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Global existence for rate-independent gradient plasticity at finite strain

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Abstract

We provide a global existence result for the time-continuous elastoplasticity problem using the energetic formulation. For this we show that the geometric nonlinearities via the multiplicative decomposition of the strain can be controlled via polyconvexity and a priori stress bounds in terms of the energy density. While temporal oscillations are controlled via the energy dissipation the spatial compactness is obtain via the regularizing terms involving gradients of the internal variables.

1 Introduction

The theory of elastostatics at finite strains has seen a rapid development within the last decades. The fundamental work on polyconvex materials developed in [Bal77] provided a basis for a general theory that allows to treat the geometric nonlinearities arising in physically correct model. In particular, a stored-energy density $W : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty \overset{\text{def}}{=} \mathbb{R} \cup \{\infty\}$ has to satisfy

$$\begin{aligned} W(RF) &= W(F) && \text{for} \quad R \in \mathrm{SO}(d), F \in \mathbb{R}^{d\times d}, \\ W(F) &= +\infty && \text{for} \quad \det F \leq 0, \\ W(F) &\to +\infty && \text{for} \quad \det F \to 0^+. \end{aligned}$$

Approximately at the same time the theory of elastoplasticity obtained a sound mathematical basis starting from [Mor74], see also [Tem85, Alb98, HaR99] for surveys on further developments. However, this theory is restricted to the case of small strains and the so-called additive split $\mathrm{e}(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) = \mathrm{e}_{\mathrm{el}} + \mathrm{e}_{\mathrm{pl}}$, as it fundamentally depends on the methods of convex analysis in Banach spaces.

Elastoplasticity at finite strain is usually based on the multiplicative decomposition $\nabla\varphi = F = F_{\mathrm{el}}F_{\mathrm{pl}}$, introduced in [Lee69]. Meanwhile, these models are heavily used in engineering and are quite successful in predicting macroscopic deformation processes like deep drawing and other forming processes, see e.g. [SiO85, MiS92, NeW03]. A major advance was the observation in [OrS99] that the time-incremental problems in rate-independent and in the viscoplastic case can be written as variational problems, viz., the sum of the increments in the stored energy and in the dissipated energy has to be minimized (at least locally) to obtain the new state at the next time level. This idea opened up the whole toolbox of the direct methods in the calculus of variations and lead, in particular, to the analysis of microstructures in elastoplasticity, see [OrR99, ORS00, CHM02, Mie03a, CoT05, GM*06].

In this paper we follow a similar spirit but our aim is to develop a theory for the time-continuous setting. Thus, we want to connect the classical formation involving the elastic equilibrium and the plastic flow law with an analytic approach that allows us to handle the associated geometric nonlinearities. For this, we introduce some notations. Let $\varphi : \Omega \to \mathbb{R}^d$ denote the deformation, $P : \Omega \to \mathrm{SL}(d) = \{ P \in \mathbb{R}^{d\times d} \mid \det P = 1 \}$ the plastic tensor, and $p : \Omega \to \mathbb{R}^m$ some hardening variables. Then, we assume that the stored-energy functional takes the form

$$\widetilde{\mathcal{E}}(t, \varphi, P, p) = \int_\Omega W(x, \nabla\varphi P^{-1}, P, p, \nabla P, \nabla p)\,\mathrm{d}x - \langle \ell(t), \varphi\rangle.$$

Here, the gradients $(\nabla P, \nabla p)$ are essential to provide compactness and prevent formation of microstructures. The plastic flow law is expressed through a dissipation distance

$$\mathcal{D}(P_0, p_0, P_1, p_1) = \int_\Omega D(x, P_0(x), p_0(x), P_1(x), p_1(x))\,\mathrm{d}x,$$

where $D(x, \cdot, \cdot)$ satisfies a triangle inequality.

If additionally at the Dirichlet boundary $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$ the boundary conditions

$$\varphi(t, x) = g_{\mathrm{Dir}}(t, x) \text{ for } (t, x) \in [0, T]\times\Gamma_{\mathrm{Dir}} \tag{1.1}$$

and an initial condition $z_0 = (P_0, p_0) \in \mathcal{Z}$ are prescribed, we want to find a so-called *energetic solution* $\boldsymbol{q} = (\varphi, P, p) : [0, T] \to \mathcal{Q} = \mathcal{Y}\times\mathcal{Z}$ satisfying for all $t \in [0, T]$ the *stability condition* (S) and the *energy balance* (E):

$$\begin{array}{ll} \text{(S)} & \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) \le \widetilde{\mathcal{E}}(t, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}(t), \widehat{\boldsymbol{q}}) \text{ for all } \widehat{\boldsymbol{q}} \in \mathcal{Q} \text{ satisfying (1.1)} \\ \text{(E)} & \widetilde{\mathcal{E}}(t, \boldsymbol{q}(t)) + \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{q}; [0, t]) = \widetilde{\mathcal{E}}(0, \boldsymbol{q}(0)) + \int_0^t \pi(s)\,\mathrm{d}s, \end{array} \tag{1.2}$$

where $\mathrm{Diss}_{\mathcal{D}}(\boldsymbol{q}; [r, s]) = \sup\{ \sum_1^N \mathcal{D}(P(\tau_{j-1}), p(\tau_{j-1}), P(\tau_j), p(\tau_j)) \mid \text{all partitions of } [r, s] \}$ and $\pi : [0, T] \to \mathbb{R}$ is the power of the external loadings:

$$\pi(t) = -\langle \dot{\ell}(t), \varphi(t)\rangle - \int_{\Gamma_{\mathrm{Dir}}} \tau(t, x) \cdot \dot{g}_{\mathrm{Dir}}(t, x)\,\mathrm{d}x, \tag{1.3}$$

with τ being the normal stress on the boundary.

In Section 2 we repeat some of the arguments in [Mie03b] that explain why the concept of energetic solutions can be seen as a weak version of the classical plasticity formulation. The major advantage of (S) and (E) is that it avoids derivatives and is based solely on the functionals $\widetilde{\mathcal{E}}$ and $\mathcal{D}$, which need not be smooth or even continuous. In Section 3 we formulate precise assumptions on W, D, and g_{Dir} that allow us to construct solutions in suitable Sobolev spaces. The main existence result is Theorem 3.1 which states the global existence of energetic solutions for a large class of models of elastoplasticity. In particular, we show that the classical cases of kinematical hardening and isotropic hardening are included, see Examples 3.3 and 3.4.

The existence theory is based on an abstract result for *rate-independent energetic systems* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$, where $\mathcal{Q} = \mathcal{Y}\times\mathcal{Z}$ are topological spaces (here: weakly closed subsets of reflexive Banach spaces $Q = Y\times Z$ equipped with the weak topology). This theory was developed in [MiT99, MTL02, MaM05, FrM06], and in Section 4 we provide the suitable version with all the abstract assumptions.

This abstract approach relies on incremental minimization. For a partition $0 = t_0 < t_1 < \cdots < t_K = T$ and an initial state one has to solve iteratively, for $k = 1, \ldots, K$,

$$\boldsymbol{q}_k \text{ minimizes } \ \boldsymbol{q} \mapsto \widetilde{\mathcal{E}}(t_k, \boldsymbol{q}) + \mathcal{D}(\boldsymbol{q}_{k-1}, \boldsymbol{q}). \tag{1.4}$$

These minimization problems are close to the ones discussed above, the difference being that we use the dissipation distance $\mathcal{D}$ whereas most other works approximate this by some explicit predictors. Here it is crucial that $\mathcal{D}$ is a quasi-metric on $\mathcal{Z}$, i.e., $\mathcal{D}(z_0, z_1) = 0$ implies $z_0 = z_1$ and the triangle inequality holds (see (4.D1) and (4.D2)). The latter property is essential to provide natural a priori bounds while the former allows to apply a generalized Helly selection principle, cf. [MaM05, FrM06].

In [Mie04] it is shown that (1.4) can be solved for arbitrarily large K even without the regularizing terms $(\nabla P, \nabla p)$, but under severe restrictions on W and D. In [MiM06], where the term $(\operatorname{curl} P)P^T$ is used for regularization, again the solvability of (1.4) is established for more general W and D. Here, we use full regularization via $(\nabla P, \nabla p)$ which is also common in engineering models, cf. [FlH97, Gur00, Gur02].

In Section 5 we then show that all assumptions are satisfied in our elastoplastic setting that includes kinematic hardening (cf. Example 3.3) as well as isotropic hardening (cf. Example 3.4). Our theory relies on sufficiently strong hardening to obtain coercivity and, thus, failure effects like localization or fracture are prevented. Similarly, the regularization via $(\nabla P, \nabla p)$ prevents the formation of microstructure, cf. [CHM02, BC*04].

At the heart of the analysis is the treatment of the nonlinearities arising through the geometric structure of finite strains. To treat the case of time-dependent boundary conditions g_{Dir} we seek for $\varphi(t, \cdot)$ in the form

$$\varphi(t, x) = g_{\mathrm{Dir}}(t, y(s)) \quad \text{with} \quad y \in \mathcal{Y} \overset{\text{def}}{=} \{\, y \in \mathrm{W}^{1,q_Y}(\Omega; \mathbb{R}^d) \mid y|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id}\,\}$$

and set $\mathcal{E}(t, y, P, p) = \widetilde{\mathcal{E}}(t, g_{\mathrm{Dir}}(t, \cdot)\circ y, P, p)$. As a consequence the integrand of $\mathcal{E}$ depends on the product

$$\nabla g_{\mathrm{Dir}}(t, y(x))\nabla y(x)P(x)^{-1}. \tag{1.5}$$

To allow for finite-strain elasticity we assume that W is polyconvex in the corresponding argument. Hence, to establish the lower semicontinuity, we need to show that the minors $\mathbb{M}_s$ of order $s \in \{1, \ldots, d\}$ of the term in (1.5) are weakly continuous in suitable Sobolev spaces. For this, we use the Cauchy–Binet formula

$$\mathbb{M}_s(GFP^{-1}) = \mathbb{M}_s(G)\mathbb{M}_s(F)\mathbb{M}_s(P^{-1}) = \mathbb{M}_s(G)\mathbb{M}_s(F)\mathbb{K}_{d-s}(P),$$

for $\det P = 1$, and the celebrated weak continuity of $\mathbb{M}_s(\nabla y)$ (cf. [Res67, Bal77]), and strong convergence of P, which is obtained from coercivity of W in ∇P.

Another important ingredient, which again relies on the multiplicative structure of $\mathrm{GL}^+(d)$, is the control of the power of the external forces $\partial_t \mathcal{E}(t, y, P, p)$. In particular, it is then allowed to replace π in (1.3) by $\partial_t \mathcal{E}(t, y(t), P(t), p(t))$. Using the condition $|\partial_F W(F)F^T| \leq c_1^W(W(F)+c_0^W)$, which was introduced in [BOP91] and popularized in [Bal02], we obtain the formula

$$\begin{aligned} &\partial_t \mathcal{E}(t,q) = \textstyle\int_\Omega (\partial_F W(\widetilde{F}, z, \nabla z)\widetilde{F}^T) : V(t,x)\,\mathrm{d}x \\ &\text{with } \widetilde{F}(t,x) = \nabla g_{\mathrm{Dir}}(t, y(x))\nabla y(x) P(x)^{-1} \\ &\text{and } \ V(t,x) = \nabla g_{\mathrm{Dir}}(t,y(x))^{-1}\nabla \dot{g}_{\mathrm{Dir}}(t, y(x)). \end{aligned}$$

Under suitable assumptions on g_{Dir} this allows for an estimate of $\partial_t \mathcal{E}(t,q)$ in terms of $\mathcal{E}(t,q)$ itself and to derive further helpful continuity properties for $\partial_t \mathcal{E}$.

