# A stochastic volatility Libor model and its robust calibration

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**Abstract**

In this paper we propose a Libor model with a high-dimensional specially structured system of driving CIR volatility processes. A stable calibration procedure which takes into account a given local correlation structure is presented. The calibration algorithm is FFT based, so fast and easy to implement.

## 1 Introduction

Since Brace, Gatarek, Musiela (1997), Jamshidian (1997), and Miltersen, Sandmann and Sondermann (1997), almost independently, initiated the development of the Libor market interest rate model, research has grown immensely towards improved models that fit market quotes of standard interest rate products such as cap and swaption prices for different strikes and maturities. As a matter of fact, while caps can be priced using a Black type formula and swaptions via closed form approximations with high accuracy, an important drawback of the market model is the impossibility of matching cap and swaption volatility smiles and skews observed in the markets. As a remedy, various alternatives to the standard Libor market model have been proposed. They can be roughly categorized into three streams: local volatility models, stochastic volatility models, and jump-diffusion models. Especially jump-diffusion and stochastic

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volatility models are popular due to their economically meaningful behavior, and the greater flexibility they offer compared to local volatility models for instance. For local volatility Libor models we refer to Brigo and Mercurio (2006). Jump-diffusion models for assets go back to Merton (1976) and Eberlein (1998). Jamshidian (2001) developed a general semimartingale framework for the Libor process which covers the possibility of incorporating jumps as well as stochastic volatility. Specific jump-diffusion Libor models are proposed, among others, by Glasserman and Kou (2003) and Belomestny and Schoenmakers (2006). Levy Libor models are studied by Eberlein and Özkan (2005). Incorporation of stochastic volatility has been proposed by Andersen and Brotherton-Ratcliffe (2001), Piterbarg (2004), Wu and Zhang (2006), Zhu (2007).

In the present article we focus on a flexible particularly structured Heston type stochastic volatility Libor model that, due to its very construction, can be calibrated to the cap/strike matrix in a robust way. In this model we incorporate a core idea from Belomestny and Schoenmakers (2006), who propose a jump-diffusion Libor model as a perturbation of a given input Libor market model. As a main issue, Belomestny and Schoenmakers (2006) furnish the jump-diffusion extension in such a way that *the (local) covariance structure of the extended model coincides with the (local) covariance structure of the market model.* The approach of perturbing a given market model while preserving its covariance structure, has turned out to be fruitfull and is carried over into the design of the stochastic volatility Libor model presented in this paper. In fact, this idea is supported by the following arguments (see also Belomestny and Schoenmakers (2006)).

1. Cap prices do not depend on the (local) correlation structure of forward Libors in a Libor market model but, typically, do depend only weakly on this in a more general model. Since this correlation structure contains important information about, for example, prices of ATM swaptions, we do not want to destroy this (input) correlation structure while calibrating the extended model to the cap(let)-strike matrix.

2. The lack of smile behavior of a Libor market model is considered a consequence of Gaussianity of the driving random sources (Wiener processes). Therefore we want to perturb this Gaussian randomness to a non-Gaussian one by incorporating a CIR volatility process, while maintaining the (local) *correlation structure* of the Libor market model we started with.

3. Preserving the correlation structure allows for robust calibration, since it significantly reduces the number of parameters to be calibrated while holding a realistic correlation structure.

Specifically, the perturbation part of the presented model will involve CIR volatility processes, and so the construction will finally resemble a Heston type Libor model (Heston (1993)). The CIR model, as developed by its founders Cox, Ingersoll, Ross (1985), was originally derived in a framework based on equilibrium assumptions.

The idea of utilizing a Heston type process has already been formulated in Wu and Zhang (2006), and Zhu (2007). However, the present article differs from these works in the following respects.

1. As opposed to a one-dimensional stochastic volatility process as in Wu & Zhang, or a (possibly) vector valued one which inhibits only one source of randomness as in Zhu (2007), we will study multi-dimensional CIR vector volatility processes with each component being driven by its own Brownian motion. This leads to a more realistic local correlation structure and renders the model more flexible for calibration.

2. We suggest a multi-dimensional partial-Gaussian and partial-Heston type model, where each forward Libor is driven by a linear combination of CIR processes.

3. While in both papers the issue of robust calibration has not been addressed, we give full consideration to this problem using novel ideas mentioned above.

Furthermore, approximative analytic pricing formulas for caplets and swaptions are derived for this new Libor model which allow for fast calibration to these products. Ultimately, complex structured Over The Counter products may be priced by Monte Carlo using guidelines for simulating Heston type models as given in Kahl and Jäckel (2006).

# 2 Dynamics of the Libor Model

Consider a fixed sequence of tenor dates $0 =: T_0 < T_1 < \ldots < T_n$, called a tenor structure, together with a sequence of so called day-count fractions $\delta_i := T_{i+1} - T_i,\ i = 1, \ldots, n-1$. With respect to this tenor structure we consider zero bond processes $B_i,\ i = 1, \ldots, n$, where each $B_i$ lives on the interval $[0, T_i]$ and ends up with its face value $B_i(T_i) = 1$. With respect to this bond system we deduce a system of forward rates, called Libor rates, which are defined by

$$L_i(t) := \frac{1}{\delta_i}\left(\frac{B_i(t)}{B_{i+1}(t)} - 1\right), \quad 0 \le t \le T_i,\ 1 \le i \le n-1.$$

Note that $L_i$ is the annualized effective forward rate to be contracted at the date $t$, for a loan over a forward period $[T_i, T_{i+1}]$. Based on this rate one has to pay at $T_{i+1}$ an interest amount of $\$\delta_i L_i(T_i)$ on a \$1 notional.

For a pre-specified volatility process $\gamma_i \in \mathbb{R}^m$, adapted to the filtration generated by some standard Brownian motion $W \in \mathbb{R}^m$, the dynamics of the corresponding Libor model have the form,

$$\frac{dL_i}{L_i} = (...)dt + \gamma_i^\top dW \tag{1}$$

$i = 1, ..., n-1$. The drift term, adumbrated by the dots, is known under different numeraire measures, such as the risk-neutral, spot, terminal and all measures induced by individual bonds taken as numeraire. If the processes $t \to \gamma_i(t)$ in (1) are deterministic, one speaks of a Libor *market* model.

