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Expansion of random boundary excitations for elliptic PDEs

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Abstract

In this paper we deal with elliptic boundary value problems with random boundary conditions. Solutions to these problems are inhomogeneous random fields which can be represented as series expansions involving a complete set of deterministic functions with corresponding random coefficients. We construct the Karhunen-Loève (K-L) series expansion which is based on the eigen-decomposition of the covariance operator. It can be applied to simulate both homogeneous and inhomogeneous random fields. We study the correlation structure of solutions to some classical elliptic equations in respond to random excitations of functions prescribed on the boundary. We analyze the stochastic solutions for Dirichlet and Neumann boundary conditions to Laplace equation, biharmonic equation, and to the Lamé system of elasticity equations. Explicit formulae for the correlation tensors of the generalized solutions are obtained when the boundary function is a white noise, or a homogeneous random field on a circle, a sphere, and a half-space. These exact results may serve as an excellent benchmark for developing numerical methods, e.g., Monte Carlo simulations, stochastic volume and boundary element methods.

1 Introduction.

Boundary value problems with random coefficients, parameters, random source terms, stochastically distributed boundary functions, or even with randomly moving boundaries are used as a powerful instrument in modern science and technology. We mention here applied fields such as structural mechanics, composite materials [2], porous media and soils [6], [33], [17], [49], biological tissues [47], geodesy [30], [40], turbulence [48], [3], [19], [31], etc.

In engineering related stochastic boundary value problems, the common computational techniques include Monte Carlo methods, stochastic finite elements, finite difference, and spectral methods. Among these methods, the finite volume and boundary element techniques are the methods most adaptable to problems in solid and structural mechanics characterized with highly irregular and complex structures [2], [9], [43]. We mention also classical potential problems dealing with random boundary conditions and sources [7] where the Monte Carlo methods are very efficient (e.g., see [31], [37], [35]), [36]). In electrical impedance tomography [13] important problem is to evaluate a global response to random boundary excitations, and to estimate local fluctuations of the solution fields. Similar analysis is made in the inverse problems of elastography [25], [32], recognition technology [10], acoustic scattering from rough surfaces [46], fluid dynamics [1], and reaction-diffusion equations with white noise boundary perturbations [42].

It should be noted that the numerical simulation methods for stationary processes and homogeneous Gaussian random fields are well developed, and the most convenient and probably most often used are methods based on the spectral representations (e.g., see [41], [9], [31], [20], [19]). The most common simulation method for inhomogeneous random processes and fields is based on the Karhunen-Loève (K-L) expansion, also known as a proper orthogonal decomposition (POD), a series representation consisting of eigen-functions as the orthogonal basis (e.g., see [2], [15], [21],

[3], [12], [26], [27]). The expansion is known to produce the most efficient representation among all orthogonal bases for the Gaussian case. According to A.M. Yaglom's personal communication, the proper orthogonal decomposition was suggested independently by Kosambi [18], Loève [21], Karhunen [15], Pougachev [28], and Obukhov [24]. We also mention a comprehensive studies by Van Trees [44], and A.M. Yaglom himself [48], and one generalization of K-L expansion for the Wiener process [38], [39].

In this paper, we construct exact proper orthogonal decomposition for some classical boundary value problems, for a disc, ball, and a half-plane, with a Dirichlet and Neumann boundary conditions, where the boundary functions are white noise or homogeneous (2π-periodic) random processes. In case the boundary function is a white noise, the solutions are treated as generalized random fields with the convergence in the proper spaces and relevant generalized treatment of boundary conditions, e.g., see [29], [30], [40].

The paper is organized as follows. After a short description of the spectral and Karhunen-Loève expansions, we consider in Section 2 the 2D Laplace equation, with Dirichlet and Neumann boundary conditions, for a disc and a half-plane. Generalizations to a three-dimensional case is given in Section 3. In Section 4 we analyze the biharmonic equation for a disc. The plane elasticity problem is presented in Section 5. For all these boundary value problems we find explicitly the correlation functions, and give the Karhunen-Loève expansion of the relevant random fields.

1.1 Spectral representations.

Let us first consider a real-valued zero mean homogeneous Gaussian l-dimensional vector random field $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), \ldots, u_l(\mathbf{x}))^T$, $\mathbf{x} \in I\!R^d$ with a given covariance tensor $B(\mathbf{r})$ with entries

$$B_{ij}(\mathbf{r}) = \langle u_i(\mathbf{x}+\mathbf{r})\, u_j(\mathbf{x})\rangle, \quad i,j = 1,\ldots l,$$

or with the corresponding spectral tensor F:

$$F_{ij}(\mathbf{k}) = \int\limits_{R^d} e^{-\mathrm{i}\,2\pi\,\mathbf{k}\cdot\mathbf{r}} B_{ij}(\mathbf{r})\, d\mathbf{r}, \quad B_{ij}(\mathbf{r}) = \int\limits_{R^d} e^{\mathrm{i}\,2\pi\,\mathbf{r}\cdot\mathbf{k}} F_{ij}(\mathbf{k})\, d\mathbf{k}\ . \tag{1}$$

We call also B_{ij} a correlation tensor which is equivalent since we assume without loss of generality that the random fields have zero means.

Often it is reasonable to assume [20] that the condition $\int\limits_{R^d} \sum_{j=1}^{l} |B_{jj}(\mathbf{r})|\, d\mathbf{r} < \infty$ is satisfied which ensures that the spectral functions F_{ij} are uniformly continuous with respect to $\mathbf{k}$. Note that a weaker assumption that B is squared integrable guarantees only the existence of the spectral tensor in the space L_2.

Let $Q(\mathbf{k})$ be an $l \times n$-matrix defined by $Q(\mathbf{k})Q^*(\mathbf{k}) = F(\mathbf{k}), \quad Q(-\mathbf{k}) = \bar{Q}(\mathbf{k})$. Here the star stands for the complex conjugate transpose which is equivalent to taking two operations, the transpose ^{T}, and the complex conjugation of each entry. Then the spectral representation of the random field is written as follows (e.g., see [48])

$$\mathbf{u}(\mathbf{x}) = \int\limits_{R^d} e^{\mathrm{i}\,2\pi\,\mathbf{k}\mathbf{x}}\, Q(\mathbf{k})\, \mathbf{Z}(d\mathbf{k}) \tag{2}$$

where the column-vector $\mathbf{Z} = (Z_1, \ldots Z_n)^T$ is a complex-valued homogeneous n-dimensional white noise on $I\!R^d$ with a unite variance and zero mean:

$$\langle \mathbf{Z}(d\mathbf{k})\rangle = 0, \quad \langle Z_i(d\mathbf{k}_1)\, \bar{Z}_j(d\mathbf{k}_2)\rangle = \delta_{ij}\, \delta(\mathbf{k}_1 - \mathbf{k}_2)\, d\mathbf{k}_1\, d\mathbf{k}_2, \quad \mathbf{Z}(-d\mathbf{k}) = \bar{\mathbf{Z}}(d\mathbf{k})\ .$$

Note that in the literature, different forms of the Fourier transform between the correlation and spectral tensors are used. Along with (1), we will mainly use

$$F_{ij}(\mathbf{k}) = \frac{1}{(2\pi)^d} \int\limits_{R^d} e^{-\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}} B_{ij}(\mathbf{r})\, d\mathbf{r}, \quad B_{ij}(\mathbf{r}) = \int\limits_{R^d} e^{\mathrm{i}\,\mathbf{r}\cdot\mathbf{k}} F_{ij}(\mathbf{k})\, d\mathbf{k}, \quad i,j = 1,\dots l\ .$$

The spectral representation (2) is used in different numerical simulation methods, through a deterministic or randomized evaluation of the stochastic integral in (2), see for instance [31], [19], [20].

A straightforward evaluation of the stochastic integral (2) is based on the Riemann sums calculation with fixed cells (see, e.g. [41]). The integral is approximated by a finite sum

$$\mathbf{u}(\mathbf{x}) \approx \sum_{i=1}^{n} \Big[\cos(2\pi \mathbf{k}_i \cdot \mathbf{x}) \boldsymbol{\xi}_i + \cos(2\pi \mathbf{k}_i \cdot \mathbf{x})\, \boldsymbol{\eta}_i \Big]$$

where $\mathbf{k}_i$ are deterministic nodes in the Fourier space, $\boldsymbol{\xi}_i$ and $\boldsymbol{\eta}_i$ are Gaussian random vectors with zero mean and relevant covariance. Efficient calculation of the above sum is usually carried out by the fast Fourier transform which assumes that the nodes are chosen uniformly. It should be mentioned that this scheme suffers from an artificially periodicity in the scale of $1/\Delta k$ where Δk is the integration step in the Fourier space. In Randomized models, the nodes are chosen at random, with an appropriate probability distribution so that the model has the desired correlation structure (e.g., see [31], [19]).

Partially homogeneous random fields present an important class of random fields where this approach can be efficiently used.

Let $\mathbf{x} = (\mathbf{y}, \mathbf{z})$, $\mathbf{y} \in \mathbf{R}^n$, $\mathbf{z} \in \mathbf{R}^m$, and let $\mathbf{V}(\mathbf{x}) = (v_1(\mathbf{x}), \dots, v_l(\mathbf{x}))^T$. Assume that the random field $\mathbf{V}(\mathbf{y}, \mathbf{z})$ is homogeneous with respect to the variable $\mathbf{y}$, i.e.,

$$\langle \mathbf{V}(\mathbf{y}_1, \mathbf{z}_1)\, \mathbf{V}^*(\mathbf{y}_2, \mathbf{z}_2) \rangle = B(\mathbf{y}_1 - \mathbf{y}_2, \mathbf{z}_1, \mathbf{z}_2) \rangle\ .$$

Random fields with this property are called partially homogeneous random fields [31]. The partial spectral tensor is defined by

$$f(\boldsymbol{\lambda}, \mathbf{z}_1, \mathbf{z}_2) = \frac{1}{(2\pi)^n} \int\limits_{\mathbf{R}^n} B(\boldsymbol{\rho}, \mathbf{z}_1, \mathbf{z}_2) \exp\{-\mathrm{i}(\boldsymbol{\lambda}, \boldsymbol{\rho})\}\, d\boldsymbol{\rho}\ .$$

It is not difficult to verify that for a general complex-valued random field $\mathbf{V}(\mathbf{x})$, which is partially homogeneous,

$$\mathbf{V}(\mathbf{y}, \mathbf{z}) = \frac{1}{[p(\boldsymbol{\lambda})]^{1/2}} \exp\{\mathrm{i}\,(\boldsymbol{\lambda}, \mathbf{y})\} \boldsymbol{\xi}_{\boldsymbol{\lambda}}(\mathbf{z})$$

its correlation tensor is equal to $B(\boldsymbol{\rho}, \mathbf{z}_1, \mathbf{z}_2)$, if $\boldsymbol{\lambda}$ is distributed according to a probability density $p(\boldsymbol{\lambda})$ which can be chosen quite arbitrarily, and $\boldsymbol{\xi}_{\boldsymbol{\lambda}}$ ($\boldsymbol{\lambda}$ fixed) is a homogeneous l-dimensional complex-valued random field with the correlation tensor $f(\boldsymbol{\lambda}, \mathbf{z}_1, \mathbf{z}_2)$. A rigorous proof of this statement is given in [31].

1.2 The Karhunen-Loève expansion.

Let us now consider a real-valued inhomogeneous random field $u(x), \quad x \in G$ defined on a probability space (Ω, A, P) and indexed on a bounded domain G. The case of unbounded domains can also be treated, in particular, if the covariance tensor belongs to a class $\mathcal{A}$ defined in [4], for which the corresponding covariance operator is compact and trace class. This important generalization is based on the result due to I.M. Novitsky [23] (see also [5]). In section 2.4 we deal with an unbounded domain when analysing the Dirichlet problem for the half-plane. To simplify the notations, we will not use here and in what follows the boldface characters to denote the vectors if not otherwise indicated. They will be essentially used in Section 5 for the vector solution to the Lamé equation.

Assume (without loss of generality) that the field has a zero mean and a variance $E\,u^2(x)$ that is bounded for all $x \in G$. The Karhunen-Loève expansion has the form [48]

$$u(x) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \xi_k\, h_k(x) \ ,$$

where λ_k and $h_k(x)$ are the eigen-values and eigen-functions of the covariance function $B(x_1, x_2) = \langle u(x_1)\, u(x_2) \rangle$, and ξ_k is a family of random variables.

By definition, $B(x_1, x_2)$ is bounded, symmetric and positive definite. For such kernels, the Hilbert-Schmidt theory says that the following spectral representation is valid

$$B(x_1, x_2) = \sum_{k=1}^{\infty} \lambda_k\, h_k(x_1)\, h_k(x_2)$$

where the eigen-values and eigen-functions are the solutions of the following eigen-value problem for the correlation operator:

$$\int_G B(x_1, x_2)\, h_k(x_1)\, dx_1 = \lambda_k\, h_k(x_2) \ .$$

The eigen-functions form a complete orthogonal set $\int_G h_i(x)\, h_j(x)\, dx = \delta_{ij}$ where δ_{ij} is the Kronecker delta-function. The family $\{\xi_k\}$ is a set of uncorrelated random variables which are obviously related to h_k by

$$\xi_k = \frac{1}{\sqrt{\lambda_k}} \int_G u(x)\, h_k(x)\, dx \ , \qquad E\,\xi_k = 0, \qquad E\xi_i\, \xi_j = \delta_{ij} \ .$$

We mention also that the assumptions of the Hilbert-Schmidt can be weakened as it is done in Mercer's theorem. This will be discussed in section 2.

It is well known that the Karhunen-Loève expansion presents an optimal (in the mean square sense) convergence for any distribution of $u(x)$. If $u(x)$ is a zero mean Gaussian random field, then $\{\xi_k\}$ is a family of standard Gaussian random variables. Some generalizations to non-gaussian random fields are reported in [27].

Consider now a case when G is unbounded, e.g., a homogeneous random process $u(x)$ is defined on the whole real line $\mathbf{R}$. The eigen-value problem reads

$$\int_{\mathbf{R}} B(x_2 - x_1)\, h_k(x_1)\, dx_1 = \lambda_k\, h_k(x_2) \ , \quad -\infty < x_2 < \infty \ . \tag{3}$$

Note that we can take $h(x) = e^{\mathrm{i}\omega x}$, then from (3) we get

$$\lambda = \int_{-\infty}^{\infty} B(x_2 - x_1)\, e^{-\mathrm{i}\omega(x_2 - x_1)}\, dx_1 \equiv S(\omega) \ .$$

To make further considerations more rigorous, we assume that G is large but finite, and u is periodic (e.g., see [22], [44]). Then, we may develop $B(x_2 - x_1)$ in a Fourier series,

$$B(x - x') = \sum_k \lambda_k\, e^{\mathrm{i}\, 2\pi k (x - x')} \ . \tag{4}$$

The eigen-value problem can then be solved via the unique representation

$$B(x - x') = \sum_k \lambda_k\, e^{\mathrm{i}\, 2\pi k\, x} e^{-\mathrm{i}\, 2\pi k\, x'} \tag{5}$$

which imply that $e^{\mathrm{i}\, 2\pi k\, x}$ are the eigen-functions with eigen-values $\lambda_k = S(\omega_k)$. And conversely, if the eigen-functions are Fourier modes we can write the equality (5) which leads to (4).

Thus the correlation function B depends on the difference $x - x'$ if and only if the eigen-functions of the correlation operator are Fourier modes.

In our considerations this fact will be used in two-dimensional regions, when G is a disc, a ball or a half-plane. The correlation function of a zero mean random process has the form $B(\mathbf{x}, \mathbf{x}') = B(x, y; x'y')$. Suppose that our random process is homogeneous with respect to one coordinate, say, $B = B(x - x'; y, y')$. Then we can perform the above procedure over the x-direction, and get a 1D eigen-value problem for every Fourier wavenumber. It means, we then work with the partial spectral density.

Assume we deal with a homogeneous real-valued process on the whole line. Then it is possible to cut-off the integration in the eigen-value problem, i.e., we have to solve the eigen-value problem

$$\int_{-a}^{a} B(x_2 - x_1)\, h(x_1)\, dx_1 = \lambda_k\, h_k(x_2),$$

where a is sufficiently large. Then it is possible to show (e.g., see [44]) that

$$\lambda_k \approx S(\omega_k) = S(\pi\, k/a) \ , \quad h_k(x) \approx \frac{1}{2\pi}\, e^{\mathrm{i}\, (\pi\, k\, x/a)} \ ,$$

which yields an approximation

$$B(x_1, x_2) \approx \tilde{B}_a(x_1, x_2) = \sum_{k=1}^{\infty} \frac{1}{a} S(\frac{\pi\, k}{a})\, \cos\Big(\frac{\pi\, k(x_2 - x_1)}{a}\Big) \ ,$$

and the K-L expansion approaches in this case to the spectral representation

$$u(x) \approx \tilde{u}_a(x) = \sum_{k=1}^{\infty} \Big[\frac{1}{2a} S(\frac{\pi\, k}{a})\Big]^{1/2} \Big\{\xi_k\, \cos[\pi\, k\, x/a] + \eta_k\, \sin[\pi\, k\, x/a]\Big\} \ .$$

The rate of convergence of the K-L expansion is closely related to the smoothness of the correlation kernel and to ratio between the length a and L, the correlation length of the process. For example, in [22] is reported that for the particular case $B(x_1, x_2) = \sigma\, e^{-|x_2 - x_1|/L}$, an upper bound for the relative error in variance ε of the process represented by its K-L expansion is given by $\varepsilon \le \frac{4}{\pi^2} \frac{1}{n} \frac{a}{L}$ where n is the number of retained terms.