In the final Section 6 we discuss some aspects of the developed theory and give several possible generalizations. The plastic tensor P can be chosen from general subset of $\mathrm{GL}^+(d)$, which does not have a group structure nor a manifold structure. Such situations occur in crystal plasticity with infinite latent hardening. Moreover, it is possible to include a condition that forbids (global) self-interpenetration.

2 Modeling via energetic solutions

We consider an elastic body $\Omega \subset \mathbb{R}^d$ which is bounded and has a Lipschitz boundary $\partial\Omega$. A deformation is a mapping $\varphi : \Omega \to \mathbb{R}^d$ such that the deformation gradient $F(x) = \nabla\varphi(x)$ exists for a.e. $x \in \Omega$ and satisfies

$$F(x) \in \mathrm{GL}^+(d) \stackrel{\mathrm{def}}{=} \{\, F \in \mathbb{R}^{d\times d} \mid \det F > 0 \,\}.$$

Moreover, in this section we simplify by assuming that φ satisfies time-independent displacement boundary conditions $\varphi(t,x) = g_{\mathrm{Dir}}(x)$ on $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$.

The internal plastic state at a material point $x \in \Omega$ is described by the plastic tensor $P = F_{\mathrm{pl}} \in \mathrm{GL}^+(d)$ and a possibly vector-valued hardening variable $p \in \mathbb{R}^m$. We shortly write $z = (P, p)$ to denote the set of all plastic variables. The major assumption in finite-strain elastoplasticity is the multiplicative decomposition of the deformation gradient F into an elastic and a plastic part: $F = F_{\mathrm{el}}P$ with $P = F_{\mathrm{pl}}$. The point of this decomposition is that the elastic properties will depend only on F_{el}, whereas previous plastic transformations through P are completely forgotten. However, the hardening variable p will remember changes in P and may influence the elastic properties (e.g., via back stresses).

The deformation process is governed by two principles. First, we have energy storage which gives rise to the equilibrium equations and, second, we have dissipation due to plastic transformations which gives rise to the plastic flow rule. Energy storage is described by the Gibbs energy

$$\mathcal{E}(t,\varphi,z)=\int_\Omega \widetilde{W}(x,\nabla\varphi(x),z(x),\nabla z(x))\,\mathrm{d}x-\langle \ell(t),\varphi\rangle, \tag{2.1}$$

where $\langle \ell(t),\varphi\rangle=\int_\Omega f_{\mathrm{ext}}(t,x)\cdot\varphi(x)\,\mathrm{d}x+\int_{\Gamma_{\mathrm{Neu}}} h_{\mathrm{ext}}(t,x)\cdot\varphi(x)\,\mathrm{d}a(x)$ denotes the process-time dependent loading and $\Gamma_{\mathrm{Neu}}=\partial\Omega/\Gamma_{\mathrm{Dir}}$. The major constitutive assumption is the multiplicative decomposition

$$\widetilde{W}(x,F,P,p,A)=W(x,FP^{-1},P,p,A). \tag{2.2}$$

It seems that the form on the right-hand side is not more specific as the one on the left-hand side. However, in Section 3 we will make different assumption on the dependence of W with respect to the variable $F_{\mathrm{el}}=FP^{-1}$ and P. Subsequently, we will drop the variable x for notational convenience, but the whole analysis works in the inhomogeneous case as well.

The dissipation effects are usually modeled by prescribing yield surfaces. For our purpose it is more convenient and mathematically clearer to start on the other side, namely with the dissipation metric. In mechanics this metric is called dissipation potential, since the dissipational friction forces are obtained from it via differentiation with respect to the plastic rates. We emphasize that the natural setup for the plastic transformation P is that of an element of a Lie group $\mathfrak{P}\subset\mathrm{GL}^+(d)$ (however see Section 6 for more general cases). A usual assumption is *plastic incompressibility*, which gives

$$\mathfrak{P}=\mathrm{SL}(d)\overset{\mathrm{def}}{=}\{\,P\mid \det P=1\,\}.$$

However, $\mathfrak{P}=\mathrm{GL}^+(d)$ or a single-slip system $\mathfrak{P}=\{\,\mathbf{1}+\gamma e_1\otimes e_2\mid\gamma\in\mathbb{R}\,\}$ may also be possible. A dissipation potential is a mapping

$$R:\Omega\times\mathrm{T}(\mathfrak{P}\times\mathbb{R}^m)\to[0,\infty], \tag{2.3}$$

which is called a dissipation metric if it is convex and positively homogeneous of degree 1 in the rate

$$R(P,p,\alpha\dot P,\alpha\dot p))=\alpha R(P,p,\dot P,\dot p)\text{ for }\alpha\geq 0. \tag{2.4}$$

(Again, we dropped the variable x for notational convenience.) This condition leads to rate-independent material behavior. Together with the multiplicative decomposition one assumes *plastic indifference*:

$$R(P\widehat{P},p,\dot P\widehat{P},\dot p)=R(P,p,\dot P,\dot p)\text{ for all }\widehat{P}\in\mathfrak{P}.$$

This property implies the existence of $\widehat{R}:\mathbb{R}^m\times\mathbb{R}^m\times\mathfrak{p}\to[0,\infty]$ with $R(P,p,\dot P,\dot p)=\widehat{R}(p,\dot p,\dot PP^{-1})$. Here $\mathfrak{p}=\mathrm{T}_{\mathbf{1}}\mathfrak{P}$ is the Lie algebra associated with the Lie group $\mathfrak{P}$, and $\dot PP^{-1}$ is strictly speaking the right translation of $\dot P(t)\in\mathrm{T}_{P(t)}\mathfrak{P}$ to $\mathfrak{p}=\mathrm{T}_{\mathbf{1}}\mathfrak{P}$.

An fundamental feature of our theory is the induced dissipation distance D on $\mathfrak{P}\times\mathbb{R}^m$ defined via (recall $z=(P,p)$)

$$D(z_0,z_1)=\inf\{\,\int_0^1 R(z(s),\dot z(s))\,\mathrm{d}s\mid z\in\mathrm{C}^1([0,1],\mathfrak{P}\times\mathbb{R}^m),z(0)=z_0,z(1)=z_1\,\}.$$

It is important to note that we didn't assume symmetry (i.e., $R(z,-\dot z)=R(z,\dot z)$) which would contradict hardening. Thus, $D(\cdot,\cdot)$ will not be symmetric either. However, the triangle inequality

$$D(z_1,z_3)\le D(z_1,z_2)+D(z_2,z_3)$$

holds as an immediate consequence from the definition. The replacement of the (local) dissipation metric R by the (global) dissipation distance in the incremental problem (1.4) is the essential step to make to avoid the linear structure and to obtain good a priori bounds. Plastic difference implies that the dissipation distance satisfies

$$D(P_1,p_1,P_2,p_2)=D(\mathbf{1},p_1,P_2P_1^{-1},p_2). \tag{2.5}$$

Integration gives the total dissipation between two internal states $z_j:\Omega\to\mathfrak{P}\times\mathbb{R}^m$ via

$$\mathcal{D}(z_0,z_1)=\int_\Omega D(z_0(x),z_1(x))\,\mathrm{d}x. \tag{2.6}$$

Our work is based on the following energetic formulation. To make the arguments more rigorous we define the set of kinematically admissible deformations via

$$\mathcal{Y}=\{\,\varphi\in \mathrm{W}^{1,q_1}(\Omega;\mathbb{R}^d)\mid \varphi|_{\Gamma_{\mathrm{Dir}}}=g_{\mathrm{Dir}}\,\}, \tag{2.7}$$

where the integrability power q_1 in W^{1,q_1} will be chosen larger than d in order to apply the theory of polyconvexity. The loading can then be considered as a function $\ell:[0,T]\to \mathrm{W}^{1,q_1}(\Omega,\mathbb{R}^d)^*$, where * denotes the dual space.

The set of admissible internal states takes the form

$$\mathcal{Z}=\{\,z\in \mathrm{L}^{q_2}(\Omega;\mathbb{R}^{d\times d}\times\mathbb{R}^m)\mid z(x)\in\mathfrak{P}\times\mathbb{R}^m \text{ a.e. on } \Omega,\ z\in \mathrm{W}^{1,r}(\Omega)\,\}. \tag{2.8}$$

Because of the image space, which is a manifold, it is not clear whether it is reasonable to equip $\mathcal{Z}$ with a Banach-space structure. It would seem more natural to equip $\mathcal{Z}$ with the metric $\mathcal{D}$ and to use arguments of general metric spaces. However, our analysis will need some underlying Banach spaces as indicated in (2.8).

As in the introduction we have now an energetic system $(\mathcal{Y}\times\mathcal{Z},\mathcal{E},\mathcal{D})$ and can define energetic solutions via (S) and (E). The major advantage of the energetic formulation is that neither derivatives of the constitutive functions W and R nor higher derivatives of the solution (φ,z) are needed (just those for calculating $\mathcal{E}$). Nevertheless, (S) and (E) are strong enough to determine the physically relevant solutions. We refer to [MiT04] for uniqueness results under additional convexity assumptions. Moreover, it is shown in [Mie03b] that sufficiently smooth solutions (φ,z) of (S) and (E) satisfy the classical equations of elastoplasticity, namely the equilibrium equation (2.9) and the plastic flow rule (2.10) in the form of an internal force balance (Biot's equation):

$$\left.\begin{array}{rcll} -\operatorname{div} T(t,x) &=& f_{\mathrm{ext}}(t,x) & \text{in } \Omega,\\ \varphi(t,x) &=& g_{\mathrm{Dir}}(x) & \text{on } \Gamma_{\mathrm{Dir}},\\ T(t,x)\nu(x) &=& h_{\mathrm{ext}}(t,x) & \text{on } \Gamma_{\mathrm{Neu}}, \end{array}\right\} \tag{2.9}$$

where $T(t,x) \overset{\text{def}}{=} \frac{\partial}{\partial F_{\text{el}}} W(\nabla\varphi(t,x)P(t,x)^{-1}, z(t,x), \nabla z(t,x))P(t,x)^{-\mathsf{T}}$, and

$$0 \in \partial^{\text{sub}}_{\dot z} R(z(t,x), \dot z(t,x)) - Q(t,x), \tag{2.10}$$

where $\partial^{\text{sub}}_{\dot z} R(z,\dot z)$ denotes the subgradient of the convex function $R(z,\cdot) : \mathrm{T}_z(\mathfrak{P}\times\mathbb{R}^m) \to [0,\infty]$ and Q is the thermodynamically conjugated driving force to z, i.e.,

$$Q \overset{\text{def}}{=} -\tfrac{\partial}{\partial z} W(FP^{-1}, z, \nabla z) + \operatorname{div}\big(\tfrac{\partial}{\partial(\nabla z)}(FP^{-1}, z, \nabla z)\big).$$

Using the elastic domain

$$\mathbb{Q}(z) = \partial^{\text{sub}}_{\dot z} R(z,0) \subset \mathrm{T}^*_z(\mathfrak{P}\times\mathbb{R}^m)$$

the Legendre-Fenchel transform shows that (2.10) is equivalent to the plastic rate equation $\dot z \in \partial\chi_{\mathbb{Q}(z)}(Q) = \mathrm{N}_Q\mathbb{Q}(z)$, where $\chi_{\mathbb{Q}}$ is the indicator function and $\mathrm{N}\mathbb{Q}$ the normal cone.

3 Assumptions and results

We formulate the precise assumption here. For notational simplicity we omit volume and surface forces, i.e., we let $\ell \equiv 0$. Instead, the process will be driven by time-dependent Dirichlet data $g_{\text{Dir}}(t,\cdot)$. See [FrM06] and Remark 3.5 for the simple changes to be done, if forces have to be included. Moreover, we will assume $\mathfrak{P} = \mathrm{SL}(d)$ as this is the most important case and as it avoids complications involving additional terms "$\det P$" appearing otherwise.

