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On Weak Solutions to the Stationary MHD-Equations Coupled to Heat Transfer with Nonlocal Radiation Boundary Conditions.

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Abstract

We study the coupling of the stationary system of magnetohydrodynamics to the heat equation. Coupling occurs on the one hand from temperature-dependent coefficients and from a temperature-dependent force term in the Navier-Stokes equations. On the other hand, the heat sources are given by the dissipation of current in the electrical conductors, and of viscous stresses in the fluid. We consider a domain occupied by several different materials, and have to take into account interface conditions for the electromagnetic fields. Since we additionally want to treat high-temperatures applications, we also take into account the effect of heat radiation, which results in nonlocal boundary conditions for the heat flux. We prove the existence of weak solutions for the coupled system, under the assumption that the imposed velocity at the boundary of the fluid remains sufficiently small. We prove a uniqueness result in the case of constant coefficients and small data. Finally, we discuss the regularity issue in a simplified setting.

Introduction

The possibility to exert a control on the motion of electrically conducting fluids with the help of magnetic fields is well known. An industrial area in which this idea is applied nowadays is crystal growth from the melt, where a too strong melt convection can result in thermal inhomogeneities, and in a loss of quality for the production. The use of magnetic fields in such applications leads to complex models in which hydrodynamical, electromagnetical, and thermodynamical phenomena closely interact with each other. The attempt to accurately model these phenomena results in strongly coupled systems of PDE, for which few mathematical results have been stated.

In the present paper, the system of magnetohydrodynamics is coupled to the energy balance, on the one hand through temperature-dependent coefficients and a temperature dependent force term in the Navier-Stokes equations, and on the other hand through the heat sources. The latter are given by the dissipation of current in the conductors, and of viscous stresses in the fluid. Since we want to deal with realistic domains of computation, for example a Czochralski furnace (see Figure 1 below), we have to take interface conditions for the electromagnetic fields and the heat flux into account. At the high temperatures involved, the effect of heat radiation results in nonlocal boundary conditions.

Our plan is as follows. In the remainder of the introduction, we describe precisely the mathematical problem that we want to consider. Then, we propose a functional setting and define what we will call a weak solution. The first section describes existence results in the case that the temperature-dependent force term in the fluid equation is bounded. In

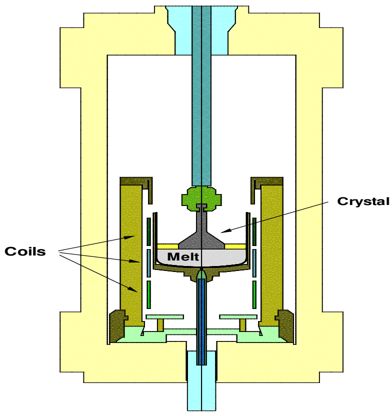

Figure 1: Schematic representation of a vapor pressure growth arrangement of the Institute of Crystal Growth (IKZ) Berlin.

the second section, we treat the case that the force term in the fluid equations has a linear growth in temperature, and prove existence under a smallness assumption on the data. The third section states a uniqueness result for constant coefficients and small data, and the last section is devoted to the regularity issue in simplified settings. In the appendix, we have recalled or proved some basic auxiliary results that we use throughout the paper.

The mathematical problem. We first need to introduce our geometrical setting. We denote by $\Omega \subseteq \tilde{\Omega} \subset \mathbb{R}^3$ two bounded domains, which respectively represent the region of interest for the computation of the temperature (that is, typically, the furnace), and for the computation of the electromagnetic fields. The domain $\tilde{\Omega}$ has the form

$$\overline{\tilde{\Omega}} = \bigcup_{i=0}^{m} \overline{\tilde{\Omega}_i} \, .$$

with disjoint domains $\tilde{\Omega}_i$ $(i = 0, \ldots, m)$ that represent heterogeneous materials. Setting $\Omega_i := \tilde{\Omega}_i \cap \Omega$ for $i = 0, \ldots, m$, we obviously have the decomposition $\overline{\Omega} = \bigcup_{i=0}^{m} \overline{\Omega_i}$. Of special signification will be throughout this paper:

$$\Omega_0 = \text{ transparent cavity in } \Omega, \quad \Omega_1 = \text{ vessel filled with fluid.}$$

Thanks to the index set $I_c \subset \{0, \ldots, m\}$ defined by $i \in I_c \iff \tilde{\Omega}_i$ is electrically conducting, we can define sets

$$\overline{\tilde{\Omega}_c} = \bigcup_{i \in I_c} \overline{\tilde{\Omega}_i} \, , \quad \overline{\Omega_c} = \bigcup_{i \in I_c} \overline{\Omega_i} \, ,$$

that represent the electrical conductors located respectively in $\tilde{\Omega}$ and in Ω. We denote by $\tilde{\Omega}_{c_0} \subset \tilde{\Omega}_c$ the conductors in which *direct* current is applied (in Figure 1, one can think of a resistance heater instead of the induction coils). Throughout the paper, we consider the simplest case that these conductors are isolated from the remainder of the conductors, i. e. $\mathrm{dist}(\tilde{\Omega}_{c_0}, \tilde{\Omega}_c \setminus \tilde{\Omega}_{c_0}) > 0$. For the sake of simplicity, we will also make the plausible supposition that apart from $\tilde{\Omega}_{c_0}$, the region $\tilde{\Omega} \setminus \Omega$ consists only of vacuum.

In order to model the heat radiation, we need to introduce the surfaces

$$\Gamma := \partial\Omega\,, \qquad \Sigma := \partial\Omega_0\,, \tag{1}$$

and we set $\partial\Omega_{\mathrm{Rad}} := \Gamma \cup \Sigma$ as the total surface where heat radiation occurs. Note that on the set Γ, only emission of radiation occurs, whereas the boundary Σ forms a cavity in which surface-to-surface heat radiation must be modeled. For the remainder of the paper, we assume that the domain Ω forms an enclosure in the sense of Definition 5.3 (see the appendix).

The model for the melt flow. We assume that the fluid flow is governed by the stationary Navier-Stokes equations for an incompressible, electrically conducting, and heat conducting fluid

$$\begin{aligned} \rho_1\,(v\cdot\nabla)v &= -\nabla p + \mathrm{div}(\eta(\theta)\,D\,v) + f + j\times B\,, \\ \mathrm{div}\,v &= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega_1\,. \end{aligned} \tag{2}$$

The mass density ρ_1 of the fluid is a given constant, and the term $j \times B$ represents the Lorentz force. For notational reasons, the dynamical viscosity η as function of temperature is scaled by a factor 2. We use the notation

$$D\,v = D_{i,j}\,(v) \;:=\; \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\;(i,j = 1,\ldots,3)\,,$$

and we set

$$D(u,v) := D\,u \,:\, D\,v := D_{i,j}\,(u)\,D_{i,j}\,(v)\,.$$

Here and trough the paper, we use the convention that repeated indices imply summation over 1, 2, 3.

The industrial applications on which we focus are usually considered as falling in the range of validity of Boussinesq's approximation of compressible fluids. Therefore, we allow for temperature-dependence of the force of gravity by setting

$$\rho = \rho(\theta) = \rho_1\,(1 - \alpha\,(\theta - \theta_{\mathrm{Ref}}))\,, \tag{3}$$

where α is the thermal expansion coefficient of the fluid, and θ_{Ref} a reference temperature. We then set

$$f = f(\theta) := \rho(\theta)\,\vec{g}\,, \tag{4}$$

with the fixed vector of gravity $\vec{g}$. At the boundary, we assume that the velocity is imposed, that is

$$v = v_0 \quad \text{on } \partial\Omega_1 \,, \tag{5}$$

and since we typically consider the case that the velocity is imposed by the rotation of an axisymmetric vessel, we impose the additional restriction that

$$v_0 \cdot \vec{n} = 0 \quad \text{on } \partial\Omega_1 \,. \tag{6}$$

Heat transfer. We assume that heat convection only occurs in the domain Ω_1. We then have:

$$\begin{aligned} \rho_1 \, c_V \, v \cdot \nabla\theta &= \mathrm{div}(\kappa(\theta)\, \nabla\theta) + \eta(\theta)\, D(v,v) + \frac{|j|^2}{\mathfrak{s}(\theta)} && \text{in } \Omega_1 \,, \\ 0 &= \mathrm{div}(\kappa(\theta)\, \nabla\theta) + \frac{|j|^2}{\mathfrak{s}(\theta)} && \text{in } \Omega_i \;\; \text{for } i = 0,\, 2,\, \ldots, m \,. \end{aligned} \tag{7}$$

The coefficients κ and $\mathfrak{s}$ denote the heat conductivity and the electrical conductivity. It is usual to consider for a Boussinesq fluid that the heat production in the fluid is negligible (that is, the viscous dissipation given by $\eta\, D(v,v)$, and the ohmic dissipation given by $|j|^2/\mathfrak{s}$). However, not neglecting these contributions gives a broader range of validity to the model, as it was shown for example in [GG76].

An important physical effect is the heat radiation that occurs at the surface Σ. We model it by using the following nonlocal boundary condition (see for example [KPS04], [Tii97], [Voi01]):

$$\left[-\kappa(\theta)\, \frac{\partial\theta}{\partial\vec{n}}\right] = R - J \quad \text{on } \Sigma \,, \tag{8}$$

where R denotes the radiosity, and J denotes the incoming radiation. The notation $[\cdot]$ represents the jump of a quantity across the surface Σ. In order to model R and J, we need to introduce the so-called *view factor* w given by

$$w(z,y) := \begin{cases} \frac{\vec{n}(z)\cdot(y-z)\;\vec{n}(y)\cdot(z-y)}{\pi|y-z|^4}\, \Theta(z,y) & \text{if } z \neq y \,, \\ 0 & \text{if } z = y \,, \end{cases} \tag{9}$$

where

$$\Theta(z,y) = \begin{cases} 1 & \text{if }]z,y[\cap \overline{\Omega \setminus \Omega_0} = \emptyset \,, \\ 0 & \text{else} \;\; . \end{cases}$$

With the symbol $]z,y[$, we denote the set $conv\{z,y\} \setminus \{z,y\}$. Our model for R, J consists in setting

$$R = \epsilon\, \sigma\, |\theta|^3\, \theta + (1-\epsilon)\, J \,, \quad J = K(R) \quad \text{on } \Sigma \,,$$

where the function ϵ is the emissivity on the surface Σ, and σ denotes the Stefan-Boltzmann constant. The linear integral operator K is defined by

$$\big(K(f)\big)(z) := \int_\Sigma w(z,y)\, f(y)\, dS_y \quad \text{for } z \in \Sigma\,.$$

At the outer boundary Γ, we consider the condition

$$\theta = \theta_0 \ \text{ on } \Gamma\,. \tag{10}$$

On other boundaries, we simply assume the continuity of the heat flux.

The electromagnetic model. In $\tilde{\Omega}_{c_0}$, a current density is given. We make the consistency assumptions

$$\operatorname{div} j_0 = 0 \ \text{ in } \tilde{\Omega}_{c_0}\,, \quad j_0 \cdot \vec{n} = 0 \ \text{ on } \partial\tilde{\Omega}_{c_0}\,, \tag{11}$$

which reflect the conservation of charge (see [Bos04]). We then consider the Ampère-Ohm relation

$$\operatorname{curl} H \;=\begin{cases} 0 & \text{in } \tilde{\Omega} \setminus \tilde{\Omega}_c \\ j_0 & \text{in } \tilde{\Omega}_{c_0} \\ \mathfrak{s}(\theta)\,(E + v \times B) & \text{in } \tilde{\Omega}_c \setminus \tilde{\Omega}_{c_0} \end{cases}\,, \tag{12}$$

with the temperature-dependent electrical conductivity $\mathfrak{s}$. The electric field E satisfies

$$\operatorname{curl} E = 0\,, \quad \text{in } \tilde{\Omega}\,, \tag{13}$$

and the displacement current D is such that

$$\operatorname{div} D = 0 \ \text{ in } \tilde{\Omega} \setminus \tilde{\Omega}_c\,. \tag{14}$$

The magnetic induction B satisfies

$$\operatorname{div} B = 0 \quad \text{in } \tilde{\Omega}\,. \tag{15}$$

We need a constitutive relation between B and H, as well as between D and E. We consider only linear materials, that is

$$B = \mu\, H\,, \quad D = \mathfrak{e}\, E\,. \tag{16}$$

with the function μ of magnetic permeability, and the function $\mathfrak{e}$ of electrical permittivity.. In the interior of the domain $\tilde{\Omega}$, these fields have to satisfy the natural interface conditions

$$\big[H \times \vec{n}\big]_{i,j} = 0\,, \quad \big[B \cdot \vec{n}\big]_{i,j} = 0\,, \quad \big[E \times \vec{n}\big]_{i,j} = 0 \quad \text{on } \partial\tilde{\Omega}_i \cap \partial\tilde{\Omega}_j\,, \tag{17}$$

where $\big[\,\cdot\,\big]_{i,j}$ denotes the jump of a quantity across the surface $\partial\tilde{\Omega}_i \cap \partial\tilde{\Omega}_j$ ($i,j = 0,\dots,m$). On the external boundary $\partial\tilde{\Omega}$, we have the conditions

$$B \cdot \vec{n} = 0\,, \quad E \times \vec{n} = 0 \ \text{ on } \partial\tilde{\Omega}\,. \tag{18}$$

which are mostly intended as an approximation of the condition of vanishing at infinity. These conditions are however physical in the case that $\partial\tilde{\Omega}$ models a magnetic shield.

Definition 0.1. We will address the problem of finding fields v, H, B, E, D, j and scalars p, θ that satisfy (2), (5), (7), (8), (10), (12), (13), (14), (15), (16), (17), (18) as Problem (P).

Situation of the paper. Since each decoupled part of the problem (P) for itself constitutes a broad topic, we shall limit ourselves to basic remarks in order to describe the specificity of our work.

The MHD system. The paper [MS96] provides a nice survey about recent developments in the mathematical theory of MHD. From the analytical viewpoint, the main difficulty of the system of magnetohydrodynamics consists in the term $j \times B$ as right-hand side of the fluid equations. In view of Ampère's law and (16), we can write $j \times B = \operatorname{curl} H \times \mu\, H$, and we see that in the natural $L^2_{\text{curl}}-$setting of Maxwell's equations, the latter term belongs *a priori* only to L^1 (see [Dru07a] for a discussion of this question). All the results available on the MHD system are based on the fact that the vector field H that solves Maxwell's equations belongs to the Sobolev space $W^{1,2}$, provided that the magnetic permeability μ and the interfaces $\partial\tilde{\Omega}_i \cap \partial\tilde{\Omega}_j$ $(i,\, j = 0, \ldots, m)$ satisfy some regularity assumptions (see e. g. Lemma 5.1 in the appendix).

In [DL72] or in [ST83], existence is proved for the MHD system in the case that the outer boundary $\partial\tilde{\Omega}$ is of class $\mathcal{C}^2$ and that the magnetic permeability is globally smooth in the domain $\tilde{\Omega}$. Note that in the latter case, no interface conditions need to be taken into account. In the paper [LS60], existence is proved under the assumption that the interfaces $\partial\tilde{\Omega}_i \cap \partial\tilde{\Omega}_j$ are closed surfaces of class $\mathcal{C}^2$. The magnetic permeability is only assumed to be smooth in each domain $\overline{\tilde{\Omega}_i}$. The latter result is revisited and completed in the more recent papers [MS96], [MS99], where existence is proved under a $\mathcal{C}^{1,1}$ regularity assumption of the interface.

Our results in the present paper will strongly rely on the techniques developed in the papers that we just mentioned for handling the MHD equations. But note that in the context of the industrial applications that we have in mind (crystal growth from the melt), we work reluctantly with strong conditions on the regularity of the interfaces since a phase transition and triple jump points of the material properties are to be expected. Thanks to our preliminary study [Dru07a], we are able to propose weaker hypotheses than in [LS60] or [MS96] under which we still obtain the required higher integrability of the Lorentz force.

Heat conducting fluids. Resistive heating. The coupling of the heat equation to the Navier-Stokes equations (*heat conductive fluids*) or to the stationary Maxwell's equations (*resistive heating*) leads also to a right-hand side in L^1. The canonical approach that would consist in obtaining the higher integrability of the viscous dissipation $D(v, v)$ (resp. of $|\operatorname{curl} H|^2$) by means of regularity estimates is however less practicable than in the case of the Lorentz force. Sufficiently strong regularity results are obtained in the Navier-Stokes equations only for smooth boundaries and coefficients, which again is very restricting with respect to the covered class of applications.

A summary of recent results concerning the coupling of the stationary, incompressible Navier-Stokes equations to the heat equation and techniques for handling systems with L^1-right-hand sides can be found in [Nau05]. These methods will be used in the present paper, but note that we have the nontrivial task to extend their validity to the case of nonlocal radiation boundary conditions. This can be accomplished thanks to the preparatory work accomplished in the paper [Dru07b] concerning the heat equation with L^1 source terms and radiation boundary conditions.

Mathematical setting and definition of a weak solution. In the context of the generalized theory of electromagnetics, we need the space

$$L^2_{\rm curl}(\tilde\Omega) \;:=\; \Big\{ H \in [L^2(\tilde\Omega)]^3 \;\Big|\; {\rm curl}\, H \in [L^2(\tilde\Omega)]^3 \Big\}\,,$$

where the differential operator *curl* is intended in its generalized sense.

It is well known that the space $L^2_{\rm curl}(\tilde\Omega)$ is a Hilbert space with respect to the scalar product

$$(H_1\,,\,H_2)_{L^2_{\rm curl}(\tilde\Omega)} := \int_{\tilde\Omega} \Big({\rm curl}\, H_1 \cdot {\rm curl}\, H_2 + H_1 \cdot H_2 \Big)\,.$$

Actually, in view of (12), the natural frame in which to search for the field H will be the space

$$\mathcal{H}(\tilde\Omega) \;:=\; \Big\{ H \in\; L^2_{\rm curl}(\tilde\Omega) \Big|\; {\rm curl}\, H = 0 \text{ in } \tilde\Omega \setminus \tilde\Omega_c \Big\}\,. \tag{19}$$

Obviously, this is a closed linear subspace of $L^2_{\rm curl}(\tilde\Omega)$. We will also need the space

$$\mathcal{H}^0(\tilde\Omega) \;:=\; \Big\{ H \in \mathcal{H}(\tilde\Omega) \;\Big|\; {\rm curl}\, H = 0 \text{ in } \tilde\Omega_{c_0} \Big\}\,. \tag{20}$$

If μ is given by (28) and satisfies (30), it is possible to deal with the divergence constraint (15) and the boundary conditions (17) by introducing

$$\mathcal{H}_\mu(\tilde\Omega) := \Big\{ H \in \mathcal{H}(\tilde\Omega) \;\Big|\; {\rm div}(\mu\, H) = 0 \text{ in } \tilde\Omega\,;\; \mu\, H \cdot \vec{n} = 0 \text{ on } \partial\tilde\Omega \Big\}\,, \tag{21}$$

and, correspondingly,

$$\mathcal{H}^0_\mu(\tilde\Omega) := \Big\{ H \in \mathcal{H}^0(\tilde\Omega) \;\Big|\; {\rm div}(\mu\, H) = 0 \text{ in } \tilde\Omega\,;\; \mu\, H \cdot \vec{n} = 0 \text{ on } \partial\tilde\Omega \Big\}\,. \tag{22}$$

Here, the divergence constraint is intended in the generalized sense of the operator *div*.

In the context of the Navier-Stokes equations, we will need the usual spaces

$$D^{1,2}(\Omega_1) := \Big\{ u \in [W^{1,2}(\Omega_1)]^3 \;\Big|\; {\rm div}\, u = 0 \text{ almost everywhere in } \Omega_1 \Big\}\,,$$

$$D^{1,2}_0(\Omega_1) := \Big\{ u \in [W^{1,2}_0(\Omega_1)]^3 \;\Big|\; {\rm div}\, u = 0 \text{ almost everywhere in } \Omega_1 \Big\}\,. \tag{23}$$

For the mathematical setting of the stationary heat equation with radiation boundary condition, we need the Banach spaces

$$V^{p,q}(\Omega) := \Big\{ \theta \in W^{1,p}(\Omega) \;\Big|\; \gamma(\theta) \in L^q(\Sigma) \Big\}\,, \quad 1 \le p, q \le \infty\,, \tag{24}$$

where γ denotes the trace operator. The subscript Γ will indicate the subspace consisting of all functions whose trace vanishes on the boundary part Γ.