2 Stochastic boundary value problems for the 2D Laplace equation.

Let us start with the two-dimensional boundary value problems for the Laplace equation. We are interested in the statistical structure of the solution when the solution (Dirichlet boundary conditions), or the normal derivative (Neumann boundary conditions) are homogeneous random functions ($g(y)$) on the boundary. The basic idea is first to establish the Karhunen-Loève expansion for the case when the boundary function g is a white noise, therefore, the solutions are considered as generalized random fields. This expansion gives a smooth representation for the solution and the correlation function inside the open disc, and the case of general homogeneous boundary functions is immediately obtained from this expansion by a simple substitution of the spectral expansion of the boundary random function $g(x)$.

Before we start with the details for the Laplace equation, let us outline shortly the general scheme. Assume we are given a stochastic Dirichlet boundary value problem for a linear elliptic equation in a domain D with a boundary $\Gamma = \partial D$:

$$Lu(x) = 0, \quad x \in D, \qquad u(x)|_{x \to y \in \Gamma} = g(y)$$

where $g(y)$ is a random field with zero mean and covariance function $B_g(y_1, y_2) = \langle g(y_1)\, g(y_2) \rangle$. We are interested in the covariance of the solution, $B_u(x_1, x_2) = \langle u(x_1)\, u(x_2) \rangle$.

Suppose that there exists a continuous normal derivative of the Green function on the boundary, $\frac{\partial G}{\partial \mathbf{n}}$, so that the solution is represented by the Green formula:

$$u(x) = \int\limits_{\Gamma} \frac{\partial G}{\partial \mathbf{n}}(x, y) g(y)\, dS(y) \ .$$

Using the Green formula representation for the solution in points x_1 and x_2 we obtain

$$B_u(x_1, x_2) = \int\limits_{\Gamma} \int\limits_{\Gamma} \frac{\partial G}{\partial \mathbf{n}}(x_1, y_1) \frac{\partial G}{\partial \mathbf{n}}(x_2, y_2)\, B_g(y_1, y_2)\, dS(y_1)\, dS(y_2) \ . \qquad (1)$$

If g is a white noise, $B_g(y_1, y_2) = \delta(y_1 - y_2)$, and we obtain formally from (1) that

$$B_u(x_1, x_2) = \int\limits_{\Gamma} \frac{\partial G}{\partial \mathbf{n}}(x_1, y) \frac{\partial G}{\partial \mathbf{n}}(x_2, y)\ dS(y) \ . \qquad (2)$$

This representation shows that the covariance function $B_u(x, x_2)$ solves the boundary value problem

$$\begin{aligned} L_x B(x, x_2) &= 0, \quad x, x_2 \in D, \\ B(x, x_2)|_{x \to y \in \Gamma} &= \frac{\partial G}{\partial \mathbf{n}}(x_2, y)|_{y \in \Gamma} \ , \end{aligned} \qquad (3)$$

so that the solution of this problem at any point $x = x_1 \in D$ yields $B_u(x_1, x_2)$ for any fixed $x_2 \in D$ which defines well the covariance function for any two points x_1 and x_2 inside the domain D. These formal considerations leave open the singularity problem of the correlation function when both points tend to one point on the boundary, but the weak convergence to the delta-function can be given in the framework of generalized solutions (e.g., see [29], [30], [40]).

2.1 Dirichlet problem for a 2D disc. White noise excitations.

Let us consider the Dirichlet boundary value problem for the Laplace equation

$$\Delta u(x) = 0, \quad x \in D, \quad u(y) = g(y) \quad y \in \Gamma = \partial D, \tag{4}$$

where the domain D is a disc $K(x_0, R)$ centered at $O = x_0$, bounded by the circle $\Gamma = S(x_0, R)$. We denote the closed disc by $\bar{K}(x_0, R) = K(x_0, R) \cup S(x_0, R)$.

The regular solution to the harmonic equation is represented by the Poisson integral formula [45]:

$$u(x) \quad = \quad \frac{R^2 - r^2}{2\pi R} \int\limits_{S(x_0,\, R)} \frac{g(y) dS_y}{|x - y|^2} \;,$$

for any point $x \in K(x_0, R)$, where $r = |x - x_0|$.

We suppose that the boundary function $g(y)$ is a zero mean Gaussian random field, homogeneous or not, defined by its correlation function $B_g(y_1, y_2) = \langle g(y_1) g(y_2) \rangle$. In case g is homogeneous, it is alternatively defined by its spectral density function $f(k)$ related to the correlation function $B_g(y),\ y = y_2 - y_1$, by the Fourier transform

$$f(k) = \frac{1}{2\pi} \int B_g(y) e^{-\mathrm{i}(y\,k)}\, dy \;, \qquad B_g(y) = \int f(k) e^{\mathrm{i}(y\,k)}\, dk \;.$$

When dealing with the homogeneous random processes $g(\varphi)$ on the circle, we assume throughout the paper that they are 2π-periodic, so the spectra are discrete, and the Fourier integral transforms become Fourier series.

Let us start with the case when the prescribed boundary function g is a Gaussian white noise, $B_g(y, y') = \delta(y - y')$, thus we deal in this paper with generalized random solutions which however are smooth in the open domain (in a disc, ball, and a half-plane). The generalized treatment of the convergence to the boundary functions can be explicitly described (e.g., see [29]) in more general cases.

Let us introduce polar coordinates centered at x_0, so that a point x is specified by (r, θ), hence, for two points, $x_1 = (r_1, \theta_1)$, $x_2 = (r_2, \theta_2)$, and $\rho_1 = r_1/R$, $\rho_2 = r_2/R$.

It is convenient then to rewrite the Poisson formula as follows

$$u(r, \theta) \quad = \quad \frac{1 - \rho^2}{2\pi} \int\limits_0^{2\pi} \frac{g(\varphi)\, d\varphi}{1 - 2\rho \cos(\theta - \varphi) + \rho^2} \tag{5}$$

where $\rho = r/R$.

Theorem 1. *The solution of the Dirichlet problem (4) in a disc $K(x_0, R)$ with the white noise boundary function $g(y)$ is an inhomogeneous 2D Gaussian random field uniquely defined by its correlation function*

$$\langle u(r_1, \theta_1)\, u(r_2, \theta_2) \rangle = B_u(\rho_1, \theta_1; \rho_2, \theta_2) = \frac{1}{2\pi} \frac{1 - \rho_1^2 \rho_2^2}{1 - 2\rho_1 \rho_2 \cos(\theta_2 - \theta_1) + \rho_1^2 \rho_2^2} \tag{6}$$

which is harmonic, and it depends only on the angular difference $\theta_2 - \theta_1$ and the product of radial coordinates $\rho_1 \rho_2 = r_1 r_2 / R^2$. The random field $u(r, \theta)$ is thus homogeneous with respect *to the angular coordinate* θ*, and its partial discrete spectral density has the form* $f_\theta(0) = 1/2\pi$, $f_\theta(k) = (\rho_1\rho_2)^k/\pi, \quad k = 1, \dots .$

Proof. We start by simple evaluations:

$$
\begin{aligned}
B_u &= \langle u(r_1,\theta_1)\, u(r_2,\theta_2)\rangle \\
&= \Big\langle \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-{r_1}^2)\, g(\varphi)\, d\varphi}{R^2 - 2\,Rr_1\, cos(\theta_1-\varphi) + {r_1}^2} \cdot \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-{r_2}^2)\, g(\varphi)\, d\varphi}{R^2 - 2\,Rr_2\, cos(\theta_2-\varphi) + {r_2}^2} \Big\rangle \\
&= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \frac{((R^2-r_1^2)(R^2-r_2^2)\,\langle g(\varphi')\, g(\varphi'')\rangle\, d\varphi'\, d\varphi''}{[R^2 - 2R\, r_1\cos(\theta_1-\varphi') + r_1^2]\,[R^2 - 2R\, r_2\cos(\theta_2-\varphi'') + r_2^2]} \\
&= \frac{1}{(2\pi)^2}\int_0^{2\pi} \frac{1-{\rho_1}^2}{1 - 2\,\rho_1\, cos(\theta_1-\varphi) + {\rho_1}^2} \cdot \frac{1-{\rho_2}^2}{1 - 2\,\rho_2\, cos(\theta_2-\varphi) + {\rho_2}^2}\, d\varphi\ . \qquad (7)
\end{aligned}
$$

Here we used the property of the white noise $\langle g(\varphi')\, g(\varphi'')\rangle = \delta(\varphi' - \varphi'')$.

This integral can be evaluated explicitly, and the result is given in (6). However we will obtain it using Fourier series expansion which not only presents a simple derivation of (6), but yields the spectrum of our random field, and the Karhunen-Loève expansion.

Indeed, we start with the well known expansion [45]

$$
K(\rho;\theta-\varphi) \equiv \frac{1}{2\pi}\cdot\frac{1-\rho^2}{1-2\rho\cos(\theta-\varphi)+\rho^2} = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\rho^k\, \cos[k(\theta-\varphi)] \qquad (8)
$$

and proceed (7) as follows

$$
\begin{aligned}
B_u &= \int_0^{2\pi}\left\{\frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\rho_1^k\, \cos[k(\theta_1-\varphi)]\right\}\left\{\frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\rho_2^k\, \cos[k(\theta_2-\varphi)]\right\} d\varphi \\
&= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\rho_1^k\int_0^{2\pi}\cos[k(\theta_1-\varphi)]K(\rho_2;\theta_2-\varphi)\, d\varphi \\
&= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\rho_1^k\int_0^{2\pi}[\cos\, k\theta_1\, \cos\, k\varphi + \sin\, k\theta_1\, \sin\, \varphi]\, K(\rho_2;\theta_2-\varphi)\, d\varphi \\
&= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\rho_1^k\rho_2^k\cos[k(\theta_1-\theta_2)] = \frac{1}{2\pi}\cdot\frac{1-\rho_1^2\rho_2^2}{1-2\rho_1\rho_2\cos(\theta_2-\theta_1)+\rho_1^2\rho_2^2}\ . \qquad (9)
\end{aligned}
$$

Here we used the nice property of the integral operator with the kernel $K(\rho;\theta-\varphi)$ that it has the following system of eigen-values $\{\lambda_k\}$ and the corresponding orthonormal eigen-functions $\{h_k(\varphi)\}$ complete in $L_2(0,2\pi)$:

$$
\begin{aligned}
&\lambda_0 = 1,\ h_0 = \frac{1}{\sqrt{2\pi}}, \quad \lambda_{2k-1} = \lambda_{2k} = \rho^k, \\
&h_{2k-1} = \pi^{-1/2}\cos(k\theta)\ ;\ h_{2k} = \pi^{-1/2}\sin(k\theta), \quad k = 1, 2, \dots . \qquad (10)
\end{aligned}
$$

This can be verified by a direct substitution of the series expansion (8) into the eigen-value problem

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-\rho^2)\;h_k(\varphi)\,d\varphi}{1-2\rho\cos(\theta-\varphi)+\rho^2}=\lambda_k\,h_k(\theta)\;. \tag{11}$$

So it remains to prove that our random field $u(\rho,\theta)$ has a discrete partial spectral density, $f_\theta(0)=1/2\pi$, and

$$f_\theta(k)=\frac{1}{2\pi}\int_0^{2\pi}B_u(\rho_1,\theta_1;\rho_2,\theta_2)\,e^{-\mathrm{i}\,k\,(\theta_2-\theta_1)}\,d(\theta_2-\theta_1)=(\rho_1\rho_2)^k/\pi,\quad k=1,\dots\;. \tag{12}$$

Actually this can be easily seen from the arguments given in (9). A direct proof follows from the Fourier transform property for convolutions. Indeed, the representation (9) shows that the correlation function B_u is written in the form of a convolution, i.e.,

$$\begin{aligned} B_u &= K(\rho_1;\psi)\,*\,K(\rho_2;\psi-(\theta_1-\theta_2)) \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\frac{1-\rho_1^2}{(1-2\rho_1\cos(\psi)+\rho_1^2)}\cdot\frac{1-\rho_2^2}{(1-2\rho_2\cos(\psi-(\theta_2-\theta_1))+\rho_2^2)}\;d\psi\;. \end{aligned}$$

Now we take the inverse Fourier transform of both parts, and use the Fourier transform property for convolutions. This yields

$$f_\theta(0)=1/2\pi,\quad f_\theta(k)=\rho_1^k\;\rho_2^k/\pi$$

which is the desired result. Here we used the property [11]

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-\rho^2)\;\cos(kx)\;dx}{1-2\rho\;\cos\,x+\rho^2}=\rho^k$$

while the sin-transform is zero. Finally, the covariance function $B_u(x_1,x_2)$ is harmonic with respect to both of its coordinates which follows from the general representation (3).

The proof is complete. □

Remark 1. The angular behaviour of the correlation function shows thus that the random field is partially homogeneous. The radial behaviour is also interesting. Let us fix a direction, say the line $y=0$, then, $B(x_1,x_2)=\frac{1}{2\pi}\cdot\frac{R^2+x_1x_2}{R^2-x_1x_2}$, where x_1 and x_2 vary between $-R$ and R. This shows that if one of the points, x_1, x_2 is in the center of the disc, the covariance equals to a constant value, $1/2\pi$.

For illustration, in Figure 1 we show the angular (left panel) and radial (right panel) behaviour of the correlation function B_u. The angular and radial functions are both plotted versus the section number k, the number of sections being 50, so that $\theta=k\,2\pi/50$ (angular behaviour, left panel), and $x=k\,2R/50$ (radial behaviour, right panel). The angular behaviour in the left panel is shown for three different choices of the radii ρ_1 and ρ_2. The radial behaviour is given for 6 different values of the value x_1, the radius of the disc was 5, see the right panel in Figure 1. As expected, a low number of eigen-modes in the K-L expansion is enough to have a good approximation; in Figure 2 we compare the K-L approximation against the exact result, taking $M=5$ and $M=10$ terms (left panel), and $M=2$ and $M=5$ terms (right panel).

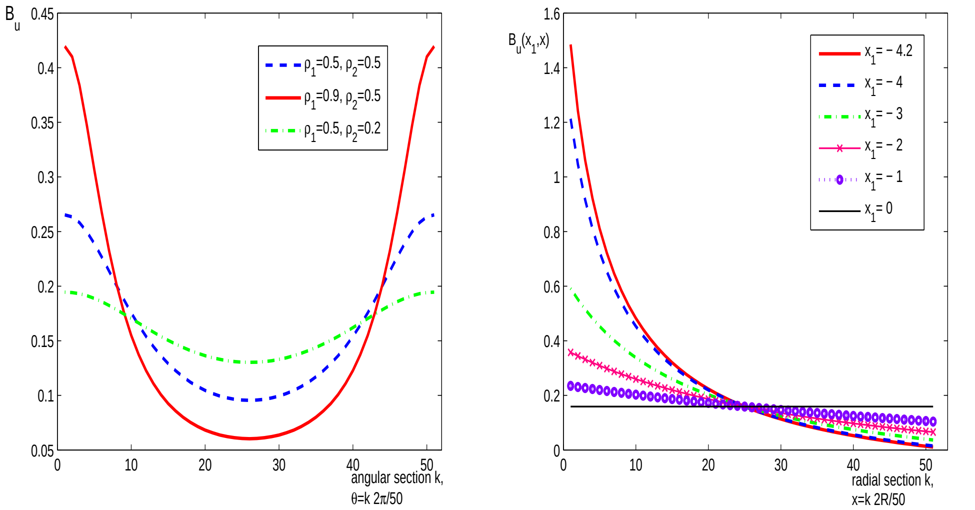

Figure 1: Laplace equation, Dirichlet boundary conditions: angular correlations B_u, for three different values of the ratio $\rho_i = r_i/R$ (left panel), and radial correlations $B(x_1, x)$, for six different values of the starting point x_1, R=5 (right panel).

