Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

A discrete variational principle for rate-independent evolution

Alexander Mielke[1,2] and Ulisse Stefanelli[3]

submitted: 30th January 2008

[1] Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstraße 39
10117 Berlin, Germany
E-Mail: mielke@wias-berlin.de
http://www.wias-berlin.de/people/mielke/

[2] Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
12489 Berlin-Adlershof, Germany

[3] IMATI - CNR
v. Ferrata 1
I-27100 Pavia, Italy
E-Mail: ulisse.stefanelli@imati.cnr.it
http://www.imati.cnr.it/ulisse/

No. 1295
Berlin 2008

2000 *Mathematics Subject Classification.* 35K55.

Key words and phrases. Variational Principle, rate-independent process, time-discretization.

Part of this research has been performed during a joint stay at the *Centro di Ricerca Matematica Ennio De Giorgi* in Pisa in occasion of the workshop *Rate-independence, Homogenization, and Multi-Scaling.* The kind hospitality of the Centro De Giorgi is gratefully acknowledged.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract. We develop a global-in-time variational approach to the time-discretization of rate-independent processes. In particular, we investigate a discrete version of the variational principle based on the *weighted energy-dissipation functional* introduced in [MO08]. We prove the conditional convergence of time-discrete approximate minimizers to energetic solutions of the time-continuous problem. Moreover, the convergence result is combined with approximation and relaxation. For a fixed partition the functional is shown to have an asymptotic development by Γ-convergence (cf. [AB93]) in the limit of vanishing viscosity.

1. Introduction

Heat conduction, quasi-static viscoelasticity under linearized kinematics, and solid-liquid phase change are examples of the many dissipative systems which can be described by means of the model problem

$$\partial\Psi(\dot{u}(t)) + \mathrm{D}\mathcal{E}(t,u(t)) \ni 0, \quad u(0) = u_0. \tag{1.1}$$

Here, $t \in [0,T] \mapsto u(t) \in U$ represents the state-trajectory of the dissipative system and U is a Banach space. The functional $\Psi : U \to [0,\infty]$ is a convex *dissipation potential* and $\mathcal{E} : [0,T] \times U \to (-\infty,\infty]$ is the energy of the system (time-dependence models external actions). A solution of (1.1) is a trajectory starting from the initial state u_0 and realizing the balance (1.1) between the *dissipative forces*, here represented by the subdifferential $\partial\Psi(\dot{u})$, and the *conservative forces* given by the Fréchet derivative $\mathrm{D}\mathcal{E}(t,u(t))$ of the energy with respect to the state u.

This note is specifically concerned with the case of rate-independent evolution. Namely, we shall assume from the very beginning that

$$\Psi \quad \text{is positively 1-homogeneous.} \tag{1.2}$$

As a consequence, solutions u of (1.1) present no intrinsic time-scale. Namely, by re-parametrizing time via a strictly increasing diffeomorphism $t \mapsto \alpha(t)$, the trajectory $t \mapsto u(\alpha(t))$ solves the re-parametrized version of (1.1) where $\mathcal{E}(t,\cdot)$ is changed into $\mathcal{E}(\alpha(t),\cdot)$. This feature appears as the distinctive character of hysteresis [Vis96] and has been addressed in the frame of (1.1) in connection with elasto-plasticity [Joh78, Suq78, HR99, Mie03, DDM06], damage [MR06], brittle fractures [DFT05], delamination [KMR06], ferro-electricity [MT05], shape-memory alloys [MT99, MTL02, MR03, Rou07, AMS08], and vortex pinning in superconductors [SM05]. The reader is referred to Mielke [Mie05] for a comprehensive survey of the mathematical theory.

Problem (1.1) is classically tackled by time discretization. Namely, by fixing a uniform partition with diameter $\tau = T/N,\ N \in \mathbb{N}$ of the interval $[0,T]$, and setting $u^0 = u_0$, one is lead to consider the family of minimum problems

$$u^i \in \operatorname{Arg\,min}_{u\in U} \left\{ \Psi\left(\frac{u-u^{i-1}}{\tau}\right) + \frac{\mathcal{E}\big(i\tau,u\big) - \mathcal{E}\big((i-1)\tau,u^{i-1}\big)}{\tau} \right\} \quad \text{for} \ \ i = 1,\ldots,N. \tag{1.3}$$

Usually the latter problems are solved *sequentially*. Following the ideas in [MO08], we wish instead to collect all of them in a single minimization problem for the entire discrete trajectory with nodal values $\boldsymbol{u} = (u^0, u^1, \ldots, u^N)$ by considering the global functional

$$\mathcal{I}_{\varepsilon\tau}(\boldsymbol{u}) = \sum_{i=1}^{N} \tau \lambda^i_{\varepsilon\tau} \left(\Psi\left(\frac{u^i - u^{i-1}}{\tau} \right) + \frac{\mathcal{E}\left(i\tau, u^i\right) - \mathcal{E}\left((i-1)\tau, u^{i-1}\right)}{\tau} \right) \tag{1.4}$$

depending on an extra parameter $\varepsilon > 0$. Here the *Pareto weights* $\boldsymbol{\lambda}_{\varepsilon\tau} = (\lambda^1_{\varepsilon\tau}, \ldots, \lambda^N_{\varepsilon\tau})$ are chosen is such a way to approximate *causality*. In particular, we will enforce $\lambda^1_{\varepsilon\tau} \gg \lambda^2_{\varepsilon\tau} \gg \cdots \gg \lambda^N_{\varepsilon\tau}$ so that a much larger priority is accorded to the first minimum problem in (1.3) with respect to the second, to the second with respect to the third, and so on. More specifically, we shall ask for

$$\lambda^i_{\varepsilon\tau} = \left(\frac{\varepsilon}{\tau + \varepsilon} \right)^i \quad \text{for } \ i = 1, \ldots, N, \tag{1.5}$$

and remark that, along with this choice, the functional $\mathcal{I}_{\varepsilon\tau}$ may be regarded as a quadrature of the functional defined on time-continuous trajectories $u : [0, T] \to U$ as

$$\mathcal{I}_\varepsilon(u) = \int_0^T \mathrm{e}^{-t/\varepsilon} \left(\Psi(\dot{u}) + \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(t, u) \right) \mathrm{d}t.$$

The latter functional is called *weighted dissipation-energy* (WED) functional and has been introduced by Mielke & Ortiz [MO08] in order to characterize variationally the trajectories of the dissipative system (1.1). To gain some insight into this variational perspective, one may compute the Euler-Lagrange equations for $\mathcal{I}_\varepsilon$ (some extra smoothness for Ψ has to be assumed) which turn out to be

$$\begin{aligned}
&- \varepsilon \mathrm{D}^2 \Psi(\dot{u}) \ddot{u} + \mathrm{D}\Psi(\dot{u}) + \mathrm{D}\mathcal{E}(t, u) = 0, \\
&u(0) = u_0, \\
&\mathrm{D}\Psi(u(T)) + \mathrm{D}\mathcal{E}(T, u(T)) = 0.
\end{aligned}$$

Namely, minimizing $\mathcal{I}_\varepsilon$ appears to be closely related to performing and elliptic-in-time regularization of the original problem (1.1).

We shall stress that, at all levels $\varepsilon > 0$, causality is lost. Hence, it turns out to be crucial to consider the *causal limit* $\varepsilon \to 0$ within a possible sequence u_ε of minimizers of $\mathcal{I}_\varepsilon$. This has been accomplished in [MO08] in the rate-independent case (1.2). In particular, a subsequence of u_ε is proved to converge to an *energetic solution* of (1.1) (see below). Moreover, the causal limit $\varepsilon \to 0$ is combined in [MO08] with relaxation, paving the way to the application of the tools of the Calculus of Variations to the evolution problem (1.1).

The purpose of this note is to reconsider this variational perspective by arguing directly at the level of the time-discrete functionals $\mathcal{I}_{\varepsilon\tau}$. The interest in the resulting discrete variational principle is threefold.

First, we provide a conditional convergence result for time-discrete trajectories as τ and ε converge to 0. This serves also as a justification of the original formal approach in [MO08] (Section 2). Moreover, the present time-discrete approach appears to be more flexible than the time-continuous one, since convergence for qualified minimizing sequences instead of exact minimizers can be obtained (Subsection 3)

Secondly, by facing the problem directly at the discretization level, we are allowed a much greater generality which in turn broadens the spectrum of possible applications

(Subsection 4). The convergence analysis may be combined with relaxation and space discretization giving rise to a complete approximation theory (Subsection 5).

Finally, the limit $\varepsilon \to 0$ for $\mathcal{I}_{\varepsilon\tau}$ is studied using the method of asymptotic development by Γ-convergence following [AB93]. It turns out that the corresponding minimizers are exactly the solutions of the (causal) incremental problem (1.3), see Theorem 6.1.

Let us however make clear that our interest in this discrete variational principle is purely theoretical. Indeed, from a computational viewpoint, we do not expect that minimizing $\mathcal{I}_{\varepsilon\tau}$ for some small ε could be preferable to solve sequentially the minimization problems in (1.3). However, the analysis here intends to contribute to the notoriously difficult question of relaxation of evolutionary problems, in particular in the rate-independent case. A *joint relaxation* of a finite sequence of time-incremental problems is certainly a good move toward a general theory and improves upon the *separate relaxation* proposed in [MTL02, Mie04] and analyzed in more detail in [MRS08].

Before closing this introduction, let us mention that an alternative variational approach to doubly nonlinear equations as (1.1) has been developed in [Ste06], specifically applied to hardening elasto-plasticity in [Ste08], and tailored to discontinuous rate-independent evolution in [Ste08]. This second approach is informed by a completely different philosophy and shows quite distinct features with respect to the present analysis. We shall compare these two variational techniques elsewhere.

2. Main results

2.1. **Assumptions.** Let us start by enlisting our assumptions. First of all, the problem is framed in a Banach space setting, see Subsection 4 for a more general metric approach. In particular, we ask that

$$V \text{ and } U \text{ are Banach spaces, } V \text{ is reflexive, and } V \subset U \text{ compactly.} \tag{2.1}$$

We denote by $\|\cdot\|_V$ and $\|\cdot\|_U$ the corresponding norms. Moreover, we indicate with $\mathrm{BV}([0,T];U)$ the space of (everywhere defined) functions $u:[0,T]\to U$ such that the corresponding *total variation in* U is finite, namely

$$\mathrm{Var}(u) = \sup\left\{\sum_{i=1}^{N} \|u(t^i) - u(t^{i-1})\|_U \ : \ 0 = t^0 < t^1 < \cdots < t^N = T\right\} < \infty.$$

As for the dissipation potential Ψ we assume:

$$\Psi : U \to [0,\infty] \ \text{ is convex, 1-homogeneous,} \tag{2.2}$$

$$\exists c_\Psi > 0 \ : \ \Psi(v) \geq c_\Psi \|v\|_U \quad \forall v \in U, \tag{2.3}$$

$$\Psi \ \text{ is lower semi-continuous.} \tag{2.4}$$

Note that (2.2) implies the triangle inequality

$$\Psi(v_1 + v_1) \leq \Psi(v_1) + \Psi(v_2) \quad \forall v_1,\, v_2 \in U. \tag{2.5}$$

We don't make any assumption concerning symmetry, i.e. $\Psi(-v) \neq \Psi(v)$ is allowed. This is important to be able to treat applications like elastoplasticity or damage. As regards to the energy $\mathcal{E} : [0,T] \times U \to (-\infty, \infty]$ we assume:

$$\mathcal{E}(t,\cdot) \text{ is l.s.c with resp. to the strong topology in } U,\ \forall t \in [0,T] \tag{2.6}$$

$$\begin{aligned} &\exists \alpha,\, c_{\mathcal{E}},\, C_{\mathcal{E}} > 0 \ : \ \mathcal{E}(t,u) \geq c_{\mathcal{E}} \|u\|_V^\alpha - C_{\mathcal{E}} \quad \forall (t,u) \in [0,T] \times U \\ &(\text{namely } \mathcal{E}(t,u) = \infty \text{ if } u \in U \setminus V), \end{aligned} \tag{2.7}$$

$$\begin{aligned} &t \mapsto \mathcal{E}(t,u) \text{ is differentiable for all } u \in V \text{ and} \\ &\exists c_0,\, c_1 > 0 \ : \ |\partial_t \mathcal{E}(t,u)| \leq c_1 \Big(\mathcal{E}(t,u) + c_0 \Big) \quad \forall (t,u) \in (0,T) \times V, \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\forall E > 0,\ \forall \varepsilon > 0,\ \exists \delta > 0 : \\ &\mathcal{E}(t,y) \leq E \ \text{ and } \ |t-s| \leq \delta \ \implies \ |\partial_t \mathcal{E}(t,y) - \partial_t \mathcal{E}(s,y)| < \varepsilon. \end{aligned} \tag{2.9}$$

Note that assumption (2.8) implies that the energy is bounded from below by the constant $-c_0$. Moreover, the following Gronwall-like estimate holds

$$\mathcal{E}(t,u) + c_0 \leq (\mathcal{E}(s,u) + c_0) \mathrm{e}^{c_1 |t-s|} \quad \forall s,t \in [0,T]. \tag{2.10}$$

Since, regardless of the smoothness of $\mathcal{E}$, an absolutely continuous solution of (1.1) may fail to exist (even locally), we are forced to deal with discontinuous solutions instead. For all $f \in \mathrm{C}([0,T];\mathbb{R})$ with $f \geq 0$ and f non-increasing, $u \in \mathrm{BV}([0,T];U)$, and an interval $J \subset [0,T]$, we use the notation

$$\int_J f(s) \Psi(\mathrm{d}u) = \sup \Bigg\{ \sum_{i=1}^N f(t^i) \Psi\Big(u(t^i) - u(t^{i-1}) \Big) \ : \ t^0, t^N \in J,\ \ t^0 < t^1 < \cdots < t^N \Bigg\}.$$

For $\varepsilon > 0$, the *weighted dissipation-energy* (WED) functionals read

$$\begin{aligned} \mathcal{I}_\varepsilon(u) &= \int_{[0,T]} \mathrm{e}^{-t/\varepsilon} \left(\Psi(\mathrm{d}u) + \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(t,u) \right) \\ &= \int_{[0,T]} \mathrm{e}^{-t/\varepsilon} \left(\Psi(\mathrm{d}u) + \frac{1}{\varepsilon} \mathcal{E}(t,u) \right) + \mathrm{e}^{-T/\varepsilon} \mathcal{E}(T, u(T)) - \mathcal{E}(0, u(0)). \end{aligned} \tag{2.11}$$

A trajectory $u : [0,T] \to U$ is called an *energetic solution of* (1.1) if $u(0) = u_0$ and the following *global stability* condition (2.12) and *energy conservation* (2.13) hold for all $t \in [0,T]$:

$$\mathcal{E}(t,u(t)) \leq \mathcal{E}(t,v) + \Psi(v - u(t)) \quad \forall v \in V, \tag{2.12}$$

$$\mathcal{E}(t,u(t)) + \int_{[0,t]} \Psi(\mathrm{d}u) = \mathcal{E}(0,u_0) + \int_0^t \partial_t \mathcal{E}(s,u(s)) \mathrm{d}s, \tag{2.13}$$

For the sake of later reference, we shall define the set of *stable states* at time $t \in [0,T]$ as

$$\mathcal{S}(t) = \{ u \in U \ : \ \mathcal{E}(t,u) < \infty \ \text{ and } \ \mathcal{E}(t,u) \leq \mathcal{E}(t,v) + \Psi(v-u) \ \ \forall v \in U \}. \tag{2.14}$$

Moreover, we will make use of the following notion of *approximately stable states* which depends on the small parameter $\alpha > 0$, namely

$$S^\alpha(t) = \Big\{ u \in U \ : \ \mathcal{E}(t,u) < \infty \ \text{ and } \\ \mathcal{E}(t,u) \le (1+\alpha)\Big(\mathcal{E}(t,v) + \Psi(v-u)\Big) + \alpha \ \forall v \in U \Big\}. \tag{2.15}$$

As for the initial datum u_0 we assume:

$$u_0 \in S(0), \tag{2.16}$$

although some of our results still hold under the weaker assumption $\mathcal{E}(0,u_0) < \infty$.

2.2. **Time-discretization.** Assume now to be given a partition of $[0,T]$ which we identify with the corresponding vector $\boldsymbol{\tau} = (\tau^1, \ldots, \tau^{N_{\boldsymbol{\tau}}})$ of strictly positive timesteps. Note that we indicate with superscripts the elements of a generic vector. In particular τ^j represents the j-th component of the vector $\boldsymbol{\tau}$ (and not the j-th power of the scalar τ).

