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Aging and quenched localization for one-dimensional random walks in random environment in the sub-ballistic regime

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Abstract

We consider transient one-dimensional random walks in random environment with zero asymptotic speed. An aging phenomenon involving the generalized Arcsine law is proved using the localization of the walk at the foot of "valleysöf height $\log t$. In the quenched setting, we also sharply estimate the distribution of the walk at time t.

1 Introduction

One-dimensional random walks in random environment have been the subject of constant interest in physics and mathematics for the last thirty years since they naturally appear in a great variety of situations in physics and biology.

In 1975, Solomon gives, in a seminal work [26], a criterion of transience-recurrence for such walks moving to the nearest neighbours, and shows that three different regimes can be distinguished: the random walk may be recurrent, or transient with a positive asymptotic speed, but it may also be transient with zero asymptotic speed. This last regime, which does not exist among usual random walks, is probably the one which is the less well understood and its study is the purpose of the present paper.

Let us first remind the main existing results concerning the other regimes. In his paper, Solomon computes the asymptotic speed of transient regimes. In 1982, Sinai states, in [25], a limit theorem in the recurrent case. It turns out that the motion in this case is unusually slow. Namely, the position of the walk at time n has to be normalized by $(\log n)^2$ in order to present a non trivial limit. In 1986, the limiting law is characterized independently by Kesten [22] and Golosov [19]. Let us notice here that, beyond the interest of his result, Sinai introduces a very powerful and intuitive tool in the study of one-dimensional random walks in random environment. This tool is the potential, which is a function on $\mathbb{Z}$ canonically associated to the random environment. It turns out to be an usual random walk when the transition probabilities at each site are independent and identically distributed (i.i.d.).

The proof by Sinai of an annealed limit law in the recurrent case is based on a quenched localization result. Namely, a notion of valley of the potential is introduced, as well as an order on the set of valleys. It is then proved that the walk is localized at time t, with a probability converging to 1, around the bottom of the smallest valley of depth bigger than $\log t$ surrounding the origin. An annealed convergence in law of this site normalized by $(\log t)^2$ implies the annealed limiting law for the walk.

In the case of transient random walks in random environment with zero asymptotic speed, the proof of the limiting law by Kesten, Kozlov and Spitzer [23] does not follow this scheme. Therefore an analogous result to Sinai's localization in the quenched setting was missing. As we will see, the answer to this question is more complicated than in the recurrent case but still very explicit.

In the setting of sub-ballistic transient random walks, the valleys we introduce are, like in [15] and [24], related to the excursions of the potential above its past minimum. Now, the key observation is that with a probability converging to 1, the particle relies at time t at the foot of a valley having depth and width of order $\log t$. Therefore, since the walk spends a random time of order t inside a valley of depth $\log t$, it is not surprising that this random walk exhibits an aging phenomenon.

What is usually called aging is a dynamical out-of-equilibrium physical phenomenon observed in disordered systems like spin-glasses at low temperature, defined by the existence of a limit of a given two-time correlation function of the system as both times diverge keeping a fixed ratio between them; the limit should be a non-trivial function of the ratio. It has been extensively studied in the physics literature, see [9] and therein references.

More precisely, in our setting, Theorem 2.1 expresses, for each given ratio $h > 1$, the probability that the particle remains confined within the same valley during the time interval $[t, th]$. This probability is expressed in terms of the generalized Arcsine law, which confirms the status of universality ascribed to this law by Ben Arous and Černý in their study of aging phenomena arising in trap models [4].

Let us remind that the trap model is a model of random walk that was first proposed by Bouchaud and Dean [8, 10] as a toy model for studying this aging phenomenon. In the mathematics litterature, much attention has recently been given to the trap model, and many aging result were derived from it, on $\mathbb{Z}$ in [17] and [3], on $\mathbb{Z}^2$ in [7], on $\mathbb{Z}^d$ ($d \geq 3$) in [5], or on the hypercube in [1, 2]. A comprehensive approach to obtaining aging results for the trap model in various settings was later developed in [6].

Let us finally mention that Theorem 2.1 generalizes the aging result obtained by heuristical methods of renormalization by Le Doussal, Fisher and Monthus in [13] in the limit case when the bias of the random walk defining the potential tends to 0 (the case when this bias is 0 corresponding to the recurrent regime for the random walk in random environment). The recurrent case leading also to an aging phenomenon was treated in the same article and rigorous arguments were later presented by Dembo, Guionnet and Zeitouni in [12].

The second aspect of our work concerns localization properties of the walk and can be considered as the analog of Sinai's localization result in the transient setting. Unlike the recurrent case, the random walk is not localized near the bottom of a single valley. Nevertheless, if one introduces a confidence threshold α, one can say that, asymptotically, at time t, with a probability converging to 1 on the environment, the walk is localized with probability bigger than α around the bottoms of a finite number of valleys having depth of order $\log t$. This number depends on t and on the environment, but is not converging to infinity with t. Moreover, in Theorem 2.3 and Corollary 2.4 we sharply estimate the probability for the walk of being at time t in each of these valleys.

2 Notations and main results

Let $\omega := (\omega_i,\, i \in \mathbb{Z})$ be a family of i.i.d. random variables taking values in $(0,1)$ defined on Ω, which stands for the random environment. Denote by P the distribution of ω and by E the corresponding expectation. Conditioning on ω (i.e. choosing an environment), we define the random walk in random environment $X = (X_n,\, n \geq 0)$ on $\mathbb{Z}^{\mathbb{N}}$ as a nearest-neighbor random walk on $\mathbb{Z}$ with transition probabilities given by ω: $(X_n,\, n \geq 0)$ is the Markov chain satisfying $X_0 = 0$ and for $n \geq 0$,

$$\begin{aligned} P_\omega\left(X_{n+1} = x+1 \,|\, X_n = x\right) &= \omega_x, \\ P_\omega\left(X_{n+1} = x-1 \,|\, X_n = x\right) &= 1-\omega_x. \end{aligned}$$

We denote by P_ω the law of $(X_n,\, n \geq 0)$ and E_ω the corresponding expectation. We denote by $\mathbb{P}$ the joint law of $(\omega, (X_n)_{n\geq 0})$. We refer to Zeitouni [27] for an overview of results on random walks in random environment. Let us introduce

$$\rho_i := \frac{1-\omega_i}{\omega_i}, \qquad i \in \mathbb{Z}.$$

Our first main result is the following theorem which shows aging phenomenon in the transient sub-ballistic regime.