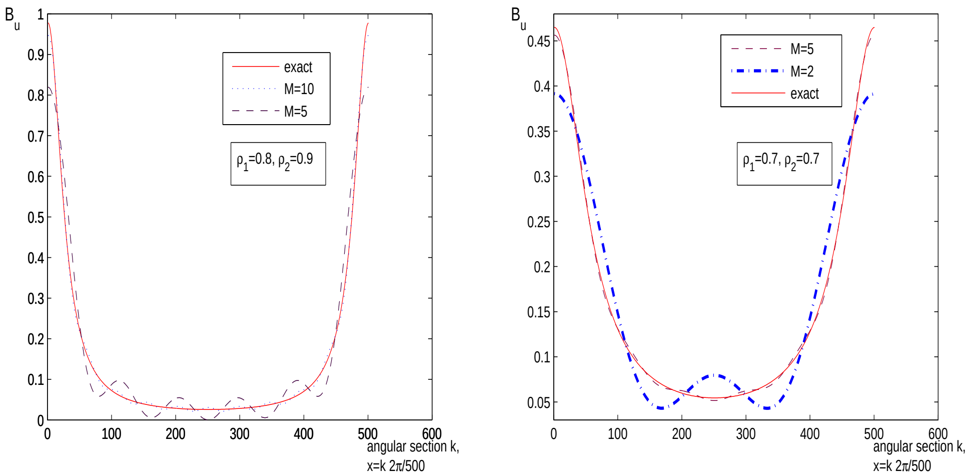

Figure 2: Laplace equation, Dirichlet boundary conditions: angular correlations B_u, for two different values of the retained number of terms M, for $\rho_1 = 0.8, \rho_2 = 0.9$ (left panel), and $\rho_1 = 0.7, \rho_2 = 0.7$ (right panel). The number of angular sections equals 500.

Theorem 2. *The Gaussian random field described in Theorem 1 has the following Karhunen-Loève type expansion*

$$u(r,\theta) = \frac{\xi_0}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \rho^k \left[\xi_k \cos(k\,\theta) + \eta_k \sin(k\,\theta)\right] \tag{13}$$

where $\{\xi_k\}$, $\{\eta_k\}$ *are sets of mutually independent standard Gaussian random variables.*

Proof. The idea of the proof appeals to Mercer's theorem which states the following (e.g., see [14]. Let U be a compact set in $\mathbb{R}^d$, and let $K(s,t)$ be a symmetric $L_2(U)$-kernel with eigen-values $\{\lambda_n\}$ and eigen-functions $h_k(t)$:

$$\int K(s,t)\, h_k(t)\, dt = \lambda_k h_k(x) \ , \quad k = 1, 2 \ldots \ .$$

Mercer's Theorem. *If a nonnull, symmetric* $L_2(U)$*-kernel* $K(s,t)$ *is quasi-definite (i.e., when all but a finite number of eigen-values are of one sign) and continuous, then the series* $\sum_{n=0}^{\infty} \lambda_n$ *is convergent, and*

$$K(s,t) = \sum_{n=0}^{\infty} \lambda_n h_n(s)\, \bar{h}_n(t) \ , \tag{14}$$

where $\bar{h}_k(t)$ *be the complex conjugate of* $h_k(t)$*, and the series converges absolutely and uniformly in* $U \times U$*.*

From this theorem, the Karhunen-Loève expansion can be obtained (e.g., see [48]):
Let $v(x)$ *be a real-valued, zero mean, gaussian random field with continuous covariance function* $K(x,y)$ *which has Mercer's expansion* $K(x,y) = \sum\limits_k \lambda_k h_k(x)\, h_k(y)$*. Then, under some regulary conditions,*

$$v(x) = \sum_{k=0}^{\infty} \sqrt{\lambda_k}\, h_k(x)\, \xi_k \ , \tag{15}$$

in L_2 *and a.s., where* $\{\xi_k\}_{k \in \mathbb{N}}$ *is a sequence of independent and identically standard normally distributed random variables.*

Note that although our correlation function (9) is continuous everywhere inside the disc, it increases infinitely as both points approach a point on the boundary, i.e., when $\theta_1 = \theta_2$, and $\rho_1 \to 1$, $\rho_2 \to 1$.

However our kernel, the covariance function (9), belongs to $L_2(\bar{K}_0)$, for each disc $\bar{K}_0(x_0, \rho_0) \subset K(x_0, 1)$, and we find from the expansion (9) that

$$\int\limits_{\bar{K}_0(x_0,\rho_0)} B_u\, dx\, dy < \infty \ ,$$

and so the weak convergence as $\rho_0 \to 1$ can be proven.

Now we consider the eigen-value problem for the covariance function B_u:

$$\int_0^1 d\rho_1 \int\limits_0^{2\pi} \frac{1}{2\pi} \frac{(1-\rho_1^2\rho_2^2)\ h_k(\rho_1,\theta_1)\, d\theta_1}{1 - 2\rho_1\rho_2 \cos(\theta_2-\theta_1) + \rho_1^2\rho_2^2} = \lambda_k\, h_k(\rho_2,\theta_2) \ .$$

Using the expansion (9) we find the eigen-functions and eigen-values:

$$\lambda_0 = 1,\ h_0 = \frac{1}{\sqrt{2\pi}}; \quad \lambda_{2k-1} = \lambda_{2k} = \frac{1}{2k+1};$$
$$h_{2k-1}(\rho,\varphi) = \sqrt{2k+1}\ \rho^k\ \frac{\cos(k\theta)}{\pi^{1/2}}\ ;\ h_{2k}(\rho,\varphi) = \sqrt{2k+1}\ \rho^k\ \frac{\sin((\theta)}{\pi^{1/2}},$$
$$k = 1, 2, 3, \dots$$

where the eigen-functions are orthonormal to one another:

$$\int_0^1 \int_0^{2\pi} h_n(\rho,\theta)\, h_m(\rho,\theta)\ d\rho\, d\theta = \delta_{nm}\ .$$

Thus the Karhunen-Loève expansion (13) follows from the representation

$$u(r,\theta) = \sum_{k=1}^{\infty} \zeta_k \sqrt{\lambda_k}\ h_k(\rho,\theta)$$

where ζ is a family of standard independent Gaussian random variables.

The proof of Theorem 2 is complete. □

The explicit representation of our random field (13) is very convenient in practical simulations, as well as in analytical evaluations of different statistical functionals.

Note that since our random field is homogeneous with respect to the angular variable, we can also write down the relevant randomized spectral representation when $\rho = \rho_1 = \rho_2$.

Indeed, we now let the discrete wave numbers k be randomly distributed with the distribution

$$p_k = \frac{1-\rho^2}{\rho^2}\,\rho^{2k}\ , \quad k = 1, 2, \dots\ .$$

Then the random field

$$u(r,\theta) = \frac{\xi_0}{\sqrt{2\pi}} + \frac{\rho}{\sqrt{\pi(1-\rho^2)}}\big[\xi\ \cos(k\,\theta) + \eta\ \sin(k\,\theta)\big] \tag{16}$$

has the desired correlation function (6). Here ξ_0, ξ and η are standard independent Gaussian variables. Further, to make the distributions close to Gaussian, in the spectral models one usually takes independent sums of models (16) (e.g., see [31]).

2.2 General homogeneous boundary excitations.

Assume now that a zero mean real-valued Gaussian random process g is defined on the circle by its spectrum f_k so that the covariance function reads

$$B_g(\varphi'' - \varphi') = \frac{f_0}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} f_k\ \cos\ k(\varphi'' - \varphi')\ .$$

Substituting this in (7) and using the series expansion of the kernel $K(\rho;\theta-\varphi)$, we arrive at the following series expansion for the covariance function B_u:

$$B_u(\rho_1,\theta_1;\rho_2,\theta_2) = \frac{f_0}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} f_k\,\rho_1^k\,\rho_2^k \cos\ k(\theta_2 - \theta_1)\ . \tag{17}$$

Thus the generalization of the random filed representation (15) has the form

$$u(r,\theta) = \frac{\sqrt{f_0}\,\xi_0}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \sqrt{f_k}\,\rho^k \left[\xi_k\, \cos(k\,\theta) + \eta_k\, \sin(k\,\theta)\right] \,. \tag{18}$$

The result (17) is an indication that there should be a simple relation between the correlation function B_u and the correlation function B_g of the homogeneous process g defined on the boundary. Indeed, we present this relation below in Theorem 3.

The correlation function of the solution in the case when g is a white noise, is given in (6). It depends on the difference $\psi = \theta_2 - \theta_1$, and on the product $\rho_1\rho_2$. Thus in the notation of Poisson kernel given in (8) the correlation function (6) reads

$$B_u = K(\rho_1\rho_2;\psi) = \frac{1}{2\pi}\frac{1-\rho_1^2\rho_2^2}{1-2\rho_1\rho_2\cos(\psi)+\rho_1^2\rho_2^2} \,. \tag{19}$$

Now we can give the desired relation between the correlation functions.

Theorem 3. *Assume the boundary function g in the Diriclet problem (4) is a homogeneous random process with a continuous correlation function $B_g(\psi)$. Then the solution of the problem (4) is partially homogeneous with respect to the angular coordinate, and its correlation function $B_u(\rho_1,\theta_1;\rho_2,\theta_2)$ depends on the angular difference $\psi = \theta_2 - \theta_1$ and the product $\rho_1\rho_2$, and is explicitly given by the convolution $B_u = K * B_g$, i.e., by the Poisson formula*

$$B_u(\rho_1\rho_2;\psi) = \frac{1}{2\pi}\int_0^{2\pi} K(\rho_1\rho_2;\psi-\psi')\,B_g(\psi')\,d\psi' \tag{20}$$

which implies that the correlation function $B_u(\rho,\theta)$ is harmonic in the unit disc, and it is the unique solution of the Dirichlet boundary value problem

$$\Delta B_u = 0, \quad B_u|_{\rho\to 1} = B_g \,. \tag{21}$$

Proof. To obtain (20), we turn to the proof of Theorem 1, and use in the double integral in (7) the change of variable $\psi = \varphi'' - \varphi'$, use there the series expansions for the both Poisson kernels, and perform the integration over φ''. This yields (20).

Remark 2. From the proof it is clear that the same convolution relation result remains true if two homogeneous and homogeneously correlated stochastic processes are given on the boundary. Indeed, let g_1 and g_2 be two homogeneous processes on the circle with zero mean and a cross-correlation $B_{g_1g_2}(\theta_2-\theta_1)$. Then the corresponding solutions u_1 and u_2 are also homogeneously correlated, and the cross-correlation function $B_{u_1u_2}$ is related to $B_{g_1g_2}$ by the same convolution formula with the kernel K as in Theorem 3: $B_{u_1u_2} = K * B_{g_1g_2}$.

Finally we note that from (18) we can derive the expressions for B_{u_x} and B_{u_y}, the correlation functions for the derivatives u_x and u_y which is our case remarkably coincide:

$$B_{u_x} = B_{u_y} = \frac{1}{\pi}\sum_{k=1}^{\infty} f_k\,k^2\,\rho^{k-1}\,\cos\left[(k-1)\,\theta\right] \,.$$

2.3 Neumann boundary conditions.

Let us study the case when on the boundary, the normal derivative is prescribed, i.e., we consider the inner problem for the disc $D = K(x_0, R)$:

$$\Delta u(x) = 0, \quad x \in D, \quad \frac{\partial u}{\partial \mathbf{n}}(y) = g(y) \quad y \in \Gamma = \partial D, \tag{22}$$

where $\mathbf{n}$ is the external normal vector.

The Poisson type formula in polar coordinates centered at x_0 has the form [45]

$$u(r,\theta) \quad = \quad -\frac{1}{2\pi}\int_0^{2\pi} \ln(1 - 2\rho\cos(\theta-\varphi) + \rho^2)\, g(\varphi)\, d\varphi + const \tag{23}$$

where $\rho = r/R$, and *const* is an arbitrary constant which we further take equal to zero.

As in the Dirichlet problem, here the eigen-value property of the kernel (see (10) plays the crucial role. By direct evaluations we can prove that

$$-\frac{1}{2\pi}\int_0^{2\pi} \ln(1 - 2\rho\cos(\theta-\varphi) + \rho^2)\, h_k(\varphi)\, d\varphi = \lambda_k\, h_k(\theta) \tag{24}$$

where

$$\lambda_{2k-1} = \lambda_{2k} = \frac{\rho^k}{k}; \;\; h_k = \pi^{-1/2}\cos(k\theta) \;;\; h_{2k} = \pi^{-1/2}\sin(k\theta), \quad k = 1,2,3,\ldots \tag{25}$$

This can be easily proved by substituting the expansion [11]

$$\ln(1 - 2\rho\cos(\theta-\varphi) + \rho^2) = -2\sum_{k=1}^{\infty}\frac{\rho^k}{k}\,\cos[k(\theta-\varphi)]$$

in (24).

From this, we can derive the following result which is a counterpart of Theorem 1.

Theorem 4. *The solution of the Neumann problem (22) in a disc $K(x_0,R)$ with the Gaussian white noise boundary function $g(y)$ is an inhomogeneous 2D Gaussian random field uniquely defined by the correlation function*

$$B_u(\rho_1,\theta_1;\rho_2,\theta_2) = K_{Neum}(\rho_1\rho_2;\theta_2-\theta_1) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\rho_1^k\rho_2^k}{k^2}\,\cos\, k(\theta_2-\theta_1)\;. \tag{26}$$

The random field $u(r,\theta)$ is homogeneous with respect to the angular coordinate θ, and its respective discrete spectral density has the form $f_\theta(k) = \frac{\rho_1^k\rho_2^k}{\pi k^2} \quad k = 1,\ldots$.
Moreover, if g is a homogeneous random process with a correlation function $B_g(\psi')$ then the correlation function of the solution is related to B_g by the convolution

$$B_u(\rho_1\rho_2;\psi) = \frac{1}{2\pi}\int_0^{2\pi} K_{Neum}(\rho_1\rho_2;\psi-\psi')\, B_g(\psi')\, d\psi'\;. \tag{27}$$

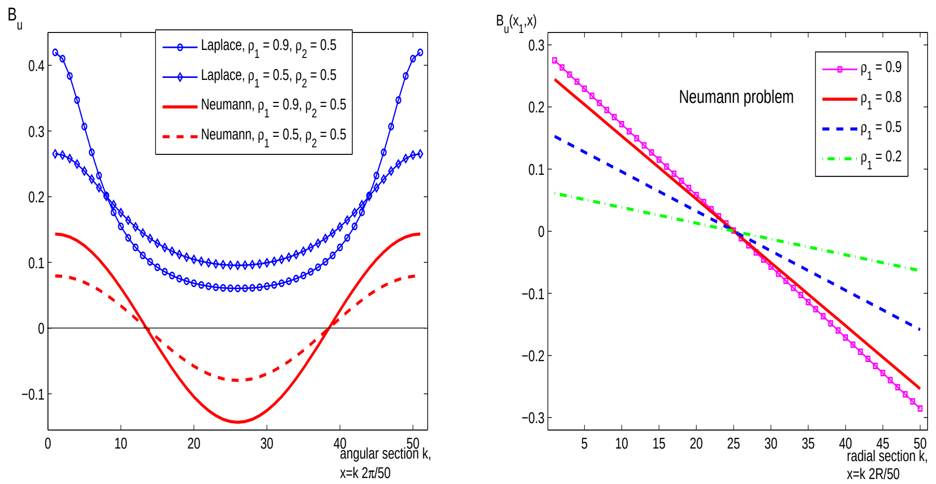

Figure 3: Comparison of angular correlations for Laplace and Neumann boundary conditions, for two different values of the radii (left panel). Radial correlation function for the Neumann boundary conditions (right panel).

Proof. The proof of (26) is essentially the same as in the case of Dirichlet problem. The correlation function B_u is written in the form of a convolution, i.e.,

$$B_u \quad = \quad K_1(\rho_1;\psi) \, * \, K_1(\rho_2;\psi-(\theta_1-\theta_2))$$

where $K_1(\rho;\psi)=\ln(1-2\rho\cos(\psi)+\rho^2)$. Then we take the Fourier transform of both parts, and use the Fourier transform property for convolutions. This yields $f_\theta(k)=\rho_1^k\;\rho_2^k/\pi k^2$ which is the desired result. The proof of (27) follows basically the same scheme, and repeats the scheme given in the proof of Theorem 3. □

From these considerations, we can find the eigen-values and eigen-functions of the correlation function. These are

$$\lambda_{2k-1}=\lambda_{2k}=\frac{1}{k^2\,(2k+1)};\;\; k=1,2,3,\ldots \tag{28}$$
$$h_{2k-1}(\rho,\varphi)=\sqrt{2k+1}\;\rho^k\,\frac{\cos(k\theta)}{\pi^{1/2}}\;;\; h_{2k}(\rho,\varphi)=\sqrt{2k+1}\;\rho^k\,\frac{\sin(k\theta)}{\pi^{1/2}}\;.$$

This leads to the Karhunen-Loève expansion (26). The random filed is therefore written as follows

$$u(r,\theta)=\frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{\rho^k}{k}\left[\xi_k\,\cos(k\,\theta)+\eta_k\,\sin(k\,\theta)\right]\;.$$

We compare in Figure 3 the angular correlations for the Laplace and Neumann boundary conditions (left panel), and show the radial behaviour of the correlation function for the solution of the Neumann problem (right panel).

