Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Higher integrability of the Lorentz force for weak solutions to Maxwell's equations in complex geometries.

Pierre-Étienne Druet [1]

submitted: 13th November 2007

[1] Weierstrass Institut für Angewandte Analysis und Stochastik
Mohrenstrasse 39
10117 Berlin
Germany
E-Mail: druet@wias-berlin.de

No. 1270
Berlin 2007

2000 *Mathematics Subject Classification.* 35D10, 35J55, 35Q60.

Key words and phrases. Maxwell's equations, natural interface conditions, Lorentz force, higher integrability.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider the stationary Maxwell system in a domain filled with different materials. The magnetic permeability being only piecewise smooth, we have to take into account the natural interface conditions for the electromagnetic fields. We present two sets of hypothesis under which we can prove the existence of weak solutions to the Maxwell system such that the Lorentz force jxB is integrable to a power larger than 6/5. This property is important for the investigation of problems in magnetohydrodynamics, with many industrial applications such as crystal growth.

1 Introduction

The purpose of this note is to formulate some consequences of recent regularity results that are relevant for the mathematical theory of Maxwell's equations. More specifically, we want to show how the theory of the papers [Zan00], [ERS07] can help us to deal with the difficulties that arise from coupling the weak formulation of Maxwell's system to the equation of momentum balance in complex geometries.

In this introduction, we first describe the type of geometrical setting that we have in mind.

We consider finitely many bounded domains $\tilde{\Omega}_0, \ldots, \tilde{\Omega}_m \subset \mathbb{R}^3$, that represent disjoint materials with different electromagnetic properties, and we define a domain $\tilde{\Omega} \subset \mathbb{R}^3$ by

$$\overline{\tilde{\Omega}} = \bigcup_{i=0}^{m} \overline{\tilde{\Omega}_i}\,. \tag{1}$$

We assume that the set $\tilde{\Omega}$ is connected. Introducing an index set $I_c \subseteq \{0, \ldots, m\}$ by

$$i \in I_c \quad \Longleftrightarrow \quad \tilde{\Omega}_i \text{ is electrically conducting,} \tag{2}$$

we can gather the electrical conductors in a domain $\tilde{\Omega}_c$ given by

$$\overline{\tilde{\Omega}_c} = \bigcup_{i\in I_c} \overline{\tilde{\Omega}_i}\,.$$

We denote by $\tilde{\Omega}_{c_0} \subset \tilde{\Omega}_c$ a part of the conductors where we the current is prescribed. Throughout the paper, we consider only the simplest case that the set $\tilde{\Omega}_{c_0}$ is isolated from the rest of the conductors. We denote by $\Omega_c := \tilde{\Omega}_c \setminus \tilde{\Omega}_{c_0}$ the part of the conductors where the current is unknown.

In the domain $\tilde{\Omega}$, we consider the stationary Maxwell's equations. Ampère's law

$$\operatorname{curl} H = j\,, \qquad \text{in } \tilde{\Omega}\,, \tag{3}$$

and Ohm's law for the electrical conductors

$$j = \sigma\,(E + v \times B)\,, \qquad \text{in } \tilde{\Omega}_c\,, \tag{4}$$

can be written in the short form

$$\operatorname{curl} H \;=\; \begin{cases} 0 & \text{in } \tilde{\Omega} \setminus \tilde{\Omega}_c\,, \\ j_0 & \text{in } \tilde{\Omega}_{c0}\,, \\ \sigma\,(E + v \times B) & \text{in } \Omega_c\,, \end{cases} \tag{5}$$

where j_0 is a given density of direct current, and the function σ represents the electrical conductivity of the medium. Further, the magnetic induction B satisfies

$$\operatorname{div} B = 0\,, \qquad \text{in } \tilde{\Omega}\,. \tag{6}$$

For the electric field E, it holds that

$$\operatorname{curl} E = 0\,, \qquad \text{in } \tilde{\Omega}\,. \tag{7}$$

In the non-conducting parts, the displacement current D has to satisfy

$$\operatorname{div} D = 0\,, \qquad \text{in } \tilde{\Omega} \setminus \tilde{\Omega}_c\,. \tag{8}$$

We still need a constitutive relation between B and H, and between E and D. We consider only linear materials, that is

$$B = \mu\, H\,, \qquad D = \mathfrak{e}\, E\,, \tag{9}$$

with the function μ of magnetic permeability and the function $\mathfrak{e}$ of electrical permittivity. In the interior of $\tilde{\Omega}$, the fields B, H, E have to satisfy the *natural interface conditions*

$$\big[H \times \vec{n}\big]_{i,j} = 0\,, \quad \big[B \cdot \vec{n}\big]_{i,j} = 0\,, \quad \big[E \times \vec{n}\big]_{i,j} = 0 \qquad \text{on } \partial\tilde{\Omega}_i \cap \partial\tilde{\Omega}_j \tag{10}$$

where $\big[\cdot\big]_{i,j}$ denotes the jump of a quantity across the surface $\partial\tilde{\Omega}_i \cap \partial\tilde{\Omega}_j$ for $i, j = 0, \ldots, m$. We consider that the outer boundary $\partial\tilde{\Omega}$ models a magnetic shield, and prescribe the conditions

$$B \cdot \vec{n} = 0\,, \quad E \times \vec{n} = 0 \qquad \text{on } \partial\tilde{\Omega}\,. \tag{11}$$

Definition 1.1. We denote the problem of finding fields H, B, E, D, j that satisfy (5), (6), (7), (8) together with the constitutive relations (9) and the boundary conditions (10) and (11) as Problem (P).

The problem of the integrability of the force $\mathbf{j} \times \mathbf{B}$. For given current source j_0 and velocity field v, the problem (P) has been successfully solved in the past, also for more general geometrical settings than (1) and for nonlinear constitutive relations (9) (see e. g. [PM99]). This was mainly achieved thanks to the theory of *generalized curl* and *div* operators, and the use of decomposition theorems of the space L^2 relying on this theory.

For the field H that weakly solves the problem (P), the generalized theory of electromagnetics gives the following basic informations (see for example [DL76], [PM99] or [Bos04]):

$$H \in V_{\mu,0}(\tilde{\Omega}) := \left\{ \psi \in [L^2(\tilde{\Omega})]^3 \;\middle|\; \operatorname{curl} \psi \in [L^2(\tilde{\Omega})]^3,\ \operatorname{div}(\mu\,\psi) = 0,\ \mu\,\psi \cdot \vec{n} = 0 \text{ on } \partial\tilde{\Omega} \right\}, \tag{12}$$

where the operators *curl*, *div*, and the normal trace $\psi \cdot \vec{n}$ are intended in the generalized sense.

However, for the study of the coupled problem involving for example the Navier-Stokes equations, the use of such weak solutions can lead to considerable difficulties. Assume for example that a subset of the described system, say $\tilde{\Omega}_1 \subset \Omega_c$, consists of a rotating cylindrical vessel filled with an electrically conducting fluid. Whenever a direct current is applied in the conductors $\tilde{\Omega}_{c_0}$, it will generate a steady state magnetic field in the entire region $\tilde{\Omega}$, and influence the fluid motion. This type of interaction is described by the stationary MHD system. The Navier-Stokes equations for an incompressible fluid are then written as follows:

$$\rho\,(v \cdot \nabla)v = \nabla p + \eta \,\triangle v + j \times B\,, \quad \operatorname{div} v = 0\,, \qquad \text{in } \tilde{\Omega}_1\,, \tag{13}$$

where ρ is the mass density of the fluid, η its kinematical viscosity, and $j \times B$ denotes the electromagnetic force (Lorentz force). Since we assume that Ampère's law (3) and the linear relation (9) are valid, we can write for the electromagnetic force in (13) also

$$j \times B = \operatorname{curl} H \times \mu\, H\,.$$

Therefore, if our knowledge about the regularity of H is limited to (12), we cannot expect in general more than $j \times B \in [L^1(\tilde{\Omega})]^3$.

