Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Multisymplectic analysis of the Short Pulse Equation and Numerical Applications

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submitted: 13th December 2007

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No. 1278
Berlin 2007

1991 *Mathematics Subject Classification.* 37K10, 78A60, 35Q60, 35Q51.

Key words and phrases. multisymplectic formalism, multisymplectic integrator, Short Pulse Equation,, ultrashort pulses, nonlinear optics.

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Abstract

The multisymplectic analysis of the Short Pulse Equation known in nonlinear optics is used in order to construct a geometric multisymplectic integrator of it. A brief comparison of its effectiveness relative to the pseudo-spectral integration scheme is presented.

1 Introduction

The multisymplectic Hamiltonian formalism has emerged from geometric theories in the calculus of variations [1]. It has been a subject of numerous investigations recently [2–9] including possible applications to field quantization [10–13]. The multisymplectic approach to the construction of geometric numerical integrators of PDEs was proposed in [14]. The application of the closely related "multi-symplectic" structure in wave propagation has been pioneered by Bridges [15].

In this contribution we apply the multisymplectic formalism to the short pulse equation (SPE) known in nonlinear optics. The short pulse equation has appeared recently [16, 17] as a description of ultra-short pulses when the standard nonlinear Schrödinger equation cannot be applied because the slowly varying envelope approximation it is based on is not valid anymore. In [18, 19] the integrability of this equation has been proven, and in [20] an example of the exact solution has been constructed. In [21] three integrable two component generalizations of SPE have been found.

Here we apply the multisymplectic formalism in order to construct a multisymplectic geometric integrator for SPE. This work is a part of the investigation of the properties of ultra-short pulses in nonlinear optics with the help of SPE and its generalizations which requires a stable and robust numerical integration scheme for SPE.

The multisymplectic formulation of SPE is discussed in Sect. 2. In Sect. 3 we construct the simplest multisymplectic integrator and briefly compare its effectiveness with the well known pseudo-spectral numerical integration [22].

2 The multisymplectic formulation of SPE

The short pulse equation

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx} \tag{1}$$

can be written in the form

$$\phi_{xt} - \phi - \frac{1}{6}(\phi_x^3)_x = 0 \tag{2}$$

if we introduce the potential ϕ

$$u := \phi_x. \tag{3}$$

This equation can be derived from the first order Lagrangian

$$L = \frac{1}{2}\phi_t\phi_x - \frac{1}{24}\phi_x^4 + \frac{1}{2}\phi^2. \tag{4}$$

Using the standard multisymplectic (De Donder-Weyl) Hamiltonian formalism, we introduce the polymomenta

$$\begin{aligned} p^t &:= \frac{\partial L}{\partial \phi_t} = \frac{1}{2}\phi_x, \\ p^x &:= \frac{\partial L}{\partial \phi_x} = \frac{1}{2}\phi_t - \frac{1}{6}\phi_x^3, \end{aligned} \tag{5}$$

and the (De Donder-Weyl) Hamiltonian

$$H_{DW} := p^t\phi_t + p^x\phi_x - L = 2p^xp^t + \frac{2}{3}(p^t)^4 - \frac{1}{2}\phi^2. \tag{6}$$

Then the multisymplectic (De Donder-Weyl) Hamiltonian equations take the form

$$\begin{aligned} \partial_x p^x + \partial_t p^t &= -\frac{\partial H}{\partial \phi} = \phi, \\ \partial_x \phi &= \frac{\partial H}{\partial p^x} = 2p^t, \\ \partial_t \phi &= \frac{\partial H}{\partial p^t} = 2p^x + \frac{8}{3}(p^t)^3. \end{aligned} \tag{7}$$

This set of first order equations is equivalent to SPE written in terms of the potential function $\phi(x,t)$, Eq. 2. It is well known that these equations can be obtained from the geometrical formulation of first order variational problems using the Poincare-Cartan form and its exterior derivative (the multisymplectic form) [1, 9].

$$\Omega = d\phi \wedge dp^x \wedge dt + d\phi \wedge dp^t \wedge dx - dH \wedge dx \wedge dt. \tag{8}$$

In order to establish a connection with the multi-symplectic formulation of Bridges [15] which has became more popular in discussions of geometric integrators of PDEs, let us introduce the set of variables $Z^v := (\phi, p^x, p^t)$. Then the DW Hamiltonian equations can be written in matrix form

$$\beta^t\partial_t Z + \beta^x\partial_x Z = \nabla_Z H, \tag{9}$$

where the β-matrices

$$\beta^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \beta^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{10}$$

can be identified with the so-called Duffin-Kemmer-Petiau matrices (in 2D) [23] which fulfill the DKP algebra relations $(a, b, c = (x, t))$.

$$\beta^a \beta^b \beta^c + \beta^c \beta^b \beta^a = -\beta^a \delta^{bc} - \beta^c \delta^{ab}. \tag{11}$$

This form of DW Hamiltonian equations generalizes the Hamiltonian equations in mechanics written in the form

$$\omega \partial_t Z = \partial_Z H,$$

where $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the symplectic matrix and $Z := (p, q)$.