In this work we study extensions of a Libor market model, which is given via a deterministic volatility structure $\gamma$, with respect to an extended Brownian filtration. In particular, we consider extensions with the following structure,

$$\frac{dL_i}{L_i} = (...)dt + \sqrt{1 - r_i^2}\gamma_i^\top dW + r_i \beta_i^\top dU, \qquad 1 \le i < n, \tag{2}$$

$$\begin{aligned} dU_k &= \sqrt{v_k} d\widetilde{W}_k \qquad 1 \le k \le d, \\ dv_k &= \kappa_k(\theta_k - v_k)dt + \sigma_k \sqrt{v_k}\left(\rho_k d\widetilde{W}_k + \sqrt{1-\rho_k^2} d\overline{W}_k\right), \end{aligned} \tag{3}$$

where $\widetilde{W}$ and $\overline{W}$ are mutually independent $d$-dimensional standard Brownian motions, both independent of $W$. In (2), $\beta_i \in \mathbb{R}^d$ are chosen to be deterministic vector functions. They will be specified later. The $r_i$ are constants that may be considered "allotment" or "proportion" factors, quantifying how much of the original input market model should be in play. For $r_i = 0$ for all $i$, it is easily seen from (2) that the classical market model is retrieved. As such, for small values of the $r_i$, the extended model may be regarded as a perturbation of the former. Finally, from a modeling point of view system (2) is obviously overparameterized in the following sense. By setting $\beta_{ik} =: \alpha_k \widetilde{\beta}_{ik}$ and $v_k =: \alpha_k^{-2}\widetilde{v}_k$, $\theta_k =: \alpha_k^{-2}\widetilde{\theta}_k$, $\sigma_k =: \alpha_k^{-1}\widetilde{\sigma}_k$, we retrieve exactly the same model. From now on we therefore normalize by setting $\theta_k \equiv 1$ without loss of generality.

It is helpful to think of the Libor model as a vector-valued stochastic process of dimension $n-1$ driven by $m + 2d$ standard Brownian motions with dynamics of the form

$$\frac{dL_i}{L_i} = (...)dt + \Gamma_i^\top d\mathcal{W}, \quad i = 1, ..., n-1,$$

where

$$\Gamma_i = \begin{pmatrix} \sqrt{1-r_i^2}\gamma_{i1} \\ \sqrt{1-r_i^2}\gamma_{i2} \\ \cdot \\ \cdot \\ \sqrt{1-r_i^2}\gamma_{im} \\ r_i\beta_{i1}\sqrt{v_1} \\ \cdot \\ \cdot \\ r_i\beta_{id}\sqrt{v_d} \end{pmatrix} \qquad d\mathcal{W} = \begin{pmatrix} dW_1 \\ dW_2 \\ \cdot \\ \cdot \\ dW_m \\ d\widetilde{W}_1 \\ \cdot \\ \cdot \\ d\widetilde{W}_d \end{pmatrix}. \tag{4}$$

In (4) the square-root processes $v_k$ are given by (3) (with $\theta_k \equiv 1$).

In our approach we will work throughout under the terminal measure $P_n$. Following Jamshidian (1997, 2001), the Libor dynamics in this measure are given by

$$\frac{dL_i}{L_i} = -\sum_{j=i+1}^{n-1} \frac{\delta L_j}{1+\delta L_j} \left( \sum_{k=1}^{m+d} \Gamma_{jk}\Gamma_{ik} \right) dt + \Gamma_i^\top d\mathcal{W}^{(n)}. \tag{5}$$

Often it turns out technically more convenient to work with the log-Libor dynamics. A straightforward application of Itô's lemma to (5) yields,

$$d\ln L_i = -\frac{1}{2}|\Gamma_i|^2 dt - \sum_{j=i+1}^{n-1} \frac{\delta L_j}{1+\delta L_j} \left( \sum_{k=1}^{m+d} \Gamma_{jk}\Gamma_{ik} \right) dt + \Gamma_i^\top d\mathcal{W}^{(n)}, \quad 1 \leq i < n. \tag{6}$$

# 3 Reduction of parameters by covariance assumption

Within the particular framework constructed above, one could interpret the second diffusion part in (2), namely $r_i\beta_i^\top dU$, as an extension or perturbation of a given Libor market model.
Let us integrate the diffusion part of (6) from zero to $t$ and define the resulting zero-mean random variable by

$$\xi_i(t) := \int_0^t \Gamma_i^\top d\mathcal{W}^{(n)}. \tag{7}$$

Recall that $\gamma_i \in \mathbb{R}^m$ is the (given) deterministic volatility structure of the input market model obtained by some calibration procedure to ATM caps and ATM swaptions. We assume further that the matrix $(\gamma_{i,j}(t))_{1\leq i<n, 1\leq j\leq m}$ has full rank $m$ for all $t$. The deterministic vector functions $\beta_i \in \mathbb{R}^d$ will allow additional degrees of freedom for the upcoming fitting to the volatility curve. We will now see that under the covariance assumption we will have to restrict ourselves to specified values for the $\beta_i$.

For the covariance function of $\xi_i(t)$ in the terminal measure we obtain

$$\begin{aligned} E_n(\xi_i(t)\xi_j(t)) &= \sqrt{1-r_i^2}\sqrt{1-r_j^2}\int_0^t \gamma_i^\top \gamma_j ds + r_i r_j E_n \int_0^t \beta_i^\top dU \cdot \int_0^t \beta_j^\top dU \\ &= \sqrt{1-r_i^2}\sqrt{1-r_j^2}\int_0^t \gamma_i^\top \gamma_j ds + r_i r_j \sum_{k=1}^d E_n \int_0^t \beta_{ik}\beta_{jk}\, d\langle U_k\rangle \\ &= \sqrt{1-r_i^2}\sqrt{1-r_j^2}\int_0^t \gamma_i^\top \gamma_j ds + r_i r_j \sum_{k=1}^d \int_0^t \beta_{ik}\beta_{jk}\, E_n v_k\, ds \\ &=: \sqrt{1-r_i^2}\sqrt{1-r_j^2}\int_0^t \gamma_i^\top \gamma_j ds + r_i r_j \int_0^t \beta_i^\top \Lambda(t)\beta_j\, ds \end{aligned} \tag{8}$$

where $\Lambda(t)$ denotes a diagonal matrix in $\mathbb{R}^{d\times d}$ whose elements are the expected values $\lambda_k = E_n v_k \in \mathbb{R}$.