Throughout the paper, we impose to the surfaces defined in (25) the geometrical restrictions that

$$\operatorname{dist}(\Gamma,\, \Sigma) \,>\, 0\,. \tag{25}$$

and that there exists a number $0 < \alpha \leq 1$ such that

$$\Sigma \in \mathcal{C}^{1,\alpha}\,. \tag{26}$$

In order to describe the situation at the interfaces of materials with heterogeneous electromagnetical properties, we now introduce different hypotheses on the pair $(\mu,\, \tilde{\Omega})$ that are going to play a role in the paper. A first property is

$$(A0) \quad \text{For } i, j \in \{0, \ldots, m\},\ i \neq j \ \text{ the boundary } \partial\tilde{\Omega}_i \cap \partial\tilde{\Omega}_j \text{ is a closed, connected surface.} \tag{27}$$

Note that the hypothesis $(A0)$ forbids for example the presence of triple jump points of the electromagnetical properties, i. e. points at which more than two subdomains of $\tilde{\Omega}$ are in contact. It forbids as well that the outer boundary $\partial\tilde{\Omega}$ is in contact with more than one subdomain. We will also discuss the complementary assumptions

$$(A1)\begin{cases} \mu_{|_{\tilde{\Omega}_i}} \in \mathcal{C}^1\left(\overline{\tilde{\Omega}_i}\right) & \text{for } i = 0, \ldots, m, \\ \partial\tilde{\Omega}_i \setminus \partial\tilde{\Omega} \in \mathcal{C}^2 & \text{for } i = 0, \ldots, m\,, \\ \partial\tilde{\Omega} \in \mathcal{C}^2, \end{cases} \qquad (A2)\begin{cases} \mu_{|_{\tilde{\Omega}_i}} \in \mathcal{C}\left(\overline{\tilde{\Omega}_i}\right) & \text{for } i = 0, \ldots, m, \\ \partial\tilde{\Omega}_i \setminus \partial\tilde{\Omega} \in \mathcal{C}^1 & \text{for } i = 0, \ldots, m \\ \partial\tilde{\Omega} \in \mathcal{C}^{0,1}. \end{cases}$$

In order to define a weak solution, we still need to introduce some assumptions on the coefficients of the problem. The coefficients of electrical conductivity, of magnetic permeability, and of heat conductivity are material-dependent. We introduce the abbreviations

$$\mathfrak{s} := \mathfrak{s}_i\,, \quad \mu := \mu_i\,, \quad \kappa := \kappa_i \qquad \text{in each } \tilde{\Omega}_i \quad \text{for } i = 0, \ldots, m\,. \tag{28}$$

Whenever we allow for a temperature dependence of these functions, we will always require that

$$\mathfrak{s}_i,\, \mu_i,\, \kappa_i,\, \eta \in C\left(\mathbb{R}\right) \quad \text{for } i = 0, \ldots, m\,. \tag{29}$$

Throughout the paper, we assume that there exist positive constants $\mathfrak{s}_l$, $\mathfrak{s}_u$, μ_l, μ_u, κ_l, κ_u, η_l, η_u such that

$$\begin{aligned} &0 < \mathfrak{s}_l \leq \mathfrak{s} \leq \mathfrak{s}_u < +\infty\,, \quad 0 < \mu_l \leq \mu \leq \mu_u < +\infty\,, \\ &0 < \kappa_l \leq \kappa \leq \kappa_u < +\infty, \quad 0 < \eta_l \leq \eta \leq \eta_u < +\infty\,. \end{aligned} \tag{30}$$

The emissivity of the surface Σ, denoted by ϵ, is a material function of the position. We assume that $\epsilon : \Sigma \longrightarrow \mathbb{R}$ is measurable and that there exists a positive number ϵ_l such that

$$0 < \epsilon_l \leq \epsilon_i \leq 1 \quad \text{on } \partial\Omega_i \cap \Sigma \quad \text{for } i = 0, \ldots, m\,. \tag{31}$$

For the sake of notational commodity, we introduce the auxiliary function

$$r := \begin{cases} \frac{1}{\mathfrak{s}} & \text{on } \tilde{\Omega}_c \\ 1 & \text{on } \tilde{\Omega} \setminus \tilde{\Omega}_c \end{cases}, \quad r_l := \mathfrak{s}_u^{-1} \quad r_u := \mathfrak{s}_l^{-1}. \tag{32}$$

Definition 0.2. Let the assumptions (28), (29), (30), (31) on the coefficients η, $\mathfrak{s}$, μ, κ be satisfied. We call *weak solution* of (P) a triple

$$\{v,\, H,\, \theta\} \in D^{1,2}(\Omega_1) \times \mathcal{H}_\mu(\tilde{\Omega}) \times \bigcap_{1 \le p < 3/2} V^{p,4}(\Omega)\,,$$

such that $v = v_0$ on $\partial\Omega_1$, $\theta = \theta_0$ on Γ, $\operatorname{curl} H = j_0$ in $\tilde{\Omega}_{c_0}$ and the integral relations

$$\int_{\Omega_1} \rho_1\,(v\cdot\nabla)v\cdot\phi + \int_{\Omega_1} \eta(\theta)\,D(v\,,\,\phi) = \int_{\Omega_1} (\operatorname{curl} H \times \mu\,H)\cdot\phi + \int_{\Omega_1} f(\theta)\cdot\phi\,, \tag{33}$$

$$\int_{\tilde{\Omega}} r(\theta)\,\operatorname{curl} H\cdot\operatorname{curl}\psi = \int_{\Omega_1} \big(v\times\mu\,H\big)\cdot\operatorname{curl}\psi\,, \tag{34}$$

$$\int_{\Omega_1} \rho_1\,c_V\,v\cdot\nabla\theta\,\xi + \int_{\Omega} \kappa(\theta)\nabla\theta\cdot\nabla\xi + \int_{\Sigma} G(\sigma\,\theta^4)\,\xi$$
$$= \int_{\Omega} r(\theta)\,|\operatorname{curl} H|^2\,\xi + \int_{\Omega_1} \eta(\theta)\,D(v\,,\,v)\,\xi\,, \tag{35}$$

are satisfied for all $\{\phi,\,\psi,\,\xi\} \in D_0^{1,2}(\Omega) \times \mathcal{H}_\mu^0(\tilde{\Omega}) \times W_\Gamma^{1,\bar{r}}(\Omega)$ with $\bar{r} > 3$. In addition, we require that $\theta \ge 0$ almost everywhere in Ω, and the vector field H has to satisfy

$$\operatorname{curl} H \times \mu\,H \in [L^{6/5}(\tilde{\Omega})]^3\,. \tag{36}$$

Remark 0.3. (1) The conditions $v = v_0$ on $\partial\Omega_1$ and $\theta = \theta_0$ on Γ, are intended in the sense of traces, whereas the equality $\operatorname{curl} H = j_0$ has to hold pointwise almost everywhere in $\tilde{\Omega}_{c_0}$.

(2) By the requirement (36), we ensure that weak solutions of (P) in the sense of Definition 0.2 satisfy an energy equality in the Navier-Stokes equations and in the Maxwell system.

1 Existence results.

Here, and for the remainder of the paper, we assume that the domain $\tilde{\Omega}$ is simply connected and has a Lipschitz boundary. Throughout this section, we will consider domains of the form $\overline{\tilde{\Omega}} = \bigcup_{i=0}^m \overline{\tilde{\Omega}_i}$ described in the introduction.

As to the force $f\,:\,\mathbb{R} \longrightarrow \mathbb{R}^3$ given by (4), we note that $\rho_1\,\vec{g} = \nabla G$ for some scalar potential G. Therefore, we can as well solve the problem (P) with the corrected pressure $\tilde{p} := p + G$, and the force

$$f = -\rho_1\,\vec{g}\,\alpha\,(\theta - \theta_M)\,, \tag{37}$$

where for the reference tempereture $\theta_{\rm Ref}$, we have chosen the mean value θ_M of θ over the set Ω_1. The term $\alpha\,(\theta-\theta_M)$ represents the rate of the density variations in the fluid. This quantity has to remain small compared to unity for the Boussinesq model to make sense. Throughout the present section, we replace the force term f by

$$f = -\rho_1\,\vec{g}\,\mathrm{sign}(\theta-\theta_M)\,\min\{\alpha\,|\theta-\theta_M|,\,M_t\}\,, \tag{38}$$

with a positive number M_t that can be interpreted as the maximal rate of allowed density variations. In the present section, we thus have

$$\max_{\mathbb{R}} |f| \leq \rho_1\,|\vec{g}|\,M_t < \infty\,. \tag{39}$$

The more complicated case (37) will be treated in the second section.

We introduce some notations. We denote by $c_{\rm Korn}$ and $c_{\mathcal{H}}$ the smallest positive constants such that for all $v \in D^{1,2}_0(\Omega_1)$ and all $\psi \in \mathcal{H}_\mu(\tilde{\Omega})$,

$$\int_{\Omega_1} |\nabla v|^2 \leq c \int_{\Omega_1} D(v,\,v), \qquad \|\psi\|^2_{[L^2(\tilde{\Omega})]^3} \leq c_{\mathcal{H}} \int_{\tilde{\Omega}} |\,\mathrm{curl}\,\psi|^2\,.$$

The existence of the constant $c_{\mathcal{H}}$ is granted in view of Lemma 5.1. In our estimates, we will use the abbreviations $\mathbf{v_0} := \max_{\partial\Omega_1}|v_0|$ and $L := \mathrm{diam}(\Omega_1)$. Throughout this section, we also suppose that v_0 is given by (6) and satisfies the smallness assumption

$$\mathbf{v_0} < c\,\min\left\{\frac{\eta_l}{\rho_1\,L}\,,\,\frac{r_l}{2\,\mu_u}\right\}\,, \tag{40}$$

with $c := \min\{c_{\rm Korn}^{-1}\,,\,c_{\mathcal{H}}^{-1}\}$. If (40) is valid, we can define the positive number

$$\gamma_0 := \min\{\eta_l\, - c_{\rm Korn}\,\rho_1\,\mathbf{v_0}\,L\,,\,r_l\, - 2\,c_{\mathcal{H}}\,\mu_u\mathbf{v_0}\}\,. \tag{41}$$

Our main result in this section is the following theorem.

Theorem 1.1. Assume that $\tilde{\Omega}$ is a simply connected Lipschitz domain that has the structure described in the introduction and satisfies $(A0)$, (25) and (26). Assume in addition that the pair $(\mu,\,\tilde{\Omega})$ satisfies the condition $(A1)$. Let the coefficients $\eta\,,\,r,\,\mu\,,\epsilon\,,\,\kappa$ satisfy the hypothesis (30), (31), (32), and let the force term f and the given current density $j_0 \in [L^2(\tilde{\Omega}_{c_0})]^3$ respectively have the properties (39) and (11). Assume finally for the boundary data that $v_0 \in D^{1,2}(\Omega_1)\cap L^\infty(\Omega_1)$ satisfies (6) and the smallness assumption (40), and that the imposed temperature θ_0 belongs to $W^{1,2}(\Omega)\cap L^\infty(\Omega)$.

Then, there exists at least one weak solution of (P) in the sense of Definition 0.2.

The statement of Theorem 1.1 remains true under weaker conditions.

Proposition 1.2. (1) If one replaces the conditions $(A0)$ and $(A1)$ by the conditions $(A0)$ and $(A2)$, then the existence result of Theorem 1.1 holds true.

(2) There exists a positive constant $C = C(\tilde{\Omega})$, such that if the function μ satisfies the condition

$$C \left(1 - \frac{\mu_l}{\mu_u}\right) < 1 \,, \tag{42}$$

then the statement of Theorem 1.1 remains true without the requirements $(A0)$ and $(A1)$.

The rest of the section is devoted to the proof of Theorem 1.1 and of Propostion 1.2. First, we need to introduce some additional notations. For vector fields $v \in D^{1,2}_0(\Omega_1)$, we use the notation

$$\hat{v} \,:=\, v + v_0 \,. \tag{43}$$

Thanks to the assumption (25), we can fix some $\phi_0 \in C^\infty(\overline{\Omega})$ such that $\phi_0 = 1$ on Γ and $\phi_0 = 0$ on Σ. For $\theta \in V^{2,5}_\Gamma(\Omega)$, we introduce the notation

$$\hat{\theta} \,:=\, \theta + \theta_0 \, \phi_0 \,. \tag{44}$$

In this way, we homogenize the problem for the temperature without perturbing the nonlocal terms on Σ. Given a current density j_0 with (11), (6), we can find by Lemma 5.1 some $H_0 \in \mathcal{H}_\mu(\tilde{\Omega})$ such that

$$\operatorname{curl} H_0 = j_0 \quad \text{in } \tilde{\Omega} \,. \tag{45}$$

For vector fields $H \in \mathcal{H}^0_\mu(\tilde{\Omega})$, we then define a reaction field

$$\hat{H} := H + H_0 \,. \tag{46}$$

For a function $g : \tilde{\Omega} \longrightarrow \mathbb{R}$ and $\delta \in \mathbb{R}^+$, we introduce the cutoff

$$[g]_{(\delta)} \,:=\, \frac{g}{1 + \delta \, |g|} \,. \tag{47}$$

Note the property stated in Lemma 5.10 for this cutoff operator. In the next propostion, we construct approximate solutions.

Proposition 1.3. Let $\delta > 0$ be an arbitrary positive number. If the assumptions of Theorem 1.1 or of Proposition 1.2 are satisfied, there exists a triple

$$\{v,\, H,\, \theta\} \in D^{1,2}(\Omega_1) \times \mathcal{H}_\mu(\tilde{\Omega}) \times V^{2,5}(\Omega) \,,$$

such that $v = v_0$ on $\partial\Omega_1$, $\theta = \theta_0$ on Γ, $\operatorname{curl} H = j_0$ in $\tilde{\Omega}_{c_0}$ and such that the relations

$$\int_{\Omega_1} \rho_1 \, (v \cdot \nabla) v \cdot \phi + \int_{\Omega_1} \eta(\theta) \, D(v \,,\, \phi) = \int_{\Omega_1} \big(\operatorname{curl} H \times \mu \, H\big) \cdot \phi + \int_{\Omega_1} f(\theta) \cdot \phi \,, \tag{48}$$

$$\int_{\tilde{\Omega}} r(\theta) \, \operatorname{curl} H \cdot \operatorname{curl} \psi = \int_{\Omega_1} \big(v \times \mu \, H\big) \cdot \operatorname{curl} \psi, \tag{49}$$

$$\int_{\Omega_1} \rho_1 \, c_V \, v \cdot \nabla\theta \, \xi + \int_{\Omega} \kappa(\theta) \, \nabla\theta \cdot \nabla\xi + \int_{\Sigma} G(\sigma \, \theta^4) \, \xi$$

$$= \int_{\Omega} \Big[r(\theta) \, |\operatorname{curl} H|^2 + \eta(\theta) \, D(v \,,\, v) \, \chi_{\Omega_1}\Big]_{(\delta)} \, \xi \,, \tag{50}$$

are satisfied for all $\{\phi,\, \psi,\, \xi\} \in D^{1,2}_0(\Omega_1) \times \mathcal{H}^0_\mu(\tilde\Omega) \times V^{2,5}_\Gamma(\Omega)$. In addition,

$$\theta \geq \operatorname*{ess\,inf}_{\Gamma} \theta_0 \qquad \text{almost everywhere in } \Omega\,. \tag{51}$$

Proof. Define $V := D^{1,2}_0(\Omega_1) \times \mathcal{H}^0_\mu(\tilde\Omega) \times V^{2,5}_\Gamma(\Omega)$. Then, the isomorphism

$$V^* \;\cong\; [D^{1,2}_0(\Omega_1)]^* \times [\mathcal{H}^0_\mu(\tilde\Omega)]^* \times [V^{2,5}_\Gamma(\Omega)]^*$$

is valid. Throughout this proof, we denote by $\langle \cdot, \cdot\rangle$ the duality pairing between V and V^*. Recalling the notations (43), (44), (46) and (47), we define an operator $A : V \to V^*$ by

$$\begin{aligned}
&\big\langle\, A\big(\{v, H, \theta\}\big)\,,\, \{\phi, \psi, \xi\}\,\big\rangle \\
&:= \int_{\Omega_1} \rho_1\, (\hat v\cdot\nabla)\hat v \cdot \phi + \int_{\Omega_1} \eta(\hat\theta)\, D(\hat v\,,\,\phi) - \int_{\Omega_1} \big(\operatorname{curl} \hat H \times \mu\, \hat H\big)\cdot \phi \\
&- \int_{\Omega_1} f(\hat\theta)\cdot\phi + \int_{\tilde\Omega} r(\hat\theta)\, \operatorname{curl}\hat H \cdot \operatorname{curl}\psi - \int_{\Omega_1} \big(\hat v \times \mu\, \hat H\big)\cdot \operatorname{curl}\psi + \int_{\Omega_1} \rho_1\, c_V\, \hat v\cdot \nabla\hat\theta\, \xi \\
&+ \int_\Omega \kappa(\hat\theta)\, \nabla\hat\theta\cdot\nabla\xi + \int_\Sigma G(\sigma\, |\hat\theta|^3\, \hat\theta)\, \xi - \int_\Omega \Big[r(\hat\theta)\, |\operatorname{curl}\hat H|^2 + \eta(\hat\theta)\, D(\hat v\,,\,\hat v)\, \chi_{\Omega_1}\Big]_{(\delta)}\, \xi\,.
\end{aligned}$$

Note that using the results of Lemma 5.1, we have under the assumption $(A0)$ and $(A1)$ for $i = 0, \ldots, m$ the continuous embedding $\mathcal{H}_\mu(\tilde\Omega) \hookrightarrow [W^{1,2}(\tilde\Omega_i)]^3$. Under the weaker assumptions $(A0)$ and $(A2)$ or (42), the embedding $\mathcal{H}_\mu(\tilde\Omega) \hookrightarrow [L^q(\tilde\Omega)]^3$ is valid for some $q > 3$. Using Sobolev's embedding relations, we can therefore prove under the assumptions of Theorem 1.1 or of Proposition 1.2 that A is well defined.

Thanks to the well-known fact that the coercivity and the pseudomonotonicity of the operator A are sufficient for its surjectivity, we now want to establish the latter property.

We first discuss the coercivity. Observe that

$$\int_{\Omega_1} \rho_1\, (\hat v\cdot\nabla) v\cdot v = \int_{\Omega_1} \rho_1\, \hat v_j\, \frac{1}{2}\, \frac{\partial}{\partial x_j} v_i^2 = 0\,, \qquad \int_{\Omega_1} \rho_1\, c_V\, \hat v\cdot\nabla\theta\,\theta = \int_{\Omega_1} \rho_1\, c_V\, \hat v_j\, \frac{1}{2}\, \frac{\partial}{\partial x_j}\theta^2 = 0\,,$$

since $v \in D^{1,2}_0(\Omega_1)$, and since v_0 is divergence free in Ω_1 and tangential on $\partial\Omega_1$. It follows that

$$\int_{\Omega_1} \rho_1\, (\hat v\cdot\nabla)\hat v\cdot v = \int_{\Omega_1} \rho_1\, \big((v\cdot\nabla)v_0 + (v_0\cdot\nabla)v_0\big)\cdot v = -\int_{\Omega_1} \rho_1\, (v_j\, v_{0,i} + v_{0,i}\, v_{0,j})\, \frac{\partial v_i}{\partial x_j}\,. \tag{52}$$

Thus, by Poincaré's and Young's inequality, we find the estimate

$$\begin{aligned}
\left|\int_{\Omega_1} \rho_1\, (\hat v\cdot\nabla)\hat v\cdot v\right| &\leq \rho_1\, \Big(\mathbf{v_0}\, L\, \|\nabla v\|_{[L^2(\Omega_1)]^9} + \mathbf{v_0}^2\, \operatorname{meas}(\Omega_1)^{1/2}\Big)\, \|\nabla v\|_{[L^2(\Omega_1)]^9} \\
&\leq (\rho_1\, \mathbf{v_0}\, L + \gamma)\, \|\nabla v\|^2_{[L^2(\Omega_1)]^9} + \frac{\rho_1^2\, \mathbf{v_0}^4\, \operatorname{meas}(\Omega_1)}{4\,\gamma}\,,
\end{aligned}$$

where γ is an arbitrary small, positive number. We also consider the estimate

$$\left|\int_{\Omega_1} \big(v_0 \times \mu\, \hat{H}\big) \cdot \operatorname{curl} H\right| \le 2\, \mathbf{v_0}\, \mu_u\, \|H + H_0\|_{[L^2(\Omega_1)]^3}\, \|\operatorname{curl} H\|_{[L^2(\Omega_1)]^3}$$

$$\le (2\, \mathbf{v_0}\, \mu_u\, c_{\mathcal{H}} + \gamma)\, \|\operatorname{curl} H\|^2_{[L^2(\Omega_1)]^3} + \frac{\mathbf{v_0}^2\, \mu_u^2}{\gamma}\, \|H_0\|^2_{[L^2(\Omega_1)]^3}\,.$$

Therefore, we can at first write

$$\begin{aligned}
&\big\langle\, A\big(\{v, H, \theta\}\big)\,,\, \{v, H, \theta\}\,\big\rangle\ \ge \int_{\Omega_1} \eta(\hat{\theta})\, D(\hat{v}\,,\, v) - \int_{\Omega_1} \big(\operatorname{curl} \hat{H} \times \mu\, \hat{H}\big) \cdot v - \int_{\Omega_1} f(\hat{\theta}) \cdot v \\
&\quad + \int_{\tilde{\Omega}} r(\hat{\theta})\, \operatorname{curl} \hat{H} \cdot \operatorname{curl} H - \int_{\Omega_1} \big(v \times \mu\, \hat{H}\big) \cdot \operatorname{curl} H + \int_{\Omega} \kappa(\hat{\theta}) \nabla\hat{\theta} \cdot \nabla\theta + \int_{\Sigma} G(\sigma\, |\hat{\theta}|^3\, \hat{\theta})\, \theta \\
&\quad - \int_{\Omega} \Big[r(\hat{\theta})\, |\mathrm{curl}\hat{H}|^2 + \eta(\hat{\theta})\, D(\hat{v}\,,\, \hat{v})\, \chi_{\Omega_1}\Big]_{(\delta)}\, \theta \\
&\quad - (\rho_1\, \mathbf{v_0}\, L + \gamma)\, \|\nabla v\|^2_{[L^2(\Omega_1)]^9} - (2\, \mathbf{v_0}\, \mu_u\, c_{\mathcal{H}} + \gamma\,)\, \|\operatorname{curl} H\|^2_{[L^2(\Omega_1)]^3} - C_\gamma\,.
\end{aligned} \tag{53}$$

where the precise value of the constant C_γ is not needed anymore. Further, we observe that

$$\int_{\Omega_1} \big(v \times \mu\, \hat{H}\big) \cdot \operatorname{curl} H = -\int_{\Omega_1} \big(\operatorname{curl} H \times \mu\, \hat{H}\big) \cdot v = -\int_{\Omega_1} \big(\operatorname{curl} \hat{H} \times \mu\, \hat{H}\big) \cdot v\,, \tag{54}$$

since $\mathit{curl}\, H_0 = j_0 = 0$ in Ω_1.