We let $t^0_{\boldsymbol{\tau}} = 0$ and

$$t^i_{\boldsymbol{\tau}} - t^{i-1}_{\boldsymbol{\tau}} = \tau^i, \quad I^i_{\boldsymbol{\tau}} = [t^{i-1}_{\boldsymbol{\tau}}, t^i_{\boldsymbol{\tau}}) \quad \text{for} \ \ i = 1, \ldots, N_{\boldsymbol{\tau}},$$

and we will use the symbols

$$\overline{\boldsymbol{\tau}} = \max_{i=1,\ldots,N_{\boldsymbol{\tau}}} \tau^i, \qquad \underline{\boldsymbol{\tau}} = \min_{i=1,\ldots,N_{\boldsymbol{\tau}}} \tau^i$$

for the maximum timestep (fineness of the partition) and the minimum timestep, respectively. Moreover, we will make use of the notation $\boldsymbol{u}_{\boldsymbol{\tau}} = (u^0_{\boldsymbol{\tau}}, \ldots, u^{N_{\boldsymbol{\tau}}}_{\boldsymbol{\tau}})$ for generic vectors in $U^{N_{\boldsymbol{\tau}}+1}$ and let $u_{\boldsymbol{\tau}} : [0,T] \to U$ be its piecewise constant interpolant on the intervals $I^i_{\boldsymbol{\tau}}$, namely $u_{\boldsymbol{\tau}}(t) = u^i_{\boldsymbol{\tau}}$ for all $t \in I^i_{\boldsymbol{\tau}}$, $i = 1, \ldots, N_{\boldsymbol{\tau}}$ and $u_{\boldsymbol{\tau}}(T) = u^{N_{\boldsymbol{\tau}}}_{\boldsymbol{\tau}}$.

Letting $\mathcal{E}^i_{\boldsymbol{\tau}}(\cdot) = \mathcal{E}(t^i_{\boldsymbol{\tau}}, \cdot)$ for brevity, we define the discrete counterparts of WED functionals (2.11) as $\mathcal{I}_{\varepsilon\boldsymbol{\tau}} : U^{N_{\boldsymbol{\tau}}+1} \to (-\infty, \infty]$

$$\begin{aligned} \mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u}) &= \sum_{i=1}^{N_{\boldsymbol{\tau}}} \tau^i \lambda^i_{\varepsilon\boldsymbol{\tau}} \left(\Psi\left(\frac{u^i - u^{i-1}}{\tau^i}\right) + \frac{\mathcal{E}^i_{\boldsymbol{\tau}}(u^i) - \mathcal{E}^{i-1}_{\boldsymbol{\tau}}(u^{i-1})}{\tau^i} \right) \\ &= \sum_{i=1}^{N_{\boldsymbol{\tau}}} \lambda^i_{\varepsilon\boldsymbol{\tau}} \left(\Psi\left(u^i - u^{i-1}\right) + \mathcal{E}^i_{\boldsymbol{\tau}}(u^i) - \mathcal{E}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}) \right) \end{aligned}$$

where the Pareto weights $\boldsymbol{\lambda}_{\varepsilon\tau}$ are given by (see (1.5))

$$\lambda^i_{\varepsilon\boldsymbol{\tau}} = \prod_{j=1}^{i} \frac{\varepsilon}{\tau^j + \varepsilon} \quad \text{for} \ \ i = 1, \ldots, N_{\boldsymbol{\tau}}. \tag{2.17}$$

In particular, $\boldsymbol{\lambda}_{\varepsilon\boldsymbol{\tau}}$ is nothing but the solution of the variable timestep implicit Euler discretization of the problem $\lambda' + \lambda/\varepsilon = 0$ with initial condition $\lambda(0) = 1$. Hence, by defining the piecewise-constant interpolant $t \mapsto \lambda_{\varepsilon\boldsymbol{\tau}}(t)$, we have that

$$\lambda_{\varepsilon\boldsymbol{\tau}}(t) \to \mathrm{e}^{-t/\varepsilon} \ \ \text{uniformly as} \ \ \tau \to 0.$$

Moreover, one can equivalently express $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u})$ as (see (2.11))

$$\begin{aligned}\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u}) &= \sum_{i=1}^{N_{\boldsymbol{\tau}}}\lambda^i_{\varepsilon\boldsymbol{\tau}}\Psi(u^i-u^{i-1}) + \sum_{i=1}^{N_{\boldsymbol{\tau}}-1}(\lambda^i_{\varepsilon\boldsymbol{\tau}}-\lambda^{i+1}_{\varepsilon\boldsymbol{\tau}})\mathcal{E}^i_{\boldsymbol{\tau}}(u^i)\\ &\quad +\lambda^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}}\mathcal{E}^{N_{\boldsymbol{\tau}}}_{\boldsymbol{\tau}}(u^{N_{\boldsymbol{\tau}}}) - \lambda^1_{\varepsilon\boldsymbol{\tau}}\mathcal{E}^0_{\boldsymbol{\tau}}(u^0)\\ &= \sum_{i=1}^{N_{\boldsymbol{\tau}}}\Big(\lambda^i_{\varepsilon\boldsymbol{\tau}}\Psi(u^i-u^{i-1}) + \sigma^i_{\varepsilon\boldsymbol{\tau}}\mathcal{E}^i_{\boldsymbol{\tau}}(u^i)\Big) - \lambda^1_{\varepsilon\boldsymbol{\tau}}\mathcal{E}^0_{\boldsymbol{\tau}}(u^0) \end{aligned}\tag{2.18}$$

where the positive weights $\boldsymbol{\sigma}_{\varepsilon\boldsymbol{\tau}}$ are given by

$$\sigma^i_{\varepsilon\boldsymbol{\tau}} = \lambda^i_{\varepsilon\boldsymbol{\tau}} - \lambda^{i+1}_{\varepsilon\boldsymbol{\tau}} = \lambda^i_{\varepsilon\boldsymbol{\tau}}\frac{\tau^{i+1}}{\tau^{i+1}+\varepsilon} \quad \text{for } \ i=1,\dots,N_{\boldsymbol{\tau}}-1, \ \text{ and } \ \sigma^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}} = \lambda^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}}.\tag{2.19}$$

The specific choice in (2.17) will be directly exploited in the computations. Let us however stress that (2.17) is not the only possible choice of weights which can be considered and that we restrict to it for the sake of simplicity only.

Proposition 2.1 (Existence of minimizers)**.** *Assume* (2.2), (2.4), (2.6)-(2.7), *and* (2.16). *Then, the functional $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$ is lower semicontinuous and coercive with respect to the strong topology on $\{\boldsymbol{v}_{\boldsymbol{\tau}} \in U^{N_{\boldsymbol{\tau}}+1} \ : \ v^0_{\boldsymbol{\tau}} = u_0\}$. Hence, minimizers exist.*

Proof. Owing to the choice (2.17), for all $\boldsymbol{u} \in \{\boldsymbol{u} \in U^{N_{\boldsymbol{\tau}}+1} \ : \ u^0 = u_0\}$ relation (2.18) gives

$$\begin{aligned}\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u}) = \sum_{i=1}^{N_{\boldsymbol{\tau}}}\lambda^i_{\varepsilon\boldsymbol{\tau}}\Psi(u^i-u^{i-1}) + \sum_{i=1}^{N_{\boldsymbol{\tau}}-1}\lambda^i_{\varepsilon\boldsymbol{\tau}}\left(\frac{\tau^{i+1}}{\tau^{i+1}+\varepsilon}\right)\mathcal{E}^i_{\boldsymbol{\tau}}(u^i)\\ + \lambda^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}}\mathcal{E}^{N_{\boldsymbol{\tau}}}_{\boldsymbol{\tau}}(u^{N_{\boldsymbol{\tau}}}) - \left(\frac{\varepsilon}{\tau^1+\varepsilon}\right)\mathcal{E}^0_{\boldsymbol{\tau}}(u_0)\end{aligned}$$

and the assertion follows, since $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\cdot)$ is a sum of lower semicontinuous functionals. □

For the sake of later reference, let us recall that the incremental problems (1.3) read in the current variable timestep setting as follows.

$$u^i \in \operatorname{Arg\,min}_{u\in U}\left\{\Psi\left(u-u^{i-1}\right) + \mathcal{E}^i_{\boldsymbol{\tau}}(u) - \mathcal{E}^{i-1}_{\boldsymbol{\tau}}\left(u^{i-1}\right)\right\}, \quad i=1,\dots,N_{\boldsymbol{\tau}}.\tag{2.20}$$

In particular, the length τ^i of the timestep does not occur explicitly, but the energy is evaluated at times $t^{i-1}_{\boldsymbol{\tau}}$ and $t^i_{\boldsymbol{\tau}}$.

Our first result concerns an upper bound on the energy of a minimizer of $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$.

Lemma 2.2 (Upper energy estimate)**.** *Assume* (2.1)-(2.2), (2.8), (2.16), (2.17), *and let $\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}$ minimize $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$ on $\{\boldsymbol{v}_{\boldsymbol{\tau}} \in U^{N_{\boldsymbol{\tau}}+1} \ : \ v^0_{\boldsymbol{\tau}} = u_0\}$. Then*

$$\begin{aligned}\mathcal{E}^i_{\boldsymbol{\tau}}(u^i_{\varepsilon\boldsymbol{\tau}}) + \Psi(u^i_{\varepsilon\boldsymbol{\tau}}-u^{i-1}_{\varepsilon\boldsymbol{\tau}}) \le \mathcal{E}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}_{\varepsilon\boldsymbol{\tau}}) + \int_{t^{i-1}_{\boldsymbol{\tau}}}^{t^i_{\boldsymbol{\tau}}}\partial_t\mathcal{E}(t,u^{i-1}_{\varepsilon\boldsymbol{\tau}})\,\mathrm{d}t\\ \forall i=1,\dots,N_{\boldsymbol{\tau}}.\end{aligned}\tag{2.21}$$

Proof. Let $i=1,\dots,N_{\boldsymbol{\tau}}$ be fixed and define

$$v^i_{\varepsilon\boldsymbol{\tau}} = u^{i-1}_{\varepsilon\boldsymbol{\tau}} \ \text{ and } \ v^j_{\varepsilon\boldsymbol{\tau}} = u^j_{\varepsilon\boldsymbol{\tau}} \ \text{ for } j \ne i.$$

For all $i \neq N_{\boldsymbol{\tau}}$, as $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \leq \mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{v}_{\varepsilon\boldsymbol{\tau}})$, we have that

$$\begin{aligned}
&\lambda^i_{\varepsilon\boldsymbol{\tau}}\Big(\Psi(u^i_{\varepsilon\boldsymbol{\tau}}-u^{i-1}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^i_{\boldsymbol{\tau}}(u^i_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}_{\varepsilon\boldsymbol{\tau}})\Big)\\
&\quad + \lambda^{i+1}_{\varepsilon\boldsymbol{\tau}}\Big(\Psi(u^{i+1}_{\varepsilon\boldsymbol{\tau}}-u^{i}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^{i+1}_{\boldsymbol{\tau}}(u^{i+1}_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{i}_{\boldsymbol{\tau}}(u^{i}_{\varepsilon\boldsymbol{\tau}})\Big)\\
&\leq \lambda^i_{\varepsilon\boldsymbol{\tau}}\Big(\Psi(u^{i-1}_{\varepsilon\boldsymbol{\tau}}-u^{i-1}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^i_{\boldsymbol{\tau}}(u^{i-1}_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}_{\varepsilon\boldsymbol{\tau}})\Big)\\
&\quad + \lambda^{i+1}_{\varepsilon\boldsymbol{\tau}}\Big(\Psi(u^{i+1}_{\varepsilon\boldsymbol{\tau}}-u^{i-1}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^{i+1}_{\boldsymbol{\tau}}(u^{i+1}_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{i}_{\boldsymbol{\tau}}(u^{i-1}_{\varepsilon\boldsymbol{\tau}})\Big).
\end{aligned}$$

By exploiting the triangle inequality we conclude that

$$(\lambda^i_{\varepsilon\boldsymbol{\tau}}-\lambda^{i+1}_{\varepsilon\boldsymbol{\tau}})\Big(\Psi(u^i_{\varepsilon\boldsymbol{\tau}}-u^{i-1}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^i_{\boldsymbol{\tau}}(u^i_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{i}_{\boldsymbol{\tau}}(u^{i-1}_{\varepsilon\boldsymbol{\tau}})\Big) \leq 0$$

$$\text{for} \quad i = 1, \dots, N_{\boldsymbol{\tau}} - 1. \tag{2.22}$$

In case $i = N_{\boldsymbol{\tau}}$, we have that

$$\begin{aligned}
&\lambda^{N_\tau}_{\varepsilon\boldsymbol{\tau}}\Big(\Psi(u^{N_\tau}_{\varepsilon\boldsymbol{\tau}}-u^{N_\tau-1}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^{N_\tau}_{\boldsymbol{\tau}}(u^{N_\tau}_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{N_\tau-1}_{\boldsymbol{\tau}}(u^{N_\tau-1}_{\varepsilon\boldsymbol{\tau}})\Big)\\
&\leq \lambda^{N_\tau}_{\varepsilon\boldsymbol{\tau}}\Big(\Psi(u^{N_\tau-1}_{\varepsilon\boldsymbol{\tau}}-u^{N_\tau-1}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^{N_\tau}_{\boldsymbol{\tau}}(u^{N_\tau-1}_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{N_\tau-1}_{\boldsymbol{\tau}}(u^{N_\tau-1}_{\varepsilon\boldsymbol{\tau}})\Big),
\end{aligned}$$

which amounts to say that

$$\lambda^{N_\tau}_{\varepsilon\boldsymbol{\tau}}\Big(\Psi(u^{N_\tau}_{\varepsilon\boldsymbol{\tau}}-u^{N_\tau-1}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^{N_\tau}_{\boldsymbol{\tau}}(u^{N_\tau}_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{N_\tau}_{\boldsymbol{\tau}}(u^{N_\tau-1}_{\varepsilon\boldsymbol{\tau}})\Big) \leq 0.$$

Finally, by collecting the latter and (2.22) and using the strict positivity and the monotonicity of the Pareto weights one has that

$$\Psi(u^i_{\varepsilon\boldsymbol{\tau}}-u^{i-1}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^i_{\boldsymbol{\tau}}(u^i_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{i}_{\boldsymbol{\tau}}(u^{i-1}_{\varepsilon\boldsymbol{\tau}}) \leq 0$$
$$\text{for} \quad i = 1, \dots, N_{\boldsymbol{\tau}},$$

and the assertion follows. ☐

The upper energy estimate (2.21) is at the basis of an a priori control on the trajectory. In particular, we have the following.

Corollary 2.3 (A priori estimates)**.** *Assume* (2.1)-(2.2), (2.8), (2.16), (2.17), *and let* $\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}$ *minimize* $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$ *on* $\{\boldsymbol{v}_{\boldsymbol{\tau}} \in U^{N_\tau+1} \ : \ v^0_{\boldsymbol{\tau}} = u_0\}$*. Then,*

$$\max_i \mathcal{E}^i_{\boldsymbol{\tau}}(u^i_{\varepsilon\boldsymbol{\tau}}) + \sum_{i=1}^{N_\tau} \Psi(u^i_{\varepsilon\boldsymbol{\tau}}-u^{i-1}_{\varepsilon\boldsymbol{\tau}}) \leq 2(\mathcal{E}(0,u_0) + c_0)\mathrm{e}^{c_1 T} =: c_{\mathrm{stab}}. \tag{2.23}$$

The latter is proved in [Mie05, Thm. 3.2]. We shall present some slightly refined version of the same argument (providing a proof of Corollary 2.3 as well) in Subsection 3 below.

Let us explicitly remark that, by requiring (2.3) and (2.7), under the assumption of Corollary 2.3 the a priori estimate (2.23) entails that

$$\|u_{\varepsilon\boldsymbol{\tau}}\|_{\mathrm{L}^\infty(0,T;V)} \ \text{ and } \ \mathrm{Var}(u_{\varepsilon\boldsymbol{\tau}}) \ \text{ are bounded independently of } \varepsilon \text{ and } \boldsymbol{\tau}, \tag{2.24}$$

the latter bound depending indeed on c_Ψ, $c_{\mathcal{E}}$, $C_{\mathcal{E}}$, $\mathcal{E}(0, u_0)$, c_0, c_1, and T.

The global stability property (2.12) of time-continuous energetic solutions is inherited by the discrete trajectories which are obtained by sequentially solving the minimum problems (2.20), see [MT99, MT04]. This is however not the case for minimizers of $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$ which, due to the causality lack, turn out to be only *approximately* stable in the sense of (2.15).

Lemma 2.4 (Approximate Stability)**.** *Assume* (2.1)-(2.2), (2.8), (2.16), (2.17), $\varepsilon/\underline{\boldsymbol{\tau}} < 1$, *and let* $\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}$ *minimize* $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$ *on* $\{\boldsymbol{v}_{\boldsymbol{\tau}} \in U^{N_{\boldsymbol{\tau}}+1} \ : \ v^0_{\boldsymbol{\tau}} = u_0\}$. *Then,* $u^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}} \in S(T)$ *and*

$$u^i_{\varepsilon\boldsymbol{\tau}} \in S^{\alpha(\varepsilon/\underline{\boldsymbol{\tau}})}(t^i_{\boldsymbol{\tau}}) \quad \forall v \in V, \ i = 1, \dots, N_{\boldsymbol{\tau}} - 1, \tag{2.25}$$

where the function $r \mapsto \alpha(r) \geq 0$ *depends on* c_0, c_1, T, *and* $\mathcal{E}(0, u_0)$, *and is such that* $\alpha(r) \to 0$ *as* $r \to 0+$.