Situation and structure of the paper. A basic message of most papers about magnetohydrodynamics is that the difficulties described in the previous paragraph can be avoided by making suitable assumptions on the regularity of the function μ, and on the structure of the domain $\tilde{\Omega}$. We briefly describe the main ideas. A *first idea*, applied e. g. in [DL72], [ST83], consists in supposing that μ is a *globally smooth* function in the domain $\tilde{\Omega}$. From (12) it then follows that

$$\operatorname{div} H = \frac{-\nabla \mu}{\mu} \cdot H \in L^2(\tilde{\Omega})\,.$$

Whenever the function μ is bounded away from zero, a vector field H that satisfies (12) then belongs to the space

$$V := \left\{ \psi \in [L^2(\tilde{\Omega})]^3 \;\middle|\; \operatorname{curl} \psi \in [L^2(\tilde{\Omega})]^3,\ \operatorname{div} \psi \in L^2(\tilde{\Omega}),\ \psi \cdot \vec{n} = 0 \text{ on } \partial\tilde{\Omega} \right\}. \tag{14}$$

In [DL76], it is shown that if the boundary $\partial\tilde{\Omega}$ is of class $\mathcal{C}^2$, then the topological identity

$$V = \left\{H \in [H^1(\tilde{\Omega})]^3 \,\middle|\, H \cdot \vec{n} = 0 \text{ on } \partial\tilde{\Omega}\right\}, \tag{15}$$

is valid. With the help of Sobolev's embedding theorems, we immediately obtain that $H \in [L^6(\tilde{\Omega})]^3$. This gives $curl\, H \times B \in L^{3/2}(\tilde{\Omega})$. Thanks to recent advances in regularity theory (see e. g. the paper [ABDG98] and the references therein), this type of result can be extended to less regular, Lipschitz domains. Still supposing that the permeability μ is globally smooth, one proves that the embedding

$$V \hookrightarrow [H^{1/2}(\tilde{\Omega})]^3, \tag{16}$$

is continuous (see [Mon03], Theorem 3.47, for a proof). This gives for a vector field H that satisfies (12) that $H \in [L^3(\tilde{\Omega})]^3$. Therefore, one obtains that $curl\, H \times B$ belongs to $[L^{6/5}(\tilde{\Omega})]^3$ even in Lipschitz domains, which is still sufficient for solving (13) via standard theory.

However, it is not always a good approximation to assume the smoothness of the permeability μ. In real-life applications, the magnetic permeability has jumps at the interfaces that separate different materials, and it is necessary to take into account transmission conditions at the interfaces.

In this sense, the authors of the paper [LS60] considered a setting with two disjoint subdomains $\tilde{\Omega} = \overline{\Omega_1} \cup \Omega_2$, where the set Ω_1 is supposed to be simply connected and compactly included in $\tilde{\Omega}$. Under the assumption that the outer boundary $\partial\tilde{\Omega}$, as well as the *interface* $\partial\Omega_1$ are of class $\mathcal{C}^2$, one can prove the topological identity

$$V_{\mu,0}(\tilde{\Omega}) = V_{\mu,0}(\tilde{\Omega}) \cap \bigcap_{i=1}^{2} [W^{1,2}(\Omega_i)]^3. \tag{17}$$

This result was confirmed by other methods in the more recent papers [MS96], [MS99]. One must note, however, that restriction to interfaces that are globally in the class $\mathcal{C}^2$ (or at least in $\mathcal{C}^{1,1}$) excludes most situations that one expects to find in complex applications, such as triple jump points of the magnetic permeability, or interfaces with corners.

In the present note, our aim is to take advantage of recent regularity results to derive weaker conditions than in [LS60] on the data pair $(\mu, \tilde{\Omega})$ under which we can obtain the higher integrability of $j \times B$.

We present two different sets of hypotheses that yield the existence of a number $q > 3$ such that the space $V_{\mu,0}(\tilde{\Omega})$ embeds continuously into $[L^q(\tilde{\Omega})]^3$. This gives that $curl\, H \times B \in [L^r(\tilde{\Omega})]^3$ for some $r > 6/5$.

First, exploiting the regularity theory of [ERS07], the higher integrability follows if we require that the interfaces $\partial\tilde{\Omega}_i \cap \partial\tilde{\Omega}_j$ are of class $\mathcal{C}^1$, that the outer boundary $\partial\tilde{\Omega}$ is of class $\mathcal{C}^{0,1}$, and that the function μ is uniformly continuous on each subdomain $\tilde{\Omega}_i$.

Second, exploiting the results of [Zan00], the higher integrability follows without further conditions on the interfaces, provided that the domain $\tilde{\Omega}$ is Lipschitzian and that the function $\mu \in L^\infty(\tilde{\Omega})$ is nearly constant (in a sense to be made more precise below). The latter situation quite often occurs in practice, since the ratio of the magnetic permeability of non-magnetic materials to the magnetic permeability of the vacuum is nearly one.

2 Embedding results for vector fields that satisfy a *curl* and a *div* constraint.

Several embedding results have been stated in the past for vector fields that satisfy a *curl* and a *div* constraint, and in general also a constraint on the normal or on the tangential values taken at the boundary. A typical example is given by the embedding (15) of the space V defined in (14), which relies on the inequality

$$\|\nabla\psi\|_{[L^2(U)]^9} \leq c\,(\|\operatorname{curl}\psi\|_{[L^2(U)]^3} + \|\operatorname{div}\psi\|_{L^2(U)})\,, \tag{18}$$

valid whenever the domain $U \subset \mathbb{R}^3$ is of class $\mathcal{C}^2$ (see [DL76], Ch. 7, Th. 6.1 for a proof). The inequality (18) is known as Gaffney's inequality, see [Pic84]). Inequalities of this type can be generalized in smooth domains to the case $1 < p < +\infty$, as was shown in [vW92], Th. 2.1, and to Sobolev spaces of fractional order. With respect to nonsmooth domains, these results mostly extend to convex polyhedra (see [GR86] and references), but examples of Lipschitz domains in three space dimensions are known for which (18) fails. One can still hope, though, to prove an embedding result in higher L^p-spaces, i. e. an inequality of the type

$$\|\psi\|_{[L^q(U)]^9} \leq c\,(\|\operatorname{curl}\psi\|_{[L^p(U)]^3} + \|\operatorname{div}\psi\|_{L^p(U)})\,,$$

with $q > p$. An example of a similar result obtained via embedding results for Sobolev spaces of fractional order is given by (16). In the following of this preliminary section, we first recall basic notions concerning the generalized operators *curl* and *div*, and then investigate embedding results that can be obtained directly.

2.1 The generalized operators *curl* and *div*.

We at first recall the definitions of the generalized differential operators *curl* and *div*.

Definition 2.1. Let $U \subset \mathbb{R}^3$ be a bounded domain, and $1 \leq p \leq \infty$.

(1) For a vector field $\psi \in [L^p(U)]^3$, we write $curl\,\psi \in [L^p(U)]^3$ if there exists a $\xi \in [L^p(U)]^3$ such that

$$\int_U \psi \cdot \operatorname{curl}\phi = \int_U \xi \cdot \phi\,,$$

for all $\phi \in [C_c^\infty(U)]^3$. The uniquely determined vector field ξ is called the generalized *curl* of ψ, and we define $curl\,\psi := \xi$.

(2) For a vector field $\psi \in [L^p(U)]^3$, we write $div\,\psi \in L^p(U)$ if there exists a function $\zeta \in L^p(U)$ such that

$$\int_U \psi \cdot \nabla\phi = -\int_U \zeta\,\phi\,,$$

for all $\phi \in C_c^\infty(U)$. The uniquely determined function ζ is called the generalized divergence of ψ, and we define $div\,\psi := \zeta$.

For a bounded domain $U \subset \mathbb{R}^3$, we then introduce

$$L^p_{\text{curl}}(U) := \left\{ \psi \in [L^p(U)]^3 \;\middle|\; \text{curl}\,\psi \in [L^p(U)]^3 \right\},$$
$$L^p_{\text{div}}(U) := \left\{ \psi \in [L^p(U)]^3 \;\middle|\; \text{div}\,\psi \in L^p(U) \right\},$$

where the operators *curl* and *div* are intended in the sense of Definition 2.1. These spaces are Banach spaces with respect to the graph topologies

$$\|\psi\|_{L^p_{\text{curl}}(U)} := \|\psi\|_{[L^p(U)]^3} + \|\,\text{curl}\,\psi\|_{[L^p(U)]^3}\,,$$
$$\|\psi\|_{L^p_{\text{div}}(U)} := \|\psi\|_{[L^p(U)]^3} + \|\,\text{div}\,\psi\|_{L^p(U)}\,. \tag{19}$$

For $p = 2$, they are Hilbert spaces. For vector fields that belong to a space (19), it is possible to define trace operators. Denoting by $\vec{n}$ the outward-pointing unit normal to ∂U, we have for $\phi,\,\psi \in [C^\infty(\overline{U})]^3$ the well-known formula

$$\int_U \psi \cdot \text{curl}\,\phi - \int_U \text{curl}\,\psi \cdot \phi = -\int_{\partial U} (\psi \times \vec{n}) \cdot \phi =: -\big\langle \gamma_\tau(\psi),\, \phi \big\rangle\,.$$

Thanks to results for the density of the smooth functions in the spaces (19), it can be shown (see for example [DL76], [PM99]) that the operator γ_τ extends to a linear bounded operator on the space $L^2_{\text{curl}}(U)$. For $\psi \in L^2_{\text{curl}}(U)$, we then call $\gamma_\tau(\psi)$ the *trace* of ψ. In general, this trace need not to be identical to an integrable function on the boundary. Nevertheless, for $\phi,\,\psi \in L^2_{\text{curl}}(U)$, we often abuse notation and write $\int_{\partial U} (\psi \times \vec{n}) \cdot \phi$ instead of $\big\langle \gamma_\tau(\psi),\, \phi \big\rangle$.