2.4 Upper half-plane.

Let us consider the Dirichlet problem in the half-plane:

$$\Delta u(x)=0,\quad x\in D^+,\quad u(y)=g(y)\quad y\in\Gamma=\partial D^+, \tag{29}$$

where the domain D^+ is the upper half-plane with the boundary $\Gamma = \{(x,y): \ y=0\}$.

The Poisson formula reads [45]

$$u(x) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(\xi)\, d\xi}{(x-\xi)^2 + y^2} \ . \tag{30}$$

Direct evaluation of the covariance function $B(x_1, y_1; x_2, y_2)$ even in the case when the function $g(\xi)$ is a white noise meets some technical difference in comparison to the disc,

$$B(x_1, y_1; x_2, y_2) = \frac{y_1\, y_2}{(\pi)^2} \int_{-\infty}^{\infty} \frac{d\xi}{[(x_1-\xi)^2 + y_1^2][(x_2-\xi)^2 + y_2^2]} \ . \tag{31}$$

Therefore, we use the Fourier transform technique. Let us introduce the notation for the kernel

$$K_p(\eta, y) = \frac{y}{\eta^2 + y^2} \tag{32}$$

so that the covariance is written in the form of convolution

$$B(x_1, y_1; x_2, y_2) = K_p(\eta, y_1) * K_p(\eta - (x_2 - x_1), y_2) \ ,$$

and the Fourier transform yields $\mathcal{F}_B = \mathcal{F}_{K(\cdot, y_1)} \cdot \mathcal{F}_{K(\cdot, y_2)}$. Since [11]

$$\mathcal{F}_{K(\cdot, y)} = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\cos(kx)\, dx}{y^2 + x^2} = e^{-|k| y} \ ,$$

we get $\mathcal{F}_B = e^{-|k|(y_1+y_2)}$. Inverse Fourier transform finally yields [11]

$$B(x_1, y_1; x_2, y_2) = \mathcal{F}^{-1}(e^{-|k|(y_1+y_2)}) = \frac{1}{\pi} \frac{y_1 + y_2}{(y_1+y_2)^2 + (x_1-x_2)^2} \ . \tag{33}$$

Now we need to solve the eigen-value problem for the covariance operator:

$$\int_0^{\infty} dy_2 \int_{-\infty}^{\infty} \frac{y_1+y_2}{\pi} \frac{h_k(x_2, y_2)\, dx_2}{(y_1+y_2)^2 + (x_1-x_2)^2} = \lambda_k\, h_k(x_1, y_1) \ . \tag{34}$$

Here we cannot apply the classical Hilbert-Schmidt theory since the process is defined on the unbounded domain D^+. Therefore, we can apply the cut-off approach described in section 1.2. Indeed, the correlation function (33) is partially homogeneous, with respect to the horizontal coordinate x.

Through a Fourier analysis we find that the partial spectrum is $S(k) = \exp(-k(y_1+y_2)$, and the eigen-functions are the Fourier modes. Thus as discussed in section 1.2,

$$\lambda_k \approx S(\omega_k) = S(\pi k/a) = \exp\{-\pi k(y_1+y_2)/a\},$$

and the spectral approximations are

$$B_u \approx \tilde{B}_a(x_1, y_1; x_2, y_2) = \sum_{k=1}^{\infty} \frac{1}{a} e^{-\frac{\pi k}{a}(y_1+y_2)} \cos\left[\frac{\pi k(x_2-x_1)}{a}\right] \ , \tag{35}$$

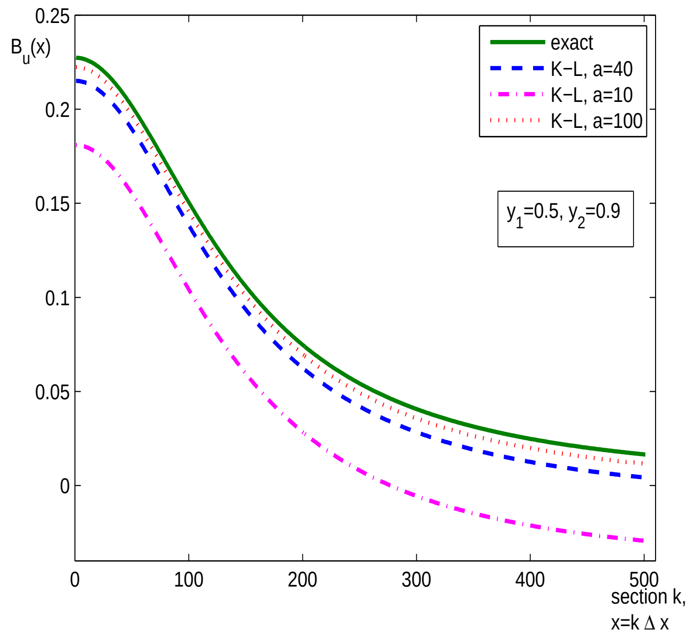

Figure 4: Comparison of the approximate correlation function with the exact result, for three different values of the cut-off parameter a, $\Delta x = 0.01$.

$$u \approx \tilde{u}_a(x,y) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{a}} \exp\{-\pi\, ky/a\}\; [\xi_k\, \cos[\pi\, k\, x/a)] + \eta_k\, \sin[\pi\, k\, x/a)] \tag{36}$$

where ξ_k and η_k are two mutually independent families of standard Gaussian random variables. Thus introducing the cut-off we can find the orthonormal set of eigen-functions of the eigen-value problem for the correlation operator (34):

$$h_{2k-1}(x,y) = \frac{\cos(\pi k x/a)}{\sqrt{a}} \sqrt{\frac{2\pi\, k}{a}}\, e^{-\frac{\pi k y}{a}}\;, \quad h_{2k}(x,y) = \frac{\sin(\pi k x/a)}{\sqrt{a}} \sqrt{\frac{2\pi\, k}{a}}\, e^{-\frac{\pi k y}{a}}\;,$$

$$\lambda_{2k-1} = \lambda_{2k} = \frac{a}{2\pi\, k}, \quad k = 1, 2, \ldots,\; .$$

Note that the cut-off parameter a should be chosen large enough.

For illustration, we present in Figure 4 the approximation (35) for 3 different values of a compared against the exact representation (33). The numerical convergence is clearly seen as the cut-off parameter increases. Obviously, as mentioned at the end of section 1.2, the larger a, the larger the number of retained terms n, so that $n \sim a/\varepsilon$ where ε is the approximation error.

Finally we notice that the Theorem 3 proved above for the case of a disc holds also for the half-plane where the kernel $K(\rho_1\rho_2;\theta_2-\theta_1)$ should be replaced by the kernel (33). Thus if the random function g defined on the axis x, $\{(x,y):\; y=0\}$ is a homogeneous random process with the correlation function B_g, then the correlation function of the solution $B_u(x_1,y_1;x_2,y_2) = B_u(x_2-x_1, y_2+y_1)$ depends on $x = x_2 - x_1$ and $y = y_2 + y_1$, so $B_u(x,y)$ is harmonic in D^+, with the boundary conditions $B_u|_{y\to 0} = B_g$. We will show now that this is true indeed for a half-space in any dimension. So let us give the result in more details. Here it is convenient to use the boldface chracter $\mathbf{x}$ for the horithontal coordinates, and y for the vertical one.

Theorem 5. *Let $u(\mathbf{x}, y)$, $\mathbf{x} = (x_1, \dots x_{n-1})$ be a random field defined in the half-space $D_+ = \mathbb{R}^n_+$ as a harmonic function with the boundary condition $u|_{y=0} = g$ where g is a zero mean homogeneous random field on the boundary $\{y = 0\}$ with the correlation function $B_g(\mathbf{x})$ which is bounded in dimension $n = 2$, or tends to zero as $|\mathbf{x}| \to \infty$ if $n > 2$. Then $B_u(\mathbf{x}, y) = B_u(\mathbf{x}_2 - \mathbf{x}_1, y_1 + y_2)$, the correlation function of the solution, is a harmonic function in $\mathbb{R}^n_+$, and is related to B_g by the Poisson type formula:*

$$B_u(\mathbf{x}_2 - \mathbf{x}_1, y_1 + y_2) = \frac{\Gamma(n/2)}{\pi^{n/2}} \int\limits_{\partial D_+} \frac{(y_1 + y_2)\, B_g(\mathbf{x}')\, dS(\mathbf{x}')}{[(\mathbf{x}' - (\mathbf{x}_2 - \mathbf{x}_1))^2 + (y_1 + y_2)^2]^{n/2}} \,. \tag{37}$$

The proof is obtained by the same Fourier transform technique we used above.

Remark 3. We remark that exactly as in the case of a disc as discussed in Remark 2 to the Theorem 3, the same convolution relation (37) is true for the cross-correlation functions, we need only to write it for the kernel K_p: $B_{u_1 u_2} = K_p * B_{g_1 g_2}$. Note that in the n-dimensional case, K_p has the form of the kernel given in (37).

In practice, it is often important to know the statistical structure of the gradient of the solution. Let us denote by $B_{u_{x_i}}(\mathbf{x}, y)$, $i = 1, \dots, n-1$, and $B_{u_y}(\mathbf{x}, y)$ the correlation functions of the partial derivatives of the solution u. They obviously also depend only on $\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$ and $y = y_1 + y_2$ by the assumption that g is homogeneous. Direct evaluation gives

$$B_{u_{x_i}} = -\frac{\partial^2 B_u}{\partial x_i}, \quad i = 1, \dots, n-1, \qquad B_{u_y} = \frac{\partial^2 B_u}{\partial^2 y} \,.$$

Note that since the correlation function B_u is harmonic, this implies the following remarkable property: $B_{u_y} = \sum\limits_{i=1}^{n-1} B_{u_{x_i}}$. So in dimension two, $B_{u_y} = B_{u_x}$.

3 3D Laplace equation.

For a ball in 3D, all considerations are quite similar, where the eigen-functions involved are the spherical harmonics. The regular solution to the harmonic equation in a 3D ball $D(x_0, R)$ of radius R centered at a point x_0 is represented by the Poisson integral formula as an integral over the sphere $S(x_0, R) = \partial D(x_0, R)$ [45]:

$$u(x) \quad = \quad \frac{R^2 - r^2}{4\pi R} \int\limits_{S(x_0,\, R)} \frac{g(y) dS_y}{|x - y|^{3/2}}$$

for any point $x \in D(x_0, R)$, where $r = |x - x_0|$.

In spherical coordinates centered at x_0 the Poisson formula reads

$$u(r, \theta, \varphi) \quad = \quad \frac{1 - \rho^2}{4\pi} \int\limits_0^{2\pi} \int\limits_0^{\pi} \frac{\sin(\theta')\, g(\theta', \varphi')\, d\theta'\, d\varphi'}{[1 - 2\rho\cos(\psi) + \rho^2]^{3/2}} \tag{38}$$

where $\rho = r/R$, and ψ is the angle between the vectors s and s', which implies,

$$cos(\psi) = \cos\theta\, \cos\theta' + \sin\theta\, \sin\theta'\, \cos(\varphi - \varphi') \,. \tag{39}$$

Let g be a zero mean random field defined on the sphere $S(x_0, R)$. It is called isotropic, if its correlation function $B_g(s, s')$ depends only on the angular distance between s and s', i.e., only on the angle ψ as defined in (39). We say that a random field defined in a ball $D(x_0, R)$ is partially isotropic in the ball if it is isotropic with respect to the angular coordinates.

The statement of Theorem 1 for the 3D case can be formulated as follows.

Theorem 6. *The solution of the Dirichlet problem in the ball $D(x_0, R)$ with the white noise boundary function $g(y)$ is an inhomogeneous 3D Gaussian random field uniquely defined by the correlation function*

$$B_u = K_3(\rho_1\rho_2; \psi_{12}) \equiv \frac{1}{4\pi} \frac{1 - \rho_1^2\rho_2^2}{[1 - 2\rho_1\rho_2 \cos(\psi_{1,2}) + \rho_1^2\rho_2^2]^{3/2}} \tag{40}$$

where $\cos(\psi_{1,2}) = \cos\theta_1\, \cos\theta_2 + \sin\theta_1\, \sin\theta_2\, \cos(\varphi_1 - \varphi_2)$. The random field $u(r, \theta, \varphi)$ is partially isotropic in the ball, and its respective discrete spectral density has the form $f_\theta(0) = 1/4\pi$, $f_\theta(k) = (\rho_1\rho_2)^k/2\pi, \quad k = 1, \ldots$. Generally, if g is an isotropic random field on the sphere, the correlation function of the solutions B_u is related to B_g by $B_u(\rho_1\rho_2; \psi) = K_3(\rho_1\rho_2; \psi - \psi') * B_g(\psi')$ which implies that the solution u is partially isotropic in the ball, and the correlation function B_u is harmonic, with the prescribed boundary function B_g.

Proof. We use here also a series expansion method. Let us recall some definitions.

The Legendre polynomials we denote by $\mathcal{P}_l(\cos\theta)$, - recall that these functions are defined on $(-1, 1)$ as follows:

$$\mathcal{P}_l(\mu) = \frac{1}{2^l\, l!} \frac{d^l}{d\mu^l} (\mu^2 - 1)^l, \quad l = 0, 1, \ldots .$$

The associated Legendre polynomials $\mathcal{P}_l^m(\mu)$, $l = 0, 1, \ldots; \quad m = 0, 1, \ldots, l$ are defined via the (m)-derivatives of $\mathcal{P}_l(\mu)$ as follows

$$\mathcal{P}_l^m(\mu) = (1 - \mu^2)^{m/2} \mathcal{P}^{(m)}(\mu), \quad l = 0, 1, \ldots; \quad m = 0, 1, \ldots, l \; .$$

Then, the system of spherical harmonics functions $\{Y_l^m(\theta, \varphi)\}$, $l = 0, 1, \ldots; \quad m = 0, \pm 1, \pm 2, \ldots, \pm l$ is defined as follows

$$\begin{aligned} Y_l^m(\theta, \varphi) &= \mathcal{P}_l^m(\cos\theta)\, \cos(m\varphi), \quad m = 0, 1, 2, \ldots, l; \\ Y_l^m(\theta, \varphi) &= \mathcal{P}_l^m(\cos\theta)\, \sin(|m\varphi|), \quad m = -1, -2, \ldots . \end{aligned} \tag{41}$$

It is well known that this is a system of orthogonal functions complete in $L_2(S)$, and

$$||Y_l^m||^2 = \int_0^\pi \int_0^{2\pi} [Y_l^m(\theta, \varphi)]^2 \, \sin\theta\, d\theta\, d\varphi = 2\pi \frac{1 + \delta_{0m}}{2l + 1} \frac{(l + |m|)!}{(l - |m|)!} \; .$$

The following expansion is well known (e.g., see [45], [8]):

$$K(\rho, \psi) \equiv \frac{1 - \rho^2}{[1 - 2\rho\cos(\psi) + \rho^2]^{3/2}} = 1 + \sum_{k=1}^{\infty} \rho^k\, (2k + 1)\, \mathcal{P}_k(\cos(\psi)) \; . \tag{42}$$

For brevity, let us introduce the notation for the unit vectors, s', s_1 and s_2 defined by its direction angles (θ', φ'), (θ_1, φ_1), and (θ_2, φ_2), respectively, and let

$$\begin{aligned} (s', s_1) = \cos(\psi_1) &= \cos\theta_1\, \cos\theta' + \sin\theta_1\, \sin\theta'\, \cos(\varphi_1 - \varphi') \\ (s', s_2) = \cos(\psi_2) &= \cos\theta_2\, \cos\theta' + \sin\theta_2\, \sin\theta'\, \cos(\varphi_2 - \varphi') \; . \end{aligned}$$

In what follows, we will sometimes use a shorter notation for the integration over a surface measure ds' on a unit sphere:

$$\int ds = \int_0^{2\pi}\int_0^{\pi} \sin(\theta')d\theta'\, d\varphi' \ .$$

We use the expansion (42) in the following explicit evaluations:

$$\begin{aligned}
&B_u(\rho_1, s_1; \rho_2, s_2) \\
&= \frac{1}{(4\pi)^2}\int_0^{2\pi}\int_0^{\pi} \frac{(1-\rho_1^2)(1-\rho_2^2)\ \sin(\theta')\, d\theta'\, d\varphi'}{[1-2\rho_1\cos(\psi_1)+\rho_1^2]^{3/2}\,[1-2\rho_2\rho_2\cos(\psi_2)+\rho_2^2]^{3/2}} \\
&= \frac{1}{(4\pi)^2}\int \Big[1+\sum_{k=1}^{\infty}\rho_1^k\,(2k+1)\,\mathcal{P}_k((s',s_1))\Big]\Big[1+\sum_{k=1}^{\infty}\rho_2^k\,(2k+1)\,\mathcal{P}_k((s',s_2))\Big]\,ds' \\
&= \frac{1}{4\pi}\Big[1+\sum_{k=1}^{\infty}(\rho_1\rho_2)^k\,(2k+1)^2\ \frac{\mathcal{P}_k((s_1,s_2))}{2k+1}\Big] \ . \qquad (43)
\end{aligned}$$

Here we used the following property:

$$\frac{1}{4\pi}\int \mathcal{P}_l((s,s_1))\mathcal{P}_k((s,s_2))ds = \frac{\mathcal{P}_k((s_1,s_2))}{2k+1}\,\delta_{kl} \ . \qquad (44)$$

This can be derived from the following property

$$\frac{1}{4\pi}\int \mathcal{P}_k((s,s'))\,Y_l^m(s')\,ds' = \frac{1}{2l+1}Y_l^m(s)\,\delta_{lk} \qquad (45)$$

which in turn follows from

$$\mathcal{P}_l((s,s')) = \sum_{m=-l}^{l}\kappa_{lm}\,Y_l^m(s)Y_l^m(s') \ . \qquad (46)$$

Here the coefficients are given by

$$\kappa_{lm} = \frac{2}{(1+\delta_{0m})}\frac{(l-|m|)!}{(l+|m|)!} \ . \qquad (47)$$

Thus the last line of (43) gives the desired result (40) and the proof is complete. □

Now we use the series representation (43) to solve the eigen-problem for the correlation function

$$\int_0^1 d\rho_2 \int B_u(\rho_1, s_1; \rho_2, s_2)\,h_l(\rho_2, s_2)\,ds_2 = \lambda_l\,h_l(\rho_1, s_1) \ . \qquad (48)$$

The next assertion follows immediately from the properties (44)-(46).