Proof. Let $i = 1, \dots, N_{\boldsymbol{\tau}}$ and $v \in V$ be fixed and define

$$v^j_{\varepsilon\boldsymbol{\tau}} = u^j_{\varepsilon\boldsymbol{\tau}} \ \text{ for } j \leq i-1 \ \text{ and } \ v^j_{\varepsilon\boldsymbol{\tau}} = v \ \text{ for } j \geq i.$$

In case $i = N_{\boldsymbol{\tau}}$, as $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \leq \mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{v}_{\varepsilon\boldsymbol{\tau}})$, we have that

$$\begin{aligned} &\lambda^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}} \Big(\Psi(u^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}} - u^{N_{\boldsymbol{\tau}}-1}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^{N_{\boldsymbol{\tau}}}_{\boldsymbol{\tau}}(u^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{N_{\boldsymbol{\tau}}-1}_{\boldsymbol{\tau}}(u^{N_{\boldsymbol{\tau}}-1}_{\varepsilon\boldsymbol{\tau}}) \Big) \\ &\leq \lambda^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}} \Big(\Psi(v - u^{N_{\boldsymbol{\tau}}-1}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^{N_{\boldsymbol{\tau}}}_{\boldsymbol{\tau}}(v) - \mathcal{E}^{N_{\boldsymbol{\tau}}-1}_{\boldsymbol{\tau}}(u^{N_{\boldsymbol{\tau}}-1}_{\varepsilon\boldsymbol{\tau}}) \Big), \end{aligned}$$

and, by the triangle inequality

$$\lambda^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}} \Big(- \Psi(v - u^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^{N_{\boldsymbol{\tau}}}_{\boldsymbol{\tau}}(u^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{N_{\boldsymbol{\tau}}}_{\boldsymbol{\tau}}(v) \Big) \leq 0.$$

Hence, the stability $u^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}} \in S(T)$ follows from $\lambda^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}} > 0$.

Let us now consider the case $i \neq N_{\boldsymbol{\tau}}$ instead. Again from $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \leq \mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{v}_{\varepsilon\boldsymbol{\tau}})$, we have that

$$\begin{aligned} &\lambda^i_{\varepsilon\boldsymbol{\tau}} \Big(\Psi(u^i_{\varepsilon\boldsymbol{\tau}} - u^{i-1}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^i_{\boldsymbol{\tau}}(u^i_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}_{\varepsilon\boldsymbol{\tau}}) \Big) \\ &\qquad + \sum_{j=i+1}^{N_{\boldsymbol{\tau}}} \lambda^j_{\varepsilon\boldsymbol{\tau}} \Big(\Psi(u^j_{\varepsilon\boldsymbol{\tau}} - u^{j-1}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^j_{\boldsymbol{\tau}}(u^j_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{j-1}_{\boldsymbol{\tau}}(u^{j-1}_{\varepsilon\boldsymbol{\tau}}) \Big) \\ &\leq \lambda^i_{\varepsilon\boldsymbol{\tau}} \Big(\Psi(v - u^{i-1}_{\varepsilon\boldsymbol{\tau}}) + \mathcal{E}^i_{\boldsymbol{\tau}}(v) - \mathcal{E}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}_{\varepsilon\boldsymbol{\tau}}) \Big) + \sum_{j=i+1}^{N_{\boldsymbol{\tau}}} \lambda^j_{\varepsilon\boldsymbol{\tau}} \Big(\mathcal{E}^j_{\boldsymbol{\tau}}(v) - \mathcal{E}^{j-1}_{\boldsymbol{\tau}}(v) \Big). \end{aligned}$$

By exploiting the triangle inequality

$$\Psi(v - u^{i-1}_{\varepsilon\boldsymbol{\tau}}) \leq \Psi(v - u^i_{\varepsilon\boldsymbol{\tau}}) + \Psi(u^i_{\varepsilon\boldsymbol{\tau}} - u^{i-1}_{\varepsilon\boldsymbol{\tau}}),$$

canceling on both sides the terms $\lambda^i_{\varepsilon\boldsymbol{\tau}} \Big(\Psi(u^i_{\varepsilon\boldsymbol{\tau}} - u^{i-1}_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}_{\varepsilon\boldsymbol{\tau}}) \Big)$, dividing by $\lambda^i_{\varepsilon\boldsymbol{\tau}} > 0$, and using the non-negativity of Ψ, we have that

$$\begin{aligned} &\mathcal{E}^i_{\boldsymbol{\tau}}(u^i_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^i_{\boldsymbol{\tau}}(v) - \Psi(v - u^i_{\varepsilon\boldsymbol{\tau}}) \\ &\leq - \sum_{j=i+1}^{N_{\boldsymbol{\tau}}} \frac{\lambda^j_{\varepsilon\boldsymbol{\tau}}}{\lambda^i_{\varepsilon\boldsymbol{\tau}}} \Big(\mathcal{E}^j_{\boldsymbol{\tau}}(u^j_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{j-1}_{\boldsymbol{\tau}}(u^{j-1}_{\varepsilon\boldsymbol{\tau}}) \Big) + \sum_{j=i+1}^{N_{\boldsymbol{\tau}}} \frac{\lambda^j_{\varepsilon\boldsymbol{\tau}}}{\lambda^i_{\varepsilon\boldsymbol{\tau}}} \Big(\mathcal{E}^j_{\boldsymbol{\tau}}(v) - \mathcal{E}^{j-1}_{\boldsymbol{\tau}}(v) \Big). \end{aligned} \tag{2.26}$$

We aim now at controlling the two sums in the right-hand side above. Owing to (2.8) and (2.23) we have that

$$- \sum_{j=i+1}^{N_{\boldsymbol{\tau}}} \frac{\lambda^j_{\varepsilon\boldsymbol{\tau}}}{\lambda^i_{\varepsilon\boldsymbol{\tau}}} \Big(\mathcal{E}^j_{\boldsymbol{\tau}}(u^j_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{j-1}_{\boldsymbol{\tau}}(u^{j-1}_{\varepsilon\boldsymbol{\tau}}) \Big) \leq (c_0 + c_{\mathrm{stab}}) \sum_{j=i+1}^{N_{\boldsymbol{\tau}}} \frac{\lambda^j_{\varepsilon\boldsymbol{\tau}}}{\lambda^i_{\varepsilon\boldsymbol{\tau}}}$$

and we easily handle this last sum as

$$\sum_{j=i+1}^{N_{\boldsymbol{\tau}}} \frac{\lambda^j_{\varepsilon\boldsymbol{\tau}}}{\lambda^i_{\varepsilon\boldsymbol{\tau}}} = \sum_{j=i+1}^{N_{\boldsymbol{\tau}}} \prod_{k=i+1}^{j} \frac{\varepsilon}{\tau^k+\varepsilon}$$

$$\leq \sum_{j=i+1}^{N_{\boldsymbol{\tau}}} \left(\frac{\varepsilon}{\underline{\boldsymbol{\tau}}}\right)^{j-i} = \sum_{k=1}^{N_{\boldsymbol{\tau}}-i} \left(\frac{\varepsilon}{\underline{\boldsymbol{\tau}}}\right)^{k} = \frac{\varepsilon}{\underline{\boldsymbol{\tau}}}\left(\frac{1-(\varepsilon/\underline{\boldsymbol{\tau}})^{N_{\boldsymbol{\tau}}-i}}{1-\varepsilon/\underline{\boldsymbol{\tau}}}\right) \leq \frac{\varepsilon/\underline{\boldsymbol{\tau}}}{1-\varepsilon/\underline{\boldsymbol{\tau}}}. \tag{2.27}$$

Consider now the second sum in the right-hand side of (2.26). Owing to (2.10) we have

$$\mathcal{E}^j_{\boldsymbol{\tau}}(v)+c_0 \leq \big(\mathcal{E}^i_{\boldsymbol{\tau}}(v)+c_0\big)\mathrm{e}^{c_1(t^j_{\boldsymbol{\tau}}-t^i_{\boldsymbol{\tau}})}.$$

Then, by using the monotonicity of the Pareto weights, we have that

$$\begin{aligned}
\sum_{j=i+1}^{N_{\boldsymbol{\tau}}} \frac{\lambda^j_{\varepsilon\boldsymbol{\tau}}}{\lambda^i_{\varepsilon\boldsymbol{\tau}}}\Big(\mathcal{E}^j_{\boldsymbol{\tau}}(v)-\mathcal{E}^{j-1}_{\boldsymbol{\tau}}(v)\Big) &\leq \frac{\lambda^{i+1}_{\varepsilon\boldsymbol{\tau}}}{\lambda^i_{\varepsilon\boldsymbol{\tau}}}\sum_{j=i+1}^{N_{\boldsymbol{\tau}}} \left(\mathcal{E}^{j-1}_{\boldsymbol{\tau}}(v)+c_0\right)\left(\mathrm{e}^{c_1(t^j_{\boldsymbol{\tau}}-t^{j-1}_{\boldsymbol{\tau}})}-1\right)\\
&\leq \frac{\varepsilon}{\underline{\boldsymbol{\tau}}}\sum_{j=i+1}^{N_{\boldsymbol{\tau}}} \left(\mathcal{E}^{i}_{\boldsymbol{\tau}}(v)+c_0\right)\left(\mathrm{e}^{c_1(t^j_{\boldsymbol{\tau}}-t^{i}_{\boldsymbol{\tau}})}-\mathrm{e}^{c_1(t^{j-1}_{\boldsymbol{\tau}}-t^{i}_{\boldsymbol{\tau}})}\right)\\
&= \frac{\varepsilon}{\underline{\boldsymbol{\tau}}}\left(\mathcal{E}^{i}_{\boldsymbol{\tau}}(v)+c_0\right)\left(\mathrm{e}^{c_1(T-t^{i}_{\boldsymbol{\tau}})}-1\right) \leq \frac{\varepsilon}{\underline{\boldsymbol{\tau}}}\left(\mathcal{E}^{i}_{\boldsymbol{\tau}}(v)+c_0\right)\left(\mathrm{e}^{c_1T}-1\right).
\end{aligned}$$

Finally, by defining for instance

$$\alpha(r) = r\left((1+c_0)\left(\mathrm{e}^{c_1T}-1\right)+\frac{c_0+c_{\mathrm{stab}}}{1-r}\right), \tag{2.28}$$

the assertion follows. □

Remark 2.5. In case Ψ is even, namely $\Psi(-v)=\Psi(v)$, the proof of (2.25) greatly simplifies. In particular, the situation $i\neq N_{\boldsymbol{\tau}}$ may be treated by letting ($v\in V$ fixed)

$$v^i_{\boldsymbol{\tau}} = v \quad\text{and}\quad v^j_{\varepsilon\boldsymbol{\tau}} = u^j_{\varepsilon\boldsymbol{\tau}} \quad \text{for } j\neq i,$$

and exploiting $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \leq \mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{v}_{\boldsymbol{\tau}})$ in order to get

$$\begin{aligned}
&\lambda^i_{\varepsilon\boldsymbol{\tau}}\Big(\Psi(u^i_{\varepsilon\boldsymbol{\tau}}-u^{i-1}_{\varepsilon\boldsymbol{\tau}})+\mathcal{E}^i_{\boldsymbol{\tau}}(u^i_{\varepsilon\boldsymbol{\tau}})-\mathcal{E}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}_{\varepsilon\boldsymbol{\tau}})\Big)\\
&\qquad+\lambda^{i+1}_{\varepsilon\boldsymbol{\tau}}\Big(\Psi(u^{i+1}_{\varepsilon\boldsymbol{\tau}}-u^{i}_{\varepsilon\boldsymbol{\tau}})+\mathcal{E}^{i+1}_{\boldsymbol{\tau}}(u^{i+1}_{\varepsilon\boldsymbol{\tau}})-\mathcal{E}^{i}_{\boldsymbol{\tau}}(u^{i}_{\varepsilon\boldsymbol{\tau}})\Big)\\
&\leq \lambda^i_{\varepsilon\boldsymbol{\tau}}\Big(\Psi(v-u^{i-1}_{\varepsilon\boldsymbol{\tau}})+\mathcal{E}^i_{\boldsymbol{\tau}}(v)-\mathcal{E}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}_{\varepsilon\boldsymbol{\tau}})\Big)\\
&\qquad+\lambda^{i+1}_{\varepsilon\boldsymbol{\tau}}\Big(\Psi(u^{i+1}_{\varepsilon\boldsymbol{\tau}}-v)+\mathcal{E}^{i+1}_{\boldsymbol{\tau}}(u^{i+1}_{\varepsilon\boldsymbol{\tau}})-\mathcal{E}^{i}_{\boldsymbol{\tau}}(v)\Big).
\end{aligned}$$

Again by the triangle inequality we deduce

$$(\lambda^i_{\varepsilon\boldsymbol{\tau}}-\lambda^{i+1}_{\varepsilon\boldsymbol{\tau}})\Big(\mathcal{E}^i_{\boldsymbol{\tau}}(u^i_{\varepsilon\boldsymbol{\tau}})-\mathcal{E}^i_{\boldsymbol{\tau}}(v)\Big) \leq (\lambda^i_{\varepsilon\boldsymbol{\tau}}+\lambda^{i+1}_{\varepsilon\boldsymbol{\tau}})\Psi(v-u^i_{\varepsilon\boldsymbol{\tau}}),$$

and, by exploiting the choice of Pareto weights (2.17),

$$\mathcal{E}^i_{\boldsymbol{\tau}}(u^i_{\varepsilon\boldsymbol{\tau}}) \leq \mathcal{E}^i_{\boldsymbol{\tau}}(v)+\left(1+\frac{2\varepsilon}{\tau^{i+1}}\right)\Psi(v-u^i_{\varepsilon\boldsymbol{\tau}})$$

and the function α can be simply taken to be $\alpha(r)=2r$.

We shall now turn to our main result.

Theorem 2.6 (Conditional convergence)**.** *Assume* (2.1)-(2.9), (2.16), *and let*

$$\Psi \text{ be continuous with respect to the strong topology in } U. \tag{2.29}$$

Moreover, let a sequence of partitions $\boldsymbol{\tau}_n$ *and parameters* ε_n *with* $(\varepsilon_n, \overline{\boldsymbol{\tau}}_n) \to (0,0)$ *and* $\varepsilon_n/\underline{\boldsymbol{\tau}}_n \to 0$ *be given, and* $\boldsymbol{\lambda}_{\varepsilon_n\boldsymbol{\tau}_n}$ *fulfill* (2.17)*. Finally, let* $\boldsymbol{u}_{\varepsilon_n\boldsymbol{\tau}_n}$ *be a minimizer of* $\mathcal{I}_{\varepsilon_n\boldsymbol{\tau}_n}$ *on* $\{\boldsymbol{v}_{\boldsymbol{\tau}_n} \in U^{N_{\boldsymbol{\tau}_n}+1} \ : \ v^0_{\boldsymbol{\tau}_n} = u_0\}$*. Then, there exists a (not relabeled) subsequence* $(\varepsilon_n, \boldsymbol{\tau}_n)$ *and* $u : [0,T] \to U$ *energetic solution of* (1.1) *such that, for all* $t \in [0,T]$*, the following convergences hold*

$$u_{\varepsilon_n\boldsymbol{\tau}_n}(t) \to u(t) \quad \textit{weakly in } V \ \textit{(and strongly in } U\textit{)}, \tag{2.30}$$

$$\int_{[0,t]} \Psi(\mathrm{d}u_{\varepsilon_n\boldsymbol{\tau}_n}) \to \int_{[0,t]} \Psi(\mathrm{d}u), \tag{2.31}$$

$$\mathcal{E}(t, u_{\varepsilon_n\boldsymbol{\tau}_n}(t)) \to \mathcal{E}(t, u(t)), \tag{2.32}$$

$$\partial_t\mathcal{E}(\cdot, u_{\varepsilon_n\boldsymbol{\tau}_n}(\cdot)) \to \partial_t\mathcal{E}(\cdot, u(\cdot)) \quad \textit{in} \ \ \mathrm{L}^1(0,T). \tag{2.33}$$

Proof. The statement follows along the lines of the convergence proof of solutions of the incremental problem to energetic solutions [Mie05, Thm. 5.2], the only difference being that the converging sequences are not stable but rather approximately stable in the sense of Lemma 2.4.

Step 1: A priori estimates and selection of subsequences. Let $u_n = u_{\varepsilon_n\boldsymbol{\tau}_n}$ and $p_n(t) = \partial_t\mathcal{E}(t, u_n(t))$ for simplicity. Owing to (2.1), the a priori estimate (2.24), and Helly's principle [Mie05, Thm. 5.1], upon extracting not relabeled subsequences, we have that, for all $t \in [0,T]$,

$$u_n(t) \to u(t) \quad \text{weakly in } V \text{ and strongly in } U, \tag{2.34}$$

$$\delta_n(t) = \int_{[0,t]} \Psi(\mathrm{d}u_n) \to \delta(t), \tag{2.35}$$

$$p_n \to p \quad \text{weakly star in } \ \mathrm{L}^\infty(0,T), \tag{2.36}$$

for some non-decreasing function $\delta : [0,T] \to [0,\infty)$.