The domain $\Omega \subset \mathbb{R}^d$ is bounded and has a Lipschitz boundary. The Dirichlet part Γ_{Dir} of the boundary is assumed to have positive surface measure. The time-dependent Dirichlet data are imposed via a function $g_{\text{Dir}} : [0,T]\times\Gamma_{\text{Dir}} \to \mathbb{R}^d$, and we assume that it can be extended to all of $\mathbb{R}^d$ as follows:

$$\begin{aligned} &g_{\text{Dir}} \in \mathrm{C}^1([0,T]\times\mathbb{R}^d;\mathbb{R}^d), \quad \nabla g_{\text{Dir}} \in \mathrm{BC}^1([0,T]\times\mathbb{R}^d, \mathrm{Lin}(\mathbb{R}^d;\mathbb{R}^d)) \\ &\text{and } |\nabla g_{\text{Dir}}(t,x)^{-1}| \le C \text{ for all } (t,x) \in [0,T]\times\mathbb{R}^d, \end{aligned} \tag{3.1}$$

where "BC" stands for bounded and continuous. Thus, for each $t \in [0,T]$ the mapping $g_{\text{Dir}}(t,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is a global diffeomorphism. The desired deformation $\varphi : [0,T]\times\Omega \to \mathbb{R}^d$ is searched in the form of a composition

$$\varphi(t,x) = g_{\text{Dir}}(t, y(t,x)) \quad \text{with } y(t,\cdot) \in \mathcal{Y},$$

where the space of admissible deformations y is given by

$$\mathcal{Y} \overset{\text{def}}{=} \{\, y \in Y \mid y|_{\Gamma_{\text{Dir}}} = \mathrm{id} \,\} \quad \text{with } Y = \mathrm{W}^{1,q_Y}(\Omega;\mathbb{R}^d),$$

where $q_Y \in\,]d,\infty[$ is specified later. The composition $\varphi = g_{\text{Dir}}(t,\cdot)\circ y$ leads to a multiplicative split of the deformation gradient

$$\nabla\varphi(t,x) = \nabla g_{\text{Dir}}(t, y(x))\nabla y(x)$$

due to the classical chain rule. It is this multiplicative decomposition of the deformation tensor that is compatible with the assumption of finite strains.

The internal variable will be $z = (P, p) \in \mathrm{SL}(d)\times\mathbb{R}^m)$, and the space $\mathcal{Z}$ is chosen as the set

$$\mathcal{Z} \stackrel{\text{def}}{=} \{\, (P, p) \in Z \mid P(x) \in \mathrm{SL}(d) \text{ a.e. in } \Omega \,\}$$
$$\text{with } Z = \big(\mathrm{L}^{q_P}(\Omega; \mathbb{R}^{d\times d}) \cap \mathrm{W}^{1,r}(\Omega; \mathbb{R}^{d\times d}\big) \times \big(\mathrm{L}^{q_p}(\Omega; \mathbb{R}^m) \cap \mathrm{W}^{1,r}(\Omega; \mathbb{R}^m)\big),$$

with $q_P, q_p, r \in\,]1, \infty[$ to be specified later (see Section 6 for the physically relevant case $r = 1$). Clearly, Y, Z, and $Q = Y\times Z$ are separable, reflexive Banach spaces, and $\mathcal{Y}, \mathcal{Z}$, and $\mathcal{Q} = \mathcal{Y}\times\mathcal{Z}$ are weakly closed subsets of the corresponding Banach spaces.

The stored-energy functional $\mathcal{E}$ and the dissipation distance $\mathcal{D}$ take the forms

$$\mathcal{E}(t, y, z) \stackrel{\text{def}}{=} \int_\Omega W(x, \nabla g_{\mathrm{Dir}}(t, y(x))\nabla y(x)P(x)^{-1}, z(x), \nabla z(x))\,\mathrm{d}x,$$
$$\mathcal{D}(z_0, z_1) \stackrel{\text{def}}{=} \int_\Omega D(x, z_0(x), z_1(x))\,\mathrm{d}x.$$

To define the conditions on D and W we use the notion of a *normal integrand.* If U is a topological space and $\mathcal{B}(U)$ its Borel σ-algebra, then a function $f : \Omega\times U \to \mathbb{R}_\infty \stackrel{\text{def}}{=} \mathbb{R}\cup\{\infty\}$ is called a normal integrand, if

(NI1) f is $\mathcal{L}_\Omega\times\mathcal{B}(U)$ measurable,

(NI2) for a.a. $x \in \Omega$: $f(x, \cdot) : U \to \mathbb{R}_\infty$ is lower semicontinuous,

where $\mathcal{L}_\Omega$ denotes the (Lebesgue) measurable subsets of Ω. Note that for each measurable mapping $u : \Omega \to U$ the composition $x \mapsto f(x, u(x))$ is measurable. We define the *domain* $\mathrm{dom}\, f$ via

$$\mathrm{dom}\, f \stackrel{\text{def}}{=} \{\, (x, u) \in \Omega\times U \mid f(x, u) < \infty \,\}.$$

For the quasi-distances $D(x, \cdot, \cdot)$ we impose the conditions

$$D : \Omega \times (\mathrm{SL}(d)\times\mathbb{R}^m)^2 \to [0, \infty] \text{ is a normal integrand;} \tag{3.2a}$$

$$\forall\, x \in \Omega,\ z_1, z_2 \in \mathrm{SL}(d)\times\mathbb{R}^m : \quad D(x, z_1, z_2) = 0 \iff z_1 = z_2; \tag{3.2b}$$

$$\forall\, x \in \Omega,\ z_1, z_2, z_3 \in \mathrm{SL}(d)\times\mathbb{R}^m : \ D(x, z_1, z_3) \le D(x, z_1, z_2) + D(x, z_2, z_3). \tag{3.2c}$$

The conditions on W are much more involved. In particular, they include coercivity assumptions and convexity assumptions to obtain lower semicontinuity. To shorten notation we let $L^{(d,m)} \stackrel{\text{def}}{=} \mathbb{R}^{d\times d\times d}\times\mathbb{R}^{m\times d}$ and use A as a placeholder for $\nabla z = (\nabla P, \nabla p) \in L^{(d,m)}$. The function $\mathbb{M} : \mathbb{R}^{d\times d} \to \mathbb{R}^{\mu_d}$ with $\mu_d = \sum_{s=1}^d \binom{d}{s}^2$ maps a matrix to all its minors (subdeterminants).

$$\begin{aligned}
&\exists\, \mathbb{W} : \Omega \times \mathbb{R}^{\mu_d} \times \mathbb{R}^{d\times d} \times \mathrm{SL}(d) \times \mathbb{R}^m \times L^{(d,m)} \to \mathbb{R}_\infty :\\
&\quad \text{(i)}\quad \mathbb{W} \text{ is a normal integrand,}\\
&\quad \text{(ii)}\quad \forall\,(x,F,z,A):\ W(x,F,z,A) = \mathbb{W}(x,\mathbb{M}(F),z,A),\\
&\quad \text{(iii)}\quad \forall\,(x,z): \mathbb{W}(x,\cdot,z,\cdot)\colon \mathbb{R}^{\mu_d}\times L^{(d,m)} \to \mathbb{R}_\infty \text{ is convex;}
\end{aligned} \tag{3.3a}$$

$$\begin{aligned}
&\exists\, c>0,\ h \in \mathrm{L}^1(\Omega),\ q_F, q_P, q_p, r > 1\ \forall\,(x,F,P,p,A) \in \operatorname{dom} W :\\
&\quad W(x,F,P,p,A) \geq h(x) + c\left(|F|^{q_F} + |P|^{q_P} + |p|^{q_p} + |A|^r\right).
\end{aligned} \tag{3.3b}$$

$$\begin{aligned}
&\exists\, c_0^W \in \mathbb{R},\ c_1^W > 0,\ \delta > 0,\ \text{modulus of continuity } \omega :]0,\delta[\to]0,\infty[\\
&\forall\,(x,F,z,A) \in \operatorname{dom} W\ \forall\, N \in \mathcal{N}_\delta \stackrel{\text{def}}{=} \{\, N \in \mathbb{R}^{d\times d} \mid |N-\mathbf{1}| < \delta \,\} :\\
&\quad W(x,\cdot,z,A) \text{ is differentiable on } \mathcal{N}_\delta F \text{ and}\\
&\quad \text{(i)}\quad |\partial_F W(x,F,z,A)F^\top| \leq c_1^W (W(x,F,z,A) + c_0^W)\\
&\quad \text{(ii)}\quad |\partial_F W(x,F,z,A)F^\top - \partial_F W(x,NF,z,A)(NF)^\top|\\
&\qquad\qquad\qquad\qquad \leq \omega(|N-\mathbf{1}|)(W(x,F,z,A) + c_0^W).
\end{aligned} \tag{3.3c}$$

Thus, (3.3a) implies that the mapping $F \mapsto W(x,F,z,A)$ is polyconvex, cf. [Bal77]. In (3.3c) a modulus of continuity ω is a nondecreasing function with $\omega(\rho) \to 0$ for $\rho \to 0^+$, and $\mathcal{N}_\delta F$ means $\{\, NF \mid N \in \mathcal{N}_\delta \,\} \subset \mathbb{R}^{d\times d}$. The usefulness of constitutive assumption (3.3c)(i) is emphasized in [Bal02] and may be called a multiplicative stress control, as the Kirchhoff stress tensor $\partial_F W(x,F,z,A)F^\top$ can be estimated uniformly in terms of the energy W. Assumption (3.3c)(ii) states that we even have uniform continuity, if we use the energy as a weight. The importance of these conditions is their full compatibility with polyconvexity and with the physically desirable condition $W(x,F,z,A) = \infty$ for $\det F \leq 0$ and $W(x,F,z,A) \to \infty$ for $\det F \to 0^+$, see our examples below.

We need one more condition that we give in two versions, one relating to kinematic hardening and the other to isotropic hardening. These alternative conditions should be seen as prototypical situations that have to be adjusted to the concrete plasticity model one wants to investigate. The first condition is simple but more restrictive concerning the applications in elastoplasticity:

$$D : \Omega \times (\mathrm{SL}(d)\times\mathbb{R}^m)^2 \to [0,\infty[\text{ is a Carathéodory function,} \tag{3.4a}$$

$$\begin{aligned}
&\exists\, h \in \mathrm{L}^1(\Omega),\ C>0,\ q_1 \in [1,q_P[\,,\ q_2 \in [1,q_p[\, :\\
&\qquad |D(x,P_0,p_0,P_1,p_1)| \leq h(x) + C\big(|P_0|^{q_1} + |P_1|^{q_1} + |p_0|^{q_2} + |p_1|^{q_2}\big).
\end{aligned} \tag{3.4b}$$

The second condition is more complicated, since it involves D and W. We set $\mathbb{D}_R(x) \stackrel{\text{def}}{=} \{\, (x,z_0,z_1) \mid D(x,z_1,z_2) < \infty,\ |z_0|, |z_1| \leq R \,\}$ and $\mathbb{D}(x) = \operatorname{dom} D(x,\cdot) = \cup_{R>0}\mathbb{D}_R(x)$, and make the following assumptions:

$$r > d; \tag{3.5a}$$

$$D(x,\cdot,\cdot) : \mathbb{D}(x) \to [0,\infty[\text{ is continuous}; \tag{3.5b}$$

$$\forall\, M>0\ \exists\, R>0\ \forall\, x\in\Omega\ \forall\, z_0,z_1\in\mathbb{D}_R(x):\ D(x,z_0,z_1)\leq M; \tag{3.5c}$$

$$\left.\begin{array}{l}\text{there exists a } v^*\in\mathbb{R}^m \text{ such that the following holds:}\\ \text{(i) } \exists\, c_0^W,\ \text{modulus of continuity } \omega\ \ \forall\,\delta>0, (x,F,P,p,A)\in\operatorname{dom}W:\\ \qquad |W(x,F,P,p{+}\delta v^*,A)-W(x,F,P,p,A)|\leq\omega(\delta)(W(x,F,P,p,A){+}c_0^W),\\ \text{(ii) } \forall\,\delta,R>0\ \exists\,\rho>0\ \forall\, x\in\Omega\ \forall z,z_0,z_1:\\ \qquad |z{-}z_0|\leq\rho \text{ and } (z_0,z_1)\in\mathbb{D}(x) \quad\Longrightarrow\quad (z,z_1{+}(0,\delta v^*))\in\mathbb{D}(x).\end{array}\right\} \tag{3.5d}$$

We are now formulate our existence result, which will be proved in Section 5.