Similarly, for $\psi \in [C^\infty(\overline{U})]^3$ and $\phi \in C^\infty(\overline{U})$, we have the formula

$$\int_U \psi \cdot \nabla\phi + \int_U \text{div}\,\psi \cdot \phi = \int_{\partial U} \psi \cdot \vec{n}\,\phi =: \big\langle \gamma_n(\psi),\, \phi \big\rangle\,.$$

The operator γ_n extends to a linear bounded operator on the space $L^2_{\text{div}}(U)$. For $\psi \in L^2_{\text{div}}(U)$, we call $\gamma_n(\psi)$ the *trace* of ψ. For $\psi \in L^2_{\text{div}}(U)$ and $\phi \in L^2_{\text{div}}(U)$, we often write $\int_{\partial U} \psi \cdot \vec{n}\,\phi$ instead of $\big\langle \gamma_n(\psi),\, \phi \big\rangle$.

2.2 Embedding of $L^p_{\rm curl}(U) \cap L^p_{\rm div}(U)$ into $L^q(U)$ for $(q > p)$.

Let $U \subset \mathbb{R}^3$ be a simply connected bounded domain. For $1 < p,\, \alpha < \infty$, we consider the spaces

$$\mathcal{W}^{p,\alpha}(U) := \left\{ u \in L^p_{\rm curl}(U) \cap L^p_{\rm div}(U) \,\middle|\, \gamma_n(u) \in L^\alpha(\partial U) \right\}. \tag{20}$$

We denote by p^* the Sobolev embedding exponent

$$p^* := \begin{cases} \frac{3p}{3-p} & \text{if } 1 \le p < 3\,, \\ 1 \le s < \infty \text{ arbitrary} & \text{if } p = 3\,, \\ \infty & \text{if } p > 3\,. \end{cases}$$

In this section, we prove:

Proposition 2.2. Let $U \subset \mathbb{R}^3$ be a simply connected bounded Lipschitz domain. Then, there exists a $q_1 > 3$ such that for all $p \in]q_1', q_1[$, the space $\mathcal{W}^{p,\alpha}(U)$ embeds continuously in $[L^\xi(U)]^3$ for $\xi := \min\left\{\frac{3\alpha}{2},\, p^*,\, q_1\right\}$. If the domain U is of class $\mathcal{C}^1$, one may take $q_1 = +\infty$.

In order to prove Proposition 2.2, we first need an extension result.

Lemma 2.3. Let $U \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let $f \in L^p_{\rm div}(\tilde\Omega)$ such that *div* $f = 0$ in the sense of the generalized *div* operator. Let $\tilde U \subset \mathbb{R}^3$ be a bounded domain such that $U \subset\subset \tilde U$.

Then, there exists a $q_1 > 3$ such that for all $p \in]q_1', q_1[$, we can find an extension $\tilde f$ of f to $\tilde U \setminus U$ such that $\tilde f \in L^p_{\rm div}(\tilde U)$ and

$$\operatorname{div} \tilde f = 0 \text{ in the weak sense in } \tilde U\,, \quad \tilde f \cdot \vec n = 0 \text{ on } \partial \tilde U\,.$$

In addition, there exists a constant $c = c(U, p)$ such that

$$\|\tilde f\|_{[L^p(\tilde U)]^3} \le c\, \|f\|_{[L^p(U)]^3}\,.$$

Proof. Denoting by p' the conjugated exponent to p, we define a functional $F \in [W^{1,p'}(\tilde U \setminus U)]^*$ by

$$F(\phi) := \int_{\partial U} f \cdot \vec n\, \phi\,.$$

for $\phi \in W^{1,p'}(\tilde U \setminus U)$. Since f is weakly divergence-free in U, we see that $F(1) = 0$, and we can find a positive constant c such that

$$\|F\|_{[W^{1,p'}(\tilde U \setminus U)]^*} \le c\, \|f\|_{[L^p_{\rm div}(U)]^3} = c\, \|f\|_{[L^p(U)]^3}\,.$$

Theorem 1.6 in [Zan00] (see also Proposition 4.5 below) provides the existence of some $q_1 > 3$ such that for all $p \in]q_1', q_1[$, the Neumann problem to find some $a \in W^{1,p}(\tilde{U} \setminus U)$ such that the relation

$$\int_{\tilde{U}\setminus U} \nabla a \cdot \nabla \phi = F(\phi)\,,$$

holds for all $\phi \in W^{1,p}(\tilde{U} \setminus U)$ possesses a weak solution, which is unique up to constants. In addition, the estimate

$$\|a\|_{W^{1,p}(\tilde{U}\setminus U)} \leq C\,\|F\|_{[W^{1,p'}(\tilde{U}\setminus U)]^*}\,.$$

is valid, with a constant C that only depends on the Lipschitz constant of the domain U. We define

$$\tilde{f} := \begin{cases} f & \text{in } U\,, \\ \nabla a & \text{in } \tilde{U} \setminus U\,. \end{cases}$$

It is then easy to verify that this extension has the required properties. $\square$

Proof of Proposition 2.2. We consider an arbitrary $u \in \mathcal{W}^{p,\alpha}(U)$.

Define $f := \operatorname{curl} u$. Since $div\, f = 0$ almost everywhere in U, we see immediately that $f \in L^p_{\mathrm{div}}(U)$ for all $1 \leq p \leq \infty$. We now choose some p in the range $]q_1', q_1[$, where q_1 is given by Lemma 2.3, and we fix some smoothly bounded domain $\tilde{U} \subset \mathbb{R}^3$ such that $U \subset\subset \tilde{U}$. Applying Lemma 2.3, we find an extension $\tilde{f} \in L^p_{\mathrm{div}}(\tilde{U})$ such that

$$\tilde{f} = f \text{ in } [L^p(U)]^3, \quad \operatorname{div} \tilde{f} = 0 \text{ weakly in } \tilde{U}, \quad \tilde{f} \cdot \vec{n} = 0 \text{ in the sense of traces on } \partial\tilde{U}\,.$$

In view of Lemma 2.3, we have the estimate

$$\|\tilde{f}\|_{[L^p(\tilde{U})]^3} \leq c\,\|f\|_{[L^p(U)]^3}\,. \tag{21}$$

Since the domain $\tilde{U}$ is regular, we can apply Theorem 3.3 in [Gri90], valid for $\mathcal{C}^1$ domains, to find a vector field $A \in [W^{1,p}_0(\tilde{U})]^3$ such that

$$\operatorname{curl} A = \tilde{f} \text{ in } \tilde{U}, \quad \|A\|_{[W^{1,p}_0(\tilde{U})]^3} \leq \bar{c}\,\|\tilde{f}\|_{[L^p(\tilde{U})]^3}\,,$$

with a constant $\bar{c}$ that depends on $\tilde{U}$ and on p. Using (21), it follows that

$$\|A\|_{[W^{1,p}_0(\tilde{U})]^3} \leq \bar{c}\,\|\operatorname{curl} u\|_{[L^p(U)]^3}\,. \tag{22}$$

Observe that $\operatorname{curl} u = f = \operatorname{curl} A$ almost everywhere in U. Since we assume that U is simply connected, we find a function $r \in W^{1,p}(U)$ such that $u - A = \nabla r$. Our goal is now to obtain an estimate on r.

