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Glauber dynamics on hyperbolic graphs: boundary conditions and mixing time

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Abstract

We study a continuous time Glauber dynamics reversible with respect to the Ising model on hyperbolic graphs and analyze the effect of boundary conditions on the mixing time. Specifically, we consider the dynamics on an n-vertex ball of the hyperbolic graph $\mathbb{H}(v,s)$, where v is the number of neighbors of each vertex and s is the number of sides of each face, conditioned on having $(+)$-boundary. If $v>4$, $s>3$ and for all low enough temperatures (phase coexistence region) we prove that the spectral gap of this dynamics is bounded below by a constant independent of n. This implies that the mixing time grows at most linearly in n, in contrast to the free boundary case where it is polynomial with exponent growing with the inverse temperature β. Such a result extends to hyperbolic graphs the work done by Martinelli, Sinclair and Weitz for the analogous system on regular tree graphs, and provides a further example of influence of the boundary condition on the mixing time.

1 Introduction

The goal of this paper is to analyze the Glauber dynamics for the Ising model defined on hyperbolic graphs. In particular we will focus on the problem of determining the influence of boundary condition on the mixing time of the dynamics. Before discussing the motivation and the formulation of the results we shall give some necessary definitions.

Given a finite graph $G=(V,E)$, we consider spin configurations $\sigma=\{\sigma_x\}_{x\in V}$ which consist of an assignment of ±1-values to each vertex of V. In the Ising model the probability of finding the system in a configuration $\sigma\in\{\pm1\}^V\equiv\Omega_G$ is given by the Gibbs measure

$$\mu_G(\sigma)\,=\,(Z_G)^{-1}\exp\left(\beta\sum_{(xy)\in E}\sigma_x\sigma_y+\beta h\sum_{x\in V}\sigma_x\right), \tag{1.1}$$

where Z_G is a normalizing constant and β and h are parameters of the model corresponding respectively to the inverse temperature and to the external field. Boundary conditions can also be taken in account by fixing the spin values at some specified boundary vertices of G; the term free boundary is used to indicate that no boundary is specified.

The Glauber dynamics for the Ising model on G is a (discrete or continuous time) Markov chain on the set of spin configurations Ω_G, reversible respect to the Gibbs measure μ_G. The correspondent generator is given by

$$(\mathcal{L}f)(\sigma)\,=\,\sum_{x\in V}c_x(\sigma)[f(\sigma^x)-f(\sigma)]\,, \tag{1.2}$$

where σ^x is the configuration obtained from σ by spin flip at the vertex x and $c_x(\sigma)$ is the jump rate from σ to σ^x.

Beyond of being the basis of Markov chain Monte Carlo algorithms, the Glauber dynamics provides a plausible model for the evolution of the underlying physical system toward the equilibrium. In both contexts, a central question is to determine the *mixing time*, i.e. the number of steps until the dynamics is close to its stationary measure.

In the past decades a lot of efforts have been devoted to the study of the dynamics for the classical Ising model, namely when $G = G_n$ is a cube of size n in the finite-dimensional lattice $\mathbb{Z}^d$, and a remarkable connection between the equilibrium and the dynamical phenomena has been pointed out. As an example, on finite n-vertex cubes with free boundary in $\mathbb{Z}^d$, when $h = 0$ and β is smaller then the critical value β_c (one-phase region), the mixing time is of order $\log n$, while for $\beta > \beta_c$ (phase coexistence region) it is $\exp(n^{(d-1)/d})$ ([28, 21, 22, 20]).

More recently an increasing attention has been devoted to the study of spin systems on graphs other than the regular lattices. Among the various motivations which are beyond this new surge of interest, we stress that many new phenomena only appear when one considers graphs different from the Euclidean lattices, thus revealing the presence of an interplay between the geometry of the graph and the behavior of statistical system.

Here we are interested in the problem of the *influence of boundary conditions* on the mixing time. It has been conjectured that in the presence of (+)-boundary condition on regular boxes of the lattice $\mathbb{Z}^d$, the mixing time should remain at most polynomial in n for all temperatures rather then $\exp(n^{(d-1)/d})$ [9]. But even if some results supporting this conjecture have been achieved [5], a formal proof for the dynamics on the lattice is still missing.

However a different scenario can appear if one replaces the classical lattice structure with different graphs. The first rigorous result along this direction, has been obtained recently by Martinelli, Sinclair and Weitz [23] when studying the Glauber dynamics for the Ising model on regular tree graphs. With this graph setting and in presence of $(+)$-boundary condition, they proved in fact that the mixing time remains of order $\log n$ also at low temperature (phase coexistence region), in contrast to the free boundary case where it grows polynomially in n [4].

In this paper we extend the above result to the Glauber dynamics on hyperbolic graphs which, roughly speaking, are a discretization of the hyperbolic plane $\mathbb{H}^2$ in the same sense as $\mathbb{Z}^d$ is a discretization of $\mathbb{R}^d$. In particular, we prove that spectral gap of the dynamics on an n-vertex ball of the hyperbolic graph with $(+)$-boundary condition is $\Omega(1)$ (i.e. bounded away from zero uniformly in n) for all low enough temperatures and zero external field. This provides, by classical argument (see, e.g., [25]), an upper bound of order n on the mixing time. Notice that, with a free boundary and zero external field, the only known bound on the mixing time is of order $n^{\alpha(\beta)}$, with exponent $\alpha(\beta)$ arbitrarily increasing with β [4].

We remark that the possibility of this extension to hyperbolic graphs is suggested by the fact that these graphs, as well as trees, have exponential growth, a property which we believe to be determinant for the result obtained in [23]. On the other hand the presence of cycles, which are absent on trees, makes their structure more

similar to the lattices. Let us finally stress that the Ising model on hyperbolic graphs has a more complex phase diagram with respect to the classical Euclidian case and exhibits extra phenomena like "double phase transition" and "existence of infinite extremal Gibbs states" [27, 29, 30].
The work is organized as follows. In section 2 we give some basic definitions and state the main result. Then in section 3 we analyze the system at the equilibrium and prove a mixing property of the plus phase. Finally in section 4 we relate this property to the spectral gap of the dynamics and we conclude the proof of our main result.

2 The model: definitions and main result

2.1 Graph setting

Before describe the hyperbolic graphs, let us fix some notation and recall a few definitions concerning the graph structure.
Let $G = (V, E)$ be a general infinite graph , where V denotes the vertex set and E the edge set. The *graph distance* between two vertices $x, y \in V$ is defined as the length of the shortest path from x to y and it is denoted by $d(x, y)$. If x and y are at distance one, i.e. if they are neighbors, we write $x \sim y$.
For a given subset $S \subset V$, we denote by $E(S)$ the set of all edges in E which have both their end vertices in S and we call $G(S) = (S, E(S))$ the *induced subgraph* on S. When it will create no confusion, we will identify $G(S)$ with its vertex set S.
For $S \subset V$ let us introduce the *vertex boundary* of S

$$\partial_V S = \{x \in V \setminus S : \exists y \in S \text{ s.t. } x \sim y\}$$

and the *edge boundary* of S

$$\partial_E S = \{e = (x, y) \in E \text{ s.t. } x \in S , y \in V \setminus S\} .$$

If $G = (V, E)$ is an infinite, locally finite, connected graph, we can define the *edge isoperimetric constant* of G (also called *Cheeger constant*) by

$$i_e(G) := \inf \left\{ \frac{|\partial_E(S)|}{|S|} ; S \subset V \text{ finite} \right\} . \tag{2.1}$$

Definition 2.1. *A graph $G = (V, E)$ is amenable if its edge isoperimetric constant is zero, i.e. if for every $\epsilon > 0$ there is a finite set of vertices S such that $|\partial_E S| < \epsilon|S|$. Otherwise G is non-amenable.*

A typical example of amenable graph is the lattice $\mathbb{Z}^d$, while one can easily show that the regular trees with branching number bigger then two are non-amenable.

In this work we consider the hyperbolic graphs, which are a family of infinite planar graphs characterized by a cycle periodic structure. They can be briefly described as follows (for their detailed construction see, e.g., [19], or Section 2 of ref. [24]). Consider a graph in which each vertex has the same number of neighbors (or *vertex-degree*) denoted by v, and each face (or *tile*) is equilateral with constant number of sides denoted by s. If the parameters v and s satisfy the relation $(v-2)(s-2) > 4$, then the graph can be embedded in the hyperbolic plane $\mathbb{H}^2$ and it is called *hyperbolic graph* (or *hyperbolic tiling*) with parameters v and s. It will be denoted by $\mathbb{H}(v,s)$. The typical representation of hyperbolic tilings make use of the Poincaré disc D^2 that is in bi-univocal correspondence with $\mathbb{H}^2$ (see Fig. 2.1).

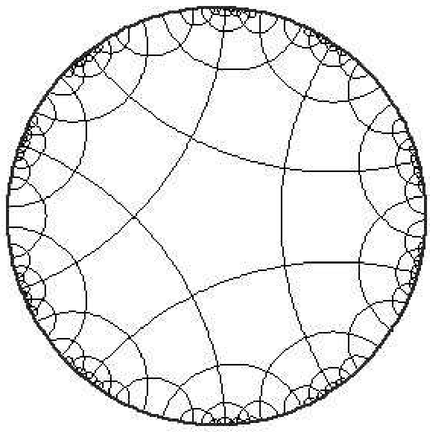

Figure 2.1: The hyperbolic graph $\mathbb{H}(4,5)$ in the Poincaré disc representation.

Between the properties characterizing the hyperbolic graphs, we first mention the non-amenability which roughly speaking, as one deduces from Definition 2.1, means that the boundary of every subset of the graph is of comparable size to its volume. The edge isoperimetric constant of $\mathbb{H}(v,s)$ has been explicitly computed in [11] as a function of v and s.
The main similarity between hyperbolic graphs and the Euclidean lattices is related to the fact of having many cycles, which are instead missing in the trees. To be more precise, let us give the following definition.

Definition 2.2. *The number of ends $\mathcal{E}(G)$ of a graph $G=(V,E)$ is defined as*

$$\mathcal{E}(G) = \sup_{\substack{K\subset V \\ K \text{finite}}} \{ \text{ number of infinite connected components of } G\setminus K \},$$

where $G\setminus K$ denotes the graph obtained from G by removing the vertices which belong to K and the edges incident to these vertices.

It is known that hyperbolic graphs, as well as all the lattices $\mathbb{Z}^d$ with $d\geq 2$, are *one-ended* graphs. The fact of having many cycles is indeed encoded in the impossibility of split the graph in more then one infinite component by simply removing a finite number of vertices. At the contrary, regular trees are *infinite-ended* graphs.
Non-amenability and the property of being one-ended seem to be strongly related to the qualitative behavior of models in statistical mechanics (see, e.g., [13, 16, 17, 26, 23] for results concerning the Ising and the Potts models, and [6, 7, 8, 10, 11, 12] for percolation and random cluster models). Non-amenability will appear in

the proof of our main result as an essential element. Beside, due to the property of being one-ended, we will need a careful analysis of the correlations between spins. This is actually the main distinction between our proof and the similar work on trees [23].

2.2 Ising model on hyperbolic graphs

The study of the Ising model on the hyperbolic graph $\mathbb{H}(v,s)$ led to the characterization of two different phase transitions appearing at inverse temperatures $\beta_c \le \beta_c'$ and zero magnetic field [27, 29, 30]. The first one, β_c, corresponds to the occurrence of a uniqueness/non-uniqueness phase transition. In particular, as for the model on $\mathbb{Z}^d$ with $d \ge 2$, when $\beta > \beta_c$ there are at least two extremal Gibbs measures which are usually denoted by μ^+ and μ^- and are obtained by imposing, respectively, $(+)$- and $(-)$-boundary condition. The second critical temperature refers to a change in the properties of the free boundary condition measure μ^f; specifically it is defined as

$$\beta_c' = \inf\{\beta \ge \beta_c : \mu^f = (\mu^+ + \mu^-)/2\}.$$

It turns out that for $\beta_c < \beta \le \beta_c'$ the measure μ^f is not a convex combination of μ^+ and μ^-, while for $\beta > \beta_c'$ the property $\mu^f = (\mu^+ + \mu^-)/2$ is recovered.
The interesting scenario appears when the strict inequality $\beta_c < \beta_c'$ (see [30] for details). In this case, for all inverse temperatures β in the nonempty interval $(\beta_c, \beta_c']$ it holds that $\mu^f \neq (\mu^+ + \mu^-)/2$, which implies the existence of a translation invariant Gibbs state different from μ^+ and μ^-. Notice that this behavior is in contrast to what happens for the Ising model on $\mathbb{Z}^d$, where the only translation invariant Gibbs states are μ^+ and μ^- [2].
Another interesting result concerning the properties of the extremal measures has been obtained by Sinai and Series in [27]. They proved that for low enough temperatures and $h = 0$, there exist uncountably many mutually singular Gibbs states which they conjectured to be extremal. Again, this result is in contrast to the properties of the model on $\mathbb{Z}^d$, where is known that the extremal measures are at most a countable number.
For that concerns the model when a magnetic field $h \neq 0$ is added to the system, we recall the result obtained by Jonasson and Steif [13] for transitive non-amenable graphs with finite vertex degree. For this class of graphs they proved the existence of a critical value β_0 and of a critical curve $h_c(\beta)$ such that, for all $\beta > \beta_0$, the Gibbs measure is not-unique when $|h| < h_c(\beta)$, and it is unique when $|h| > h_c(\beta)$. This result applies to the hyperbolic graphs and shows the existence of a uniqueness/non-uniqueness phase transition for $h \neq 0$.