Associated with the above two antisymmetric matrices β are two pre-symplectic forms

$$\begin{aligned} \kappa^x &= \frac{1}{2} dz \wedge \beta^x dz = -dp^x \wedge d\phi, \\ \kappa^t &= \frac{1}{2} dz \wedge \beta^t dz = dp^t \wedge d\phi. \end{aligned} \tag{12}$$

The structure given by two pre-symplectic forms κ^x and κ^t is called multi-symplectic by Bridges [15]. In the notations introduced by Bridges (1997) $\beta^x = K$ and $\beta^t = M$ and $H = -S$. These notations are now standard in the papers devoted to the geometric (multisymplectic) integrators of PDEs [24–28]. In this notation the fundamental *multisymplectic conservation law* is written in the form:

$$d/dt \kappa^t + d/dx \kappa^x = 0. \tag{13}$$

3 A multisymplectic integrator for SPE

The simplest realization of the multisymplectic integrator is constructed by the discretization of DW Hamiltonian equations using the midpoint method in both x and t directions:

$$\begin{aligned} &\phi_{i,j} \approx \phi(i\Delta x, j\Delta t), \\ &\phi_{i+1,j+1/2} := \frac{1}{2}(\phi_{i,j} + \phi_{i,j+1}), \\ &\phi_{i+1/2,j+1/2} := \frac{1}{4}(\phi_{i,j} + \phi_{i,j+1} + \phi_{i+1,j} + \phi_{i+1,j+1}). \end{aligned} \tag{14}$$

We obtain:

$$\frac{p^x_{i+1,j+\frac{1}{2}} - p^x_{i,j+\frac{1}{2}}}{\Delta x} + \frac{p^t_{i+\frac{1}{2},j+1} - p^t_{i+\frac{1}{2},j}}{\Delta t} = \phi_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{15a}$$

$$\frac{\phi_{i+1,j+\frac{1}{2}} - \phi_{i,j+\frac{1}{2}}}{\Delta x} = 2p^t_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{15b}$$

$$\frac{\phi_{i+\frac{1}{2},j+1} - \phi_{i+\frac{1}{2},j}}{\Delta t} = 2p^x_{i+\frac{1}{2},j+\frac{1}{2}} + \frac{8}{3}(p^t_{i+\frac{1}{2},j+\frac{1}{2}})^3. \tag{15c}$$

As a consequence of the statement proven by Bridges and Reich [24] this integrator fulfills the discretized multisymplectic conservation law.

3.1 The numerical implementation

We solve the initial boundary value problem for Eq. 1 using the above multisymplectic integrator. We know the initial value $u(x, t = 0)$, its discretization $u_{i,j=0}, i = 1, ..., N$, and the vanishing values of the solutions on the right boundary (the wave propagates from the right to the left). Hence, we know $p^t_{i,j=0}$ and $\phi_{i,j=0}, i = 1, N$, and $p^t_{N,j} = \phi_{N,j} = p^x_{N,j} + p^x_{N,j+1} = 0, j = 0, ..., M$. From the known values at three mesh points $(i+1, j)$, $(i+1, j+1)$, and (i, j) (see Fig. 1) we calculate new values at the point $(i, j + 1)$, i.e. given $p^t_{i+1,j}$, $p^t_{i+1,j+1}$, $p^t_{i,j}$, $\phi_{i+1,j}$, $\phi_{i+1,j+1}$, $\phi_{i,j}$, and $p^x_{i+1,j+1} + p^x_{i+1,j}$, we obtain $p^t_{i,j+1}$, $\phi_{i,j+1}$ and $p^x_{i,j+1} + p^t_{i,j}$. Then the integrator yields

$$\begin{aligned}
&(p^t_{i,j+1})^3 + 3(p^t_{i+1,j} + p^t_{i,j} + p^t_{i+1,j+1})(p^t_{i,j+1})^2 \\
&+3\left((p^t_{i+1,j} + p^t_{i,j} + p^t_{i+1,j+1})^2 + 4\left(\frac{2\Delta x}{\Delta t} + \frac{(\Delta x)^2}{2}\right)\right)p^t_{i,j+1} \\
&-\frac{24}{\Delta t}(\phi_{i+1,j+1} - \phi_{i,j}) - 12\Delta x(\phi_{i+1,j} + \phi_{i+1,j+1}) \\
&+12\left(\frac{2\Delta x}{\Delta t} + \frac{(\Delta x)^2}{2}\right)p^t_{i+1,j+1} + 6(\Delta x)^2(p^t_{i+1,j} + p^t_{i,j}) \\
&+24(p^x_{i+1,j} + p^x_{i+1,j+1}) + (p^t_{i+1,j} + p^t_{i,j} + p^t_{i+1,j+1})^3 = 0,
\end{aligned} \tag{16a}$$