We define $g := -\operatorname{div}(u - A)$. Since $u \in L^p_{\mathrm{div}}(U)$, we have $g \in L^p(U)$. The function r satisfies

$$\int_U \nabla r \cdot \nabla \phi = F(\phi)\,, \tag{23}$$

for all $\phi \in W^{1,p'}(U)$, where F is the functional

$$F(\phi) := \int_U g\,\phi + \int_{\partial U} (A-u)\cdot \vec{n}\,\phi\,.$$

Using Gausses formula, we see that $F(1) = 0$. On the other hand,

$$\Big|F(\phi)\Big| \leq \|g\|_{L^p(U)}\,\|\phi\|_{L^{p'}(U)} + \|A\cdot\vec{n}\|_{L^{\frac{2p}{3-p}}(\partial U)}\,\|\phi\|_{L^{\frac{2p'}{3}}(\partial U)} + \|u\cdot\vec{n}\|_{L^\alpha(\partial U)}\,\|\phi\|_{L^{\alpha'}(\partial U)}\,. \tag{24}$$

For α and p fixed, we now consider a real number $3 > q > 1$ such that

$$\frac{2\,q}{3-q} \geq \max\left\{\alpha',\, \frac{2\,p'}{3}\right\}\,. \tag{25}$$

This choice of q ensures, on the one hand, the continuity of the embeddings

$$W^{1,q}(U) \hookrightarrow L^{\alpha'}(\partial U), \quad W^{1,q}(U) \hookrightarrow L^{\frac{2p'}{3}}(\partial U)\,.$$

On the other hand, we see that for this choice also $\frac{3\,q}{3-q} > p'$, so that the embedding $W^{1,q}(U) \hookrightarrow L^{p'}(U)$ is continuous. From (24), we then deduce that

$$\Big|F(\phi)\Big| \leq c\,(\|g\|_{L^p(U)} + \|A\|_{[W^{1,p}(U)]^3} + \|u\cdot\vec{n}\|_{L^\alpha(\partial U)})\,\|\phi\|_{W^{1,q}(U)}\,.$$

With the help of (22), it now follows that

$$\Big|F(\phi)\Big| \leq c\,(\|\operatorname{div} u\|_{L^p(U)} + \|\operatorname{curl} u\|_{[L^p(U)]^3} + \|u\cdot\vec{n}\|_{L^\alpha(\partial U)})\,\|\phi\|_{W^{1,q}(U)}\,,$$

Applying Proposition 4.5 (see the appendix), we find the existence of some $q_1 > 3$ such that for all $q \in]q_1',\, q_1[$, the solution r of (23) belongs to $W^{1,q'}(U)$. In addition, the estimate

$$\|r\|_{W^{1,q'}(U)} \leq c\,\|F\|_{[W^{1,q}(U)]^*} \leq C\,(\|\operatorname{div} u\|_{L^p(U)} + \|\operatorname{curl} u\|_{[L^p(U)]^3} + \|u\cdot\vec{n}\|_{L^\alpha(\partial U)})\,,$$

is valid. Note that q_1 is the same as in Lemma 2.3. Setting $\xi := \min\{p^*,\, q'\}$, we have

$$\begin{aligned}\|u\|_{[L^\xi(U)]^3} &\leq c\,(\|A\|_{[L^{p^*}(U)]^3} + \|\nabla r\|_{[L^{q'}(U)]^3})\\ &\leq \bar{C}\,(\|\operatorname{div} u\|_{L^p(U)} + \|\operatorname{curl} u\|_{[L^p(U)]^3} + \|u\cdot\vec{n}\|_{L^\alpha(\partial U)})\,.\end{aligned}$$

It remains to compute the optimal exponent $q' \leq q_1$ by taking into account the condition (25). We obtain that $q' := \min\left\{\frac{3\alpha}{2},\, p^*,\, q_1\right\}$, and the claim follows. If the domain U is of class $\mathcal{C}^1$, we can apply Theorem 3.3 of [Gri90] directly in U, and we do not need Lemma 2.3. By the results of [SS92], Theorem 1.4, the solution r of (23) belongs to $W^{1,q'}(U)$ for $1 < q' < \infty$. Therefore, $q_1 = +\infty$. □

2.3 Functional spaces for the problem (P).

We now want to study functional spaces more specifically needed for the analysis of the problem (P). From now on, we assume that the domain $\tilde{\Omega}$ is simply connected and has a Lipschitz boundary. Denoting by μ the magnetic permeability in $\tilde{\Omega}$, we assume through the remainder of the paper that μ is a measurable function such that

$$0 < \mu_l \le \mu(x) \le \mu_u < \infty \text{ for all } x \in \tilde{\Omega}\,, \tag{26}$$

with positive constants μ_l , μ_u.

Consider the spaces

$$V_\mu(\tilde{\Omega}) := \left\{ \psi \in [L^2(\tilde{\Omega})]^3 \,\middle|\, \operatorname{curl}\psi \in [L^2(\tilde{\Omega})]^3,\ \operatorname{div}(\mu\,\psi) \in L^2(\tilde{\Omega}),\ \gamma_n(\mu\,\psi) = 0 \text{ on } \partial\tilde{\Omega} \right\}, \tag{27}$$

$$V_{\mu,0}(\tilde{\Omega}) := \left\{ \psi \in V_\mu(\tilde{\Omega}) \,\middle|\, \operatorname{div}(\mu\,\psi) = 0 \right\} \tag{28}$$

We endow $V_\mu(\tilde{\Omega})$ with the norm of the graph

$$\|\psi\|_{V_\mu(\tilde{\Omega})} := \|\psi\|_{[L^2(\tilde{\Omega})]^3} + \|\operatorname{curl}\psi\|_{[L^2(\tilde{\Omega})]^3} + \|\operatorname{div}(\mu\,\psi)\|_{L^2(\tilde{\Omega})}\,.$$

Obviously, $V_\mu(\tilde{\Omega})$ is a Hilbert space in this topology.

In the introduction, we have emphasized the importance of additional hypotheses on the pair $(\mu,\,\tilde{\Omega})$ for embedding results concerning space of the type of $V_\mu(\tilde{\Omega})$. In this respect, an important class of domains consists of the domains $\overline{\tilde{\Omega}} = \bigcup_{i=0}^{m} \overline{\tilde{\Omega}_i}$ having the following property:

$$(A0) \quad \text{For } 0 \le i, j \le m,\ i \ne j, \text{ the boundary } \partial\tilde{\Omega}_i \cap \partial\tilde{\Omega}_j \text{ is a closed, connected surface.} \tag{29}$$

For the pair $(\mu,\,\tilde{\Omega})$ we want to discuss the following cases:

$$(A1) \begin{cases} \mu_{|\tilde{\Omega}_i} \in \mathcal{C}^1\left(\overline{\tilde{\Omega}_i}\right) & \text{for } i = 0, \dots, m, \\ \partial\tilde{\Omega}_i,\ \partial\tilde{\Omega} \in \mathcal{C}^2 & \text{for } i = 0, \dots, m. \end{cases} \qquad (A2) \begin{cases} \mu_{|\tilde{\Omega}_i} \in \mathcal{C}\left(\overline{\tilde{\Omega}_i}\right) & \text{for } i = 0, \dots, m, \\ \partial\tilde{\Omega}_i \setminus \partial\tilde{\Omega} \in \mathcal{C}^1 & \text{for } i = 0, \dots, m, \\ \partial\tilde{\Omega} \in \mathcal{C}^{0,1} & . \end{cases}$$

The main result of this section is an embedding result for the space $V_\mu(\tilde{\Omega})$. In order to complete its proof, we first need two auxiliary statements. For a real number $q \in]1, \infty[$, we recall that q' denotes the conjugated exponent to q.

Lemma 2.4. Let $\tilde{\Omega}$ be a simply connected Lipschitz domain. Let $\psi \in [L^2(\tilde{\Omega})]^3$ be given, and assume that $p \in W^{1,2}(\tilde{\Omega})$ satisfies the integral relation

$$\int_{\tilde{\Omega}} \mu\,\nabla p \cdot \nabla\phi = \int_{\tilde{\Omega}} \mu\,\psi \cdot \nabla\phi\,, \tag{30}$$

for all $\phi \in W^{1,2}(\tilde{\Omega})$.

(1) If the pair $(\mu, \tilde{\Omega})$ satisfies $(A0)$ and $(A1)$, and if $\psi \in [W^{1,2}(\tilde{\Omega})]^3$, then for $i = 0, \ldots, m$, one has

$$p \in W^{2,2}(\tilde{\Omega}_i), \quad \|p\|_{W^{2,2}(\tilde{\Omega}_i)} \le c\, \|\psi\|_{[W^{1,2}(\tilde{\Omega})]^3} .$$

(2) Suppose that $(\mu, \tilde{\Omega})$ satisfies $(A0)$ and $(A2)$. Then there exists some $q_1 > 3$ such that if $\psi \in [L^q(\tilde{\Omega})]^3$ for a $q \in]q_1', q_1[$, then

$$p \in W^{1,q}(\tilde{\Omega}), \quad \|p\|_{W^{1,q}(\tilde{\Omega})} \le \bar{c}\, \|\psi\|_{[L^q(\tilde{\Omega})]^3} .$$

If $\partial\tilde{\Omega} \in \mathcal{C}^1$, then one may take $q_1 = +\infty$.

(3) If the number $1 - \mu_l/\mu_u$ is sufficiently small, then the same as in (2) is valid without further assumption on the function μ and on the domain $\tilde{\Omega}$.

Proof. The assertion (1) was proved in the paper [LS60], Lemma 1.

(2): Obviously, the functional

$$F(\zeta) := \int_{\tilde{\Omega}} \mu\, \psi \cdot \nabla\zeta \,,$$

is a well-defined element of $[W^{1,q'}(\tilde{\Omega})]^*$. Under our geometrical assumption on the domain $\tilde{\Omega}$, the remark 3.16 in [ERS07] shows that the operator

$$\nabla \cdot (\mu\, \nabla) : W^{1,q}(\tilde{\Omega}) \longrightarrow [W^{1,q'}(\tilde{\Omega})]^* \,,$$

is a topological isomorphism. This proves the claim.