The square-root diffusions in (2) have a limiting stationary distribution. The transition law of the general CIR process

$$v(t) = v(u) + \int_u^t \left(\kappa(\theta - v(s))ds + \sigma\sqrt{v(s)}dW(s)\right),$$

is known. In particular, we have the representation

$$v(t) = \frac{\sigma^2\left(1 - e^{-\kappa(t-u)}\right)}{4\kappa}\chi^2_{\alpha,c}, \qquad t > u,$$

where $\chi^2_{\alpha,c}$ is a noncentral chi-square random variable with $\alpha$ degrees of freedom and noncentrality $c$, where

$$\alpha := \frac{4\theta\kappa}{\sigma^2}, \quad c := \frac{4\kappa e^{-\kappa(t-u)}}{\sigma^2\left(1 - e^{-\kappa(t-u)}\right)}v(u).$$

For the expectation we have

$$E[v(t) \mid \mathcal{F}_u] = (v(u) - \theta)e^{-\kappa(t-u)} + \theta, \quad t \geq u, \tag{9}$$

e.g. see Glasserman (2003) for details. It is natural to take the limit expectation as the starting value of the process. Thus, we set

$$v_k(0) = \theta_k = 1, \quad \text{for} \quad k = 1, \dots, d,$$

to obtain $Ev_k(t) \equiv 1$, hence $\Lambda = I$ is constant.

Let us now introduce the covariance restriction mentioned in the introduction, which will be in fact a modified version of the covariance restriction in Belomestny and Schoenmakers (2006). In the latter article one requires (in a jump-diffusion context)

$$E_n(\xi_i(t)\xi_j(t)) = \int_0^t \gamma_i^\top \gamma_j ds. \tag{10}$$

In view of (8) and as a next simplification, we set $r_i \equiv r$, to yield from (10),

$$\int_0^t \gamma_i^\top \gamma_j ds = \int_0^t \beta_i^\top \beta_j ds, \tag{11}$$

which is obviously satisfied by taking $\beta \equiv \gamma$, and then, in particular, we have $d = m$. However, in order to obtain closed-form expressions for characteristic functions later on, we would like $\beta(t)$ to be piecewise constant in time. For a better tractability we even assume $\beta(t)$ to be time independent. In either case this means that (11) has to be relaxed. As a first relaxation of (11) we require only

$$\int_0^{T_k} \gamma_i^\top \gamma_j dt = \int_0^{T_k} \beta_i^\top \beta_j dt, \quad k \leq \min(i,j), \tag{12}$$

which can be satisfied by taking $\beta(t)$ suitably piecewise constant. Unfortunately, for time independent $\beta$, (12) can still not be matched in general. As a pragmatic solution for this case, we therefore relax (12) further to

$$\beta_i^\top \beta_j = \frac{1}{\min(i,j)} \sum_{k=1}^{\min(i,j)} \frac{1}{T_k} \int_0^{T_k} \gamma_i^\top \gamma_j dt, \tag{13}$$

or as an alternative,

$$\beta_i^\top \beta_j = \frac{1}{T_{\min(i,j)}} \int_0^{T_{\min(i,j)}} \gamma_i^\top \gamma_j dt. \tag{14}$$

It can be shown that in both cases the matrix $(\beta_i^\top \beta_j)$ is positive definite and so defines a covariance structure.

Of course there are further variations possible. Note that even when $m < n-1$, exact fitting of (13) or (14), respectively, may require $d = n-1$. Depending on the readers preferences however, one may choose any $d$, $d < n-1$, and then fit (13) or (14) after dimension reduction via principal component analysis of the respective symmetric right-hand-sides.

# 4 Dynamics under various measures

## 4.1 Dynamics under forward measures

So far the Libor dynamics have been considered under the terminal measure. In order to price caplets later on, however, we will need to represent the above process under various forward measures. In what follows we denote the time independent solution for $\beta$ of either (13), (14), or any other sensible choice of the reader for the covariance constraint, by $\overline{\gamma} \in \mathbb{R}^{(n-1)\times d}$. Thus, spelling out (5) with $r_i \equiv r$ yields

$$\begin{aligned} \frac{dL_i}{L_i} = \quad & -\sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j} \left[ (1-r^2)\gamma_i^\top \gamma_j + r^2 \sum_{k=1}^{d} \overline{\gamma}_{ik} \overline{\gamma}_{jk} v_k \right] dt \\ & + \sqrt{1-r^2}\gamma_i^\top dW^{(n)} + r \sum_{k=1}^{d} \sqrt{v_k} \overline{\gamma}_{ik} d\widetilde{W}_k^{(n)} \end{aligned} \tag{15}$$

with corresponding volatility processes

$$dv_k = \kappa_k (1-v_k) dt + \sigma_k \sqrt{v_k} \left( \rho_k d\widetilde{W}_k^{(n)} + \sqrt{1-\rho_k^2} d\overline{W}_k^{(n)} \right), \tag{16}$$

under the measure $P_n$. By rearranging terms we may write,

$$\begin{aligned}\frac{dL_i}{L_i} &= \sqrt{1-r^2}\gamma_i^\top \left( dW^{(n)} - \sqrt{1-r^2} \sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j} \gamma_j dt \right) \\ &\quad + r \sum_{k=1}^{d} \overline{\gamma}_{ik} \sqrt{v_k} \left( d\widetilde{W}_k^{(n)} - r \sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j} \overline{\gamma}_{jk} \sqrt{v_k} dt \right) \\ &=: \sqrt{1-r^2}\gamma_i^\top dW^{(i+1)} + r \sum_{k=1}^{d} \overline{\gamma}_{ik} \sqrt{v_k} d\widetilde{W}_k^{(i+1)}. \end{aligned} \tag{17}$$

Since $L_i$ is a martingale under $P_{i+1}$, we have that both $W^{(i+1)}$ and $\widetilde{W}^{(i+1)}$ in (17) are standard Brownian motions under $P_{i+1}$. In terms of these new Brownian motions the volatility dynamics becomes

$$\begin{aligned} dv_k &= \kappa_k (1 - v_k) dt + r\sigma_k \rho_k \sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j} \overline{\gamma}_{jk} v_k dt \\ &\quad + \rho_k \sigma_k \sqrt{v_k} d\widetilde{W}_k^{(i+1)} + \sqrt{1-\rho_k^2} \sigma_k \sqrt{v_k} d\overline{W}_k^{(n,i+1)}. \end{aligned} \tag{18}$$

As shown in the Appendix, the process $\overline{W}^{(n,i+1)}$ in (18) is standard Brownian motion under both measures $P_{i+1}$ and $P_n$.