Step 2: Stability of the limit process. We may assume with no loss of generality that $\varepsilon_n/\underline{\boldsymbol{\tau}}_n < 1$. Let us fix $t \in [0,T)$ and denote by $t_n = \max\{t^i_{\boldsymbol{\tau}_n} \ : \ t^i_{\boldsymbol{\tau}_n} \leq t\}$. Hence, owing to the convergence (2.34) and Lemma 2.4, we have that $u_n(t_n) \in S^{\alpha_n}(t_n)$ for some $\alpha_n \to 0$ and, for all $v \in V$ with $\mathcal{E}(t,v) < \infty$,

$$\begin{aligned}
\mathcal{E}(t,u(t)) &\overset{(2.6)}{\leq} \liminf_{n\to\infty} \mathcal{E}(t, u_n(t)) \\
&= \liminf_{n\to\infty} \left(\mathcal{E}(t_n, u_n(t)) + \int_{t_n}^t \partial_t\mathcal{E}(s, u_n(t))\,\mathrm{d}s \right) \\
&\overset{(2.8)+(2.23)}{=} \liminf_{n\to\infty} \mathcal{E}(t_n, u_n(t_n)) \\
&\overset{(2.25)}{\leq} \liminf_{n\to\infty} \Big((1+\alpha_n)\big(\mathcal{E}(t_n, v) + \Psi(v - u_n(t_n))\big) + \alpha_n \Big) \\
&\overset{(2.29)}{=} \mathcal{E}(t,v) + \Psi(v - u(t)).
\end{aligned}$$

Namely, $u(t) \in S(t)$ for all $t \in [0,T]$ (the case $t = T$ is classical [Mie05, Thm. 5.2]).

Step 3: Upper energy estimate. From (2.21) we deduce that, for all $[s,t] \subset [0,T]$,

$$\mathcal{E}(t,u_n(t)) + \int_{[0,t]} \Psi(\mathrm{d}u_n) \le \mathcal{E}(0,u_0) + \int_0^t \partial_t \mathcal{E}(r,u_n(r))\,\mathrm{d}r + c\overline{\boldsymbol{\tau}}_n,$$

where the constant c bounds $\partial_t \mathcal{E}(\cdot,u_n(\cdot))$ uniformly in time and n. Hence, passing to the lim inf and using (2.6) and the convergences (2.34)-(2.36), we have

$$\mathcal{E}(t,u(t)) + \int_{[0,t]} \Psi(\mathrm{d}u) \le \liminf_{n\to\infty} \mathcal{E}(t,u_n(t)) + \delta(t) \le \mathcal{E}(0,u_0) + \int_0^t p(s)\,\mathrm{d}s. \tag{2.37}$$

In fact $\mathcal{E}(t,u_n(t)) \to \mathcal{E}(t,u(t))$ since, for all $t \in [0,T]$,

$$\begin{aligned}
\mathcal{E}(t,u(t)) &\overset{(2.8)-(2.9)}{=} \lim_{n\to\infty} \left(\mathcal{E}(t_n,u(t)) + \int_{t_n}^t \partial_t \mathcal{E}(s,u(t))\,\mathrm{d}s \right) \\
&\overset{(2.29)}{=} \lim_{n\to\infty} \Big((1+\alpha_n)\big(\mathcal{E}(t_n,u(t)) + \Psi(u(t)-u_n(t))\big) + \alpha_n \Big) \\
&\overset{(2.25)}{\ge} \limsup_{n\to\infty} \mathcal{E}(t_n,u_n(t)) \\
&= \limsup_{n\to\infty} \left(\mathcal{E}(t,u_n(t)) - \int_{t_n}^t \partial_t \mathcal{E}(s,u_n(t))\,\mathrm{d}s \right) \\
&\overset{(2.6)}{\ge} \mathcal{E}(t,u(t))
\end{aligned}$$

and (2.32) follows. Moreover, owing to (2.4) and (2.7)-(2.9), we may apply [Mie05, Prop. 5.6] and deduce that $p(t) = \partial_t \mathcal{E}(t,u(t))$ and (2.33) holds.

Step 4: Lower energy estimate. This follows at once from (2.2)-(2.3), and (2.8)-(2.9), since $u(t) \in S(t)$ for all $t \in [0,T]$ and $t \mapsto \partial_t \mathcal{E}(t,u(t)) \in \mathrm{L}^\infty(0,T)$. Applying [Mie05, Prop. 5.7] we have that

$$\mathcal{E}(t,u(t)) + \int_{[0,t]} \Psi(\mathrm{d}u) \ge \mathcal{E}(0,u_0) + \int_0^t \partial_t \mathcal{E}(s,u(s))\,\mathrm{d}s. \tag{2.38}$$

In particular, (2.13) follows.

Step 5: Improved convergence. By collecting (2.37) and (2.38) we have that

$$\begin{aligned}
\mathcal{E}(0,u_0) + \int_0^t \partial_t \mathcal{E}(s,u(s))\,\mathrm{d}s &\overset{(2.38)}{\le} \mathcal{E}(t,u(t)) + \int_{[0,t]} \Psi(\mathrm{d}u) \\
\overset{(2.6)}{\le} \liminf_{n\to\infty} \mathcal{E}(t,u_n(t)) + \delta(t) &\overset{(2.37)}{\le} \mathcal{E}(0,u_0) + \int_0^t \partial_t \mathcal{E}(s,u(s))\,\mathrm{d}s.
\end{aligned}$$

Hence, all inequalities are actually equalities and (2.31) follows from (2.35). ☐

Remark 2.7. Owing to rate-independence, for any given non-uniform partition $\boldsymbol{\tau}$, one is always allowed to reduce to a uniform partition by time-rescaling. This strategy is however little suited for studying convergence for a family of variable timestep partition with diameters tending to 0. Hence, it is not pursued here.

3. Minimizing sequences

An interesting issue that differentiates the present time-discrete approach from the time-continuous one of [MO08] is that, in order for conditional convergence to hold, the minimality of $\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}$ is actually not required and one could ask $\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}$ to be a *θ-approximate minimizer*, namely

$$\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \leq \mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{v}_{\boldsymbol{\tau}}) + \theta \quad \forall \boldsymbol{v}_{\boldsymbol{\tau}} \in U^{N_{\boldsymbol{\tau}_n}+1}, \ v^0_{\boldsymbol{\tau}_n} = u_0, \tag{3.39}$$

where the tolerance $\theta > 0$ is given. This possibility is particularly interesting as one may consider situations where the minimum of $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$ on $\{\boldsymbol{v}_{\boldsymbol{\tau}_n} \in U^{N_{\boldsymbol{\tau}_n}+1} \ : \ v^0_{\boldsymbol{\tau}_n} = u_0\}$ is not attained (see Subsection 5 below). In particular, given any $\theta > 0$, the existence of θ-approximate minimizers for $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$ on $\{\boldsymbol{v}_{\boldsymbol{\tau}} \in U^{N_{\boldsymbol{\tau}}+1} \ : \ v^0_{\boldsymbol{\tau}} = u_0\}$ is straightforward whenever Ψ and $\mathcal{E}$ are bounded from below (no coercivity and lower semicontinuity are of course needed, see Proposition 2.1). Nevertheless, even in the case of (3.39), an upper energy estimate, a global a priori estimate, the approximate stability, and a conditional convergence result can be proved. Let us start from the estimates.

Theorem 3.1 (Approximate minimality, a priori estimates)**.** *Assume* (2.1)-(2.2), (2.8), (2.16), (2.17), *let $\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}$ be a θ-approximate minimizer of $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$ on $\{\boldsymbol{v}_{\boldsymbol{\tau}} \in U^{N_{\boldsymbol{\tau}}+1} \ : \ v^0_{\boldsymbol{\tau}} = u_0\}$, and $\boldsymbol{\rho}_{\varepsilon\boldsymbol{\tau}}$ be defined as*

$$\rho^i_{\varepsilon\boldsymbol{\tau}} = 1/\sigma^i_{\varepsilon\boldsymbol{\tau}}, \tag{3.40}$$

see (2.19) *for $\sigma^i_{\varepsilon\boldsymbol{\tau}}$. Finally, assume*

$$\frac{\varepsilon}{\underline{\boldsymbol{\tau}}} < 1 \quad \text{and} \quad \theta \sum_{i=1}^{N_{\boldsymbol{\tau}}} \rho^i_{\varepsilon\boldsymbol{\tau}} < 1. \tag{3.41}$$

Then, we have that

$$\begin{aligned} &\mathcal{E}^i_{\boldsymbol{\tau}}(u^i_{\varepsilon\boldsymbol{\tau}}) + \Psi(u^i_{\varepsilon\boldsymbol{\tau}} - u^{i-1}_{\varepsilon\boldsymbol{\tau}}) \leq \mathcal{E}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}_{\varepsilon\boldsymbol{\tau}}) + \int_{t^{i-1}_{\boldsymbol{\tau}}}^{t^i_{\boldsymbol{\tau}}} \partial_t \mathcal{E}(t, u^{i-1}_{\varepsilon\boldsymbol{\tau}}) + \theta \rho^i_{\varepsilon\boldsymbol{\tau}} \\ &\forall i = 1, \ldots, N_{\boldsymbol{\tau}}, \end{aligned} \tag{3.42}$$

$$\begin{aligned} &\max_i \mathcal{E}^i_{\boldsymbol{\tau}}(u^i_{\varepsilon\boldsymbol{\tau}}) + \sum_{i=1}^{N_{\boldsymbol{\tau}}} \Psi(u^i_{\varepsilon\boldsymbol{\tau}} - u^{i-1}_{\varepsilon\boldsymbol{\tau}}) \\ &\quad \leq 2\left(\mathcal{E}(0, u_0) + c_0 + 1\right) \mathrm{e}^{c_1 T} + 1 = c_{\text{stab}} + 2\mathrm{e}^{c_1 T} + 1, \end{aligned} \tag{3.43}$$

$$\begin{aligned} &u^i_{\varepsilon\boldsymbol{\tau}} \in S^{\alpha(\boldsymbol{\tau},\varepsilon,\theta)}(t^i_{\boldsymbol{\tau}}) \quad \text{for} \ \ i = 1, \ldots, N_{\boldsymbol{\tau}} - 1 \qquad \text{where} \\ &\quad \alpha(\boldsymbol{\tau}, \varepsilon, \theta) = \frac{\varepsilon}{\underline{\boldsymbol{\tau}}} \left((1{+}c_0)\left(\mathrm{e}^{c_1 T}{-}1\right) + \frac{c_0{+}c_{\text{stab}}{+}2\mathrm{e}^{c_1 T}{+}1}{1{-}\varepsilon/\underline{\boldsymbol{\tau}}} \right) + \theta \rho^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}}, \end{aligned} \tag{3.44}$$

$$u^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}} \in S^{\theta \rho^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}}}(t^{N_{\boldsymbol{\tau}}}_{\boldsymbol{\tau}}). \tag{3.45}$$

Sketch of the proof. This result is simply obtained by reconsidering the arguments of Section 2 by keeping track of the extra error term depending on θ. In particular, relations (3.42) and (3.45) are immediate. As for the expression in the right-hand side of (3.43), we shall reconsider here the argument of [Mie05, Thm. 3.2] and, letting for notational simplicity $\mathcal{E}^i = \mathcal{E}^i_{\boldsymbol{\tau}}(u^i_{\varepsilon\boldsymbol{\tau}})$ and $\Psi^i = \Psi(u^{i-1}_{\varepsilon\boldsymbol{\tau}} - u^i_{\varepsilon\boldsymbol{\tau}})$, start from (3.42) and exploit (2.8) in order to get that

$$\mathcal{E}^i + \Psi^i \le \mathcal{E}^{i-1} + (\mathcal{E}^{i-1} + c_0)(\mathrm{e}^{c_1\tau^i} - 1) + \theta\rho^i_{\varepsilon\boldsymbol{\tau}} \qquad \text{for } i = 1, \dots, N_{\boldsymbol{\tau}}. \tag{3.46}$$

In the latter computation we have used (2.10) and

$$\begin{aligned}\int_{t^{i-1}_{\boldsymbol{\tau}}}^{t^i_{\boldsymbol{\tau}}} \partial_t\mathcal{E}(t, u^{i-1}_{\varepsilon\boldsymbol{\tau}})\mathrm{d}t &\overset{(2.8)}{\le} \int_{t^{i-1}_{\boldsymbol{\tau}}}^{t^i_{\boldsymbol{\tau}}} c_1\Big(\mathcal{E}(t, u^{i-1}_{\varepsilon\boldsymbol{\tau}}) + c_0\Big)\mathrm{d}t \\ &\overset{(2.10)}{\le} \int_{t^{i-1}_{\boldsymbol{\tau}}}^{t^i_{\boldsymbol{\tau}}} c_1(\mathcal{E}^{i-1} + c_0)\mathrm{e}^{c_1(t-t^{i-1}_{\boldsymbol{\tau}})}\mathrm{d}t \;=\; (\mathcal{E}^{i-1}+c_0)(\mathrm{e}^{c_1\tau^i}-1).\end{aligned}$$

Hence, as $\Psi^i \ge 0$, we check by induction that for $i = 1, \dots, N_{\boldsymbol{\tau}}$

$$\mathcal{E}^i + c_0 \le (\mathcal{E}^0+c_0)\mathrm{e}^{c_1t^i_{\boldsymbol{\tau}}} + \theta\mathrm{e}^{c_1t^i_{\boldsymbol{\tau}}}\sum_{j=1}^{i}\rho^j_{\varepsilon\boldsymbol{\tau}} \le \left(\mathcal{E}^0+c_0+\theta\sum_{j=1}^{i}\rho^j_{\varepsilon\boldsymbol{\tau}}\right)\mathrm{e}^{c_1t^i_{\boldsymbol{\tau}}}. \tag{3.47}$$

Moreover, taking the sum for $i = 1, \dots, k$ in (3.46),

$$\begin{aligned}\sum_{i=1}^{k}\Psi^i &\overset{(3.46)}{\le} \mathcal{E}^0 - \mathcal{E}^k + \sum_{i=1}^{k}(\mathcal{E}^{i-1}+c_0)\big(\mathrm{e}^{c_1\tau^i}-1\big) + \theta\sum_{i=1}^{k}\rho^i_{\varepsilon\boldsymbol{\tau}} \\ &\overset{(3.47)}{\le} \left(\mathcal{E}^0+c_0+\theta\sum_{i=1}^{N_{\boldsymbol{\tau}}}\rho^i_{\varepsilon\boldsymbol{\tau}}\right) - (\mathcal{E}^k+c_0) \\ &\qquad + \left(\mathcal{E}^0+c_0+\theta\sum_{i=1}^{N_{\boldsymbol{\tau}}}\rho^i_{\varepsilon\boldsymbol{\tau}}\right)\sum_{i=1}^{k}(\mathrm{e}^{c_1t^i_{\boldsymbol{\tau}}}-\mathrm{e}^{c_1t^{i-1}_{\boldsymbol{\tau}}}) + \theta\sum_{i=1}^{k}\rho^i_{\varepsilon\boldsymbol{\tau}} \\ &\le \left(\mathcal{E}^0+c_0+\theta\sum_{i=1}^{N_{\boldsymbol{\tau}}}\rho^i_{\varepsilon\boldsymbol{\tau}}\right) + \left(\mathcal{E}^0+c_0+\theta\sum_{i=1}^{N_{\boldsymbol{\tau}}}\rho^i_{\varepsilon\boldsymbol{\tau}}\right)(\mathrm{e}^{c_1T}-1) \\ &\qquad + \theta\sum_{i=1}^{N_{\boldsymbol{\tau}}}\rho^i_{\varepsilon\boldsymbol{\tau}},\end{aligned}$$

where we also exploited $\mathcal{E}_k + c_0 \ge 0$ and we have the a priori bound (3.43). Now it suffices to reproduce the argument of Lemma 2.4 by keeping track of the extra error term due to the tolerance θ. In particular, one proves that (see (2.26))

$$\begin{aligned}&\mathcal{E}^i_{\boldsymbol{\tau}}(u^i_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^i_{\boldsymbol{\tau}}(v) - \Psi(v-u^i_{\varepsilon\boldsymbol{\tau}}) \\ &\le -\sum_{j=i+1}^{N_{\boldsymbol{\tau}}}\frac{\lambda^j_{\varepsilon\boldsymbol{\tau}}}{\lambda^i_{\varepsilon\boldsymbol{\tau}}}\Big(\mathcal{E}^j_{\boldsymbol{\tau}}(u^j_{\varepsilon\boldsymbol{\tau}}) - \mathcal{E}^{j-1}_{\boldsymbol{\tau}}(u^{j-1}_{\varepsilon\boldsymbol{\tau}})\Big) + \sum_{j=i+1}^{N_{\boldsymbol{\tau}}}\frac{\lambda^j_{\varepsilon\boldsymbol{\tau}}}{\lambda^i_{\varepsilon\boldsymbol{\tau}}}\Big(\mathcal{E}^j_{\boldsymbol{\tau}}(v) - \mathcal{E}^{j-1}_{\boldsymbol{\tau}}(v)\Big) + \frac{\theta}{\lambda^i_{\varepsilon\boldsymbol{\tau}}} \\ &\le \big(c_0+c_{\mathrm{stab}}+2\mathrm{e}^{c_1T}+1\big)\,\frac{\varepsilon/\underline{\boldsymbol{\tau}}}{1-\varepsilon/\underline{\boldsymbol{\tau}}} + \frac{\varepsilon}{\underline{\boldsymbol{\tau}}}\,\big(\mathcal{E}^i_{\boldsymbol{\tau}}(v)+c_0\big)\,\big(\mathrm{e}^{c_1T}-1\big) + \theta\rho^{N_{\boldsymbol{\tau}}}_{\varepsilon\boldsymbol{\tau}}\end{aligned}$$

and (3.44) follows. $\square$

Once the estimates (3.42)-(3.45) are established, it is a standard matter to follow the very same argument for the proof of Theorem 2.6 above and deduce the following.