In this paper we are interested in the region of the phase diagram where the dynamics is highly sensitive to the boundary condition, namely when the temperature is low and the magnetic field is zero (phase coexistence region). In particular, for a given sequence of subgraphs $\{B_r\}_{r\in\mathbb{N}}$ converging to $\mathbb{H}(v,s)$ as $r \to \infty$, we will focus on the Ising model on B_r conditioned on having $(+)$-spins on the boundary $\partial_V B_r$. Let us explain in details the model and give the necessary definitions and notation.

Consider the hyperbolic graph $\mathbb{H}(v,s)$, with vertex set V and edge set E. Let $o \in V$ be a distinguished vertex (*root*) and for any $r \in \mathbb{N}$, denote by $B_r = (V_r, E_r) \subset \mathbb{H}(v,s)$ the ball centered in o and with radius r, namely the finite subgraph induced on $V_r = \{x \in V : d(o,x) \le r\}$. When it does not create confusion, we identify the subgraphs of $\mathbb{H}(v,s)$ with their vertex sets.
Given a finite ball $B \equiv B_m$ and an Ising spin configuration τ on the hyperbolic graph $\mathbb{H}(v,s)$, let $\Omega_B^\tau \subset \{\pm 1\}^{B\cup\partial_V B}$ be the set of configurations that agree with τ on $\partial_V B$. Analogously, for any subset $A \subseteq V_m$ and any $\eta \in \Omega_B^\tau$, we denote by $\Omega_A^\eta \subset \{\pm 1\}^{A\cup\partial_V A}$ the set of configurations that agree with η on $\partial_V A$. The Ising model on A with η-boundary condition (b.c.) and zero external field is thus specified by the Gibbs probability measure μ_A^η, with support on Ω_A^η, defined as

$$\mu_A^\eta(\sigma) = \frac{1}{Z(\beta)} \exp\big(\beta \sum_{(x,y)\in E(\overline{A})} \sigma_x \sigma_y\big)\,, \tag{2.2}$$

where $Z(\beta)$ is a normalizing constant and the sum runs over every couples of nearest neighbors in the induced subgraph of $\overline{A} = A \cup \partial_V A$.
Similarly, the Ising model on A with free boundary condition is specified by the Gibbs measure μ_A supported on the set of configurations $\Omega_A := \{\pm 1\}^A$ and defined as in (2.2) by replacing the sum over $E(\overline{A})$ in a sum over $E(A)$, namely cutting away the influence from the boundary $\partial_V A$. Notice that when $A = V_m$, $\mu_{V_m}^\eta$ is simply the Gibbs measure on B with boundary condition τ (η agrees with τ on $\partial_V V_m \equiv \partial_V B$) and μ_A is the Gibbs measure on B with free boundary condition.
We denote by $\mathcal{F}_A$ the σ-algebra generated by the set of projections $\{\pi_x\}_{x\in A}$ from $\{\pm 1\}^A$ to $\{\pm 1\}$, where $\pi_x : \sigma \mapsto \sigma_x$, and write $f \in \mathcal{F}_A$ to indicate that f is $\mathcal{F}_A$-measurable. Finally, we recall that if $f : \Omega_B^\tau \to \mathbb{R}$ is a measurable function, the *expectation of* f w.r.t. μ_A^η is given by $\mu_A^\eta(f) = \sum_{\sigma\in\Omega} \mu_A^\eta(\sigma) f(\sigma)$ and the *variance of* f w.r.t. μ_A^η is given by $\mathrm{Var}_A^\eta = \mu_A^\eta(f^2) - \mu_A^\eta(f)^2$. We usually think of them as functions of η, that is $\mu_A(f)(\eta) = \mu_A^\eta(f)$ and $\mathrm{Var}_A(f)(\eta) = \mathrm{Var}_A^\eta(f)$; in particular $\mu_A(f)\,, \mathrm{Var}_A(f) \in \mathcal{F}_{A^c}$.
In the following discussion we will be concerned with the Ising model on B with $(+)$-b.c. and we will use the abbreviations Ω^+, $\mathcal{F}$ and μ instead of Ω_B^+, $\mathcal{F}_{B\cup\partial_V B}$ and μ_B^+, and thus $\mu(f)$ and $\mathrm{Var}(f)$ instead of $\mu_B^+(f)$ and $\mathrm{Var}_B^+(f)$.

2.3 Glauber dynamics and mixing time

The Glauber dynamics on B with $(+)$-boundary condition is a continuous time Markov chain $(\sigma(t))_{t\ge 0}$ on Ω^+ with Markov generator $\mathcal{L}$ given by

$$(\mathcal{L}f)(\sigma) = \sum_{x\in B} c_x(\sigma)\,[f(\sigma^x) - f(\sigma)]\,, \tag{2.3}$$

where σ^x denotes the configuration obtained from σ by flipping the spin at the site x and $c_x(\sigma)$ is the jump rate from σ to σ^x. We sometimes prefer the short notation $\nabla_x f(\sigma) = [f(\sigma^x) - f(\sigma)]$. The jump rates are required to be of finite-range, uniformly positive, bounded and they should satisfy the detailed balance condition w.r.t. the Gibbs measure μ. Although all our results apply to any choice of jump rates satisfying these hypothesis, for simplicity we will work with a specific choice called *heat-bath dynamics*:

$$c_x(\sigma) \;=\; \mu_x^\sigma(\sigma^x) = \frac{1}{1+\omega_x(\sigma)} \quad \text{where} \quad \omega_x(\sigma) \;=\; \exp(2\beta\sigma_x \sum_{y\sim x}\sigma_y\,). \tag{2.4}$$

It is easy to check that the Glauber dynamics is ergodic and reversible w.r.t. the Gibbs measure μ, and so converges to μ by the Perron-Frobenius Theorem. The key point is now to determine the rate of convergence of the dynamics.
A useful tool to approach this problem is the *spectral gap* of the generator $\mathcal{L}$, that can be defined as the inverse of the first nonzero eigenvalue of $\mathcal{L}$.

Remark 2.3. *Notice that the generator $\mathcal{L}$ is a non-positive self-adjoint operator on $\ell^2(\Omega^+,\mu)$. Its spectrum thus consists of discrete eigenvalues of finite multiplicity that can be arranged as $0=\lambda_0 \ge -\lambda_1 \ge -\lambda_2 \ge \ldots, \ge -\lambda_{N-1}$, if $|\Omega^+|=N$, with $\lambda_i \ge 0$.*

An equivalent definition of spectral gap is given through the so called *Poincaré inequality* for the measure μ. For a function $f:\Omega^+ \mapsto \mathbb{R}$, define the *Dirichlet form* of f associated to $\mathcal{L}$ by

$$\mathcal{D}(f) \;:=\; \frac{1}{2}\sum_{x\in B}\mu\left(c_x[\nabla_x f]^2\right) \;=\; \sum_{x\in B}\mu(\mathrm{Var}_x(f))\,, \tag{2.5}$$

where the second equality holds under our specific choice of jump rates. The *spectral gap* $c_{gap}(\mu)$ is then defined as the inverse of the best constant c in the *Poincaré inequality*

$$\mathrm{Var}(f) \;\le\; c\mathcal{D}(f)\,, \quad \forall f\in\ell^2(\Omega^+,\,\mu), \tag{2.6}$$

or equivalently

$$c_{gap}(\mu) \;:=\; \inf\left\{\frac{\mathcal{D}(f)}{\mathrm{Var}(f)};\mathrm{Var}(f)\neq 0\right\}\,. \tag{2.7}$$

Denoting by P_t the Markov semigroup associated to $\mathcal{L}$, with transition kernel $P_t(\sigma,\eta)=e^{t\mathcal{L}}(\sigma,\eta)$, it easy to show that

$$\mathrm{Var}(P_t f) \;\le\; e^{-2c_{gap}(\mu)t}\,\mathrm{Var}(f)\,. \tag{2.8}$$

The last inequality shows that the spectral gap gives a measure of the exponential decay of the variance, and justifies the name *relaxation time* for the inverse of the spectral gap.
Moreover, let h_t^σ denote the density of the distribution at time t of the process starting at σ w.r.t. μ, i.e. $h_t^\sigma(\eta) = \frac{P_t(\sigma,\eta)}{\mu(\eta)}$. For $1\le p\le\infty$ and a function $f\in\ell^p(\Omega^+,\,\mu)$, let $\|f\|_p$ denote the ℓ^p norm of f and define the time of convergence

$$\tau_p \;=\min\left\{t>0\;:\;\sup_\sigma \|h_t^\sigma-1\|_p\le e^{-1}\right\}\,, \tag{2.9}$$

that for $p=1$ is called *mixing time*. A well known and useful result relating τ_p to the spectral gap (see, e.g., [25]), when specializing to the Glauber dynamics yield the following:

Theorem 2.4. *On an n-vertex ball $B \subset \mathbb{H}(v,s)$ with (τ)-boundary condition, it holds*

$$c_{gap}(\mu)^{-1} \le \tau_1 \le c_{gap}(\mu)^{-1} \times cn\,, \tag{2.10}$$

where $\mu = \mu_B^\tau$ and $c = c(\beta, v, s)$ is a constant independent of n. □

We stress that a different choice of jump rates (here we considered the heat-bath dynamics) only affects the spectral gap by at most a constant factor. The bound stated in Theorem 2.4 is thus equivalent, apart for a multiplicative constant, for any choice of the Glauber dynamics.
Before presenting our main result, we recall the Glauber dynamics for the Ising model on hyperbolic graphs has been recently investigated by Peres et al. in [4]. They consider the dynamics on a finite ball $B \in \mathbb{H}(v,s)$ with free boundary condition and zero external field, and prove that at all temperatures the inverse of the spectral gap (relaxation time) scales at most polynomially in the size of B, with exponent $\alpha(\beta) \uparrow \infty$ as $\beta \to \infty$. Again, let us stress that under the same conditions, the dynamics on a cube of size n in the d-dimensional lattice can relax in a time exponentially large in the surface area $n^{(d-1)/d}$.

2.4 Main result

We are finally in position to state our main result.

Theorem 2.5. *Let $\mathbb{H}(v,s)$ such that $v>4$ and $s>3$. Then, for all $\beta \gg 1$, the Glauber dynamics on an n-vertex ball B with (+)-boundary condition and zero external field has spectral gap $\Omega(1)$.*

As a corollary we obtain that, under the same hypothesis of the theorem above, the mixing time of the dynamics is bounded linearly in n (see Theorem 2.4).
This result provides a convincing example of the influence of the boundary condition on the mixing time. Indeed, as just recalled, for free boundary conditions the only known estimate on the spectral gap for balls in the hyperbolic graph is a lower bound of order $n^{-\alpha(\beta)}$, with $\alpha(\beta) \uparrow \infty$ as $\beta \to \infty$ [4]. The presence of the $(+)$-boundary condition thus gives rise to an abrupt jump of the spectral gap from $n^{-\alpha(\beta)}$ to a constant, and consequently it speeds up the dynamics.
Remarks.