(3): In view of Lemma 4.1 (see the appendix), there exists a constant C such that if the smallness assumption $C\left(1 - \frac{\mu_l}{\mu_u}\right) < 1$ is satisfied, the assertion follows. □

We still need another auxiliary result concerning the possibility to find vector potentials in the space $V_{\mu,0}$.

Lemma 2.5. We consider a simply connected Lipschitz domain $\tilde{\Omega}$. Let $j \in L^2_{\mathrm{div}}(\tilde{\Omega})$ be such that $div\, j = 0$ in $\tilde{\Omega}$ in the generalized sense. Then we can find a vector potential $B \in V_{\mu,0}(\tilde{\Omega})$ such that

$$\operatorname{curl} B = j \,, \quad \|B\|_{V_\mu(\tilde{\Omega})} \le c\, \|j\|_{[L^2(\tilde{\Omega})]^3} . \tag{31}$$

In addition, the following results are valid:

(1) If $(A0)$ and $(A1)$ are satisfied, then for $i = 0, \ldots, m$ we have

$$B \in [W^{1,2}(\tilde{\Omega}_i)]^3 \,, \quad \|B\|_{[W^{1,2}(\tilde{\Omega}_i)]^3} \le c\, \|j\|_{[L^2(\tilde{\Omega})]^3} .$$

(2) If $(A0)$ and $(A2)$ are satisfied, or if the number $1 - \mu_l/\mu_u$ is sufficiently small, then there exists some $\tilde{\xi} > 3$ such that

$$B \in [L^{\tilde{\xi}}(\tilde{\Omega})]^3\,, \quad \|B\|_{[L^{\tilde{\xi}}(\tilde{\Omega})]^3} \leq c\,\|j\|_{[L^2(\tilde{\Omega})]^3}\,.$$

If $\partial\tilde{\Omega} \in \mathcal{C}^1$, then we can choose $\tilde{\xi} = 6$.

Proof. By Lemma I.3.6 in [GR86], we find a potential A in the space $L^2_{\text{curl}}(\tilde{\Omega}) \cap L^2_{\text{div}}(\tilde{\Omega})$ such that

$$\begin{aligned} &\operatorname{div} A = 0\,, \quad \operatorname{curl} A = j\,, \quad \text{ in } \tilde{\Omega}\,, \\ &\gamma_n(A) = 0 \quad \text{ on } \partial\tilde{\Omega}\,. \end{aligned} \tag{32}$$

In addition, there exists a positive constant C independent of j such that

$$\|A\|_{[L^2(\tilde{\Omega})]^3} \leq C\,\|j\|_{[L^2(\tilde{\Omega})]^3}\,.$$

We consider the (up to constants) unique function $p \in W^{1,2}(\tilde{\Omega})$ that satisfies

$$\int_{\tilde{\Omega}} \mu\,\nabla p \cdot \nabla\phi = \int_{\tilde{\Omega}} \mu\,A \cdot \nabla\phi$$

for all $\phi \in W^{1,2}(\tilde{\Omega})$, and we set $B := A - \nabla p$. We verify easily that $B \in V_{\mu,0}(\tilde{\Omega})$, and that $curl\,B = j$.

(1): If $(A0)$ and $(A1)$ are satisfied, then Theorem I.3.8 in [GR86] even gives that the potential A in (32) belongs to $[W^{1,2}(\tilde{\Omega})]^3$, and that

$$\|A\|_{[W^{1,2}(\tilde{\Omega})]^3} \leq \bar{c}\,\|j\|_{[L^2(\tilde{\Omega})]^3}\,.$$

Then, Lemma 2.4 implies that $p \in W^{2,2}(\tilde{\Omega}_i)$ for $i = 0, \ldots, m$, and that

$$\|p\|_{W^{2,2}(\tilde{\Omega}_i)} \leq C\,\|A\|_{[W^{1,2}(\tilde{\Omega})]^3} \leq \bar{C}\,\|j\|_{[L^2(\tilde{\Omega})]^3}\,.$$

Thus, B belongs to $[W^{1,2}(\tilde{\Omega}_i)]^3$ and satisfies the assertion.

(2): If $\partial\tilde{\Omega} \in \mathcal{C}^{0,1}$, we see that $A \in \mathcal{W}^{2,\infty}(\tilde{\Omega})$ (c. p. (20)). Proposition 2.2 then implies the existence of a number $\xi > 3$ such that

$$\|A\|_{[L^{\xi}(\tilde{\Omega})]^3} \leq \bar{c}\,\|j\|_{[L^2(\tilde{\Omega})]^3}\,.$$

If $\partial\tilde{\Omega} \in \mathcal{C}^1$, the choice $\xi = 6$ is possible. Under the hypothesis of the present lemma, it follows from Lemma 2.4 that there exists some $\tilde{\xi} \in]3, \xi]$ such that $p \in W^{1,\tilde{\xi}}(\tilde{\Omega})$, and that

$$\|p\|_{W^{1,\tilde{\xi}}(\tilde{\Omega})} \leq C\,\|A\|_{[L^{\xi}(\tilde{\Omega})]^3} \leq \hat{C}\,\|j\|_{[L^2(\tilde{\Omega})]^3}\,.$$

Therefore, $B = A + \nabla p$ belongs to $[L^{\tilde{\xi}}(\tilde{\Omega})]^3$ and satisfies the assertion. Again, if $\partial\tilde{\Omega} \in \mathcal{C}^1$, one can prove that the choice $\tilde{\xi} = 6$ is possible. $\square$

Proposition 2.6. Let $\tilde{\Omega}$ be a simply connected Lipschitz domain .

(1) If $(\mu, \tilde{\Omega})$ satisfy $(A0)$ and $(A1)$, then the topological identity

$$V_\mu(\tilde{\Omega}) = V_\mu(\tilde{\Omega}) \cap \bigcap_{i=0}^{m} [W^{1,2}(\tilde{\Omega}_i)]^3$$

is valid.

(2) If the pair $(\mu, \tilde{\Omega})$ satisfies $(A0)$ and $(A2)$, then there exists a number $\tilde{\xi} > 3$ such that $V_\mu(\tilde{\Omega}) \hookrightarrow [L^{\tilde{\xi}}(\tilde{\Omega})]^3$ with continuous embedding. If $\partial\tilde{\Omega} \in \mathcal{C}^1$, then one can choose $\tilde{\xi} = 6$.

(3) If the number $1 - \mu_l/\mu_u$ is sufficiently small, then the same as in (2) is valid, without further assumption on the function μ and on the domain $\tilde{\Omega}$.

Proof. We consider an arbitrary $\psi \in V_\mu(\tilde{\Omega})$. Since $curl\, \psi$ is divergence-free almost everywhere in $\tilde{\Omega}$, we find by Lemma 2.5 a $B \in V_{\mu,0}(\tilde{\Omega})$ such that

$$\operatorname{curl} B = \operatorname{curl} \psi \quad \text{ in } \tilde{\Omega}\,.$$

Since $\tilde{\Omega}$ is simply connected, we conclude from the fact that $curl\,(\psi - B) = 0$ that

$$\psi = B + \nabla p\,, \tag{33}$$

for some $p \in W^{1,2}(\tilde{\Omega})$. The function p is a weak solution to the transmission problem

$$\int_{\tilde{\Omega}} \mu \nabla p \cdot \nabla \phi = -\int_{\tilde{\Omega}} \operatorname{div}(\mu\, \psi)\, \phi + \sum_{i=0}^{m} \int_{\partial\tilde{\Omega}_i} \left[\mu\, \psi \cdot \vec{n}\right] \phi\,,$$

for all $\phi \in W^{1,2}(\tilde{\Omega})$. (1): Suppose that $(A0)$ and $(A1)$ are satisfied. Then it is shown in [LS60], Lemma 1, that $p \in W^{2,2}(\tilde{\Omega}_i)$ for $i = 0, \ldots, m$, and that

$$\|p\|_{W^{2,2}(\tilde{\Omega}_i)} \le c\, \|\operatorname{div}(\mu\, \psi)\|_{L^2(\tilde{\Omega})}\,.$$

Since by Lemma 2.5, we know that $B \in [W^{1,2}(\tilde{\Omega}_i)]^3$ for $i = 0, \ldots, m$, we obtain from (33) the norm estimate

$$\|\psi\|_{[W^{1,2}(\tilde{\Omega}_i)]^3} \le \|p\|_{W^{2,2}(\tilde{\Omega}_i)} + \|B\|_{[W^{1,2}(\tilde{\Omega}_i)]^3} \le c\,(\|\operatorname{div}(\mu\, \psi)\|_{L^2(\tilde{\Omega})} + \|\operatorname{curl} \psi\|_{[L^2(\tilde{\Omega})]^3})\,.$$

(2): If $(A0)$ and $(A2)$ are satisfied, resp. if the number $1 - \mu_l/\mu_u$ is sufficiently small, we can use the arguments of [ERS07], resp. of the appendix, to obtain the existence of some $\tilde{\xi} > 3$ such that $p \in W^{1,\tilde{\xi}}(\tilde{\Omega})$. In addition, we find the norm estimate

$$\|p\|_{W^{1,\tilde{\xi}}(\tilde{\Omega})} \le c\, \|\operatorname{div}(\mu\, \psi)\|_{L^2(\tilde{\Omega})}\,.$$

Using Lemma 2.5 and (33), the claim follows. $\square$

Finally, we introduce functional spaces that will help us to deal with the constraint on *curl* H in (P).