Theorem 3.1 *Let the spaces $\mathcal{Q}=\mathcal{Y}{\times}\mathcal{Z}\subset Y{\times}Z=Q$ and the functionals $\mathcal{E}$ and $\mathcal{D}$ be defined as above. The integrability powers q_Y,q_F,q_P,q_p, and r satisfy*

$$\frac{1}{q_F}+\frac{1}{q_P}=\frac{1}{q_Y}<\frac{1}{d},\quad q_p>1,\quad \text{and } r>1. \tag{3.6}$$

Moreover, the conditions (3.1), (3.2), (3.3) *hold and, additionally, either* (3.4) *or* (3.5). *Let $\boldsymbol{q}_0=(y_0,z_0)\in\mathcal{Q}$ be a stable initial condition, i.e.,*

$$\mathcal{E}(0,\boldsymbol{q}_0)<\infty \quad \text{and} \quad \forall\,\widehat{\boldsymbol{q}}\in\mathcal{Q}\colon \mathcal{E}(0,\boldsymbol{q}_0)\leq\mathcal{E}(0,\widehat{\boldsymbol{q}})+\mathcal{D}(q_0,\widehat{\boldsymbol{q}}).$$

Then, there exists an energetic solution $\boldsymbol{q}:[0,T]\to\mathcal{Q}$ for $(\mathcal{Q},\mathcal{E},\mathcal{D})$ with $\boldsymbol{q}(0)=\boldsymbol{q}_0$ such that $\boldsymbol{q}:[0,T]\to Q$ is measurable.

Remark 3.2 *The condition $r>d$ in* (3.5a) *is of technical nature. Using the ideas of [Tho08], which are developed for a damage model, it is expected that the result can be extended to all $r>1$.*

We provide two typical examples, which show that the theory is applicable to rate-independent elastoplasticity at finite strain. In particular, all the geometric nonlinearities arising from the multiplicative decomposition can be handled in this framework.

Example 3.3 (Kinematic hardening) We do not need the variable $p\in\mathbb{R}^m$ here, i.e., we set $m=0$ in the above:

$$W(x,F,P,A)=W_{\mathrm{el}}(F\,P^{-1})+c_1|P|^{q_P}+c_2|\nabla P|^r,$$

where the elastic part is chosen to be polyconvex in the form

$$W_{\mathrm{el}}(F) = \begin{cases} c_1|F|^{q_F} + \frac{c_2}{(\det F)^\gamma} & \text{for } \det F_{\mathrm{el}} > 0, \\ \infty & \text{otherwise,} \end{cases} \qquad \text{with } c_1, c_2, \gamma > 0.$$

It is easy to see that (3.3a) and (3.3b) are satisfied. The Kirchhoff tensor $K = \partial_F W F^\mathsf{T}$ in (3.3c) only depends on F and takes the simple form

$$K(F) = c_3 q_F |F|^{q_F-2} F\, F^\mathsf{T} - \frac{c_4 \gamma}{(\det F)^\gamma}\mathbf{1}.$$

Hence, (3.3c)(i) immediately holds with $c_0^W = 0$ and $c_1^W = \max\{q_F, \gamma\sqrt{d}\}$. Moreover, condition (ii) also holds, since K can be differentiated once again giving $|\partial_F K(F)[HF]| \leq CW(F)|H|$.

For the dissipation density D we choose any left-invariant distance on the Lie group $\mathrm{SL}(d)$, viz.,

$$D(x, P_0, P_1) = d_{\mathrm{SL}}(P_1 P_0^{-1}) \text{ with } d_{\mathrm{SL}} : \mathrm{SL}(d) \to [0, \infty[\,,$$

where d_{SL} is generated by a norm R on the Lie algebra $\mathrm{sl}(d) \overset{\mathrm{def}}{=} \mathrm{T}_{\mathbf{1}}\mathrm{SL}(d)$ via

$$d_{\mathrm{SL}}(P_1) = \inf\{\, \textstyle\int_0^1 R(\dot P(s) P(s)^{-1})\,\mathrm{d}s \mid P \in \mathrm{C}^1([0,1], \mathrm{SL}(d)),\ P(0) = \mathbf{1},\ P(1) = P_1 \,\}.$$

Clearly this D satisfies the plastic indifference condition (2.5). According to [Mie02] the mapping d_{SL} is continuous, is strictly positive for $P \neq \mathbf{1}$, satisfies the triangle inequality $d_{\mathrm{SL}}(P_1 P_0) \leq d_{\mathrm{SL}}(P_0) + d_{\mathrm{SL}}(P_1)$, and allows for the bounds

$$\delta|\Sigma| \leq R(\Sigma) \leq d_{\mathrm{SL}}(Q\,\mathrm{e}^{\Sigma}) \leq C + R(\Sigma) \quad \text{for } \Sigma = \Sigma^T \text{ and } Q \in \mathrm{SO}(d), \tag{3.7}$$

with $\delta, C > 0$, see [Mie02, HMM03]. Thus, conditions (3.2) and (3.4) are fulfilled.

This shows that Theorem 3.1 is applicable for the case of kinematic hardening.

Example 3.4 (Isotropic hardening) We now use the scalar parameter $p \in \mathbb{R}$ to measure the amount of hardening, i.e., we have $m = 1$ in the abstract setting of Section 2. For the stored-energy density we take the form

$$W(x, F, P, p, A) = W_{\mathrm{el}}(F\,P^{-1}) + c_1 \exp(c_2\, p) + c_5|\nabla P|^r + c_6|\nabla p|^r + \chi_{\mathcal{P}}((P, p)),$$

where W_{el} is as in Example 3.3. Here $\chi_{\mathcal{A}}$ is the characteristic function with $\chi_{\mathcal{P}}((P,p)) = 0$ for $(P, p) \in \mathcal{P}$ and ∞ otherwise. Before we specify the set $\mathcal{P}$, we define, using d_{SL} from above, the dissipation distance

$$D(x, P_0, p_1, P_1, p_1) = \begin{cases} d_{\mathrm{SL}}(P_1 P_0^{-1}) & \text{for } p_1 \geq p_0 + d_{\mathrm{SL}}(P_1 P_0^{-1}), \\ \infty & \text{otherwise,} \end{cases}$$

which again satisfies (2.5).

We let $\mathcal{P} \stackrel{\text{def}}{=} \{\, (P,p) \mid D(x,\mathbf{1},0,P,p) < \infty \,\}$ and obtain $\mathcal{P} = \{\, (R\,\mathrm{e}^{\Sigma},p) \mid p \geq d_{\mathrm{SL}}(R\,\mathrm{e}^{\Sigma}) \,\}$. Using (3.7) and $|R\,\mathrm{e}^{\Sigma}| = |\mathrm{e}^{\Sigma}| \leq \mathrm{e}^{|\Sigma|}$ we find, for $P = R\,\mathrm{e}^{\Sigma}$, the coercivity estimate

$$c_1 \mathrm{e}^{c_2 p} \;\geq\; \tfrac{1}{2}\big(c_1 \mathrm{e}^{c_2 p} + c_1 \mathrm{e}^{c_2 \delta |\Sigma|}\big) \;\geq\; c_7 |p|^{q_p} + c_8 |P|^{q_P}$$

with arbitrary $q_p > 1$ and $q_P = c_2 \delta$. Thus, conditions (3.2) and (3.3) hold.

We now show condition (3.5). Part (a) can be achieved by taking $r > d$, and (3.5b) and (3.5c) hold automatically. For (3.5d) the vector $v^* = 1$ is the obvious choice. In fact, in (i) the estimate reduces to

$$\big|\, c_1 \exp(c_2\,(p{+}\delta)) - c_1 \exp(c_2\, p) \,\big| = \omega(\delta) c_1 \exp(c_2\, p) \quad \text{with } \omega(\delta) = |\mathrm{e}^{c_2\delta}{-}1|.$$

Condition (ii) is also valid, since d_{SL} is continuous.

Thus, we have shown that elastoplasticity with isotropic hardening and gradient regularization is covered by Theorem 3.1, and, hence the existence of energetic solutions is guaranteed.

Remark 3.5 *Time-dependent loading can also be added to $\mathcal{E}$ in the form indicated in* (2.1). *However, we again have to substitute $\varphi(x) = g_{\mathrm{Dir}}(t, y(x))$ and assume that f_{ext} and h_{ext} such that $\ell \in \mathrm{C}^1([0,T], (\mathrm{W}^{1,q_Y}(\Omega;\mathbb{R}^d))^*)$ holds. Then, the above theorem remains true without any change, see [FrM06].*

4 Abstract existence result

Our existence theory for elastoplasticity is based on the abstract theory of energetic solutions for rate-independent processes on topological spaces. This theory is developed in [MaM05, FrM06, Mai07], see also the survey [Mie05] and the further developments in [MRS08]. We use a slightly adapted version as we are in a concrete Banach space setting. Thus, we will use notions like weak and strong convergence, coercivity, and boundedness, which are not available in the general topological setting.

We consider two reflexive and separable Banach spaces Y and Z and weakly closed subsets $\mathcal{Y}$ and $\mathcal{Z}$, respectively. The state space for the full system is then given by $\mathcal{Q} = \mathcal{Y}{\times}\mathcal{Z} \subset Q \stackrel{\text{def}}{=} Y{\times}Z$, and the states are denoted by $\boldsymbol{q} = (y,z)$. The evolution is described in terms of the stored-energy functional $\mathcal{E} : [0,T]{\times}\mathcal{Q} \to \mathbb{R}_\infty \stackrel{\text{def}}{=} \mathbb{R} \cup \{\infty\}$ and the dissipation distance $\mathcal{D} : \mathcal{Z}{\times}\mathcal{Z} \to [0,\infty]$. The triple $(\mathcal{Q},\mathcal{E},\mathcal{D})$ is called a *rate-independent energetic system.*

For the stored-energy functional $\mathcal{E}$ impose the general conditions

$$\begin{aligned}&\textit{Compactness of energy sublevels:}\\&\forall\, t \in [0,T]\ \forall\, E \in \mathbb{R}:\ \ L_{t,E} := \{\, \boldsymbol{q} \in \mathcal{Q} \mid \mathcal{E}(t,\boldsymbol{q}) \leq E \,\} \text{ is bounded}\\&\qquad\qquad\qquad\qquad\quad \text{and weakly closed in } Q;\end{aligned} \tag{4.E1}$$

$$\begin{aligned}&\textit{Uniform control of the power } \partial_t\mathcal{E}:\\&\exists\, c_0^E \in \mathbb{R}\ \exists\, c_1^E > 0\ \forall\, (t_*,\boldsymbol{q}) \in \operatorname{dom}\mathcal{E}:\\&\mathcal{E}(\cdot,\boldsymbol{q}) \in \mathrm{C}^1([0,T]) \text{ and } |\partial_t\mathcal{E}(t,\boldsymbol{q})| \leq c_1^E(c_0^E{+}\mathcal{E}(t,\boldsymbol{q})) \text{ for all } t;\end{aligned} \tag{4.E2}$$

$$\begin{aligned}&\textit{Uniform time-continuity of the power } \partial_t\mathcal{E}:\\&\forall\, \varepsilon > 0\ \forall\, E \in \mathbb{R}\ \exists\, \delta > 0\ \forall\, t_1,t_2,\boldsymbol{q}:\\&\mathcal{E}(t_1,\boldsymbol{q}) \leq E \text{ and } |t_1{-}t_2| < \delta \implies |\partial_t\mathcal{E}(t_1,\boldsymbol{q}){-}\partial_t\mathcal{E}(t_2,\boldsymbol{q})| < \varepsilon.\end{aligned} \tag{4.E3}$$