Theorem 2.1. *Let $\omega := (\omega_i,\, i \in \mathbb{Z})$ be a family of independent and identically distributed random variables such that*

(a) there exists $0 < \kappa < 1$ for which $E\left[\rho_0^\kappa\right] = 1$ and $E\left[\rho_0^\kappa \log^+ \rho_0\right] < \infty$,

(b) the distribution of $\log \rho_0$ is non-lattice.

Then, for all $h > 1$ and all $\eta > 0$, we have

$$\lim_{t\to\infty} \mathbb{P}(|X_{th} - X_t| \leq \eta \log t) = \frac{\sin(\kappa\pi)}{\pi} \int_0^{1/h} y^{\kappa-1}(1-y)^{-\kappa}\,\mathrm{d}y.$$

Let us make some comments about the concentration of the particle inside a region of size $\eta \log t$ in Theorem 2.1. Let us first mention that a convergence of the processes $(n^{-\kappa}X_{nt};\, t \geq 0)$ towards the inverse of a stable subordinator of index κ, when n goes to infinity, is proved in [15]. Conjugating this result with standard facts about the jumps of a stable subordinator, one can get a weaker version of Theorem 2.1, where the term $\eta \log t$ is replaced by ηt^κ. As we will see, the proof of a confinement inside a region of order $\log t$ which corresponds to the width of the trapping valley at time t, requires a finer analysis. Finally, in the trap models considered in [6], the confinement occurs on a single attracting site, but this comes from the nature of this model, and in our setting the role of the attracting site of the trap model is played by the attracting valley.

Let us now remind some basic result about X_n : under the same assumptions (a)-(b), Kesten, Kozlov and Spitzer [23] proved that X_n/n^κ converges in law to $C(\frac{1}{\mathcal{S}_\kappa^{ca}})^\kappa$ where

C is a positive parameter and $\mathcal{S}_\kappa^{ca}$ is the normalized positive stable law of index κ, i.e. with Laplace transform

$$E[\mathrm{e}^{-\lambda \mathcal{S}_\kappa^{ca}}] = \mathrm{e}^{-\lambda^\kappa}, \qquad \forall \lambda > 0.$$

In [14, 15] we gave a different proof of this result and we were able to give an explicit expression for the constant C.

The proof was based on a precise analysis of the potential associated with the environment, as it was defined by Sinai for its analysis of the recurrent case, see [25]. In this paper, we use the technics developed in [14, 15] to prove Theorem 2.1. This potential, denoted by $V = (V(x),\ x \in \mathbb{Z})$, is a function of the environment ω. It is defined as follows:

$$V(x) := \begin{cases} \sum_{i=1}^{x} \log \rho_i & \text{if } x \geq 1, \\ 0 & \text{if } x = 0, \\ -\sum_{i=x+1}^{0} \log \rho_i & \text{if } x \leq -1. \end{cases}$$

Furthermore, we consider the weak descending ladder epochs for the potential defined by $e_0 := 0$ and

$$e_i := \inf\{k > e_{i-1} :\ V(k) \leq V(e_{i-1})\}, \qquad i \geq 1,$$

which play a crucial role in our proof. Observe that the sequence $(e_i - e_{i-1})_{i\geq 1}$ is a family of i.i.d. random variables. Moreover, classical results of fluctuation theory (see [16], p. 396), tell us that, under assumptions (a)-(b) of Theorem 2.1,

$$E[e_1] < \infty. \tag{2.1}$$

Now, observe that the sequence $((e_i, e_{i+1}])_{i\geq 0}$ stands for the set of excursions of the potential above its past minimum. Let us introduce H_i, the height of the excursion $[e_i, e_{i+1}]$ defined by $H_i := \max_{e_i \leq k \leq e_{i+1}} (V(k) - V(e_i))$, for $i \geq 0$. Note that the $(H_i)_{i\geq 0}$'s are i.i.d. random variables.

For $t \in \mathbb{N}$, we introduce the critical height

$$h_t := \log t - \log\log t. \tag{2.2}$$

As in [15] we define the deep valleys from the excursions which are higher than the critical height h_t. Let $(\sigma(j))_{j\geq 1}$ be the successive indexes of excursions, whose heights are greater than h_t. More precisely,

$$\begin{aligned} \sigma(1) &:= \inf\{i \geq 0 : H_i \geq h_t\}, \\ \sigma(j) &:= \inf\{i > \sigma(j-1) : H_i \geq h_t\}, \qquad j \geq 2. \end{aligned}$$

We consider now some random variables depending only on the environment, which define the deep valleys.

Definition 2.2. For all $j \geq 1$, let us introduce

$$
\begin{array}{rcl}
b_j & := & e_{\sigma(j)}, \\
a_j & := & \sup\{k \leq b_j : V(k) - V(b_j) \geq D_t\}, \\
T_j^{\uparrow} & := & \inf\{k \geq b_j : V(k) - V(b_j) \geq h_t\}, \\
\overline{d}_j & := & e_{\sigma(j)+1}, \\
c_j & := & \inf\{k \geq b_j : V(k) = \max_{b_j \leq x \leq \overline{d}_j} V(x)\}, \\
d_j & := & \inf\{k \geq \overline{d}_j : V(k) - V(\overline{d}_j) \leq -D_t\}.
\end{array}
$$

where $D_t := (1 + \frac{1}{\kappa}) \log n_t$. We call (a_j, b_j, c_j, d_j) a deep valley and denote by $H^{(j)}$ the height of the j-th deep valley.

Moreover, let us introduce the index of the last visited deep valley at time t, denoted by

$$\ell_t := \sup\{n \geq 0 : \ \tau(b_n) \leq t\}.$$

Before stating the quenched localization result, recall that X is defined on the sample probability space $\mathbb{Z}^{\mathbb{N}}$. Then, let us introduce $\mathbf{e} = (\mathbf{e}_i, \ i \geq 1)$ a sequence of i.i.d. exponential random variables with parameter 1, independent of X. We define $\mathbf{e}$ on a probability space Ξ and denote its law by $P^{(\mathbf{e})}$. In order to express the independence between X and $\mathbf{e}$, we consider for each environment ω, the probability space $(\mathbb{Z}^{\mathbb{N}} \times \Xi, P_\omega \times P^{(\mathbf{e})})$ on which we define $(X, \mathbf{e})$.