Theorem 3.2 (Approximate minimality, conditional convergence)**.** *Assume* (2.1)-(2.9), (2.16), (2.29), *let a sequence of partitions* $\boldsymbol{\tau}_n$ *and parameters* ε_n, θ_n *with* $(\varepsilon_n, \overline{\boldsymbol{\tau}}_n, \theta_n) \to (0,0,0)$ *and* $\varepsilon_n/\underline{\boldsymbol{\tau}}_n \to 0$ *be given, and* $\boldsymbol{\lambda}_{\varepsilon_n \boldsymbol{\tau}_n}$ *fulfill* (2.17)*. Moreover, let*

$$\theta_n \sum_{i=1}^{N_{\boldsymbol{\tau}_n}} \rho^i_{\varepsilon_n \boldsymbol{\tau}_n} \to 0 \tag{3.48}$$

where $\boldsymbol{\rho}_{\varepsilon_n \boldsymbol{\tau}_n}$ *is defined in* (3.40)*. Finally, let* $\boldsymbol{u}_{\varepsilon_n \boldsymbol{\tau}_n}$ *be a* θ_n-*approximate minimizer of* $\mathcal{I}_{\varepsilon_n \boldsymbol{\tau}_n}$ *on* $\{\boldsymbol{v}_{\boldsymbol{\tau}_n} \in U^{N_{\boldsymbol{\tau}_n}+1} \ : \ v^0_{\boldsymbol{\tau}_n} = u_0\}$*. Then, the conclusions of Theorem* 2.6 *hold.*

Let us mention that the latter convergence result is not ensuring that all minimizing sequences such that $\mathcal{I}_{\varepsilon \boldsymbol{\tau}}(\boldsymbol{u}_{\varepsilon \boldsymbol{\tau}}) \to 0$ admit a convergent subsequence as the qualification (3.48) is crucially required. On the other hand, the latter condition, which indeed relates the limiting behavior of ε_n, $\boldsymbol{\tau}_n$, and θ_n, is tailored to the concrete situation where ε_n and $\boldsymbol{\tau}_n$ (and thence $\boldsymbol{\rho}_{\varepsilon_n \boldsymbol{\tau}_n}$) may be considered to be fixed and then θ_n can be chosen very small in order (3.48) to hold.

4. A metric formulation

The results of Section 2 make no essential use of the linear structure of U and can therefore be reformulated in a more abstract setting. To this aim, let

$$(U, d) \quad \text{be a complete quasi-metric space.} \tag{4.49}$$

Here $d : U \times U \to [0, \infty]$ is called a quasi-metric, if $d(u, v) = 0$ implies $u = v$ and if the triangle inequality holds. We explicitly allow here the situation $d(u, v) = \infty$ as well as $d(u, v) \neq d(v, u)$ as this is needed in applications. For every given trajectory $u : [0, T] \to U$ we define the total dissipation on $[s, t] \subset [0, T]$ via

$$\mathrm{Diss}\,(u, [s, t]) = \sup \left\{ \sum_{i=1}^{N} d\big(u(t^{i-1}), u(t^i)\big) \ : \ s = t^0 < \cdots < t^N = t \right\}.$$

As for the energy functional $\mathcal{E} : [0, T] \times U \to (-\infty, \infty]$ we assume the former (2.8)-(2.9) and reformulate the compactness assumption in (2.7) as

$$\mathcal{E}(t, \cdot) : U \to (-\infty, \infty] \quad \text{has compact sublevels} \ \ \forall t \in [0, T]. \tag{4.50}$$

Given these assumptions, an energetic solution of (1.1) is now a trajectory $u : [0, T] \to U$ such that $u(0) = u_0$ and the following hold for all $t \in [0, T]$:

$$\mathcal{E}(t, u(t)) \leq \mathcal{E}(t, v) + d(u(t), v) \quad \forall v \in V, \tag{4.51}$$

$$\mathcal{E}(t, u(t)) + \mathrm{Diss}\,(u, [0, t]) = \mathcal{E}(0, u_0) + \int_0^t \partial_t \mathcal{E}(s, u(s)) \mathrm{d}s. \tag{4.52}$$

Of course the set $S(t)$ of stable states at time t is now defined as

$$S(t) = \{u \in U \ : \ \mathcal{E}(t, u) \leq \mathcal{E}(t, v) + d(u, v) \ \ \forall v \in U\}. \tag{4.53}$$

By reconsidering the proofs of Section 2, it is clear that we can reproduce the very same results in this metric setting. In particular, the convergence result reads now as follows.

Theorem 4.1 (Conditional convergence, metric setting)**.** *Assume* (4.49), (4.50), (2.7), (2.8), *and* (2.16)*. Moreover, let a sequence of partitions $\boldsymbol{\tau}_n$ and parameters ε_n with $(\varepsilon_n, \overline{\boldsymbol{\tau}}_n) \to (0,0)$ and $\varepsilon_n/\underline{\boldsymbol{\tau}}_n \to 0$ be given, and $\boldsymbol{\lambda}_{\varepsilon_n \boldsymbol{\tau}_n}$ fulfill* (2.17)*. Finally, let $\boldsymbol{u}_{\varepsilon_n \boldsymbol{\tau}_n}$ be a minimizer of $\mathcal{I}_{\varepsilon_n \boldsymbol{\tau}_n}$ on $\{\boldsymbol{v}_{\boldsymbol{\tau}_n} \in U^{N_{\boldsymbol{\tau}_n}+1} \ : \ v^0_{\boldsymbol{\tau}_n} = u_0\}$. Then, there exists a (not relabeled) subsequence $(\varepsilon_n, \boldsymbol{\tau}_n)$ and $u : [0,T] \to U$ energetic solution of* (1.1) *such that, for all $t \in [0,T]$, the following convergences hold*

$$u_{\varepsilon_n \boldsymbol{\tau}_n}(t) \to u(t) \quad \text{in } U, \tag{4.54}$$

$$\mathrm{Diss}\,(u_{\varepsilon_n \boldsymbol{\tau}_n}, [0,t]) \to \mathrm{Diss}\,(u, [0,t]), \tag{4.55}$$

$$\mathcal{E}(t, u_{\varepsilon_n \boldsymbol{\tau}_n}(t)) \to \mathcal{E}(t, u(t)), \tag{4.56}$$

$$\partial_t \mathcal{E}(\cdot, u_{\varepsilon_n \boldsymbol{\tau}_n}(\cdot)) \to \partial_t \mathcal{E}(\cdot, u(\cdot)) \quad \text{in} \ \ \mathrm{L}^1(0,T). \tag{4.57}$$

In particular, even in the present metric setting, energetic solutions may be recovered by passing to the joint (and conditional) limit in the time-discretization and in ε.

5. Approximation and relaxation

The convergence results of Section 2 are quite flexible and may be adapted to the situation of a sequence of pairs $(\Psi_k, \mathcal{E}_k)$ which are Γ-converging to a limiting pair $(\Psi, \mathcal{E})$ [Dal93, GF75]. In particular, we address the question under which conditions θ-approximate minimizers $\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}k}$ of the approximating functionals $\mathcal{I}_{\varepsilon\boldsymbol{\tau}k}$

$$\mathcal{I}_{\varepsilon\boldsymbol{\tau}k}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}k}) = \sum_{i=1}^{N_{\boldsymbol{\tau}}} \lambda^i_{\varepsilon\boldsymbol{\tau}} \Big(\Psi_k \left(u^i_{\varepsilon\boldsymbol{\tau}k} - u^{i-1}_{\varepsilon\boldsymbol{\tau}k} \right) + \mathcal{E}^i_{\boldsymbol{\tau}k}(u^i_{\varepsilon\boldsymbol{\tau}k}) - \mathcal{E}^{i-1}_{\boldsymbol{\tau}k}(u^{i-1}_{\varepsilon\boldsymbol{\tau}}) \Big)$$

are converging to an energetic solution for $(\Psi, \mathcal{E})$, namely solving (2.12)-(2.13).

A closely related issue has already been considered by Mielke, Roubíček, & Stefanelli [MRS08] for the case of energetic solutions at level k converging to a limiting energetic solution as $k \to \infty$. The crucial point in [MRS08] is the observation that the two *disjoint* Γ-convergences

$$\Psi = \Gamma\text{-}\lim_{k\to\infty} \Psi_k \quad \text{and} \quad \mathcal{E} = \Gamma\text{-}\lim_{k\to\infty} \mathcal{E}_\infty,$$

are not sufficient in order to ensure convergence of solutions and some extra condition has to be additionally required. A quite natural choice is that of asking for the *conditional upper semi-continuity of the approximately stable states*:

$$\begin{gathered} \Big(u_k \in S_k^{\alpha_k}(t_k), \ \ \sup_k \mathcal{E}_k(t_k, u_k) < \infty, \ \ (t_k, \alpha_k) \to (t, 0) \ \text{ and } \ u_k \to u \ \text{ in } \ U, \Big) \\ \Rightarrow u \in S(t), \end{gathered} \tag{5.58}$$

where we have denoted by $S_k^{\alpha_k}$ the set of α_k-approximately stable states (2.15) referred to the pair $(\Psi_k, \mathcal{E}_k)$ and the parameter $\alpha_k \geq 0$. A full hierarchy of conditions implying (5.58) is presented in [MRS08]. Here, for the sake of simplicity, we just mention the strongest, namely continuous convergence for Ψ_k and Γ-convergence for $\mathcal{E}_k$

$$u_k \to u \ \Rightarrow \Psi_k(u_k) \to \Psi(u) \qquad \text{and} \qquad \mathcal{E} = \Gamma\text{-}\lim_{k\to\infty} \mathcal{E}_k. \tag{5.59}$$

As we aim at relaxation, we shall consider the situation when Ψ_k and/or $\mathcal{E}_k$ are not lower semi-continuous. In this case, minimizers of $\mathcal{I}_{\varepsilon\boldsymbol{\tau}k}$ may fail to exist and one is forced to focus on $\theta-$approximate minimizers from the very beginning. Our convergence result reads as follows.

Theorem 5.1 (Approximation and relaxation)**.** *Assume* (2.1)-(2.3), (2.8)-(2.9) *and let* $(\Psi_k, \mathcal{E}_k)$ *fulfill* (2.2)-(2.3) *and* (2.8) *uniformly with respect to* k *and* u_0^k *fulfill* (2.16) *for all* k*. Moreover, assume* (5.58) *and the following:*

$$\Psi \leq \Gamma\text{-}\liminf_{k\to\infty} \Psi_k, \qquad \mathcal{E}(t,\cdot) \leq \Gamma\text{-}\liminf_{k\to\infty} \mathcal{E}_k(t,\cdot) \quad \forall t \in [0,T]; \tag{5.60}$$

$$\bigcup_{k=1}^{\infty} \{u \in U \ : \ \mathcal{E}_k(t,u) \leq E\} \ \ \textit{is relatively compact in} \ \ U; \tag{5.61}$$

$$\forall t \in [0,T], \ k \in \mathbb{N}, \ E > 0,$$

$$\Big(u_k \in S_k^{\alpha_k}(t_k), \quad \sup_k \mathcal{E}_k(t_k,u_k) < \infty, \quad (t_k,u_k,\alpha_k) \to (t,u,0) \Big)$$

$$\Rightarrow \partial_t \mathcal{E}_k(t_k,u_k) \to \partial_t \mathcal{E}(t,u). \tag{5.62}$$

$$u_0^k \to u_0 \quad \textit{and} \quad \mathcal{E}_k(0,u_0^k) \to \mathcal{E}(0,u_0). \tag{5.63}$$

Finally, let a sequence of partitions $\boldsymbol{\tau}_n$*, parameters* ε_n*,* θ_n*, and* k_n *be given such that* $(\varepsilon_n, \overline{\boldsymbol{\tau}}_n, k_n, \theta_n) \to (0,0,\infty,0)$*,* $\varepsilon_n/\underline{\boldsymbol{\tau}}_n \to 0$*, and* $\boldsymbol{\lambda}_{\varepsilon_n\boldsymbol{\tau}_n}$ *fulfill* (2.17) *and* (3.48)*. Let* $\boldsymbol{u}_{\varepsilon_n\boldsymbol{\tau}_n k_n}$ *be* θ_n*-approximate minimizers of* $\mathcal{I}_{\varepsilon_n\boldsymbol{\tau}_n k_n}$ *on* $\{\boldsymbol{v}_{\boldsymbol{\tau}_n k_n} \in U^{N_{\boldsymbol{\tau}_n}+1} \ : \ v^0_{\boldsymbol{\tau}_n k_n} = u_0^{k_n}\}$ *and* $u^{(n)} : [0,T] \to U$ *the piecewise constant interpolant of* $\boldsymbol{u}_{\varepsilon_n\boldsymbol{\tau}_n k_n}$*. Then, there exists a subsequence* $(u^{(n_j)})_{j\in\mathbb{N}}$ *and an energetic solution* $u : [0,T] \to U$ *of* (1.1) *such that, for all* $t \in [0,T]$*, the following convergences hold*

$$u^{(n_j)}(t) \to u(t) \quad \textit{weakly in } V \textit{ and strongly in } U, \tag{5.64}$$

$$\int_{[0,t]} \Psi_{k_n}(\mathrm{d}u^{(n_j)}) \to \int_{[0,t]} \Psi(\mathrm{d}u), \tag{5.65}$$

$$\mathcal{E}_{k_n}(t, u^{(n_j)}(t)) \to \mathcal{E}(t,u(t)), \tag{5.66}$$

$$\partial_t \mathcal{E}_{k_n}(\cdot, u^{(n_j)}(\cdot)) \to \partial_t \mathcal{E}(\cdot, u(\cdot)) \quad \textit{in} \ \ \mathrm{L}^1(0,T). \tag{5.67}$$

Sketch of the proof. We shall assume from the very beginning that

$$\frac{\varepsilon_n}{\underline{\boldsymbol{\tau}}_n} < 1 \quad \text{and} \quad \theta_n \sum_{i=1}^{N_{\boldsymbol{\tau}_n}} \rho^i_{\varepsilon_n\boldsymbol{\tau}_n} < 1.$$

As Ψ_k are uniformly coercive and (5.61) holds, the a priori estimate (3.43) entails the possibility of extracting a (not relabeled) subsequence $u_n = u_{\varepsilon_n\boldsymbol{\tau}_n k_n}$ such that the following convergences hold (see [MRS08, Thm. A.1]) for all $t \in [0,T]$

$$u_n(t) \to u(t) \quad \text{weakly in } V \text{ (and strongly in } U\text{)},$$

$$\delta_n(t) = \int_{[0,t]} \Psi_{k_n}(\mathrm{d}u_n) \to \delta(t),$$

$$p_n = \partial_t \mathcal{E}_{k_n}(\cdot, u_n(\cdot)) \to p \quad \text{weakly star in} \ \ \mathrm{L}^\infty(0,T).$$

We shall denote by $p^*(t) = \limsup_{n\to\infty} \partial_t \mathcal{E}_{k_n}(t,u_n(t))$ and explicitly observe that $p \leq p^*$. Note in particular that there exists a constant c bounding $\partial_t \mathcal{E}_{k_n}$ independently of n.

As for the stability of the limit trajectory u we just fix $t \in [0,T]$ and exploit (5.58) with the choice $u_{k_n} = u_{k_n}(t)$ by recalling that $u_{k_n}(t) \in S^{\alpha_{k_n}}_{k_n}(t)$ for all $t \in (0,T]$ where

$$\begin{aligned}\alpha_{k_n} &= \alpha(\boldsymbol{\tau}_n, \varepsilon_n, \theta_n)\\ &= \frac{\varepsilon_n}{\underline{\boldsymbol{\tau}}_n}\left((1{+}c_0)\left(\mathrm{e}^{c_1 T}{-}1\right) + \frac{(c_0{+}c_{\mathrm{stab}}{+}2\mathrm{e}^{c_1 T}{+}1)}{1{-}\varepsilon_n/\underline{\boldsymbol{\tau}}_n}\right) + \theta_n \rho^{N_{\boldsymbol{\tau}_n}}_{\boldsymbol{\tau}_n}.\end{aligned}$$

Note that $\alpha_{k_n} \to 0$ as $n \to \infty$. In particular, by (5.62), $p^*(t) = \partial_t \mathcal{E}(t, u(t))$.