(i) *We recall that on $\mathbb{Z}^d$ not much is known about the spectral gap when $\beta > \beta_c$, $h = 0$ and the boundary condition is $(+)$, though it has been conjectured that in high enough dimensions ($d \ge 3$) the spectral gap should remain away from zero uniformly in n (see [9] and [5]).*

(ii) *A result similar to Theorem 2.5 has been obtained for the spectral gap, and thus for the mixing time, of the dynamics on a regular b-ary tree (see [23]). In particular it has been proved that while under free-boundary condition the mixing time on a tree of size n jumps from $\log n$ to $n^{\Theta(\beta)}$ when passing a certain critical temperature, it remains of order $\log n$ at all temperatures and at all values of the magnetic field under $(+)$-boundary condition. However we stress* *that while trees do not have any cycle and belong to the class of infinite-ended graphs, hyperbolic graphs, as well as the Euclidean lattices, have many cycles and belong to the class of one-ended graphs (see section 2.1). The theorem above can thus be looked upon as an extension of this result to a class of graphs which in many respects are similar to Euclidean lattices.*

We now proceed to sketch briefly the ideas and techniques used along the paper. The proof of our main result is based on the variational definition of the spectral gap and it is aimed to show that the Gibbs measure relative to the system satisfies a Poincaré inequality with constant c independent of the size of B. We will first analyze the equilibrium properties of the system conditioned on having $(+)$-boundary, and under this condition we will deduce a peculiar notion of correlation decay between spins. The proof of this kind of spatial mixing property rests on a disagreement argument which is then concluded by a Peierls type argument together with some isoperimetric estimates.
The second main step of the proof is deriving a suitable Poincaré inequality for the Gibbs measure describing the system, from the deduced notion of spatial mixing. This will be achieved by first deducing, via coupling techniques, a Poincaré inequality for the marginal Gibbs measure with support on suitable subsets, and then iterating the argument to recover the required estimate on the variance.

3 Mixing properties of the plus phase

In this section we analyze the effect of the $(+)$-boundary condition on the equilibrium properties of the system. In particular, we prove that the Gibbs measure $\mu \equiv \mu_B^+$ satisfies a kind of *spatial mixing property*, i.e. a form of weak dependence between spins placed at distant sites.
Before presenting the main result of this section, we need some more notation and definitions. Recall that for every integer i, we denoted by $B_i = (V_i, E_i)$ the ball of radius i centered in o. Let us define the following objects:

(i) the i-th *level* $L_i = \{x \in V \,:\, d(x,o) = i\} \equiv \partial_V B_{i-1}$;

(ii) the vertex-set $F_i \subseteq B$ given by $F_i := \{v \in B_{i-1}^c \cap B\}$;

(iii) the σ-algebra $\mathcal{F}_i$ generated by the functions π_x for $x \in F_i^c = B_{i-1}$.

We will be mainly concerned with the Gibbs distribution on F_i with boundary condition $\eta \in \Omega^+$, which we will shortly denote by $\mu_i^\eta = \mu_{F_i}^\eta = \mu(\cdot\,|\eta \in \mathcal{F}_i)$; analogously we will denote by Var_i^η the variance w.r.t. μ_i^η. Notice that $\{F_i\}_{i=0}^{m+1}$ is a decreasing sequence of subsets such that $V_m = F_0 \supset F_1 \supset \ldots \supset F_{m+1} = \emptyset$, and in particular it holds that $\mu_i(\mu_{i+1}(f)) = \mu_i(f)$, for all finite i, and $\mu_{m+1}(f) = f$.
We then introduce a linear order on the levels L_i as follows: let T_B be a shortest path spanning tree of B, namely such that for every $x \in V_m$ the path from o to x in T_B is a shortest path in B. Clearly the i-th level of T_B is equal to the level L_i of B. We thus choose, for every $i \in \{0, \ldots, m\}$, a vertex $x_0^i \in L_i$ and order in counterclockwise sense all the vertices in L_i along T_B. This order depends on the

choice of x_0^i, but it does not affect the next computations.
We set $x^i_{|L_i|} = x^i_0$ for all $i \in \{0, \dots, m\}$, and notice that for all $k \in \{0, \dots, |L_i|\}$ the vertices x^i_k and x^i_{k+1} belong to the same tile of B. We will call a pair of vertices in the same level and with this property *level-neighboring vertices*.
We can now define the following distance on L_i:

Definition 3.1 (L_i-distance)**.** *Given $n, m \in \{0, 1, \cdots, |L_i|\}$ such that $n \geq m$, the L_i-distance between x^i_n and x^i_m in L_i is given by*

$$d_i(x^i_n, x^i_m) = \min\{n - m \, , \, m + (|L_i| - n)\} \, .$$

Remark 3.2. *Let us remark that $d_i(x^i_n, x^i_m)$ is just the minimal number of jumps between L_i-neighboring vertices from x^i_n to x^i_m. Notice also that the definition of L_i-distance doesn't depend on the choice of the ordering on L_i. In general, for $x, y \in L_i$, we have $d_i(x, y) \neq d(x, y)$, where $d(\cdot, \cdot)$ is the usual graph distance.*

Finally, for a given $i \in \{0, \dots, m\}$ and a given vertex $x \in L_i$, we consider the Gibbs measure on the set $K_x = F_{i+1} \cup \{x\}$ conditioned on the configuration outside K_x being $\sigma \in \Omega^+$, which as usually will be denoted by $\mu^\sigma_{K_x}$. We are now able to state the following:

Proposition 3.3. *Let $B \subset \mathbb{H}(v, s)$ such that $v > 4$ and $s > 3$. Then there exist two positive constants c_1 and c_2 dependent on the parameters of the hyperbolic graph such that, for every $\beta > \beta_0 = \frac{c_2}{c_1}$, every $\sigma \in \Omega^+$ and every couples of vertices $x, y \in L_i$, $i \in \{0, \dots, m\}$, it holds*

$$|\mu^\sigma_{K_x}(\sigma_x = +) - \mu^{\sigma^y}_{K_x}(\sigma_x = +)| \leq c e^{-\beta' d_i(x,y)} \, , \tag{3.1}$$

with $\beta' := c_1\beta - c_2 > 0$.

Let us briefly justify the above result. Due to the non-amenability of hyperbolic graphs, namely to the fact that the boundary of any set is proportional to its volume, the $(+)$-b.c. on B turns out to be strong enough to influence sites at arbitrary distance. In particular, as we will prove, the effect of the boundary on a given site x weakens the influence on x coming from other sites (arbitrary near to x) and gives rise to the decay correlation stated in Proposition 3.3. Notice that the correlation decay increases with β.
The proof of Proposition 3.3, which will be presented in the rest of this section, is divided in two parts. First, we define a suitable event and show that the correlation between two spins is controlled by the probability of this event. Then, in the second part, we estimate this probability first using a Peierls type argument and then deducing some isoperimetric inequalities which yield the exponential factor in formula 3.1.

3.1 Proof of Proposition 3.3

Let us consider two vertices $x, y \in L_i$ such that $d_i(x, y) = \ell$, and a configuration $\sigma \in \Omega^+$. When $\ell = 0$ (x and y coincident) inequality (3.1) is trivial, thus we assume $\ell \geq 1$. Let $\sigma^{y,+}$ be the configuration that agrees with σ in all sites but y and has a

$(+)$-spin on y; define analogously $\sigma^{y,-}$ and denote by $\mu_{K_x}^{y,+}$ and $\mu_{K_x}^{y,-}$ the measures conditioned on having respectively $\sigma^{y,+}$- and $\sigma^{y,-}$-b.c.. With this notation and from the obvious fact that the event $\{\sigma : \sigma_x = +\}$ is increasing, we get that

$$|\mu_{K_x}^{\sigma}(\sigma_x = +) - \mu_{K_x}^{\sigma^y}(\sigma_x = +)| \; = \; \mu_{K_x}^{y,+}(\sigma_x = +) - \mu_{K_x}^{y,-}(\sigma_x = +) \,. \tag{3.2}$$

In the rest of the proof we will focus on the correlation in the r.h.s. of (3.2).
In order to introduce and have a better understanding of the ideas and techniques that we will use along the proof, we first consider the case $\ell = 1$, which is simpler but with a similar structure respect to the general case $\ell > 1$.

3.1.1 Correlation decay: the case $\ell = 1$

Suppose that $\ell = 1$, which means that possibly x and y are neighboring sites. Denoting by $\mu_{K_x}^{-}$ the measure with $(-)$-b.c. on $K_x^c = B_i \setminus \{x\}$, we get

$$\begin{aligned} \mu_{K_x}^{y,+}(\sigma_x = +) - \mu_{K_x}^{y,-}(\sigma_x = +) \;&=\; \mu_{K_x}^{y,-}(\sigma_x = -) - \mu_{K_x}^{y,+}(\sigma_x = -) \\ &\le\; \mu_{K_x}^{y,-}(\sigma_x = -) \\ &\le\; \mu_{K_x}^{-}(\sigma_x = -) \,, \end{aligned} \tag{3.3}$$

where the last inequality follows by monotonicity. The problem is thus reduced to estimate the probability of the event $\{\sigma : \sigma_x = -\}$ w.r.t. $\mu_{K_x}^{-}$.
Let $\mathcal{K}$ be the set of connected subsets in K_x containing x and write

$$\mathcal{K} \;=\; \bigsqcup_{m\ge1} \mathcal{K}_m \qquad \text{with} \quad \mathcal{K}_m = \{C \in \mathcal{K} \ \text{s.t.}\ |C| = m\} \,.$$

For any configuration $\sigma \in \Omega^+$, we denote by $K^{(\sigma)}$ the maximal negative component in $\mathcal{K}$ admitted by σ, i.e.

$$K^{(\sigma)} \in \mathcal{K} \quad \text{s.t.} \; \begin{cases} \sigma_z = - & \forall\, z \in K^{(\sigma)} \\ \sigma_z = + & \forall\, z \in \partial_V K^{(\sigma)} \cap K_x \end{cases} \tag{3.4}$$

With this notation the event $\{\sigma : \sigma_x = -\}$ can be expressed by means of disjoint events as

$$\{\sigma : \sigma_x = -\} \;=\; \bigsqcup_{m\ge1} \bigsqcup_{C\in\mathcal{K}_m} \{\sigma : K^{(\sigma)} \;=\; C\} \,. \tag{3.5}$$

and then we get

$$\mu_{K_x}^{-}(\sigma_x = -) \;=\; \sum_{m\ge1} \; \sum_{C\in\mathcal{K}_m} \mu_{K_x}^{-}(K^{(\sigma)} = C) \,. \tag{3.6}$$

Let us introduce the symbol $\sigma \sim C$ for a configuration σ such that $\sigma_C = -$ and $\sigma_{\partial_V C \cap K_x} = +$. The main step in the proof is to show the following claim:

Claim 3.4. *Let B a ball in the hyperbolic graph $\mathbb{H}(v,s)$ and assume that $v > 4$ and $s > 3$. For any subset $C \subset K_x$ then it holds*

$$\mu_{K_x}^{-}(\sigma \sim C) \le e^{-c_1 \beta |C|} \,, \tag{3.7}$$

with $c_1 = c_1(v,s)$ positive finite constant.

The proof of Claim 3.4 is postponed to section 3.2. Let us assume for the moment its validity and complete the proof of the case $\ell = 1$. By Claim 3.4 and from the definition of $K^{(\sigma)}$, we get

$$\mu^-_{K_x}(K^{(\sigma)} = C) \leq e^{-c_1 \beta |C|} \,. \tag{3.8}$$

We now recall the following Lemma due to Kesten (see [14]).

Lemma 3.5. *Let G an infinite graph with maximum degree Δ and let $\mathcal{C}_m$ be the set of connected sets with m vertices containing a fixed vertex v. Then $|\mathcal{C}_m| \leq (e(\Delta+1))^m$.*

Applying Lemma 3.5 to the set $\mathcal{K}_m$, we obtain the bound $|\mathcal{K}_m| \leq e^{c_2 m}$, with $c_2 = 1 + \log(v+1)$. Continuing from (3.6), we finally get that for all $\beta' = c_1\beta - c_2 > 0$, i.e. for all $\beta > \frac{c_2}{c_1}$,

$$\begin{aligned} \mu^-_{K_x}(\sigma_x = -) &\leq \sum_{m \geq 1} \sum_{C \in \mathcal{C}^x_m} e^{-c_1 \beta m} \\ &\leq \sum_{m \geq 1} e^{-c_1 \beta m} e^{c_2 m} \\ &\leq c e^{-\beta'} \end{aligned} \tag{3.9}$$

which concludes the proof of (3.3) in the case $\ell = 1$.
Notice that the argument above only involves the spin at x and thus applies for all couples of $x, y \in L_i$, independently for their d_i-distance. Anyway, when $d_i(x,y) > 1$ this method does not provide the decay with the distance stated in Proposition 3.3, and thus a different approach is required.