By freezing the Libors at their initial values in (18), we obtain an approximative CIR dynamics

$$dv_k \approx \kappa_k^{(i+1)} \left( \theta_k^{(i+1)} - v_k \right) dt + \sigma_k \sqrt{v_k} \left( \rho_k d\widetilde{W}_k^{(i+1)} + \sqrt{1-\rho_k^2} d\overline{W}_k^{(i+1)} \right) \tag{19}$$

with reversion speed parameter

$$\kappa_k^{(i+1)} := \kappa_k - r\sigma_k \rho_k \sum_{j=i+1}^{n-1} \frac{\delta_j L_j(0)}{1+\delta_j L_j(0)} \overline{\gamma}_{jk}, \tag{20}$$

and mean reversion level

$$\theta_k^{(i+1)} := \frac{\kappa_k}{\kappa_k^{(i+1)}}. \tag{21}$$

The approximative dynamics (19) for the volatility process will be used for calibration in Section 5.

## 4.2 Dynamics under swap measures

An interest rate swap is a contract to exchange a series of floating interest payments in return for a series of fixed rate payments. Consider a series of payment dates between $T_{p+1}$ and $T_q$, $q > p$. The fixed leg of the swap pays $\delta_j K$ at each time $T_{j+1}$, $j = p, \ldots, q-1$ where $\delta_j = T_{j+1} - T_j$. In return, the floating leg pays $\delta_j L_j(T_j)$ at time $T_{j+1}$, where $L_j(T_j)$ is the rate fixed at time $T_j$ for payment at $T_{j+1}$. Thus, the time $t$ value of the interest rate swap is

$$\sum_{j=p}^{q-1} \delta_j B_{j+1}(t)(L_j(t) - K).$$

The swap rate $S_{p,q}(t)$ is the value of the fixed rate $K$, such that the present value of the contract is zero, hence after some rearranging

$$S_{p,q}(t) = \frac{\sum_{j=p}^{q-1} \delta_j B_{j+1}(t) L_j(t)}{\sum_{j=p}^{q-1} \delta_j B_{j+1}(t)} = \frac{B_p(t) - B_q(t)}{\sum_{j=p}^{q-1} \delta_j B_{j+1}(t)}. \tag{22}$$

So $S_{p,q}$ is a martingale under the probability measure $P_{p,q}$, induced by the annuity numeraire $B_{p,q} = \sum_{j=p}^{q-1} \delta_j B_{j+1}(t)$. Therefore we may write

$$dS_{p,q}(t) = \sigma_{p,q}(t) S_{p,q}(t) d\mathcal{W}^{(p,q)}(t), \tag{23}$$

where $d\mathcal{W}^{(p,q)}(t)$ is standard Brownian motion under $P_{p,q}$. From (22) we see that the swap rate can be expressed as a weighted sum of the constituent forwards rates,

$$S_{p,q}(t) = \sum_{j=p}^{q-1} w_j(t) L_j(t)$$

with

$$w_j(t) = \frac{\delta_j B_{j+1}(t)}{B_{p,q}}.$$

An application of Ito's Lemma yields

$$\begin{aligned} dS_{p,q}(t) &= \sum_{j=p}^{q-1} \frac{\partial S_{p,q}(t)}{\partial L_j(t)} dL_j(t) + \sum_{j=p}^{q-1} \sum_{i=p}^{q-1} \frac{\partial^2 S_{p,q}}{\partial L_j(t) \partial L_i(t)} dL_j(t) dL_i(t) \\ &= \sum_{j=p}^{q-1} \frac{\partial S_{p,q}(t)}{\partial L_j(t)} L_j(t) \Gamma_j^\top \left[ d\mathcal{W}^{(n)} + (\ldots) dt \right]. \end{aligned} \tag{24}$$

Equating (23) and (24), gives

$$dS_{p,q}(t) = S_{p,q}(t) \left[ \sum_{j=p}^{q-1} \nu_j(t) \Gamma_j^\top \right] d\mathcal{W}^{(p,q)}(t)$$

with $\mathcal{W}^{(p,q)} = (W^{(p,q)}, \widetilde{W}^{(p,q)})$ and

$$\nu_j(t) := \frac{\partial S_{p,q}(t)}{\partial L_j(t)} \frac{L_j(t)}{S_{p,q}(t)}.$$

The change of measure from $\mathcal{W}^{(n)}$ to $\mathcal{W}^{(p,q)}$ can be found in Schoenmakers (2005). In particular,

$$dW^{(p,q)} = dW^{(n)} - \sqrt{1-r^2}\sum_{i=p}^{q-1} w_i \sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j}\gamma_j dt$$

and

$$d\widetilde{W}^{(p,q)} = d\widetilde{W}_k^{(n)} - r\sum_{i=p}^{q-1} w_i \sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j}\overline{\gamma}_{jk}\sqrt{v_k}dt.$$

In terms of these new Brownian motions the volatility processes read,

$$\begin{aligned} dv_k &= \kappa_k(1-v_k)dt + r\sigma_k\rho_k \sum_{i=p}^{q-1} w_i(t) \sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j}\overline{\gamma}_{jk} v_k dt \\ &\quad + \rho_k\sigma_k\sqrt{v_k}d\widetilde{W}_k^{(p,q)} + \sqrt{1-\rho_k^2}\sigma_k\sqrt{v_k}d\overline{W}_k^{(p,q,n)}. \end{aligned} \tag{25}$$

As shown in the Appendix, the process $\overline{W}^{(p,q,n)}$ in (25) is standard Brownian motion under both measures $P_{p,q}$ and $P_n$. Assuming now that $\frac{\partial S_{p,q}(t)}{\partial L_j(t)}$ and $\frac{L_j(t)}{S_{p,q}(t)}$ are approximately constant in time, we freeze the weights at their initial time $t = 0$. Then the swap rate dynamic is approximately given by

$$dS_{p,q}(t) \approx S_{p,q}(t)\left[\sum_{j=p}^{q-1} \nu_j(0)\Gamma_j^\top\right] d\mathcal{W}^{(p,q)}(t). \tag{26}$$