$$\begin{aligned}
\phi_{i,j+1} &= (\phi_{i+1,j} - \phi_{i,j} + \phi_{i+1,j+1}) \\
&-\Delta x\,(p^t_{i+1,j} + p^t_{i,j} + p^t_{i+1,j+1}) - \Delta x\, p^t_{i,j+1},
\end{aligned} \tag{16b}$$

$$\begin{aligned}
(p^x_{i,j+1} + p^x_{i,j}) &= (p^x_{i+1,j+1} + p^x_{i+1,j}) - \frac{\Delta x}{\Delta t}(p^t_{i,j} + p^t_{i+1,j} - p^t_{i+1,j+1}) \\
&+\frac{\Delta x}{\Delta t}p^t_{i,j+1} - \frac{\Delta x}{2}(\phi_{i+1,j} + \phi_{i,j} + \phi_{i+1,j+1} - \frac{\Delta x}{2}\phi_{i,j+1}.
\end{aligned} \tag{16c}$$

We first calculate $p^t_{i,j+1}$ from the cubic Eq. (16a) (the root which ensures the continuity of the solution is selected). Then Eq. (16b) yields $\phi_{i,j+1}$ and Eq. (16c) yields $p^x_{i,j+1} + p^x_{i,j}$ (see Fig. 1). Thus, we obtain $u_{i,j+1} = 2p^t_{i,j+1}$.

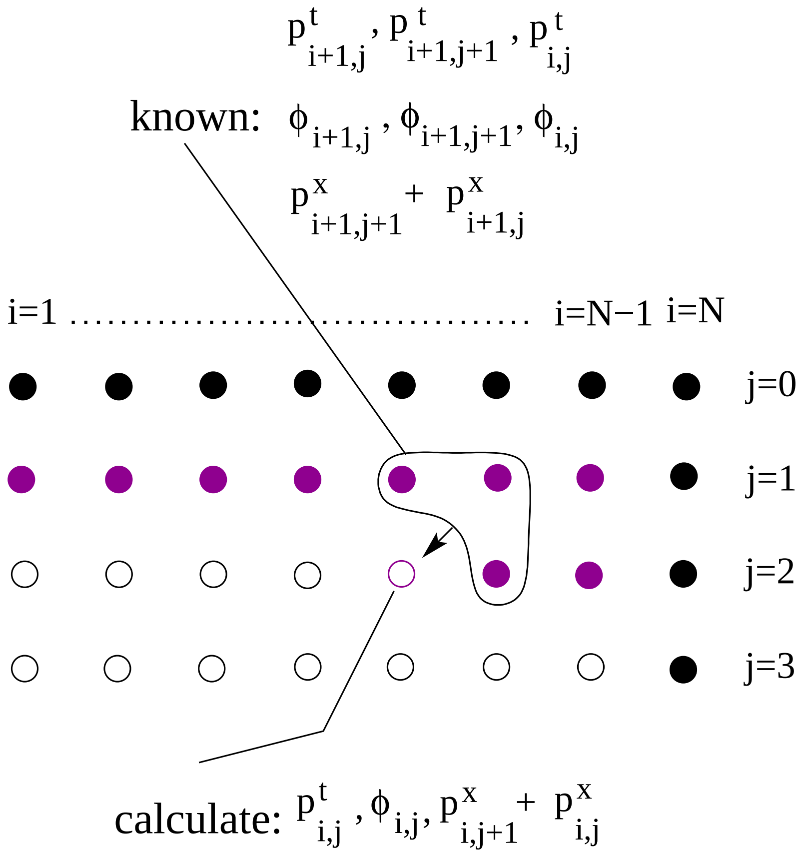

Figure 1: The discretization mesh.

In order to test the effectiveness of the method, we numerically propagate the known Sakovich' exact solution of SPE [20] to $t = 100$. The evolution of the Sakovich exact solution (with $m = 0.2$) is shown on Fig. 2 at $t = 0$ and $t = 100$.

The exact solution is compared with the numerical solutions obtained using the multisymplectic scheme and the pseudo-spectral scheme. We compare the error of the methods, and the CPU time required to reach $t = 100$ at different values of discretization steps Δt and Δx. The error of numerical integration is given by the standard deviation:

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1} N \left(u_{i,j} - \bar{u}_{i,j}\right)^2}, \tag{17}$$

where $u_{i,j}$ is the numerical solution and $\bar{u}_{i,j}$ is the exact Sakovich' solution at time $t = j\Delta t$.