3.1.2 Correlation decay: the case $\ell > 1$

Let us now consider two vertices $x, y \in L_i$ such that $d_i(x,y) = \ell > 1$. Before defining new objects, we want to clarify the main idea beyond the proof. Since the measure $\mu^\sigma_{K_x}$ fixes the configuration on all the sites in $K^c_x \equiv B_i \setminus \{x\}$, the vertex y can communicate with x only through paths going from x to y and crossing vertices in K_x. However, the effect of this communication can be very small respect to the information arriving to x from the $(+)$-boundary. In particular if every path starting from y crosses a $(+)$-spin before arriving to x, then the communication between them is interrupted. Let us formalize this assertion.
We denote by $\mathcal{C}$ the set of connected subsets $C \in K_x \cup \{y\}$ such that $y \in C$, and call an element $C \in \mathcal{C}$ a *component* of y. For every vertex $z \in L_i$, we then denote by N_z the set of nearest neighbors of z belonging to the level L_{i+1}, and introduce the set $\mathcal{C}^\emptyset := \{C \in \mathcal{C} \text{ s.t. } C \cap N_x = \emptyset\}$. Again, for every configuration $\sigma \in \Omega^+$, we define $C^{(\sigma)}$ as the maximal component of y which is negative on $C^{(\sigma)} \cap K_x$, i.e

$$C^{(\sigma)} \in \mathcal{C} \quad \text{s.t.} \begin{cases} \sigma_z = - & \forall\, z \in C^{(\sigma)} \cap K_x \\ \sigma_z = + & \forall\, z \in \partial_V C^{(\sigma)} \cap K_x \end{cases} \tag{3.10}$$

and we observe that the spin on y is not fixed under the event $\{\sigma : \, C^{(\sigma)} = C\}$.

Finally we define the event

$$A := \{\sigma : C^{(\sigma)} \in \mathcal{C}^{\emptyset}\} = \bigsqcup_{C\in\mathcal{C}^{\emptyset}} \{\sigma : C^{(\sigma)} = C\}, \tag{3.11}$$

and perform the following computation

$$\begin{aligned}
\mu_{K_x}^{y,-}(\sigma_x = + \,|\, A) &= \sum_{C\in\mathcal{C}^{\emptyset}} \mu_{K_x}^{y,-}(\sigma_x = +, C^{(\sigma)} = C \,|\, A) \\
&= \frac{\sum_{C\in\mathcal{C}^{\emptyset}} \mu_{K_x}^{y,-}(\sigma_x = +, C^{(\sigma)} = C\,)}{\sum_{C\in\mathcal{C}^{\emptyset}} \mu_{K_x}^{y,-}(\,C^{(\sigma)} = C\,)} \\
&= \frac{\sum_{C\in\mathcal{C}^{\emptyset}} \mu_{K_x}^{y,-}(\sigma_x = + \,|\, C^{(\sigma)} = C\,)\mu_{K_x}^{y,-}(\,C^{(\sigma)} = C\,)}{\sum_{C\in\mathcal{C}^{\emptyset}} \mu_{K_x}^{y,-}(\,C^{(\sigma)} = C\,)} \\
&\geq \min_{C\in\mathcal{C}^{\emptyset}} \mu_{K_x}^{y,-}(\sigma_x = + \,|\, C^{(\sigma)} = C)\,.
\end{aligned} \tag{3.12}$$

Notice that when the measure $\mu_{K_x}^{y,-}$ is conditioned on the event $\{\sigma : C^{(\sigma)} = C\}$, the spin configuration on $\partial_V C$ is completely determined by the boundary condition: on $\partial_V C \cup K_x$ it is given by all $(+)$-spins and on $\partial_V C \cup K_x^c$ it corresponds to $\sigma^{y,-}$. It follows that the spins on $K_x \setminus (C \cup \partial_V C)$ become independent from the spins on C and then we have

$$\begin{aligned}
\mu_{K_x}^{y,-}(\,\cdot\,|\, C^{(\sigma)} = C) &= \mu_{K_x}^{y,-}(\,\cdot\,|\sigma_z = +\,,\ z \in \partial_V C) \\
&= \mu_{K_x}^{y,+}(\,\cdot\,|\sigma_z = +\,,\ z \in C \cup \partial_V C) \\
&\geq \mu_{K_x}^{y,+}(\,\cdot\,)
\end{aligned} \tag{3.13}$$

where the last inequality follows by stochastic domination. Being $\{\sigma : \sigma_x = +\}$ an increasing event, from (3.12) and (3.13) we get

$$\mu_{K_x}^{y,-}(\sigma_x = + \,|\, A) \geq \mu_{K_x}^{y,+}(\sigma_x = +)\,,$$

which with the obvious fact that $\mu_{K_x}^{y,-}(\sigma_x = +) \geq \mu_{K_x}^{y,-}(\sigma_x = + \,|\, A)\, \mu_{K_x}^{y,-}(A)$, implies

$$\mu_{K_x}^{y,+}(\sigma_x = +) - \mu_{K_x}^{y,-}(\sigma_x = +) \leq \mu_{K_x}^{y,-}(A^c)\,. \tag{3.14}$$

Because A^c is a decreasing event, it holds by monotonicity that $\mu_{K_x}^{y,-}(A^c) \leq \mu_{K_x}^{-}(A^c)$, where we recall that $\mu_{K_x}^{-}$ denotes the measure on K_x conditioned on having all $(-)$-spins on K_x^c. We now focus on $\mu_{K_x}^{-}(A^c)$.

Let $\mathcal{C}^{\neq\emptyset}$ denote the set of components of y with nonempty intersection with N_x, and for every $m \in \mathbb{N}$, let $\mathcal{C}_m$ be the set of components in $C^{\neq\emptyset}$ with m vertices, i.e

$$\mathcal{C}_m := \{C \in \mathcal{C}^{\neq\emptyset} \text{ s.t. } |C| = m\} \qquad \mathcal{C}^{\neq\emptyset} := \bigsqcup_{m>0} \mathcal{C}_m.$$

Notice that a component of y containing a vertex in N_x has at least cardinality $\ell+1$, since $d_i(x,y) = \ell$. Thus A^c can be expressed by means of disjoint events as

$$A^c = \bigsqcup_{m\geq\ell+1} \bigsqcup_{C\in\mathcal{C}_m} \{\sigma : C^{(\sigma)} = C\}, \tag{3.15}$$

and we get

$$\mu^-_{K_x}(A^c) \;=\; \sum_{m\geq \ell+1}\;\sum_{C\in\mathcal{C}_m} \mu^-_{K_x}(C^{(\sigma)}=C)\,. \tag{3.16}$$

Now we observe that the event $\{\sigma : C^{(\sigma)} = C\} \equiv \{\sigma :\ \sigma_{C\setminus\{y\}} = - \,,\ \sigma_{\partial_V C\cap K_x} = +\}$ is a subset of $\{\sigma :\ \sigma_{C\setminus\{y\}} = - \,,\ \sigma_{\partial_V (C\setminus\{y\})\cap K_x} = +\} \equiv \{\sigma : \sigma \sim C\setminus\{y\}\}$. Applying again the result stated in Claim 3.4 to the set $C\setminus\{y\}$, we obtain the bound

$$\mu^-_{K_x}(C^{(\sigma)}=C) \leq e^{-c_1\,\beta(|C|-1)}\,, \tag{3.17}$$

which holds under the same hypothesis of the claim. Continuing from (3.16) we get that for all $\beta' = c_1\beta - c_2 > 0$, i.e. for all $\beta > c_2/c_1$,

$$\begin{aligned}\mu^-_{K_x}(A^c) \;&\leq\; \sum_{m\geq \ell+1}\;\sum_{C\in\mathcal{C}_m} e^{-c_1\,\beta(m-1)}\\ &\leq\; e^{c_2}\sum_{m\geq\ell} e^{-c_1\beta m}e^{c_2 m}\\ &\leq\; ce^{-\beta'\ell}\end{aligned} \tag{3.18}$$

where in the second line we used the bound $|\mathcal{C}_m| \leq e^{c_2 m}$ due to Lemma 3.5. This conclude the proof of Proposition 3.3. In the next section we will go back and prove Claim 3.4.

3.2 Proof of Claim 3.4

To estimate the probability $\mu^-_{K_x}(\sigma \sim C)$, we now appeal to a kind of Peierls argument that runs as follows (see also [13]). For a given subset $C\subset K_x$, we first consider the edge boundary $\partial_E C$ and define

$$\begin{aligned}\partial_+ C \;&:=\; \{e=(z,w)\in\partial_E C \;:\; z,\, w\in K_x\}\\ \partial_- C \;&:=\; \{e=(z,w)\in\partial_E C \;:\; z \text{ or } w\in K_x^c\}\end{aligned} \tag{3.19}$$

with $\partial_E C \;=\; \partial_+ C\cup\partial_- C$. The meaning of this notation can be better understood if we consider a configuration $\sigma\in\Omega^-_{K_x}$ such that $C^{(\sigma)} = C$ (see (3.10)). In this case σ has $(-)$-spins on both the end-vertices of every edge in $\partial_- C$ and a $(+)$-spin in one end-vertex of every edge in $\partial_+ C$. Similarly if we consider σ such that $K^{(\sigma)} = C$ (see (3.4)).

For every $\sigma\in\Omega^-_{K_x}$ such that $\sigma\sim C$, let $\sigma^*\in\Omega^-_{K_x}$ denote the configuration obtained by a global spin flip of σ on the subset C, and observe that the map $\sigma\to\sigma^*$ is injective. This flipping changes the Hamiltonian contribute of the interactions just along the edges in $\partial_E C$; in particular σ^* loses the positive contribute of the edges in $\partial_+ C$ and gains the contribute of the edges in $\partial_- C$ and then we get

$$H^-_{K_x}(\sigma^*) \;=\; H^-_{K_x}(\sigma) - 2(|\partial_+ C| - |\partial_- C|)\,. \tag{3.20}$$

Finally, we perform the following computation

$$
\begin{aligned}
\mu_{K_x}^{-}(\sigma \sim C) &= \sum_{\{\sigma:\sigma\sim C\}} \frac{e^{-\beta H_{K_x}^{-}(\sigma)}}{Z_{K_x}^{-}} \\
&\leq \frac{\sum_{\{\sigma:\sigma\sim C\}} e^{-\beta H_{K_x}^{-}(\sigma)}}{\sum_{\{\sigma:\sigma\sim C\}} e^{-\beta H_{K_x}^{-}(\sigma^*)}} \\
&= e^{-2\beta(|\partial_+ C|-|\partial_- C|)}, \qquad (3.21)
\end{aligned}
$$

where in the first inequality we reduced the partition function to a summation over $\{\sigma : \sigma \sim C\}$ and then we applied (3.20).
To complete the proof of Claim 3.4, we have to verify the bound

$$|\partial_+ C| - |\partial_- C| \leq \frac{c_1}{2}|C|\,. \qquad (3.22)$$

To establish inequality (3.22), we make use of the following lemmas. The first one concerns the growth properties of the nearest neighborhood of a vertex in $B \subset \mathbb{H}(v,s)$, while the second one is intrinsic related to our formulation of the problem.

Lemma 3.6 (link-property)**.** *Consider the hyperbolic graph $\mathbb{H}(v,s)$ and assume that $s > 3$. Then for any vertex $x \in L_i$, respect to some reference point $o \in V$, the number of neighbors of x in L_{i+1} is at least $v-2$.*

Proof. Being v the vertex degree of the graph, Lemma 3.6 can be equivalently stated by saying that each vertex $x \in L_i$, with respect to a given root o, is linked to the vertices in $L_i \cup L_{i-1}$ (same or previous level) by at most 2 edges. Indeed, as can be directly verified from figure (see 2.1), only three situations can appear regarding these edges (see fig. 3.2):

1. x is linked with two ancestors and none vertex on the same level;

2. x is linked with one ancestor and one vertex on the same level;

3. x is linked with one ancestor.

The exclusion of the other possibilities comes from the planarity of the graph together with the requirement $s > 3$, which restrict the result to non-triangular tilings. □

Lemma 3.7. *Consider the hyperbolic graph $\mathbb{H}(v,s)$ and assume that $v > 4$ and $s > 3$. For every subset $C \subset K_x$, we then have*

$$|\partial_+ C| \geq (1+\delta)|\partial_- C|\,, \qquad (3.23)$$

where $\delta = \frac{v-4}{2} > 0$.