Similarly, freezing the Libors in the drift term of (25) leads to an approximated volatility process $v_k$ given by

$$dv_k \approx \kappa_k^{(p,q)}\left(\theta_k^{(p,q)} - v_k\right)dt + \sigma_k\sqrt{v_k}\left(\rho_k d\widetilde{W}_k^{(p,q)} + \sqrt{1-\rho_k^2}d\overline{W}_k^{(p,q,n)}\right) \tag{27}$$

with reversion speed parameter

$$\kappa_k^{(p,q)} := \kappa_k - r\sigma_k\rho_k \sum_{i=p}^{q-1} w_i(0) \sum_{j=i+1}^{n-1} \frac{\delta_j L_j(0)}{1+\delta_j L_j(0)}\overline{\gamma}_{jk}, \tag{28}$$

and mean reversion level

$$\theta_k^{(p,q)} := \frac{\kappa_k}{\kappa_k^{(p,q)}}. \tag{29}$$

# 5 Calibration to Caplet prices

A caplet for the period $[T_j, T_{j+1}]$ with strike $K$ is an option that pays $(L_j(T_j) - K)^+\delta_j$ at time $T_{j+1}$, where $1 \le j < n$. It is well-known that under the forward measure $P_{j+1}$ the $j$-th caplet price at time zero is given by

$$C_j(K) = \delta_j B_{j+1}(0)E_{j+1}(L_j(T_j) - K)^+.$$

Thus, under $P_{j+1}$ the $j$-th caplet price is determined by the dynamics of $L_j$ only. The FFT-method of Carr and Madan (1999) can be straightforwardly adapted to the caplet pricing problem as done in Belomestny and Schoenmakers (2006). We here recap the main results.

In terms of the log-moneyness variable

$$v := \ln \frac{K}{L_j(0)} \tag{30}$$

the $j$-th caplet price can be expressed as

$$\mathcal{C}_j(v) := C_j(e^v L_j(0)) = \delta_j B_{j+1}(0) L_j(0) E_{j+1} \left( e^{X_j(T_j)} - e^v \right)^+ ,$$

where $X_j(t) = \ln L_j(t) - \ln L_j(0)$. One then defines the auxiliary function

$$\mathcal{O}_j(v) := \delta_j^{-1} B_{j+1}^{-1}(0) L_j^{-1}(0) \mathcal{C}_j(v) - (1 - e^v)^+ \tag{31}$$

and can show the following proposition.

**Proposition 1** *For the Fourier transform of the function $\mathcal{O}_j$ defined above and $\varphi_{j+1}(\cdot; t)$ denoting the characteristic function of the process $X_j(t)$ under $P_{j+1}$ we have*

$$\mathcal{F}\left\{\mathcal{O}_j\right\}(z) = \int_{-\infty}^{\infty} \mathcal{O}_j(v) e^{\mathrm{i}vz} dv = \frac{1 - \varphi_{j+1}(z - \mathrm{i}; T_j)}{z(z - \mathrm{i})}. \tag{32}$$

The proof can be found in Belomestny/Reiß (2006). Next, combining (30), (31), and (32) yields

$$C_j(K) = \delta B_{j+1}(0) \left(L_j(0) - K\right)^+ \tag{33}$$

$$+ \frac{\delta B_{j+1}(0) L_j(0)}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \varphi_{j+1}(z - \mathrm{i}; T_j)}{z(z - \mathrm{i})} e^{-\mathrm{i}z \ln \frac{K}{L_j(0)}} dz.$$

We now outline a calibration procedure for the Libor structure (2), under the following additional assumptions.

(i) The input market Libor volatility structure $\gamma \in \mathbb{R}^{(n-1)\times m}$ is assumed to be of full rank, that is $m = n - 1$. (Strictly speaking it would be enough to require the right-hand-sides of (13) or (14) to be of full rank.)

(ii) The terminal log-Libor increment $d \ln L_{n-1}$ is influenced by a single stochastic volatility shock $dU_{n-1}$, the one but last, hence $d \ln L_{n-2}$, by only $dU_{n-1}$ and $dU_{n-2}$, and so forth. Put differently, we assume $\beta \in \mathbb{R}^{(n-1)\times d}$ to be a squared upper triangular matrix of rank $n - 1$, hence $d = n - 1$.

(iii) The $r_i$ are taken to be constant, that is $r_i \equiv r$, and the matrix $\beta$ is determined as the time independent upper triangular solution $\overline{\gamma}$ of the covariance condition (13) or (14), depending on the readers preference.

(iv) Recall that $v_k(0) \equiv \theta_k \equiv 1$, $1 \leq k < n$.

For the Libor dynamics structured in the above way we thus have

$$d \ln L_i(t) = -\frac{1}{2}\left[(1-r^2)\,|\gamma_i|^2 + r^2 \sum_{k=i}^{n-1} \overline{\gamma}_{ik}^2 v_k\right] dt$$
$$+ \sqrt{1-r^2}\gamma_i^\top dW^{(i+1)}$$
$$+ r \sum_{k=i}^{n-1} \overline{\gamma}_{ik} \sqrt{v_k} d\widetilde{W}_k^{(i+1)}, \quad 1 \leq i < n, \tag{34}$$

where for $i = n-1$ the dynamics of $v_{n-1}$ is given by (16), and for $i < n-1$ the dynamics of $v_k$, $i \leq k < n$, is approximately given by (19).

We will calibrate the structure to prices of caplets according to the following roadmap.

1. First step $i = n-1$. Calibrate $r$ and the parameter set $(\kappa_{n-1}, \theta_{n-1} = 1, \sigma_{n-1}, \rho_{n-1})$ to the $T_{n-1}$ column of the cap-strike matrix via (33) using the explicitly known characteristic function $\varphi_n$ of $\ln[L_{n-1}(T_{n-1})/L_{n-1}(0)]$ (see Appendix (8.0.1)).

2. For $i = n-2$ down to 1 carry out the next iteration step:

3. The k-th step $i = n-k$. Transform the yet known parameter set $(\kappa_j, \sigma_j, \rho_j)$ $i < j < n$, via (20) and (21) into the corresponding set $(\kappa_j^{(i+1)}, \sigma_j^{(i+1)}, \rho_j^{(i+1)}, \theta_j^{(i+1)})$, $i < j < n$. By the upper triangular structure of the square matrix $\overline{\gamma}$ we obviously have $\kappa_i^{(i+1)} = \kappa_i$, hence by (21) $\theta_i^{(i+1)} = 1$. Then calibrate the at this stage unknown parameter set $(\kappa_i, \sigma_i, \rho_i)$ to the $T_i$ column of the cap-strike matrix via (33) using the explicitly known characteristic function $\varphi_{i+1}$ of $\ln[L_i(T_i)/L_i(0)]$ under the approximation (17)-(19) (see Appendix (8.0.1)).