We introduce the space

$$\mathcal{H}(\tilde{\Omega}) := \left\{ H \in L^2_{\text{curl}}(\tilde{\Omega}) \middle| \, \text{curl}\, H = 0 \text{ in } \tilde{\Omega} \setminus \tilde{\Omega}_c \right\}.$$

Naturally, $\mathcal{H}(\tilde{\Omega})$ is a Hilbert space with respect to the topology of $L^2_{\text{curl}}(\tilde{\Omega})$. We also need the space of homogenized fields

$$\mathcal{H}^0(\tilde{\Omega}) := \left\{ H \in L^2_{\text{curl}}(\tilde{\Omega}) \middle| \, \text{curl}\, H = 0 \text{ in } \tilde{\Omega} \setminus \Omega_c \right\}.$$

In order to satisfy the divergence constraint, we introduce the space

$$\mathcal{H}_\mu(\tilde{\Omega}) := \left\{ \psi \in \mathcal{H}(\tilde{\Omega}) \middle| \, \text{div}(\mu\, \psi) = 0 \;\text{ in } \tilde{\Omega}\,, \quad \mu\, \psi \cdot \vec{n} = 0 \;\text{ on } \partial\tilde{\Omega} \right\},$$

and

$$\mathcal{H}^0_\mu(\tilde{\Omega}) := \left\{ \psi \in \mathcal{H}^0(\tilde{\Omega}) \middle| \, \text{div}(\mu\, \psi) = 0 \;\text{ in } \tilde{\Omega}\,, \quad \mu\, \psi \cdot \vec{n} = 0 \;\text{ on } \partial\tilde{\Omega} \right\}.$$

Naturally, the space $\mathcal{H}_\mu(\tilde{\Omega})$ is a closed subspace of $V_\mu(\tilde{\Omega})$.

Lemma 2.7. Let the assumption (26) on the function μ be satisfied. Then we have:

(1) The embedding $\mathcal{H}_\mu(\tilde{\Omega}) \hookrightarrow [L^2(\tilde{\Omega})]^3$ is compact.

(2) There exists a constant $C > 0$ such that for all $\psi \in \mathcal{H}_\mu(\tilde{\Omega})$,

$$\int_{\tilde{\Omega}} |\,\text{curl}\, \psi|^2 \geq C\, \|\psi\|^2_{V_\mu(\tilde{\Omega})}\,.$$

Proof. (1): This is only a special case of more general results (see [PM99]).

(2): In view of (1), we use the usual contradiction argument to prove the existence of a constant $c > 0$ such that for all $\psi \in \mathcal{H}_\mu(\tilde{\Omega})$, $\|\psi\|^2_{[L^2(\tilde{\Omega})]^3} \leq c \int_{\tilde{\Omega}} |\,\text{curl}\, \psi|^2$. □

3 Solution of (P).

3.1 Definition of a weak solution.

We first make precise the assumptions on the coefficients and data under which we intend to solve (P).

For $i = 0, \ldots, m$, we denote by σ_i the electrical conductivity of the medium $\tilde{\Omega}_i$. We assume that σ_i is a Lebesgue measurable function. We additionally assume that there exist positive constants σ_l , σ_u such that for $i \in I_c$,

$$0 < \sigma_l \leq \sigma_i(x) \leq \sigma_u < \infty \quad \text{ for almost all } x \in \tilde{\Omega}_i\,. \tag{34}$$

We recall that (26) is assumed to hold. For notational commodity, we introduce the auxiliary function $r : \tilde{\Omega} \to \mathbb{R}$ defined by

$$r := \begin{cases} \frac{1}{\sigma} & \text{in } \tilde{\Omega}_c\,, \\ 1 & \text{elsewhere.} \end{cases} \tag{35}$$

Since modeling the current source is not our main concern, we suppose that the current is imposed in the conductor $\tilde{\Omega}_{c_0}$ and that the given current density j_0 satisfies

$$\operatorname{div} j_0 = 0 \quad \text{in } \tilde{\Omega}_{c_0}, \qquad j_0 \cdot \vec{n} = 0 \quad \text{on } \partial\Omega_{c_0}\,, \tag{36}$$

which ensures the conservation of charge. Since we want to allow for the motion of some of the conductors, we suppose that a velocity vector $v : \Omega_c \longrightarrow \mathbb{R}^3$ is given. We denote by $\bar{v}$ the extension by zero of v to $\tilde{\Omega}$. In order to derive a variational formulation of (P), we start from equation (5):

$$\operatorname{curl} H = \sigma\,(E + v \times B) \quad \text{in } \Omega_c\,.$$

For an arbitrary $\psi \in \mathcal{H}^0(\tilde{\Omega})$, we then deduce that

$$\int_{\Omega_c} \frac{1}{\sigma} \operatorname{curl} H \cdot \operatorname{curl} \psi = \int_{\Omega_c} E \cdot \operatorname{curl} \psi + \int_{\Omega_c} (v \times B) \cdot \operatorname{curl} \psi\,.$$

Since $\psi \in \mathcal{H}^0(\tilde{\Omega})$, we have in view of (7) and of (11) that

$$\int_{\Omega_c} E \cdot \operatorname{curl} \psi = \int_{\tilde{\Omega}} E \cdot \operatorname{curl} \psi = \int_{\tilde{\Omega}} \operatorname{curl} E \cdot \psi + \int_{\partial\tilde{\Omega}} (E \times \psi) \cdot \vec{n} = 0\,.$$

Therefore, we can write for all $\psi \in \mathcal{H}^0(\tilde{\Omega})$ the integral identity

$$\int_{\tilde{\Omega}} r \operatorname{curl} H \cdot \operatorname{curl} \psi = \int_{\tilde{\Omega}} (\bar{v} \times B) \cdot \operatorname{curl} \psi\,, \tag{37}$$

which is in particular true for all $\psi \in \mathcal{H}^0_\mu(\tilde{\Omega})$.

Definition 3.1. Assume that the conditions (34), (26) and (36) on σ, μ and j_0 are satisfied. We call a vector field $H \in \mathcal{H}_\mu(\tilde{\Omega})$ a *weak solution* to the problem (P) if $\mathit{curl}\, H = j_0$ almost everywhere in the set $\tilde{\Omega}_{c_0}$, and if the relation

$$\int_{\tilde{\Omega}} (r \operatorname{curl} H - \bar{v} \times \mu\, H) \cdot \operatorname{curl} \psi = 0 \tag{38}$$

is satisfied for all $\psi \in \mathcal{H}^0_\mu(\tilde{\Omega})$.

Remark 3.2. We briefly show in which sense Definition 3.1 states an equivalent formulation of the problem (P). Assume that $H \in \mathcal{H}_\mu(\tilde{\Omega})$ satisfies (38) for all $\psi \in \mathcal{H}^0_\mu(\tilde{\Omega})$. Then, the integral relation

$$\int_{\tilde{\Omega}} (r \operatorname{curl} H - \bar{v} \times \mu\, H) \cdot \operatorname{curl} \psi = 0\,, \tag{39}$$

is even satisfied for $\psi \in \mathcal{H}^0(\tilde{\Omega})$. (As a matter of fact, to every $\psi \in \mathcal{H}^0(\tilde{\Omega})$, we can find a function $p \in W^{1,2}(\tilde{\Omega})$ such that $\psi - \nabla p$ belongs to $\mathcal{H}_\mu(\tilde{\Omega})$.) In particular, we can choose $\psi \in [C_c^\infty(\Omega_c)]^3$, and obtain

$$\int_{\Omega_c} (r \,\mathrm{curl}\, H - v \times \mu\, H) \cdot \mathrm{curl}\, \psi = 0\,.$$

This means nothing else but

$$\mathrm{curl}\,(r \,\mathrm{curl}\, H - v \times \mu\, H) = 0 \quad \text{in the sense of the generalized curl operator in } \Omega_c.$$

Define the electric field in Ω_c by

$$E_{\Omega_c} := r \,\mathrm{curl}\, H - v \times \mu\, H \quad \text{in } \Omega_c\,.$$