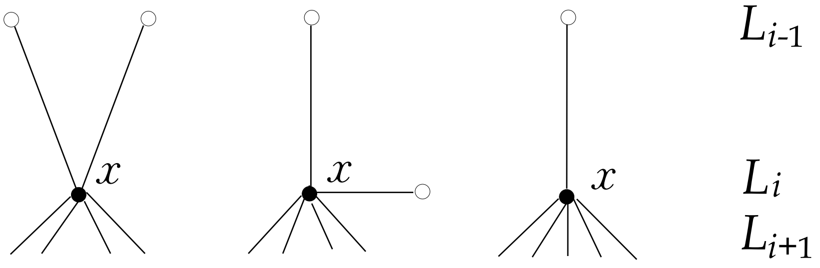

Figure 3.2: The three possible connections between a site on a level and the neighbors

Proof. If $\partial_- C = \emptyset$, Lemma 3.7 follows trivially. Thus let us suppose that $\partial_- C \neq \emptyset$ and consider the subset $S := C \cap \partial_V K_x^c$, corresponding to the set of the end-vertices in C of the edges in $\partial_- C$. For every vertex z, let us also introduce the notation P_z for the set of nearest neighbors of z belonging to the same or previous level of z, so that $P_z + N_z = v$. Notice that by Lemma 3.6 $P_z \leq 2$ and $N_z \geq v - 2$. We now assert that every edge in $\partial_- C$ can be written as $e = (z, w)$, with $z \in S$ and $w \in P_z$. Indeed, the end-vertices of $\partial_- C$ which do not belong to S, belong by definition to $K_x^c = B_i$, while due to the shape of K_x, the set S is a subset of $L_i \cup L_{i+1}$.
From this observation and using the link property, we get

$$\begin{aligned} |\partial_- C| &= \sum_{z \in S} |\{w \in P_z \,, w \notin C\}| \\ &= \sum_{z \in S} |\{w \in P_z\}| - \sum_{z \in S} |\{w \in P_z \,, w \in C\}| \\ &\leq 2|S| - \sum_{z \in S} |\{w \in P_z \,, w \in S\}| \,, \end{aligned} \tag{3.24}$$

which corresponds to the inequality

$$|S| \geq \frac{1}{2} \left(|\partial_- C| + \sum_{z \in S} |\{w \in P_z \,, w \in S\}| \right) . \tag{3.25}$$

Now we want to find a suitable relation between $|\partial_+ C|$ and $|S|$. To this aim, let us consider the increasing sequence of subsets of C defined as follows

$$C_0 = S \quad \text{and} \quad C_j = C_{j-1} \cup \{z \in C \cap L_{i+j}\} \;\; \forall j \geq 1 \,, \tag{3.26}$$

and notice that for some finite integer k, depending on C, $C_k \equiv C$. We then introduce the notation $\partial_+ C_j := \{e = (z, w) \in \partial_E(C_j) \,:\, z, w \in K_x\}$ and state that

(i) $|\partial_+ C_0| \geq \sum_{z \in S} |\{w \in N_z \,, w \notin S\}|$;

(ii) $|\partial_+ C_j| \geq |\partial_+ C_{j-1}| + (v - 4) \, |\{z \in C \cap L_{i+j}\}| \,, \;\; \forall j \geq 1$.

Inequality (i) is due to the trivial fact that $\{w \sim z \,, w \notin S \cup K_x^c\} \supset \{w \in N_z \,, w \notin S\}$. To understand inequality (ii), first notice that by construction $C_j \subseteq B_{i+j}$ and

$C_0 \subset B_{i+1}$. Thus, for every $z \in C \cap L_{i+j}$ and $j \geq 1$, there are at most $|P_z|$ edges between z and C_{j-1}, i.e. edges in $\partial_+ C_{j-1} \setminus \partial_+ C_j$, and at least $|N_z|$ edges between z and $K_x \setminus C_j$, i.e. edges in $\partial_+ C_j \setminus \partial_+ C_{j-1}$. Inequality (ii) then follows from the link-property.
From these last inequalities and being $v > 4$ by hypothesis, we get

$$|\partial_+ C| \geq |\partial_+ C_0| \geq \sum_{z \in S} |\{w \in N_z \,, w \notin S\}| \,. \tag{3.27}$$

Remark 3.8. *Since $S \subset \{x\} \cup L_{i+1}$, x is the only vertex in K_x which can satisfies both the conditions $x \in S$ and $N_x \cap S \neq \emptyset$, and then $|\{w \in N_z \,, w \in S\}| = 0$ for all $z \in S$ different from x. However, we prefer to use this more general notation in order to facilitate the extension of this computation to similar object. For example, it easy to verify that all the above construction continues to holds if we consider, instead of K_x, a set $U = F_{i+1} \cup V$ with $V \subset L_i$. This can be useful, for example, to compute the correlation $\mu_U^{y,+}(\sigma_x = +) - \mu_U^{y,-}(\sigma_x = +)$ between two sites $x \in U$ and $y \in U^c$, or simply the probability $\mu_U^-(\sigma_x = -)$. As a special case we can take $V = \emptyset$, so that $U \equiv F_{i+1}$ and $\mu_U \equiv \mu_{i+1}$.*

From (3.25) and(3.27), and again by the link-property, it holds

$$\begin{aligned}
|\partial_+ C| &\geq \sum_{z \in S} |\{w \in N_z \,, w \notin S\}| \\
&= \sum_{z \in S} (v - |\{w \in P_z\}|) - \sum_{z \in S} |\{w \in N_z \,, w \in S\}| \\
&\geq (v-2)|S| - \sum_{z \in S} |\{w \in N_z \,, w \in S\}| \\
&= (v-2)|S| - \sum_{z \in S} |\{w \in P_z \,, w \in S\}| \\
&\geq \frac{(v-2)}{2} |\partial_- C| + \frac{(v-4)}{2} \sum_{z \in S} |\{w \in P_z \,, w \in S\}| \\
&\geq (1 + \frac{(v-4)}{2}) |\partial_- C| \,,
\end{aligned} \tag{3.28}$$

which concludes the proof of Lemma 3.7. □

The proof of Claim 3.4 now follows straightforwardly. From(3.23) and with some trivial computations, we obtain

$$\begin{cases} |\partial_+ C| - |\partial_- C| \geq \frac{\delta}{1+\delta} |\partial_+ C| \\ |\partial_+ C| + |\partial_- C| \leq \frac{2+\delta}{1+\delta} |\partial_+ C| \end{cases} \Longrightarrow |\partial_+ C| - |\partial_- C| \geq \frac{\delta}{2+\delta} |\partial_E C| \,.$$

To bound $|\partial_E C|$, we use the isoperimetric inequality $|\partial_E C| \geq i_e \, |C| \,$, where $i_e \equiv i_e(\mathbb{H}(v,s))$ is the isoperimetric constant of $\mathbb{H}(v,s)$ explicitly computed in [11] as a function of v and s (see Section 2.1). We thus obtain

$$|\partial_+ C| - |\partial_- C| \geq \frac{\delta}{2+\delta} i_e \, |C| \,, \tag{3.29}$$

which together (3.21) yields the inequality (3.7) with constant $c_1 = \frac{\delta i_e}{2(2+\delta)}$. This conclude the proof of Claim 3.4, and thus of Proposition 3.3.

4 Fast mixing inside the plus phase

In this section we will prove that the spectral gap of the Glauber dynamics, in the situation described by Theorem 2.5, is bounded from zero uniformly in the size of the system. From the Definition 2.7 of spectral gap, this is equivalent to show that for all inverse temperature $\beta \gg 1$, it holds the Poincaré inequality

$$\mathrm{Var}(f) \leq c\,\mathcal{D}(f)\,, \quad \forall f \in L^2(\Omega^+, \mathcal{F}, \mu)$$

with constant $c = c(\beta, B)$ independent of the size of B.
First, we give a brief sketch of the proof. The rest of the section is divided into two parts. In the first part, from the mixing property deduced in section 3 and by means of coupling techniques, we derive a Poincaré inequality for some suitable marginal Gibbs measures. Then, in the second part, we will run a recursive argument that together with some estimates, also derived from Proposition 3.3, will yield the Poincaré inequality for the global Gibbs measure μ.

4.1 Plan of the Proof

Let us first recall the following decomposition property of the variance which holds for all subsets $D \subseteq C \subseteq B$

$$\mathrm{Var}^\eta_C(f) \;=\; \mu^\eta_C[\mathrm{Var}_D(f)] + \mathrm{Var}^\eta_C[\mu_D(f)]\,. \tag{4.1}$$

Applying recursively (4.1) to the subsets $B \equiv F_0 \supset F_1 \supset \ldots \supset F_{m+1} \;=\; \emptyset$ and recalling the relations $\mu_i(\mu_{i+1}(f)) \;=\; \mu_i(f)$ and $\mu_{m+1}(f) = f$, we obtain

$$\begin{aligned}
\mathrm{Var}(f) &= \mu[\mathrm{Var}_m(f)] + \mathrm{Var}[\mu_m(f)] \\
&= \mu[\mathrm{Var}_m(\mu_{m+1}(f))] + \mu[\mathrm{Var}_{m-1}(\mu_m(f))] + \mathrm{Var}[\mu_{m-1}(\mu_m(f))] \\
&= \vdots \\
&= \sum_{i=0}^{m} \mu[\mathrm{Var}_i(\mu_{i+1}(f))]\,.
\end{aligned}$$

To simplify notation we define $g_i \;:=\; \mu_i(f)$ for all $i \;=\; 0, \ldots, m+1$; notice that $g_i \in \mathcal{F}_i$. Inserting g_i in (4.2) we then have

$$\mathrm{Var}(f) = \sum_{i=0}^{m} \mu[\mathrm{Var}_i(g_{i+1})]\,. \tag{4.2}$$

The proof of the Poincaré inequality for μ, with constant independent of the size of the system, is given in the following two steps:

1. proving that $\forall\, \tau \in \Omega^+$ and $i \in \{0, \ldots, m\}$, it holds the Poincaré inequality

$$\mathrm{Var}^\tau_i(g_{i+1}) \leq c \sum_{x \in L_i} \mu^\tau_i(\mathrm{Var}_x(g_{i+1})) \tag{4.3}$$

   with constant c uniformly bounded in the size of L_i;

2. relating the local variance of $g_i = \mu_i(f)$ to the local variance of f in order to get an inequality of the kind

$$\sum_{i=0}^{m} \sum_{x \in L_i} \mu(\mathrm{Var}_x(g_{i+1})) \leq \mathcal{D}(f) + \varepsilon \sum_{i=0}^{m} \sum_{x \in L_i} \mu(\mathrm{Var}_x(g_{i+1})) \tag{4.4}$$

with ε a small quantity for $\beta \gg 1$.

Notice that from (4.4) it follows the inequality

$$\sum_{i=0}^{m} \sum_{x \in L_i} \mu(\mathrm{Var}_x(g_{i+1})) \leq (1-\varepsilon)^{-1} \mathcal{D}(f)\,,$$

with $(1-\varepsilon)^{-1} = \Omega(1)$ for all $\beta \gg 1$. Together with Eqs. (4.2) and (4.3), this will establish the required Poincaré inequality for μ and therefore thus conclude of Theorem 2.5.

4.2 Step 1: From correlation decay to Poincaré inequality

In this section we prove that under the same hypothesis of Proposition 3.3, the marginal of the Gibbs measure on suitable defined subsets satisfies a Poincaré inequality with constant independent of the size of the subsets.
In order to state the result in its main generality, let us give a few definitions.

Definition 4.1 (Interval). *A subset $S \subseteq L_i$ is called an interval if its vertices can be ordered as $x_{i_1}, x_{i_2}, \cdots, x_{i_k}$, with $d_i(x_{i_j}, x_{i_{j+1}}) = 1$ for all $j = 1, \cdots, k-1$.*

Let us fix an interval $S \subset L_i$. For every configuration $\tau \in \Omega^+$, we define the measure

$$\nu_S^\tau(\sigma) = \sum_{\eta : \eta_S = \sigma_S} \mu(\eta \,|\, \tau \in \mathcal{F}_{B_i \setminus S})\,, \tag{4.5}$$

which is the marginal of the Gibbs measure $\mu^\tau_{F_{i+1} \cup S}$ on S. We denote by $\mathrm{Var}_{\nu_S^\tau}$ the variance w.r.t. ν_S^τ and then state the following:

Theorem 4.2. *For all $\beta \gg 1$ and for every interval $S \subseteq L_i$, $\tau \in \Omega^+$ and $f \in L^2(\Omega, \mathcal{F}_S, \nu_S^\tau)$, the measure ν_S^τ satisfies the Poincaré inequality*

$$Var_{\nu_S^\tau}(f) \leq c \sum_{x \in S} \nu_S^\tau(\,Var_x(f))\,. \tag{4.6}$$

with $c = c(\beta) = 1 + O(e^{-c\beta})$.