Furthermore, let us define the *weight* of the k-th deep valley by

$$W_k(\omega) := \sum_{\substack{a_k \leq m \leq n \\ b_k \leq n \leq d_k}} e^{V_\omega(n) - V_\omega(m)}.$$

Moreover, let us introduce the following integer, for any $t \geq 0$,

$$\ell_{t,\omega}^{(\mathbf{e})} := \sup\Big\{i \geq 0 : \ \sum_{k=1}^{i} W_k(\omega)\mathbf{e}_k \leq t\Big\}.$$

We are now able to state our second main result.

Theorem 2.3. *Under assumptions* (a)-(b) *of Theorem 2.1, we have,*

(i) for all $\eta > 0$,

$$\lim_{t \to \infty} \mathbb{P}(|X_t - b_{\ell_t}| \leq \eta \log t) = 1,$$

(ii) for all $\delta > 0$,

$$\lim_{t \to \infty} P\Big(d_{TV}(\ell_t, \ell_{t,\omega}^{(\mathbf{e})} + 1) > \delta\Big) = 0,$$

where d_{TV} *denotes the distance in total variation.*

Observe that we can easily deduce the following quenched localization result by assembling part (i) and part (ii) of Theorem 2.3.

Corollary 2.4. *Under assumptions (a)-(b) of Theorem 2.1, we have, for all $\delta, \eta > 0$, that*

$$P\left(\sum_{i\geq 1}\left|P_{0,\omega}(|X_t - b_i| \leq \eta \log t) - P^{(\mathbf{e})}\left(\sum_{k=1}^{i-1} W_k(\omega)\mathbf{e}_k \leq t < \sum_{k=1}^{i} W_k(\omega)\mathbf{e}_k\right)\right| > \delta\right)$$

converges to 0, when t tends to ∞.

The content of this result is twofold. It first says that, with a probability converging to 1, the process at time t is concentrated near the bottom of a valley of depth of order $\log t$. It also determines, for each of these valleys, the probability that, at time t, the particle lies at the bottom of it. This probability is driven by a renewal Poisson process which is skewed by the weights of each of these valleys.

This result may be of big interest when trying to get informations on the environment on the basis of the observation of a sample of trajectories of the particle, like it is done, in this setting, in recent works about DNA reconstruction, see [11].

3 Notations

A result of Iglehart [21] which will be of constant use, says that, under assumptions (a)-(b) of Theorem 2.1, the tail of the height H_i of an excursion above its past minimum is given by

$$P(H_1 > h) \sim C_I\,\mathrm{e}^{-\kappa h}, \qquad h \to \infty, \tag{3.1}$$

for a positive constant C_I (we will not need its explicit value).

The analysis done in [14, 15] shows that on the interval $[0, t]$, $t \in \mathbb{N}$, the walk X_n spends asymptotically all its time trying to climb excursions of height of order $\log t + C$ for a real C. Let us now introduce the integer

$$n_t := \lfloor t^\kappa \log\log t \rfloor.$$

The integer n_t will be use to bound the number of excursions the walk can cross before time t. The strategy will be to show that we can neglect the time spent between two excursions of size smaller than h_t, and to show that at time t the walk X_t is closed to the foot of an excursion of height larger than h_t.

3.1 The deep valleys

Let us define the number of deep valleys in the n_t first excursions by

$$K_t := \sup\{j \geq 0 :\ \sigma(j) \leq n_t\},$$

which is the number of excursions higher than the critical height h_t in the n_t first excursions.

Remark. This definition corresponds to the definition of deep valleys introduced in [15] with $n = n_t$, but with a different critical height. In [15] the critical height was $h_n = \frac{1-\varepsilon}{\kappa}\log n$, for ε such that $0 < \varepsilon < 1$. Here, we see that h_{n_t} would be equal to $(1-\varepsilon)\log t + \frac{1-\varepsilon}{\kappa}\log\log\log t$ which is smaller than our critical height $h_t = \log t - \log\log t$. This means that the deep valleys are higher and less numerous in the present paper than in [15].

3.2 The $*$-valleys

Let us first define the maximal variations of the potential before site x by:

$$V^{\uparrow}(x) := \max_{0\le i\le j\le x}(V(j)-V(i)), \qquad x\in\mathbb{N},$$
$$V^{\downarrow}(x) := \min_{0\le i\le j\le x}(V(j)-V(i)), \qquad x\in\mathbb{N}.$$

By extension, we introduce

$$V^{\uparrow}(x,y) := \max_{x\le i\le j\le y}(V(j)-V(i)), \qquad x<y,$$
$$V^{\downarrow}(x,y) := \min_{x\le i\le j\le y}(V(j)-V(i)), \qquad x<y.$$

The deep valleys defined above are not necessarily made of disjoint portions of the environment. To overcome this difficulty we defined another type of valleys, called $*$-valleys, which form a subsequence of the previous valleys, which by construction are made of disjoint portions of environment, and which will coincide with high probability with the previous valleys on the portion of the environment visited by the walk before time t.

$$\begin{aligned}
\gamma_1^* &:= \inf\{k\ge 0:\ V(k)\le -D_t\},\\
T_1^* &:= \inf\{k\ge \gamma_1^*:\ V^{\uparrow}(\gamma_1^*,k)\ge h_t\},\\
b_1^* &:= \sup\{k\le T_1^*:\ V(k)=\min_{0\le x\le T_1^*}V(x)\},\\
a_1^* &:= \sup\{k\le b_1^*:\ V(k)-V(b_1^*)\ge D_t\},\\
\overline{d}_1^* &:= \inf\{k\ge T_1^*:\ V(k)\le V(b_1^*)\},\\
c_1^* &:= \inf\{k\ge b_1^*:\ V(k)=\max_{b_1^*\le x\le \overline{d}_1^*}V(x)\},\\
d_1^* &:= \inf\{k\ge \overline{d}_1^*:\ V(k)-V(\overline{d}_1^*)\le -D_t\}.
\end{aligned}$$

Let us define the following sextuplets of points by iteration

$$(\gamma_j^*,a_j^*,b_j^*,T_j^*,c_j^*,\overline{d}_j^*,d_j^*) := (\gamma_1^*,a_1^*,b_1^*,T_1^*,c_1^*,\overline{d}_1^*,d_1^*)\circ\theta_{d_{j-1}^*}, \qquad j\ge 2,$$

where θ_i denotes the i-shift operator.