By the homogenization (44) and the coercivity result of Lemma 5.5, (1) we have on the other hand that

$$\int_{\Omega} \kappa(\hat{\theta})\, \nabla\hat{\theta} \cdot \nabla\theta + \int_{\Sigma} G(\sigma\, |\hat{\theta}|^3\, \hat{\theta})\, \theta = \int_{\Omega} \kappa(\hat{\theta})\, |\nabla\theta|^2 + \int_{\Sigma} G(\sigma\, |\theta|^3\, \theta)\, \theta - \int_{\Omega} \kappa(\hat{\theta})\, \nabla(\theta_0\, \phi_0) \cdot \nabla\theta$$

$$\ge c\, \min\Big\{\|\theta\|^2_{V^{2,5}_\Gamma(\Omega)}\,,\, \|\theta\|^5_{V^{2,5}_\Gamma(\Omega)}\Big\} - \int_{\Omega} \kappa(\hat{\theta})\, \nabla(\theta_0\, \phi_0) \cdot \nabla\theta\,.$$

By Young's inequality, this implies that

$$\begin{aligned}
&\int_{\Omega} \kappa(\hat{\theta})\, \nabla\hat{\theta} \cdot \nabla\theta + \int_{\Sigma} G(\sigma\, |\hat{\theta}|^3\, \hat{\theta})\, \theta \\
&\quad \ge c\, \min\Big\{\|\theta\|^2_{V^{2,5}_\Gamma(\Omega)}\,,\, \|\theta\|^5_{V^{2,5}_\Gamma(\Omega)}\Big\} - \gamma\, \|\nabla\theta\|^2_{L^2(\Omega)}\ - c_\gamma\, \|\nabla\theta_0\|^2_{L^2(\Omega)} \ge \bar{c}\, \|\theta\|^2_{V^{2,5}_\Gamma(\Omega)} - C\,.
\end{aligned}$$

If we additionally consider the facts

$$\left|\int_{\Omega_1} f(\hat{\theta}) \cdot v\right| \le \rho_1\, |\vec{g}|\, M_t\, \|v\|_{[L^1(\Omega_1)]^3}\,, \qquad \left|\int_{\Omega} \Big[r(\hat{\theta})\, |\mathrm{curl}\hat{H}|^2 + \eta(\hat{\theta})\, D(\hat{v}\,,\, \hat{v})\Big]_{(\delta)}\, \theta\right| \le \frac{\|\theta\|_{L^1(\Omega_1)}}{\delta}\,,$$

we find by (53) and Young's inequality that

$$\big\langle\, A\big(\{v, H, \theta\}\big)\,,\, \{v, H, \theta\}\,\big\rangle\ \ge \frac{\gamma_0}{2}\, \|\{v, H, \theta\}\|^2_V - C\,,$$

with the number γ_0 given by (41). This proves the coercivity.

In order to prove that A is pseudomonotone, we consider an arbitrary sequence $\{v_k, H_k, \theta_k\} \subset V$ such that

$$v_k \rightharpoonup v \text{ in } D_0^{1,2}(\Omega_1)\,, \qquad H_k \rightharpoonup H \text{ in } \mathcal{H}^0_\mu(\tilde{\Omega})\,, \qquad \theta_k \rightharpoonup \theta \text{ in } V_\Gamma^{2,5}(\Omega)\,, \tag{55}$$

and we assume that

$$\limsup_{k\to\infty} \big\langle\, A\big(\{v_k, H_k, \theta_k\}\big)\,,\, \{v_k, H_k, \theta_k\} - \{v, H, \theta\}\,\big\rangle \le 0\,. \tag{56}$$

By well-known compactness properties and Lemma 5.1, we find a subsequence, that we do not relabel, such that

$$\begin{aligned} &v_k \longrightarrow v \text{ in } L^4(\Omega_1)\,, \qquad H_k \longrightarrow H \text{ in } L^2(\tilde{\Omega})\,, \\ &\theta_k \longrightarrow \theta \text{ in } L^2(\Omega)\,, \qquad \theta_k \longrightarrow \theta \text{ in } L^2(\Sigma)\,, \qquad \theta_k \longrightarrow \theta \text{ almost everywhere in } \Omega\,. \end{aligned} \tag{57}$$

Using straightforward rearrangements of terms, we can write

$$\begin{aligned} &\int_{\Omega_1} \eta(\hat{\theta}_k)\, D(v_k - v\,,\, v_k - v) + \int_{\tilde{\Omega}} r(\hat{\theta}_k)\,|\operatorname{curl}(H_k - H)|^2 + \int_\Omega \kappa(\hat{\theta}_k)\,|\nabla(\theta_k - \theta)|^2 \\ &\quad = \big\langle\, A\big(\{v_k, H_k, \theta_k\}\big)\,,\, \{v_k, H_k, \theta_k\} - \{v, H, \theta\}\,\big\rangle - \int_{\Omega_1} \eta(\hat{\theta}_k)\, D(\hat{v}\,,\, v_k - v) \\ &\quad - \int_{\tilde{\Omega}} r(\hat{\theta}_k)\operatorname{curl}\hat{H}\cdot\operatorname{curl}(H_k - H) - \int_\Omega \kappa(\hat{\theta}_k)\,\nabla\hat{\theta}\cdot\nabla(\theta_k - \theta) - \int_{\Omega_1} \rho_1\,(\hat{v}_k\cdot\nabla)\hat{v}_k\cdot(v_k - v) \\ &\quad + \int_{\Omega_1} \big(\operatorname{curl}\hat{H}_k \times \mu\,\hat{H}_k\big)\cdot(v_k - v) + \int_{\Omega_1} f(\hat{\theta}_k)\cdot(v_k - v) + \int_{\Omega_1} \big(\hat{v}_k \times \mu\,\hat{H}_k\big)\cdot\operatorname{curl}(H_k - H) \\ &\quad - \int_{\Omega_1} \rho_1\, c_V\, \hat{v}_k\cdot\nabla\hat{\theta}_k\,(\theta_k - \theta) - \int_\Sigma G(\sigma\,|\hat{\theta}_k|^3\,\hat{\theta}_k)\,(\theta_k - \theta) \\ &\quad + \int_\Omega \Big[r(\hat{\theta}_k)\,|\operatorname{curl}\hat{H}_k|^2 + \eta(\hat{\theta}_k)\,D(v_k\,,\, v_k)\,\chi_{\Omega_1}\Big]_{(\delta)}(\theta_k - \theta)\,. \end{aligned}$$

Observe that

$$\begin{aligned} \int_\Sigma G(\sigma\,|\hat{\theta}_k|^3\,\hat{\theta}_k)\,(\theta_k - \theta) &= \int_\Sigma G(\sigma\,|\theta_k|^3\,\theta_k)\,(\theta_k - \theta) = \int_\Sigma \sigma\,|\theta_k|^3\,\theta_k\, G(\theta_k - \theta) \\ &= \int_\Sigma \epsilon\,\sigma\,|\theta_k|^3\,\theta_k\,(\theta_k - \theta) - \int_\Sigma \epsilon\,\sigma\,|\theta_k|^3\,\theta_k\,\tilde{\mathbf{H}}(\theta_k - \theta)\,, \end{aligned}$$

where the operator $\tilde{\mathbf{H}}$ is compact from $L^{5/4}(\Sigma)$ into itself, according to Lemma 5.7, (1). Thus, passing to subsequences if necessary, we find that

$$\liminf_{k\to\infty} \int_\Sigma G(\sigma\,|\hat{\theta}_k|^3\,\hat{\theta}_k)\,(\theta_k - \theta) = \liminf_{k\to\infty} \int_\Sigma \epsilon\,\sigma\,|\theta_k|^3\,\theta_k\,(\theta_k - \theta) \ge 0\,. \tag{58}$$

By (56) and (57) and (58), we see immediately that

$$\limsup_{k\to\infty}\left(\int_{\Omega_1} D(v_k - v\,,\, v_k - v) + \int_{\tilde\Omega} |\operatorname{curl}(H_k - H)|^2 + \int_\Omega |\nabla(\theta_k-\theta)|^2\right) \le 0\,.$$

We thus find (not relabelled) subsequences with the properties

$$v_k \longrightarrow v \text{ in } D_0^{1,2}(\Omega_1)\,, \qquad H_k \longrightarrow H \text{ in } \mathcal{H}^0_\mu(\tilde\Omega)\,. \tag{59}$$

By the dominated convergence theorem, this implies for a subsequence and for all $1 \le q < \infty$ that

$$\Big[r(\hat\theta_k)\,|\operatorname{curl}\hat H_k|^2 + \eta(\hat\theta_k)\,D(v_k\,,\,v_k)\,\chi_{\Omega_1}\Big]_{(\delta)} \longrightarrow \Big[r(\hat\theta)\,|\operatorname{curl}\hat H|^2 + \eta(\hat\theta)\,D(v\,,\,v)\,\chi_{\Omega_1}\Big]_{(\delta)}\,,$$

in $L^q(\Omega)$. We observe that by the compactness of the non local operator $\tilde{\mathbf{H}}$ and (55), we have generally

$$\liminf_{k\to\infty}\int_\Sigma G(\sigma\,|\theta_k|^3\,\theta_k)\,(\theta_k-\xi) \ge \int_\Sigma G(\sigma\,|\theta|^3\,\theta)\,(\theta-\xi)\,,$$

for all $\xi \in V^{2,5}_\Gamma(\Omega)$. By this property and (59), we can easily show that

$$\liminf_{k\to\infty}\big\langle\, A\big(\{v_k, H_k, \theta_k\}\big)\,,\, \{v_k, H_k, \theta_k\} - \{\phi,\psi,\xi\}\,\big\rangle \;\ge \big\langle\, A\big(\{v, H, \theta\}\big)\,,\, \{v, H, \theta\} - \{\phi,\psi,\xi\}\,\big\rangle\,,$$

for all $\{\phi,\psi,\xi\} \in V$, proving the preudomonotonicity of A. By the results of [Lio69], Ch. 2, Th. 2.7., the equation $A\big(\{v, H, \theta\}\big) = 0$ has at least one solution in V.

We at last prove that (51) is valid. By the previous considerations, we have obtained in particular the relation

$$\begin{aligned}\int_{\Omega_1}\rho_1\,c_V\,\hat v\cdot\nabla\hat\theta\,\xi + \int_\Omega \kappa(\hat\theta)\,\nabla\hat\theta\cdot\nabla\xi + \int_\Sigma G(\sigma\,|\hat\theta|^3\,\hat\theta)\,\xi&\\ = \int_\Omega \Big[r(\hat\theta)\,|\operatorname{curl}\hat H|^2 + \eta(\hat\theta)\,D(\hat v\,,\,\hat v)\,\chi_{\Omega_1}\Big]_{(\delta)}\,\xi\,,&\end{aligned} \tag{60}$$

for all $\xi \in V^{2,5}_\Gamma(\Omega)$. We define $k_0 := \operatorname*{ess\,inf}_\Gamma \theta_0$, and we test with the function $\xi = (\hat\theta - k_0)^-$ in the relation (60). We observe that

$$\int_{\Omega_1}\rho_1\,c_V\,\hat v\cdot\nabla\hat\theta\,(\hat\theta-k_0)^- = \int_{\Omega_1}\rho_1\,c_V\,\hat v\cdot\frac{1}{2}\nabla(\hat\theta-k_0)^{-^2} = 0$$

$$\int_\Sigma G(\sigma\,|\hat\theta|^3\,\hat\theta)\,(\hat\theta-k_0)^- = \int_\Sigma G(\sigma\,|\hat\theta|^3\,\hat\theta)\,[(\hat\theta-k_0)^- + k_0] \ge 0\,.$$

Here, we used the fact that $G(1) = 0$ and the elementary properties of the operator G in enclosures (see Lemma 5.4, (2)). In order to obtain the inequality, we applied Lemma 5.6. We get $\int_\Omega \kappa(\hat\theta)\,\big|\nabla(\hat\theta-k_0)^-\big|^2 \le 0$, and since $\hat\theta \ge k_0$ on Γ, it follows that $\hat\theta \ge k_0$ almost everywhere in Ω. We can replace the term $|\hat\theta|^3\,\hat\theta$ by $\hat\theta^4$ in (60). We obtain (50). Writing from now on $\{v,\, H,\, \theta\}$ instead of $\{\hat v,\, \hat H,\, \hat\theta\}$, this finishes the proof of the proposition. □

For sequences of approximate solutions according to Proposition 1.3, we introduce the notation

$$M_\delta := \frac{1}{\text{meas}(\Sigma)} \int_\Sigma \theta_\delta^4 \,. \tag{61}$$

Proposition 1.4. For any sequence of approximations $\{v_\delta,\, H_\delta,\, \theta_\delta\}$ according to Proposition 1.3, the following uniform estimates are valid:

(1) For the MHD energy, we have the estimate

$$\| \, v_\delta \, \|_{D^{1,2}(\Omega_1)} + \| \, H_\delta \, \|_{\mathcal{H}_\mu(\tilde\Omega)} \leq c \, (\|f(\theta_\delta)\|_{[L^2(\Omega_1)]^3} + \| \, j_0 \, \|_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_0\|_{D^{1,2}(\Omega_1)}) \,.$$

(2) For the temperature, we find the uniform bound

$$\begin{aligned} &\| \, \theta_\delta \, \|_{W^{1,p}_\Gamma(\Omega)} + \| \, \theta_\delta^4 - M_\delta \, \|_{L^1(\Sigma)} \\ &\qquad \leq P_p(\|f(\theta_\delta)\|_{[L^2(\Omega_1)]^3}, \, \| \, j_0 \, \|_{[L^2(\tilde\Omega_{c_0})]^3}, \, \|v_0\|_{D^{1,2}(\Omega_1)}, \, \|\nabla\theta_0\|_{L^2(\Omega)}, \, \|\theta_0\|_{L^\infty(\Omega)}) \,, \end{aligned}$$

with a continuous function P_p for all $1 \leq p < \frac{3}{2}$.

Proof. For the sake of notational simplicity, we write throughout this proof v instead of v_δ etc.

(1): We test in (48) with the vector field $v - v_0$, and in (49) with $H - H_0$. Recalling (52) and (54), we obtain, after adding both relations, that

$$\begin{aligned} \int_{\Omega_1} \eta(\theta) \, D(v \,,\, v) + \int_{\tilde\Omega} r(\theta) \, | \, \text{curl}\, H|^2 &= \int_{\Omega_1} \rho_1 \, v_j \, v_{0,i} \, \frac{\partial v_i}{\partial x_j} + \int_{\Omega_1} \eta(\theta) \, D(v \,,\, v_0) \\ &+ \int_{\Omega_1} f(\theta) \cdot (v - v_0) + \int_{\Omega_1} \big(v_0 \times \mu \, H\big) \cdot \text{curl}\, H + \int_{\tilde\Omega} r(\theta) \, j_0 \cdot \text{curl}\, H \,. \end{aligned} \tag{62}$$

By standard inequalities, we find for the absolute value of the right-hand side of (62) the upper bound

$$\begin{aligned} &\int_{\Omega_1} \eta(\theta) \, D(v \,,\, v) + \int_{\tilde\Omega} r(\theta) \, | \, \text{curl}\, H|^2 \leq \rho_1 \, \mathbf{v_0} \, L \, \|\nabla v\|^2_{[L^2(\Omega_1)]^9} + \gamma \int_{\Omega_1} \eta(\theta) \, D(v \,,\, v) \\ &\quad + \gamma_2 \, \|\nabla v\|^2_{[L^2(\Omega_1)]^9} + \frac{L^2 \, \|f(\theta)\|^2_{[L^2(\Omega_1)]^3}}{4 \, \gamma_2} + \frac{1}{4\gamma} \int_{\Omega_1} \eta(\theta) \, D(v_0 \,,\, v_0) + \|f(\theta) \cdot v_0\|_{[L^1(\Omega_1)]^3} \\ &+ 2 \, \mathbf{v_0} \, \mu_u \, c_{\mathcal{H}} \, \| \, \text{curl}\, H\|^2_{[L^2(\Omega_1)]^3} + \gamma \int_{\tilde\Omega} r(\theta) \, | \, \text{curl}\, H|^2 + \frac{1}{4\gamma} \int_{\tilde\Omega} r(\theta) \, |j_0|^2 \,, \end{aligned}$$

where γ, γ_2 are arbitrary small positive numbers. We obtain that

$$\begin{aligned} &[(1-\gamma) \, \eta_l \, c_{\text{Korn}}^{-1} - \rho_1 \, \mathbf{v_0} \, L - \gamma_2] \int_{\Omega_1} |\nabla v|^2 + [(1-\gamma) \, r_l - c_{\mathcal{H}} \, \mathbf{v_0} \, \mu_u] \int_{\tilde\Omega} | \, \text{curl}\, H|^2 \\ &\quad \leq \frac{L^2}{4\gamma_2} \, \|f(\theta)\|^2_{[L^2(\Omega_1)]^3} + \|f(\theta) \cdot v_0\|_{[L^1(\Omega_1)]^3} + \frac{1}{4\gamma} \int_{\tilde\Omega} r(\theta) \, |j_0|^2 \,. \end{aligned}$$

where we can choose γ, γ_2 arbitrary small, and the estimate (1) follows from the assumption (41).

(2): For a parameter $\gamma > 0$ to be fixed later, we introduce the continuous function

$$g_\gamma(t) := \mathrm{sign}(t) \left(1 - \frac{1}{(1+|t|)^\gamma}\right) \qquad \text{for } t \in \mathbb{R}\,.$$

In (50) we use the test function

$$\xi = \xi_\gamma := \; g_\gamma(\theta - \tilde{\theta}_0) = \mathrm{sign}(\theta - \tilde{\theta}_0) \left(1 - \frac{1}{(1+|\theta - \tilde{\theta}_0|)^\gamma}\right)\,.$$

Here, we have set $\tilde{\theta}_0 := \theta_0\,\phi_0$, with a smooth function ϕ_0 such that $\phi_0 = 0$ on Σ and $\phi_0 = 1$ on Γ. Note that ξ vanishes on the boundary Γ, that $0 \leq \xi \leq 1$ in Ω, and that

$$\nabla\xi \;=\; \gamma\,\frac{\nabla(\theta - \tilde{\theta}_0)}{(1+|\theta - \tilde{\theta}_0|)^{1+\gamma}}\,,$$

so that we are allowed to test the relation (50) with this function.