Remark 4.3. *Before proceeding with the proof of Theorem 4.2, we point out that this result includes, as a particular case, inequality (4.3), and thus conclude the first part of the proof of Theorem 2.5. Indeed, taking $S = L_i$ and $f = g_{i+1}$ and observing that $\mu(\,\cdot\,|\mathcal{F}_{B_i \setminus S}) \equiv \mu_i$ and $\mu_i^\tau(g_{i+1}) \equiv \nu_{L_i}^\tau(g_{i+1})$, we can apply the result of Theorem 4.2 to obtain the Poincaré inequality*

$$Var_i^\tau(g_{i+1}) \leq c \sum_{x \in L_i} \mu_i^\tau(\,Var_x(g_{i+1}))\,.$$

4.2.1 Proof of Theorem 4.2

The proof of Theorem 4.2 rests on the so called *coupling technique*. This is useful method to bound from above the mixing time of Markov processes, introduced for the first time in this setting by Aldous [1] and subsequently refined to the *path coupling* [3, 18]. See also [15] for a wider discussion on the coupling method.
A *coupling* of two measure μ_1 and μ_2 on Ω is any joint distribution ρ on $\Omega \times \Omega$ whose marginal are μ_1 and μ_2 respectively. Here, we want to construct a coupling of two Glauber dynamics on Ω_S with same reversible measure ν_S^τ but different initial configurations. We denote by $\mathcal{L}_S$ the generator of this dynamics. We also recall that for all $\sigma \in \Omega_S^\tau$, $x \in S$ and $a \in \{\pm 1\}$, the jump rates of the heat bath version of the dynamics (see Def. 2.4) are given by

$$\begin{aligned} c_x(\sigma, a) &= \nu_S^\tau(\sigma_x = a \,|\, \sigma \in \mathcal{F}_{S\setminus x}) \\ &= \mu(\sigma_x = a \,|\, \sigma \in \mathcal{F}_{S\setminus x}, \tau \in \mathcal{F}_{B_i \setminus S}) \\ &= \mu^\sigma_{K_x}(\sigma_x = a)\,, \end{aligned} \tag{4.7}$$

where in the second line we applied the Definition 4.5 of ν_S^τ and exploited the fact that $\{\sigma_x = a\} \in \mathcal{F}_S$, and in the third line we introduced the notation $K_x = \{x\} \cup F_{i+1}$ as in section 3.
We now consider the coupled process $(\sigma(t), \eta(t))_{t \geq 0}$ on $\Omega_S \times \Omega_S$ defined as follows. Given the initial configurations (σ, η), we let the two dynamics evolve at the same time and update the configurations at the same vertex. We then chose the coupling jump rates $\tilde{c}_x((\sigma, a), (\eta, b))$ to go from (σ, η) to $(\sigma^{x,a}, \eta^{x,b})$, with $a, b \in \{\pm 1\}$, as the optimal coupling (see [15]) between the jump rates $\mu^\sigma_{K_x}(\sigma_x = a)$ and $\mu^\eta_{K_x}(\sigma_x = b)$. More explicitly, for $a \in \{\pm 1\}$, they are given by

$$\begin{cases} \tilde{c}_x((\sigma, a), (\eta, a)) = \min\{\mu^\sigma_{K_x}(\sigma_x = a)\,;\, \mu^\eta_{K_x}(\sigma_x = a)\} \\ \tilde{c}_x((\sigma, a), (\eta, -a)) = \max\{0\,;\, \mu^\sigma_{K_x}(\sigma_x = a) - \mu^\eta_{K_x}(\sigma_x = a)\} \end{cases} \tag{4.8}$$

We denote by $\widetilde{\mathcal{L}}$ the generator of the coupled process, and by $\widetilde{P}_t$ the correspondent Markov semigroup. Notice that from the choice of coupling jump rates as in (4.8), we get that the probability of disagreement in x after one update in x of (σ, η), is given by

$$P^x_{dis}(\sigma, \eta) := |\mu^\sigma_{K_x}(\sigma_x = +) - \mu^\eta_{K_x}(\sigma_x = +)|\,. \tag{4.9}$$

Let us now consider the subset $H \subset \Omega_S \times \Omega_S$ given by all couples of configurations which differ by a single spin flip in some vertex of S. One can easily verify that the graph $(\Omega_S \times \Omega_S,\, H)$ is connected and that the induced graph distance $D(\sigma, \tau)$ between configurations $(\sigma, \tau) \in \Omega_S \times \Omega_S$ just corresponds to their Hamming distance. Let us also denote by $\mathbb{E}_{\sigma,\eta}[D(\sigma(t), \eta(t))] \equiv \mathbb{E}[D(\sigma(t), \eta(t)) | (\sigma, \eta)]$ the average distance at time t between two coupled configurations of the process starting at (σ, η). We claim the following:

Claim 4.4. *For all $\beta \gg 1$ there exists a positive constant $\alpha \equiv \alpha(\beta)$ such that, for every initial configurations $(\sigma, \eta) \in H$, the coupling process $(\sigma(t), \eta(t))_{t \geq 0}$ verifies the inequality*

$$\frac{d}{dt}\, \mathbb{E}_{\sigma,\eta}[D(\sigma(t), \eta(t))]\,|_{t=0} \leq -\,\alpha\,. \tag{4.10}$$

Proof of Claim 4.4. Let us first explicit the derivative in t of the average distance as

$$\begin{aligned}\frac{d}{dt}\,\mathbb{E}_{\sigma,\eta}[D(\sigma(t),\eta(t))]\,|_{t=0} &= \frac{d}{dt}\,\left(\widetilde{P}_t D\right)(\sigma,\eta)\,|_{t=0} \;=\; (\widetilde{\mathcal{L}}\,D)(\sigma,\eta)\\ &= \sum_{x\in S}\;\sum_{a,b\in\{\pm1\}} \tilde{c}_x((\sigma,a)(\eta,b))[D(\sigma^{x,a},\eta^{x,b})-D(\sigma,\eta)]\,. \end{aligned}\tag{4.11}$$

Since $(\sigma,\eta)\in H$, there exists a vertex $y\in S$ such that $\eta=\sigma^y$. If $x=y$, then $P^x_{dis}(\sigma,\sigma^x)=0$ and the distance between the updated configurations decreases of one. While if $x\neq y$, with probability $P^x_{dis}(\sigma,\sigma^y)$ the updated configurations have different spin at x and their distance increases of one. Continuing from (4.11), we get

$$\begin{aligned}\frac{d}{dt}\,\mathbb{E}_{\sigma,\eta}[D(\sigma(t),\eta(t))]\,|_{t=0} &= -1+\sum_{\substack{x\in S\\ x\neq y}} P^x_{dis}(\sigma,\sigma^y)\\ &\le -1+c\sum_{\ell\ge1}e^{-\beta'\ell}\\ &\le -(1-ce^{-\beta'})\,, \end{aligned}\tag{4.12}$$

where in the second line we used the bound

$$P^x_{dis}(\sigma,\sigma^y)=|\mu^{\sigma}_{K_x}(\sigma_x=+)-\mu^{\sigma^y}_{K_x}(\sigma_x=+)|\le ce^{-\beta' d_i(x,y)}\,,\tag{4.13}$$

which holds for all $\beta'=c_1\beta-c_2>0$ as stated in Proposition 3.3. Claim 4.4 follows taking $\alpha=(1-ce^{-\beta'})$ and β sufficiently large. ☐

Using the *path coupling* technique (see [3]), we can extend the result of Claim 4.4 to arbitrary initial configurations $(\sigma,\eta)\in\Omega_S\times\Omega_S$, to obtain

$$\frac{d}{dt}\,\mathbb{E}_{\sigma,\eta}[D(\sigma(t),\eta(t))]\,|_{t=0}\le -\,\alpha D(\sigma,\eta)\,.\tag{4.14}$$

From (4.14) it now follows straightforwardly that $\mathbb{E}_{\sigma,\eta}[D(\sigma(t),\eta(t))]\le e^{-\alpha\,t}D(\sigma,\eta)$, and then we get

$$\mathbb{P}(\sigma(t)\neq\eta(t))\le\mathbb{E}_{\sigma,\eta}(D(\sigma(t),\eta(t)))\le e^{-\alpha\,t}D(\sigma,\eta)\,.\tag{4.15}$$

To bound the spectral gap $c_{gap}(\nu^\tau_S)$ of the dynamics on S, we then consider an eigenfunction f of $\mathcal{L}_S$ with eigenvalue $-c_{gap}(\nu^\tau_S)$, so that

$$\mathbb{E}_\sigma f(\sigma(t))=e^{t\mathcal{L}_S}f(\sigma)\;=\;e^{-c_{gap}(\nu^\tau_S)\,t}f(\sigma)\,.$$

Since the identity function has eigenvalue zero, and therefore is orthogonal to f, it holds that $\nu^\tau_S(f)=0$ and $\nu^\tau_S(\mathbb{E}_\eta f(\eta(t)))=0$, where ν^τ_S is the invariant measure for $\mathcal{L}_S$. From these considerations and inequality (4.15), we obtain

$$\begin{aligned} e^{t\mathcal{L}_S}f(\sigma) &= \mathbb{E}_\sigma f(\sigma(t))-\nu^\tau_S(\mathbb{E}_\eta f(\eta(t)))\\ &= \sum_\eta \nu^\tau_S(\eta)[\,E_\sigma f(\sigma(t))-E_\eta f(\eta(t))\,]\\ &\le 2\|f\|_\infty \sup_{\sigma,\eta}\mathbb{P}(\sigma(t)\neq\eta(t))\\ &\le 2\|f\|_\infty|S|e^{-\alpha\,t}\,. \end{aligned}\tag{4.16}$$

From the last computation, which holds for all $\sigma \in \Omega_S^\tau$ and for all t, we finally obtain that $c_{gap}(\nu_S^\tau) \geq \alpha$ independently of the size of S, which implies the Poincaré inequality (4.6) with constant $c = \alpha^{-1} = 1 + O(e^{-c\beta})$. This concludes the proof of Theorem 4.2. $\square$

4.3 Step 2: Poincaré inequality for the global Gibbs measure

With the previous analysis we obtained a Poincaré inequality for the marginal of the measure μ_i on the level L_i, which inserted in formula (4.2) provides the bound

$$\mathrm{Var}(f) \;\leq\; c \sum_{i=0}^{m} \sum_{x \in L_i} \mu\left[\mu_i(\mathrm{Var}_x(g_{i+1}))\right] . \tag{4.17}$$

Using the same notation as in [23], let us denote the sum in the r.h.s. of (4.17) by $\mathrm{P}_{\mathrm{var}}(f)$. The aim of the following analysis is to analyze $\mathrm{P}_{\mathrm{var}}(f)$ in order to find an inequality of the kind $\mathrm{P}_{\mathrm{var}}(f) \leq D(f) + \varepsilon \mathrm{P}_{\mathrm{var}}(f)$, with $\varepsilon = \varepsilon(\beta) < 1$ independent of the size of the system. This would imply that

$$\mathrm{Var}(f) \leq c \cdot \mathrm{P}_{\mathrm{var}}(f) \leq \frac{c}{1-\varepsilon} \mathcal{D}(f)$$

and then, from the remark made at the end of section 4.1, the proof of Theorem 2.5 would follow.

In the next sections, we will first relate the local variance of $g_i = \mu_i(f)$ with the local variance of f. This will produce a covariance term that will be then analyzed using a recursive argument.

4.3.1 Reduction to covariance

In order to reconstruct from (4.17) the Dirichlet form of f, we want to extract the local variance of f from the local variance of g_{i+1}. For $x \in L_i$ and $\tau \in \Omega^+$, let $p(\tau) = \mu_x^\tau(\sigma_x = +)$ and $q(\tau) = \mu_x^\tau(\sigma_x = -)$, and then consider the quantity

$$\mu_i\left(\mathrm{Var}_x(g_{i+1})\right) \;=\; \sum_\tau \mu_i(\tau) p(\tau) q(\tau) \left(\nabla_x \, g_{i+1}(\tau)\right)^2 . \tag{4.18}$$

Using the martingale property $g_{i+1} = \mu_{i+1}(g_{i+2})$, the local variance $\mathrm{Var}_x(g_{i+1})$ can be split in two terms stressing the dependence on x of g_{i+2} and of the conditioned measure μ_{i+1}. Let us formalize this idea.