Definition 3.1. We call a $*$-valley any quadruplet $(a_j^*,b_j^*,c_j^*,d_j^*)$ for $j\ge 1$. Moreover, we shall denote by K_t^* the number of such $*$-valleys before e_{n_t}, i.e. $K_t^* := \sup\{j\ge 0 : T_j^*\le e_{n_t}\}$.

It will be made of independent and identically distributed portions of potential (up to some translation).

4 Preliminary estimates

4.1 Introducing good environments

As in [15], we introduce the following series of events, which will occur with high probability when t tends to infinity.

$$\begin{aligned}
A_1(t) &:= \{e_{n_t} \le C' n_t\}, \\
A_2(t) &:= \left\{K_t \le (\log t)^{\frac{1+\kappa}{2}}\right\}, \\
A_3(t) &:= \cap_{j=0}^{K_t} \left\{\sigma(j+1) - \sigma(j) \ge t^{\kappa/2}\right\}, \\
A_4(t) &:= \cap_{j=1}^{K_t+1} \{d_j - a_j \le C'' \log t\},
\end{aligned}$$

where $\sigma(0) := 0$ (for convenience of notation) and C', C'' stand for positive constants which will be specified below.

Lemma 4.1. *Let $A(t) := A_1(t) \cap A_2(t) \cap A_3(t) \cap A_4(t)$, then*

$$\lim_{t\to\infty} P(A(t)) = 1.$$

Proof. Concerning $A_2(t)$, we know that the number of excursions higher than h_t in the first n_t excursions is a binomial with parameter (n_t, q_t) where $q_t := P(H_1 \ge h_t)$. Since (3.1) implies $q_t \sim C_I e^{-\kappa h_t}$, $t \to \infty$, we have that $E[K_t] = n_t q_t \sim C_I \log\log t (\log t)^\kappa$. Using Markov inequality we get that $P(A_2(t))$ tends to 1. The fact that $P(A_1(t) \cap A_3(t) \cap A_4(t))$ converges to 1 is a consequence of Lemma 1, Lemma 3 and Lemma 4 of [15] since the deep valleys with h_t are less numerous than with h_{n_t} (cf Remark 3.1). □

The following lemma tells us that the $*$-valleys coincide with the sequence of deep valleys with an overwhelming probability when t goes to infinity.

Lemma 4.2. *If $A^*(t) := \{K_t = K_t^*; (a_j, b_j, c_j, d_j) = (a_j^*, b_j^*, c_j^*, d_j^*), 1 \le j \le K_t^*\}$, then we have that the probability $P(A^*(t))$ converges to 1, when t goes to infinity.*

Proof. By definition, the $*$-valleys constitute a subsequence of the deep valleys, and $A^*(t)$ occurs as soon as the valleys (a_j, b_j, c_j, d_j) are disjoint for $1 \le j \le K_t$. Hence, we see that $A_3(t) \cap A_4(t) \subset A^*(t)$. Then, Lemma 4.2 is a consequence of Lemma 4.1. □

4.2 Directed traps

Let us introduce, for any $x, y \in \mathbb{Z}$,

$$\tau(x, y) := \inf\{k \ge 0 : X_{\tau(x)+k} = y\}.$$

We recall from [15] the following lemmas.

Lemma 4.3. *Defining* $DT(t) := A(t) \cap \bigcap_{j=1}^{K_t} \left\{\tau(d_j, b_{j+1}) < \tau(d_j, \overline{d}_j)\right\}$, *we have*

$$\mathbb{P}\left(DT(t)\right) \to 1, \qquad t \to \infty.$$

Proof. The proof is exactly the same as in [15], but easier since the deep valleys with h_t are less numerous than with h_{n_t} (cf Remark 3.1). □

Lemma 4.4. *Defining* $DT^*(t) := \bigcap_{j=1}^{K_t^*} \left\{\tau(b_j^*, d_j^*) < \tau(b_j^*, a_j^*)\right\}$, *we have*

$$\mathbb{P}(DT^*(t)) \to 1, \qquad t \to \infty.$$

Proof. The proof is close to be the same as in [15], except that the deep valleys with h_t are still less numerous than with h_{n_t} and that the γ_i's are remplaced by the a_i's. This does not modify the proof of [15] since we only have to check that $V(a_i) - V(b_i) \geq D_t$, which is true by definition of the a_i's (see Definition 2.2). □

Finally, we need to know that the time spent between the deep valleys is small. This a consequence of Lemma 7 in [15].

Lemma 4.5. *Let us introduce the following event*

$$IA(t) := A(t) \cap \left\{ \tau(b_1) + \sum_{j=1}^{K_t} \tau(d_j, b_{j+1}) < \frac{t}{\log \log t} \right\}.$$

Then, we have

$$\mathbb{P}(IA(t)) \to 1, \qquad t \to \infty.$$

Proof. The arguments are the same as in the proof of Lemma 10 in [15]. The main tool is Lemma 7 of [15], which says that there exists $C > 0$, such that for all $h \geq 0$,

$$\mathbb{E}_{|0}\left[\tau(T^{\uparrow}(h) - 1)\right] \leq C\mathrm{e}^{h},$$

where $\mathbb{E}_{|0}$ denotes the expectation under the annealed law $\mathbb{P}_{|0}$ associated with the random walk in random environment reflected at 0. □

4.3 Localization in traps

In a first step, we give a technical result, which will be very useful to control the localization of the particle in a valley.

Lemma 4.6. *If* $F_\gamma(t) := \left\{\max\{V^{\uparrow}(a_1, b_1)\,;\, -V^{\downarrow}(b_1, c_1)\,;\, V^{\uparrow}(c_1, d_1)\} \leq \gamma \log t\right\}$, *then we have, for any* $\gamma > 0$ *and any* $0 < \varepsilon < \gamma$,

$$P(F_\gamma(t)) = 1 - o(t^{-\kappa\varepsilon}), \qquad t \to \infty.$$

In words, $F_\gamma(t)$ ensures that the potential does not have excessive fluctuations in a typical box.

Proof. The arguments are the same as in the proof of Lemma 13 in [15]. □

For each deep valley, let us introduce the position $\overline{c}_i$

$$\overline{c}_i := \inf\{n \geq c_i : \ V(n) \leq V(c_i) - h_t/3\}.$$

We first need to know that during its sojourn time inside a deep valley, the random walk spends almost all its time inside the interval (a_i, c_i). This is a consequence of the following lemma.