Denoting by Ψ the primitive function of g_γ that vanishes at zero, we observe that

$$\int_{\Omega_1} \rho_1\, c_V\, v \cdot \nabla(\theta - \tilde{\theta}_0)\,\xi = \int_{\Omega_1} \rho_1\, c_V\, v \cdot \nabla\Psi(\theta - \tilde{\theta}_0) = 0\,.$$

By Lemma 5.6 and the fact that $\tilde{\theta}_0$ vanishes on Σ, we obtain on the other hand that

$$\int_\Sigma G(\sigma\,\theta^4)\,\xi = \int_\Sigma G(\sigma\,\theta^4)\,\left(1 - \frac{1}{(1+\theta)^\gamma}\right) \geq 0\,.$$

Thus, the inequality

$$\gamma\int_\Omega \frac{\kappa(\theta)\,|\nabla(\theta - \tilde{\theta}_0)|^2}{(1+|\theta - \tilde{\theta}_0|)^{1+\gamma}} \leq \gamma\int_\Omega \kappa(\theta)\frac{|\nabla\theta_0|\,|\nabla(\theta - \tilde{\theta}_0)|}{(1+|\theta - \tilde{\theta}_0|)^{1+\gamma}} + \int_{\Omega_1} \rho_1\, c_V\, |v \cdot \nabla\tilde{\theta}_0|$$
$$+ \int_\Omega \Big[r(\theta)\,|\,\mathrm{curl}\, H|^2 + \eta(\theta)\, D(v\,,\, v)\,\chi_{\Omega_1}\Big]_{(\delta)}\,,$$

is readily verified. By Young's inequality, it follows that

$$\frac{\kappa_l\,\gamma}{2}\int_\Omega \frac{|\nabla(\theta - \tilde{\theta}_0)|^2}{(1+|\theta - \tilde{\theta}_0|)^{1+\gamma}} \leq \frac{\gamma\,\kappa_u}{2\,\kappa_l}\,\|\nabla\tilde{\theta}_0\|^2_{L^2(\Omega)} + \rho_1\, c_V\, L\,\|\nabla\tilde{\theta}_0\|_{L^2(\Omega_1)}\,\|\nabla v\|_{[L^2(\Omega_1)]^9}$$
$$+ \int_\Omega r(\theta)\,|\,\mathrm{curl}\, H|^2 + \int_{\Omega_1} \eta(\theta)\, D(v\,,\, v)\,.$$

Making use of (1), we obtain for arbitrary $\gamma \in]0\,,\,1[$ that

$$\gamma\int_\Omega \frac{|\nabla(\theta - \tilde{\theta}_0)|^2}{(1+|\theta - \tilde{\theta}_0|)^{1+\gamma}} \leq C_1\,,$$

where the constant C_1 depends on the data trough the previous estimate (1). By the arguments of Lemma 5.9, we obtain that

$$\|\theta-\tilde{\theta}_0\|_{W^{1,p}_\Gamma(\Omega)} \leq P_p(\|f(\theta)\|_{[L^2(\Omega_1)]^3},\, \|j_0\|_{[L^2(\tilde{\Omega}_c)]^3},\, \|v_0\|_{D^{1,2}(\Omega_1)},\, \|\nabla\tilde{\theta}_0\|_{L^2(\Omega)})\,.$$

for all $1 \leq p < \frac{3}{2}$.

In order to derive the complete estimate (2), we now want to estimate θ on the boundary Σ. We define $\bar{k}_0 := \operatorname*{ess\,sup}_\Gamma \theta_0$, and we recall the definition (61) of the numbers M_δ.

Observe that in the case that $M_\delta \leq \bar{k}_0^4$, the estimate

$$\|\theta^4 - M_\delta\|_{L^1(\Sigma)} \leq (\mathrm{meas}(\Sigma)+1)\, M_\delta \leq 2\,\bar{k}_0^4\,\mathrm{meas}(\Sigma)\,, \tag{63}$$

is valid. Suppose now that $M_\delta > \bar{k}_0^4$. For $\gamma > 0$, we introduce the function

$$g_\gamma(t) := \frac{1}{\gamma}\,\mathrm{sign}\big(t\big)\,\min\Big\{\big|t\big|\,,\,\gamma\Big\} + 1\,, \qquad \text{for } t \in \mathbb{R}\,.$$

In (50) we choose the test function

$$\xi = \xi_{\delta,\gamma} := g_\gamma(\theta - M_\delta) \;=\; \frac{1}{\gamma}\,\mathrm{sign}\big(\theta^4 - M_\delta\big)\,\min\Big\{\big|\theta^4 - M_\delta\big|\,,\,\gamma\Big\} + 1\,.$$

Note that for all $0 < \gamma < M_\delta - \bar{k}_0^4$, the function ξ vanishes on Γ, and observe that $0 \leq \xi \leq 2$ in Ω. On the other hand, since

$$\nabla\xi = \frac{4}{\gamma}\,|\theta|^3\,\chi_{\{x\in\Omega\,:\,|\theta(x)^4 - M_\delta|<\gamma\}}\,\nabla\theta\,,$$

we can verify that

$$|\nabla\xi|^2 \leq \left(\frac{4}{\gamma}\right)^2\,(M_\delta+\gamma)^{\frac{3}{2}}\,|\nabla\theta|^2 \in L^1(\Omega)\,,$$

so that we can test with this function in (50). Since g_γ is nondecreasing, we have $\nabla\theta\cdot\nabla g_\gamma(\theta) = g_\gamma'(\theta)\,|\nabla\theta|^2 \geq 0$, and we obtain that

$$\begin{aligned}\int_\Sigma G(\sigma\,|\theta|^4)\,g_\gamma(\theta) \leq &-\int_{\Omega_1}\rho_1\,c_V\,|v\cdot\nabla\tilde{\theta}_0|\,g_\gamma(\theta)\\ &+\int_\Omega\Big[r(\theta)\,|\,\mathrm{curl}\,H|^2 + \eta(\theta)\,D(v\,,\,v)\,\chi_{\Omega_1}\Big]_{(\delta)}\,g_\gamma(\theta)\,.\end{aligned} \tag{64}$$

Now, since Ω is an enclosure and $G(1)\equiv 0$ (see Lemma 5.4), we can write

$$\begin{aligned}&\int_\Sigma G(\sigma\,|\theta|^4)\,\left[\frac{1}{\gamma}\,\mathrm{sign}(\theta^4 - M_\delta)\,\min\{|\theta^4 - M_\delta|,\,\gamma\} + 1\right]\\ &\quad= \int_\Sigma G\Big(\sigma\,\big[|\theta|^4 - M_\delta\big]\Big)\,\frac{1}{\gamma}\,\mathrm{sign}(\theta^4 - M_\delta)\,\min\{|\theta^4 - M_\delta|,\,\gamma\}\,.\end{aligned}$$

Letting $\gamma \to 0$ in (64), it follows that

$$\int_\Sigma G\Big(\sigma\,\big[|\theta|^4 - M_\delta\big]\Big)\,\mathrm{sign}(\theta^4 - M_\delta) \le 2\ \Big(\int_{\Omega_1} \rho_1\, c_V\, |v\cdot\nabla\tilde\theta_0|$$
$$+ \int_\Omega \Big[r(\theta)\,|\mathrm{curl} H|^2 + \eta(\theta)\, D(v,v)\,\chi_{\Omega_1}\Big]\Big)\ .$$

By the previous estimates and Lemma 5.5, we get

$$\|\theta^4 - M_\delta\|_{L^1(\Sigma)} \le c\,\Big(\|f(\theta)\|^2_{[L^2(\Omega_1)]^3} + \|j_0\|^2_{[L^2(\tilde\Omega_c)]^3} + \|v_0\|^2_{D^{1,2}(\Omega_1)} + \|\nabla\tilde\theta_0\|^2_{L^2(\Omega)}\Big)\,. \tag{65}$$

Putting together (63) and (65), we obtain for all $\delta > 0$ that

$$\|\theta^4 - M_\delta\|_{L^1(\Sigma)} \le c\,\Big(\|f(\theta)\|^2_{[L^2(\Omega_1)]^3} + \|j_0\|^2_{[L^2(\tilde\Omega_c)]^3} + \|v_0\|^2_{D^{1,2}(\Omega_1)} + \|\nabla\tilde\theta_0\|^2_{L^2(\Omega)}\Big) + 2\,\bar k_0^4\,\mathrm{meas}(\Sigma)\,,$$

which finally proves (2). □

Proposition 1.5. Let $\big\{v_\delta,\, H_\delta,\, \theta_\delta\big\}$ be any sequence of approximate solutions according to Proposition 1.3. Then, there exists some $\{v,\, H,\, \theta\} \in D^{1,2}(\Omega)\times\mathcal{H}_\mu(\tilde\Omega)\times V^{p,4}(\Omega)$ $(1 \le p < 3/2)$ and some subsequence $\delta \to 0$ such that

$$v_\delta \longrightarrow v \text{ in } D^{1,2}(\Omega_1)\,,\quad H_\delta \longrightarrow H \text{ in } \mathcal{H}_\mu(\tilde\Omega)\,,$$
$$\theta_\delta \rightharpoonup \theta \text{ in } W^{1,p}(\Omega)\,,\quad \theta_\delta^4 \longrightarrow \theta^4 \text{ in } L^1(\Sigma)\,.$$

Proof. By the estimates of Proposition 1.4, we at first find a sequence

$$v_\delta \rightharpoonup v \text{ in } D^{1,2}(\Omega_1)\,,\qquad H_\delta \rightharpoonup H \text{ in } \mathcal{H}_\mu(\tilde\Omega)\,,\qquad \theta_\delta \rightharpoonup \theta \text{ in } W^{1,p}(\Omega)\,. \tag{66}$$

By well-known compactness properties, we now find a (not relabelled) subsequence such that

$$v_\delta \longrightarrow v \text{ in } L^4(\Omega_1)\,,\quad H_\delta \longrightarrow H \text{ in } L^2(\tilde\Omega)\,,\quad \theta_\delta \longrightarrow \theta \text{ in } L^p(\Omega)\,,$$
$$\theta_\delta \longrightarrow \theta \text{ in } L^p(\Sigma)\,,\quad \theta_\delta \longrightarrow \theta \text{ almost everywhere in } \Omega\,. \tag{67}$$

Passing to the limit $\delta \to 0$ in (48), (49), by the same arguments as in the proof of Propsition 1.3, we see that the pair $\{v,\, H\}$ satisfies the relations

$$\int_{\Omega_1} \rho_1\,(v\cdot\nabla)v\cdot\phi + \int_{\Omega_1}\eta(\theta)\,D(v\,,\,\phi) = \int_{\Omega_1}\big(\mathrm{curl}\,H\times\mu\,H\big)\cdot\phi + \int_{\Omega_1} f(\theta)\cdot\phi\,,$$
$$\int_{\tilde\Omega} r(\theta)\ \mathrm{curl}\,H\cdot\mathrm{curl}\,\psi = \int_{\tilde\Omega}\big(v\times\mu\,H\big)\cdot\mathrm{curl}\,\psi\,.$$

In these relations, we now use the test functions $\phi = v_\delta - v$, $\psi = H_\delta - H$. We do the same in the identities (48) and (49). Subtracting the two arising integral relations, we can write

$$\begin{aligned}
&\int_{\Omega_1} \eta(\theta_\delta)\, D(v_\delta - v\,,\, v_\delta - v) \\
&\quad = - \int_{\Omega_1} [\eta(\theta_\delta) - \eta(\theta)]\, D(v\,,\, v_\delta - v) - \int_{\Omega_1} \rho_1\, (v_\delta \cdot \nabla v_\delta - v \cdot \nabla v) \cdot (v_\delta - v) \\
&\quad - \int_{\Omega_1} \Big(\big(\operatorname{curl} H_\delta \times \mu\, H_\delta \big) - \big(\operatorname{curl} H \times \mu\, H \big) \Big) \cdot (v_\delta - v) - \int_{\Omega_1} [f(\theta_\delta) - f(\theta)] \cdot (v_\delta - v)\,,
\end{aligned}$$

as well as

$$\begin{aligned}
\int_{\tilde\Omega} r(\theta_\delta)\, |\operatorname{curl}(H_\delta - H)|^2 &= - \int_{\tilde\Omega} [r(\theta_\delta) - r(\theta)]\, \operatorname{curl} H \cdot \operatorname{curl}(H_\delta - H) \\
&+ \int_{\Omega_1} \Big(\big(v_\delta \times \mu\, H_\delta \big) - \big(v \times \mu\, H \big) \Big) \cdot \operatorname{curl}(H_\delta - H)\,.
\end{aligned}$$

Now, by (66) and (67), it is not difficult to see that the right-hand sides of both relations converge to zero for $\delta \to 0$, proving that

$$v_\delta \longrightarrow v \text{ in } D^{1,2}(\Omega_1)\,, \qquad H_\delta \longrightarrow H \text{ in } \mathcal{H}_\mu(\tilde\Omega)\,.$$

Thus, we have also

$$D(v_\delta\,,\, v_\delta) \longrightarrow D(v\,,\, v) \text{ in } L^1(\Omega_1)\,, \qquad |\operatorname{curl} H_\delta|^2 \longrightarrow |\operatorname{curl} H|^2 \text{ in } L^1(\tilde\Omega)\,,$$

which, in view of Lemma 5.10, yields

$$\Big[r(\theta_\delta)\, |\operatorname{curl} H_\delta|^2 + \eta(\theta_\delta)\, D(v_\delta\,,\, v_\delta)\, \chi_{\Omega_1} \Big]_{(\delta)} \longrightarrow r(\theta)\, |\operatorname{curl} H|^2 + \eta(\theta)\, D(v\,,\, v)\, \chi_{\Omega_1} \text{ in } L^1(\Omega)\,. \tag{68}$$

Now we prove the convergence property for the boundary integral. Since the employed techniques are similar to the ones used in [Dru07b], we will only give the main ideas.

First, we prove that the sequence of numbers M_δ given by (61) is bounded. Using estimate (2) and Fatou's lemma, we can write that

$$\int_\Sigma \liminf_{\delta\to 0} |\theta_\delta^4 - M_\delta| \le \liminf_{\delta\to 0} \int_\Sigma |\theta_\delta^4 - M_\delta| \le C\,. \tag{69}$$

Seeking a contradiction, we suppose that there exists a subsequence such that $M_\delta \to \infty$. For this subsequence, we obtain that

$$\liminf_{\delta\to 0} |\theta_\delta^4 - M_\delta| = \lim_{\delta\to 0} |\theta_\delta^4 - M_\delta| = \lim_{\delta\to 0} |\theta^4 - M_\delta| = +\infty \quad \text{almost everywhere on } \Sigma\,,$$

since the pointwise limit θ^4 is almost everywhere finite. This contradicts (69).

Thus, the sequence $\{M_\delta\}$ is bounded, which by definition also implies a uniform bound $\| \theta_\delta^4 \|_{L^1(\Sigma)} \leq C$. By Lemma 5.7, (3), it follows that

$$\tilde{\mathbf{H}}(\theta_\delta^4) \rightharpoonup u \text{ in } L^1(\Sigma)\,,$$

for some $u \in L^1(\Sigma)$.

Now, for an arbitrary $\xi \in C_c^\infty(\Omega)$, we take the limit $\delta \to 0$ in relation (50). Considering (68), we obtain that

$$\begin{aligned}\int_{\Omega_1} \rho_1\, c_V\, v \cdot \nabla\theta\, \xi + \int_\Omega \kappa(\theta)\, \nabla\theta \cdot \nabla\xi + \lim_{\delta\to 0} \int_\Sigma \epsilon\, \sigma\, |\theta_\delta|^4\, \xi - \int_\Sigma \epsilon\, \sigma\, u\, \xi \\ = \int_\Omega r(\theta)\, |\operatorname{curl} H|^2\, \xi + \int_{\Omega_1} \eta(\theta)\, D(v\,,\, v)\, \xi\,. \qquad (70)\end{aligned}$$

In order to compute $\lim_{\delta\to 0} \int_\Sigma G(\sigma\, |\theta_\delta|^4)\, \xi$, we now test in (50) with the function $g_\gamma(\theta_\delta)\, \xi$, where ξ is an arbitrary $C_c^\infty(\Omega)$-function which is nonnegative in Ω, and g_γ is for $\gamma > 0$ the nondecreasing function defined by $g_\gamma(t) := \frac{1}{1+\gamma\, t^4}$. For a while, we do not indicate the indices γ.

Using the techniques of the proof of [Dru07b], Th. 6.1, we can prove the inequality

$$\begin{aligned}\int_{\Omega_1} \rho_1\, c_V\, v_\delta \cdot \nabla\theta_\delta\, g(\theta_\delta)\, \xi + \int_\Omega \kappa(\theta_\delta)\, \nabla\theta_\delta \cdot \nabla\xi\, g(\theta_\delta) + \int_\Sigma G(\sigma\, |\theta_\delta|^4)\, \xi\, g(\theta_\delta) \\ \geq \int_\Omega \Big[r(\theta_\delta)\, |\operatorname{curl} H_\delta|^2 + \eta(\theta_\delta)\, D(v_\delta, v_\delta)\, \chi_{\Omega_1}\Big]_{(\delta)}\, \xi\, g(\theta_\delta)\,, \qquad (71)\end{aligned}$$

in which it is, by the same arguments, possible to take the limit $\delta \to 0$ to obtain the relation

$$\begin{aligned}\int_{\Omega_1} \rho_1\, c_V\, v \cdot \nabla\theta\, g(\theta)\, \xi + \int_\Omega \kappa(\theta)\, \nabla\theta \cdot \nabla\xi\, g(\theta) + \int_\Sigma \epsilon\, \sigma\, \frac{\theta^4}{1+\gamma\, \theta^4}\, \xi - \int_\Sigma \epsilon\, \sigma\, u\, \xi\, g(\theta) \\ \geq \int_\Omega r(\theta)\, |\operatorname{curl} H|^2\, \xi\, g(\theta) + \int_{\Omega_1} \eta(\theta)\, D(v, v)\, \xi\, g(\theta)\,.\end{aligned}$$

At this point, recalling that $g = g_\gamma$, we observe that for all $t \in \mathbb{R}^+$, the monotone convergence $g_\gamma(t) \nearrow 1$ as $\gamma \to 0$ takes place. Therefore, taking the limit in the last inequality yields

$$\begin{aligned}\int_{\Omega_1} \rho_1\, c_V\, v \cdot \nabla\theta\, \xi + \int_\Omega \kappa(\theta)\, \nabla\theta \cdot \nabla\xi + \int_\Sigma \epsilon\, \sigma\, |\theta|^4\, \xi - \int_\Sigma \epsilon\, \sigma\, u\, \xi \\ \geq \int_\Omega r(\theta)\, |\operatorname{curl} H|^2\, \xi + \int_{\Omega_1} \eta(\theta)\, D(v, v)\, \xi\,. \qquad (72)\end{aligned}$$

Comparing the relations (70) and (72), we find that

$$\int_\Sigma \epsilon\, \sigma\, |\theta|^4\, \xi \geq \lim_{\delta\to 0} \int_\Sigma \epsilon\, \sigma\, |\theta_\delta|^4\, \xi\,,$$

for all $\xi \in C_c^\infty(\Omega)$ such that $\xi \geq 0$ in Ω. With the help of Fatou's lemma, we even have

$$\lim_{\delta \to 0} \int_\Sigma \epsilon\, \sigma\, |\theta_\delta|^4 \xi = \int_\Sigma \epsilon\, \sigma\, |\theta|^4\, \xi\,. \tag{73}$$

But in view of (25), it is possible to choose $\xi \in C_c^\infty(\Omega)$ such that $\xi \geq 0$ in Ω and $\xi = 1$ on Σ. It then follows from (73) and Lemma 5.11 that $\theta_\delta^4 \longrightarrow \theta^4$ in $L^1(\Sigma)$, proving the last assertion and the proposition. □

We are now able to prove the main result of this section.

Proof of Theorem 1.1. Thanks to the convergence properties stated by Proposition 1.5, we find a triple $\{v,\, H,\, \theta\} \in D^{1,2}(\Omega_1) \times \mathcal{H}_\mu(\tilde{\Omega}) \times V^{p,4}(\Omega)$, with $1 \leq p < \frac{3}{2}$ arbitrary, such that $v = v_0$ on $\partial\Omega_1$, $\theta = \theta_0$ on Γ, $curl\, H = j_0$ in $\tilde{\Omega}_{c_0}$, and the relations

$$\begin{aligned}
&\int_{\Omega_1} \rho_1\,(v\cdot\nabla)v\cdot\phi + \int_{\Omega_1}\eta(\theta)\,D(v\,,\,\phi) = \int_{\Omega_1}\big(\mathrm{curl} H \times \mu\, H\big)\cdot\phi + \int_{\Omega_1} f(\theta)\cdot\phi\,,\\
&\int_{\tilde{\Omega}} r(\theta)\ \mathrm{curl}\, H\cdot \mathrm{curl}\,\psi = \int_{\Omega_1}\big(v\times\mu\, H\big)\cdot\ \mathrm{curl}\,\psi\,,\\
&\int_{\Omega_1}\rho_1\, c_V\, v\cdot\nabla\theta\,\xi + \int_\Omega \kappa(\theta)\,\nabla\theta\cdot\ \nabla\xi + \int_\Sigma G(\sigma\,\theta^4)\,\xi\\
&\qquad\qquad\qquad = \int_\Omega r(\theta)\,|\,\mathrm{curl}\, H|^2\,\xi + \int_{\Omega_1}\eta(\theta)\,D(v\,,\,v)\,\xi\,,
\end{aligned}\tag{74}$$

are satisfied for all $\{\phi,\, \psi,\, \xi\} \in D_0^{1,2}(\Omega_1) \times \mathcal{H}^0_\mu(\tilde{\Omega}) \times V_\Gamma^{p',\infty}(\Omega)$. By the result of Lemma 5.1, (3) and Sobolev's embedding theorems, we find that $curl\, H \times \mu\, H \in L^{3/2}(\tilde{\Omega})$. This proves the claim. □

Proof of Proposition 1.2. If the conditions $(A0)$ and $(A2)$ are satisfied instead of $(A0)$ and $(A1)$, or if (42) is valid, the proof of Theorem 1.1 can simply be repeated. Using the continuity of the embedding $\mathcal{H}_\mu(\tilde{\Omega}) \hookrightarrow [L^\xi(\tilde{\Omega})]^3$ for a $\xi > 3$ (see [Dru07a]), we see that the Lorentz force $curl\, H \times \mu\, H$ still belongs to $[L^s(\tilde{\Omega})]^3$ for some $s > 6/5$. □

2 Boussinesq approximation.

In the first section, we replaced the Boussinesq approximation of the gravitational force (37) by the *bounded* term (38). We can argue in favor of this choice by observing that the Boussinesq approximation is valid only in the range of *small* density variations, that is,

$$0 \leq \alpha\,(\theta - \theta_M) \ll 1\,. \tag{75}$$

This approach would be fully justify if we could prove *a posteriori* that the weak solutions obtained in the first section actually satisfy (75). We cannot give a proof of such a full justification. Instead, we have a weaker result.