The proof of the upper energy estimate follows exactly as in Step 3 of Theorem 2.6 as the extra error term $\theta_n \sum_{i=1}^{N_{\boldsymbol{\tau}_n}} \rho^i_{\varepsilon_n \boldsymbol{\tau}_n}$ goes to 0 by assumption (3.48) and we have convergence at time 0 (5.63).

Owing to (2.8)-(2.9) and the stability $u(t) \in S(t)$ for all $t \in [0,T]$ we may apply [MRS08, Prop. 2.4] and deduce the lower energy estimate as well.

Finally, just as in the proof Theorem 2.6, we have that

$$\begin{aligned}\mathcal{E}(0,u_0) + \int_0^t p^*(s)\,\mathrm{d}s \quad &\leq \quad \mathcal{E}(t,u(t)) + \int_{[0,t]} \Psi(\mathrm{d}u)\\ &\overset{(5.60)}{\leq} \quad \liminf_{n\to\infty} \mathcal{E}_{k_n}(t,u_n(t)) + \delta(t)\\ &\leq \quad \mathcal{E}(0,u_0) + \int_0^t p(s)\,\mathrm{d}s\\ &\leq \quad \mathcal{E}(0,u_0) + \int_0^t p^*(s)\,\mathrm{d}s\end{aligned}$$

and all inequalities are actually equalities. Hence

$$\mathcal{E}(t,u(t)) + \int_{[0,t]} \Psi(\mathrm{d}u) = \liminf_{n\to\infty} \mathcal{E}_{k_n}(t,u_n(t)) + \delta(t)$$

for all $t \in [0,T]$ and $p = p^*$. Since by (5.60) one has that

$$\mathcal{E}(t,u(t)) \leq \liminf_{n\to\infty} \mathcal{E}_{k_n}(t,u_n(t)) \quad \text{and} \quad \int_{[0,t]} \Psi(\mathrm{d}u) \leq \delta(t),$$

the convergences (5.65)-(5.66) follow. □

6. The causal limit $\varepsilon \to 0$ for a fixed partition

Let $\boldsymbol{\tau}$ be fixed throughout this subsection. We shall comment on the limit $\varepsilon \to 0$ within the minimization of $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$. The natural guess is that a sequence of (approximate) minimizers $\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}$ of $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$ admits a subsequence converging to a solution $\boldsymbol{u}_{\boldsymbol{\tau}}$ of the corresponding relaxed incremental problems

$$(6.68)\begin{cases} u^0_{\boldsymbol{\tau}} = u_0,\\ u^1_{\boldsymbol{\tau}} \in \operatorname{Arg\,min}_{u\in U} \left\{\Psi\left(u{-}u_0\right) + \widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u) - \mathcal{E}^0_{\boldsymbol{\tau}}(u_0)\right\},\\ u^i_{\boldsymbol{\tau}} \in \operatorname{Arg\,min}_{u\in U} \left\{\Psi\left(u{-}u^{i-1}\right) + \widehat{\mathcal{E}}^i_{\boldsymbol{\tau}}(u) - \widehat{\mathcal{E}}^{i-1}_{\boldsymbol{\tau}}(u^{i-1})\right\} \ i = 2,\dots,N_{\boldsymbol{\tau}},\end{cases}$$

where

$$\widehat{\mathcal{E}}^i_{\boldsymbol{\tau}}(v) = \inf\left\{ \liminf_{n\to\infty} \mathcal{E}^i_{\boldsymbol{\tau}}(v_n) \ : \ v_n \to v \ \text{ strongly in } \ U \right\} \quad i = 1, \ldots, N_{\boldsymbol{\tau}},$$

This is indeed the case. Namely, by letting $\varepsilon \to 0$ for a fixed partition, we restore causality at the discrete level, since (6.68) is the classical incremental procedure (1.3), however with $\mathcal{E}(t, \cdot)$ replaced by its Γ-$\liminf$, namely $\widehat{\mathcal{E}}(t, \cdot)$.

Theorem 6.1 (Causality at the discrete level)**.** *Assume* (2.1)-(2.2), (2.7)-(2.8), (2.16), (2.17), *and* (2.29). *Moreover let $\boldsymbol{\tau}$ be given and $\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}$ be a $o(\varepsilon^{N_{\boldsymbol{\tau}}})$-approximate minimizer of $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$ on $\{\boldsymbol{v}_{\boldsymbol{\tau}} \in U^{N_{\boldsymbol{\tau}}+1} \ : \ v^0_{\boldsymbol{\tau}} = u_0\}$, namely $u^0_{\varepsilon\boldsymbol{\tau}} = u_0$ and*

$$\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \le \mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{v}_{\boldsymbol{\tau}}) + o(\varepsilon^{N_{\boldsymbol{\tau}}}) \quad \forall \boldsymbol{v}_{\boldsymbol{\tau}} \in U^{N_{\boldsymbol{\tau}}+1} \quad \textit{with} \ \ v^0_{\boldsymbol{\tau}} = u_0. \tag{6.69}$$

Then, there exists a not relabeled subsequence $\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}$ and $\boldsymbol{u}_{\boldsymbol{\tau}} \in U^{N_{\boldsymbol{\tau}}+1}$ such that $\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}} \to \boldsymbol{u}_{\boldsymbol{\tau}}$ strongly in $U^{N_{\boldsymbol{\tau}}+1}$ as $\varepsilon \to 0$ and $\boldsymbol{u}_{\boldsymbol{\tau}}$ fulfills (6.68).

Let us now comment on a possible strategy for a proof. First of all, a priori bounds for approximate minimizers are available due to Theorem 3.1 (note that (3.41) follows as $\boldsymbol{\tau}$ is fixed and $\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}$ is a $o(\varepsilon^{N_{\boldsymbol{\tau}}})$-approximate minimizer of $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$). Hence, some (not relabeled) subsequence $\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}$ strongly converging to $\boldsymbol{u}_{\boldsymbol{\tau}}$ in $U^{N_{\boldsymbol{\tau}}+1}$ as $\varepsilon \to 0$ exists and clearly $u^0_{\boldsymbol{\tau}} = u_0$. By defining the functionals $\mathcal{F}^i$ and $\widehat{\mathcal{F}}^i$ on $U^{N_{\boldsymbol{\tau}}+1}$ as

$$\mathcal{F}^i(\boldsymbol{u}) = \Psi\left(u^i - u^{i-1}\right) + \mathcal{E}^i_{\boldsymbol{\tau}}(u^i) - \mathcal{E}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}) \quad \text{for } \ i = 1, \ldots, N_{\boldsymbol{\tau}},$$

and

$$\begin{aligned} \widehat{\mathcal{F}}^1(\boldsymbol{u}) &= \Psi\left(u^1 - u^0\right) + \widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^0) - \mathcal{E}^0_{\boldsymbol{\tau}}(u^0), \\ \widehat{\mathcal{F}}^i(\boldsymbol{u}) &= \Psi\left(u^i - u^{i-1}\right) + \widehat{\mathcal{E}}^i_{\boldsymbol{\tau}}(u^i) - \widehat{\mathcal{E}}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}) \quad \text{for } \ i = 2, \ldots, N_{\boldsymbol{\tau}}, \end{aligned}$$

we have that (see (2.17))

$$\begin{aligned} \mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) &= \sum_{i=1}^{N_{\boldsymbol{\tau}}} \lambda^i_{\varepsilon\boldsymbol{\tau}} \mathcal{F}^i(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) = \sum_{i=1}^{N_{\boldsymbol{\tau}}} \left(\prod_{j=1}^{i} \frac{\varepsilon}{\tau^j - \varepsilon} \right) \mathcal{F}^i(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \\ &= \left(\frac{\varepsilon}{\tau^1 + \varepsilon} \right) \mathcal{F}^1(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) + \left(\frac{\varepsilon^2}{(\tau^1+\varepsilon)(\tau^2+\varepsilon)} \right) \mathcal{F}^2(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}) \\ &\quad + \ldots \ + \left(\frac{\varepsilon^{N_{\boldsymbol{\tau}}}}{(\tau^1+\varepsilon)\ldots(\tau^{N_{\boldsymbol{\tau}}}+\varepsilon)} \right) \mathcal{F}^{N_{\boldsymbol{\tau}}}(\boldsymbol{u}_{\varepsilon\boldsymbol{\tau}}), \end{aligned} \tag{6.70}$$

and we can check that (see the proof of Theorem (6.4) below)

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \mathcal{I}_{\varepsilon\boldsymbol{\tau}} = \frac{1}{\tau^1} \widehat{\mathcal{F}}^1$$

so that, in particular, $u^1_{\boldsymbol{\tau}}$ solves the first minimum problem in (6.68) by virtue of the Fundamental Theorem of Γ-convergence.

This argument can indeed be iterated for $i = 2, \ldots, N_{\boldsymbol{\tau}}$ leading to the proof of Theorem 6.1 by arguing on the the subsequent powers of ε in (6.70). In particular, we investigate the *asymptotic development by Γ-convergence* of the functional $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$. This technique has been originally introduced by Anzellotti & Baldo in [AB93] and is aimed at characterizing the Γ-limit of a functional by means of a sort of asymptotic expansion.

Let us start by recalling some definition and the main result from [AB93]. Given a first-countable topological space X and a sequence of functionals $\mathcal{G}_\varepsilon : X \to (-\infty, \infty]$, the notation

$$\mathcal{G} = \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{G}_\varepsilon \quad \text{in} \quad M \subset X$$

defines Γ-*convergence of* $\mathcal{G}_\varepsilon$ *to* $\mathcal{G}$ *on the subset* M *in the ambient space* X via the following two conditions:

$$\mathcal{G}(x) \le \inf\left\{ \liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(x_\varepsilon) \ : \ x_\varepsilon \to x \right\} \quad \forall x \in M,$$
$$\forall x \in M,\ \exists x_\varepsilon \to x \ : \ \mathcal{G}(x) = \lim_{\varepsilon\to 0} \mathcal{G}_\varepsilon(x_\varepsilon).$$

Here it is important that the approximating sequences x_ε in both lines above may be taken from the full ambient space X and are not restricted to the set M. Hence the notion of Γ-convergence on $M \subset X$ intrinsically depend on M and on X.

Definition 6.2 (Development by Γ-convergence [AB93])**.** *We say that the* development in Γ-convergence

$$\mathcal{G}_\varepsilon \overset{\Gamma}{=} \mathcal{G}^0 + \varepsilon\mathcal{G}^1 + \varepsilon^2\mathcal{G}^2 + \cdots + \varepsilon^{N_{\boldsymbol{\tau}}}\mathcal{G}^{N_{\boldsymbol{\tau}}} + o(\varepsilon^{N_{\boldsymbol{\tau}}}) \quad \textit{in} \quad X \tag{6.71}$$

holds true if, letting M^0 be the set of minimizers of $\mathcal{G}^0$ on X, m^0 be the corresponding minimum, and, for all $i = 1, \ldots, N_{\boldsymbol{\tau}}$,

$$m^i = \inf_{M^{i-1}} \mathcal{G}^i, \quad M^i = \{u \in M^{i-1} \ : \ \mathcal{G}^i(u) = m^i\},$$
$$\mathcal{G}^0 = \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{G}_\varepsilon \quad \textit{in} \quad X, \quad \mathcal{G}^1_\varepsilon = \frac{\mathcal{G}_\varepsilon - m^0}{\varepsilon}, \quad \mathcal{G}^i_\varepsilon = \frac{\mathcal{G}^{i-1}_\varepsilon - m^{i-1}}{\varepsilon},$$

we have that, for all $i = 1, \ldots, N_{\boldsymbol{\tau}}$,

$$m^{i-1} < \infty, \quad M^{i-1} \neq \emptyset, \quad \textit{and}$$
$$\mathcal{G}^i = \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{G}^i_\varepsilon \quad \textit{in} \quad M^{i-1} \subset X \quad \textit{for} \quad i = 1, \ldots, N_{\boldsymbol{\tau}}. \tag{6.72}$$

The main result on asymptotic developments in Γ-convergence reads as follows.

Theorem 6.3 ([AB93, Thm. 1.2])**.** *Let* (6.71) *hold, x_ε be a $o(\varepsilon^{N_{\boldsymbol{\tau}}})$-approximate minimizer of $\mathcal{G}_\varepsilon$, and $x_\varepsilon \to x$ in X. Then x minimizes $\mathcal{G}^{N_{\boldsymbol{\tau}}}$ on $M^{N_{\boldsymbol{\tau}}-1}$ (that is, it minimizes $\mathcal{G}^i$ on M^{i-1} for all $i = 1, \ldots, N_{\boldsymbol{\tau}}$).*

In fact, the cited result is concerned with exact minimizers x_ε only. On the other hand, the corresponding result for $o(\varepsilon^{N_{\boldsymbol{\tau}}})$-approximate minimizers turns out to be an immediate extension of the original proof.

Our aim now is to prove is that $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$ admits a development in Γ-convergence in terms of the functionals $\widehat{\mathcal{F}}^i$ as suggested by (6.70).

Theorem 6.4 (Development in Γ-convergence of $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$)**.** *Assume* (2.1)-(2.2), (2.7)-(2.8), (2.16), (2.17), *and* (2.29). *Then the following development holds*

$$\begin{aligned} &\mathcal{I}_{\varepsilon\boldsymbol{\tau}} \overset{\Gamma}{=} \mathcal{G}^0 + \varepsilon\mathcal{G}^1 + \varepsilon^2\mathcal{G}^2 + \cdots + \varepsilon^{N_{\boldsymbol{\tau}}}\mathcal{G}^{N_{\boldsymbol{\tau}}} + o(\varepsilon^{N_{\boldsymbol{\tau}}}) \\ &\textit{in} \quad X = \left\{ \boldsymbol{v}_{\boldsymbol{\tau}} \in U^{N_{\boldsymbol{\tau}}+1} \ : \ v^0_{\boldsymbol{\tau}} = u_0 \right\} \end{aligned} \tag{6.73}$$

where $\mathcal{G}^0$ is the indicator function $I_{\mathcal{D}}$ of $\mathcal{D} = \{u_0\} \times D(\widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}) \times \cdots \times D(\widehat{\mathcal{E}}^{N_{\boldsymbol{\tau}}}_{\boldsymbol{\tau}})$, the functionals $\mathcal{G}^i$ are given by

$$\mathcal{G}^i = a^i + \frac{1}{\tau^1 \ldots \tau^i} \widehat{\mathcal{F}}^i + I_{\mathcal{D}} \quad \textit{for} \ \ i = 1, \ldots, N_{\boldsymbol{\tau}},$$

and the constants a^i depend on $(\tau^1, \ldots, \tau^{i-1})$ and on the minimum values

$$\mu^i = \min_{M^{i-1}} \widehat{\mathcal{F}}^i \quad \textit{for} \ \ i = 1, \ldots N_{\boldsymbol{\tau}} \quad \textit{where} \quad M^0 = \mathcal{D}$$
$$\textit{and} \ \ M^i = \{\boldsymbol{u} \in M^{i-1} \ : \ \widehat{\mathcal{F}}^i(\boldsymbol{u}) = \mu^i\} \quad \textit{for} \ \ i = 1, \ldots N_{\boldsymbol{\tau}}. \tag{6.74}$$

In particular, starting from $a^1_\varepsilon = 0$, the constants a^i are recursively defined for $i = 1, \ldots, N_{\boldsymbol{\tau}}$ as

$$a^i = \lim_{\varepsilon \to 0} a^i_\varepsilon \qquad a^{i+1}_\varepsilon = \frac{1}{\varepsilon} \left(\frac{\lambda^i_{\varepsilon \boldsymbol{\tau}}}{\varepsilon^i} \mu^i - \frac{\mu^i}{\tau^1 \ldots \tau^i} - a^i + \frac{a^i_\varepsilon}{\varepsilon} \right). \tag{6.75}$$

Before moving to the proof of Theorem 6.4, let us firstly check by induction that the definition of the values μ^i in (6.74) makes sense. Indeed, taking into account (2.2), (2.7)-(2.8), (2.16), and (2.29), one clearly has that $\min\{\widehat{\mathcal{F}}^1(\boldsymbol{u}) \ : \ \boldsymbol{u} \in M^0\}$ has at least a solution (note that $\widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}$ fulfills (2.7) as well). In particular, the set of minimizers M^1 is non-empty and closed with respect to the strong topology of $U^{N_{\boldsymbol{\tau}}+1}$. Moreover, $\{u^1 \ : \ \boldsymbol{u} \in M^1\}$ is compact in U by (2.7). Then, we compute for $\boldsymbol{u} \in M^1$ that

$$\begin{aligned} \widehat{\mathcal{F}}^2(\boldsymbol{u}) \quad &= \quad \Psi(u^2 - u^1) + \widehat{\mathcal{E}}^2_{\boldsymbol{\tau}}(u^2) - \widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1) \\ &\overset{\boldsymbol{u} \in M^1}{=} \quad \Psi(u^2 - u^1) + \widehat{\mathcal{E}}^2_{\boldsymbol{\tau}}(u^2) + \Psi(u^1 - u_0) - \mathcal{E}^0_{\boldsymbol{\tau}}(u_0) - \mu^1. \end{aligned}$$

Hence, $\widehat{\mathcal{F}}^2$ is lower semicontinuous and coercive on M^1 and $\min\{\widehat{\mathcal{F}}^2(\boldsymbol{u}) \ : \ \boldsymbol{u} \in M^1\}$ has a solution. Namely $\mu^2 < \infty$, M^2 is non-empty and closed, and $\{(u^1, u^2) \ : \ \boldsymbol{u} \in M^2\}$ is compact in U^2.