Lemma 4.7. *Let $LT(t)$ be the event*

$$LT(t) := \bigcap_{i=1}^{K_t} \left\{ \tau(\overline{c}_i, d_i) \leq \frac{t}{\log t} \right\}.$$

Then,

$$\mathbb{P}(LT(t)) \to 1, \qquad t \to \infty.$$

This result just means that at the time scale t, if the walk meet $\overline{c}_i$, then soon after it exits the deep valley (a_i, d_i).

Proof. Since $P(K_t \leq (\log t)^{\frac{1+\kappa}{2}}) \to 1$, when $t \to \infty$, we only have to prove that

$$\mathbb{P}\left(\tau(\overline{c}_1, d_1) > \frac{t}{\log t} \right) = o((\log t)^{-\frac{1+\kappa}{2}}), \qquad t \to \infty.$$

Now, applying the strong Markov property at $\tau(\overline{c}_1)$, we get

$$\mathbb{P}\left(\tau(\overline{c}_1, d_1) > \frac{t}{\log t} \right) \leq E\left[P_{\omega,|c_1}^{\overline{c}_1} \left(\tau(d_1) > t/\log t \right) \right] + E\left[P_{\omega}^{\overline{c}_1} \left(\tau(c_1) < \tau(d_1) \right) \right].$$

Considering the first term, using the fact that $E_{\omega,|c_1}^{\overline{c}_1}[\tau(d_1)] \leq \sum_{c_1 \leq i \leq j \leq d_1} \mathrm{e}^{V(j)-V(i)}$ (see (A1) in [18]) and Chebychev inequality, we obtain

$$P_{\omega,|c_1}^{\overline{c}_1} \left(\tau(d_1) > t/\log t \right) \leq \frac{\log t}{t} \sum_{c_1 \leq i \leq j \leq d_1} \mathrm{e}^{V(j)-V(i)} \leq \frac{\log t}{t} (d_1 - c_1) \mathrm{e}^{\gamma \log t},$$

on $F_\gamma(t)$. Since the proof of Lemma 4 in [15] contains the fact that $P\{d_1 - c_1 \geq C \log t\} = o((\log t)^{-\frac{1+\kappa}{2}})$, when $t \to \infty$, we only have to choose $\gamma < 1$, which implies

$$E\left[P_{\omega,|c_1}^{\overline{c}_1} \left(\tau(d_1) > t/\log t \right) \right] = o((\log t)^{-\frac{1+\kappa}{2}}), \qquad t \to \infty.$$

In order to treat the second term, by (Zeitouni [27], formula (2.1.4)), we get

$$P_{\omega}^{\overline{c}_1} \left(\tau(c_1) < \tau(d_1) \right) \ \leq \ \frac{\sum_{k=\overline{c}_1}^{d_1-1} \mathrm{e}^{V(k)}}{\sum_{k=c_1}^{d_1-1} \mathrm{e}^{V(k)}} \leq (d_1 - c_1) \mathrm{e}^{V(\overline{c}_1) + \gamma \log t - V(c_1)} \leq (d_1 - c_1) \mathrm{e}^{\gamma \log t - \frac{h_t}{3}},$$

on $F_\gamma(t)$. Now, let us choose $\gamma < 1/3$.

Recalling Lemma 4.6, and since we have $P(d_1 - c_1 \geq C \log t) = o((\log t)^{-\frac{1+\kappa}{2}})$, when $t \to \infty$, we get

$$E\left[P_{\omega}^{\overline{c}_1} \left(\tau(c_1) < \tau(d_1) \right) \right] = o((\log t)^{-\frac{1+\kappa}{2}}), \qquad t \to \infty,$$

which concludes the proof of Lemma 4.7. □

Now, we need to be sure that the bottom of the deep valleys are sharp. For $\eta > 0$, we introduce the following subsets of the deep valleys

$$O_i := [a_i + 1, \overline{c}_i - 1] \setminus (b_i - \eta \log t + 1, b_i + \eta \log t - 1), \qquad i \in \mathbb{N},$$

and the event

$$A_5(t) := \bigcap_{i=1}^{K_t} \left\{ \min_{k \in O_i \cap \mathbb{Z}} (V(k) - V(b_i)) \geq C''' \eta \log t \right\},$$

for a constant C''' (small enough) to be defined later. Then, we have the following result.

Lemma 4.8. *For all $\eta > 0$,*

$$\lim_{t \to \infty} P(A_5(t)) = 1.$$

Proof. Observe first that if $\eta > C''$, then the sets $(O_i, 1 \leq i \leq K_t)$ are empty on $A_4(t)$. Therefore, Lemma 4.8 is a consequence of Lemma 4.1.

Now, let us assume $\eta \leq C''$. The definition of $\overline{c}_i$ implies that $\min_{c_i \leq k < \overline{c}_i} (V(k) - V(b_i)) \geq \frac{2}{3} h_t$. Then, choosing C''' such that $C''' C'' < 2/3$ implies that $C''' \eta \log t < \frac{2}{3} h_t$ for all large t, which yields

$$P\left(\bigcap_{i=1}^{K_t} \left\{ \min_{c_i \leq k < \overline{c}_i} (V(k) - V(b_i)) \geq C''' \eta \log t \right\} \right) = 1, \tag{4.1}$$

for all large t. Then, let us introduce the sets

$$O_i' := O_i \cap [b_i, c_i], \qquad O_i'' := O_i \cap [a_i, b_i], \qquad i \in \mathbb{Z},$$

and the events

$$\begin{aligned} A_5'(t) &:= \bigcap_{i=1}^{K_t} \left\{ \min_{k \in O_i' \cap \mathbb{Z}} (V(k) - V(b_i)) \geq C''' \eta \log t \right\}, \\ A_5''(t) &:= \bigcap_{i=1}^{K_t} \left\{ \min_{k \in O_i'' \cap \mathbb{Z}} (V(k) - V(b_i)) \geq C''' \eta \log t \right\}. \end{aligned}$$

Now, recalling (4.1), the proof of Lemma 4.8 boils down to showing that

$$\lim_{t \to \infty} P(A_5'(t)) = 1, \tag{4.2}$$

$$\lim_{t \to \infty} P(A_5''(t)) = 1. \tag{4.3}$$

Let us first prove (4.2). Since $P(K_t \leq (\log t)^{\frac{1+\kappa}{2}}) \to 1$, when $t \to \infty$, we only have to prove that it is possible to choose C''' small enough such that

$$P\left(\min_{k \in O_1' \cap \mathbb{Z}} (V(k) - V(b_1)) < C''' \eta \log t \right) = o((\log t)^{-\frac{1+\kappa}{2}}), \qquad t \to \infty. \tag{4.4}$$