Using a simple Gronwall argument we see that (4.E2) implies the bound

$$(t,\boldsymbol{q}) \in \operatorname{dom}\mathcal{E} \implies \forall\, t_1,t_2 \in [0,T]:\ \mathcal{E}(t_1,\boldsymbol{q}){+}c_0^E \leq \mathrm{e}^{c_1^E|t_2-t_1|}\big(\mathcal{E}(t_1,\boldsymbol{q}){+}c_0^E\big). \tag{4.1}$$

For the dissipation distance $\mathcal{D} : \mathcal{Z}{\times}\mathcal{Z} \to [0,\infty]$ we impose the general conditions

$$\begin{aligned}&\textit{Positivity of } \mathcal{D}:\\&\forall\, z_1,z_2 \in \mathcal{Z}:\ \mathcal{D}(z_1,z_2) = 0 \Longleftrightarrow z_1 = z_2.\end{aligned} \tag{4.D1}$$

$$\begin{aligned}&\textit{Triangle inequality:}\\&\forall\, z_1,z_2,z_3 \in \mathcal{Z}:\ \mathcal{D}(z_1,z_3) \leq \mathcal{D}(z_1,z_2) + \mathcal{D}(z_2,z_3).\end{aligned} \tag{4.D2}$$

$$\begin{aligned}&\textit{Weak lower semi-continuity:}\\&z_k \rightharpoonup z,\ \widehat{z}_k \rightharpoonup \widehat{z} \implies \mathcal{D}(z,\widehat{z}) \leq \liminf_{k\to\infty} \mathcal{D}(z_k,\widehat{z}_k).\end{aligned} \tag{4.D3}$$

Note that (4.D1) and (4.D3) imply the following condition (4.2), which is used in [FrM06].

Lemma 4.1 *If* (4.D1) *and* (4.D3) *hold, then we also have the following:*

$$\textit{if } (z_k)_{k\in\mathbb{N}} \textit{ is bounded and if } \min\{\mathcal{D}(z_k,z),\mathcal{D}(z,z_k)\} \to 0,\ \textit{ then } z_k \rightharpoonup z. \tag{4.2}$$

Proof: To prove condition (4.2) we take any bounded sequence $(z_k)_{k\in\mathbb{N}}$ and z in $\mathcal{Z}$. By choosing a subsequence we find $\widehat{z} \in \mathcal{Z}$ with $z_{k_n} \rightharpoonup \widehat{z}$ and either (i) $\mathcal{D}(z_{k_n},z) \to 0$ or (ii) $\mathcal{D}(z,z_{k_n}) \to 0$. Let us consider the case (i), the case (ii) is analogous. Using (4.D3) we find

$$0 \leq \mathcal{D}(\widehat{z},z) \leq \liminf_{n\to\infty} \mathcal{D}(z_{k_n},z) = 0.$$

Hence, we have $\mathcal{D}(\widehat{z},z) = 0$, and the positivity (4.D1) gives $z = \widehat{z}$ and $z_{k_n} \rightharpoonup z = \widehat{z}$. As the limit z is unique, we conclude that even the whole sequence converges (without taking a subsequence). ∎

To formulate the existence result we need to impose additional conditions which provide a suitable compatibility between the two functionals $\mathcal{E}$ and $\mathcal{D}$. For this we define the *set of stable states at time t* via

$$\begin{aligned}
&\mathcal{S}(t) \stackrel{\text{def}}{=} \{\, \boldsymbol{q} \in \mathcal{Q} \mid \mathcal{E}(t,\boldsymbol{q}) < \infty,\ \forall\, \widehat{\boldsymbol{q}} \in \mathcal{Q}\colon \mathcal{E}(t,\boldsymbol{q}) \leq \mathcal{E}(t,\widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q},\widehat{\boldsymbol{q}}) \,\},\\
&\mathcal{S}_{[0,T]} \stackrel{\text{def}}{=} \textstyle\bigcup_{t\in[0,T]}\{t\}\times\mathcal{S}(t) \subset [0,T]\times\mathcal{Q}.
\end{aligned}$$

Moreover, we define the notion of a *stable sequence* $(t_k,\boldsymbol{q}_k)_{k\in\mathbb{N}}$ via

$$\sup_{k\in\mathbb{N}} \mathcal{E}_k(t_k,\boldsymbol{q}_k) \ < \infty \quad \text{and} \quad \boldsymbol{q}_k \in \mathcal{S}(t_k) \text{ for all } k \in \mathbb{N}. \tag{4.3}$$

A function $q : [0,T] \to \mathcal{Q}$ is called an *energetic solution* of $(\mathcal{Q},\mathcal{E},\mathcal{D})$, if $t \mapsto \partial_t\mathcal{E}(t,\boldsymbol{q}(t))$ is integrable and if for all $t \in [0,T]$ we have global stability (S) and energy balance (E):

$$\begin{aligned}
&\text{(S)} \quad \boldsymbol{q}(t) \in \mathcal{S}(t);\\
&\text{(E)} \quad \mathcal{E}(t,\boldsymbol{q}(t)) + \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{q},[0,t]) = \mathcal{E}(0,\boldsymbol{q}(0)) + \textstyle\int_0^t \partial_s\mathcal{E}(s,\boldsymbol{q}(s))\,\mathrm{d}s.
\end{aligned}$$

For the proof of the following existence result we refer to [FrM06, MRS08] and remark that it is based on abstract versions of ideas developed in [DFT05]. The measurability results was first obtained in [Mai05]. The proof is based on time-incremental minimization as indicated in Section 1, but to keep this paper short we will not go into details here and refer to [Mie05] for a survey.

Theorem 4.2 *Let $\mathcal{E}$ and $\mathcal{D}$ satisfy conditions* (4.E) *and* (4.D). *Moreover, let the following compatibility condition hold:*

$$\begin{aligned}
&\forall \text{ stable seq. } (t_k,\boldsymbol{q}_k)_{k\in\mathbb{N}} \text{ with } (t_k,\boldsymbol{q}_k) \rightharpoonup (t_*,\boldsymbol{q}_*) :\\
&\partial_t\mathcal{E}(t_k,\boldsymbol{q}_k) \to \partial_t\mathcal{E}(t_*,\boldsymbol{q}_*), && \text{(4.C1)}\\
&\boldsymbol{q}_* \in \mathcal{S}(t_*). && \text{(4.C2)}
\end{aligned}$$

Then, for each $\boldsymbol{q}_0 \in \mathcal{S}(0)$ there exists a solution $\boldsymbol{q} : [0,T] \to \mathcal{Q}$ of the rate-independent energetic system $(\mathcal{Q},\mathcal{E},\mathcal{D})$ satisfying $\boldsymbol{q}(0) = \boldsymbol{q}_0$. Moreover, the solution can be chosen such that $\boldsymbol{q} : [0,T] \to Q$ is measurable.

Note that condition (4.C1) is slightly weaker than weak continuity of $\partial_t\mathcal{E} : \mathcal{S}_{[0,T]} \to \mathbb{R}$, since stable sequences have bounded energies. We may reformulate (4.C1) as weak continuity of $\partial_t\mathcal{E}$ when restricted to $\mathcal{S}_{[0,T]} \cap \{\, (t,\boldsymbol{q}) \mid \mathcal{E}(t,\boldsymbol{q}) \leq E \,\}$ for any $E \in \mathbb{R}$. Similarly, (4.C2) is slightly weaker than weak closedness of $\mathcal{S}_{[0,T]}$.

The following abstract results are sufficient to establish the compatibility conditions (4.C1) and (4.C2) for our application to elastoplasticity. The first result is implicitly contained in [MaM05] but for the readers we provide a direct short proof. See [MRS08] for a discussion of more general *joint recovery sequence conditions*.

Proposition 4.3 *Let $\mathcal{E}$ and $\mathcal{D}$ satisfy* (4.E) *and* (4.D), *respectively. Moreover assume that*

$$\begin{aligned}&\forall \text{ stable seq. } (t_j, \boldsymbol{q}_j)_{j\in\mathbb{N}} \text{ with } (t_j, \boldsymbol{q}_j) \rightharpoonup (t_*, \boldsymbol{q}_*)\ \forall \widehat{\boldsymbol{q}} \in \mathcal{Q}\ \exists\, (\widehat{\boldsymbol{q}}_j)_{j\in\mathbb{N}} :\\ &\limsup_{j\to\infty} \big(\mathcal{E}(t_j, \widehat{\boldsymbol{q}}_j) + \mathcal{D}(\boldsymbol{q}_j, \widehat{\boldsymbol{q}}_j)\big) \ \le\ \mathcal{E}(t_*, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}_*, \widehat{\boldsymbol{q}}),\end{aligned} \tag{4.4}$$

then $\mathcal{E}$ is weakly continuous along stable sequences and (4.C2) *holds.*

Proof: Let $(t_j, \boldsymbol{q}_j)_{j\in\mathbb{N}}$ be any stable sequence with $(t_j, \boldsymbol{q}_*) \rightharpoonup (t_*, \boldsymbol{q}_*)$. First take $\widehat{\boldsymbol{q}} = \boldsymbol{q}_*$ in (4.4) and obtain a sequence $(\boldsymbol{q}_j)_j$. Using stability of $\boldsymbol{q}_j$ we find

$$\limsup_{j\to\infty} \mathcal{E}(t_j, \boldsymbol{q}_j) \le \limsup_{j\to\infty} \mathcal{E}(t_j, \widehat{\boldsymbol{q}}_j) + \mathcal{D}(\boldsymbol{q}_j, \widehat{\boldsymbol{q}}_j) \le \mathcal{E}(t_*, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}_*, \widehat{\boldsymbol{q}}) = \mathcal{E}(t_*, \boldsymbol{q}_*).$$

Using $\sup_j \mathcal{E}(t_j, \boldsymbol{q}_j) \le C_0$ and (4.1) we find

$$|\mathcal{E}(t_j, \boldsymbol{q}_j) - \mathcal{E}(t_*, \boldsymbol{q}_j)| \le \big(\mathrm{e}^{c_1^E |t_j - t_j|} - 1\big)\mathrm{e}^{c_1^E |t_j - t_j|}\big(\mathcal{E}(t_j, \boldsymbol{q}_j) + c_0^E\big) \le \big(\mathrm{e}^{c_1^E |t_j - t_j|} - 1\big)\mathrm{e}^{c_1^E T}\big(C_0 + c_0^E\big).$$

Since $\mathcal{E}$ is lower semicontinuous, we find

$$\liminf_{j\to\infty} \mathcal{E}(t_j, \boldsymbol{q}_j) = \lim_{j\to\infty} \mathcal{E}(t_j, \boldsymbol{q}_j) - \mathcal{E}(t_*, \boldsymbol{q}_j) + \liminf_{j\to\infty} \mathcal{E}(t_*, \boldsymbol{q}_j) \ge 0 + \mathcal{E}(t_*, \boldsymbol{q}_*).$$

Together with the lim sup-estimate from above we have $\mathcal{E}(t_j, \boldsymbol{q}_j) \to \mathcal{E}(t_*, \boldsymbol{q}_*)$, as desired.