Lemma 2.1. Assume that the hypotheses of Theorem 1.1 or of Proposition 1.2 are satisfied, and assume in addition that θ_0 is a constant. Let the numbers α, M_t in (38) be such that

$$1 - \bar{c}\,\frac{\mathrm{meas}(\Omega_1)\,\rho_1^2\,|\vec{g}|^2}{\kappa_l}\,M_t\,\alpha > 0\,,$$

where $\bar{c} = \sqrt{2}\,c\,c_0^2$, with the constant c that appears in Proposition 1.4, (1) and the constant c_0 of the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$.

Then, for the weak solutions of (P) constructed as in Theorem 1.1 or in Proposition 1.2, the estimate

$$\left(\frac{1}{\mathrm{meas}(\Omega_1)}\int_{\Omega_1} \alpha^2\,|\theta - \theta_M|^2\right)^{1/2} \le \frac{\bar{c}\,\alpha\,(\|j_0\|^2_{[L^2(\tilde{\Omega}_{c_0})]^3} + \|v_0\|^2_{D^{1,2}(\Omega_1)})}{\kappa_l - \bar{c}\,\mathrm{meas}(\Omega_1)\,\rho_1^2\,|\vec{g}|^2\,\alpha\,M_t}\,,$$

is valid.

Proof. We consider some sequence of approximate solutions $\{v_\delta,\,H_\delta,\,\theta_\delta\}$ according to Proposition 1.3 and derive an additional uniform estimate. We start from (50), and we for a while write v, H, θ instead of v_δ, H_δ, θ_δ.

For a parameter $\lambda > 0$, we are allowed to use the test function

$$\xi = (\theta - \theta_0)^{(\lambda)} = \mathrm{sign}(\theta - \theta_0)\,\min\{|\theta - \theta_0|,\,\lambda\}\,.$$

Denoting by Ψ a primitive of the function $s \mapsto (s - \theta_0)^{(\lambda)}$ $(s \in \mathbb{R})$, we can write

$$\int_{\Omega_1} \rho_1\,c_V\,v \cdot \nabla\theta\,\xi = \int_{\Omega_1} \rho_1\,c_V\,v \cdot \nabla\Psi(\theta) = 0\,,$$

since v is divergence free in Ω_1 and tangential on $\partial\Omega_1$. It follows that

$$\int_\Omega \kappa(\theta)\,|\nabla(\theta - \theta_0)^{(\lambda)}|^2 + \int_\Sigma G(\sigma\,\theta^4)\,(\theta - \theta_0)^{(\lambda)}$$
$$= \int_\Omega \Big[\eta(\theta)\,D(v, v)\,\chi_{\Omega_1} + r(\theta)\,|\,\mathrm{curl}\,H|^2\Big]_{(\delta)}\,(\theta - \theta_0)^{(\lambda)}\,. \tag{76}$$

Using the selfadjointness of the operator G and the fact that $G(1) \equiv 0$ on Σ, we can write

$$\int_\Sigma G(\sigma\,\theta^4)\,(\theta - \theta_0)^{(\lambda)} = \int_\Sigma G(\sigma\,\theta^4)\,[(\theta - \theta_0)^{(\lambda)} + \min\{\theta_0,\,\lambda\}]\,.$$

We see that the function

$$F(s) := [(s - \theta_0)^{(\lambda)} + \min\{\theta_0,\,\lambda\}] \quad \text{for } s \in \mathbb{R}\,,$$

satisfies the assumptions of Lemma 5.6 below. Therefore, (76) leads to the inequality

$$\int_\Omega \kappa(\theta)\,|\nabla(\theta - \theta_0)^{(\lambda)}|^2 \le \int_\Omega \Big[\eta(\theta)\,D(v, v)\,\chi_{\Omega_1} + r(\theta)\,|\,\mathrm{curl}\,H|^2\Big]_{(\delta)}\,(\theta - \theta_0)^{(\lambda)}\,.$$

Using (1), we find that

$$\int_\Omega \kappa(\theta)\,|\nabla(\theta-\theta_0)^{(\lambda)}|^2 \le \lambda\Big(\int_{\Omega_1} \eta(\theta)\,D(v,v) + \int_\Omega r(\theta)\,|\,\mathrm{curl}\,H|^2\Big)$$
$$\le c\ (\|f(\theta)\|^2_{[L^2(\Omega_1)]^3} + \|\ j_0\ \|^2_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_0\|^2_{D^{1,2}(\Omega_1)})\,\lambda\,. \tag{77}$$

On the other hand, using the continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, we find that

$$\int_\Omega \kappa(\theta)\,|\nabla(\theta-\theta_0)^{(\lambda)}|^2 \ge c_0^{-2}\,\kappa_l\,\|(\theta-\theta_0)^{(\lambda)}\|^2_{L^6(\Omega)}\,.$$

This together with (77) obviously gives that

$$c_0^{-2}\,\kappa_l\,\lambda^2\ \mathrm{meas}\Big(\{x\in\Omega\ :\ |\theta-\theta_0|\ge\lambda\}\Big)^{1/3}$$
$$\le c\ (\|f(\theta)\|^2_{[L^2(\Omega_1)]^3} + \|\ j_0\ \|^2_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_0\|^2_{D^{1,2}(\Omega_1)})\,\lambda\,.$$

Therefore,

$$\sup_{\lambda>0}\Big\{\lambda\ \mathrm{meas}\Big(\{x\in\Omega_1\ :\ |\theta-\theta_0|\ge\lambda\}\Big)^{1/3}\Big\}$$
$$\le \frac{c\,c_0^2}{\kappa_l}\ (\|f(\theta_\delta)\|^2_{[L^2(\Omega_1)]^3} + \|\ j_0\ \|^2_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_0\|^2_{D^{1,2}(\Omega_1)})\,.$$

Now, we apply the embedding properties of the weak L^p-spaces (see Lemma 5.8) in order to obtain that

$$\|\theta-\theta_0\|_{L^2(\Omega_1)} \le \sqrt{2}\ \mathrm{meas}(\Omega_1)^{1/2}\sup_{\lambda>0}\Big\{\lambda\ \mathrm{meas}\Big(\{x\in\Omega_1\ :\ |\theta-\theta_0|\ge\lambda\}\Big)^{1/3}\Big\}$$
$$\le \frac{\bar c\ \mathrm{meas}(\Omega_1)^{1/2}}{\kappa_l}\,(\|f(\theta)\|^2_{[L^2(\Omega_1)]^3} + \|\ j_0\ \|^2_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_0\|^2_{D^{1,2}(\Omega_1)})\,. \tag{78}$$

On the other hand, we use the estimate (39), and can write

$$\|f(\theta)\|^2_{[L^2(\Omega_1)]^3} \le \rho_1^2\,|\vec g|^2\,M_t\int_{\Omega_1}\alpha\,|\theta-\theta_M| \le \rho_1^2\,|\vec g|^2\,M_t\,\alpha\ \mathrm{meas}(\Omega_1)^{1/2}\,\|\theta-\theta_M\|_{L^2(\Omega_1)}\,. \tag{79}$$

In view of (78), we then have

$$\Big(1-\frac{\bar c\ \mathrm{meas}(\Omega_1)\,\rho_1^2\,|\vec g|^2}{\kappa_l}\,M_t\,\alpha\Big)\ \|\theta-\theta_M\|_{L^2(\Omega_1)}$$
$$\le \frac{\bar c\ \mathrm{meas}(\Omega_1)^{1/2}\ (\|\ j_0\ \|^2_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_0\|^2_{D^{1,2}(\Omega_1)})}{\kappa_l}\,.$$

We recall that $\theta=\theta_\delta$. The claim follows, since the last estimate is preserved in the limit $\delta\to 0$. $\square$

Remark 2.2. Note that at the expense of technical complications, a slightly modified result holds if θ_0 is not a constant. Lemma 2.1 shows that the density variations in the fluid are controlled by the data in a weaker norm than the $L^\infty-$Norm. That is the reason why replacing (37) by (38) as in the first section is only partially justified. However, the proof of Lemma 2.1 shows a very simple way to deal with the linear growth condition (37) by means of a smallness assumption and a fixed-point procedure, as we will show in the remainder of this section.

Assuming that the hypotheses of Theorem 1.1 are satisfied, we prove the following result. It is easy to see that this result remains valid under the weaker hypothesis of Proposition 1.2.

Theorem 2.3. Let the assumptions of Theorem 1.1 or of Proposition 1.2 be satisfied, but let f be given by (37). If the coefficient α is sufficiently small with respect to the other data, the existence result of Theorem 1.1 holds true.

The rest of the section is devoted to the proof of this theorem. We will use the same notations as in Propostion 1.3. We additionally introduce the notation

$$J_n(\Omega_1) := \left\{ u \in [L^2(\Omega_1)]^3 \,\middle|\, \operatorname{div} u = 0 \text{ in } \Omega_1\,,\ u \cdot \vec{n} = 0 \text{ on } \partial\Omega_1 \right\},$$

where the constraints are intended in the sense of the generalized *div* operator.

Proposition 2.4. Let $\delta > 0$ be an arbitrary number. Suppose that the assumptions of Theorem 2.3 are satisfied. If $\{\tilde{v},\, \tilde{H},\, \tilde{\theta}\}$ is an arbitrary element of $J_n(\Omega_1) \times [L^2(\tilde{\Omega})]^3 \times L^2(\Omega)$, then there exists a unique triple

$$\{v,\, H,\, \theta\} \in D^{1,2}(\Omega_1) \times \mathcal{H}_\mu(\tilde{\Omega}) \times V^{2,5}(\Omega)\,,$$

such that $v = v_0$ on $\partial\Omega_1$, $\theta = \theta_0$ on Γ, $\operatorname{curl} H = j_0$ in $\tilde{\Omega}_{c_0}$, and

$$\int_{\Omega_1} \rho_1\, (\tilde{v} \cdot \nabla) v \cdot \phi + \int_{\Omega_1} \eta(\tilde{\theta})\, D(v\,,\, \phi) = \int_{\Omega_1} \left(\operatorname{curl} H \times [\mu\, \tilde{H}]_{(\delta)} \right) \cdot \phi + \int_{\Omega_1} f(\tilde{\theta}) \cdot \phi\,, \tag{80}$$

$$\int_{\tilde{\Omega}} r(\tilde{\theta})\, \operatorname{curl} H \cdot \operatorname{curl} \psi = \int_{\Omega_1} \left(v \times [\mu\, \tilde{H}]_{(\delta)} \right) \cdot\, \operatorname{curl} \psi\,, \tag{81}$$

$$\int_{\Omega_1} \rho_1\, c_V\, \tilde{v} \cdot \nabla\theta\, \xi + \int_{\Omega} \kappa(\tilde{\theta})\, \nabla\theta \cdot\, \nabla\xi + \int_{\Sigma} G(\sigma\, \theta^4)\, \xi$$
$$= \int_{\Omega} \left[r(\tilde{\theta})\, |\operatorname{curl} H|^2 + \eta(\tilde{\theta})\, D(v\,,\, v)\, \chi_{\Omega_1} \right]_{(\delta)} \xi\,, \tag{82}$$

are satisfied for all $\{\phi,\, \psi,\, \xi\} \in D^{1,2}_0(\Omega_1) \times \mathcal{H}^0_\mu(\tilde{\Omega}) \times V^{2,5}_\Gamma(\Omega)$. In addition, $\theta \geq \operatorname{ess\,inf}_{\Gamma} \theta_0$ almost everywhere in Ω.

Proof. Existence is a routine matter and is proved, for example, by the method of Proposition 1.3.

We prove the uniqueness. Suppose that both $\{v_1,\, H_1,\, \theta_1\}$ and $\{v_2,\, H_2,\, \theta_2\}$ satisfy the integral relations (80), (81) and (82). Then, in (80) written alternatively for v_1 and v_2, we test with $v_1 - v_2$ and subtract both results. We do the same in (81). We observe that

$$\int_{\Omega_1} \rho_1 \Big(\tilde{v} \cdot \nabla(v_1 - v_2)\Big) \cdot (v_1 - v_2) = 0\,.$$

We obtain the two relations

$$\int_{\Omega_1} \eta(\tilde{\theta})\, D(v_1 - v_2\,,\, v_1 - v_2) = \int_{\Omega_1} \big(\mathrm{curl}(H_1 - H_2) \times [\mu\, \tilde{H}]_{(\delta)}\big) \cdot (v_1 - v_2)\,,$$

$$\int_{\tilde{\Omega}} r(\tilde{\theta})\, |\, \mathrm{curl}(H_1 - H_2)|^2 = \int_{\Omega_1} \big((v_1 - v_2) \times [\mu\, \tilde{H}]_{(\delta)}\big) \cdot \mathrm{curl}(H_1 - H_2)\,,$$

which clearly imply, after addition, that $v_1 = v_2$ and $H_1 = H_2$. Now, for $\gamma > 0$, we use in (82) the test function $g_\gamma := \min\{(\theta_1 - \theta_2)^+\,,\, \gamma\}$, and observing that $\int_{\Omega_1} \rho_1\, c_V\, \tilde{v} \cdot \nabla(\theta_1 - \theta_2)\, g_\gamma = 0$, we obtain the relation

$$\int_{\Omega} \kappa(\tilde{\theta})\, \nabla(\theta_1 - \theta_2) \cdot \nabla g_\gamma + \int_{\Sigma} G\Big(\sigma\, \big[\theta_1^4 - \theta_2^4\big]\Big)\, g_\gamma = 0\,.$$

By the arguments of [LT01] (see also [Dru07b]), this leads to the uniqueness. □

Proposition 2.4 provides us with a well-defined, obviously compact mapping

$$\begin{aligned} T_\delta :\; J_n(\Omega_1) \times [L^2(\tilde{\Omega})]^3 \times L^2(\Omega) &\longrightarrow J_n(\Omega_1) \times [L^2(\tilde{\Omega})]^3 \times L^2(\Omega) \\ \{\tilde{v},\, \tilde{H},\, \tilde{\theta}\} &\longmapsto \{v,\, H,\, \theta\}\,. \end{aligned} \tag{83}$$

Next we show the

Lemma 2.5. If the coefficient α is sufficiently small with respect to the other data, the mapping T_δ given by (83) satisfies the assumptions of the Schauder fixed point principle. (In the simplified case of constant coefficients and boundary data, the smallness assumption on α is formulated more precisely in the equation (87) below.)

Proof. To prove the continuity of T_δ is, again, a routine matter. We have to consider an arbitrary sequence $\{\tilde{v}_k,\, \tilde{H}_k,\, \tilde{\theta}_k\}$ in $J_n(\Omega_1) \times [L^2(\tilde{\Omega})]^3 \times L^2(\Omega)$ such that

$$\{\tilde{v}_k,\, \tilde{H}_k,\, \tilde{\theta}_k\} \longrightarrow \{\tilde{v},\, \tilde{H},\, \tilde{\theta}\} \;\text{ in } J_n(\Omega_1) \times [L^2(\tilde{\Omega})]^3 \times L^2(\Omega)\,.$$

Choosing an arbitrary subsequence, that we not relabel, we will find by the compactness properties of T_δ a sub-subsequence such that $T_\delta\Big(\{\tilde{v}_k,\, \tilde{H}_k,\, \tilde{\theta}_k\}\Big) \longrightarrow w$ in $J_n(\Omega_1) \times [L^2(\tilde{\Omega})]^3 \times L^2(\Omega)$. By arguments similar to the proof of Proposition 1.3, that we do not want to repeat in detail, and the uniqueness obtained in Proposition 2.4, we show that $w = T_\delta\Big(\{\tilde{v},\, \tilde{H},\, \tilde{\theta}\}\Big)$. Then, strong convergence follows for the entire sequence.

We finally prove that T_δ maps some closed, bounded convex set into itself. In order to easier arrive at an estimate, we prove the claim in the simplified case that $v_0 = 0$, that

θ_0 is constant, and, all coefficients are piecewise constants. At the expense of technical complications, one verifies that the result is qualitatively preserved in the general case. Inserting v in (80) and H in (81), we obtain the estimate (cp. (1))

$$\int_{\Omega_1} \eta\, D(v\,,\, v) + \int_{\tilde\Omega} r\,|\,\mathrm{curl}\, H|^2 \leq \frac{L^2}{\eta}\, \|f(\tilde\theta)\|^2_{[L^2(\Omega_1)]^3} + \int_{\tilde\Omega} r\,|j_0|^2\,. \tag{84}$$

Arguing now as in the proof of Lemma 2.1, we verify that the solution $T_\delta\{\tilde v,\, \tilde H,\, \tilde\theta\}$ satisfies

$$\|\theta - \theta_0\|_{L^2(\Omega_1)} \leq \frac{\bar c\ \mathrm{meas}(\Omega_1)^{1/2}}{\kappa_l}\, (\|f(\tilde\theta)\|^2_{[L^2(\Omega_1)]^3} + \parallel j_0 \parallel^2_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_0\|^2_{D^{1,2}(\Omega_1)})\,. \tag{85}$$

To estimate $\|f(\tilde\theta)\|_{[L^2(\Omega_1)]^3}$ as in (79) is not possible anymore. Instead, we simply assume that $\tilde\theta - \tilde\theta_M \in \overline{B_X(0)}(\subset L^2(\Omega_1))$ for some $X > 0$, and we obtain that

$$\|\theta - \theta_M\|_{L^2(\Omega_1)} \leq \frac{\bar c\ \mathrm{meas}(\Omega_1)^{1/2}}{\kappa_l}\, (\rho_1^2\, |\vec g|^2\, \alpha^2\, X^2 + \parallel j_0 \parallel^2_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_0\|^2_{D^{1,2}(\Omega_1)})\,. \tag{86}$$

We introduce

$$a_1 := \frac{\bar c\ \mathrm{meas}(\Omega_1)^{1/2}}{\kappa_l}\, \rho_1^2\, |\vec g|^2\, \alpha^2, \quad a_0 := \frac{\bar c\ \mathrm{meas}(\Omega_1)^{1/2}}{\kappa_l}\, (\parallel j_0 \parallel^2_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_0\|^2_{D^{1,2}(\Omega_1)})\,.$$

Under the condition

$$1 - 4\, \frac{\bar c^2\ \mathrm{meas}(\Omega_1)}{\kappa_l^2}\, \rho_1^2\, |\vec g|^2\, \alpha^2\, (\parallel j_0 \parallel^2_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_0\|^2_{D^{1,2}(\Omega_1)}) > 0\,, \tag{87}$$

we see that the equation $X = a_1\, X^2 + a_0$ has the positive solution

$$X = \frac{2\, a_0}{1 + \sqrt{1 - 4\, a_0\, a_1}} \leq 2\, \frac{\bar c\ \mathrm{meas}(\Omega_1)^{1/2}}{\kappa_l}\, (\parallel j_0 \parallel^2_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_0\|^2_{D^{1,2}(\Omega_1)})\,. \tag{88}$$

We then define a closed convex set $M = M(X) \subset L^2(\Omega)$ by

$$M := \left\{\tilde\theta \in L^2(\Omega) \,\middle|\, \tilde\theta - \tilde\theta_M \in \overline{B_X(0)}(\subset L^2(\Omega_1))\right\}.$$

Note in view of (86) that $\tilde\theta \in M$ implies $\theta \in M$. In view of (84) and of the uniform estimates available for θ, we then easily find numbers Y_1, Y_2, Y_3 depending on X and on the data such that T_δ maps the closed, convex and bounded set

$$\overline{B_{Y_1}(0)} \times \overline{B_{Y_2}(0)} \times \overline{M \cap B_{Y_3}(\theta_0)}\ \subset\ J_n(\Omega_1) \times [L^2(\tilde\Omega)]^3 \times L^2(\Omega)\,,$$

into itself. $\square$

Now, we prove the main result of this section.