Recalling assumption (a) of Theorem 2.1 and denoting by μ the law of $\log \rho_0$, we can define the law $\tilde{\mu} = \rho_0^\kappa \mu$, and the law $\tilde{P} = \tilde{\mu}^{\otimes \mathbb{Z}}$ which is the law of a sequence of i.i.d. random variables with law $\tilde{\mu}$. The definition of κ implies that $\int \log \rho \, \tilde{\mu}(d\rho) > 0$. Now, observe that the probability term in (4.4) can be written

$$
\begin{aligned}
& P\left(\min_{\lfloor \eta \log t \rfloor \le k \le T_H} V(k) < C''' \eta \log t | H \ge h_t \right) \\
\le \; & C \mathrm{e}^{\kappa h_t} P\left(\min_{\lfloor \eta \log t \rfloor \le k \le T_H} V(k) < C''' \eta \log t \,;\, H \ge h_t \right) \\
\le \; & C \tilde{E}\left[\mathrm{e}^{-\kappa (V(T_H) - h_t)} \mathbf{1}_{\{\min_{\lfloor \eta \log t \rfloor \le k \le T_H} V(k) < C''' \eta \log t \,;\, H \ge h_t\}} \right] \\
\le \; & C \tilde{P}\left(\min_{\lfloor \eta \log t \rfloor \le k \le T_H} V(k) < C''' \eta \log t \,;\, H \ge h_t \right), \qquad (4.5)
\end{aligned}
$$

the first inequality being a consequence of (3.1) and the second deduced from Girsanov property. Now, let us introduce $\alpha = \alpha(\eta) := c\eta$ with $0 < c < \min\{\tilde{E}\,[V(1)]\,; 1/C''\}$ and $\gamma = \gamma(\eta) := c\eta/2$. Observe that $\alpha \log t < h_t$ for all large t, such that $T_{\alpha \log t} \le T_{h_t} \le T_H < \infty$ on $\{H \ge h_t\}$. Now since $c < \tilde{E}\,[V(1)]$, we obtain from Cramer's theory, see [20], that $\tilde{P}\,(V(\lfloor \eta \log t \rfloor) < \alpha \log t) \le C \exp\{-\eta \tilde{I}(c) \log t\} = o((\log t)^{-\frac{1+\kappa}{2}})$, where $\tilde{I}(\cdot)$ denotes the convex rate function associated with V under $\tilde{P}$. This yields $\tilde{P}\,(T_{\alpha \log t} \le \lfloor \eta \log t \rfloor) = 1 - o((\log t)^{-\frac{1+\kappa}{2}})$, when t tends to infinity. Therefore, we get that

$$
\begin{aligned}
& P\left(\min_{\lfloor \eta \log t \rfloor \le k \le T_H} V(k) < C''' \eta \log t \,;\, H \ge h_t \right) \\
\le \; & P\left(\min_{T_{\alpha \log t} \le k \le T_H} V(k) < C''' \eta \log t \,;\, H \ge h_t \right) + o((\log t)^{-\frac{1+\kappa}{2}}). \qquad (4.6)
\end{aligned}
$$

Then, recalling that Lemma 4.6 implies that $P(F_\gamma(t)) = 1 - o((\log t)^{-\frac{1+\kappa}{2}})$, $t \to \infty$, let us write

$$
\begin{aligned}
& P\left(\min_{T_{\alpha \log t} \le k \le T_H} V(k) < C''' \eta \log t \,;\, H \ge h_t \right) \\
= \; & P\left(\min_{T_{\alpha \log t} \le k \le T_H} V(k) < C''' \eta \log t \,;\, H \ge h_t \,;\, F_\gamma(t) \right) + o((\log t)^{-\frac{1+\kappa}{2}}). \qquad (4.7)
\end{aligned}
$$

Furthermore, observe that on $F_\gamma(t)$, we have $\min_{T_{\alpha \log t} \le k \le T_H} V(k) \ge (\alpha - \gamma) \log t$, which yields $\min_{T_{\alpha \log t} \le k \le T_H} V(k) \ge C''' \eta \log t$, if we choose C''' smaller than $c/2$. Therefore, for C''' small enough (independently of $\eta \le C''$), we get that the probability term in (4.7) is null for all large t. Now, assembling (4.5) , (4.6) and (4.7) implies (4.4) and concludes the proof of (4.2) .

The proof of (4.3) is similar but easier. Indeed, we do not have to use Girsanov property to study the potential on $[a_i, b_i]$. $\square$

5 Two versions of a Dynkin type renewal result

We define the sequence of random times $(\tau_i^*)_{i \ge 1}$ as follows: conditioning on the environment ω, $(\tau_i^*)_{i \ge 1}$ is defined as an independent sequence of random variables with the law of $\tau(d_i^*)$ under $P^{b_i^*}_{\omega,|a_i^*}$, where $\tau(d_i^*)$ denotes the first hitting time of d_i^* and $P^{b_i^*}_{\omega,|a_i^*}$ is the law of the Markov chain in environment ω, starting from b_i^* and reflected at a_i^*. Hence, under the annealed law $\mathbb{P}$, $(\tau_i^*)_{i\geq 1}$ is an i.i.d. sequence since the $*$-valleys are independent and identically distributed. The first step in our proof is to derive the following result.

Proposition 5.1. *Let ℓ_t^* be the random integer defined by*

$$\ell_t^* := \sup\{n \geq 0 : \ \tau_1^* + \cdots + \tau_n^* \leq t\}.$$

For all $0 \leq x_1 < x_2 \leq 1$, we have

$$\lim_{t\to\infty} \mathbb{P}(t(1-x_2) \leq \tau_1^* + \cdots + \tau_{\ell_t^*}^* \leq t(1-x_1)) = \frac{\sin(\kappa\pi)}{\pi}\int_{x_1}^{x_2} \frac{x^{-\kappa}}{(1-x)^{\kappa-1}}\,\mathrm{d}x.$$

For all $0 \leq x_1 < x_2$, we have

$$\lim_{t\to\infty} \mathbb{P}(t(1+x_1) \leq \tau_1^* + \cdots + \tau_{\ell_t^*+1}^* \leq t(1+x_2)) = \frac{\sin(\kappa\pi)}{\pi}\int_{x_1}^{x_2} \frac{\mathrm{d}x}{x^{\kappa}(1+x)}.$$