To prove $\boldsymbol{q}_* \in \mathcal{S}(t_*)$, we now choose $\widehat{\boldsymbol{q}}$ in (4.4) arbitrary and take $(\widehat{\boldsymbol{q}}_j)_j$ as stated there. Using stability of $\boldsymbol{q}_j$ we obtain

$$\mathcal{E}(t_*, \boldsymbol{q}_*) = \lim_{j\to\infty} \mathcal{E}(t_j, \boldsymbol{q}_j) \le \liminf_{j\to\infty} \mathcal{E}(t_j, \widehat{\boldsymbol{q}}_j) + \mathcal{D}(\boldsymbol{q}_j, \widehat{\boldsymbol{q}}_j) \le \mathcal{E}(t_*, \widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}_*, \widehat{\boldsymbol{q}}),$$

which is the desired stability. ■

The following Proposition 4.4, which is proved in [FrM06], shows that condition (4.C1) on the continuity of the power is a direct consequence of Proposition 4.3. Thus, to apply the abstract existence result it suffices to satisfy (4.E), (4.D), and (4.4).

Proposition 4.4 *Let $\mathcal{E}$ satisfy conditions* (4.E). *Then, we have*

$$\left.\begin{array}{c} t_j \to t_*,\ q_j \rightharpoonup q_*, \\ \mathcal{E}(t_j, \boldsymbol{q}_j) \to \mathcal{E}(t_*, \boldsymbol{q}_*) < \infty \end{array}\right\} \quad\Longrightarrow\quad \partial_t \mathcal{E}(t_j, \boldsymbol{q}_j) \to \partial_t \mathcal{E}(t_*, \boldsymbol{q}_*).$$

5 Coercivity and lower semicontinuity

The aim of this section is to show that the assumptions in Section 3 for the elastoplastic problem are sufficient to establish the abstract assumption (4.E) for the stored-energy functional $\mathcal{E}$, (4.D) for the dissipation distance $\mathcal{D}$, and the compatibility conditions (4.C). Having done this, the Existence Theorem 3.1 for the elastoplastic problem is a direct consequence of the abstract existence result of Section 4.

5.1 Stored energy potential

To establish the coercivity of $\mathcal{E}$ we note that we always use the matrix norm $|F| \stackrel{\text{def}}{=} (F{:}F)^{1/2}$, where the matrix scalar product is defined as $A{:}B \stackrel{\text{def}}{=} \operatorname{tr}(A^\mathsf{T}B) = \sum_{i,j=1}^d A_{ij}B_{ij}$. In particular, we have $|AB| \leq |A|\,|B|$, which implies

$$|FP^{-1}| \geq |F|/|P| \geq r\delta^{r/(r-1)}|F|^{1/r} - (r{-}1)\delta|P|^{1/(r-1)} \text{ for } \det P \neq 0,$$

where $\delta > 0$ and $r > 1$ are arbitrary. In particular, Hölder's inequality applied to $P \in \mathrm{L}^{q_P}(\Omega;\mathbb{R}^{d\times d})$ and $AP^{-1} \in \mathrm{L}^{q_F}(\Omega;\mathbb{R}^{d\times d})$ gives, with $\frac{1}{q_Y} = \frac{1}{q_F} + \frac{1}{q_P}$,

$$\begin{aligned} \|AP^{-1}\|^{q_F}_{\mathrm{L}^{q_F}(\Omega;\mathbb{R}^{d\times d})} &\geq \|A\|^{q_F}_{\mathrm{L}^{q_Y}(\Omega;\mathbb{R}^{d\times d})}/\|P\|^{q_F}_{\mathrm{L}^{q_P}(\Omega;\mathbb{R}^{d\times d})} \\ &\geq r\delta^{1/(r-1)}\|A\|^{q_Y}_{\mathrm{L}^{q_Y}(\Omega;\mathbb{R}^{d\times d})} - (r{-}1)\delta\|P\|^{q_P}_{\mathrm{L}^{q_P}(\Omega;\mathbb{R}^{d\times d})}, \end{aligned} \tag{5.1}$$

where $r = q_F/q_Y$.

We now integrate the coercivity assumption (3.3b) over Ω. Exploiting the bound on $\nabla g_{\mathrm{Dir}}^{-1}$ in (3.1) and (5.1) with $\delta > 0$ sufficiently small we obtain

$$\begin{aligned} \mathcal{E}(t,y,P,p) &\geq \int_\Omega h\,\mathrm{d}x + c\big(\|\nabla g_{\mathrm{Dir}}\nabla yP^{-1}\|^{q_F}_{\mathrm{L}^{q_F}} + \|P\|^{q_P}_{\mathrm{L}^{q_P}} + \|p\|^{q_p}_{\mathrm{L}^{q_p}} + \|(\nabla P,\nabla p)\|^{q_r}_{\mathrm{L}^r}\big) \\ &\geq \widetilde{c}\big(\|\nabla g_{\mathrm{Dir}}\nabla y\|^{q_Y}_{\mathrm{L}^{q_Y}} + \|(P,p)\|^{q_Z}_{Z}\big) - C \\ &\geq \widehat{c}\big(\|\nabla y\|^{q_Y}_{\mathrm{L}^{q_Y}}/\|\nabla g_{\mathrm{Dir}}^{-1}\|^{q_Y}_{\mathrm{L}^\infty} + \|(P,p)\|^{q_Z}_{Z}\big) - C, \end{aligned} \tag{5.2}$$

where $q_Z = \min\{q_P, q_p, r\}$. This shows that $\mathcal{E}(t,\boldsymbol{q}_k) \to \infty$ whenever $\|\boldsymbol{q}_k\|_Q \to \infty$. Hence, all sublevels of $\mathcal{E}(t,\cdot)$ are bounded uniformly in $t \in [0,T]$.

Second, we establish the lower semicontinuity of $\mathcal{E}(t,\cdot)$. For this we use the following result that relies on the weak continuity of the minors of gradients, cf. [Res67, Bal77].

Proposition 5.1 (Convergence of minors) *The three sequences $(G_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ and $(P_k)_{k\in\mathbb{N}}$ satisfy*

$$\begin{gathered} G_k \to G \quad \text{in } \mathrm{L}^\infty(\Omega;\mathbb{R}^{d\times d}), \qquad y_k \rightharpoonup y \quad \text{in } \mathrm{W}^{1,q_Y}(\Omega;\mathbb{R}^d), \\ P_k \to P \quad \text{in } \mathrm{L}^{\widehat{q}}(\Omega;\mathbb{R}^{d\times d}) \text{ and } \det P_k \equiv 1. \end{gathered}$$

If $q_Y > d$, $\widehat{q} \geq 1$, and $\frac{1}{q_Y} + \frac{d-1}{\widehat{q}} \leq 1$, then all minors of the product $G_k\nabla y_k P_k^{-1}$ converge weakly, i.e.,

$$\mathbb{M}(G_k\nabla y_k P_k^{-1}) \rightharpoonup \mathbb{M}(G\nabla y\, P^{-1}) \quad \text{in } \mathrm{L}^1(\Omega;\mathbb{R}^{\mu_d}).$$

Proof: For a matrix $F \in \mathbb{R}^{d\times d}$ we introduce the matrix $\mathbb{M}_s(F) \in \mathbb{R}^{\binom{d}{s}\times\binom{d}{s}}$ consisting of all minors of order s. Then, the weak continuity of minors of gradients gives

$$\mathbb{M}_s(\nabla y_k) \rightharpoonup \mathbb{M}_s(\nabla y) \quad \text{in } \mathrm{L}^{q_Y/s}(\Omega;\mathbb{R}^{\binom{d}{s}\times\binom{d}{s}}). \tag{5.3}$$

Of course, strong convergence $H_k \to H$ in $\mathrm{L}^{q_H}(\Omega;\mathbb{R}^{d\times d})$ and $1 \leq s \leq q_H$ imply strong convergence of the minors, i.e., $\mathbb{M}_s(H_k) \to \mathbb{M}_s(H)$ in $\mathrm{L}^{q_H/s}$.

We prove the statement for $d \in \{1,2,3\}$ first and then the general case. For $d = 1$ the result is trivial as the product of a weakly convergent sequence times several strongly convergent sequences is again weakly convergent.

For $d = 2$ the result is again trivial for $s = 1$ as $\mathbb{M}_1(F) = F$. Since $\mathbb{M}_2(F) = \det F$ and $\det P_k \equiv 1$, we have $\mathbb{M}_2(G_k \nabla y_k P_k^{-1}) = \det G_k \, \det \nabla y_k$ and the result follows again.

For $d = 3$ we have $\mathbb{M}_1(F) = F$ and $\mathbb{M}_3(F) = \det F$, and we may identify $\mathbb{M}_2(F)$ with the cofactor matrix $\operatorname{cof} F$, which satisfies $\operatorname{cof} F = (\det F)\, F^{-\mathsf{T}}$ for invertible F. Using $\det P \equiv 1$ we have $P^{-1} = (\operatorname{cof} P)^{\mathsf{T}}$. Thus, we have

$$\mathbb{M}_1(G_k \nabla y_k P_k^{-1}) = G_k \nabla y_k \, (\operatorname{cof} P_k)^{\mathsf{T}}, \quad \operatorname{cof}(G_k \nabla y_k P_k^{-1}) = \operatorname{cof} G_k \, \operatorname{cof} \nabla y_k \, P_k^{\mathsf{T}},$$
$$\det(G_k \nabla y_k P_k^{-1}) = \det G_k \, \det \nabla y_k.$$

We again see that in all cases $s \in \{1,2,3\}$ we have the desired weak convergence in $\mathrm{L}^{\sigma_s} > 1$ where $\frac{1}{\sigma_s} = \frac{s}{q_Y} + \frac{d-s}{\widehat{q}}$.

For $d \geq 4$ one needs the general definitions of the minor matrix $\mathbb{M}_s$ and the cofactor matrix $\mathbb{K}_s$, see [Šil02, App. A] or [MiM06, Lem. 2.4]. In particular, we have $\mathbb{M}_s(AB) = \mathbb{M}_s(A)\mathbb{M}_s(B)$ and $\mathbb{M}(P^{-1}) = \frac{1}{\det P}\mathbb{K}_{d-s}(P)^{\mathsf{T}}$ if $\det P \neq 0$. Moreover, $\mathbb{M}_s$ and $\mathbb{K}_s$ are polynomial and are homogeneous of degree s. Thus, we obtain the desired convergence as above from $\mathbb{M}_s(G_k \nabla y_k P_k^{-1}) = \mathbb{M}_s(G_k)\mathbb{M}_s(\nabla y_k)\mathbb{K}_{d-s}(P)^{\mathsf{T}}$ and the weak and strong convergence properties. ∎

Theorem 5.2 (Weak lower semicontinuity) The assumptions (3.1), (3.3a), (3.3b), and (3.6) hold. Then, $\mathcal{E}(t,\cdot) : \mathcal{Q} \to \mathbb{R}_\infty$ is weakly lower semicontinuous with respect to the topology of $Q = Y \times Z$.

Proof: We take a sequence $q_k = (y_k, P_k, p_k) \rightharpoonup (y, P, p)$ in Q. The weak convergence of P_k in $\mathrm{L}^{q_P} \cap \mathrm{W}^{1,r}$ implies by the compact embedding of $\mathrm{W}^{1,r}$ into L^r strong convergence in L^r. As weak convergence in L^{q_P} implies boundedness, the classical interpolation yields strong convergence in L^q for all $q \in [1, q_P[$. Similarly, p_k strongly convergences in L^σ for all $\sigma \in [1, q_p[$.