Assume now $i \geq 3$ and M^{i-1} non-empty and closed with $\{(u^1, \ldots, u^{i-1}) \ : \ \boldsymbol{u} \in M^{i-1}\}$ compact in U^{i-1}. Then, for all $\boldsymbol{u} \in M^{i-1}$ we have that

$$\begin{aligned} \mu^1 \quad &= \quad \Psi(u^1 - u_0) + \widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1) - \mathcal{E}^0_{\boldsymbol{\tau}}(u_0), \\ \mu^j \quad &= \quad \Psi(u^j - u^{j-1}) + \widehat{\mathcal{E}}^j_{\boldsymbol{\tau}}(u^j) - \widehat{\mathcal{E}}^{j-1}_{\boldsymbol{\tau}}(u^{j-1}) \quad \text{for} \ \ j = 2, \ldots, i-1. \end{aligned}$$

By adding up the above equalities we easily obtain that

$$\begin{aligned} \widehat{\mathcal{F}}^i(\boldsymbol{u}) \quad &= \quad \Psi(u^i - u^{i-1}) + \widehat{\mathcal{E}}^i_{\boldsymbol{\tau}}(u^i) - \widehat{\mathcal{E}}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}) \\ &\overset{\boldsymbol{u} \in M^{i-1}}{=} \quad \sum_{j=1}^{i} \Psi(u^j - u^{j-1}) + \widehat{\mathcal{E}}^i_{\boldsymbol{\tau}}(u^i) - \mathcal{E}^0_{\boldsymbol{\tau}}(u_0) - \sum_{j=1}^{i-1} \mu^j \end{aligned}$$

and $\widehat{\mathcal{F}}^i$ is lower semicontinuous and coercive on M^{i-1}. Hence, $\min\{\widehat{\mathcal{F}}^i(\boldsymbol{u}) \ : \ \boldsymbol{u} \in M^{i-1}\}$ has at least a solution and the set of minimizers M^i is non-empty and closed. Moreover, $\{(u^1, \ldots, u^i) \ : \ \boldsymbol{u} \in M^i\}$ compact in U^i.

Once the values $\mu^1, \ldots, \mu^{N_{\boldsymbol{\tau}}}$ are given, a trivial but tedious computation based on the explicit form of the Pareto weights $\boldsymbol{\lambda}_{\varepsilon \boldsymbol{\tau}}$ leads to the check that the definition in (6.75) makes sense as well. Let us stress that (6.75) allows us to explicitly recover all terms in the development (6.73). For instance, the first five terms in the development (6.73) of $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$ read as follows

$$\begin{aligned}
\mathcal{G}^0 &= I_{\mathcal{D}},\\
\mathcal{G}^1 &= \frac{1}{\tau^1}\widehat{\mathcal{F}}^1 + I_{\mathcal{D}},\\
\mathcal{G}^2 &= -\frac{1}{(\tau^1)^2}\mu^1 + \frac{1}{\tau^1\tau^2}\widehat{\mathcal{F}}^2 + I_{\mathcal{D}},\\
\mathcal{G}^3 &= \frac{1}{(\tau^1)^3}\mu^1 - \frac{\tau^1+\tau^2}{(\tau^1\tau^2)^2}\mu^2 + \frac{1}{\tau^1\tau^2\tau^3}\widehat{\mathcal{F}}^3 + I_{\mathcal{D}},\\
\mathcal{G}^4 &= \frac{1}{(\tau^1)^4}\mu^1 + \frac{(\tau^1+\tau^2)^2-\tau^1\tau^2}{(\tau^1\tau^2)^3}\mu^2 - \frac{\tau^1\tau^2+\tau^2\tau^3+\tau^3\tau^1}{(\tau^1\tau^2\tau^3)^2}\mu^3 + \frac{1}{\tau^1\tau^2\tau^3\tau^4}\widehat{\mathcal{F}}^4 + I_{\mathcal{D}}.
\end{aligned}$$

Proof. We shall proceed by induction and, for the sake of clarifying the argument, we directly work out the first three terms in the development (6.73) before showing the induction step.

The 0-order term. At first, let us check that $\Gamma\text{-}\lim_{\varepsilon\to 0}\mathcal{I}_{\varepsilon\boldsymbol{\tau}} = I_{\mathcal{D}}$ in X.

Let $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ in X and assume that $\liminf_{\varepsilon\to0}\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u}_\varepsilon) < \infty$. By recalling (2.18) and possibly re-extracting, as the weights $\lambda^i_{\varepsilon\boldsymbol{\tau}}$ and $\sigma^i_{\varepsilon\boldsymbol{\tau}}$ are non-negative we have that $\boldsymbol{u}_\varepsilon \in \{u_0\}\times D(\mathcal{E}^1_{\boldsymbol{\tau}})\times\cdots\times D(\mathcal{E}^{N_{\boldsymbol{\tau}}}_{\boldsymbol{\tau}})$. Hence, $\boldsymbol{u}$ belongs to the corresponding closure (conditioned by $\sup_\varepsilon \mathcal{E}^i_{\boldsymbol{\tau}}(u^i_\varepsilon) < \infty$) which is nothing but $\mathcal{D}$. Moreover, we surely have that $\liminf_{\varepsilon\to0}\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(u_\varepsilon)\geq 0$ as $\lambda^1_{\varepsilon\boldsymbol{\tau}} = o(1)$ (see again (2.18)).

Fix now $\boldsymbol{u}\in\mathcal{D}$ and let $\boldsymbol{u}_\varepsilon\to\boldsymbol{u}$ be a *recovery sequence* in the following sense

$$u^0_\varepsilon = u_0 \quad\text{and}\quad \sup_\varepsilon \mathcal{E}^i_{\boldsymbol{\tau}}(u^i_\varepsilon) < \infty \quad \text{for}\quad i=1,\dots,N_{\boldsymbol{\tau}},$$

(such a sequence surely exists). We compute from (2.18) that $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(u_\varepsilon)\to 0$.

The first-order term. We have checked that $m^0=0$ and $M^0=\mathcal{D}$. Hence, recalling (2.18), we have that

$$\begin{aligned}
\mathcal{G}^1_\varepsilon(\boldsymbol{u}) &= \frac{\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u})-0}{\varepsilon} = \frac{1}{\varepsilon}\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u})\\
&= \sum_{i=1}^{N_{\boldsymbol{\tau}}}\left(\frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\Psi(u^i-u^{i-1}) + \frac{\sigma^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\mathcal{E}^i_{\boldsymbol{\tau}}(u^i)\right) - \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\mathcal{E}^0_{\boldsymbol{\tau}}(u^0).
\end{aligned}\tag{6.76}$$

Let $\boldsymbol{u}_\varepsilon\to\boldsymbol{u}$ in X and assume that $\liminf_{\varepsilon\to0}\mathcal{G}^1_\varepsilon(\boldsymbol{u}_\varepsilon)<\infty$. By using (6.76) and the non-negativity of $\lambda^i_{\varepsilon\boldsymbol{\tau}}$ and $\sigma^i_{\varepsilon\boldsymbol{\tau}}$, we have that

$$\mathcal{G}^1_\varepsilon(\boldsymbol{u}_\varepsilon) \geq \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\Psi(u^1_\varepsilon-u_0) + \frac{\sigma^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\mathcal{E}^1_{\boldsymbol{\tau}}(u^1_\varepsilon) - \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\mathcal{E}^0_{\boldsymbol{\tau}}(u_0),$$

so that, passing to the lim inf,

$$\liminf_{\varepsilon\to0}\mathcal{G}^1_\varepsilon(\boldsymbol{u}_\varepsilon) \geq \frac{1}{\tau^1}\widehat{\mathcal{F}}^1(\boldsymbol{u}).$$

On the other hand, fix $\boldsymbol{u}\in M^0$ and let $\boldsymbol{u}_\varepsilon\to\boldsymbol{u}$ be such that

$$u^0_\varepsilon = u_0, \quad |\mathcal{E}^1_{\boldsymbol{\tau}}(u^1_\varepsilon) - \widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1)| = o(1), \quad\text{and}\quad \sup_\varepsilon \mathcal{E}^j_{\boldsymbol{\tau}}(u^j_\varepsilon)<\infty \quad\text{for}\quad i=2,\dots,N_{\boldsymbol{\tau}}.$$

Then, we have that

$$\begin{aligned}\mathcal{G}^1_\varepsilon(\boldsymbol{u}_\varepsilon) &= \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\Psi(u^1_\varepsilon - u_0) + \frac{\sigma^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\mathcal{E}^1_{\boldsymbol{\tau}}(u^1_\varepsilon) - \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\mathcal{E}^0_{\boldsymbol{\tau}}(u_0)\\ &\quad + \sum_{i=2}^{N_{\boldsymbol{\tau}}}\left(\frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\Psi(u^i_\varepsilon - u^{i-1}_\varepsilon) + \frac{\sigma^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\mathcal{E}^i_{\boldsymbol{\tau}}(u^i)\right)\\ &= \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\Psi(u^1_\varepsilon - u_0) + \frac{\sigma^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1) - \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\mathcal{E}^0_{\boldsymbol{\tau}}(u_0) + o(1)\end{aligned}$$

so that passing to the limit one has $\mathcal{G}^1_\varepsilon(\boldsymbol{u}_\varepsilon) \to \widehat{\mathcal{F}}^1(\boldsymbol{u})/\tau^1$. Namely, we have checked that $\Gamma\text{-}\limsup_{\varepsilon\to 0}\mathcal{G}^1_\varepsilon \le \widehat{\mathcal{F}}^1/\tau^1$ and we conclude that

$$\mathcal{G}^1 = \Gamma\text{-}\lim_{\varepsilon\to 0}\mathcal{G}^1_\varepsilon = \frac{1}{\tau^1}\mathcal{F}^1 \quad\text{in}\quad M^0.$$

In particular $\min_{M^0}\mathcal{G}^1 = \mu^1/\tau^1$ and $M^1 = \{\boldsymbol{u} \in M^0 \ : \ \mathcal{G}^1(\boldsymbol{u}) = \mu^1/\tau^1\}$ is non-empty.

The second-order term. We shall refine these techniques in order to compute $\Gamma\text{-}\lim_{\varepsilon\to 0}\mathcal{G}^2_\varepsilon$ where

$$\begin{aligned}\mathcal{G}^2_\varepsilon(\boldsymbol{u}) &= \frac{\mathcal{G}^1_\varepsilon(\boldsymbol{u}) - \mu^1/\tau^1}{\varepsilon} = \frac{\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u})}{\varepsilon^2} - \frac{\mu^1}{\varepsilon\tau^1}\\ &= \frac{1}{\varepsilon}\left(\frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\Psi(u^1 - u_0) + \frac{\sigma^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1) - \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\mathcal{E}^0_{\boldsymbol{\tau}}(u_0) - \frac{\mu^1}{\tau^1}\right)\\ &\quad + \sum_{i=2}^{N_{\boldsymbol{\tau}}}\left(\frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\Psi(u^i - u^{i-1}) + \frac{\sigma^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\mathcal{E}^i_{\boldsymbol{\tau}}(u^i)\right).\end{aligned} \tag{6.77}$$

In particular, let $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ in X, $\boldsymbol{u} \in M^1$, and assume that $\liminf_{\varepsilon\to 0}\mathcal{G}^2_\varepsilon(\boldsymbol{u}_\varepsilon) < \infty$. As we have that

$$\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\Psi(u^1_\varepsilon - u_0) + \frac{\sigma^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\mathcal{E}^1_{\boldsymbol{\tau}}(u^1_\varepsilon) - \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\mathcal{E}^0_{\boldsymbol{\tau}}(u_0) - \frac{\mu^1}{\tau^1}\right) < \infty,$$

by possibly extracting not relabeled subsequences, we can compute

$$\begin{aligned}\limsup_{\varepsilon\to 0}\mathcal{E}^1_{\boldsymbol{\tau}}(u^1_\varepsilon) &\le \limsup_{\varepsilon\to 0}\frac{\varepsilon}{\sigma^1_{\varepsilon\boldsymbol{\tau}}}\left(-\frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\Psi(u^1_\varepsilon - u_0) + \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\mathcal{E}^0_{\boldsymbol{\tau}}(u_0) + \frac{\mu^1}{\tau^1}\right)\\ &\overset{(2.29)}{=} -\Psi(u^1 - u_0) + \mathcal{E}^0_{\boldsymbol{\tau}}(u_0) + \mu^1\\ &= \mu^1 - \widehat{\mathcal{F}}^1(\boldsymbol{u}) + \widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1)\\ &\overset{\boldsymbol{u}\in M^1}{=} \widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1).\end{aligned}$$

Namely, as clearly $\widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1) \le \liminf_{\varepsilon\to 0}\mathcal{E}^1_{\boldsymbol{\tau}}(u^1_\varepsilon)$, we have checked that

$$\mathcal{E}^1_{\boldsymbol{\tau}}(u^1_\varepsilon) \to \widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1). \tag{6.78}$$

Let us now compute

$$\begin{aligned}
\mathcal{G}^2_\varepsilon(\boldsymbol{u}_\varepsilon) &\overset{(2.19)}{\geq} \frac{1}{\varepsilon}\left(\frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\Psi(u^1_\varepsilon - u_0) + \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\mathcal{E}^1_{\boldsymbol{\tau}}(u^1_\varepsilon) - \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon}\mathcal{E}^0_{\boldsymbol{\tau}}(u_0) - \frac{\mu^1}{\tau^1}\right) \\
&\qquad + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\Psi(u^2_\varepsilon - u^1_\varepsilon) + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\mathcal{E}^2_{\boldsymbol{\tau}}(u^2_\varepsilon) - \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\mathcal{E}^1_{\boldsymbol{\tau}}(u^1_\varepsilon) \\
&\geq \left(\frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\widehat{\mathcal{F}}^1(\boldsymbol{u}_\varepsilon) - \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}\mu^1}{\varepsilon^2}\right) + \left(\frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}\mu^1}{\varepsilon^2} - \frac{\mu^1}{\varepsilon\tau^1}\right) \\
&\qquad + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\Psi(u^2_\varepsilon - u^1_\varepsilon) + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\widehat{\mathcal{E}}^2_{\boldsymbol{\tau}}(u^2_\varepsilon) - \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\mathcal{E}^1_{\boldsymbol{\tau}}(u^1_\varepsilon) \\
&\overset{(6.78)}{\geq} 0 - \frac{\mu^1}{(\tau^1)^2} + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\Psi(u^2_\varepsilon - u^1_\varepsilon) + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\widehat{\mathcal{E}}^2_{\boldsymbol{\tau}}(u^2_\varepsilon) - \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1) + o(1).
\end{aligned}$$

Hence, passing to the lim inf and using (6.78) we have that

$$\liminf_{\varepsilon\to 0}\mathcal{G}^2_\varepsilon(\boldsymbol{u}_\varepsilon) \geq -\frac{\mu^1}{(\tau^1)^2} + \frac{1}{\tau^1\tau^2}\widehat{\mathcal{F}}^2(\boldsymbol{u}).$$

We now fix $\boldsymbol{u} \in M^1$ and let $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ be such that

$$\begin{aligned}
&u^0_\varepsilon = u_0, \\
&|\mathcal{E}^1_{\boldsymbol{\tau}}(u^1_\varepsilon) - \widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1)| = o(\varepsilon), \quad \Psi(u^1_\varepsilon - u^1) = o(\varepsilon), \\
&|\mathcal{E}^2_{\boldsymbol{\tau}}(u^2_\varepsilon) - \widehat{\mathcal{E}}^2_{\boldsymbol{\tau}}(u^2)| = o(1), \\
&\sup_\varepsilon \mathcal{E}^i_{\boldsymbol{\tau}}(u^i_\varepsilon) < \infty \quad \text{for } \ i = 3, \dots, N_{\boldsymbol{\tau}}.
\end{aligned}$$