The results of the multisymplectic integration for different values of Δt and $\Delta x = X_{max}/N$ ($X_{max} = 400$) are shown of Fig. 3. As expected, the error decreases with Δx and Δt decreasing. The multisymplectic method appears to be more effective than the pseudo-spectral method. For example, for $N = 2^{17}$ and $\Delta t = 0.0001$ the error of the multisymplectic scheme $\sigma \approx 6.5 \times 10^{-6}$, while for the pseudo-spectral

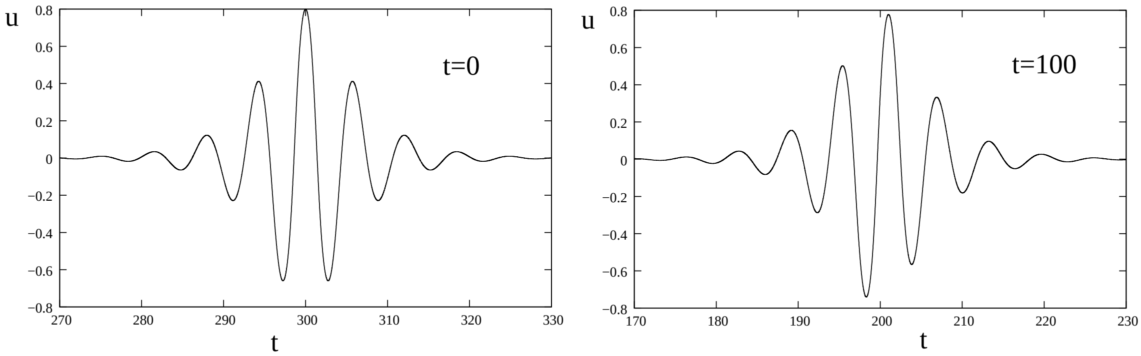

Figure 2: The evolution of the Sakovich' solution (with $m = 0.2$) for $t = 0$ and $t = 100$.

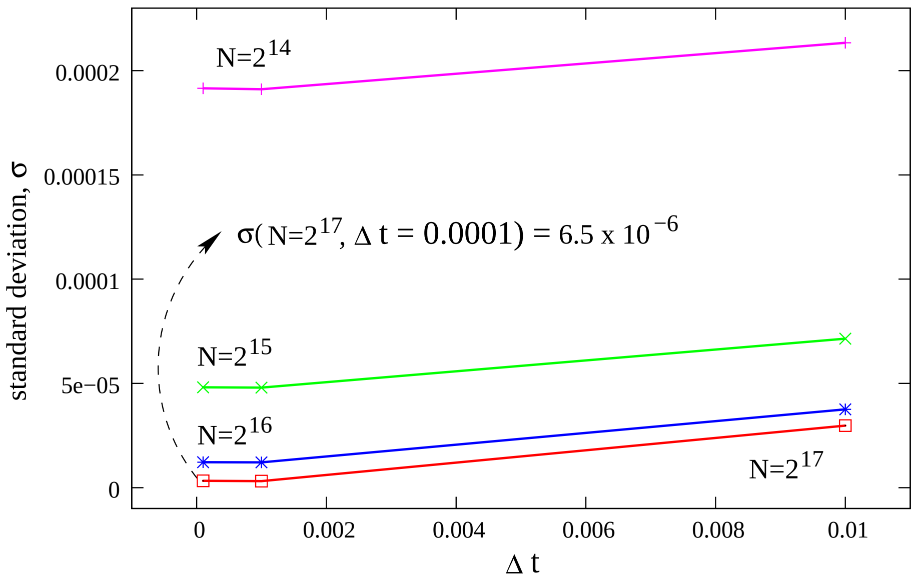

Figure 3: The dependence of the error of the multisymplectic integrator from Δt for different values of Δx.

method $\sigma \approx 7 \times 10^{-5}$. The CPU time required by the multisymplectic methods is 40000 sec, while the pseudo-spectral method requires $\approx$ 100000 sec (on 3GHz Pentium 4 PC).

In conclusion, we have used the multisymplectic formulation of SPE in order to construct the geometric multisymplectic integrator of SPE. We have compared the effectiveness of the corresponding numerical scheme with the pseudo-spectral method which uses the Runge-Kutta integration. The multisymplectic integration appears to be an order of magnitude more precise and approximately 2.5 times faster at long propagation times than the pseudo-spectral method. A comparison with the exact solution of SPE shows that the multisymplectic integration is stable and robust and preserves the energy functional.

Acknowledgments

The authors thank the Organizers of DGA 2007 Conference for their invitation and kind hospitality. The work of M.P. was supported by Deutsche Forshungsgemeinschaft (DFG) as a project D20 of the Research Center MATHEON (Berlin, Germany).

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