Theorem 7. *The eigen-value problem (48) has a complete set of orthonormal eigen-functions and the relevant eigen-values ($l = 0, 1, \ldots,\quad m = -l, \ldots, l$):*

$$\lambda_l = \frac{1}{2l+1}, \quad h_l(\rho, s) = \sqrt{\frac{\kappa_{lm}(2l+1)}{4\pi}}\ Y_l^m \cdot \sqrt{2l+1}\,\rho^l \ . \qquad (49)$$

The KL-expansions of the correlation function and the random field are given by

$$B_u(\rho_1, s_1; \rho_2, s_2) = \frac{1}{4\pi} + \frac{1}{4\pi} \sum_{k=1}^{\infty} (2k+1)\, \rho_1^k\, \rho_2^k \Big\{ \sum_{m=-k}^{k} \kappa_{km}\, Y_k^m(s_1)\, Y_k^m(s_2) \Big\} \ ,$$

$$u(r, s) = \frac{\xi_0}{\sqrt{4\pi}} + \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} \sqrt{2k+1}\, \rho^k \Big\{ \sum_{m=-k}^{k} \xi_{km} \sqrt{\kappa_{km}}\, Y_k^m(s) \Big\} \ ,$$

where ξ_0, $\{\xi_{km}\}$ are independent standard Gaussian random variables.

4 Biharmonic equation.

Let us consider the following problem for a biharmonic equation in a disc $D = K(x_0, R)$, governing a slow viscous motion inside a circular cylinder of radius R [16]:

$$\Delta u^2(x) = 0, \quad x \in D, \quad u(y) = g_0(y), \quad \frac{\partial u}{\partial \mathbf{n}}(y) = g_n(y) \quad y \in \Gamma = \partial D, \tag{50}$$

where $\mathbf{n}$ is the external normal vector.

In polar coordinates centered at x_0 with $\rho = r/R$ the Poisson type integral formula reads [31]

$$\begin{aligned} u(r, \theta) = \frac{(1-\rho^2)^2}{2\pi} \int_0^{2\pi} \Big\{ -\frac{R}{2[1 - 2\rho\cos(\theta - \varphi) + \rho^2]} \Big\} g_n(\varphi)\, d\varphi \\ + \frac{(1-\rho^2)^2}{2\pi} \int_0^{2\pi} \frac{[1 - \rho\, \cos(\theta - \varphi)]}{[1 - 2\rho\cos(\theta - \varphi) + \rho^2]^2}\, g_0(\varphi)\, d\varphi \ . \end{aligned} \tag{51}$$

Assuming the random white noise excitations g_0 and g_n are independent, we decompose the random field into two independent components: $u = u^{(1)} + u^{(2)}$. Then, the covariance of u is the sum of covariances of $u^{(1)}$ and $u^{(2)}$. From (51) we obtain

$$\begin{aligned} B_u = \langle u(r_1, \theta_1)\, u(r_2, \theta_2) \rangle = \frac{R^2}{4} (1 - \rho_1^2)(1 - \rho_2^2)\, B_\Delta(\rho_1, \theta_1; \rho_2, \theta_2) \\ + \frac{1}{(2\pi)^2} \int_0^{2\pi} \frac{(1-\rho_1^2)^2\, (1 - \rho_1 \cos(\theta_1 - \varphi))}{[1 + \rho_1^2 - 2\rho_1 \cos(\theta_1 - \varphi)]^2} \cdot \frac{(1-\rho_2^2)^2\, (1 - \rho_2 \cos(\theta_2 - \varphi))}{[1 + \rho_2^2 - 2\rho_2 \cos(\theta_2 - \varphi)]^2}\, d\varphi \end{aligned} \tag{52}$$

where B_Δ is the covariance of the solution of the Dirichlet problem for the Laplace equation given in Theorem 1.

To tackle the second term which represents the covariance of the second component, $u^{(2)}$, we first remark that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{(1-\rho^2)^2\, (1 - \rho\cos(\theta - \varphi))}{[1 + \rho^2 - 2\rho\cos(\theta - \varphi)]^2} \cos(k\varphi)\, d\varphi = \big[1 + \frac{k}{2}\, ((1 - \rho^2)\big]\, \rho^k\, \cos(k\theta) \tag{53}$$

$$\frac{1}{(2\pi)} \int_0^{2\pi} \frac{(1-\rho^2)^2\, (1 - \rho\cos(\theta - \varphi))}{[1 + \rho^2 - 2\rho\cos(\theta - \varphi)]^2} \sin(k\varphi)\, d\varphi = \big[1 + \frac{k}{2}\, ((1 - \rho^2)\big]\, \rho^k\, \sin(k\theta) \ . \tag{54}$$

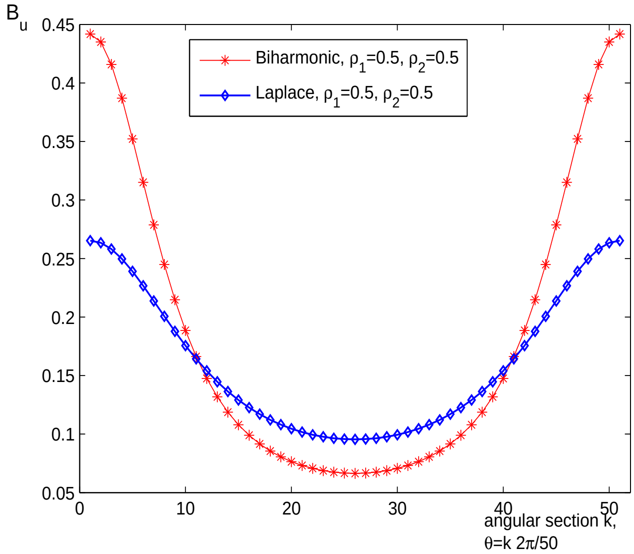

Figure 5: Angular correlation function for the biharmonic equation, compared against the correlation function for the Laplace equation with Dirichlet boundary conditions.

This can be shown as follows. First we note that by differentiating with respect to ρ we obtain the following useful equality

$$2\sum_{k=1}^{\infty} k\,\rho^k\,\cos(k\theta) = \rho\,\left(2\sum_{k=1}^{\infty}\rho^k\,\cos(k\theta)\right)' = \frac{-4\rho^2 + 2\rho^3\,\cos\,\theta + 2\,\rho\,\cos\,\theta}{[1+\rho^2-2\rho\,\cos\,\theta]^2}\ .$$

Now, combining with the expansion (8) we find that the kernel in the eigen-value problem (53), (54) is represented as the following series

$$K(\rho;\theta-\varphi) = \frac{1}{2\pi}\frac{(1-\rho^2)^2\,(1-\rho\cos(\theta-\varphi))}{[1+\rho^2-2\rho\cos(\theta-\varphi)]^2} = \frac{1-\rho^2}{2\pi}\sum_{k=1}^{\infty} k\,\rho^k\,\cos[k(\theta-\varphi)]$$
$$+\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\rho^k\,\cos[k(\theta-\varphi)]$$
$$= \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\left[\frac{(1-\rho^2)\,k}{2}+1\right]\rho^k\,\cos[k(\theta-\varphi)]\ . \qquad (55)$$

Substituting this representation in the eigen-value problem we arrive at (53), (54). The covariance can be evaluated by substituting the series expansion (55) in (52). This yields

$$B_{u^{(2)}} = \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\left[\frac{(1-\rho_1^2)\,k}{2}+1\right]\left[\frac{(1-\rho_2^2)\,k}{2}+1\right]\rho_1^k\,\rho_2^k\,\cos[k(\theta_1-\theta_2)]\ .$$

Analogously to the case of Laplace equation, we consider the eigen-value problem for the covariance kernel:

$$\int_0^{2\pi} d\theta_1 \int_0^1 d\rho_1\; B_{u^{(2)}}(\rho_1,\theta_1;\rho_2,\theta_2)\,h_k(\rho_1,\theta_1) = \lambda_k\,h_k(\rho_2,\theta_2)\ . \qquad (56)$$

Let us introduce the notation:

$$\Delta_k = \int_0^1 [\frac{(1-\rho^2)\,k}{2} + 1]^2 \,\rho^{2k}\; d\rho \;.$$

Using the series expansion of the kernel, it is not difficult to find that the eigen-value problem has the following system of eigen-functions and eigen-values:

$$\lambda_0 = 1,\;\; h_0 = \frac{1}{\sqrt{2\pi}}; \quad \lambda_{2k-1} = \lambda_{2k} = \Delta_k;$$
$$h_{2k-1}(\rho,\varphi) = \Big[\frac{(1-\rho^2)k}{2} + 1\Big]\, \frac{\rho^k}{\sqrt{\Delta_k}} \cdot \frac{\cos[k(\theta)]}{\pi^{1/2}} \;;$$
$$h_{2k}(\rho,\varphi) = \Big[\frac{(1-\rho^2)k}{2} + 1\Big]\, \frac{\rho^k}{\sqrt{\Delta_k}} \cdot \frac{\sin[k(\theta)]}{\pi^{1/2}} \;; \quad k = 1, 2, \ldots$$

where the eigen-functions are orthonormal to one another:

$$\int_0^1 \int_0^{2\pi} h_n(\rho,\theta)\, h_m(\rho,\theta)\; d\rho\, d\theta = \delta_{nm} \;.$$

From this we finally arrive at the Karhunen-Loève expansion

$$u^{(2)}(r,\theta) = \frac{\xi_0}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \Big[\frac{(1-\rho^2)\,k}{2} + 1\Big]\rho^k \,\big[\xi_k \cos k\theta + \eta_k \,\sin k\theta\big] \;. \tag{57}$$

The first component is obviously represented as $u^{(1)}(r,\theta) = \frac{R(1-\rho^2)}{2}\, u(r,\theta)$, where $u(r,\theta)$ is modeled by the KL-expansion given in (13).

In Figure 5 we show the angular behaviour of the correlation function of the solution to the biharmonic equation compared against the correlation function for the Laplace equation. In both cases, the correlations are plotted for the fixed values of ρ taken equal to 0.5, and $R = 1$.

5 Lamé equation. Plane elasticity problem.

5.1 White noise excitations.

Let us consider the plane elasticity problem in the disc $K(x_0, R)$:

$$\mu\, \Delta\mathbf{u}(x) + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\, \mathbf{u}(x) = 0, \quad x \in K(x_0, R),$$
$$\mathbf{u}(y) = \mathbf{g}(y), \;\; y \in S(x_0, R) \tag{58}$$

where $\mathbf{u} = (u_1, u_2)^T$ is the displacement column-vector which is prescribed on the boundary as a column-vector $\mathbf{g} = (g_1, g_2)^T$.

Let us work in polar coordinates centered at x_0, so that the point x is $r\, e^{i\,\theta}$, and on the boundary, $y = R\, e^{i\,\varphi}$, and as everywhere above, $\rho = r/R$.

Let us recall that the kernel in the Poisson integral formula (5) for the Laplace equation given explicitly by (8), has in the polar coordinates the form

$$K(\rho; \theta - \varphi) = \frac{1}{2\pi}\;\frac{1-\rho^2}{1 - 2\rho\cos(\theta-\varphi) + \rho^2} \;. \tag{59}$$

The Poisson type integral formula for the solution to the Lamé equation (58), derived in [34], can be rewritten as follows:

$$\mathbf{u}(r\,e^{\mathrm{i}\,\theta}) = \int_0^{2\pi} K(\rho;\theta-\varphi)\,B(\rho;\theta,\varphi)\,\mathbf{g}(R\,e^{\mathrm{i}\,\varphi})\,d\varphi \tag{60}$$

where the matrix B has the form

$$B = \mathrm{I} + \frac{\lambda+\mu}{\lambda+3\mu}\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}\,, \tag{61}$$

with the entries given explicitly by

$$Q_{11} = \cos(2\varphi) - \rho\,\cos(\theta+\varphi) + \frac{\cos(2\varphi) - 2\,\rho\,\cos(\theta+\varphi) + \rho^2\,\cos(2\theta)}{1+\rho^2-2\rho\,\cos(\theta-\varphi)}\,, \tag{62}$$

$$Q_{12} = \sin(2\varphi) - \rho\,\sin(\theta+\varphi) + \frac{\sin(2\varphi) - 2\,\rho\,\sin(\theta+\varphi) + \rho^2\,\sin(2\theta)}{1+\rho^2-2\rho\,\cos(\theta-\varphi)}\,, \tag{63}$$

and $Q_{22} = -Q_{11}, \quad Q_{21} = Q_{12}$, I being an identity matrix.

This form of the Poisson type integral formula is simple and convenient to use in numerical simulations (e.g., see [35]). However it is seen that in contrast to all of the above considered cases, the matrix kernel has loosed the nice property of depending only on the difference of the angles θ and φ. This property is crucial for our analysis. This in turn is related to the probabilistic property of the solutions considered as random fields, namely, that these solutions are homogeneous with respect to the angular variable.

From the physical and probabilistic points of view, it is clear that the solution of the Lamé equation should be homogeneous with respect to the angular variable if the boundary functions are homogeneous random functions, in particular, when they are white noises. This means, we can try to find a transformation which leads to a Poisson integral formula with a matrix kernel depending only on the difference $\theta-\varphi$. It turns out that this can be done by a proper transformation of the vector $\mathbf{u} = (u_1, u_2)^T$ to polar coordinates.

So let us turn to the expansion of our displacement vector $\mathbf{u}$ in polar coordinates

$$\mathbf{u} = u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta\,,$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$ are unit vectors in directions r and θ, respectively. Then, the vectors $(u_1, u_2)^T$ and $(u_r, u_\theta)^T$ are related through a rotation,

$$\begin{pmatrix} u_1(r,\theta) \\ u_2(r,\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_r(r,\theta) \\ u_\theta(r,\theta) \end{pmatrix}\,, \tag{64}$$

and conversely,

$$\begin{pmatrix} u_r(r,\theta) \\ u_\theta(r,\theta) \end{pmatrix} = \mathcal{R}_\theta^T \begin{pmatrix} u_1(r,\theta) \\ u_2(r,\theta) \end{pmatrix}$$

where we use the notation for the rotation matrix

$$\mathcal{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\,, \tag{65}$$

and $\mathcal{R}_\theta^T$ means the transpose to $\mathcal{R}_\theta$.

Then, the Poisson integral formula (60) can be obviously rewritten as follows

$$\begin{pmatrix} u_r(r,\theta) \\ u_\theta(r,\theta) \end{pmatrix} = \mathcal{R}_\theta^T \int\limits_0^{2\pi} K(\rho;\theta-\varphi) \begin{pmatrix} 1+\beta Q_{11} & \beta Q_{12} \\ \beta\, Q_{12} & 1-\beta\, Q_{11} \end{pmatrix} \mathcal{R}_\varphi \begin{pmatrix} g_r(R\,e^{\mathrm{i}\,\varphi}) \\ g_\theta(R\,e^{\mathrm{i}\,\varphi}) \end{pmatrix} \, d\varphi \tag{66}$$

where $\beta = \frac{\lambda+\mu}{\lambda+3\mu}$.