Then, we have that

$$\begin{aligned}
\mathcal{G}^2_\varepsilon(\boldsymbol{u}_\varepsilon) &\overset{(2.19)}{=} \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\Psi(u^1_\varepsilon - u_0) + \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\mathcal{E}^1_{\boldsymbol{\tau}}(u^1_\varepsilon) - \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\mathcal{E}^0_{\boldsymbol{\tau}}(u_0) - \frac{\mu^1}{\varepsilon\tau^1} \\
&\qquad + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\Psi(u^2_\varepsilon - u^1_\varepsilon) + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\mathcal{E}^2_{\boldsymbol{\tau}}(u^2_\varepsilon) - \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\mathcal{E}^1_{\boldsymbol{\tau}}(u^1_\varepsilon) + o(1) \\
&\leq \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\Psi(u^1 - u_0) + \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1) - \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\mathcal{E}^0_{\boldsymbol{\tau}}(u_0) - \frac{\mu^1}{\varepsilon\tau^1} \\
&\qquad + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^2}\left(\Psi(u^2_\varepsilon - u^1_\varepsilon) + \widehat{\mathcal{E}}^2_{\boldsymbol{\tau}}(u^2) - \widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1)\right) + o(1) \\
&\overset{\boldsymbol{u}\in M^1}{=} \left(\frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}\mu^1}{\varepsilon^2} - \frac{\mu^1}{\varepsilon\tau^1}\right) + \frac{1}{\tau^1\tau^2}\widehat{\mathcal{F}}^2(\boldsymbol{u}) + o(1)
\end{aligned}$$

and, passing to the lim sup, one obtains

$$\limsup_{\varepsilon\to 0}\mathcal{G}^2_\varepsilon(\boldsymbol{u}_\varepsilon) \leq -\frac{\mu^1}{(\tau^1)^2} + \frac{1}{\tau^1\tau^2}\widehat{\mathcal{F}}^2(\boldsymbol{u}).$$

The i-th order term. Let us now come to the induction step. Assume that the development (6.73) holds up to the $(i-1)$-th term. In particular, let

$$m^j = a^j + \frac{\mu^j}{\tau^1 \dots \tau^j} \quad \text{for } \ j \leq i-1 \tag{6.79}$$

with a^j defined as in (6.75). We start by noting that

$$\begin{aligned}
\mathcal{G}^i_\varepsilon(\boldsymbol{u}) &= \frac{\mathcal{G}^{i-1}_\varepsilon(\boldsymbol{u}) - m^{i-1}}{\varepsilon} = \frac{1}{\varepsilon}\left(\frac{\mathcal{G}^{i-2}_\varepsilon(\boldsymbol{u}) - m^{i-2}}{\varepsilon} - m^{i-1}\right) = \dots \\
&= \frac{\mathcal{I}_{\varepsilon\boldsymbol{\tau}}(\boldsymbol{u})}{\varepsilon^i} - \sum_{j=1}^{i-1}\frac{m^j}{\varepsilon^{i-j}} \\
&= \sum_{i=1}^{N_{\boldsymbol{\tau}}}\left(\frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^i - u^{i-1}) + \frac{\sigma^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\mathcal{E}^i_{\boldsymbol{\tau}}(u^i)\right) - \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\mathcal{E}^0_{\boldsymbol{\tau}}(u^0) - \sum_{j=1}^{i-1}\frac{m^j}{\varepsilon^{i-j}}.
\end{aligned} \tag{6.80}$$

In particular, owing to (2.19), we have that

$$\begin{aligned}
\mathcal{G}^i_\varepsilon(\boldsymbol{u}) &= \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^1 - u_0) + \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\mathcal{E}^1_{\boldsymbol{\tau}}(u^1) - \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\mathcal{E}^0_{\boldsymbol{\tau}}(u_0) - \frac{m^1}{\varepsilon^{i-1}} \\
&\quad + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^2 - u^1) + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\mathcal{E}^2_{\boldsymbol{\tau}}(u^2) - \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\mathcal{E}^1_{\boldsymbol{\tau}}(u^1) - \frac{m^2}{\varepsilon^{i-2}} \\
&\quad \vdots \\
&\quad + \frac{\lambda^{i-1}_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^{i-1} - u^{i-2}) + \frac{\lambda^{i-1}_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\mathcal{E}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}) - \frac{\lambda^{i-1}_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\mathcal{E}^{i-2}_{\boldsymbol{\tau}}(u^{i-2}) - \frac{m^{i-1}}{\varepsilon} \\
&\quad + \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^i - u^{i-1}) + \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\mathcal{E}^i_{\boldsymbol{\tau}}(u^i) - \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\mathcal{E}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}) \\
&\quad + \sum_{j=i+1}^{N_{\boldsymbol{\tau}}}\left(\frac{\lambda^j_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^j - u^{j-1}) + \frac{\sigma^j_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\mathcal{E}^j_{\boldsymbol{\tau}}(u^j)\right).
\end{aligned}$$

Let now $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ in X and assume that $\liminf_{\varepsilon\to 0}\mathcal{G}^i_\varepsilon(\boldsymbol{u}_\varepsilon) < \infty$. Arguing as above, and in particular along the same lines of the proof of (6.78), we can check that

$$\mathcal{E}^j_{\boldsymbol{\tau}}(u^j_\varepsilon) \to \widehat{\mathcal{E}}^j_{\boldsymbol{\tau}}(u^j) \quad \text{for} \quad j \leq i-1. \tag{6.81}$$

and, by using (6.80), we have

$$\begin{aligned}
\mathcal{G}^i_\varepsilon(\boldsymbol{u}_\varepsilon) &\geq \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^1_\varepsilon - u_0) + \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1_\varepsilon) - \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\mathcal{E}^0_{\boldsymbol{\tau}}(u_0) - \frac{m^1}{\varepsilon^{i-1}} \\
&\quad + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^2_\varepsilon - u^1_\varepsilon) + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^2_{\boldsymbol{\tau}}(u^2_\varepsilon) - \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1_\varepsilon) - \frac{m^2}{\varepsilon^{i-2}} \\
&\quad \vdots \\
&\quad + \frac{\lambda^{i-1}_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^{i-1}_\varepsilon - u^{i-2}_\varepsilon) + \frac{\lambda^{i-1}_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}_\varepsilon) - \frac{\lambda^{i-1}_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^{i-2}_{\boldsymbol{\tau}}(u^{i-2}_\varepsilon) - \frac{m^{i-1}}{\varepsilon} \\
&\quad + \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^i_\varepsilon - u^{i-1}_\varepsilon) + \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^i_{\boldsymbol{\tau}}(u^i_\varepsilon) - \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}_\varepsilon) \\
&\geq \left(\frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}\mu^1}{\varepsilon^i} - \frac{m^1}{\varepsilon^{i-1}}\right) + \left(\frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}\mu^2}{\varepsilon^i} - \frac{m^2}{\varepsilon^{i-2}}\right) + \dots + \left(\frac{\lambda^{i-1}_{\varepsilon\boldsymbol{\tau}}\mu^{i-1}}{\varepsilon^i} - \frac{m^{i-1}}{\varepsilon}\right) \\
&\quad + \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^i_\varepsilon - u^{i-1}_\varepsilon) + \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^i_{\boldsymbol{\tau}}(u^i_\varepsilon) - \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}_\varepsilon).
\end{aligned} \tag{6.82}$$

By using (6.81) we directly check that

$$\liminf_{\varepsilon\to 0}\left(\frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^i_\varepsilon - u^{i-1}_\varepsilon) + \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^i_{\boldsymbol{\tau}}(u^i_\varepsilon) - \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}_\varepsilon)\right) \geq \frac{1}{\tau^1\dots\tau^i}\widehat{\mathcal{F}}^i(\boldsymbol{u}).$$

As for the remainder terms in the right-hand side of (6.82), by exploiting (6.79) and the definition (6.75) we have that

$$\left(\frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}\mu^1}{\varepsilon^i} - \frac{m^1}{\varepsilon^{i-1}}\right) + \left(\frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}\mu^2}{\varepsilon^i} - \frac{m^2}{\varepsilon^{i-2}}\right) + \dots + \left(\frac{\lambda^{i-1}_{\varepsilon\boldsymbol{\tau}}\mu^{i-1}}{\varepsilon^i} - \frac{m^{i-1}}{\varepsilon}\right) = a^i_\varepsilon. \tag{6.83}$$

Hence, we have finally checked that

$$\liminf_{\varepsilon\to 0}\mathcal{G}^i_\varepsilon(\boldsymbol{u}_\varepsilon) \geq a^i + \frac{1}{\tau^1\dots\tau^i}\widehat{\mathcal{F}}^i(\boldsymbol{u}).$$

Fix now $\boldsymbol{u}\in M^{i-1}$ and choose $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ such that

$$\begin{aligned}
&\text{(i)} && u^0_\varepsilon = u_0,\\
&\text{(ii)} && \Psi(u^j_\varepsilon - u^j) = o(\varepsilon^{i-j}) \quad \text{for} \quad j\leq i-1,\\
&\text{(iii)} && \Psi(u^j - u^j_\varepsilon) = o(\varepsilon^{i-j-1}) \quad \text{for} \quad j\leq i-1,\\
&\text{(iv)} && |\mathcal{E}^j_{\boldsymbol{\tau}}(u^j_\varepsilon) - \widehat{\mathcal{E}}^j_{\boldsymbol{\tau}}(u^j)| = o(\varepsilon^{i-j}) \quad \text{for} \quad j\leq i,\\
&\text{(v)} && \sup_\varepsilon \mathcal{E}^j_{\boldsymbol{\tau}}(u^j_\varepsilon) < \infty \quad \text{for} \quad j = i+1,\dots,N_{\boldsymbol{\tau}}.
\end{aligned}$$

To find $\boldsymbol{u}_\varepsilon$, first take a recovery sequence satisfying (iv) and (v), then (2.29) can be used to establish (ii) and (iii). Then, we compute that (see (6.80))

$$\begin{aligned}
\mathcal{G}^i_\varepsilon(\boldsymbol{u}_\varepsilon) \;=\; & \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^1 - u_0) + \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1) - \frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\mathcal{E}^0_{\boldsymbol{\tau}}(u_0) - \frac{m^1}{\varepsilon^{i-1}}\\
& + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^2 - u^1) + \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^2_{\boldsymbol{\tau}}(u^2) - \frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^1_{\boldsymbol{\tau}}(u^1) - \frac{m^2}{\varepsilon^{i-2}}\\
& \vdots\\
& + \frac{\lambda^{i-1}_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^{i-1} - u^{i-2}) + \frac{\lambda^{i-1}_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}) - \frac{\lambda^{i-1}_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^{i-2}_{\boldsymbol{\tau}}(u^{i-2}) - \frac{m^{i-1}}{\varepsilon}\\
& + \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^i_\varepsilon - u^{i-1}) + \frac{\lambda^i_{\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^i_{\boldsymbol{\tau}}(u^i) - \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}) + o(1).
\end{aligned} \tag{6.84}$$

By recalling that $\boldsymbol{u}\in M^{i-1}$, we have that

$$\begin{aligned}
\mathcal{G}^i_\varepsilon(\boldsymbol{u}_\varepsilon) \;=\; & \left(\frac{\lambda^1_{\varepsilon\boldsymbol{\tau}}\mu^1}{\varepsilon^i} - \frac{m^1}{\varepsilon^{i-1}}\right) + \left(\frac{\lambda^2_{\varepsilon\boldsymbol{\tau}}\mu^2}{\varepsilon^i} - \frac{m^2}{\varepsilon^{i-2}}\right) + \dots\\
& + \left(\frac{\lambda^{i-1}_{\varepsilon\boldsymbol{\tau}}\mu^{i-1}}{\varepsilon^i} - \frac{m^{i-1}}{\varepsilon}\right)\\
& + \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\Psi(u^i_\varepsilon - u^{i-1}) + \frac{\lambda^i_{\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^i_{\boldsymbol{\tau}}(u^i) - \frac{\lambda^i_{\varepsilon\boldsymbol{\tau}}}{\varepsilon^i}\widehat{\mathcal{E}}^{i-1}_{\boldsymbol{\tau}}(u^{i-1}) + o(1),
\end{aligned} \tag{6.85}$$

so that, by passing to the lim sup and using (6.83), we get that

$$\limsup_{\varepsilon\to 0}\mathcal{G}^i_\varepsilon(\boldsymbol{u}_\varepsilon) \leq a^i + \frac{1}{\tau^1\dots\tau^i}\widehat{\mathcal{F}}^i(\boldsymbol{u}),$$

and the assertion follows. $\square$

Once the asymptotic development in Γ-convergence of $\mathcal{I}_{\varepsilon\boldsymbol{\tau}}$ of Theorem 6.4 is established, the proof of Theorem 6.1 follows directly from Theorem 6.3.

References

[AB93] G. Anzellotti and S. Baldo, *Asymptotic development by γ-convergence*, Appl. Math. Optim. (1993), 195–123.

[AMS08] F. Auricchio, A. Mielke, and U. Stefanelli, *A rate-independent model for the isothermal quasi-static evolution of shape-memory materials*, Math. Models Meth. Appl. Sci. **18** (2008), no. 1, 125–164, Preprint availble at `http://arxiv.org/abs/0708.4378`.

[Dal93] G. Dal Maso, *An introduction to Γ-convergence*, Progress in Nonlinear Differential Equations and their Applications, 8, Birkhäuser Boston Inc., Boston, MA, 1993.

[DDM06] G. Dal Maso, A. DeSimone, and M. G. Mora, *Quasistatic evolution problems for linearly elastic-perfectly plastic materials*, Arch. Ration. Mech. Anal. **180** (2006), no. 2, 237–291

[DFT05] G. Dal Maso, G. A. Francfort, and R. Toader, *Quasistatic crack growth in nonlinear elasticity*, Arch. Ration. Mech. Anal. **176** (2005), no. 2, 165–225.

[GF75] E. De Giorgi and T. Franzoni, *Su un tipo di convergenza variazionale*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **58** (1975), no. 6, 842–850.

[HR99] W. Han and B.D. Reddy, *Plasticity, mathematical theory and numerical analysis*, Springer-Verlag, New York, 1999.

[Joh78] C. Johnson, *On plasticity with hardening*, J. Math. Anal. Appl. **62** (1978), no. 2, 325–336.

[KMR06] M. Kočvara, A. Mielke, and T. Roubíček, *A rate-independent approach to the delamination problem*, Math. Mech. Solids **11** (2006), no. 4, 423–447.

[Mie03] A. Mielke, *Energetic formulation of multiplicative elasto-plasticity using dissipation distances*, Contin. Mech. Thermodyn. **15** (2003), no. 4, 351–382.

[Mie04] A. Mielke, *Deriving new evolution equations for microstructures via relaxation of variational incremental problems*, Comput. Methods Appl. Mech. Engrg. **193** (2004), no. 48–51, 5095–5127.

[Mie05] ———, *Evolution of rate-independent systems*, Handbook of Differential Equations, evolutionary equations (C. Dafermos and E. Feireisl, eds.), vol. 2, Elsevier, 2005, pp. 461–559.

[MO08] A. Mielke and M. Ortiz, *A class of minimum principles for characterizing the trajectories and the relaxation of dissipative systems*, ESAIM Control Optim. Calc. Var., (2008), to appear. Preprint available at: `http://www.wias-berlin.de/main/publications/wias-publ/index.cgi.en`.

[MR03] A. Mielke and T. Roubíček, *A rate-independent model for inelastic behavior of shape-memory alloys*, Multiscale Model. Simul. **1** (2003), no. 4, 571–597 (electronic).

[MR06] ———, *Rate-independent damage processes in nonlinear elasticity*, Math. Models Methods Appl. Sci. **16** (2006), no. 2, 177–209.

[MRS08] A. Mielke, T. Roubíček, and U. Stefanelli, *Γ-limits and relaxations for rate-independent evolutionary problems*, Calc. Var. Partial Differential Equations, (2008), to appear.

[MT99] A. Mielke and F. Theil, *A mathematical model for rate-independent phase transformations with hysteresis*, Proceedings of the workshop on "Models of Continuum Mechanics in Analysis and Engineering" (H.-D. Alber, R. Balean, and R. Farwig, eds.), Shaker-Verlag, 1999, pp. 117–129.

[MT04] ———, *On rate-independent hysteresis models*, NoDEA, Nonlinear Diff. Equations Applications **11** (2004), 151–189.

[MT05] A. Mielke and A. M. Timofte, *An energetic material model for time-dependent ferroelectric behavior: existence and uniqueness.*, Math. Models Appl. Sci. **29** (2005), 1393–1410.

[MTL02] A. Mielke, F. Theil, and V. I. Levitas, *A variational formulation of rate-independent phase transformations using an extremum principle*, Arch. Ration. Mech. Anal. **162** (2002), no. 2, 137–177.

[Rou07] T. Roubíček, *Modelling of thermodynamics of martensitic transformation in shape-memory alloys*, Discrete Contin. Dyn. Sys., (2008), to appear.

[SM05] F. Schmid and A. Mielke, *Vortex pinning in super-conductivity as a rate-independent process*, European J. Appl. Math. **16** (2005), no. 6, 799–808.

[Ste06] U. Stefanelli, *The Brezis-Ekeland principle for doubly nonlinear equations*, SIAM J. Control Optim. (2008), to appear. Preprint available at: `http://www.imati.cnr.it/ulisse/pubbl.html`.

[Ste08] ______, *A variational principle for hardening elasto-plasticity*, SIAM J. Math. Anal. (2008), to appear. Preprint available at: `http://www.imati.cnr.it/ulisse/pubbl.html`.

[Ste08] ______, *A variational characterization of rate-independent evolution*, In preparation (2008).

[Suq78] P.-M. Suquet, *Existence et régularité des solutions des équations de la plasticité*, C. R. Acad. Sci. Paris Sér. A-B **286** (1978), no. 24, A1201–A1204.

[Vis96] A. Visintin, *Models of phase transitions*, Progress in Nonlinear Differential Equations and Their Applications, vol. 28, Birkhäuser, 1996.