Using $y_k \rightharpoonup y$ in $\mathrm{W}^{1,q_Y}(\Omega)$ and $q_Y > d$ we have $y_k \to y$ in $\mathrm{C}^0(\overline{\Omega};\mathbb{R}^d)$. Hence, for $G_k \overset{\text{def}}{=} \nabla g(t, y_k(x))$ assumption (3.1) gives $G_k \to G$ in $\mathrm{C}^0(\overline{\Omega};\mathbb{R}^d)$. Since from $q_Y > d$ and (3.6) we have $q_P > d$, we can apply Proposition 5.1 by choosing $\widehat{q} \in [d, q_P[$.

Now, we use that $(P_k, p_k) \to (P, p)$ strongly in $\mathrm{L}^d \times \mathrm{L}^r$ and that

$$\Big(\mathbb{M}\big(\nabla g_{\mathrm{Dir}}(t, y_k(\cdot))\nabla y_k \, P_k^{-1}\big), \nabla P_k, \nabla p_k\Big) \rightharpoonup \Big(\mathbb{M}\big(\nabla g_{\mathrm{Dir}}(t, y(\cdot))\nabla y \, P^{-1}\big), \nabla P_k, \nabla p_k\Big).$$

Property (3.3a) states that the integrand has the form $W : (x, F, z, A) \mapsto \mathbb{W}(x, \mathbb{M}(F), z, A)$ where $\mathbb{W}$ is a normal integrand that is convex in $(\mathbb{M}, A)$. Hence, together with the lower bound (3.3b) the classical lower semicontinuity results (cf. e.g. [Eis79] or [Str90, Ch. I.1]) give the desired result.

Since the last mentioned references only treat the case that $\mathbb{W}$ is a Carathéodory function, we may obtain our more general result for normal integrands as follows. For $\varepsilon > 0$ define the Yosida-Moreau regularization

$$\mathbb{W}^\varepsilon(x, M, z, A) = \inf\{\, \mathbb{W}(x, U) + \tfrac{1}{\varepsilon}|(M, z, A)-U|^2 \mid U \in \mathbb{R}^{\mu_d}\times\mathbb{R}^m\times L^{(m,d)} \,\}.$$

Now, $\mathbb{W}^\varepsilon$ is a Carathéodory function, and it approximates $\mathbb{W}$ pointwise, monotonically from below. Moreover, the convexity property in (M, A) is maintained. Thus, we may define functionals $\mathcal{J}_\varepsilon : \mathcal{Q} \to \mathbb{R}_\infty$ by replacing $\mathbb{W}$ in $\mathcal{E}(t, \cdot)$ by $\mathbb{W}^\varepsilon$. Each $\mathcal{J}_\varepsilon$ is weakly lower semicontinuous and $\mathcal{J}_\varepsilon(\boldsymbol{q})$ is nondecreasing in ε. Using the monotone convergence lemma of Beppo Levi, we find $\mathcal{J}_\varepsilon(\boldsymbol{q}) \to \mathcal{E}(t, \boldsymbol{q})$. Thus, for $\boldsymbol{q}_k \rightharpoonup \boldsymbol{q}_*$ and each $\varepsilon > 0$ we have

$$\mathcal{J}_\varepsilon(\boldsymbol{q}_*) \le \liminf_{k\to\infty} \mathcal{J}_\varepsilon(\boldsymbol{q}_k) \le \liminf_{k\to\infty} \mathcal{E}(t, \boldsymbol{q}_k) \stackrel{\text{def}}{=} \alpha.$$

In the limit $\varepsilon \to 0^+$ we find $\mathcal{E}(t, \boldsymbol{q}_*) \le \alpha$ as desired. ■

Combining the coercivity estimate (5.2) with this weak lower semicontinuity result we have established the abstract condition (4.E1).

Third, we investigate the differentiability of $\mathcal{E}(t, \boldsymbol{q})$ with respect to time. For this we fix $\boldsymbol{q} = (y, P, p) \in \mathcal{Q}$ such that $\mathcal{E}(0, \boldsymbol{q}) < \infty$ and introduce the Kirchhoff tensor

$$K_{\boldsymbol{q}}(x, F) \stackrel{\text{def}}{=} \partial_F W(x, FP(x)^{-1}, P(x), p(x), \nabla P(x), \nabla p(x))(FP^{-1})^{\mathsf{T}} \in \mathbb{R}^{d\times d}.$$

Theorem 5.3 (Power of the boundary conditions) *If assumption* (3.1) *and* (3.3) *hold, then* $\mathcal{E}$ *satisfies* (4.E2) *and* (4.E3), *i.e., there exist constants* $c_0^E \in \mathbb{R}$ *and* $c_1^E > 0$ *and a modulus of continuity* ω *such that the following holds:*

For $\boldsymbol{q} \in \mathcal{Q}$ *with* $\mathcal{E}(0, \boldsymbol{q}) < \infty$, *we have* $\mathcal{E}(\cdot, \boldsymbol{q}) \in \mathrm{C}^1([0, T])$ *with*

$$\begin{aligned} &\partial_t\mathcal{E}(t, \boldsymbol{q}) = \textstyle\int_\Omega K_{\boldsymbol{q}}(x, \nabla g_{\mathrm{Dir}}(t, y(x))\nabla y(x)){:}V(t, y(x))\,\mathrm{d}x, \\ &\text{where } V(t, y) = \big(\nabla g_{\mathrm{Dir}}(t, y)\big)^{-1}\tfrac{\partial}{\partial t}\nabla g_{\mathrm{Dir}}(t, y), \end{aligned} \tag{5.4}$$

and the estimates

$$|\partial_t\mathcal{E}(t, \boldsymbol{q})| \le c_1^{\mathcal{E}}\big(\mathcal{E}(t, \boldsymbol{q})+c_0^E\big) \quad\text{and}\quad |\partial_t\mathcal{E}(t_1, \boldsymbol{q})-\partial_t\mathcal{E}(t_2, \boldsymbol{q})| \le \omega(|t_2-t_1|)\big(\mathcal{E}(t_1, \boldsymbol{q})+c_0^E\big).$$

Proof: First, observe that (i) in (3.3c) provides $\delta > 0$ and $C > 1$ such that

$$\begin{aligned} &\forall\,(x, F, z, A) \in \Omega\times\mathbb{R}^{d\times d}\times\mathrm{SL}(d)\times\mathbb{R}^m\times L^{(m,d)}\ \forall\, N \in \mathcal{N}_\delta : \\ &(W(x, NF, z, A)+c_0^W) + |\partial_F W(x, NF, z, A)F^T| \le C\big(W(x, F, z, A)+c_0^W\big), \end{aligned} \tag{5.5}$$

see [Bal02, Lem. 2.5]. We fix $(t_*, \boldsymbol{q}) \in [0, T]\times\mathcal{Q}$ with $\mathcal{E}(t_*, \boldsymbol{q}) < \infty$. Hence, the function

$$w(t, \cdot) : \Omega \to \mathbb{R}_\infty; x \mapsto W(x, \nabla g_{\mathrm{Dir}}(t, y(x))\nabla y(x)P(x)^{-1}, P(x), p(x), \nabla P(x), \nabla p(x))$$

is finite for $t = t_*$ and $x \in \widetilde{\Omega}$, where $\Omega \backslash \widetilde{\Omega}$ has measure 0. For $x \in \widetilde{\Omega}$, (5.5) and (3.1) shows that $t \mapsto w(t,x)$ is differentiable near t_* with derivative

$$\dot{w}(t,x) = K_{\boldsymbol{q}}(x, \nabla g_{\mathrm{Dir}}(t, y(x)) \nabla y(x) P(x)^{-1}) {:} V(t, y(x)).$$

Because of (3.1) V is uniformly bounded on $[0,T] \times \Omega$ and by (5.5) we conclude that $\mathcal{E}(t, \boldsymbol{q})$ is differentiable for t near t_*, see e.g. [Rou05, Thm. 1.29].

Using (3.3c)(i) we find the estimate

$$\begin{aligned}
&|\partial_t \mathcal{E}(t, \boldsymbol{q})| \leq \textstyle\int_\Omega |K_{\boldsymbol{q}}(x, \nabla g_{\mathrm{Dir}}(t, y(x)) \nabla y(x) P(x)^{-1})|\, |V(t, y(x))| \,\mathrm{d}x \\
&\leq \textstyle\int_\Omega c_1^W \big(W(x, \nabla g_{\mathrm{Dir}}(t, y(x)) \nabla y(x) P(x)^{-1}, z, A) + c_0^W \big) \,\mathrm{d}x \, \mathbb{V} \ \leq \ c_1^E \big(\mathcal{E}(t, \boldsymbol{q}) + c_0^E \big),
\end{aligned}$$

where $\mathbb{V} = \|V(\cdot,\cdot)\|_{\mathrm{L}^\infty([0,T]\times\Omega)}$, $c_1^E = \mathbb{V} c_1^W$, and $c_0^E = \mathbb{V} c_1^W c_0^W |\Omega|$. As this estimate is independent of t, a simple Gronwall estimate shows that $\mathcal{E}(t, \boldsymbol{q})$ is finite for all $t \in [0,T]$ if it is finite for one t, cf. (4.1). Thus we have proved (4.E2).

To show (4.E3), we use formula (5.4) and that the sublevel $L_{0,E} = \{\, \boldsymbol{q} \mid \mathcal{E}(0, \boldsymbol{q}) \leq E \,\}$ is bounded in Q. In particular, there exists R_E such that all $\boldsymbol{q} = (y, z) \in L_{0,E}$ satisfy $\|y\|_{\mathrm{L}^\infty} \leq R_E$. We set $B_E = \{\, \widehat{y} \mid |\widehat{y}| \leq R_E \,\} \subset \mathbb{R}^d$ and denote by ω_V the modulus of continuity of the mapping $V : [0,T] \to \mathrm{L}^\infty(B_E; \mathbb{R}^{d\times d})$. Moreover, (3.1) guarantees that there is a modulus of continuity ω_G such that

$$\big\| \nabla g_{\mathrm{Dir}}(t_2, y(\cdot)) \nabla g_{\mathrm{Dir}}(t_1, y(\cdot))^{-1} - \mathbf{1} \big\|_{\mathrm{L}^\infty(B_E; \mathbb{R}^{d\times d})} \leq \omega_G(|t_2 - t_1|).$$

For $t_1, t_2 \in [0,T]$ and $\boldsymbol{q} \in L_{0,E}$ we now estimate

$$\begin{aligned}
&|\partial_t \mathcal{E}(t_1, \boldsymbol{q}) - \partial_t \mathcal{E}(t_2, \boldsymbol{q})| \\
&\leq \textstyle\int_\Omega |K_{\boldsymbol{q}}(x, \nabla g_{\mathrm{Dir}}(t_1, y(\cdot)) \nabla y P^{-1}) - K_{\boldsymbol{q}}(x, \nabla g_{\mathrm{Dir}}(t_2, y(\cdot)) \nabla y P^{-1})|\, |V(t_1, y(\cdot)| \,\mathrm{d}x \\
&\quad + \textstyle\int_\Omega |K_{\boldsymbol{q}}(x, \nabla g_{\mathrm{Dir}}(t_2, y(\cdot)) \nabla y P^{-1})| |V(t_1, y(\cdot) - V(t_2, y(\cdot)| \,\mathrm{d}x \\
&\leq \textstyle\int_\Omega \omega \Big(\big| \nabla g_{\mathrm{Dir}}(t_2, y(\cdot)) \nabla g_{\mathrm{Dir}}(t_1, y(\cdot))^{-1} - \mathbf{1} \big| \Big) \big(W_{\boldsymbol{q}} + c_0^W \big) \,\mathrm{d}x \, \mathbb{V} \\
&\quad + \textstyle\int_\Omega c_1^W \big(W_{\boldsymbol{q}} + c_0^W \big) \omega_V(|t_2 - t_1|) \,\mathrm{d}x \\
&\leq \big(\mathcal{E}(t_1, \boldsymbol{q}) + c_0^W |\Omega| \big) \Big(\mathbb{V}\, \omega \big(\omega_G(|t_2 - t_1|) \big) + c_1^W \omega_V(|t_2 - t_1|) \Big),
\end{aligned}$$

where ω is defined in (3.3c)(ii). This is the desired result. ■

5.2 Dissipation potential

The dissipation distance $\mathcal{D}$ on $\mathcal{Z}$ is defined via $D(x, z_0, z_1)$. Condition (3.2a) implies that $\mathcal{D}$ is well defined and the positivity (4.D1) follows from (3.2b). Integrating the pointwise triangle inequality (3.2c) we see that (4.D2) holds.