The above calibration algorithm includes at each step, as usual, the minimization of some objective function. As such function we take the weighted sum of squares of the corresponding differences between observed market prices and prices induced by the model. The weights are taken to be proportional to Black-Scholes vegas. As an initial values for the local optimization routine at time step $i+1$ the values of estimated parameters at time step $i$ are used.

# 6 Calibration to swaption prices

A European swaption over a period $[T_p, T_q]$ gives the right to enter at $T_p$ into an interest rate swap with strike rate $K$. The swaption value at time $t \leq T_p$ is given by

$$Swpn_{p,q}(t) = B_{p,q}(t) E_{p,q}^{\mathcal{F}_t} (S_{p,q}(T_p) - K)^+.$$

Since the approximative model (26)-(27) for $S_{p,q}$ has an affine structure with constant coefficients one can write down the characteristic function of $S_{p,q}$ analytically under $P_{p,q}$ and follow the lines of the previous section to calibrate the model.

**Remark 2** *Due to the covariance restrictions* (13)-(14)*, one can expect that the model prices of ATM swaptions are not far from market prices because our model employs a covariance structure of LMM calibrated to the market prices of ATM swaptions.*

# 7 Calibration to real data

In this section we calibrate the model (17)-(19) to market data available on 14.08.2007. The caplet-strike volatility matrix is partially shown in Table 1. The corresponding implied volatility surface is shown in Figure 1.

Pronounced smiles are clearly observable. Due to the structure of the given data we are going to calibrate the jump diffusion model based on semi-annual tenors, i.e. $\delta_j \equiv 0.5$, with $n = 41$, and where the initial calibration date 14.08.2007 is identified with $T_0 = 0$.

In a pre-calibration a standard market model is calibrated to ATM caps and ATM swaptions using Schoenmakers (2005). However, we emphasize that the method by which this input market model is obtained is not essential nor a discussion point for this paper. For the pre-calibration we have used a volatility structure of the form

$$\gamma_i(t) = c_i g(T_i - t) e_i, \quad 0 \le t \le T_i, \quad 1 \le i < n,$$

where $g$ is a simple parametric function and $e_i$ are unit vectors. The pre-calibration routine returns $e_i \in \mathbb{R}^{n-1}$ such that $(e_{i,k})$ is upper triangular and

$$\begin{aligned} e_i^\top e_j = \rho_{ij} \quad &= \quad \exp\Bigg[-\frac{|j-i|}{m-1}\Bigg(-\ln\rho_\infty \\ &\qquad -\eta\frac{i^2+j^2+ij-mi-mj-3i-3j+3m+2}{(m-2)(m-3)}\Bigg)\Bigg], \qquad (35) \\ &\qquad i,j = 1,\ldots,m := n-1,\ 0 \le \eta \le -\ln\rho_\infty, \end{aligned}$$

with $n = 41$, $\rho_\infty = 0.23$, $\eta = 1.42$. The function $g$ is given by

$$g(s) = g_\infty + (1 - g_\infty + as)e^{-bs}.$$

with $a = 0.32$, $b = 0.07$, and $g_\infty = 0.58$. The loading factors $c_i$ can be readily computed from

$$(\sigma_{T_i}^{ATM})^2 T_i = c_i^2 \int_0^{T_i} g^2(s)\, ds, \quad i = 1, \ldots, n-1,$$

| T/K | 2 | 2.25 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|---|
| 0 | 24.80 | 22.60 | 21.00 | 18.50 | 14.90 | 10.60 |
| 0.5 | 24.80 | 22.60 | 21.00 | 18.50 | 14.90 | 10.60 |
| 1.0 | 22.71 | 20.81 | 19.45 | 17.25 | 14.90 | 12.47 |
| 1.5 | 20.62 | 19.02 | 17.89 | 16.01 | 14.90 | 14.34 |
| 2 | 21.31 | 19.59 | 18.51 | 16.67 | 15.40 | 14.63 |
| 2.5 | 21.99 | 20.17 | 19.13 | 17.34 | 15.91 | 14.92 |
| 3.0 | 21.72 | 20.27 | 19.21 | 17.40 | 15.99 | 14.98 |
| 3.5 | 21.45 | 20.37 | 19.29 | 17.46 | 16.08 | 15.05 |
| 4.0 | 20.85 | 20.26 | 19.17 | 17.44 | 16.01 | 15.06 |
| 4.5 | 20.23 | 20.15 | 19.04 | 17.43 | 15.95 | 15.06 |
| 5.0 | 20.46 | 19.92 | 18.87 | 17.15 | 15.71 | 14.78 |
| 5.5 | 20.69 | 19.69 | 18.69 | 16.87 | 15.48 | 14.49 |
| 6.0 | 20.92 | 19.46 | 18.51 | 16.59 | 15.25 | 14.22 |
| 6.5 | 21.16 | 19.22 | 18.33 | 16.31 | 15.01 | 13.93 |
| 7.0 | 20.81 | 19.09 | 18.15 | 16.21 | 14.92 | 13.88 |
| 7.5 | 20.46 | 18.95 | 17.97 | 16.11 | 14.82 | 13.82 |
| 8.0 | 20.11 | 18.82 | 17.78 | 16.01 | 14.72 | 13.76 |
| 8.5 | 19.75 | 18.68 | 17.60 | 15.91 | 14.62 | 13.70 |
| 9.0 | 19.40 | 18.54 | 17.41 | 15.81 | 14.52 | 13.64 |

Table 1: Caplet volatilities $\sigma_T^K$(in %) for different strikes and different tenor dates (in years).

using the initial Libor curve, which is obtained by a standard stripping procedure from the yield curve at 14.08.2007. Table 2 shows the calibrated values of $c_i$. Finally, the calibration procedure presented in Section 5 delivers the following parameter values: $r = 0.18$ and $\rho$, $\sigma$, $\kappa$ varying across several chosen maturities as shown in Table 3. The quality of the calibration can be seen in Figure 2, where calibrated volatility curves are shown for several caplet periods (corresponding to Table 7) together with the market caplet volas. The overall relative root-mean-square fit we have reached shows to be 0.5%-5%, when the caplet maturity ranges from 0.5 to 20.