Proof. Observe that the result would exactly be Dynkin's theorem (cf e.g. Feller, vol II, [16], p. 472) if the sequence $(\tau_i^*)_{i\geq 1}$ was an independent sequence of random variables in the domain of attraction of a stable law with index κ. Here, the sequence $(\tau_i^*)_{i\geq 1}$ implicitly depends on the time t, since the $*$-valleys are defined from the critical height h_t. In [15], using [14], we obtained that the Laplace transform of τ_1^* satisfies

$$\mathbb{E}\Big[1 - e^{-\lambda\frac{\tau_1^*}{t}}\Big] \sim \frac{P(H \geq h_t)^{-1}}{t}\frac{\pi\kappa}{\sin(\kappa\pi)}2^{\kappa}C_U\lambda^{\kappa}, \qquad t \to \infty, \tag{5.1}$$

for all $\lambda > 0$. The constant C_U was made explicit in [14] but we will not need this value here.

The proof is essentially the same as in [16]. Let us introduce $S_0^* = 0$ and $S_n^* := \sum_{i=1}^n \tau_i^*$, for $n \geq 1$. Then, the inequality $t(1-x_2) \leq \tau_1^* + \cdots + \tau_{\ell_t^*} \leq t(1-x_1)$ occurs iff $S_n^* = ty$ and $\tau_{n+1}^* > t(1-y)$ for some combination n, y such that $1-x_2 < y < 1-x_1$. Summing over all n and possible y we get

$$\mathbb{P}(t(1-x_2) \leq S_{\ell_t^*}^* \leq t(1-x_1)) = \int_{1-x_2}^{1-x_1} \frac{G_t(1-y)}{P(H \geq h_t)} U_t\{\mathrm{d}y\}, \tag{5.2}$$

where $G_t(x) := P(H \geq h_t)\mathbb{P}(t^{-1}\tau_1^* \geq x)$, and $U_t\{\mathrm{d}x\}$ denotes the measure associated with $U_t(x) := \sum_{n\geq 0}\mathbb{P}(t^{-1}S_n^* \leq x)$. We introduce the measure $\mathrm{d}H_t(u)$ such that $\int_x^\infty \mathrm{d}H_t(u) = G_t(x)$, for all $x \geq 0$.

Lemma 5.2. *For any $x > 0$, we have*

$$\lim_{t\to\infty} x^{\kappa} t\, G_t(x) = 2^{\kappa}\Gamma(1+\kappa)C_U. \tag{5.3}$$

Moreover, the convergence is uniform on any compact set.

Proof. In a first step, observe that $\mathbb{E}[1-e^{-\lambda\frac{\tau_1^*}{t}}] = P(H \geq h_t)^{-1}\int_0^\infty (1-\mathrm{e}^{-\lambda u})\,\mathrm{d}H_t(u)$. Recalling (5.1), we obtain

$$\lim_{t\to\infty} t\int_0^\infty (1-\mathrm{e}^{-\lambda u})\,\mathrm{d}H_t(u) = 2^\kappa\Gamma(1+\kappa)C_U\Gamma(1-\kappa)\lambda^\kappa.$$

Since $\Gamma(1-\kappa)\lambda^\kappa = \lambda\int_0^\infty \mathrm{e}^{-\lambda u}u^{-\kappa}\,\mathrm{d}u$, this implies

$$\lim_{t\to\infty} t\int_0^\infty (1-\mathrm{e}^{-\lambda u})\,\mathrm{d}H_t(u) = 2^\kappa\Gamma(1+\kappa)C_U\lambda\int_0^\infty \mathrm{e}^{-\lambda u}u^{-\kappa}\,\mathrm{d}u. \tag{5.4}$$

To the other hand, integrating by parts, we get, for any $t \geq 0$,

$$\int_0^\infty (1-\mathrm{e}^{-\lambda u})\,\mathrm{d}H_t(u) = \lambda\int_0^\infty \mathrm{e}^{-\lambda u}G_t(u)\,\mathrm{d}u. \tag{5.5}$$

Combining (5.4) and (5.5) implies that the measure $t\,G_t(u)\,\mathrm{d}u$ tends to the measure with density $2^\kappa\Gamma(1+\kappa)C_U u^{-\kappa}$. Therefore, we have for all $x \geq 0$,

$$\lim_{t\to\infty} t\int_0^x G_t(u)\,\mathrm{d}u = 2^\kappa\Gamma(1+\kappa)C_U\frac{x^{1-\kappa}}{1-\kappa}, \tag{5.6}$$

which yields

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty}\frac{\int_x^{(1+\varepsilon)x} G_t(u)\,\mathrm{d}u}{\varepsilon\int_0^x G_t(u)\,\mathrm{d}u} = 1-\kappa. \tag{5.7}$$

Moreover, observe that the monotonicity of $G_t(\cdot)$ implies

$$\frac{xG_t((1+\varepsilon)x)}{\int_0^x G_t(u)\,\mathrm{d}u} \leq \frac{\int_x^{(1+\varepsilon)x} G_t(u)\,\mathrm{d}u}{\varepsilon\int_0^x G_t(u)\,\mathrm{d}u} \leq \frac{xG_t(x)}{\int_0^x G_t(u)\,\mathrm{d}u}. \tag{5.8}$$

Now, combining (5.7) and (5.8), we obtain

$$\liminf_{t\to\infty}\frac{xG_t(x)}{\int_0^x G_t(u)\,\mathrm{d}u} \geq 1-\kappa.$$

Recalling (5.6), this yields

$$\liminf_{t\to\infty} x^\kappa t\,G_t(x) \geq 2^\kappa\Gamma(1+\kappa)C_U. \tag{5.9}$$

Similarly, we obtain, for any $\varepsilon > 0$,

$$\limsup_{t\to\infty} x^\kappa t\,G_t((1+\varepsilon)x) \leq 2^\kappa\Gamma(1+\kappa)C_U. \tag{5.10}$$

Assembling (5.9) and (5.10) concludes the proof of (5.3).

Furthermore, observe that the uniform convergence on any compact set is a consequence of the monotonicity of $x \mapsto G_t(x)$, the continuity of the limit and Dini's theorem. ☐

Lemma 5.3. *The measure* $\frac{P(H\geq h_t)^{-1}}{t}U_t\{\mathrm{d}x\}$ *converges vaguely to the measure defined by* $\frac{1}{\Gamma(\kappa)\Gamma(1+\kappa)\Gamma(1-\kappa)2^\kappa C_U}x^{\kappa-1}\mathrm{d}x$.

