewandte Analysis
und Stochastik
Hausvogteiplatz 5-7
D – O 1086 Berlin
Germany

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G.B. Di Masi
Università di Padova-Dipartimento di Matematica
Pura ed Applicata and CNR-LADSEB
I-35100 Padova, Italy

E. Platen
Australian National University
GPO Box 4, Canberra ACT,Australia
and IAAS, PF 1304, D-1086 Berlin, Germany

W.J. Runggaldier
Università di Padova-Dipartimento di Matematica
Pura ed Applicata
I-35100 Padova

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Abstract

The paper considers the hedging of contingent claims on assets with stochastic volatilities when the asset price is only observable at discrete time instants. Explicit formulae are given for risk-minimizing hedging strategies.

1 Introduction

In most practical cases a trader has to base the hedging of contingent claims on discrete observations of a risky asset whose price is characterized by continuous-time dynamics. In this paper we consider the hedging of a contingent claim on a continuous-time asset observable only at discrete times. Additionally, this asset may have a stochastic volatility.

Discrete-time observations cannot be considered in the framework of [2]; in fact they induce an additional risk. By a modification of the methods of Föllmer and his coauthors [2] [3], we shall prove a result that allows the determination of a risk-minimizing hedging strategy also under discrete-time observations.

2 Model and Main Result

Let the stochastic process $S = \{S_t, 0 \leq t \leq T\}$ describe the price of a risky asset (e.g. a stock or a currency) as square integrable solution of the Ito stochastic differential equation

$$dS_t = \sigma_{t-} S_t dW_t, \qquad 0 \leq t \leq T, \tag{1}$$

with initial condition S_0 and where $W = \{W_t, 0 \leq t \leq T\}$ is a given Wiener process. For simplicity we assume zero interest rate. The volatility $\sigma = \{\sigma_t, 0 \leq t \leq T\}$ is supposed to form a positive, cadlag, square integrable semi-martingale independent of W. At this point we do not further specify the stochastic volatility, but one may think of σ as proposed e.g. in Hull and White [8] or Hofmann, Platen, Schweizer [7]. At the end of the paper we shall discuss an example, where σ represents a discrete-time, finite-state, inhomogeneous Markov chain. We remark that we interpret our asset price evolution in a risk-neutral world (see [5], [6]) which could in particular be the one corresponding to the minimal equivalent martingale measure proposed in [2] (see also Hofmann, Platen, Schweizer [7]). Besides the risky asset S, we assume that there is a bond with constant unit value.

We denote the underlying probability space by $(\Omega, \underline{\mathcal{F}}, \mathcal{F}, P)$, where $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is the filtration generated by the flow of σ-algebras

$$\mathcal{F}_t = \sigma\{\sigma_r, S_s; 0 \leq r \leq T, 0 \leq s \leq t\},$$

for $t \in [0, T]$. We also set $\underline{\mathcal{F}} = \mathcal{F}_T$. The process $S = \{S_t, 0 \leq t \leq T\}$ is thus an $(\mathcal{F}, P)$-martingale.

Now let us consider the problem of hedging a contingent claim of the form $H = f(S_T)$ on the price of the risky asset at maturity T, which is traded at time $t = 0$.

A *dynamical trading strategy* $\phi = \{\phi_t = (\xi_t, \eta_t), 0 \leq t \leq T\}$ is a strategy to build a portfolio consisting at time t of the amount ξ_t of the risky asset and the amount η_t of the bond. Here ξ is assumed to be $\mathcal{F}$ -predictable and η is $\mathcal{F}$-adapted.

Given a trading strategy ϕ, we shall define its *value process* $V(\phi) = \{V_t(\phi), 0 \leq t \leq T\}$ by

$$V_t(\phi) = \xi_t S_t + \eta_t \cdot 1 \tag{2}$$

and its *cost process* $C(\phi) = \{C_t(\phi), 0 \leq t \leq T\}$ (see [3]) by

$$C_t(\phi) = V_t(\phi) - \int_0^t \xi_s dS_s. \tag{3}$$

Given a contingent claim H at maturity T, we shall say that ϕ *hedges against* H if $V_T(\phi) = H$.

The problem that we address in this paper is that of determining a suitable trading strategy that hedges against H, when the trader does not have full information about the underlying price process. We are going to model lack of information by working with a subfiltration to which S is possibly not adapted. The latter is the case if the asset is observed only at discrete times.

A right-continuous filtration $\mathcal{A} = (\mathcal{A}_t, 0 \leq t \leq T)$ with $\mathcal{A}_t \subseteq \mathcal{F}_t, t \in [0, T]$ will be called a *subfiltration* if $\mathcal{A}_0$ measures S_0 and $\mathcal{A}_T$ measures S_T.

Examples for subfiltrations are easily obtained from the σ-algebras generated by the observations S_{τ_n} at some given time instants $0 = \tau_0 < \tau_1 < \ldots < \tau_T = T$, which could represent e.g., daily stock data excluding weekends. We remark that such subfiltrations do not satisfy the assumptions in [2] since the asset S is not adapted with respect to them.