After some transformations we come to the desired form of the Poisson integral formula

$$\begin{pmatrix} u_r(r,\theta) \\ u_\theta(r,\theta) \end{pmatrix} = \frac{1}{\lambda+3\mu} \int\limits_0^{2\pi} K(\rho;\theta-\varphi) \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} g_r(R\,e^{\mathrm{i}\,\varphi}) \\ g_\theta(R\,e^{\mathrm{i}\,\varphi}) \end{pmatrix} \, d\varphi \tag{67}$$

where the entries of the new matrix kernel $G = G(\theta-\varphi)$ are

$$\begin{aligned} G_{11} &= [2(\lambda+2\mu)\,\cos(\theta-\varphi) - (\lambda+\mu)\rho] + (\lambda+\mu)\,\frac{\cos(\theta-\varphi) - 2\rho + \rho^2\,\cos(\theta-\varphi)}{1+\rho^2-2\rho\,\cos(\theta-\varphi)}\,, \\ G_{12} &= 2\mu\,\sin(\theta-\varphi) - (\lambda+\mu)\,\frac{(1-\rho^2)\,\sin(\theta-\varphi)}{1+\rho^2-2\rho\,\cos(\theta-\varphi)}\,, \\ G_{21} &= -\,2(\lambda+2\mu)\,\sin(\theta-\varphi) - (\lambda+\mu)\,\frac{(1-\rho^2)\,\sin(\theta-\varphi)}{1+\rho^2-2\rho\,\cos(\theta-\varphi)}\,, \\ G_{22} &= [2\mu\,\cos(\theta-\varphi) + (\lambda+\mu)\rho] - (\lambda+\mu)\,\frac{\cos(\theta-\varphi) - 2\rho + \rho^2\,\cos(\theta-\varphi)}{1+\rho^2-2\rho\,\cos(\theta-\varphi)}\,. \end{aligned} \tag{68}$$

We are now in a position to formulate and solve the eigen-value problem for the integral operator with the matrix-kernel of the Poisson type integral (67)

$$L(\rho;\theta-\varphi) = \frac{1}{\lambda+3\mu} K(\rho;\theta-\varphi)\,G(\rho;\theta-\varphi)\,. \tag{69}$$

The eigen-value problem is written as the following system

$$\int\limits_0^{2\pi} L(\rho;\theta-\varphi) \begin{pmatrix} h_1(\varphi) \\ h_2(\varphi) \end{pmatrix} \, d\varphi = \lambda \begin{pmatrix} h_1(\theta) \\ h2(\theta) \end{pmatrix} \,. \tag{70}$$

Theorem 7. *The eigen-value problem (70) has the following system of eigen-values and eigen-functions ($k = 1, 2, \dots$) :*

$$\lambda_{2k-1} = \lambda_{2k} = \rho^{k-1}, \quad \begin{pmatrix} h_{1,2k-1} \\ h_{2,2k-1} \end{pmatrix} = \begin{pmatrix} \sin\, k\theta \\ \cos\, k\theta \end{pmatrix}, \quad \begin{pmatrix} h_{1,2k} \\ h_{2,2k} \end{pmatrix} = \begin{pmatrix} -\cos\, k\theta \\ \sin\, k\theta \end{pmatrix},$$

and for the case $k = 2$, for $\lambda_3 = \lambda_4 = \rho$, there is a third eigen-function $\begin{pmatrix} h'_{1,3} \\ h'_{2,3} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Proof. In the proof, we expand the matrix kernel in the Fourier series. In the expansion, we will use the following formulae simply obtained via differentiations:

$$\frac{1-\rho^2}{1+\rho^2-2\rho\cos\,\theta} = 1 + 2\sum_{k=1}^{\infty}\rho^k\ \cos{(k\theta)}\ ,$$

$$\frac{\rho\,\sin\theta}{1+\rho^2-2\rho\cos\,\theta} = \sum_{k=1}^{\infty}\rho^k\ \sin{(k\theta)}\ ,$$

$$\frac{\rho\,(\cos\theta-2\rho+\rho^2\ \cos\,\theta)}{(1+\rho^2-2\rho\cos\,\theta)^2} = \sum_{k=1}^{\infty}k\,\rho^k\ \cos{(k\theta)}\ ,$$

$$\frac{\rho\,\sin\theta\,(1-\rho^2)}{(1+\rho^2-2\rho\cos\,\theta)^2} = \sum_{k=1}^{\infty}k\,\rho^k\ \sin{(k\theta)}\ .$$

Substituting these representations in the functions (68), after some evaluations we obtain the following series expansions for the kernel L:

$$L_{11} = \frac{\rho}{2\pi} + \frac{1}{2\pi\,(\lambda+3\mu)}\ \sum_{k=1}^{\infty}\Big[2\mu\,\rho + \frac{2(\lambda+2\mu)}{\rho} + \frac{k\,(\lambda+\mu)(1-\rho^2)}{\rho}\Big]\,\rho^k\ \cos[k(\theta-\varphi)]\ ,$$

$$L_{12} = \frac{1}{2\pi\,(\lambda+3\mu)}\ \sum_{k=1}^{\infty}\Big[\ \frac{2\mu}{\rho} - 2\mu\rho - \frac{k\,(\lambda+\mu)(1-\rho^2)}{\rho}\Big]\,\rho^k\ \sin[k(\theta-\varphi)]\ ,$$

$$L_{21} = \frac{1}{2\pi\,(\lambda+3\mu)}\ \sum_{k=1}^{\infty}\Big[2(\lambda+2\mu)\,\rho - \frac{2(\lambda+2\mu)}{\rho} - \frac{k\,(\lambda+\mu)(1-\rho^2)}{\rho}\Big]\,\rho^k\ \sin[k(\theta-\varphi)]\ ,$$

$$L_{22} = \frac{\rho}{2\pi} + \frac{1}{2\pi\,(\lambda+3\mu)}\ \sum_{k=1}^{\infty}\Big[2(\lambda+2\mu)\,\rho + \frac{2\mu}{\rho} - \frac{k\,(\lambda+\mu)(1-\rho^2)}{\rho}\Big]\,\rho^k\ \cos[k(\theta-\varphi)]\ . \qquad (71)$$

Note that each of these series could be written in the form of a power series $a_1\rho + a_2\rho_2 + a_3\rho^3 + \ldots$, however as we will see below, the form (71) is very convenient when solving the eigen-value problem for the correlation operator.

Let us introduce the notations

$$\begin{aligned}
\lambda_{11}(\rho,k) &= \frac{1}{2(\lambda+3\mu)}\Big[2\mu\,\rho + \frac{2(\lambda+2\mu)}{\rho} + \frac{k\,(\lambda+\mu)(1-\rho^2)}{\rho}\Big]\ ,\\
\lambda_{12}(\rho,k) &= \frac{1}{2(\lambda+3\mu)}\Big[\ \frac{2\mu}{\rho} - 2\mu\rho - \frac{k\,(\lambda+\mu)(1-\rho^2)}{\rho}\Big]\ ,\\
\lambda_{21}(\rho,k) &= \frac{1}{2(\lambda+3\mu)}\Big[2(\lambda+2\mu)\,\rho - \frac{2(\lambda+2\mu)}{\rho} - \frac{k\,(\lambda+\mu)(1-\rho^2)}{\rho}\Big]\ ,\\
\lambda_{22}(\rho,k) &= \frac{1}{2(\lambda+3\mu)}\Big[2(\lambda+2\mu)\,\rho + \frac{2\mu}{\rho} - \frac{k\,(\lambda+\mu)(1-\rho^2)}{\rho}\Big]\ . \qquad (72)
\end{aligned}$$

From the expansions (71) we find that

$$\begin{aligned}
\int_0^{2\pi} L_{11}(\rho;\theta-\varphi) \begin{pmatrix}\sin k\varphi\\ \cos k\varphi\end{pmatrix} d\varphi &= \lambda_{11}(\rho,k)\ \rho^k \begin{pmatrix}\sin k\theta\\ \cos k\theta\end{pmatrix} ,\\
\int_0^{2\pi} L_{12}(\rho;\theta-\varphi) \begin{pmatrix}\cos k\varphi\\ \sin k\varphi\end{pmatrix} d\varphi &= \lambda_{12}(\rho,k)\ \rho^k \begin{pmatrix}\sin k\theta\\ -\cos k\theta\end{pmatrix} ,\\
\int_0^{2\pi} L_{21}(\rho;\theta-\varphi) \begin{pmatrix}\sin k\varphi\\ \cos k\varphi\end{pmatrix} d\varphi &= \lambda_{21}(\rho,k)\ \rho^k \begin{pmatrix}-\cos k\theta\\ \sin k\theta\end{pmatrix} ,\\
\int_0^{2\pi} L_{22}(\rho;\theta-\varphi) \begin{pmatrix}\cos k\varphi\\ \sin k\varphi\end{pmatrix} d\varphi &= \lambda_{22}(\rho,k)\ \rho^k \begin{pmatrix}\cos k\theta\\ \sin k\theta\end{pmatrix} . \qquad (73)
\end{aligned}$$

Now, by substituting these equalities in the eigen-value problem (70) and taking into account that $\lambda_{11}(\rho,k)+\lambda_{12}(\rho,k)=\rho^{-1},\ \ -\lambda_{21}(\rho,k)+\lambda_{22}(\rho,k)=\rho^{-1}$, we find the solution of the eigen-value problem for $k=1,2\ldots$. The existence of the eigen-function $(1,1)^T$ for $\lambda_3=\rho$ follows from the properties

$$\int_0^{2\pi} L_{11}(\rho;\theta-\varphi)\ \cdot 1\, d\varphi=\rho, \quad \int_0^{2\pi} L_{22}(\rho;\theta-\varphi)\ \cdot 1\, d\varphi=\rho,$$
$$\int_0^{2\pi} L_{12}(\rho;\theta-\varphi)\ \cdot 1\, d\varphi=0, \quad \int_0^{2\pi} L_{21}(\rho;\theta-\varphi)\ \cdot 1\, d\varphi=0\ .$$

The proof is complete. □

We turn now to the derivation of the correlation tensor of the solution,

$$B_u(\rho_1,\theta_1;\rho_2,\theta_2)=\mathbf{u}(r_1,\theta_1)\otimes\mathbf{u}(r_2,\theta_2)\equiv\Big\langle \begin{pmatrix}u_r(r_1,\theta_1)\\ u_\theta(r_1,\theta_1)\end{pmatrix}\ (u_r(r_2,\theta_2)\,,\ u_\theta(r_2,\theta_2))\Big\rangle \qquad (74)$$

assuming the boundary random vector-function $\mathbf{g}$ has a Gaussian distribution specified by the zero mean and covariance tensor

$$B_g(\varphi_1,\varphi_2)=\Big\langle \begin{pmatrix}g_r(\varphi_1)\\ g_\theta(\varphi_1)\end{pmatrix}\ \ (g_r(\varphi_2)\,,\ g_\theta(\varphi_2)\ \Big\rangle\ .$$

We use here and in what follows the following notation for $\mathbf{v}\otimes\mathbf{u}$, a tensor product of two vectors: $\mathbf{v}\otimes\mathbf{u}=\mathbf{v}\,\mathbf{u}^T$.

The Poisson integral formula (67) reads

$$\begin{pmatrix}u_r(r,\theta)\\ u_\theta(r,\theta)\end{pmatrix}=\int_0^{2\pi}\begin{pmatrix}L_{11}(\rho;\theta-\varphi) & L_{12}(\rho;\theta-\varphi)\\ L_{21}(\rho;\theta-\varphi) & L_{22}(\rho;\theta-\varphi)\end{pmatrix}\begin{pmatrix}g_r(R\,e^{\mathrm{i}\,\varphi})\\ g_\theta(R\,e^{\mathrm{i}\,\varphi})\end{pmatrix}d\varphi\ . \qquad (75)$$

Substituting this representation in (74) and changing the relevant product of integral expressions by double integrals, we arrive at the following representation

$$B_u(\rho_1,\theta_1;\rho_2,\theta_2) = \int_0^{2\pi}\int_0^{2\pi} L(\rho_1;\theta_1-\varphi')\, B_g(\varphi',\varphi'')\, L^T(\rho_2;\theta_2-\varphi'')\, d\varphi'\, d\varphi'' \ . \tag{76}$$

Let us again first consider the case when the boundary vector-function g is a white noise, namely, assume that

$$B_g(\varphi_1,\varphi_2) = \begin{pmatrix} \delta(\varphi_1-\varphi_2) & 0 \\ 0 & \delta(\varphi_1-\varphi_2) \end{pmatrix} \ . \tag{77}$$

Note that this property then holds also in rectangular coordinates (see (95) below). Then, from (76) we obtain

$$B_u(\rho_1,\theta_1;\rho_2,\theta_2) = \int_0^{2\pi} L(\rho_1;\theta_1-\varphi)\ L^T(\rho_2;\theta_2-\varphi)\, d\varphi \ . \tag{78}$$

Theorem 8. *The exact Karhunen-Loève representations for the covariance tensor and the random field $(u_r,u_\theta)^T$ which solves the Lamé equation under the boundary white noise excitations with the covariance tensor (77) are given by*

$$B_u(\rho_1,\theta_1;\rho_2,\theta_2) = \tag{79}$$
$$\begin{pmatrix} \frac{\rho_1\rho_2}{2\pi} + \frac{1}{\pi}\sum\limits_{k=1}^{\infty}\Lambda_{11}\,\rho_1^k\,\rho_2^k\,\cos\left[k(\theta_2-\theta_1)\right] & \frac{1}{\pi}\sum\limits_{k=1}^{\infty}\Lambda_{12}\,\rho_1^k\,\rho_2^k\,\sin\left[k(\theta_2-\theta_1)\right] \\ \frac{1}{\pi}\sum\limits_{k=1}^{\infty}\Lambda_{21}\,\rho_1^k\,\rho_2^k\,\sin\left[k(\theta_2-\theta_1)\right] & \frac{\rho_1\rho_2}{2\pi} + \frac{1}{\pi}\sum\limits_{k=1}^{\infty}\Lambda_{22}\,\rho_1^k\,\rho_2^k\,\cos\left[k(\theta_2-\theta_1)\right] \end{pmatrix}$$

$$\begin{aligned} \Lambda_{11} &= \lambda_{11}(\rho_1,k)\,\lambda_{11}(\rho_2,k) + \lambda_{12}(\rho_1,k)\,\lambda_{12}(\rho_2,k), \\ \Lambda_{12} &= \lambda_{11}(\rho_1,k)\,\lambda_{21}(\rho_2,k) - \lambda_{12}(\rho_1,k)\,\lambda_{22}(\rho_2,k), \\ \Lambda_{21} &= \lambda_{22}(\rho_1,k)\,\lambda_{12}(\rho_2,k) - \lambda_{21}(\rho_1,k)\,\lambda_{11}(\rho_2,k), \\ \Lambda_{22} &= \lambda_{22}(\rho_1,k)\,\lambda_{22}(\rho_2,k) + \lambda_{21}(\rho_1,k)\,\lambda_{21}(\rho_2,k) \ , \end{aligned} \tag{80}$$

and

$$\begin{aligned} u_r(r,\theta) = \frac{\xi_0\,\rho}{2\pi} &+ \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{11}\,\rho^k\left[\xi_k\,\cos k\theta + \eta_k\,\sin k\theta\right] \\ &+ \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{12}\,\rho^k\left[-\eta'_k\,\cos k\theta + \xi'_k\,\sin k\theta\right] \ , \end{aligned} \tag{81}$$

$$\begin{aligned} u_\theta(r,\theta) = \frac{\xi'_0\,\rho}{2\pi} &+ \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{21}\,\rho^k\left[-\eta_k\,\cos k\theta + \xi_k\,\sin k\theta\right] \\ &+ \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{22}\,\rho^k\left[\xi'_k\,\cos k\theta + \eta'_k\,\sin k\theta\right] \ , \end{aligned} \tag{82}$$

where $\{\xi_k,\eta_k\}$ and $\{\xi'_k,\eta'_k\}$, $k=0,1,2,\ldots$ are two independent families of standard independent gaussian random variables. Thus the random field is homogeneous with respect to the angular variable, and the respective partial spectra are: $S_{mm}(k) = \frac{1}{\pi}\Lambda_{mm}\rho_1^k\rho_2^k$, $S_{mm}(0) = \rho_1\rho_2/2\pi$, and for $n \neq m$ the spectrum is pure imaginary: $S_{mn}(k) = \mathrm{i}\frac{1}{\pi}\Lambda_{mn}\rho_1^k\rho_2^k$.

Proof. To get the expansion of the correlation tensor (79), we substitute the expansions (71) in (78) and use the eigen-function properties (73).

To construct the explicit simulation formula (81), (82) for our random field, we first split it into two independent Gaussian random fields:

$$\mathbf{u}(r,\theta) = \mathbf{V_1}(r,\theta) + \mathbf{V_2}(r,\theta) \ .$$

We will show now that for each of these random fields we can obtain a Karhunen-Loève expansion.

We introduce four single mode vector functions

$$\mathbf{h}_{1k}(\rho,\theta) = \begin{pmatrix} \lambda_{11}(\rho,k)\ \cos\, k\theta \\ \lambda_{21}(\rho,k)\ \sin\, k\theta \end{pmatrix} \ , \quad \tilde{\mathbf{h}}_{1k}(\rho,\theta) = \begin{pmatrix} \lambda_{11}(\rho,k)\ \sin\, k\theta \\ -\,\lambda_{21}(\rho,k)\ \cos\, k\theta \end{pmatrix} \ , \tag{83}$$

$$\mathbf{h}_{2k}(\rho,\theta) = \begin{pmatrix} -\,\lambda_{12}(\rho,k)\ \cos\, k\theta \\ \lambda_{22}(\rho,k)\ \sin\, k\theta \end{pmatrix} \ , \quad \tilde{\mathbf{h}}_{2k}(\rho,\theta) = \begin{pmatrix} \lambda_{12}(\rho,k)\ \sin\, k\theta \\ \lambda_{22}(\rho,k)\ \cos\, k\theta \end{pmatrix} \ . \tag{84}$$

Here the modes are indexed by $k = 1, 2 \ldots$, while the subindexes $_1$ and $_2$ stand for the first and second series of eigen-functions.