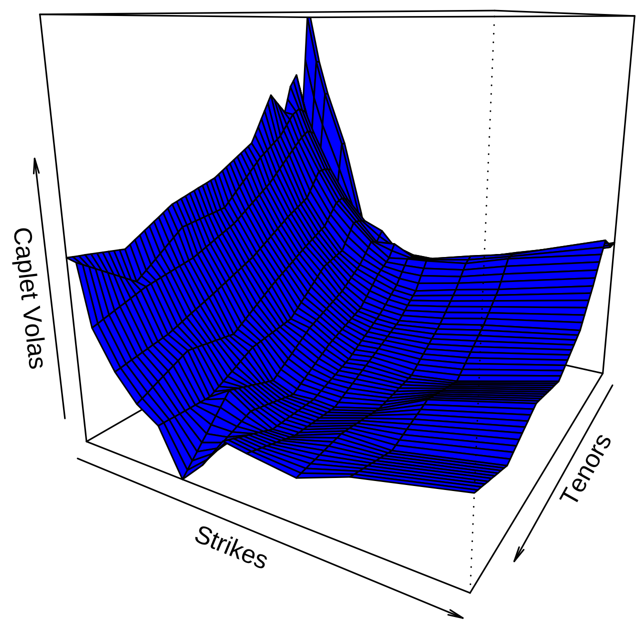


Figure 1: Caplet implied volatility surface $\sigma_T^K$.

# 8 Appendix

### 8.0.1 The Conditional Characteristic Function

We need to determine the conditional characteristic function of $\ln L_j(T)$ given $L_j(0)$ for all $j = 1, ..., n-1$. under the relevant measure $P_{j+1}$ when the Heston CIR-process has for each component $k = 1, ..., n-1$ the general form

$$dv_k = \kappa_k^{(j+1)}(\theta_k^{(j+1)} - v_k)dt + \sigma_k \rho_k \sqrt{v_k} d\widetilde{W}_k^{(j+1)} + \sigma_k \sqrt{(1-\rho_k^2)} \sqrt{v_k} d\overline{W}_k^{(j+1)}, \tag{36}$$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 0.096 | 0.090 | 0.101 | 0.111 | 0.106 | 0.101 | 0.099 | 0.097 | 0.092 | 0.087 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 0.084 | 0.081 | 0.078 | 0.076 | 0.073 | 0.071 | 0.068 | 0.066 | 0.064 | 0.062 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 0.060 | 0.059 | 0.058 | 0.057 | 0.056 | 0.055 | 0.054 | 0.0534 | 0.0526 | 0.0518 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 0.051 | 0.050 | 0.050 | 0.049 | 0.049 | 0.048 | 0.049 | 0.048 | 0.047 | 0.047 |

Table 2: The values of loadings factors $c_i$ calibrated to ATM caplets volatilities.

| Tenor | 20 | 19 | 18 | 17 |
|---|---|---|---|---|
| $\rho$ | -0.7832 | -0.7832 | -0.7832 | -0.7832 |
| $\sigma$ | 7.4920 | 7.4920 | 6.2427 | 5.0198 |
| $\kappa$ | 2.3376 | 2.3376 | 3.9385 | 4.5590 |

Table 3: Parameters estimates for chosen tenors.

In this case and a forward Libor dynamic given by (34) , with general $v \in \mathbb{R}^{n-1}$ , the solution is of the form

$$\begin{aligned}\varphi_{j+1}(z\,;T,l,v) &= E_{j+1}\left[ e^{\mathrm{i}z\ln L_j(T)} \middle| L_j(0)=l, v_k(0)=v_k, k=1,...,n-1\right] \\ &= \varphi_{j+1,0}\left(z\,;T\right)\exp(\mathrm{i}z\ln l)\prod_{k=j}^{n-1}\varphi_{j+1,k}\left(z;T\right) \qquad (37)\end{aligned}$$

where

$$\varphi_{j+1,0}(z\,;T) = \exp\left(-\frac{1}{2}(1-r^2)\eta_j^2(T)\left(z^2+\mathrm{i}z\right)\right), \quad \eta_j^2(T) = \int_0^T |\gamma_j|^2\, dt$$

and each $\varphi_{j+1,k}\left(z\,;T\right) = \varphi_{j+1,k}\left(z\,;T,l,v_k\right)$ satisfies the parabolic equation

$$\begin{aligned}\frac{\partial \varphi_{j+1,k}}{\partial T} &= \kappa_k^{(j+1)}(\theta_k^{(j+1)} - v_k)\frac{\partial \varphi_{j+1,k}}{\partial v_k} - \frac{1}{2}r^2\overline{\gamma}_{jk}^2 v_k\frac{\partial \varphi_{j+1,k}}{\partial l} + \frac{1}{2}\sigma_k^2 v_k\frac{\partial^2 \varphi_{j+1,k}}{\partial v_k^2} \\ &\quad + \frac{1}{2}r^2\overline{\gamma}_{jk}^2 v_k\frac{\partial^2 \varphi_{j+1,k}}{\partial l^2} + \sigma_k\rho_k r\overline{\gamma}_{jk}v_k\frac{\partial^2 \varphi_{j+1,k}}{\partial v_k \partial l}\end{aligned}$$

with the terminal condition

$$\varphi_{j+1,k}(z\,;0,l,v_k) = 1,$$

as can be easily verified by the Feynman-Kac formula.