Proof of Theorem 2.3. By Proposition 2.4 and Lemma 2.5, the Schauder fixed point theorem gives the existence of a triple $\{v_\delta,\, H_\delta,\, \theta_\delta\} \in D^{1,2}(\Omega_1) \times \mathcal{H}_\mu(\tilde\Omega) \times V^{2,5}(\Omega)$ such that $v = v_0$ on $\partial\Omega_1$, $\theta = \theta_0$ on Γ, $\mathrm{curl} H = j_0$ in $\tilde\Omega_{c_0}$, and

$$\int_{\Omega_1} \rho_1\, (v_\delta \cdot \nabla) v_\delta \cdot \phi + \int_{\Omega_1} \eta(\theta_\delta)\, D(v_\delta\,,\, \phi) = \int_{\Omega_1} \big(\mathrm{curl} H_\delta \times [\mu\, H_\delta]_{(\delta)}\big) \cdot \phi + \int_{\Omega_1} f(\theta_\delta) \cdot \phi\,,$$

$$\int_{\tilde\Omega} r(\theta_\delta)\, \mathrm{curl}\, H_\delta \cdot \mathrm{curl}\, \psi = \int_{\Omega_1} \big(v_\delta \times [\mu\, H_\delta]_{(\delta)}\big) \cdot\, \mathrm{curl}\, \psi\,,$$

$$\int_{\Omega_1} \rho_1\, c_V\, v_\delta \cdot \nabla\theta_\delta\, \xi + \int_{\Omega} \kappa(\theta_\delta)\, \nabla\theta_\delta \cdot\, \nabla\xi + \int_{\Sigma} G(\sigma\, |\theta_\delta|^4)\, \xi$$

$$= \int_{\Omega} \Big[r(\theta_\delta)\, |\, \mathrm{curl}\, H_\delta|^2 + \eta(\theta_\delta)\, D(v_\delta\,,\, v_\delta)\, \chi_{\Omega_1}\Big]_{(\delta)}\, \xi\,.$$

We pass to the limit with the same strategy as in the first section. In order to obtain the strong convergence

$$v_\delta \longrightarrow v \;\text{ in } D^{1,2}(\Omega_1)\,, \quad H_\delta \longrightarrow H \;\text{ in } \mathcal{H}_\mu(\tilde\Omega)\,,$$

the form $f(\theta_\delta) \,=\, -\,\rho_1\, \vec{g}\, \alpha\, (\theta_\delta - \theta_{M,\delta})$ means no particular difficulty. In the limit, we prove the existence of a weak solution. In addition, we can control the L^2-norm of the density fluctuations by a continuous function of the data. In the simplified case that $v_0 = 0$ and that θ_0 is constant, we obtain in view of (88) that

$$\left(\frac{1}{\mathrm{meas}(\Omega_1)} \int_{\Omega_1} \alpha^2\, |\theta - \theta_M|^2\right)^{1/2} \leq \frac{2\, \bar c\, \alpha}{\kappa_l}\, (\|\, j_0\, \|^2_{[L^2(\tilde\Omega_{c_0})]^3} + \|v_0\|^2_{D^{1,2}(\Omega_1)})\,. \tag{89}$$

□

3 A uniqueness result for small data.

Throughout the section, we assume for simplicity that the temperature-dependent force term f in the Navier-Stokes equations has the form (38) and is bounded. We need to make the following technical assumption:

$$\begin{cases} \text{The coefficients } \eta,\, r,\, \kappa \text{ are piecewise constant.} \\ \text{There exists } 1 \leq \tilde p < 2 \text{ such that } \partial\Omega_i \text{ is of class } \mathcal{C}^{1,1/\tilde p} \text{ for } i = 0, \ldots, m\,. \end{cases} \tag{90}$$

We introduce additional notations. We define a Banach space X for the data $\{v_0,\, j_0,\, \theta_0\}$ of (P) by

$$X := D^{1,2}(\Omega_1) \cap L^\infty(\Omega_1) \times [L^2(\tilde\Omega_{c_0})]^3 \times W^{1,2}(\Omega) \cap L^\infty(\Omega)\,,$$

and for the weak solutions $\{v,\, H,\, \theta\}$ of (P) a Banach space Y by

$$Y := D^{1,2}(\Omega_1) \times \mathcal{H}_\mu(\tilde\Omega) \times V^{p,4}(\Omega)\,,$$

for some $1 < p < 3/2$. For each given triple of data $\{v_0,\, j_0,\, \theta_0\} \in X$, we can define a subset S of the space Y by

$$S\big(\{v_0,\, j_0,\, \theta_0\}\big) := \Big\{\{v,\, H,\, \theta\} \in Y \,\Big|\, \{v,\, H,\, \theta\} \text{ solves } (P) \text{ for the data } \{v_0,\, j_0,\, \theta_0\}\Big\}\,.$$

Further, we set

$$\hat{K}_0 = \hat{K}_0\big(\{v_0,\, j_0,\, \theta_0\}\big) := \|j_0\|_{[L^2(\tilde{\Omega}_{c_0})]^3} + \|v_0\|_{D^{1,2}(\Omega_1)} + \|\nabla\theta_0\|_{[L^2(\Omega)]^3}\,, \tag{91}$$

which represents a norm of the data of the problem (P). For numbers $\theta_{\max} > 0$, we introduce the set

$$X_{\theta_{\max}} := \Big\{\{v_0,\, j_0,\, \theta_0\} \in X \,\Big|\, \|\theta_0\|_{L^\infty(\Omega)} \le \theta_{\max}\Big\}\,.$$

We denote by c_0 the constant of the continuous embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$.

Theorem 3.1. Assume that the assumption (90) is satisfied, and let $\theta_{\max} > 0$ be arbitrary, but fixed. If the number M_t in (38) satisfies

$$M_t < \frac{\eta^2\,\kappa_l}{\bar{c}\,L^2\,(1 + \operatorname{diam}(\Omega))^2\,\operatorname{meas}(\Omega_1)\,|\vec{g}|^2\,\rho_1^2\,\alpha}\,, \tag{92}$$

with $\bar{c} = \sqrt{2}\,c_{\text{Korn}}\,c_0^2$ then, there exists $\epsilon > 0$ such that for all $\{v_0,\, j_0,\, \theta_0\} \in X_{\theta_{\max}}$ that satisfy $\hat{K}_0 \le \epsilon$, the set $S\big(\{v_0,\, j_0,\, \theta_0\}\big) \subset Y$ consists of at most one element.

Remark 3.2. Whenever the stationary Navier-Stokes system is involved, the uniqueness of weak solutions can be proved only for *small external forces* (see for example [Tem77], Ch. II, Paragraphs 1 and 4). In the case of the coupled model presently under study, Theorem 3.1 shows that the uniqueness issue is related to two additional parameters: the importance of the *temperature fluctuations* in the fluid, measured by the number M_t, and the maximal imposed temperature $\theta_{\max}$.

The proof of Theorem 3.1 is based on several auxiliary results.

Lemma 3.3. Let θ_0 be a positive constant. Assume that $\{v,\, H,\, \theta\} \in S\big(\{0,\, 0,\, \theta_0\}\big)$.
If the assumptions of Theorem 3.1 are satisfied, then $\{v,\, H\} = 0$ and $\theta \equiv \theta_0$.

Proof. We test the integral relations (33) and (34) respectively with v and H, and obtain, after addition, the energy equality

$$\int_{\Omega_1} \eta\, D(v, v) + \int_{\tilde{\Omega}} r\,|\operatorname{curl} H|^2 = \int_{\Omega_1} f(\theta)\cdot v\,.$$

Using standard inequalities, we derive the estimate

$$\int_{\Omega_1} \eta\, D(v, v) + \int_{\tilde{\Omega}} r\,|\operatorname{curl} H|^2 \le c_{\text{Korn}}\,\frac{L^2}{\eta^2}\,\|f(\theta)\|^2_{[L^2(\Omega_1)]^3}\,. \tag{93}$$

For a parameter $\lambda > 0$ and $s \in \mathbb{R}$, we introduce the function $s^{(\lambda)} := \mathrm{sign}(s)\, \min\{|s|, \lambda\}$. Denoting by Ψ a primitive of the function $s \mapsto (s)^{(\lambda)}$, we observe that

$$\int_{\Omega_1} \rho_1\, c_V\, v \cdot \nabla\theta\, (\theta - \theta_0)^{(\lambda)} = \int_{\Omega_1} \rho_1\, c_V\, v \cdot \nabla\Psi(\theta - \theta_0) = 0\,.$$

In view of Lemma 5.6, we have also $\int_\Sigma G(\sigma\, \theta^4)\, (\theta - \theta_0)^{(\lambda)} \geq 0$. Using (93), we obtain from (35) and Lemma 5.12 the inequality

$$\begin{aligned}\int_\Omega \kappa\, |\nabla(\theta - \theta_0)^{(\lambda)}|^2 &\leq \int_{\Omega_1} \eta\, D(v, v)\, (\theta - \theta_0)^{(\lambda)} + \int_\Omega r\, |\operatorname{curl} H|^2\, (\theta - \theta_0)^{(\lambda)} \\ &\leq \lambda\, c_{\mathrm{Korn}}\, \frac{L^2}{\eta^2}\, \|f(\theta)\|^2_{[L^2(\Omega_1)]^3}\,. \end{aligned} \tag{94}$$

Now, we can write

$$\begin{aligned}\int_\Omega \kappa\, |\nabla(\theta - \theta_0)^{(\lambda)}|^2 &\geq \frac{\kappa_l}{(1 + \mathrm{diam}(\Omega))^2}\, \|(\theta - \theta_0)^{(\lambda)}\|^2_{W^{1,2}(\Omega)} \\ &\geq \frac{\kappa_l}{c_0^2\, (1 + \mathrm{diam}(\Omega))^2}\, \|(\theta - \theta_0)^{(\lambda)}\|^2_{L^6(\Omega)} \\ &\geq \frac{\kappa_l}{c_0^2\, (1 + \mathrm{diam}(\Omega))^2}\, \|(\theta - \theta_0)^{(\lambda)}\|^2_{L^6(\Omega_1)} \\ &\geq \frac{\kappa_l}{c_0^2\, (1 + \mathrm{diam}(\Omega))^2}\, \lambda^2\ \mathrm{meas}(\{x \in \Omega_1\ :\ |\theta - \theta_0| > \lambda\})^{\frac{1}{3}}\,.\end{aligned}$$

From (94), it now follows that for all $\lambda > 0$,

$$\frac{\kappa_l}{c_0^2\, (1 + \mathrm{diam}(\Omega))^2}\, \lambda\ \mathrm{meas}(\{x \in \Omega_1\ :\ |\theta - \theta_0| > \lambda\})^{\frac{1}{3}} \leq c_{\mathrm{Korn}}\, \frac{L^2}{\eta^2}\, \|f(\theta)\|^2_{[L^2(\Omega_1)]^3}\,.$$

By Lemma 5.8, we find that

$$\|\theta - \theta_0\|_{L^2(\Omega_1)} \leq \sqrt{2}\, c_{\mathrm{Korn}}\, c_0^2\, (1 + \mathrm{diam}(\Omega))^2\, \frac{L^2\ \mathrm{meas}(\Omega_1)^{1/2}}{\eta^2\, \kappa_l}\, \|f(\theta)\|^2_{[L^2(\Omega_1)]^3}\,.$$

Now, in view of (38), we have, on the one hand $\|f(\theta)\|_{[L^2(\Omega_1)]^3} \leq |\vec{g}|\, \rho_1\, M_t\ \mathrm{meas}(\Omega_1)^{1/2}$, and on the other hand $\|f(\theta)\|_{[L^2(\Omega_1)]^3} \leq |\vec{g}|\, \rho_1\, \alpha\, \|\theta - \theta_M\|_{L^2(\Omega_1)}$. We obtain

$$\|\theta - \theta_M\|_{L^2(\Omega_1)} \leq \sqrt{2}\, c_{\mathrm{Korn}}\, c_0^2\, (1 + \mathrm{diam}(\Omega))^2\, \frac{L^2\ \mathrm{meas}(\Omega_1)\, M_t\, |\vec{g}|^2\, \rho_1^2\, \alpha}{\eta^2\, \kappa_1}\, \|\theta - \theta_M\|_{L^2(\Omega_1)}\,.$$

The claim follows. ☐

Lemma 3.4. For $n \in \mathbb{N}$, consider an arbitrary sequence $\{j_{0,n},\, v_{0,n},\, \theta_{0,n}\} \subset X_{\theta_{\max}}$, and assume that $\hat{K}_{0,n} := \hat{K}_0(\{j_{0,n},\, v_{0,n},\, \theta_{0,n}\}) \to 0$ for $n \to \infty$. Let $\theta_0 \leq \theta_{\max}$ be the positive constant such that $\theta_{0,n} \to \theta_0$ in $W^{1,2}(\Omega)$, and assume in addition that the hypothesis of Theorem 3.1 are satisfied.

Then, for every $\{v_n,\, H_n,\, \theta_n\} \in S\big(\{j_{0,n},\, v_{0,n},\, \theta_{0,n}\}\big)$, and for every $1 \le p < 3/2$, one has

$$v_n \longrightarrow 0 \text{ in } D^{1,2}(\Omega_1)\,, \quad H_n \longrightarrow 0 \text{ in } \mathcal{H}_\mu(\tilde{\Omega})\,,$$
$$\theta_n \longrightarrow \theta_0 \text{ in } W^{1,p}(\Omega)\,, \quad \theta_n^4 \longrightarrow \theta_0^4 \text{ in } L^1(\Sigma)\,.$$

Proof. Note first that every test function used in the proof of Proposition 1.4 for estimating θ has the form $g(\theta)$ with a continuous, bounded and increasing function g such that $g(0) = 0$. Thanks to Lemma 5.12, we can estimate θ_n in exactly the same way. Applying the techniques of Proposition 1.5, we then obtain the existence of $\{v,\, H,\, \theta\} \in S\big(\{0,\, 0,\, \theta_0\}\big)$ such that

$$v_n \longrightarrow v \text{ in } D^{1,2}(\Omega_1)\,, \quad H_n \longrightarrow H \text{ in } \mathcal{H}_\mu(\tilde{\Omega})\,, \quad \theta_n^4 \longrightarrow \theta^4 \text{ in } L^1(\Sigma)\,, \tag{95}$$

and for $1 \le p < 3/2$,

$$\theta_n \rightharpoonup \theta \text{ in } W^{1,p}(\Omega)\,. \tag{96}$$

Because of Lemma 3.3, we can verify that $\{v,\, H,\, \theta\} = \{0,\, 0,\, \theta_0\}$. Therefore, in view of (95), only the strong convergence

$$\theta_n \longrightarrow \theta_0 \text{ in } W^{1,p}(\Omega)\,, \tag{97}$$

remains to be proved. For some $\gamma \in]0, 1[$, we test the relation (35) with the function

$$\xi_n := \operatorname{sign}(\theta_n - \theta_{0,n}) \left(1 - \frac{1}{(1 + |\theta_n - \theta_{0,n}|)^\gamma}\right)\,.$$

By the result (95), (96), we easily verify that

$$\lim_{n\to\infty} \int_\Omega \kappa\, \gamma\, \frac{|\nabla \theta_n|^2}{(1 + |\theta_n - \theta_{0,n}|)^{1+\gamma}} = 0\,.$$

Therefore, we can find a subsequence such that

$$\frac{|\nabla \theta_n|^2}{(1 + |\theta_n - \theta_{0,n}|)^{1+\gamma}} \longrightarrow 0 \quad \text{almost everywhere in } \Omega\,.$$

Since $(1 + |\theta_n - \theta_{0,n}|)^{1+\gamma} \longrightarrow 1$ almost everywhere in Ω, it follows that $|\nabla \theta_n| \longrightarrow 0$ almost everywhere in Ω. In view of (96) the sequence $\{\nabla \theta_n\}$ is bounded in the space $[L^p(\Omega)]^3$ for $1 \le p < 3/2$. Thus, we obtain for all $1 \le q < p < 3/2$ the strong convergence $\|\nabla \theta_n\|_{[L^q(\Omega)]^3} \longrightarrow 0$. Finally, we observe that the considerations of the present proof can be applied to each subsequence of the sequence $\{v_n,\, H_n,\, \theta_n\}$. This proves the claim. ☐

Corollary 3.5. Under the assumptions of Theorem 3.1, it holds that

$$\lim_{\gamma\to 0} \sup_{\substack{\{v_0, j_0, \theta_0\} \in X_{\theta_{\max}} \\ \hat{K}_0(\{v_0, j_0, \theta_0\}) \le \gamma}} \sup_{\{v, H, \theta\} \in S\left(\{v_0, j_0, \theta_0\}\right)} \|\{v,\, H,\, \theta - \theta_0\}\|_Y = 0\,.$$

Proof. This is only a reformulation of the statement of Lemma 3.4. ☐

We now want to prove the main result of this section. Because of the numerous estimates involved, we split the proof into five steps.

Proof of Theorem 3.1. For some data $\{v_0,\, j_0,\, \theta_0\} \in X_{\theta_{\max}}$, assume that $\{v_1,\, H_1,\, \theta_1\}$ and $\{v_2,\, H_2,\, \theta_2\}$ both belong to the set $S\big(\{v_0,\, j_0,\, \theta_0\}\big)$.

We define a number $S_0 > 0$ by

$$S_0 := \|\{v_1,\, H_1,\, \theta_1 - \theta_0\}\|_Y + \|\{v_2,\, H_2,\, \theta_2 - \theta_0\}\|_Y\,. \tag{98}$$

In view of Corollary 3.5, we have $\lim_{\hat{K}_0 \to 0} S_0 = 0$.

First step: estimates on $\mathbf{v_1} - \mathbf{v_2}$ and $\mathbf{H_1} - \mathbf{H_2}$.

Using the test functions $v_1 - v_2$ and $H_1 - H_2$ in the integral identities (33) and (34) written respectively for v_1, H_1 and v_2, H_2, we find

$$\int_{\Omega_1} \eta\, D(v_1 - v_2\,,\, v_1 - v_2) = -\int_{\Omega_1} \rho_1 \Big((v_1 \cdot \nabla) v_1 - (v_2 \cdot \nabla) v_2 \Big) \cdot (v_1 - v_2)$$
$$+ \int_{\Omega_1} \Big(\big(\operatorname{curl} H_1 \times \mu\, H_1\big) - \big(\operatorname{curl} H_2 \times \mu\, H_2\big) \Big) \cdot (v_1 - v_2) + \int_{\Omega_1} [f(\theta_1) - f(\theta_2)] \cdot (v_1 - v_2)\,,$$

and

$$\int_{\tilde{\Omega}} r\, |\operatorname{curl}(H_1 - H_2)|^2 = \int_{\Omega_1} \Big(\big(v_1 \times \mu\, H_1\big) - \big(v_2 \times \mu\, H_2\big) \Big) \cdot \operatorname{curl}(H_1 - H_2)\,.$$

We add both relations, and by straightforward rearrangements of terms, we get the estimate

$$\int_{\Omega_1} \eta\, D(v_1 - v_2\,,\, v_1 - v_2) + \int_{\tilde{\Omega}} r\, |\operatorname{curl}(H_1 - H_2)|^2 \le \rho_1 \int_{\Omega_1} |\nabla v_2|\, |v_1 - v_2|^2$$
$$+ 2\, \mu_u \int_{\Omega_1} |H_1 - H_2| \Big(|\operatorname{curl} H_2|\, |v_1 - v_2| + |v_2|\, |\operatorname{curl}(H_1 - H_2)| \Big)$$
$$+ \int_{\Omega_1} |f(\theta_1) - f(\theta_2)|\, |v_1 - v_2|\,.$$

We denote by c_0 the constant of the continuous embedding $W^{1,2}(\Omega_1) \hookrightarrow L^4(\Omega_1)$. Applying Lemma 5.1, we obtain the estimate

$$\Big(\frac{\eta}{2} - \rho_1\, c_0^2\, \|\nabla v_2\|_{[L^2(\Omega_1)]^9} \Big) \int_{\Omega_1} D(v_1 - v_2\,,\, v_1 - v_2)$$
$$+ \left(r_l - c_0^2\, c_{\mathcal{H}} \left[\rho_1\, \mu_u\, \|\nabla v_2\|_{[L^2(\Omega_1)]^9} - \frac{\mu_u^2}{\eta}\, \|\operatorname{curl} H_2\|^2_{[L^2(\tilde{\Omega})]^3} \right] \right) \int_{\tilde{\Omega}} |\operatorname{curl}(H_1 - H_2)|^2$$
$$\le \frac{L^2}{\eta}\, \|f(\theta_1) - f(\theta_2)\|^2_{[L^2(\Omega_1)]^3}\,.$$

Using Corollary 3.5, we see that we can choose the number $\hat{K}_0$ so small as to achieve that

$$\left(\frac{1}{2} - \frac{\rho_1\, c_0^2\, S_0}{\eta}\right) > 0\,, \qquad \left(1 - c_0^2\, c_{\mathcal{H}}\, \frac{1}{r_l}\, \left[\rho_1\, \mu_u\, S_0 - \frac{\mu_u^2}{\eta}\, S_0^2\right]\right) > 0\,.$$

Setting

$$\beta := \max\left\{ \left(\frac{1}{2} - \frac{\rho_1\, c_0^2\, S_0}{\eta}\right)^{-1}\,,\, \left(1 - c_0^2\, c_{\mathcal{H}}\, \frac{1}{r_l}\, \left[\rho_1\, \mu_u\, S_0 - \frac{\mu_u^2}{\eta}\, S_0^2\right]\right)^{-1} \right\}\,,$$

we can write

$$\int_{\Omega_1} \eta\, D(v_1 - v_2\,,\, v_1 - v_2) + \int_{\tilde{\Omega}} r\, |\operatorname{curl}(H_1 - H_2)|^2 \leq \frac{L^2\,\beta}{\eta}\, \|f(\theta_1) - f(\theta_2)\|^2_{[L^2(\Omega_1)]^3}\,. \tag{99}$$

Second step: estimates of volume integrals involving $\theta_1 - \theta_2$.