Note that these vectors are pairwise orthogonal:

$$\int_0^1 d\rho \int_0^{2\pi} d\theta\ \mathbf{h}_{1k} \cdot \tilde{\mathbf{h}}_{1k} = 0, \quad \int_0^1 d\rho \int_0^{2\pi} d\theta\ \mathbf{h}_{2k} \cdot \tilde{\mathbf{h}}_{2k} = 0,$$

as well as the two following vectors are orthogonal:

$$\mathbf{h}_0 = \begin{pmatrix} \frac{\rho}{\sqrt{2\pi}} \\ 0 \end{pmatrix} \ . \quad \tilde{\mathbf{h}}_0 = \begin{pmatrix} 0 \\ \frac{\rho}{\sqrt{2\pi}} \end{pmatrix} \ .$$

It is now a matter of technical evaluations to find that the correlation tensor can be represented in the form:

$$\begin{aligned} B_u(\rho_1,\theta_1;\rho_2,\theta_2) &= \mathbf{h}_0(\rho_1)\cdot \mathbf{h}_0^T(\rho_2) &(85)\\ &+\frac{1}{\pi^2}\sum_{k=1}^{\infty}\{\mathbf{h}_{1k}(\rho_1,\theta_1)\mathbf{h}_{1k}^T(\rho_2,\theta_2) + \tilde{\mathbf{h}}_{1k}(\rho_1,\theta_1)\,\tilde{\mathbf{h}}_{1k}^T(\rho_2,\theta_2)\}\,\rho_1^k\ \rho_2^k \\ +\tilde{\mathbf{h}}_0(\rho_1)\cdot\tilde{\mathbf{h}}_0^T(\rho_2) & &(86)\\ &+\frac{1}{\pi^2}\sum_{k=1}^{\infty}\{\mathbf{h}_{2k}(\rho_1,\theta_1)\mathbf{h}_{2k}^T(\rho_2,\theta_2) + \tilde{\mathbf{h}}_{2k}(\rho_1,\theta_1)\,\tilde{\mathbf{h}}_{2k}^T(\rho_2,\theta_2)\}\rho_1^k\ \rho_2^k \ . \end{aligned}$$

This follows from the easily verified representation

$$\mathbf{h}_{1k}(\rho_1,\theta_1)\,\mathbf{h}_{1k}^T(\rho_2,\theta_2) + \tilde{\mathbf{h}}_{1k}(\rho_1,\theta_1)\,\tilde{\mathbf{h}}_{1k}^T(\rho_2,\theta_2) =$$
$$\begin{pmatrix} \lambda_{11}(\rho_1,\theta_1)\,\lambda_{11}(\rho_2,\theta_2)\ \cos[k(\theta_2-\theta_1)] & \lambda_{11}(\rho_1,\theta_1)\,\lambda_{21}(\rho_2,\theta_2)\ \sin[k(\theta_2-\theta_1)] \\ -\,\lambda_{21}(\rho_1,\theta_1)\,\lambda_{11}(\rho_2,\theta_2)\ \sin[k(\theta_2-\theta_1)] & \lambda_{21}(\rho_1,\theta_1)\,\lambda_{22}(\rho_2,\theta_2)\ \cos[k(\theta_2-\theta_1)] \end{pmatrix}$$

and

$$\mathbf{h}_{2k}(\rho_1,\theta_1)\,\mathbf{h}_{2k}^T(\rho_2,\theta_2) + \tilde{\mathbf{h}}_{2k}(\rho_1,\theta_1)\,\tilde{\mathbf{h}}_{2k}^T(\rho_2,\theta_2) =$$
$$\begin{pmatrix} \lambda_{12}(\rho_1,\theta_1)\,\lambda_{12}(\rho_2,\theta_2)\ \cos[k(\theta_2-\theta_1)] & -\,\lambda_{12}(\rho_1,\theta_1)\,\lambda_{22}(\rho_2,\theta_2)\ \sin[k(\theta_2-\theta_1)] \\ \lambda_{22}(\rho_1,\theta_1)\,\lambda_{12}(\rho_2,\theta_2)\ \sin[k(\theta_2-\theta_1)] & \lambda_{22}(\rho_1,\theta_1)\,\lambda_{22}(\rho_2,\theta_2)\ \cos[k(\theta_2-\theta_1)] \end{pmatrix} \ .$$

So we can see from (85) that the first and the second pairs of lines present the covariances of the first and second vectors in our splitting, respectively:

$$B_u = \langle \mathbf{u}(r_1,\theta_1)\ \cdot \mathbf{u}^T(r_2,\theta_2)\rangle = \langle \mathbf{V_1}(r_1,\theta_1)\cdot \mathbf{V_1^T}(r_2,\theta_2)\rangle + \langle \mathbf{V_2}(r_1,\theta_1)\mathbf{V_2^T}(r_2,\theta_2)\rangle\ ,$$

thus,

$$\begin{aligned} B_{V_1} &= \mathbf{h}_0(\rho_1)\cdot \mathbf{h}_0^T(\rho_2) \\ &\quad + \sum_{k=1}^{\infty}\{\mathbf{h}_{1k}(\rho_1,\theta_1)\,\mathbf{h}_{1k}^T(\rho_2,\theta_2)\tilde{\mathbf{h}}_{1k}(\rho_1,\theta_1)\,\tilde{\mathbf{h}}_{1k}^T(\rho_2,\theta_2)\}\,\rho_1^k\ \rho_2^k\ , \\ B_{V_2} &= \tilde{\mathbf{h}}_0(\rho_1)\cdot \tilde{\mathbf{h}}_0^T(\rho_2) \\ &\quad + \sum_{k=1}^{\infty}\{\mathbf{h}_{2k}(\rho_1,\theta_1)\,\mathbf{h}_{2k}^T(\rho_2,\theta_2) + \tilde{\mathbf{h}}_{2k}(\rho_1,\theta_1)\,\tilde{\mathbf{h}}_{2k}^T(\rho_2,\theta_2)\}\rho_1^k\ \rho_2^k \end{aligned}$$

where $B_{V_1} = \langle \mathbf{V_1}(r_1,\theta_1)\cdot \mathbf{V_1^T}(r_2,\theta_2)\rangle \quad B_{V_2} = \langle \mathbf{V_2}(r_1,\theta_1)\mathbf{V_2^T}(r_2,\theta_2)\rangle$.

Note that each part, i.e., B_{V_1} and B_{V_2}, is represented as an orthogonal-mode expansion. Therefore, we can construct a KL-expansion for our random fields $\mathbf{V}_1$ and $\mathbf{V}_2$.

We have not yet normalized the eigen-functions. We can do it through dividing the angular modes by $\sqrt{\pi}$, and the radial modes by $\Delta_1(k) = \int_0^1(\lambda_{11}^2+\lambda_{21}^2)\,\rho^{2k}\,d\rho$, the first family of eigen-functions (83), and by $\Delta_2(k) = \int_0^1(\lambda_{12}^2+\lambda_{22}^2)\,\rho^{2k}\,d\rho$, the second family of eigen-functions (84). We then collect the orthonormal eigen-modes in one family:

$$\mathcal{H}_{2k-1}^{(1)} = \frac{1}{\sqrt{\Delta_1(k)\,\pi}}\ \mathbf{h}_{1k}(\rho,\theta),\quad \mathcal{H}_{2k}^{(1)} = \frac{1}{\sqrt{\Delta_1(k)\,\pi}}\ \tilde{\mathbf{h}}_{1k}(\rho,\theta),\quad k=1,2,\ldots$$

and

$$\mathcal{H}_{2k-1}^{(2)} = \frac{1}{\sqrt{\Delta_2(k)\,\pi}}\ \mathbf{h}_{2k}(\rho,\theta),\quad \mathcal{H}_{2k}^{(2)} = \frac{1}{\sqrt{\Delta_2(k)\,\pi}}\ \tilde{\mathbf{h}}_{2k}(\rho,\theta),\quad k=1,2,\ldots$$

Then, the orthonormal functions $\mathcal{H}_k^{(1)}$ and $\mathcal{H}_k^{(2)}$ are eigen-functions of the covariance tensors B_{V_1} and B_{V_2}, respectively, with the corresponding eigen-values $\Delta_1(k)$ and $\Delta_2(k)$:

$$\int_0^1\int_0^{2\pi} B_{V_m}\cdot \mathcal{H}_k^{(m)}(\rho_2,\theta_2)\,d\rho_2\,d\theta_2 = \Delta_m(k)\,\mathcal{H}_k^{(m)}(\rho_1,\theta_1)\ ,\quad m=1,2\ .$$

We can now construct a KL-expansion for the random field $\mathbf{V_1}(r,\theta)$ in the form

$$\mathbf{V_1}(r,\theta) = \sum_{k=1}^{\infty}\zeta_k\mathcal{H}_k^{(1)}(\rho,\theta)$$

where ζ_k are gaussian random variables such that

$$\langle\zeta_k\zeta_j\rangle = \Delta_1(k)\,\delta_{jk}\ ,$$

and the same for $\mathbf{V_2}(r,\theta)$.

Putting these expansions together we finally arrive at the desired representation

$$\begin{aligned} u_r(r,\theta) &= \frac{\xi_0\,\rho}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{11}\,\rho^k\left[\xi_k\ \cos\,k\theta + \eta_k\ \sin\,k\theta\right] \\ &\quad + \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{12}\,\rho^k\left[-\eta_k'\ \cos\,k\theta + \xi_k'\ \sin\,k\theta\right]\ , \end{aligned}$$

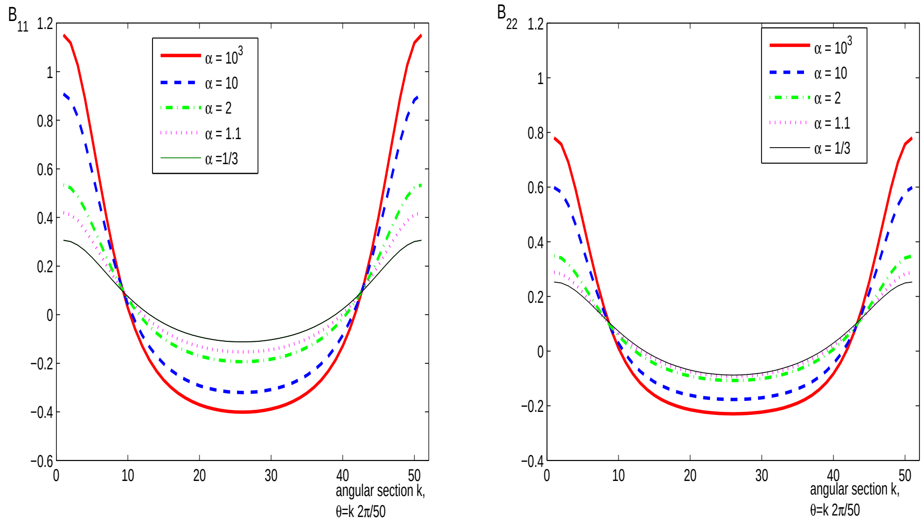

Figure 6: Correlations B_{11} (left panel) and B_{22} (right panel) for the Lamé equation, for different values of the elasticity parameter α; $\rho_1 = \rho_2 = 0.3$.

$$u_\theta(r,\theta) = \frac{\xi_0'\,\rho}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{21}\,\rho^k \left[-\eta_k \cos k\theta + \xi_k \sin k\theta \right] + \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{22}\,\rho^k \left[\xi_k' \cos k\theta + \eta_k' \sin k\theta \right]$$

where $\{\xi_k, \eta_k\}$ and $\{\xi_k', \eta_k'\}$, $k = 0, 1, 2, \ldots$ are two independent families of standard independent gaussian random variables.

Finally note that the spectra given in the theorem are obtained immediately from the representation (79). This completes the proof of Theorem 8. □

It is interesting to note that we could obtain these expressions by substituting formally a generalized representation of the boundary white noises on the circle

$$g_1(\varphi) = \frac{\xi_0}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \left[\xi_k \cos k\varphi + \eta_k \sin k\varphi\right]$$

$$g_2(\varphi) = \frac{\xi_0'}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \left[\xi_k' \cos k\varphi + \eta_k' \sin k\varphi\right]$$

into the Poisson formula (67) with the kernels given by the series expansions (71). But the justification would then need to work with generalized stochastic processes.

In the Figure 6 - 10 presented below we show the longitudinal correlation function B_{11}, the transverse correlation function B_{22}, and the cross-correlation functions B_{12} and B_{21}, in polar coordinates, as well as in rectangular coordinates. Figure 6 presents the angular behaviour of B_{11} for 5 different values of the elasticity constant α (left panel), and the same for B_{22} (right panel). The relevant cross-correlations are shown in Figure 7. The radial behaviour of B_{11} and B_{22} is shown in Figure 8. As is clearly seen from all these curves, the angular behaviour is periodic. When plotting these functions in rectangular coordinates, we get a complicated behaviour shown

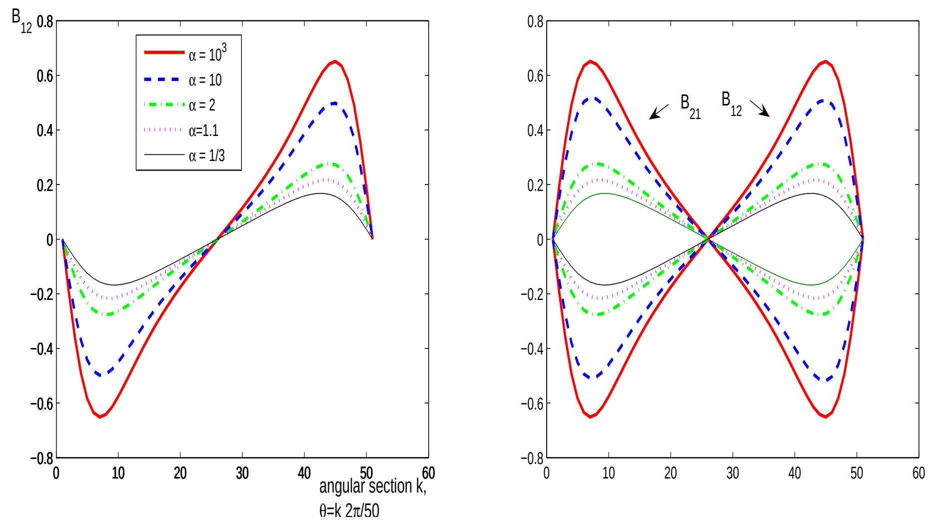

Figure 7: Correlations B_{12} (left panel), for different values of the elasticity parameter α. The same curves are shown in the right panel, superimposed by the relevant correlations B_{21}; $\rho_1 = \rho_2 = 0.3$.

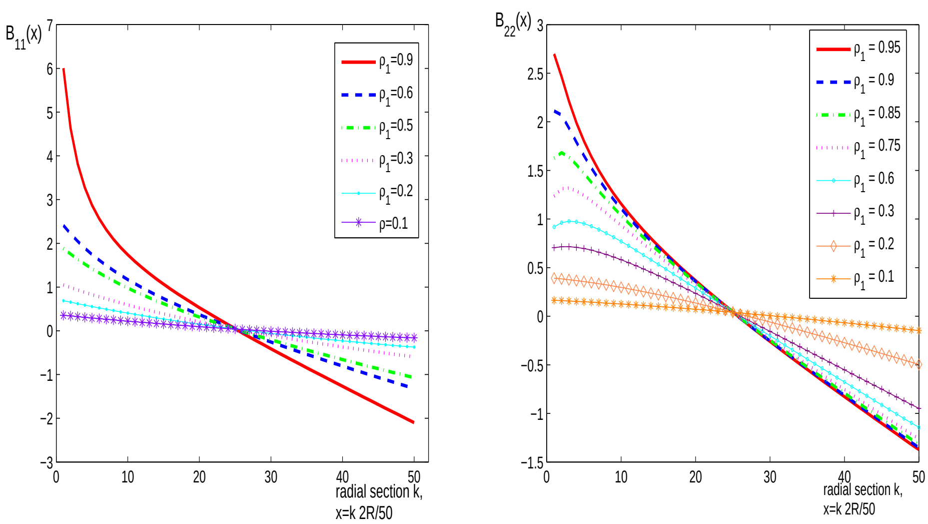

Figure 8: Radial correlations B_{11} (left panel), and B_{22} (right panel), for different values of the starting point ρ_1. Lamé equation, $\lambda = 2222, \mu = 2$.