Using again that $z_k \rightharpoonup z$ in Z implies $z_k \to z$ in $\mathrm{L}^r(\Omega)$ and that D is nonnegative and lower semicontinuous in both z-variables the classical lower semicontinuity theory implies the lower semicontinuity of $\mathcal{D}$, namely (4.D3).

5.3 Compatibility conditions (4.C)

The compatibility conditions (4.C) are derived via Proposition 4.3. Hence, it remains to show that (4.4) can be derived from the alternative conditions (3.4) or (3.5).

Case (3.4) is conceptually simpler than the other one. Since D is a Carathéodory function it is continuous in the variables (z_0, z_1). Moreover, the upper bounds on D imposed in (3.4b) implies that $\mathcal{D}$ maps $\mathcal{Z}\times\mathcal{Z}$ into $[0,\infty[$. Since weak convergence of $z_k = (P_k, p_k)$ in Z implies strong convergence of P_k in $\mathrm{L}^{q_1}(\Omega;\mathbb{R}^{d\times d})$ and of p_k in $\mathrm{L}^{q_2}(\Omega;\mathbb{R}^m)$, a classical argument shows that $\mathcal{D}$ is weakly continuous:

$$z_k \rightharpoonup z,\ \widehat{z}_k \rightharpoonup \widehat{z} \qquad \Longrightarrow \qquad \mathcal{D}(z_k,\widehat{z}_k) \to \mathcal{D}(z,\widehat{z}).$$

Now (4.4) is obviously satisfied by letting $\widehat{\boldsymbol{q}}_j = \widehat{\boldsymbol{q}}$.

Case (3.5) is more involved. We consider a weakly convergent stable sequence $(t_k, \boldsymbol{q}_k) \rightharpoonup (t_*, \boldsymbol{q}_*)$ and arbitrary test state $\widehat{\boldsymbol{q}} \in \mathcal{Q}$. If $\mathcal{E}(t_*, \widehat{\boldsymbol{q}}) = \infty$ or $\mathcal{D}(\boldsymbol{q}_*, \widehat{\boldsymbol{q}}) = \infty$ there is nothing to show as we may take any sequence $\widehat{\boldsymbol{q}}_j$. Hence, we assume $\mathcal{D}(\boldsymbol{q}_*, \widehat{\boldsymbol{q}}) < \infty$ from now on. From $r > d$ we know that Z embeds into $\mathrm{C}^0(\overline{\Omega})$; hence there exists $R > 0$ such that z_j, z_*, and $\widehat{z}$ lie in the ball of radius $R{-}1$ in $\mathbb{R}^{d\times d}\times\mathbb{R}^m$. From condition (ii) of (3.5d) we obtain a function $a :]0,1[\to]0,\delta_0[$ with $a(\rho) \to 0$ for $\rho \to 0^+$, such that for $\rho \in]0,1[$ estimate (ii) in (3.5d) holds for this R and $\delta \geq a(\rho)$.

Now, we set $\rho_k = \|P_k{-}P_*\|_{\mathrm{L}^\infty}{+}\|p_k{-}p_*\|_{\mathrm{L}^\infty}$, $\delta_k = a(\rho_k)$, and $\widehat{\boldsymbol{q}}_k = (\widehat{y}, \widehat{P}, \widehat{p}{+}\delta_k v^*)$. Using $r > d$ and $\boldsymbol{q}_k \rightharpoonup \boldsymbol{q}_*$ we find $\rho_k, \delta_k \to 0$, and by construction we have $(z_k(x), \widehat{z}_k(x)) \in \mathbb{D}_R(x)$ on Ω. Hence, the continuity of D on $\mathbb{D}_R$ (cf. (3.5b)) gives $D(x, z_k(x), \widehat{z}_k(x)) \to D(x, z_*(x), \widehat{z}(x))$ pointwise. Exploiting the uniform bound (3.5c) we find (b) in the following statement:

$$\text{(a)} \quad \mathcal{E}(t_k, \widehat{\boldsymbol{q}}_k) \to \mathcal{E}(t_*, \widehat{\boldsymbol{q}}_*), \qquad \text{(b)} \quad \mathcal{D}(z_k, \widehat{z}_k) \to \mathcal{D}(z_*, \widehat{z}). \tag{5.6}$$

It remains to establish (a), then (4.4) holds with "lim sup" being a "lim" and with equality.

For (a) first note that as above we may consider $t_k = t_*$ by the uniform Lipschitz continuity on sublevels of $\mathcal{E}$, cf. (4.1). Since $\widehat{\boldsymbol{q}}$ and $\widehat{\boldsymbol{q}}_k$ differ only by the term $(0, 0, \delta_k v^*)$, we can employ part (i) of (3.5d) to obtain

$$|\mathcal{E}(t_*, \widehat{\boldsymbol{q}}_k){-}\mathcal{E}(t_*, \widehat{\boldsymbol{q}})| \leq \int_\Omega \omega(\delta_k)\big(W_{\widehat{\boldsymbol{q}}}{+}c_0^W\big)\,\mathrm{d}x \leq \omega(\delta_k)\big(\mathcal{E}(t_*, \widehat{\boldsymbol{q}}){+}c_0^W|\Omega|\big),$$

which is the desired convergence (a).

6 Generalizations and discussion

The conditions (3.4) and (3.5) are given to fit the Examples 3.3 and 3.4, respectively. Of course, these conditions can be modified to match other constitutive assumptions.

The essential point for the mathematical analysis is that the stored-energy density W is coercive in the gradients of the internal variables $z = (P, p)$ while the dissipation distance D only depends on the point values of z. Thus, it is easily possible to include models of crystal plasticity as discussed in [OrR99, Gur00, Sve02] and formulated in the present framework in [HMM03, Mie03b] (see Ex. 3.3 and Sect. 3.4.4 in the latter work).

Of course, the regularization via the gradient $\nabla z = (\nabla P, \nabla p)$ could be replaced by coercivity in a weaker norm that still guarantees a compact embedding into $\mathrm{L}^q(\Omega)$. On the one hand we may use the physically more desirable growth rate $r = 1$ (cf. [CoO05]) by using the space $\mathrm{BV}(\Omega)$ instead of $\mathrm{W}^{1,r}(\Omega)$. Using the compact embedding of $\mathrm{BV}(\Omega)$ into $\mathrm{L}^q(\Omega)$ for each $q \in [1, d/(d-1)[$ the proof of Theorem 3.1 still works for the case that (3.4) holds. On the other hand, we may use a regularizing term like

$$\int_\Omega \kappa \frac{|\nabla z(x) - \nabla z(\widetilde{x})|^r}{|x - \widetilde{x}|^{d+rs}} \,\mathrm{d}x \,\mathrm{d}\widetilde{x},$$

where $\kappa > 0$, $s \in]0, 1[$, and $r > 1$. Then, we have coercivity in the Sobolev-Slobodetsky space $\mathrm{W}^{s,r}(\Omega)$. For $s - d/r > 0$ we have a compact embedding into $\mathrm{C}(\overline{\Omega})$ and Theorem 3.1 still holds with either (3.4) or (3.5), if (3.5a) is strengthened to $r > d/s$.

However, it remains an open problem to generalize our result to the case treated in [MiM06], where only the term $G = (\operatorname{curl} P)P^\mathsf{T}$ is used for regularization. At the moment the best we can do in this direction is to use a regularizing term in the form

$$\int_\Omega \kappa_1 |(\operatorname{curl} P)P^\mathsf{T}|^r + \kappa_0 |\nabla P|^r \,\mathrm{d}x \qquad \text{with } 0 < \kappa_0 \ll \kappa_1.$$

To treat the more interesting case κ_0 new ideas have to be developed.

From the general theory of energetic solutions for rate-independent systems (cf. [MaM05, FrM06] it is clear that the Lie-group structure of $\mathfrak{P}$ is not essential at all. In the contrary, it just makes the analysis more difficult. The only importance is that the dissipation distance D is a quasi-metric (i.e., it satisfies (3.2)). Some engineering models don't take the plastic spin into account and assume that P represents a "plastic metric" taken from

$$\mathfrak{S}(d) \stackrel{\text{def}}{=} \{\, G \in \mathbb{R}^{d\times d} \mid G = G^\mathsf{T},\ \det G > 0 \,\} \subset \mathrm{GL}^+(d),$$

which may be considered as a symmetric space but not a Lie group. Introducing a dissipation potential $R(G, \dot{G}) = \widehat{R}(R^{-1/2}\dot{G}R^{-1/2})$ for some convex and 1-homogeneous functional $\widehat{R} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0, \infty[$, the dissipation distance reads $D(G_1, G_2) = \widehat{R}\big(\log(G_1^{-1/2} G_2 G_1^{-1/2})\big)$ and our theory is again applicable.

In cases of single-crystal plasticity with infinite latent hardening the set of plastic tensors does not even have a manifold structure. Let $S_a = s_a \otimes m_a$, $a = 1, \ldots, N$, be the N glide systems with $s_a, m_a \in \mathbb{R}^d$ and $s_a \cdot m_a = 0$. Then, we choose

$$P \in \mathbf{S} \stackrel{\text{def}}{=} \bigcup_{a=1}^{N} \{\, \mathbf{1} + \gamma_a S_a \mid \gamma_a \geq 0 \,\},$$

and the dissipation distance $D : \mathbf{S}\times\mathbf{S} \to [0,\infty]$ with

$$D(\mathbf{1}+\gamma S_a, \mathbf{1}+\widetilde{\gamma} S_b) = \begin{cases} \kappa_b(\widetilde{\gamma}-\gamma) & \text{for } a=b \text{ and } \widetilde{\gamma}\geq\gamma, \\ \infty & \text{elsewhere.} \end{cases}$$

Our theory is again applicable since the set $\mathcal{Z} = \{\, P \in \mathrm{L}^{q_P}(\Omega) \cap \mathrm{W}^{1,r}(\Omega) \mid R \in \mathbf{S} \text{ a.e.} \,\}$ is weakly closed.

Finally we address the question of self-interpenetration. The property $W(F,\cdots) = \infty$ for $\det F \leq 0$ implies $\det \nabla y(t,x) > 0$ a.e. in Ω which means that there is no local self-interpenetration. Following [CiN87] (see also [MaM07] for a similar approach using currents), we may define the "non-self-interpenetration" version of the space $\mathcal{Y}$ of admissible deformations via

$$\mathcal{Y}_{\mathrm{nsi}} \overset{\mathrm{def}}{=} \{\, y \in \mathrm{W}^{1,q_Y}(\Omega;\mathbb{R}^d) \mid y|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id},\ \det\nabla \geq 0 \text{ a.e. in } \Omega,\ \int_\Omega \det\nabla y \,\mathrm{d}x \leq \mathrm{vol}(y(\Omega)) \,\}.$$

In [CiN87] it is shown that $\mathcal{Y}_{\mathrm{nsi}}$ is weakly closed in $\mathrm{W}^{1,q_Y}(\Omega;\mathbb{R}^d)$ if $q_Y > d$ (see also [Bal02] for further discussion), and hence our theory works exactly the same way if $\mathcal{Y}$ is replaced by $\mathcal{Y}_{\mathrm{nsi}}$.

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