For $\lambda > 0$ arbitrary, we now consider the test function

$$\xi_\lambda := (\theta_1 - \theta_2)^{(\lambda)} := \operatorname{sign}(\theta_1 - \theta_2)\, \min\{|\theta_1 - \theta_2|\,,\, \lambda\}\,.$$

We substract the integral identities (35), written respectively for θ_1 and θ_2. In view of Lemma 5.12, we can test the resulting relation with ξ_λ, and obtain

$$\begin{aligned}
&\int_{\Omega_1} \rho_1\, c_V\, \Big(v_1 \cdot \nabla\theta_1 - v_2 \cdot \nabla\theta_2\Big)\, (\theta_1 - \theta_2)^{(\lambda)} + \int_{\Omega} \kappa\, |\nabla(\theta_1 - \theta_2)^{(\lambda)}|^2 \\
&+ \int_{\Sigma} G\big(\sigma\, [\theta_1^4 - \theta_2^4]\big)\, (\theta_1 - \theta_2)^{(\lambda)} \leq \int_{\Omega} r\, \big(|\operatorname{curl} H_1|^2 - |\operatorname{curl} H_2|^2\big)\, (\theta_1 - \theta_2)^{(\lambda)} \\
&\quad + \int_{\Omega_1} \eta\, \big(D(v_1\,,\, v_1) - D(v_2\,,\, v_2)\big)\, (\theta_1 - \theta_2)^{(\lambda)}\,.
\end{aligned} \tag{100}$$

We want to estimate each term appearing in this relation. Since by the usual arguments

$$\int_{\Omega_1} \rho_1\, c_V\, v_1 \cdot \nabla(\theta_1 - \theta_2)\, (\theta_1 - \theta_2)^{(\lambda)} = 0\,,$$

we have

$$\begin{aligned}
\left|\int_{\Omega_1} \rho_1\, c_V\, \Big(v_1 \cdot \nabla\theta_1 - v_2 \cdot \nabla\theta_2\Big)\, (\theta_1 - \theta_2)^{(\lambda)}\right| &= \left|\int_{\Omega_1} \rho_1\, c_V\, (v_1 - v_2) \cdot \nabla\theta_2\Big)\, (\theta_1 - \theta_2)^{(\lambda)}\right| \\
&\leq \rho_1\, c_V\, \|v_1 - v_2\|_{[L^4(\Omega_1)]^3}\, \|\nabla\theta_2\|_{L^{4/3}(\Omega_1)}\, \lambda\,.
\end{aligned} \tag{101}$$

By the triangle inequality, we can write $\|\nabla\theta_2\|_{L^{4/3}(\Omega_1)} \leq \|\nabla(\theta_2 - \theta_0)\|_{L^{4/3}(\Omega_1)} + \|\nabla\theta_0\|_{L^{4/3}(\Omega_1)} \leq S_0 + \hat{K}_0$. In view of (101) and (99), we then can write

$$\begin{aligned}
&\left|\int_{\Omega_1} \rho_1\, c_V\, \Big(v_1 \cdot \nabla\theta_1 - v_2 \cdot \nabla\theta_2\Big)\, (\theta_1 - \theta_2)^{(\lambda)}\right| \\
&\leq c_0\, (1 + \operatorname{diam}(\Omega_1))\, c_{\text{Korn}}\, \frac{\rho_1\, c_V\, L\, \sqrt{\beta}}{\eta}\, (S_0 + \hat{K}_0)\, \|f(\theta_1) - f(\theta_2)\|_{[L^2(\Omega_1)]^3}\, \lambda\,.
\end{aligned} \tag{102}$$

Turning our attention to the term on the right-hand side of (100), we have in view of (99) that

$$\begin{aligned}
&\left| \int_\Omega r \left(|\operatorname{curl} H_1|^2 - |\operatorname{curl} H_2|^2 \right) (\theta_1 - \theta_2)^{(\lambda)} + \int_{\Omega_1} \eta \left(D(v_1\,,\, v_1) - D(v_2\,,\, v_2) \right) (\theta_1 - \theta_2)^{(\lambda)} \right| \\
&\quad \leq \left\| r^{1/2} \operatorname{curl}(H_1 + H_2) \right\|_{[L^2(\tilde{\Omega})]^3} \left\| r^{1/2} \operatorname{curl}(H_1 - H_2) \right\|_{[L^2(\tilde{\Omega})]^3} \lambda \\
&\qquad + \left\| \eta^{1/2} D(v_1 + v_2, v_1 + v_2) \right\|_{[L^2(\Omega_1)]^9} \left\| \eta^{1/2} D(v_1 - v_2, v_1 - v_2) \right\|_{[L^2(\Omega_1)]^9} \lambda \\
&\quad \leq \sqrt{r_u + \eta_u}\, S_0 \, \frac{L\sqrt{\beta}}{\sqrt{\eta}} \, \| f(\theta_1) - f(\theta_2) \|_{[L^2(\Omega_1)]^3} \, \lambda \,. \qquad (103)
\end{aligned}$$

Third step: estimates of surface integrals involving $\theta_1 - \theta_2$.

Using the fact that G is selfadjoint, we first write

$$\begin{aligned}
\int_\Sigma G\big(\sigma\, [\theta_1^4 - \theta_2^4]\big)\, (\theta_1 - \theta_2)^{(\lambda)} &= \int_\Sigma \sigma\, [\theta_1^4 - \theta_2^4]\, G\big((\theta_1 - \theta_2)^{(\lambda)}\big) \\
&= \int_\Sigma \sigma\, (\theta_1^2 + \theta_2^2)\, (\theta_1 + \theta_2)\, (\theta_1 - \theta_2)\, G\big((\theta_1 - \theta_2)^{(\lambda)}\big)\,.
\end{aligned}$$

With the abbreviations $F(\theta_1, \theta_2) := (\theta_1^2 + \theta_2^2)\, (\theta_1 + \theta_2)$ and $F_0 = F(\theta_0, \theta_0)$, we can also write

$$\int_\Sigma G\big(\sigma\, [\theta_1^4 - \theta_2^4]\big)\, (\theta_1 - \theta_2)^{(\lambda)} = \int_\Sigma \sigma\, F(\theta_1, \theta_2)\, (\theta_1 - \theta_2)\, G\big((\theta_1 - \theta_2)^{(\lambda)}\big)\,.$$

Using the decomposition $G = I - \mathbf{H}$, it is easy to prove the inequality

$$\begin{aligned}
&\int_\Sigma \sigma\, F(\theta_1, \theta_2)\, (\theta_1 - \theta_2)\, G\big((\theta_1 - \theta_2)^{(\lambda)}\big) \geq \int_\Sigma \sigma\, F(\theta_1, \theta_2)\, (\theta_1 - \theta_2)^{(\lambda)}\, G\big((\theta_1 - \theta_2)^{(\lambda)}\big) \\
&= \int_\Sigma \sigma\, \big[F(\theta_1, \theta_2) - F_0\big]\, (\theta_1 - \theta_2)^{(\lambda)}\, G\big((\theta_1 - \theta_2)^{(\lambda)}\big) + \int_\Sigma \sigma\, F_0\, (\theta_1 - \theta_2)^{(\lambda)}\, G\big((\theta_1 - \theta_2)^{(\lambda)}\big)\,. \qquad (104)
\end{aligned}$$

We want to estimate from below the right-hand side of (104). By Lemma 5.7, we have

$$\begin{aligned}
\int_\Sigma \sigma\, \big[F(\theta_1, \theta_2) - F_0\big]\, (\theta_1 - \theta_2)^{(\lambda)}\, G\big((\theta_1 - \theta_2)^{(\lambda)}\big) &= \int_\Sigma \epsilon\, \sigma\, \big[F(\theta_1, \theta_2) - F_0\big]\, |(\theta_1 - \theta_2)^{(\lambda)}|^2 \\
&\quad - \int_\Sigma \epsilon\, \sigma\, \big[F(\theta_1, \theta_2) - F_0\big]\, (\theta_1 - \theta_2)^{(\lambda)}\, \tilde{\mathbf{H}}\big((\theta_1 - \theta_2)^{(\lambda)}\big)\,. \qquad (105)
\end{aligned}$$

We introduce a measurable set $A_0 \subseteq \Sigma$ defined by

$$A_0 := \{ z \in \Sigma \,:\, F(\theta_1(z), \theta_2(z)) - F_0(z) \geq 0 \}\,.$$

We can decompose

$$
\begin{aligned}
&\int_\Sigma \epsilon\,\sigma\,\big[F(\theta_1,\theta_2)-F_0\big]\,|(\theta_1-\theta_2)^{(\lambda)}|^2 \\
&\quad = \int_{A_0} \epsilon\,\sigma\,\big[F(\theta_1,\theta_2)-F_0\big]\,|(\theta_1-\theta_2)^{(\lambda)}|^2 + \int_{A_0^c} \epsilon\,\sigma\,\big[F(\theta_1,\theta_2)-F_0\big]\,|(\theta_1-\theta_2)^{(\lambda)}|^2 \,.
\end{aligned}
$$

Observing that the term to the left is nonnegative, we do not need to consider it anymore in the following estimates. For all $1<q<4$, we choose q' such that $1/q+1/q'=1$, and we can estimate

$$
\begin{aligned}
&\left|\int_{A_0^c} \epsilon\,\sigma\,\big[F(\theta_1,\theta_2)-F_0\big]\,|(\theta_1-\theta_2)^{(\lambda)}|^2\right| \le \sigma\,\lambda\,\|\theta_1-\theta_2\|_{L^q(\Sigma)}\,\left(\int_{A_0^c}|F(\theta_1,\theta_2)-F_0|^{q'}\right)^{\frac{1}{q'}} \\
&\le \sigma\,\lambda\,\|\theta_1-\theta_2\|_{L^q(\Sigma)}\,(2\,F_0)^{\frac{q'-4/3}{q'}}\,\left(\int_{A_0^c}|F(\theta_1,\theta_2)-F_0|^{4/3}\right)^{\frac{1}{q'}} \\
&\le \sigma\,\lambda\,\|\theta_1-\theta_2\|_{L^q(\Sigma)}\,[2\,F(\theta_{\max},\,\theta_{\max})]^{\frac{q'-4/3}{q'}}\,\|F(\theta_1,\theta_2)-F_0\|_{L^{4/3}(\Sigma)}^{\frac{4}{3\,q'}}\,. \qquad (106)
\end{aligned}
$$

For the second term in (105), we have the estimate

$$
\begin{aligned}
&\left|\int_\Sigma \epsilon\,\sigma\,\big[F(\theta_1,\theta_2)-F_0\big]\,(\theta_1-\theta_2)^{(\lambda)}\,\tilde{\mathbf{H}}\big((\theta_1-\theta_2)^{(\lambda)}\big)\right| \\
&\quad \le \sigma\,\lambda\,\|\epsilon^{3/4}\,(F(\theta_1,\theta_2)-F_0)\|_{L^{4/3}(\Sigma)}\,\|\epsilon^{1/4}\,\tilde{\mathbf{H}}\big((\theta_1-\theta_2)^{(\lambda)}\big)\|_{L^4(\Sigma)}\,. \qquad (107)
\end{aligned}
$$

Observe that according to Lemma 5.7, (2), the operator $\tilde{H}$ belongs to $\mathcal{L}(L^{\tilde{p}}(\Sigma),C(\Sigma))$ for all $\tilde{p}<2$. In view of the assumption (90), we therefore have

$$
\|\epsilon^{1/4}\,\tilde{\mathbf{H}}\big((\theta_1-\theta_2)^{(\lambda)}\big)\|_{L^4(\Sigma)} \le \operatorname{meas}(\Sigma)^{\frac{1}{4}}\,\|\tilde{\mathbf{H}}\big((\theta_1-\theta_2)^{(\lambda)}\big)\|_{C(\Sigma)} \le c\,\operatorname{meas}(\Sigma)^{\frac{1}{4}}\,\|\theta_1-\theta_2\|_{L^{\tilde{p}}(\Sigma)}\,. \qquad (108)
$$

From the results (106) with the choice $q=\tilde{p}$, (107), (108), we conclude that

$$
\int_\Sigma \sigma\,\big[F(\theta_1,\theta_2)-F_0\big]\,(\theta_1-\theta_2)^{(\lambda)}\,G\big((\theta_1-\theta_2)^{(\lambda)}\big) \ge -f(\hat{K}_0)\,\|\theta_1-\theta_2\|_{L^{\tilde{p}}(\Sigma)}\,\lambda\,,
$$

with a positive number $f(\hat{K}_0)$ that can be estimated as follows:

$$
f(\hat{K}_0) \le \tilde{c}\,(\|F(\theta_1,\theta_2)-F_0\|_{L^{4/3}(\Sigma)}^{\frac{4}{3\,\tilde{p}'}} + \|(F(\theta_1,\theta_2)-F_0)\|_{L^{4/3}(\Sigma)})\,.
$$

Observe that Corollary 3.5 implies that $f(\hat{K}_0)\to 0$ as $\hat{K}_0$ converges to zero.

The second term on the right-hand side of (104) can be estimated in quite similar matter, so that we finally obtain

$$
\int_\Sigma G\big(\sigma\,[\theta_1^4-\theta_2^4]\big)\,(\theta_1-\theta_2)^{(\lambda)} \ge -\tilde{f}(\hat{K}_0)\,\lambda\,\|\theta_1-\theta_2\|_{L^{\tilde{p}}(\Sigma)}\,, \qquad (109)
$$

with $\tilde{p} < 2$ and a sequence of numbers $\tilde{f}(\hat{K}_0)$ which converges to zero together with $\hat{K}_0$.

Fourth Step: final estimate

The results (102), (103) and (109) give the inequality

$$\int_\Omega \kappa\, |\nabla(\theta_1 - \theta_2)^{(\lambda)}|^2 \leq c_1\, \bar{f}(\hat{K}_0)\, (\|f(\theta_1) - f(\theta_2)\|_{[L^2(\Omega_1)]^3} + \|\theta_1 - \theta_2\|_{L^{\tilde{p}}(\Sigma)})\, \lambda\,. \tag{110}$$

By the inequality (110), we obtain that

$$\big\|(\theta_1 - \theta_2)^{(\lambda)}\big\|^2_{W^{1,2}_\Gamma(\Omega)} \leq \bar{C}\ \bar{f}(\hat{K}_0)\, (\|f(\theta_1) - f(\theta_2)\|_{[L^2(\Omega_1)]^3} + \|\theta_1 - \theta_2\|_{L^{\tilde{p}}(\Sigma)})\, \lambda\,. \tag{111}$$

Using the continuity of the embeddings $W^{1,2}_\Gamma(\Omega) \hookrightarrow L^6(\Omega)$ and $W^{1,2}_\Gamma(\Omega) \hookrightarrow L^4(\Sigma)$, it follows that

$$\begin{aligned}
\lambda^2 \operatorname{meas}\Big(\{x \in \Omega\,:\,|\theta_1(x) - \theta_2(x)| > \lambda\}\Big)^{1/3} &+ \lambda^2 \operatorname{meas}\Big(\{z \in \Sigma\,:\,|\theta_1(z) - \theta_2(z)| > \lambda\}\Big)^{1/2} \\
&\leq \big\|(\theta_1 - \theta_2)^{(\lambda)}\big\|^2_{L^6(\Omega)} + \big\|(\theta_1 - \theta_2)^{(\lambda)}\big\|^2_{L^4(\Sigma)} \\
&\leq c\,\bar{C}\,\bar{f}(\hat{K}_0)\, (\|f(\theta_1) - f(\theta_2)\|_{[L^2(\Omega_1)]^3} + \|\theta_1 - \theta_2\|_{L^{\tilde{p}}(\Sigma)})\, \lambda\,.
\end{aligned}$$

On the other hand, we can use the the result of Lemma 5.8 to obtain that

$$\begin{aligned}
\big\|\theta_1 - \theta_2\big\|_{L^2(\Omega)} + \|\theta_1 - \theta_2\|_{L^{\tilde{p}}(\Sigma)}) \leq c\,\Big(&\sup_{\lambda>0}\Big\{\lambda \operatorname{meas}\Big(\{x \in \Omega\,:\,|\theta_1(x) - \theta_2(x)| > \lambda\}\Big)^{1/3}\Big\} \\
&+ \sup_{\lambda>0}\Big\{\lambda \operatorname{meas}\Big(\{z \in \Sigma\,:\,|\theta_1(z) - \theta_2(z)| > \lambda\}\Big)^{1/2}\Big\}\Big)\,.
\end{aligned}$$

The estimate (111) finally yields

$$\begin{aligned}
\big\|\theta_1 - \theta_2\big\|_{L^2(\Omega)} + \big\|\theta_1 - \theta_2\big\|_{L^{\tilde{p}}(\Sigma)} &\leq C\,\bar{f}(\hat{K}_0)\, (\|f(\theta_1) - f(\theta_2)\|_{[L^2(\Omega_1)]^3} + \|\theta_1 - \theta_2\|_{L^{\tilde{p}}(\Sigma)}) \\
&\leq\ C\,\bar{f}(\hat{K}_0)\, (\rho_1\, |\vec{g}|\, \alpha\, \|\theta_1 - \theta_2\|_{L^2(\Omega_1)} + \|\theta_1 - \theta_2\|_{L^{\tilde{p}}(\Sigma)})
\end{aligned} \tag{112}$$

The claim follows. ☐

4 A regularity result.

We recall the following result proved in [Dru07a]. Let $1 < \alpha, p < \infty$. For a bounded Lipschitz domain $U \subset \mathbb{R}^3$, we consider the space

$$\mathcal{W}^{p,\alpha}(U) := \Big\{u \in L^p_{\text{curl}}(U) \cap L^p_{\text{div}}(U)\,\Big|\, u \cdot \vec{n} \in L^\alpha(\partial U)\Big\}\,. \tag{113}$$

Define the Sobolev embedding exponent p^* by

$$p^* := \begin{cases} \frac{3p}{3-p} & \text{if } p < 3\,, \\ 1 \leq s < \infty \text{ arbitrary} & \text{if } p = 3\,, \\ \infty & \text{if } p > 3\,. \end{cases}$$

Lemma 4.1. Let $U \subset \mathbb{R}^3$ be a bounded Lipschitz domain. There exists some $q_1 > 3$ such that, for $\xi := \min\left\{\frac{3\alpha}{2},\, p^*,\, q_1\right\}$, we have $\mathcal{W}^{p,\alpha}(U) \hookrightarrow [L^\xi(U)]^3$ with continuous embedding. If in addition ∂U is of class $\mathcal{C}^1$, then q_1 can be chosen equal to $+\infty$.

Proposition 4.2. Let $\tilde{\Omega} \subset \mathbb{R}^3$ be a simply connected Lipschitz domain. Assume that the function $\mathfrak{s}_1$ of electrical conductivity is a function of the position that belongs to $C^1\left(\overline{\tilde{\Omega}_1}\right)$. Let the assumptions of Theorem 1.1 be satisfied.

Then, there exists a weak solution of (P) such that

(1) The vector field *curl* H belongs to the space $\mathcal{W}^{\frac{3}{2},\infty}(\tilde{\Omega}_1)$. In particular, *curl* $H \in [L^3(\Omega_1)]^3$, and there exists a constant $C = C(\tilde{\Omega},\, \mathfrak{s})$ such that

$$\|\operatorname{curl} H\|_{[L^3(\Omega_1)]^3} \leq C\left(\|v\|_{[W^{1,2}(\Omega_1)]^3}\,\|H\|_{[W^{1,2}(\Omega_1)]^3} + \|\operatorname{curl} H\|_{[L^{3/2}(\Omega_1)]^3}\right).$$

(2) If, in addition, the function η is a smooth function of the position in Ω_1, the temperature θ belongs to the space $V^{2,5}(\Omega) \cap L^\infty(\Omega)$.