Given a subfiltration $\mathcal{A}$, a trading strategy $\phi = \{\phi_t = (\xi_t, \eta_t), 0 \leq t \leq T\}$ will be called $\mathcal{A}$*-admissible* if it hedges against H, η is $\mathcal{A}$-adapted and ξ is $\mathcal{A}$-predictable with

$$E\{\int_0^T \mid \xi_t \mid^2 \sigma_{t-}^2 S_t^2 dt \mid \mathcal{A}_0\} < \infty. \tag{4}$$

We denote by $\psi(\mathcal{A})$ the set of $\mathcal{A}$ -admissible trading strategies.

A trading strategy ϕ will be called *$\mathcal{A}$-mean self financing* if it is $\mathcal{A}$-admissible and its *conditional cost process*

$$B(\phi, \mathcal{A}) = \{B_t(\phi, \mathcal{A}) = E\{C_t(\phi) \mid \mathcal{A}_t\}, 0 \leq t \leq T\} \tag{5}$$

is an $(\mathcal{A}, P)$-martingale.

We can now adapt a lemma from [11]

Lemma 1 *There exists a bijection between the $\mathcal{A}$-predictable processes $\xi = \{\xi_t, 0 \leq t \leq T\}$ satisfying (4) and the $\mathcal{A}$-mean self-financing strategies $\phi = \{\phi_t = (\xi_t, \eta_t), 0 \leq t \leq T\}$ by putting*

$$\eta_t = E\{H \mid \mathcal{A}_t\} - \xi_t E\{S_t \mid \mathcal{A}_t\}. \tag{6}$$

Proof It is enough to note that with the choice (6) we obtain from (5), (3) and (2) the conditional cost

$$\begin{aligned} B_t(\phi, \mathcal{A}) &= E\left\{V_t(\phi) - \int_0^t \xi_s dS_s \mid \mathcal{A}_t\right\} \\ &= E\left\{H - \int_0^T \xi_s dS_s \mid \mathcal{A}_t\right\} = E\{B_T(\phi, \mathcal{A}) \mid \mathcal{A}_t\} \end{aligned}$$

as an $(\mathcal{A}, P)$-martingale.

Given a subfiltration $\mathcal{A}$ and a trading strategy $\phi \in \psi(\mathcal{A})$, we recall from [3] the notion of *remaining risk process* $R(\phi, \mathcal{A}) = \{R_t(\phi, \mathcal{A}), 0 \leq t \leq T\}$ given by

$$R_t(\phi, \mathcal{A}) = E\{(C_T(\phi) - C_t(\phi))^2 \mid \mathcal{A}_t\}. \tag{7}$$

A trading strategy $\phi^* \in \psi(\mathcal{A})$ will be called *$\mathcal{A}$-risk-minimizing* if for all $\phi \in \psi(\mathcal{A})$ and all $t \in [0, T]$ for which S_t is $\mathcal{A}_t$-measurable

$$R_t(\phi^*, \mathcal{A}) \leq R_t(\phi, \mathcal{A}) \tag{8}$$

In this paper we address the following

Problem. Given any subfiltration $\mathcal{A}$, determine an $\mathcal{A}$-mean self financing and $\mathcal{A}$-risk-minimizing strategy ϕ^* that hedges against H, together with its value process $V_t(\phi^*)$.

To obtain an answer to this problem take into account that under our filtration $\mathcal{F}$, using a well known decomposition (see [10]) the contingent claim H can be represented in the form

$$H = E\{H \mid \mathcal{F}_t\} + \int_t^T \mu_s dS_s, \tag{9}$$

where $\mu = \{\mu_\tau, t \leq \tau \leq T\}$ is $\mathcal{F}$-predictable.

Theorem 1 *Given a subfiltration $\mathcal{A}$ and a contingent claim H with $E\{H^2 \mid \mathcal{A}_0\} < \infty$ we consider the $\mathcal{A}$-predictable process $\mu^* = \{\mu_t^*, 0 \leq t \leq T\}$ with*

$$\mu_t^* = \frac{E\{\mu_t \sigma_{t-}^2 S_t^2 \mid \mathcal{A}_t\}}{E\{\sigma_{t-}^2 S_t^2 \mid \mathcal{A}_t\}}. \tag{10}$$

Then $\phi = (\xi, \eta) \in \psi(\mathcal{A})$ is an $\mathcal{A}$-risk-minimizing and $\mathcal{A}$-mean self financing strategy that hedges against H if and only if $\xi_t = \mu_t^*$ with equality holding in $L_2(dP \times d\langle S \rangle)$ and η_t is choosen according to (6) for all $t \in [0, T]$. Furthermore, we have $v_0 = V_0(\phi) = E\{H \mid \mathcal{A}_0\}$.