In $\tilde{\Omega}_{c_0}$, consistency with Ohm's law requires that the electric field $E_{\tilde{\Omega}_{c_0}}$ satisfies $E_{\tilde{\Omega}_{c_0}} = j_0/\sigma$. Thus, the electric field

$$E_c := \begin{cases} E_{\Omega_c} & \text{in } \Omega_c\,, \\ E_{\tilde{\Omega}_{c_0}} & \text{in } \tilde{\Omega}_{c_0}\,, \end{cases}$$

is known in the electrical conductors of the system. Then, the electric field in $\tilde{\Omega} \setminus \tilde{\Omega}_c$ is determined as the solution of the problem

$$\begin{aligned} &\mathrm{curl}\, E = 0, \quad \mathrm{div}\, D = 0 \qquad \text{in } \tilde{\Omega} \setminus \tilde{\Omega}_c \\ &E \times \vec{n} = E_c \times \vec{n} \quad \text{on } \partial\tilde{\Omega} \setminus \tilde{\Omega}_c\,, \quad E \times \vec{n} = 0 \quad \text{on } \partial\tilde{\Omega}\,. \end{aligned}$$

□

3.2 An existence result with higher integrability of the Lorentz force.

Proposition 3.3. Let $\tilde{\Omega} \in \mathcal{C}^{0,1}$ be a simply connected domain. Let $j_0 \in [L^2(\tilde{\Omega}_{c_0})]^3$ satisfy (36), let σ satisfy (34), and let $v \in [L^\infty(\Omega_c)]^3$. Assume that μ satisfies (26).

Then, the problem (P) possesses at least one weak solution H. In addition, we have:

(1) If $(\mu, \tilde{\Omega})$ satisfies $(A0)$ and $(A1)$, then $\mathit{curl}\, H \times B \in [L^{3/2}(\tilde{\Omega})]^3$.

(2) If $(\mu, \tilde{\Omega})$ satisfies $(A0)$ and $(A2)$, then $\mathit{curl}\, H \times B \in [L^r(\tilde{\Omega})]^3$ for some $r > 6/5$.

(3) If the number $1 - \mu_l/\mu_u$ is sufficiently small, then the result (2) holds true without further assumptions on $(\mu, \tilde{\Omega})$.

Proof. First, we can homogenize the problem (P) by finding a field $H_0 \in \mathcal{H}_\mu(\tilde{\Omega})$ such that

$$\operatorname{curl} H_0 = j_0 \quad \text{in } \tilde{\Omega}\,.$$

Given $j_0 \in [L^2(\tilde{\Omega}_{c_0})]^3$ having the property (36), the existence of H_0 follows from Lemma 2.5. We then reformulate (P) as problem for the *reaction field* $\tilde{H} = H - H_0$. If H is a weak solution of (P), then $\tilde{H} \in \mathcal{H}^0_\mu(\tilde{\Omega})$, and the integral relation

$$\begin{aligned}\int_{\tilde{\Omega}} r \,\operatorname{curl} \tilde{H} \cdot \operatorname{curl} \psi - \int_{\tilde{\Omega}} (\bar{v} \times \mu\, \tilde{H}) \cdot \operatorname{curl} \psi \\ = - \int_{\tilde{\Omega}} r \,\operatorname{curl} H_0 \cdot \operatorname{curl} \psi + \int_{\tilde{\Omega}} (\bar{v} \times \mu\, H_0) \cdot \operatorname{curl} \psi \end{aligned} \tag{40}$$

is satisfied for all $\psi \in \mathcal{H}^0_\mu(\tilde{\Omega})$. Using Young's inequality, we find that

$$\left| \int_{\tilde{\Omega}} \left(\bar{v} \times \mu\, \tilde{H}\right) \cdot \operatorname{curl} \tilde{H} \right| \le \frac{1}{2\,\sigma_u} \int_{\tilde{\Omega}} |\operatorname{curl} \tilde{H}|^2 + 2\,\mu_1^2\,\sigma_u \|v\|^2_{[L^\infty(\Omega_c)]^3}\, \|\tilde{H}\|^2_{[L^2(\tilde{\Omega})]^3}\,,$$

for all $\tilde{H} \in \mathcal{H}^0_\mu(\tilde{\Omega})$. Thus,

$$\begin{aligned}\int_{\tilde{\Omega}} r\, |\operatorname{curl} \tilde{H}|^2 - \int_{\tilde{\Omega}} \left(\bar{v} \times \mu\, \tilde{H}\right) \cdot \operatorname{curl} \tilde{H} \\ \ge \frac{1}{2\,\sigma_u} \int_{\tilde{\Omega}} |\operatorname{curl} \tilde{H}|^2 - 2\,\mu_1^2\,\sigma_u\, \|v\|^2_{[L^\infty(\Omega_c)]^3}\, \|\tilde{H}\|^2_{[L^2(\tilde{\Omega})]^3}\end{aligned}$$

Since according to Lemma 2.7, (1), the injection $\mathcal{H}^0_\mu(\tilde{\Omega}) \hookrightarrow [L^2(\tilde{\Omega})]^3$ is compact, we find by the generalized Lax-Milgram theorem the existence of a $\tilde{H} \in \mathcal{H}^0_\mu(\tilde{\Omega})$ such that (40) holds for all $\psi \in \mathcal{H}^0_\mu(\tilde{\Omega})$. The additional claims follow from the properties stated in Proposition 2.6. ☐

4 Appendix

In this appendix, we prove the auxiliary result needed for the proof of Proposition 2.6, (3).

Let $U \subset \mathbb{R}^3$ be a simply connected Lipschitz domain. Let $1 < q < \infty$ and $g \in [L^q(U)]^3$ be given. We assume that the function $\mu : U \to \mathbb{R}$ is measurable and satisfies (c. p. (26))

$$0 < \mu_l \le \mu(x) \le \mu_u < \infty \text{ for all } x \in U\,. \tag{41}$$

We consider the problem of finding $p \in W^{1,q}(U)$ that satisfies the integral relation

$$\int_U \mu\, \nabla p \cdot \nabla \phi = \int_U g \cdot \nabla \phi\,, \tag{42}$$

for all $\phi \in W^{1,q'}(U)$.

Lemma 4.1. Assume that the domain U is Lipschitzian, and let $1 < q < \infty$. Let μ be a measurable function that satisfies (41). Then, there exists some $q_1 > 3$ such that for all $q \in]q_1', q_1[$, we can find a positive constant $C = C(U, q)$, so that if in addition the assumption

$$C\left(1 - \frac{\mu_l}{\mu_u}\right) < 1 \tag{43}$$

is satisfied, then the problem (42) possesses a (up to constants) unique solution $p \in W^{1,q}(U)$, and the estimate

$$\left(1 - C\left(1 - \frac{\mu_l}{\mu_u}\right)\right) \|\nabla p\|_{[L^q(U)]^3} \le \|g\|_{[L^q(U)]^3}$$

is valid.

Proof. In view of Proposition 4.5 below, there exists some $q_1 > 3$ such that for all $q \in]q_1', q_1[$ and $u \in [L^q(U)]^3$, there exists a (up to constants) unique $\zeta \in W^{1,q}(U)$ such that

$$\int_U \nabla\zeta \cdot \nabla\phi = \int_U u \cdot \nabla\phi\,,$$

for all $\phi \in W^{1,q'}(U)$. In addition, the solution ζ satisfies a continuous estimate

$$\|\nabla\zeta\|_{[L^q(U)]^3} \le C\,\|u\|_{[L^q(U)]^3}\,.$$

For $w \in W^{1,q}(U)$ arbitrary, we can thus find a unique $\zeta \in W^{1,q}(U)$ such that

$$\int_U \nabla\zeta \cdot \nabla\phi = \int_U \left(1 - \frac{\mu}{\mu_u}\right) \nabla w \cdot \nabla\phi + \int_U \frac{1}{\mu_u}\, g \cdot \nabla\phi\,, \tag{44}$$

for all $\phi \in W^{1,q'}(U)$. In addition, in view of Propostion 4.5, we find the norm estimate

$$\|\nabla\zeta\|_{W^{1,q}(U)} \le C\left(1 - \frac{\mu_l}{\mu_u}\right) \|\nabla w\|_{[L^q(U)]^3} + \frac{1}{\mu_u}\, C\,\|g\|_{[L^q(U)]^3}\,. \tag{45}$$