Proof. (1): Observe first that under the assumptions of Theorem 1.1, we have by Lemma 5.1 for $i = 0, \ldots, m$ that $H \in [W^{1,2}(\tilde{\Omega}_i)]^3$ whenever $\{v,\, H,\, \theta\}$ is a weak solution of (P). We can therefore write almost everywhere in Ω_1 that

$$\operatorname{curl}(v \times \mu\, H) = \big(\mu\, H \cdot \nabla\big)v - \big(v \cdot \nabla\big)(\mu\, H) = \big(\mu\, H \cdot \nabla\big)v - \mu\,\big(v \cdot \nabla\big)H - \,v \cdot \nabla\mu\, H\,.$$

By means of Sobolev's embedding theorems, we get

$$\begin{aligned}\|\operatorname{curl}(v \times \mu\, H)\|_{[L^{3/2}(\Omega_1)]^3} &\leq \mu_u\,\|\nabla v\|_{[L^2(\Omega_1)]^9}\,\|H\|_{[L^6(\Omega_1)]^3} + \mu_u\,\|\nabla H\|_{[L^2(\Omega_1)]^9}\,\|v\|_{[L^6(\Omega_1)]^3}\\ &\quad + \|\mu\|_{C^1(\overline{\Omega_1})}\,\|H\|_{[L^3(\Omega_1)]^3}\,\|v\|_{[L^3(\Omega_1)]^3}\\ &\leq c\,\|v\|_{[W^{1,2}(\Omega_1)]^3}\,\|H\|_{[W^{1,2}(\Omega_1)]^3}\,. \end{aligned} \tag{114}$$

If $\{v,\, H,\, \theta\}$ is a weak solution of (P), then the relation

$$\int_{\tilde{\Omega}} r\, \operatorname{curl} H \cdot \operatorname{curl}\psi = \int_{\Omega_1} (v \times \mu\, H) \cdot \operatorname{curl}\psi\,, \tag{115}$$

is valid for all $\psi \in \mathcal{H}^0_\mu(\tilde{\Omega})$. With the arguments of [Dru07a], we can readily show that (115) even holds for all $\psi \in [C^\infty_c(\Omega_c)]^3$. We in particular choose an arbitrary $\psi \in [C^\infty_c(\Omega_1)]^3$. We can integrate by parts the right-hand side of (115) to obtain that

$$\int_{\Omega_1} r\, \operatorname{curl} H \cdot \operatorname{curl}\psi = \int_{\Omega_1} \operatorname{curl}(v \times \mu\, H) \cdot \psi\,.$$

This means that $\operatorname{curl}(r\, \operatorname{curl} H) = \operatorname{curl}(v \times \mu\, H)$, in the sense of the generalized *curl* operator in Ω_1. Define $w := r\, \operatorname{curl} H$. In view of (114), we can write in the generalized sense of the operator *curl* that

$$\operatorname{curl} w = \operatorname{curl}(v \times \mu\, H) \in [L^{3/2}(\Omega_1)]^3\,. \tag{116}$$

On the other hand, since $r = 1/\mathfrak{s} \in C^1\left(\overline{\Omega_1}\right)$, we easily compute that

$$\operatorname{div} w = \nabla r \cdot \operatorname{curl} H \in L^2(\Omega_1)\,. \tag{117}$$

Finally, we know from Lemma 5.1, (6) that

$$w \cdot \vec{n} = 0 \text{ on } \partial\Omega_1\,. \tag{118}$$

By (116), (117) and (118), we obtain that $w \in \mathcal{W}^{\frac{3}{2},\infty}(\Omega_1)$. Applying Lemma 4.1 with $\alpha = \infty$ and $p = 3/2$, we prove that

$$\begin{aligned}\|w\|_{[L^3(\Omega_1)]^3} &\leq c\,(\|\operatorname{curl} w\|_{[L^{3/2}(\Omega_1)]^3} + \|\operatorname{div} w\|_{L^{3/2}(\Omega_1)} + \|w\cdot\vec{n}\|_{L^\infty(\partial\Omega_1)})\\ &\leq C\,(\|v\|_{[W^{1,2}(\Omega_1)]^3}\,\|H\|_{[W^{1,2}(\Omega_1)]^3} + \|\operatorname{curl} H\|_{[L^{3/2}(\Omega_1)]^3})\,.\end{aligned}$$

(2): If the viscosity η is smooth, then the classical regularity theory for the Navier-Stokes equations (see [Tem77], Ch. 2, Paragraph 1) gives that $v \in [W^{2,2}(\Omega_1)]^3$. This allows to estimate in (114) the L^2-norm of $\operatorname{curl}(v \times \mu\, H)$. We obtain that $H \in \mathcal{W}^{2,\infty}(\Omega_1)$. By Sobolev's embedding theorems and Lemma 4.1, the right-hand side of the heat equation is given by

$$r\,|\operatorname{curl} H|^2 + \eta\, D(v, v) \in L^3(\Omega)\,.$$

We only have to apply the results of [Dru07b], Theorem 4.1 about the solution operator of the heat equation with nonlocal radiation terms in order to prove the claim. ☐

5 Appendix

5.1 Tools for Maxwell's equations.

Lemma 5.1. Let the assumption (28) be satisfied for the function μ and let $\tilde\Omega \subset \mathbb{R}^3$ be a simply connected Lipschitz domain. Then, the following results hold true:

(1) The embedding $\mathcal{H}_\mu(\tilde\Omega) \hookrightarrow [L^2(\tilde\Omega)]^3$ is compact.

(2) There exists a constant $C > 0$ such that, for all $\psi \in \mathcal{H}_\mu(\tilde\Omega)$, $\int_{\tilde\Omega} |\operatorname{curl}\psi|^2 \geq C\,\|\psi\|^2_{[L^2(\tilde\Omega)]^3}$.

(3) If the pair $(\mu, \tilde\Omega)$ satisfies $(A0)$ and $(A1)$, then the topological identity $\mathcal{H}_\mu(\tilde\Omega) = \mathcal{H}_\mu(\tilde\Omega) \cap \bigcap_{i=0}^m [W^{1,2}(\tilde\Omega_i)]^3$ is valid.

(4) If the pair $(\mu, \tilde\Omega)$ satisfies $(A0)$ and $(A2)$, then there exist a number $\tilde\xi > 3$ such that $\mathcal{H}_\mu(\tilde\Omega) \hookrightarrow [L^{\tilde\xi}(\tilde\Omega)]^3$ with continuous embedding.

(5) There exist $\tilde\xi > 3$ and a constant $C = C(\tilde\Omega,\, \tilde\xi)$ such that if the condition $C\,(1-\mu_l/\mu_u) < 1$ is satisfied, then $\mathcal{H}_\mu(\tilde\Omega) \hookrightarrow [L^{\tilde\xi}(\tilde\Omega)]^3$ continuously, without further assumptions on the pair $(\mu,\, \tilde\Omega)$.

(6) Every vector field $j_0 \in [L^2(\tilde{\Omega})]^3$ such that

$$\operatorname{div} j_0 = 0 \text{ in the generalized sense in } \tilde{\Omega}, \quad j_0 = 0 \text{ a. e. in } \tilde{\Omega} \setminus \tilde{\Omega}_c\,,$$

is uniquely representable as *curl* ψ with some $\psi \in \mathcal{H}_\mu(\tilde{\Omega})$. If $\psi \in \mathcal{H}_\mu(\tilde{\Omega})$, then *curl* $H \cdot \vec{n} = 0$ in the sense of traces on $\partial\tilde{\Omega}_c$.

Proof. See [Dru07a] Lemma 2.4, Lemma 2.7 and Lemma 4.2. ☐

5.2 Tools for the Navier-Stokes equations.

Lemma 5.2. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in \mathcal{C}^{0,1}$. Let $1 < q < \infty$ be arbitrary. Then it holds that:

1. Let $F \in [W^{1,q}_0(\Omega)]^*$ satisfy $F(v) = 0$ for all $v \in D^{1,q}_0(\Omega)$. Then there exists a unique $p \in L^{q'}_M(\Omega)$ such that F has the representation $F(v) = \int_\Omega p \operatorname{div} v$ for all $v \in [W^{1,q}_0(\Omega)]^3$. Here, the subscript M denotes the subspace of functions having vanishing mean-value over Ω.

2. For all $f \in L^q_M(\Omega)$, the problem $\operatorname{div} v = f$ in Ω has at least one solution in the space $[W^{1,q}_0(\Omega)]^3$, and there exists a constant $c > 0$, that depends only on q, Ω, such that $\|v\|_{[W^{1,q}_0(\Omega)]^3} \leq c\,\|f\|_{L^q(\Omega)}$.

Proof. See [Gal94], III. 3. ☐

5.3 Tools for the energy equation.

We recall some basics about the nonlocal radiation operator G.

Definition 5.3. Let $w : \Sigma \times \Sigma \to \mathbb{R}$ be given by (9).

(1) We say that two points $z, y \in \Sigma$ *see each other* if and only if $w(z, y) \neq 0$.

(2) We call Ω an *enclosure* if and only if for $S-$almost all $z \in \Sigma$ we have $\int_\Sigma w(z, y)\, dS_y = 1$. Here, S denotes the surface measure.

For handling the nonlocal radiation operator G, we need the following results.

Lemma 5.4. Let $\Sigma \in \mathcal{C}^{1,\delta}$ piecewise.

(1) If Ω is not an enclosure, then G has the representation $G = I - \mathbf{H}$, where the operator $\mathbf{H}$ satisfies for $1 \leq p \leq +\infty$ the norm estimate $\|\mathbf{H}\|_{\mathcal{L}(L^p(\Sigma), L^p(\Sigma))} < 1$.

(2) If Ω is an enclosure, then G is selfadjoint from $L^2(\Sigma)$ into itself. In addition, for any constant function C, it holds that $G(C) = 0$.

Proof. See, for example, [Tii97]. □

Lemma 5.5. Let $\Sigma \in \mathcal{C}^{1,\delta}$.

(1) There exists a positive constant $\tilde{c}$ such that for all $\psi \in V_\Gamma^{2,5}(\Omega)$,

$$\int_\Omega |\nabla \psi|^2 + \int_\Sigma G(|\psi|^3\,\psi)\,\psi \geq \tilde{c}\,\min\left\{\|\psi\|^2_{V_\Gamma^{2,5}(\Omega)}\,,\,\|\psi\|^5_{V_\Gamma^{2,5}(\Omega)}\right\}.$$

(2) There exists a positive constant c such that

$$\int_\Sigma G(\psi)\,\mathrm{sign}(\psi) \geq c\,\parallel \psi \parallel_{L^1(\Sigma)}\,,$$

for all $\psi \in L^1(\Sigma)$ such that $\int_\Sigma \psi\,dS = 0$.

Proof. See [Dru07b]. □

We also have the following result.

Lemma 5.6. Let Ω be an enclosure. Let $F : \mathbb{R} \to \mathbb{R}$ be a nondecreasing, continuous function with $F(0) = 0$ and $|F(t)| \leq C_0\,(1 + |t|^s)$ as $|t| \to \infty$ $(0 \leq s < \infty)$. Let $0 \leq r < \infty$ be an arbitrary number. Then for all $\psi \in L^{r+s}(\Sigma)$, $\int_\Sigma G(|\psi|^{r-1}\,\psi)\,F(\psi) \geq 0$.

Proof. See [Dru07b]. □

For Banach spaces X, Y, we denote by $\mathcal{L}(X, Y)$ the set of all linear bounded operators from X into Y. We write $\mathcal{K}(X, Y)$ for the subspace of the compact operators of $\mathcal{L}(X, Y)$.

Lemma 5.7. Let $\Sigma \in \mathcal{C}^{1,\delta}$. Then the operator G has the representation $G = \epsilon\,(I - \tilde{\mathbf{H}})$.

(1) For $1 < p < \infty$, the operator $\tilde{\mathbf{H}}$ belongs to $\mathcal{K}(L^p(\Sigma)\,,\,L^p(\Sigma))$.

(2) If $p > \frac{1}{\delta}$, then $\tilde{\mathbf{H}}$ belongs to $\mathcal{K}(L^p(\Sigma)\,,\,C(\Sigma))$.

(3) The operator $\tilde{\mathbf{H}}$ has the following weak compactness property. If the sequence $\{\psi_k\}$ is bounded in the space $L^1(\Sigma)$, then we can find a subsequence $\{kj\}$ and some $u \in L^1(\Sigma)$ such that $\tilde{\mathbf{H}}(\psi_{kj}) \rightharpoonup u$ in $L^1(\Sigma)$.

Proof. See [Dru07b]. □

Lemma 5.8. Let $(X,\,\mathcal{A},\,\mu)$ be a measurable space such that $\mu(X) < \infty$. For a measurable function $u : X \to \mathbb{R}$ and $1 < p < \infty$, define

$$[u]_{L^p_w(\Omega)} := \sup_{t>0}\left\{t\,\mu\big(\{x \in X\,:\,|u(x)| > t\}\big)^{\frac{1}{p}}\right\}.$$

Then for all $1 < p < \infty$ and all $0 < \epsilon < p - 1$, one has the inequality

$$\|u\|_{L^p(X,\mathcal{A},\mu)} \leq \left(\frac{p}{\epsilon}\right)^{\frac{1}{p-\epsilon}}\,\Big(\mu(X)\Big)^{\frac{\epsilon}{p\,(p-\epsilon)}}\,[u]_{L^p_w}\,.$$

Proof. See [DL90], Appendix, Paragraph 6. □

The following Lemma is useful for obtaining estimates in the L^1-norm.

Lemma 5.9. For a $p < 2$, let $\theta \in W^{1,p}_\Gamma(\Omega)$. Assume that there exists a constant $C_1 > 0$ such that for all $\delta \in]\,0\,,1\,[$ one has $\int_\Omega \frac{|\nabla\theta|^2}{(1+\theta)^{1+\delta}} \le \frac{C_1}{\delta}$. Then the estimate

$$\int_\Omega |\nabla\theta|^p \le 2\,\mathrm{meas}(\Omega)^{\frac{2-p}{2}}\, c_p\, C_1^{\frac{p}{2}} + \tilde{c}_p\, c_0^{6-3p}\, C_1^{3-p}\,, \tag{119}$$

is valid, where the constants c_p, $\tilde{c}_p$ depend only on p and c_0 is the embedding constant of $W^{1,p}_\Gamma(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

Proof. We follow the ideas of [Rak91], [Nau05]. We can write

$$\int_\Omega |\nabla\theta|^p = \int_\Omega \frac{|\nabla\theta|^p}{(1+\theta)^{(1+\delta)\,\frac{p}{2}}}\,(1+\theta)^{(1+\delta)\,\frac{p}{2}} \le \left(\frac{C_1}{\delta}\right)^{\frac{p}{2}} \left(\int_\Omega |1+\theta|^{(1+\delta)\,\frac{p}{2}\,\frac{2}{2-p}}\right)^{\frac{2-p}{2}}\,.$$

If we denote by p^* the Sobolev embedding exponent, we find that for the choice $\delta = \frac{3-2p}{3-p}$, we have

$$\left(\int_\Omega |1+\theta|^{(1+\delta)\,\frac{p}{2}\,\frac{2}{2-p}}\right)^{\frac{2-p}{2}} = \|1+\theta\|_{L^{p^*}(\Omega)}^{\frac{(2-p)\,p^*}{2}} \le \mathrm{meas}(\Omega)^{\frac{2-p}{2}} + c_0^{\frac{(2-p)\,p^*}{2}}\, \|\nabla\theta\|_{L^p(\Omega)}^{\frac{(2-p)\,p^*}{2}}\,.$$

Defining $r := \frac{6-2p}{6-3p} > 1$ and applying Young's inequality, we obtain that

$$\int_\Omega |\nabla\theta|^p \le \left(\frac{C_1}{\delta}\right)^{\frac{p}{2}} \mathrm{meas}(\Omega)^{\frac{2-p}{2}} + \frac{p}{6-2p}\left(\frac{3\,(2-p)}{3-p}\right)^{\frac{6-3p}{p}} \left(\frac{C_1}{\delta}\right)^{3-p} c_0^{6-3p} + \frac{1}{2}\,\|\nabla\theta\|^p_{L^p(\Omega)}\,.$$

□

The next two Lemmas will help us to shorten our proofs. We recall the notation (47).

Lemma 5.10. If $g_\delta \longrightarrow g$ in $L^1(\tilde{\Omega})$, then also $[g_\delta]_{(\delta)} \longrightarrow g$ in $L^1(\tilde{\Omega})$ as $\delta \to 0$.

Proof. We have $|[g_\delta]_{(\delta)} - g| \le |(g_\delta - g)/(1+\delta\, g_\delta)| + \delta\,|g_\delta|\,|g|/(1+\delta\, g_\delta)$ so that the assertion directly follows by dominated convergence. □

Lemma 5.11. Let u_k, $u \in L^1(\Omega)$ be such that $u_k \to u$ almost everywhere and such that $\|u_k\|_{L^1(\Omega)} \to \|u\|_{L^1(\Omega)}$. Then $u_k \to u$ strongly in $L^1(\Omega)$.

Proof. See [GMS98], I.2.3 Proposition 4. □

For proving uniqueness results, we need the following Lemma.

Lemma 5.12. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with the structure described in the introduction. Assume that for $i = 0, \ldots, m$, the boundary $\partial\Omega_i \setminus \partial\Omega$ belongs to $\mathcal{C}^1$ and that the outer boundary $\partial\Omega$ belongs to $\mathcal{C}^{0,1}$. For $i = 0, \ldots, m$, let κ_i be constant. Let $f \in L^1(\Omega)$ and $h \in L^1(\Sigma)$ be given.

Then, there exists a unique $u \in W^{1,p}_0(\Omega)$, $p < 3/2$ arbitrary, such that

$$\int_\Omega \kappa \, \nabla u \cdot \nabla \xi = \int_\Omega f \, \xi + \int_\Sigma h \, \xi \,, \tag{120}$$

for all $\xi \in W^{1,p'}_0(\Omega)$, where $1/p + 1/p' = 1$. In addition, for every monotonely increasing, bounded real-valued function g with $g(0) = 0$, one has

$$\int_\Omega \kappa \, g'(u) \, |\nabla u|^2 \leq \int_\Omega f \, g(u) + \int_\Sigma h \, g(u) \,, \tag{121}$$

Proof. Existence in the class $\bigcap_{1 \leq p < 3/2} W^{1,p}_0(\Omega)$ was already proved in [Sta65], and confirmed for example in [Rak91]. The solution is also unique. Suppose that u_1, u_2 both satisfy (120). Then the difference satisfies $\int_\Omega \kappa \, \nabla(u_1 - u_2) \cdot \nabla \xi = 0$, for all $\xi \in W^{1,r}(\Omega)$ with $r > 3$. Under the assumptions of the lemma, the main theorem of the paper [ERS07] gives the existence of a $q > 3$ such that $u_1 - u_2 \in W^{1,q}_0(\Omega)$. The uniqueness clearly follows. We prove the last claim. For $\delta > 0$, consider the function $u_\delta \in W^{1,2}_0(\Omega)$ that satisfies

$$\int_\Omega \kappa \, \nabla u_\delta \cdot \nabla \xi = \int_\Omega [f]_{(\delta)} \, \xi + \int_\Sigma [h]_\delta \, \xi \,, \tag{122}$$

for all $\xi \in W^{1,2}_0(\Omega)$. By the usual uniform estimates for linear elliptic problems with L^1-right-hand sides, we find for all $1 \leq p < 3/2$ a subsequence such that

$$u_\delta \rightharpoonup w \text{ in } W^{1,p}_0(\Omega) \,, \quad u_\delta \longrightarrow w \text{ in } L^p(\Omega) \,, \tag{123}$$

as $\delta \to 0$. By the uniqueness of u obtained above, $w = u$.

Let $g : \mathbb{R} \to \mathbb{R}$ be monotonely increasing and bounded such that $g(0) = 0$. We test the relation (122) with $\xi = g(u_\delta)$ and obtain

$$\int_\Omega \kappa \, g'(u_\delta) \, |\nabla u_\delta|^2 = \int_\Omega [f]_{(\delta)} \, g(u_\delta) + \int_\Sigma [h]_\delta \, g(u_\delta) \,.$$

Since g' is positive, we can introduce the function $F(s) := \int_0^s \sqrt{g'(\tau)} \, d\tau$, and write

$$\int_\Omega \kappa \, |\nabla F(u_\delta)|^2 = \int_\Omega [f]_{(\delta)} \, g(u_\delta) + \int_\Sigma [h]_\delta \, g(u_\delta) \,. \tag{124}$$

Clearly, by the last relation, $F(u_\delta) \rightharpoonup \tilde{w}$ in $W^{1,2}_0(\Omega)$ for a subsequence. But in view of (123), we immediately find that $\tilde{w} = F(u)$. Passing to the limit in (124) for this subsequence, we obtain the inequality (121), and the lemma is proved. $\square$

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