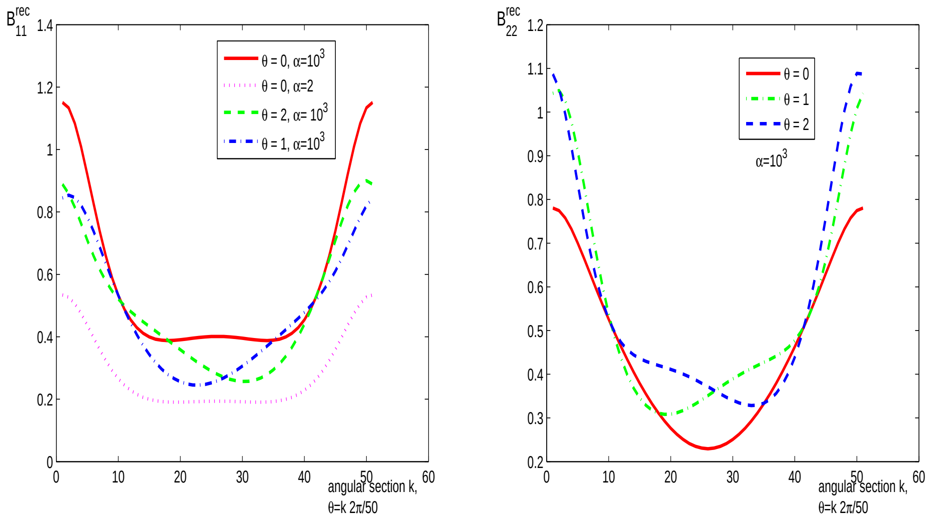

Figure 9: Angular correlations in rectangular coordinates, B_{11}^{rec} (left panel), and B_{22}^{rec} (right panel), for different values of the starting angle θ; $\rho_1 = \rho_2 = 0.3$.

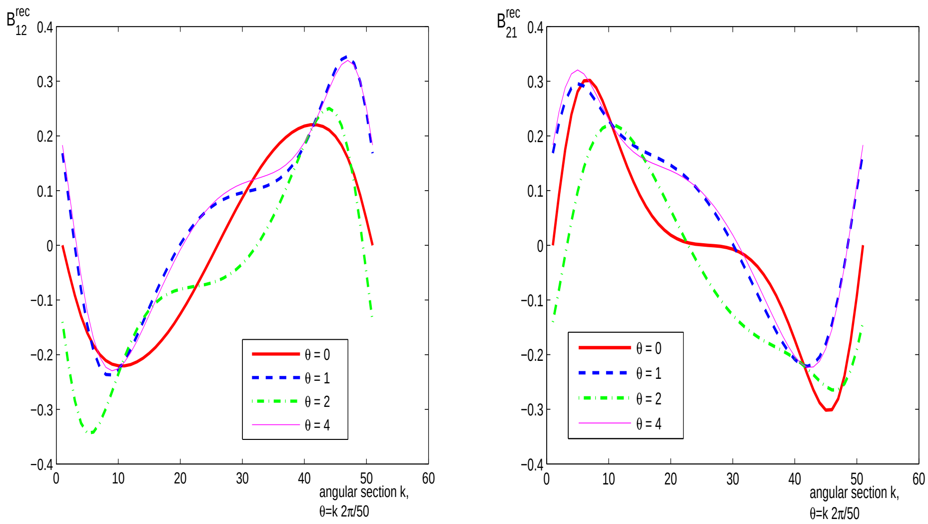

Figure 10: Angular correlations in rectangular coordinates, B_{12}^{rec} (left panel), and B_{21}^{rec} (right panel), for different values of the starting angle θ; $\rho_1 = \rho_2 = 0.3$.

in Figures 9 -10, where the correlations depend on the starting angle θ; we present the curves for different values of θ, see Figures 9 and 10.

5.2 General case of homogeneous excitations.

We have so far considered the case when the boundary functions g_1 and g_2 are two independent white noise processes. We will see now that the general case when g_1 and g_2 are some arbitrary dependent homogeneous processes, is basically derived from the white noise case.

Thus assume we are given two homogeneous zero mean processes g_1 and g_2 with the correlation tensor $B_g(\varphi_2 - \varphi_2)$, with the entries $B_{g,ij}, \quad i,j = 1,2$. As shown above, the correlation tensor of the solution B_u is related to B_g by the double integral representation (76). Changing the integration variable φ'' to a new integration variable ψ by $\varphi'' - \varphi' = \psi$ we obtain from (76) for $\mathbf{u} = (u_\rho, u_\theta)^T$:

$$B_u(\rho_1,\theta_1;\rho_2,\theta_2) = \int_0^{2\pi}\int_0^{2\pi} L(\rho_1;\theta_1-\varphi')\, B_g(\psi)\, L^T(\rho_2;\theta_2-\psi-\varphi')\, d\varphi'\, d\psi\ . \tag{87}$$

The idea is now to evaluate explicitly the inner integral with respect to φ' using the series expansions for the kernel $L(\rho,\theta)$ given above in (71). We now rewrite the relation (87) in a different form. We construct from the correlation tensor B_u a column-vector function $\hat{\mathbf{B}}_u$ as follows $\hat{\mathbf{B}}_u = (B_{u,11}, B_{u,12}, B_{u,21}, B_{u,22})^T$. Analogously, we use the notation $\hat{\mathbf{B}}_g$ for the column-vector $\hat{\mathbf{B}}_g = (B_{g,11}, B_{g,12}, B_{g,21}, B_{g,22})^T$.

Using this notation, we can rewrite (87) as follows

$$\hat{\mathbf{B}}_u(\rho_1,\theta_1;\rho_2,\theta_2) = \int_0^{2\pi}\int_0^{2\pi} L(\rho_1;\theta_1-\varphi') \otimes L(\rho_2;\theta_2-\psi-\varphi')\, \hat{\mathbf{B}}_g(\psi)\, d\varphi'\, d\psi\ . \tag{88}$$

Here we denote by $\otimes$ a tensor product of two matrices which is defined in our case as a 4×4 matrix, represented as a 2×2-block matrix each block being a 2×2 matrix of the form $L_{ij}(\rho_1;\theta_1 - \varphi')L(\rho_2;\theta_2-\psi-\varphi'),\ i,j=1,2$.

We will now evaluate explicitly all the 16 entries a_{ij} of the matrix

$$A = \int_0^{2\pi} L(\rho_1;\theta_1-\varphi') \otimes L(\rho_2;\theta_2-\psi-\varphi')\, d\varphi'\ . \tag{89}$$

Substituting the series representation of the matrix L given by (71) in (89) we obtain after a long but simple calculations

$$\begin{aligned}
a_{11} &= \frac{\rho_1\rho_2}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{11}(\rho_1,k)\lambda_{11}(\rho_2,k)\,\rho_1^k\rho_2^k\,\cos\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{12} &= \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{11}(\rho_1,k)\lambda_{12}(\rho_2,k)\,\rho_1^k\rho_2^k\,\sin\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{13} &= -\frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{12}(\rho_1,k)\lambda_{11}(\rho_2,k)\,\rho_1^k\rho_2^k\,\sin\left[k(\theta_2-\theta_1-\psi)\right] \\
a_{14} &= \frac{1}{\pi}\sum_{k=1}^{\infty} \lambda_{12}(\rho_1,k)\lambda_{12}(\rho_2,k)\,\rho_1^k\rho_2^k\,\cos\left[k(\theta_2-\theta_1-\psi)\right]
\end{aligned} \tag{90}$$

$$a_{21} = \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{11}(\rho_1,k)\lambda_{21}(\rho_2,k)\,\rho_1^k\rho_2^k\,\sin[k(\theta_2-\theta_1-\psi)]$$

$$a_{22} = \frac{\rho_1\rho_2}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{11}(\rho_1,k)\lambda_{22}(\rho_2,k)\,\rho_1^k\rho_2^k\,\cos[k(\theta_2-\theta_1-\psi)]$$

$$a_{23} = \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{12}(\rho_1,k)\lambda_{21}(\rho_2,k)\,\rho_1^k\rho_2^k\,\cos[k(\theta_2-\theta_1-\psi)]$$

$$a_{24} = -\frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{12}(\rho_1,k)\lambda_{22}(\rho_2,k)\,\rho_1^k\rho_2^k\,\sin[k(\theta_2-\theta_1-\psi)] \tag{91}$$

$$a_{31} = -\frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{21}(\rho_1,k)\lambda_{11}(\rho_2,k)\,\rho_1^k\rho_2^k\,\sin[k(\theta_2-\theta_1-\psi)]$$

$$a_{32} = \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{21}(\rho_1,k)\lambda_{12}(\rho_2,k)\,\rho_1^k\rho_2^k\,\cos[k(\theta_2-\theta_1-\psi)]$$

$$a_{33} = \frac{\rho_1\rho_2}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{22}(\rho_1,k)\lambda_{11}(\rho_2,k)\,\rho_1^k\rho_2^k\,\cos[k(\theta_2-\theta_1-\psi)]$$

$$a_{34} = \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{22}(\rho_1,k)\lambda_{12}(\rho_2,k)\,\rho_1^k\rho_2^k\,\sin[k(\theta_2-\theta_1-\psi)] \tag{92}$$

$$a_{41} = \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{21}(\rho_1,k)\lambda_{21}(\rho_2,k)\,\rho_1^k\rho_2^k\,\cos[k(\theta_2-\theta_1-\psi)]$$

$$a_{42} = -\frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{21}(\rho_1,k)\lambda_{22}(\rho_2,k)\,\rho_1^k\rho_2^k\,\sin[k(\theta_2-\theta_1-\psi)]$$

$$a_{43} = \frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{22}(\rho_1,k)\lambda_{21}(\rho_2,k)\,\rho_1^k\rho_2^k\,\sin[k(\theta_2-\theta_1-\psi)]$$

$$a_{44} = \frac{\rho_1\rho_2}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\lambda_{22}(\rho_1,k)\lambda_{22}(\rho_2,k)\,\rho_1^k\rho_2^k\,\cos[k(\theta_2-\theta_1-\psi)]\ . \tag{93}$$

Thus we see from these formulae that the entries of the matrix A depend on the difference $\theta = \theta_2 - \theta_1$, hence the correlation tensor B_u also depends on $\theta = \theta_2 - \theta_1$, and from (88), (89) we arrive at the desired convolution representation

$$\hat{\mathbf{B}}_u(\rho_1,\rho_2;\theta) = \int_0^{2\pi} A(\rho_1,\rho_2;\theta-\psi)\,\hat{\mathbf{B}}_g(\psi)\,d\psi\ . \tag{94}$$

Note that if the boundary correlation tensor B_g is given by its spectral expansion, we can express the correlation tensor of the solution through the spectra. For instance, assuming the spectral tensor is real-valued, so that

$$B_{g,ij}(\varphi''-\varphi) = \frac{f_{ij}(0)}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty} f_{ij}(k)\,\cos k(\varphi''-\varphi')\ ,\quad i,j=1,2\ ,$$

we can derive a general formula for the covariance tensor by substituting this expansion in (87). After routine evaluations we obtain the general formulae

$$
\begin{aligned}
B_{11} &= \frac{f_{11}(0)\,\rho_1\,\rho_2}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \rho_1^k\,\rho_2^k \left(\Lambda_{11}^c\,\cos[k(\theta_2-\theta_1)] + \Lambda_{11}^s\,\sin[k(\theta_2-\theta_1)]\right) ,\\
B_{12} &= \frac{1}{\pi}\sum_{k=1}^{\infty} \rho_1^k\,\rho_2^k \left(\Lambda_{12}^c\,\cos[k(\theta_2-\theta_1)] + \Lambda_{12}^s\,\sin[k(\theta_2-\theta_1)]\right) ,\\
B_{21} &= \frac{1}{\pi}\sum_{k=1}^{\infty} \rho_1^k\,\rho_2^k \left(\Lambda_{21}^c\,\cos[k(\theta_2-\theta_1)] + \Lambda_{21}^s\,\sin[k(\theta_2-\theta_1)]\right) ,\\
B_{22} &= \frac{f_{22}(0)\,\rho_1\,\rho_2}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \rho_1^k\,\rho_2^k \left(\Lambda_{22}^c\,\cos[k(\theta_2-\theta_1)] + \Lambda_{22}^s\,\sin[k(\theta_2-\theta_1)]\right) ,
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda_{11}^c &= f_{11}\lambda_{11}^1\lambda_{11}^2 + f_{22}\lambda_{12}^1\lambda_{12}^2\ , \quad & \Lambda_{11}^s &= f_{21}\lambda_{11}^1\lambda_{12}^2 - f_{12}\lambda_{12}^1\lambda_{11}^2\ ,\\
\Lambda_{12}^c &= f_{21}\lambda_{12}^1\lambda_{21}^2 + f_{12}\lambda_{11}^1\lambda_{22}^2\ , \quad & \Lambda_{12}^s &= f_{11}\lambda_{11}^1\lambda_{21}^2 - f_{22}\lambda_{12}^1\lambda_{22}^2\ ,\\
\Lambda_{21}^c &= f_{21}\lambda_{22}^1\lambda_{11}^2 + f_{12}\lambda_{21}^1\lambda_{12}^2\ , \quad & \Lambda_{21}^s &= f_{22}\lambda_{22}^1\lambda_{12}^2 - f_{11}\lambda_{21}^1\lambda_{11}^2\ ,\\
\Lambda_{22}^c &= f_{22}\lambda_{22}^1\lambda_{22}^2 + f_{11}\lambda_{21}^1\lambda_{21}^2\ , \quad & \Lambda_{22}^s &= f_{21}\lambda_{22}^1\lambda_{21}^2 - f_{12}\lambda_{21}^1\lambda_{22}^2\ .
\end{aligned}
$$

Here we use the notations $\lambda_{ij}^m = \lambda_{ij}(\rho_m, k),\ \ m = 1, 2.$

Remark 3.

Note that using the relation between the vectors in polar and rectangular coordinates,

$$
\begin{pmatrix} u_1(r,\theta) \\ u_2(r,\theta) \end{pmatrix} = \mathcal{R}_\theta \begin{pmatrix} u_\rho(r,\theta) \\ u_\theta(r,\theta \end{pmatrix}
$$

we can easily relate the desired statistical characteristics in these two coordinate systems. For example, the covariance tensors are related as follows

$$
B_{(u_1,u_2)}(\rho_1,\rho_2;\theta_1,\theta_2) = \mathcal{R}_{\theta_1}\, B_{(u_r,u_\theta)}(\rho_1,\rho_2;\theta_1,\theta_2)\,\mathcal{R}_{\theta_2}^T \tag{95}
$$

The KL-expansion in the rectangular coordinates is also obtained directly from the KL-expansion of the random field in the polar coordinates on the basis that the eigen-functions are related by $h_{rectangular} = \mathcal{R}_\theta\, h_{polar}$ and $\tilde{h}_{rectangular} = \mathcal{R}_\theta\, \tilde{h}_{polar}$.

Let us write down here the relation (95) in details. We denote the entries of the covariance matrix $B_{(u_1,u_2)}$ by B_{ij}^{rec}, and the entries of the covariance matrix $B_{(u_r,u_\theta)}$ by B_{ij}^{pol}. From (95) we obtain

$$
\begin{aligned}
B_{11}^{rec} &= \cos\theta_1\cos\theta_2\, B_{11}^{pol} - \cos\theta_1\sin\theta_2\, B_{12}^{pol} - \sin\theta_1\cos\theta_2\, B_{21}^{pol} + \sin\theta_1\sin\theta_2\, B_{22}^{pol}\ ,\\
B_{12}^{rec} &= \cos\theta_1\sin\theta_2\, B_{11}^{pol} + \cos\theta_1\cos\theta_2\, B_{12}^{pol} - \sin\theta_1\sin\theta_2\, B_{21}^{pol} - \sin\theta_1\cos\theta_2\, B_{22}^{pol}\ ,\\
B_{21}^{rec} &= \sin\theta_1\cos\theta_2\, B_{11}^{pol} - \sin\theta_1\sin\theta_2\, B_{12}^{pol} + \cos\theta_1\cos\theta_2\, B_{21}^{pol} - \cos\theta_1\sin\theta_2\, B_{22}^{pol}\ ,\\
B_{22}^{rec} &= \sin\theta_1\sin\theta_2\, B_{11}^{pol} + \sin\theta_1\cos\theta_2\, B_{12}^{pol} + \sin\theta_2\cos\theta_1\, B_{21}^{pol} + \cos\theta_1\cos\theta_2\, B_{22}^{pol}\ .
\end{aligned}
$$

This representation clearly shows that the property that the covariance functions B_{ij}^{pol} all depend only on the angle difference $\theta_2 - \theta_1$ does not generally hold for the covariance functions B_{ij}^{rec}. It is however seen that B_{ij}^{rec} will depend only on $\theta_2 - \theta_1$ if (u_r, u_θ) is homogeneous, and $B_{11}^{pol} = B_{22}^{pol}$.

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