For a given configuration $\tau \in \Omega^+$ we introduce the symbols

$$\tau^+ \;:=\; \begin{cases} \tau_y^+ = \tau_y & \text{if } y \neq x \\ \tau_y^+ = + & \text{if } y = x \end{cases} \qquad \tau^- \;:=\; \begin{cases} \tau_y^- = \tau_y & \text{if } y \neq x \\ \tau_y^- = - & \text{if } y = x \end{cases}$$

and then define the density

$$h_x(\sigma) := \frac{\mu_{i+1}^{\tau^+}(\sigma)}{\mu_{i+1}^{\tau^-}(\sigma)} , \qquad \text{with} \qquad \mu_{i+1}^{\tau^-}(h_x) = 1 . \tag{4.19}$$

Whit this notation and continuing from (4.18), it holds

$$\begin{aligned}\mu_i(\mathrm{Var}_x(g_{i+1})) &= \sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left[\nabla_x\,\mu_{i+1}(g_{i+2})(\tau)\right]^2 \\ &= \sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left[\mu_{i+1}^{\tau^-}(g_{i+2}) - \mu_{i+1}^{\tau^+}(g_{i+2})\right]^2 \\ &= \sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left[\mu_{i+1}^{\tau^+}(\nabla_x g_{i+2}) - \mu_{i+1}^{\tau^-}(h_x, g_{i+2})\right]^2 \\ &\le 2\sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left[\left(\mu_{i+1}^{\tau^+}(\nabla_x g_{i+2})\right)^2 + \left(\mu_{i+1}^{\tau^-}(h_x, g_{i+2})\right)^2\right] \end{aligned} \tag{4.20}$$

Now it is simple to verify that $\mu_{i+1}^{\tau^+}(\nabla_x g_{i+2}) = \mu_{i+1}^{\tau^+}(\nabla_x f)$. To understand this fact it is enough to observe that the dependence on x of $g_{i+2} = \mu_{i+2}(f)$ comes only from f, since the b.c. on B_{i+1} are fixed equal to τ^+. Substituting $\mu_{i+1}^{\tau^+}(\nabla_x f)$ and applying the Jensen inequality, the first term of (4.20) can be bounded as

$$\sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left(\mu_{i+1}^{\tau^+}(\nabla_x g_{i+2})\right)^2 \le \mu_i\left(q(\tau)(\,\nabla_x f\,)^2\right) \le \mu_i\left(\mathrm{Var}_x(f)\right)\,. \tag{4.21}$$

Summing both sides of (4.20) over $x \in L_i$ and $i \in \{0, \dots, m\}$, and taking in account inequality (4.21), we obtain

$$\mathrm{Pvar}(f) \;\le\; 2\,\mathcal{D}(f) + 2\sum_{i=0}^{m-1}\sum_{x\in L_i}\mu\left[\sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left(\mu_{i+1}^{\tau^-}(h_x, g_{i+2})\right)^2\right], \tag{4.22}$$

where we excluded the value m in the summation over i because $g_{m+2} \equiv f$ is constant w.r.t. μ_{m+1} and thus $\mu_{m+1}^{\tau^-}(h_x, g_{m+2}) \equiv 0$.
The more involved analysis of the covariance $\mu_{i+1}^{\tau^-}(h_x, g_{i+2})$ will be discuss in the next section.

4.3.2 Recursive argument

Before going on with the proof, we need some more definitions and notation. Recall that for every $x \in L_i$, we denoted by N_x the set of nearest neighbors of x in the level L_{i+1}. Given $x \in L_i$ and $\ell \in \mathbb{N}$, let us define the following objects:

(i) $N_{x,\ell} := \{y \in L_{i+1} \,:\, d_{i+1}(y, N_z) \le \ell\}$ is the ℓ-neighborhood of N_x in L_{i+1};

(ii) $\mathcal{F}_{x,\ell} := \sigma\left(\sigma_y \,:\, y \in B_{i+1} \setminus N_{x,\ell}\right)$ is the σ-algebra generated by the spins on $B_{i+1} \setminus N_{x,\ell}$;

(iii) $\mu_{x,\ell}(\,\cdot\,) := \mu\left(\cdot\,|\mathcal{F}_{x,\ell}\right)$ is the Gibbs measure conditioned on the σ-algebra $\mathcal{F}_{x,\ell}$.

We remark that $N_{x,0} = N_x$ and that there exists some $\ell_0 \le |L_{i+1}|$ such that, for all integers $\ell \ge \ell_0$, it holds that $N_{x,\ell} = L_{i+1}$ and $\mu_{x,\ell} = \mu_{i+1}$.
We also remark that the family of σ-algebras $\{\mathcal{F}_{x,\ell}\}_{\ell=0,1,\dots,\ell_0}$ is a filtration. In particular, for any function $f \in L^1(\Omega, \mathcal{F}_{i+1}, \mu)$, the set of variables $\{\mu_{x,\ell}(f)\}_{\ell\in\mathbb{N}}$ is a Martingale.

Let us now come back to our proof and recall the following property of the covariance (the analogous of property (4.1) for the variance)

$$\mu^\eta_C(f,g) \;=\; \mu^\eta_C(\mu_D(f,g)) + \mu^\eta_C(\mu_D(f),\mu_D(f)) \quad , \quad \text{for } D\subseteq C\subseteq B\;. \tag{4.23}$$

Since the support of μ_{i+1} strictly contains the support of $\mu_{x,0}$, we can apply the property (4.23) to the square covariance $(\mu^{\tau^-}_{i+1}(h_x,g_{i+2}))^2$ appearing in (4.22) in order to get

$$(\mu^{\tau^-}_{i+1}(h_x,g_{i+2}))^2 \;\le\; 2(\mu^{\tau^-}_{i+1}(\mu_{x,0}(h_x,g_{i+2})))^2 + 2(\mu^{\tau^-}_{i+1}(\mu_{x,0}(h_x),\mu_{x,0}(g_{i+2})))^2\,. \tag{4.24}$$

The first term in the r.h.s. of (4.24) can be bounded by the Schwartz inequality as

$$(\mu^{\tau^-}_{i+1}(\mu_{x,0}(h_x,g_{i+2})))^2 \le \mu^{\tau^-}_{i+1}(\mathrm{Var}_{x,0}(h_x))\cdot\mu^{\tau^-}_{i+1}(\mathrm{Var}_{x,0}(g_{i+2}))\,. \tag{4.25}$$

The second term can be rearranged and bounded as follows:

$$\begin{aligned}
[\mu^{\tau^-}_{i+1}(\mu_{x,0}(h_x),\mu_{x,0}(g_{i+2}))]^2 \;&=\; \Big[\mu^{\tau^-}_{i+1}\,(\,\mu_{x,0}(h_x)-\mu_{i+1}(h_x),g_{i+2}\,)\Big]^2\\
&=\; \left[\mu^{\tau^-}_{i+1}\left(\sum_{\ell=1}^{\ell_0}(\mu_{x,\ell-1}(h_x)-\mu_{x,\ell}(h_x))\,,g_{i+2}\right)\right]^2\\
&\le\; \sum_{\ell=1}^{\ell_0}\ell^2\,\Big[\mu^{\tau^-}_{i+1}(\mu_{x,\ell-1}(h_x)-\mu_{x,\ell}(h_x),g_{i+2})\Big]^2\\
&=\; \sum_{\ell=1}^{\ell_0}\ell^2\,\Big[\mu^{\tau^-}_{i+1}\,(\mu_{x,\ell}(\mu_{x,\ell-1}(h_x),g_{i+2}))\Big]^2\;,
\end{aligned} \tag{4.26}$$

where in the second line, due to the fact that $\mu_{x,\ell_0}=\mu_{i+1}$ for some ℓ_0, we substituted $\mu_{x,0}(h_x)-\mu_{i+1}(h_x)$ by the telescopic sum $\sum_{\ell=1}^{\ell_0}(\mu_{x,\ell-1}(h_x)-\mu_{x,\ell}(h_x))$. Applying again the Cauchy-Schwartz inequality to the last term in (4.26), we get

$$\begin{aligned}
&[\mu^{\tau^-}_{i+1}(\mu_{x,0}(h_x),\mu_{x,0}(g_{i+2}))]^2 \le\\
&\qquad \le \sum_{\ell=1}^{\ell_0}\ell^2\,\mu^{\tau^-}_{i+1}\,(\mathrm{Var}_{x,\ell}(\mu_{x,\ell-1}(h_x)))\cdot\mu^{\tau^-}_{i+1}\,(\mathrm{Var}_{x,\ell}(g_{i+2}))
\end{aligned} \tag{4.27}$$

To conclude the estimate on the covariance, it thus remain to analyze the three quantities appearing in (4.25) and (4.27):

(i) $\mu^{\tau^-}_{i+1}\,(\mathrm{Var}_{x,\ell}(g_{i+2}))$, for all $\ell=0,1,\dots,\ell_0$;

(ii) $\mu^{\tau^-}_{i+1}(\mathrm{Var}_{x,0}(h_x))$;

(iii) $\mu^{\tau^-}_{i+1}\,(\mathrm{Var}_{x,\ell}(\mu_{x,\ell-1}(h_x)))$, for all $\ell=1,\dots,\ell_0\,$,

We proceed in estimating separately these three terms; at the end we will come back to Eqs. (4.22), (4.25) and (4.27).

First term: Poincaré inequality for the marginal measure on $N_{x,\ell}$.
Let us consider the variance $\mathrm{Var}_{x,\ell}(g_{i+2})$ appearing in (i). From definition the function g_{i+2} depends on the spin configuration on B_{i+1}, but under the measure $\mu^{\eta}_{x,\ell}$ it only depends on $N_{x,\ell}$ and then it holds

$$\mu^{\eta}_{x,\ell}(g_{i+2}) = \mu^{\eta}_{x,\ell|_{N_{x,\ell}}}(g_{i+2})\,.$$

For every configuration $\eta \in \Omega^{+}$ we can apply the Poincaré inequality stated in Theorem (4.2) to $\mathrm{Var}^{\eta}_{x,\ell}(g_{i+2})$ and then obtain

$$\mu^{\tau^-}_{i+1}\left(\mathrm{Var}_{x,\ell}(g_{i+2})\right) \;\leq\; c \sum_{y\in N_{x,\ell}} \mu^{\tau^-}_{i+1}(\mathrm{Var}_y(g_{i+2})), \tag{4.28}$$

with $c = 1 + O(e^{-c\beta})$ independent of the size of system.

Second term: computation of the variance of h_x.
We first notice that from definition (4.19) of h_x, it is easy to show that h_x is a variable with mean one w.r.t. $\mu^{\tau^-}_{i+1}$ and only dependent on the vertices $y \in N_x$. In particular it can be expressed as

$$h_x(\sigma) \;=\; \frac{\exp(2\beta\sum_{y\in N_x}\sigma_y)}{\mu^{\tau^-}_{i+1}(\,\exp(2\beta\sum_{y\in N_x}\sigma_y)\,)} \;=\; \frac{\exp(2\beta\sum_{y\in N_x}(\sigma_y-1))}{\mu^{\tau^-}_{i+1}(\,\exp(2\beta\sum_{y\in N_x}(\sigma_y-1))\,)}\,, \tag{4.29}$$

where in the second equality we introduced a constant in the exponent in order to get the next computations easier.
Let us consider the (mean) variance $\mu^{\tau^-}_{i+1}(\mathrm{Var}_{x,\ell}(h_x))$ with $\ell \geq 0$. By the DLR equations and the Jensen inequality, we get

$$\mu^{\tau^-}_{i+1}(\mathrm{Var}_{x,\ell}(h_x)) \leq \mu^{\tau^-}_{i+1}(h_x^2) - (\mu^{\tau^-}_{i+1}(h_x))^2 = \mu^{\tau^-}_{i+1}(h_x^2) - 1\,. \tag{4.30}$$

Using the expression (4.29) for h_x, we then obtain the following bound

$$\begin{aligned}
\mu^{\tau^-}_{i+1}(h_x^2) \;&=\; \frac{\mu^{\tau^-}_{i+1}(\exp(4\beta\sum_{y\in N_x}(\sigma_y-1)))}{[\mu^{\tau^-}_{i+1}(\exp(2\beta\sum_{y\in N_x}(\sigma_y-1)))]^2}\\
&\leq\; 1/\exp(4\beta\sum_{y\in N_x}\mu^{\tau^-}_{i+1}(\sigma_y-1))\\
&\leq\; 1/\exp(-8\beta\sum_{y\in N_x}\mu^{\tau^-}_{i+1}(\sigma_y=-))\\
&\leq\; \exp\left(8\beta\, v \max_{y\in N_x}\{\mu^{\tau^-}_{i+1}(\sigma_y=-)\}\right)\,,
\end{aligned} \tag{4.31}$$

where in the second line we used that $\sigma_y - 1 \leq 0$ and the Jensen inequality to bound numerator and denominator respectively, in the third line we used that $\mu^{\tau^-}_{i+1}(\sigma_y - 1) \leq -2\mu^{\tau^-}_{i+1}(\sigma_y = -)$, and in the last line we bounded the cardinality of N_x by v, the vertex degree of B.