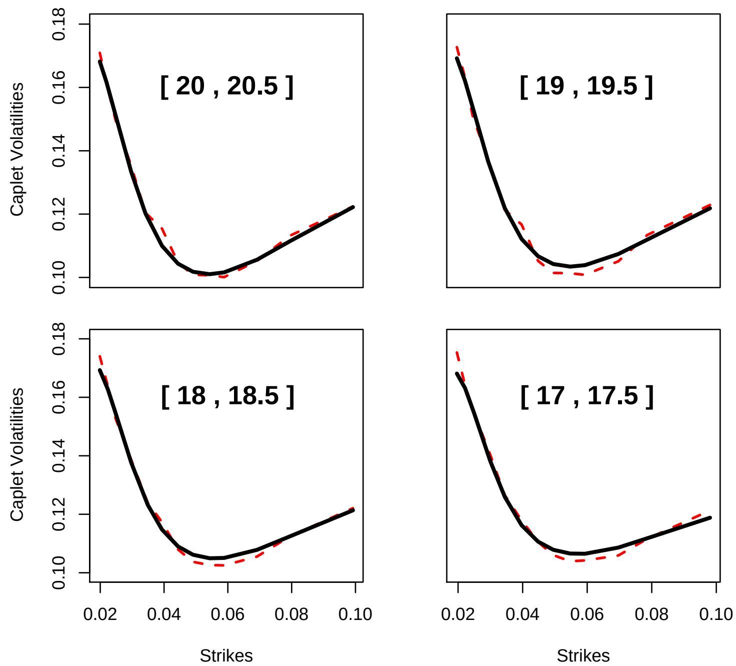


Figure 2: Caplet volas from the calibrated model (solid lines) and market caplets volas $\sigma_T^K$ (dashed lines) for different caplet periods.

Since $\overline{\gamma}_j$ are constant, the above equation can be solved explicitly. The ansatz

$$\varphi_{j+1,k}(z;T,l,v_k) = \exp\left(A_{j,k}(z;T) + v_k B_{j,k}(z;T)\right)$$

will yield

$$A_{j,k}(z;T) = \frac{\kappa_k^{(j+1)}\theta_k^{(j+1)}}{\sigma_k^2}\left\{(a_{j,k}+d_{j,k})T - 2\ln\left[\frac{1-g_{j,k}e^{d_{j,k}T}}{1-g_{j,k}}\right]\right\}$$

$$B_{j,k}(z;T) = \frac{(a_{j,k}+d_{j,k})(1-e^{d_{j,k}T})}{\sigma_k^2(1-g_{j,k}e^{d_{j,k}T})},$$

where

$$\begin{aligned} a_{j,k} &= \kappa_k^{(j+1)} - ir\rho_k\sigma_k\overline{\gamma}_{jk}z \\ d_{j,k} &= \sqrt{a_{j,k}^2 + r^2\overline{\gamma}_{jk}^2\sigma_k^2(z^2+iz)} \\ g_{j,k} &= \frac{a_{j,k}+d_{j,k}}{a_{j,k}-d_{j,k}}. \end{aligned}$$

Note that the first lower index $j+1$ at the characteristic function refers to the measure, whereas the first index $j$ at the introduced coefficients refers to relevant forward Libor. The second index refers to the component.
It is again the choice of $\overline{\gamma}$ that enables the product in (37) to be startet at $j$. This crucial feature will show to be beneficial in the calibration part. When $j = n-1$, for example, only the last ln-Libor will contribute a non-trivial factor to the characteristic function. For all others we have

$$\varphi_{n,k} \equiv 1\,, \quad k = 1, ..., n-2\,.$$

#### 8.0.2 CIR

Consider a CIR model of the form

$$dv(t) = \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}dW(t), \quad \kappa, \theta, \sigma > 0.$$

Given $v(u)$, $v(t)$ with $t > u$ is distributed with density

$$\nu\chi_d^2(\nu x, \xi)$$

where $\chi_d^2(x, \xi)$ is the density of a noncentral chi-square random variable with $d$ degrees of freedom and noncentrality parameter $\xi$ and

$$\begin{aligned}
\nu &= \frac{4\kappa}{\sigma^2(1 - e^{-\kappa(t-u)})} \\
\xi &= \frac{4\kappa e^{-\kappa(t-u)}}{\sigma^2(1 - e^{-\kappa(t-u)})}v(u) \\
d &= \frac{4\theta\kappa}{\sigma^2}.
\end{aligned}$$

The conditional mean of $v(t)$ is given by

$$E(v(t)|v(u)) = \nu^{-1}(\xi + d) = (v(u) - \theta)e^{-\kappa(t-u)} + \theta$$

and the conditional second moment is

$$\begin{aligned}
E(v^2(t)|v(u)) &= \frac{(2(d + 2\xi) + (\xi + d)^2)}{\nu^2} \\
&= \left(1 + \frac{2}{d}\right)\left[E(v(t)|v(u))\right]^2 - \frac{2}{d}e^{-2\kappa(t-u)}v^2(u).
\end{aligned}$$

### 8.0.3 Measure Invariance

Why is $d\overline{W}_k^{(n,i+1)}$ invariant under the various measures?

See Jamshidian for the compensator, which is given by

$$\mu^{i+1}_{\overline{W}_k^{(n)}} = \langle \overline{W}_k^{(n)}, \ln M \rangle.$$

with

$$M = \Pi_{j=i+1}^{n-1}(1+\delta L_j).$$

That is, we have

$$\langle \overline{W}_k^{(n)}, \ln M \rangle = d\overline{W}_k^{(n)} d\ln M = d\overline{W}_k^{(n)} d\left(\sum_{j=i+1}^{n-1} \ln\left(1+\delta L_j\right)\right)$$

$$= \sum_{j=i+1}^{n-1} d\overline{W}_k^{(n)} d\ln(1+\delta L_j)$$

$$= \sum_{j=i+1}^{n-1} \frac{\delta L_j}{1+\delta L_j} d\overline{W}_k^{(n)} d\ln L_j$$

A closer look at (15) reveils that all terms are negligible, since of higher order than $dt$, or zero due to independence of $\overline{W}$ and $W$ or $\widetilde{W}$, respectively. We thus have

$$\langle \overline{W}_k^{(n)}, \ln M \rangle = 0$$

or in other words, as indicated by $d\overline{W}_k^{(n,i+1)}$:

$$d\overline{W}_k^{(n)} = d\overline{W}_k^{(i+1)}.$$

Analoguosly we obtain by exchanging $\overline{W}_k$ with $\widetilde{W}_k$ that

$$\langle \widetilde{W}_k^{(n)}, \ln M \rangle = d\widetilde{W}_k^{(n)} d\ln M$$

$$= \sum_{j=i+1}^{n-1} \frac{\delta L_j}{1+\delta L_j} d\widetilde{W}_k^{(n)} d\ln L_j$$

$$= \sum_{j=i+1}^{n-1} \frac{r\delta L_j}{1+\delta L_j} \beta_{jk}\sqrt{v_t^k}dt$$

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