Proof Given any $\phi \in \psi(\mathcal{A})$ we can compute the remaining risk for times t for which S_t is $\mathcal{A}_t$-measurable in the form

$$\begin{aligned} R_t(\phi, \mathcal{A}) &= E\{(C_T(\phi) - C_t(\phi))^2 \mid \mathcal{A}_t\} \\ &= E\left\{\left(H - E\{H \mid \mathcal{A}_t\} - \int_t^T \xi_s dS_s\right)^2 \mid \mathcal{A}_t\right\} \\ &= E\left\{\left(\int_t^T (\mu_s - \xi_s) dS_s + \theta_t\right)^2 \mid \mathcal{A}_t\right\}, \end{aligned}$$

with

$$\theta_t = E\{H \mid \mathcal{F}_t\} - E\{H \mid \mathcal{A}_t\}$$

We get

$$\begin{aligned} Z_t : &= R_t(\phi, \mathcal{A}) - E\{\theta_t^2 \mid \mathcal{A}_t\} \\ &= E\left\{\left(\int_t^T (\mu_s - \xi_s\right)^2 \sigma_{s-}^2 S_s^2 ds \mid \mathcal{A}_t\right\} \\ &= E\left\{\int_t^T E\{(\mu_s - \mu_s^*)^2 \sigma_{s-}^2 S_s^2 \mid \mathcal{A}_s\} ds \mid \mathcal{A}_t\right\} + \\ & \quad E\left\{\int_t^T E\{[2(\mu_s - \mu_s^*)(\mu_s^* - \xi_s) + (\mu_s^* - \xi_s)^2] \sigma_{s-}^2 S_s^2 \mid \mathcal{A}_s\} ds \mid \mathcal{A}_t\right\} \end{aligned}$$

Finally, we obtain using (10)

$$Z_t = E\{\int_t^T E\{[(\mu_s - \mu_s^*)^2 + (\mu_s^* - \xi_s)^2]\sigma_{s-}^2 S_s^2 \mid \mathcal{A}_s\}ds \mid \mathcal{A}_t\}.$$

Taking into account that $(\mu_s^* - \xi_s)^2\sigma_{s-}^2 S_s^2$ is nonnegative it then follows that $R_t(\phi, \mathcal{A})$ is minimized for all $t \in [0,T]$ for which S_t is $\mathcal{A}_t$-measurable if and only if

$$\int_0^T E\{(\mu_s^* - \xi_s)^2\sigma_{s-}^2 S_s^2 \mid \mathcal{A}_s\}ds = 0,$$

which corresponds to the statement of the theorem recalling that by (1)

$$d\langle S\rangle_t = \sigma_{t-}^2 S_t^2 dt.$$

The above proof uses mainly the fact that the stochastic integral $\int(\mu_s - \mu_s^*)dS_s$ is orthogonal to other stochastic integrals $\int \alpha_s dS_s$ where α_s is $\mathcal{A}$-adapted.Process μ^* appears in this way as a conditional expectation of μ with respect to $dP \times d\langle S\rangle$.

3 Computation of a risk-minimizing strategy under discrete observation

To get an explicit example we now assume the volatility σ to be a discrete-time finite-state inhomogeneous and right-continuous Markov chain which is sometimes suggested by the analysis of historical volatilities as in Galai [4]. We shall assume that the jump times are the points $\tau_n, n = 0, 1, \ldots, N$, at which we also observe the asset price S_{τ_n}. Thus our subfiltration is

$$\mathcal{A}_t = \mathcal{F}_t^* = \sigma\{S_{\tau_n}, 0 \le \tau_n \le t\}.$$

We denote by $J = \{a_1, \ldots, a_k\}$ the finite state space of the Markov chain σ and by

$$p_{\tau_n}(j,i) = P\{\sigma_{\tau_{n+1}} = a_j \mid \sigma_{\tau_n} = a_i\}$$

its transition probabilities at time τ_n. Then we obtain from Theorem 1

$$\xi_t^* = \frac{E\{\mu_t \sigma_{\tau_n}^2 S_t^2 \mid \mathcal{F}_{\tau_n}^*\}}{E\{\sigma_{\tau_n}^2 S_t^2 \mid \mathcal{F}_{\tau_n}^*\}},$$

for $t \in (\tau_n, \tau_{n+1}], n = 1, \ldots, N-1$, where μ_t is $\mathcal{F}_t$-measurable and follows from a Black-Scholes type formula as described in Hull, White [8].

In principle there is no problem to compute explicitly ξ_t^* in the above case. We omit these formulae. They depend on the conditional probabilities $P(\sigma_{\tau_n} = a_j \mid \mathcal{F}^*_{\tau_n})$ for $t \in (\tau_n, \tau_{n+1}]$ and $j = 1, \ldots, k$. These probabilities can be estimated by filtering techniques, see e.g. Di Masi, Runggaldier [1].

In many cases, when one finds another specific structure for the volatility, one can use stochastic numerical methods as described in Kloeden, Platen [9] or used in Hofmann, Platen, Schweizer [7] to compute the hedging strategy and the option price.

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