We define the space $W^{1,q}_M(U)$ as the closed subspace of $W^{1,q}(U)$ that contains the functions with vanishing mean value over U. This space is a Banach space with respect to the norm $\|u\|_{W^{1,q}_M(U)} := \|\nabla u\|_{[L^q(U)]^3}$. We define a mapping $\mathcal{T} : W^{1,q}_M(U) \to W^{1,q}_M(U)$ by setting $\mathcal{T}(w) := \zeta$, where ζ satisfies (44). From existence and unicity of ζ, we conclude that $\mathcal{T}$ is well defined. We prove easily that $\mathcal{T}$ is strictly contractive. As a matter of fact, we can write

$$\begin{aligned} \|\mathcal{T}(w_1) - \mathcal{T}(w_2)\|_{W^{1,q}_M(U)} &\le C \sup_{\|\nabla\phi\|_{[L^{q'}(U)]^3} \le 1} \left| \int_U \left(1 - \frac{\mu}{\mu_u}\right) \nabla(w_1 - w_2) \cdot \nabla\phi \right| \\ &\le C\left(1 - \frac{\mu_l}{\mu_u}\right) \|w_1 - w_2\|_{W^{1,q}_M(U)}\,. \end{aligned} \tag{46}$$

Now, the Banach fixed point theorem proves the existence of a unique fixed point of $\mathcal{T}$. In view of (44), this fixed point is the unique solution of (42). □

In order to interpret the assumption (43), it would be interesting to know the exact dependence both on the domain U and on the exponent q of the constant C appearing in Lemma 4.1. Still having at this time to restrict ourselves to qualitative considerations, we can make the statement somewhat more precise. To this aim, we first recall some well-known notions. Let $U \subset \mathbb{R}^3$ be a bounded domain. We define

$$\mathcal{D}(U) := \left\{ \phi \in [C_c^\infty(U)]^3 \,\middle|\, \operatorname{div}\phi = 0 \text{ in } U \right\}.$$

For $1 < q < \infty$, we introduce closed subspaces $H_q(U)$, $G_q(U)$ of $[L^q(U)]^3$ defined by

$$H_q(U) := \text{ closure of } \mathcal{D}(U) \text{ in the norm } \|\cdot\|_{[L^q(U)]^3},$$

$$G_q(U) := \left\{ \phi \in [L^q(U)]^3 \,\middle|\, \phi = \nabla\zeta,\, \zeta \in W^{1,q}(U) \right\}.$$

Definition 4.2. If the decomposition

$$[L^q(U)]^3 = H_q(U) \oplus G_q(U), \tag{47}$$

is valid, it is called *Helmholtz-Weyl decomposition* of the space $[L^q(U)]^3$.

Lemma 4.3. Let the assumptions of Lemma 4.1 be satisfied. Then, for all $q \in]q_1', q_1[$, the Helmholtz-Weyl decomposition of $[L^q(U)]^3$ is valid, and the smallest constant C for which (43) holds is given by $C = \|P_{Gq}\|_{\mathcal{L}([L^q(U)]^3,\,[L^q(U)]^3)}$, where P_{Gq} is the projection onto the space $G_q(U)$.

Proof. The validity of (47) in the range $]q_1', q_1[$ follows from the equivalent characterization of the Helmholtz-Weyl decomposition recalled in Lemma 4.4 below, and of Theorem 1.6 in [Zan00].

If the decomposition (47) is valid in $[L^q(U)]^3$, then one can show that for all $p \in W^{1,q}(U)$

$$\|\nabla p\|_{[L^q(U)]^3} \leq \|P_{Gq}\|_{\mathcal{L}([L^q(U)]^3,\,[L^q(U)]^3)} \sup_{\|\nabla\phi\|_{[L^{q'}(U)]^3} \leq 1} \left| \int_U \nabla p \cdot \nabla\phi \right|.$$

This was proved for example in [SS92], Th. 1.3., Th. 6.1. Applying this result to estimate (46), the claim follows. □

The following equivalent characterization is well known.

Lemma 4.4. The following statements are equivalent.

(1) The Helmholtz-Weyl decomposition of the space $[L^q(U)]^3$ is valid.

(2) For $u \in [L^q(U)]^3$, there exists an (up to constants) unique $\zeta \in W^{1,q}(U)$ such that

$$\int_U \nabla\zeta \cdot \nabla\phi = \int_U u \cdot \nabla\phi,$$

for all $\phi \in W^{1,q'}(U)$.

Proof. See [Gal94], III. 1, Lemma 1.2, or [SS92], Th. 6.1. □

Assume that $U \subset \mathbb{R}^3$ is a bounded Lipschitz domain. For numbers $1 < q < \infty$, we denote by $q' := q/(q-1)$ the conjugated exponent. For some linear functional $F \in [W^{1,q'}(U)]^*$ such that $F(1) = 0$, we consider the variational problem to find a function $w \in W^{1,q}(\Omega)$ such that

$$\int_U \nabla w \cdot \nabla \phi = F(\phi)\,, \tag{48}$$

for all $\phi \in W^{1,q'}(U)]^*$. Thanks to the results of the paper [Zan00], we can state a very general result on the solvability of (48).

Proposition 4.5. Assume that $U \subset \mathbb{R}^3$ is a bounded Lipschitz domain. Then, there exists some number $3 < q_1 < \infty$, such that for all $q \in]q_1', q_1[$, the problem (48) possesses an up to a constant unique weak solution $w \in W^{1,q}(U)$ whenever the right-hand side F belongs to $[W^{1,q'}(U)]^*$ and satisfies $F(1) = 0$. In addition, the estimate

$$\|w\|_{W^{1,q}(U)} \leq C\,\|F\|_{[W^{1,q'}(U)]^*}\,,$$

is valid.

Proof. We apply Theorem 1.6 in [Zan00] with $\alpha = 1$ therein. □

Acknowledgment

We would like to thank Prof. J. Naumann for having called our attention to the fixed point argument in the proof of Lemma 4.1.

References

[ABDG98] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault. Vector potentials in three-dimensional non-smooth domains. *Math. Meth. Appl. Sci.*, 21:823–864, 1998.

[Bos04] A. Bossavit. *Electromagnétisme en vue de la modélisation.* Springer, Berlin, Heidelberg, New York, 2004.

[DL72] G. Duvaut and J.-L. Lions. Inéquations en thermoélasticité et magnétohydrodynamique. *Archs ration. Mech. Analysis*, 46:241–279, 1972.

[DL76] G. Duvaut and J.-L. Lions. *Inequalities in Mechanics and Physics.* Springer, Berlin, 1976.

[ERS07] J. Elschner, J. Rehberg, and G. Schmidt. Optimal regularity for elliptic transmission problems including C^1 interfaces. *Interfaces Free Bound.*, 9:233–252, 2007.

[Gal94] G.-P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations.* Springer, New York, 1994.

[GR86] V. Girault and P.-A. Raviart. *Finite element Methods for Navier-Stokes Equations.* Springer Verlag, Berlin, Heidelberg, 1986.

[Gri90] R. Griessinger. The boundary value problem rot $u = f$, u vanishing at the boundary and the related decomposition of L^q and $H_0^{1,q}$. *Ann. Univ. Ferrara - Sez. VII - Sc. Mat.*, 26:15–43, 1990.

[LS60] O.A. Ladyzhenskaja and V.A. Solonnikov. Solutions of some non-stationary problems of magnetohydrodynamics for a viscous incompressible fluid. *Trudy Mat. Inst. Steklov*, 59:115–173, 1960.

[Mon03] P. Monk. *Finite element methods for Maxwell's equations.* Clarendon press, Oxford, 2003.

[MS96] A.J. Meir and P.G. Schmidt. Variational methods for stationary MHD flow under natural interface conditions. *Nonlinear Analysis. Theory, Methods and Applications*, 26:659–689, 1996.

[MS99] A.J. Meir and P.G. Schmidt. Analysis and numerical approximation of a stationary MHD flow problem with nonideal boundary. *SIAM J. Numer. Anal.*, 36:1304–1332, 1999.

[Pic84] R. Picard. An elementary proof for a compact embedding result in generalized electromagnetic theory. *Math. Z.*, 187:151–164, 1984.

[PM99] R. Picard and A. Milani. Decomposition theorems and their application to non-linear electro- and magneto-static boundary value problems. In *Partial differential equations and calculus of variations*, volume 1357 of *Lect. Notes Math.*, pages 317–340. Springer, 1999.

[SS92] C.G. Simader and H. Sohr. The Helmholtz decomposition in L^q and related topics. In *Mathematical Problems Related to the Navier-Stokes Equations*, volume 11 of *Advances in Mathematics for Applied Sciences*, pages 1–35. World Scientific, 1992.

[ST83] M. Sermange and R. Temam. Some mathematical questions related to the MHD equations. *Archs ration. Mech. Analysis*, 36:635–664, 1983.

[vW92] W. von Wahl. Estimating ∇u by div u and curl u. *Math. Meth. Appl. Sci.*, 15:123–143, 1992.

[Zan00] Daniel D. Zanger. The inhomogeneous Neumann problem in Lipschitz domains. *Commun. in Partial Differential Equations*, 25:1771–1808, 2000.