The problem is thus reduced to the computation of the probability $\mu_{i+1}^{\tau^-}(\sigma_y = -)$, for $y \in N_x$. Denoting by μ_{i+1}^- the measure conditioned on having all minus spins in B_i and plus spins in $\partial_V B$, by monotonicity it holds

$$\mu_{i+1}^{\tau^-}(\sigma_y = -) \leq \mu_{i+1}^-(\sigma_y = -)\,.$$

Notice that the event $\{\sigma \in \Omega_{F_{i+1}}^- \,:\, \sigma_y = -\}$ corresponds to the set of configurations $\sigma \in \Omega_{F_{i+1}}^-$ such that, for some subset $C \in F_{i+1}$ containing y, $\sigma_C = -$ and $\sigma_{\partial_V C \cap F_{i+1}} = +$. Then, by the same argument developed in section 3.1 (see also Remark 3.8), it holds

$$\mu_{i+1}^-(\sigma_y = -) \leq c e^{-\beta'}\,, \tag{4.32}$$

with $\beta' = c_1\beta - c_2$ as in Proposition 3.3.
Combining (4.30), (4.31) and (4.32), we get that for $\beta \gg 1$ and $\ell \geq 0$

$$\mu_{i+1}^{\tau^-}(\mathrm{Var}_{x,\ell}(h_x)) \leq \exp(c\,\beta\, e^{-\beta'}) - 1 \sim c\,\beta e^{-\beta'} =: c_\beta\,. \tag{4.33}$$

We keep in mind this result and proceed analyzing the last term.

Third term: the variance of $\mu_{x,\ell-1}(h_x)$.
We now consider the variance $\mathrm{Var}_{x,\ell}^\eta(\mu_{x,\ell-1}(h_x))$ with $\eta \in \Omega^+$ and $\ell \geq 1$. Applying the Poincaré inequality stated in Theorem 4.2, we obtain

$$\begin{aligned}\mathrm{Var}_{x,\ell}^\eta(\mu_{x,\ell-1}(h_x)) &\leq \sum_{z \in N_{x,\ell}} \mu_{x,\ell}^\eta(\mathrm{Var}_z(\mu_{x,\ell-1}(h_x))) \\ &= \sum_{z \in N_{x,\ell}\setminus N_{x,\ell-1}} \mu_{x,\ell}^\eta(\mathrm{Var}_z(\mu_{x,\ell-1}(h_x)))\,,\end{aligned} \tag{4.34}$$

where the last inequality is due to the fact that the function $\mu_{x,\ell-1}(h_x)$ does not depend on the spin configuration on $N_{x,\ell-1}$.
Let $z \in N_{x,\ell}\setminus N_{x,\ell-1}$, and for any configuration $\zeta \in \Omega_{N_{x,\ell}}^\eta$, let us denote by ζ^+ and ζ^- the configurations that agree with ζ in all sites but z, and have respectively a $(+)$-spin and a $(-)$-spin on z. The summand in (4.34) can then be trivially bounded as

$$\mu_{x,\ell}^\eta(\mathrm{Var}_z(\mu_{x,\ell-1}(h_x))) \leq \frac{1}{2} \sup_{\zeta \in \Omega_{x\ell}^\eta} ((\mu_{x,\ell-1}^{\zeta^+}(h_x) - \mu_{x,\ell-1}^{\zeta^-}(h_x))^2\,. \tag{4.35}$$

Notice that from the stochastic domination $\mu_{x,\ell-1}^{\zeta^+} \geq \mu_{x,\ell-1}^{\zeta^-}$ and the fact that h_x is an increasing function, it holds $\mu_{x,\ell-1}^{\zeta^+}(h_x) \geq \mu_{x,\ell-1}^{\zeta^-}(h_x)$. Now, let $\nu(\sigma,\sigma')$ denote a monotone coupling with marginal measure $\mu_{x,\ell-1}^{\zeta^+}$ and $\mu_{x,\ell-1}^{\zeta^-}$. We then have

$$\begin{aligned}\mu_{x,\ell-1}^{\zeta^+}(h_x) - \mu_{x,\ell-1}^{\zeta^-}(h_x) &= \sum_{\sigma,\sigma'} \nu(\sigma,\sigma')\,(h_x(\sigma) - h_x(\sigma')) \\ &\leq 2\|h_x\|_\infty\, \nu(\sigma_y \neq \sigma'_y\,, y \in N_x) \\ &\leq 2\,v\|h_x\|_\infty \max_{y \in N_x} \big(\nu(\sigma_y = +) - \nu(\sigma'_y = +)\big) \\ &= 2\,v\|h_x\|_\infty \max_{y \in N_x} (\mu_{x,\ell-1}^{\zeta^+}(\sigma_y = +) - \mu_{x,\ell-1}^{\zeta^-}(\sigma_y = +)),\end{aligned} \tag{4.36}$$

where we used the fact that the function h_x only depends on the spins on N_x.
The quantity $\|h_x\|_\infty$ can be easily bounded using the same procedure as in (4.33). For all $\sigma \in \Omega^+$, it holds

$$\begin{aligned} h_x(\sigma) &= \frac{\exp(2\beta \sum_{y\in N_x}(\sigma_y - 1))}{\mu^{\tau^-}_{i+1}(\exp(2\beta \sum_{y\in N_x}(\sigma_y - 1)))} \\ &\leq 1/\exp(2\beta \sum_{y\in N_x} \mu^{\tau^-}_{i+1}(\sigma_y - 1)) \\ &\leq \exp(4\beta\, v\, \mu^{\tau^-}_{i+1}(\sigma_y = -)) \\ &\leq \exp(c\beta e^{-\beta'}) =: k_\beta\,, \end{aligned} \tag{4.37}$$

which implies that $\|h_x\|_\infty \leq k_\beta$.
To bound the probability of disagreement appearing in (4.36), we refer again to Proposition 3.3 and to its proof. Proceeding as in section 3.1.2, we denote by E the event that there exists a negative component of z with nonempty intersection with N_y (analogous to the event A^c defined in (3.15)) in order to obtain the bound

$$\mu^{\zeta^+}_{x,\ell-1}(\sigma_y = +) - \mu^{\zeta^-}_{x,\ell-1}(\sigma_y = +) \leq \mu^{\zeta^-}_{x,\ell-1}(E)\,. \tag{4.38}$$

Since $d_i(z, y) \geq d_i(z, N_x) = \ell$, a component of z intersecting N_y has at least cardinality $\ell + 1$. From (4.38) and performing the same computation as in Section 3.1, see in particular (3.16)-(3.17) and Remark 3.8, we get

$$\mu^{\zeta^+}_{x,\ell-1}(\sigma_y = +) - \mu^{\zeta^-}_{x,\ell-1}(\sigma_y = +) \leq c\, e^{-\beta'\ell}\,. \tag{4.39}$$

Putting together formulas (4.34)-(4.39), we finally obtain

$$\mathrm{Var}_{x,\ell}(\mu_{x,\ell-1}(h_x)) \;\leq\; k'_\beta e^{-2\beta'\,\ell} \tag{4.40}$$

with $k'_\beta = c\, k^2_\beta = c(1 + O(e^{-c\beta}))\,.$

Conclusion.
Let us go back to inequalities (4.25) and (4.27). Applying the bounds (4.28),(4.33) and (4.40), we get respectively

- $(\mu^{\tau^-}_{i+1}(\mu_{x,0}(h_x, g_{i+2})))^2 \leq c_\beta \sum_{y\in N_x} \mu^{\tau^-}_{i+1}(\mathrm{Var}_y(g_{i+2}))$;
- $[\mu^{\tau^-}_{i+1}(\mu_{x,0}(h_x), \mu_{x,0}(g_{i+2}))]^2 \leq k'_\beta \sum_{\ell=1}^{\ell_0} \ell^2 e^{-2\beta'\,\ell} \sum_{y\in N_{x,\ell}} \mu^{\tau^-}_{i+1}(\mathrm{Var}_y(g_{i+2}))$,

where we included in c_β and k'_β all constants non depending on β.
For all $\beta \gg 1$, there exists a constant $\varepsilon \equiv \varepsilon(\beta) = O(e^{-c\beta})$ such that $c_\beta \leq \varepsilon$ and $k'_\beta \ell^2 e^{-\beta'\ell} \leq k'_\beta e^{-\beta'} \leq \varepsilon$. Substituting ε in the inequalities above and summing the two terms as in (4.24), we thus obtain

$$\left(\mu^{\tau^-}_{i+1}(h_x, g_{i+2})\right)^2 \leq \varepsilon \sum_{\ell=0}^{\ell_0} e^{-\beta'\ell} \sum_{y\in N_{x,\ell}} \mu^{\tau^-}_{i+1}(\mathrm{Var}_y(g_{i+2}))\,.$$

Inserting this result in the second term of formula (4.22) and rearranging the summation, we get

$$\sum_{i=0}^{m-1}\sum_{x\in L_i}\mu\left[\sum_{\tau}\mu_i(\tau)p(\tau)q(\tau)\left(\mu_{i+1}^{\tau^-}(h_x,g_{i+2})\right)^2\right] \leq$$
$$\leq \varepsilon\sum_{i=0}^{m-1}\sum_{x\in L_i}\sum_{\ell=0}^{\ell_0}\sum_{y\in N_{x,\ell}} e^{-\beta'\ell}\mu(\mathrm{Var}_y(g_{i+2}))$$
$$\leq \varepsilon\sum_{i=0}^{m-1}\sum_{y\in L_{i+1}}\mu(\mathrm{Var}_y(g_{i+2}))\sum_{\ell=0}^{\ell_0}e^{-\beta'\ell}n(\ell)\,, \tag{4.41}$$

where in the last line we denoted by $n(\ell)$ the factor which bounds the number of vertices x such that a fixed vertex y belongs to N_x. Since $n(\ell)$ growth linearly with ℓ, the product $e^{-\beta\ell}n(\ell)$ decays exponentially with ℓ for all $\beta \gg 1$. Thus the sum over $\ell \in \{0,\dots,\ell_0\}$ can be bounded by a finite constant c which will be included in the factor ε in front of the summations. Continuing from (4.41), we get

$$\sum_{i=0}^{m-1}\sum_{x\in L_i}\mu\left[\sum_{\tau}\mu_i(\tau)p(\tau)q(\tau)\left(\mu_{i+1}^{\tau^-}(h_x,g_{i+2})\right)^2\right] \leq \varepsilon\sum_{i=1}^{m}\sum_{y\in L_i}\mu(\mathrm{Var}_y(g_{i+1}))$$
$$\leq \varepsilon\,\mathrm{Pvar}(f)\,. \tag{4.42}$$

Inserting this result in (4.22) and noticing that $\varepsilon = O(e^{-c\beta}) < 1$ for β large enough, we obtain

$$\mathrm{Pvar}(f) \leq 2\mathcal{D}(f) + \varepsilon\mathrm{Pvar}(f) \Longrightarrow \mathrm{Pvar}(f) \leq \frac{2}{1-\varepsilon}\,\mathcal{D}(f)\,,$$

and from inequality (4.17) we finally get

$$\mathrm{Var}(f) \leq c\mathrm{Pvar}(f) \leq c'\mathcal{D}(f)\,,$$

that is the desired Poincaré inequality with $c' = 2c\,/(1-\varepsilon) = \Omega(1)$, independent of the size of the system. This conclude the proof of Theorem (2.5). $\square$

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