Proof. Observe first that the Laplace transform $\widehat{U}_t(\lambda) := \int_0^\infty \mathrm{e}^{-\lambda u}U_t\{\mathrm{d}u\}$ satisfies $\widehat{U}_t(\lambda) = \sum_{n\geq 0}\mathbb{E}[\mathrm{e}^{-\lambda\frac{S_n^*}{t}}] = (1-\mathbb{E}[\mathrm{e}^{-\lambda\frac{\tau_1^*}{t}}])^{-1}$. Therefore, (5.1) yields

$$\lim_{t\to\infty}\frac{P(H\geq h_t)^{-1}}{t}\widehat{U}_t(\lambda) = \frac{\lambda^{-\kappa}}{\Gamma(1+\kappa)\Gamma(1-\kappa)2^\kappa C_U}.$$

Furthermore, since $\Gamma(\kappa)\lambda^{-\kappa} = \int_0^\infty \mathrm{e}^{-\lambda u}u^{\kappa-1}\,\mathrm{d}u$, we deduce the vague convergence of the measure from the pointwise convergence of the Laplace transforms. □

Now, recalling (5.2), we observe that Lemma 5.2 together with Lemma 5.3 imply

$$\begin{aligned}\lim_{t\to\infty}\mathbb{P}(t(1-x_2)\leq S^*_{\ell_t^*}\leq t(1-x_1)) &= \frac{1}{\Gamma(\kappa)\Gamma(1-\kappa)}\int_{1-x_2}^{1-x_1}(1-y)^{-\kappa}y^{\kappa-1}\,\mathrm{d}y,\\ &= \frac{\sin(\kappa\pi)}{\pi}\int_{x_1}^{x_2}\frac{y^{-\kappa}}{(1-y)^{\kappa-1}}\,\mathrm{d}y.\end{aligned}$$

This concludes the proof of the first part of Proposition 5.1. The second part of Proposition 5.1 is obtained using similar arguments. □

Recall Lemma 4.5 which tells that the inter-arrival times are negligible. Now, we will prove that the results of Proposition 5.1 are still true if we consider, in addition, these inter-arrival times. Let $\delta_1 := \tau(b_1)$, $\tau_1 := \tau(b_1, d_1)$ and

$$\delta_k := \tau(d_{k-1}, b_k), \qquad \tau_k := \tau(b_k, d_k), \qquad k\geq 2.$$

Moreover, we set

$$T_k := \delta_1 + \tau_1 + \cdots + \tau_{k-1} + \delta_k, \qquad k\geq 1,$$

the entering time in the k-th deep valley.

Proposition 5.4. *Recall* $\ell_t = \sup\{n\geq 0:\ \tau(b_n)\leq t\}$. *Then, we have*

$$\mathbb{P}(T_{\ell_t}\leq t < T_{\ell_t}+\tau_{\ell_t}) \to 1, \qquad t\to\infty.$$

For all $0\leq x_1 < x_2\leq 1$, *we have*

$$\lim_{t\to\infty}\mathbb{P}(t(1-x_2)\leq T_{\ell_t}\leq t(1-x_1)) = \frac{\sin(\kappa\pi)}{\pi}\int_{x_1}^{x_2}\frac{x^{-\kappa}}{(1-x)^{\kappa-1}}\,\mathrm{d}x.$$

For all $0\leq x_1 < x_2$, *we have*

$$\lim_{t\to\infty}\mathbb{P}(t(1+x_1)\leq T_{\ell_t+1}\leq t(1+x_2)) = \frac{\sin(\kappa\pi)}{\pi}\int_{x_1}^{x_2}\frac{\mathrm{d}x}{x^\kappa(1+x)}.$$

Proof. On the event $A(t) \cap DT^*(t)$, we know that the random times $(\tau_i)_{1\le i\le K_t^*}$ have the same law as the random times $(\tau_i^*)_{1\le i\le K_t^*}$ defined in Section 5. If we define $\tilde{\ell}_t := \sup\{n \ge 0 : \ \tau_1 + \cdots + \tau_n \le t\}$, then, using Proposition 5.1 and Lemma 4.3, we get that the result of Proposition 5.1 is true with τ and $\tilde{\ell}_t$ in place of τ^* and ℓ_t^*. Now, using Lemma 4.5 we see that

$$\begin{aligned} & \liminf_{t\to\infty} \mathbb{P}(\tilde{\ell}_t = \ell_t - 1\,;\, T_{\ell_t} \le t < T_{\ell_t} + \tau_{\ell_t}) \\ \ge\ & \liminf_{t\to\infty} \mathbb{P}(IA(t)\,;\, |t - (\tau_1 + \cdots + \tau_{\tilde{\ell}_t})| \ge \xi t), \end{aligned}$$

for all $\xi > 0$. Thus, using Proposition 5.1 (for $\tilde{\ell}_t$ and τ_i) and letting ξ tends to 0, we get that

$$\lim_{t\to\infty} \mathbb{P}(\tilde{\ell}_t = \ell_t - 1\,;\, T_{\ell_t} \le t < T_{\ell_t} + \tau_{\ell_t}) = 1.$$

We conclude the proof by the same type of arguments. □

6 Proof of part (i) of Theorem 2.3: a localization result

We follow the strategy developed by Sinai for the recurrent case. For each valley we denote by π_i the invariant measure of the random walk on $[a_i, \bar{c}_i]$ in environment ω, reflected at a_i and $\bar{c}_i$ and normalized so that $\pi_i(b_i) = 1$. Clearly, π_i is the reversible measure given, for $k \in [b_i + 1, \bar{c}_i - 1]$, by

$$\begin{aligned} \pi_i(k) &= \frac{\omega_{b_i}}{1 - \omega_{b_i+1}} \cdots \frac{\omega_{k-1}}{1 - \omega_k} \\ &= \omega_{b_i} \rho_{b_i+1}^{-1} \cdots \rho_{k-1}^{-1} (\rho_k^{-1} + 1) \\ &\le \mathrm{e}^{-(V(k) - V(b_i))} + \mathrm{e}^{-(V(k-1) - V(b_i))}. \end{aligned}$$

Similarly, $\pi_i(k) \le \mathrm{e}^{-(V(k)-V(b_i))} + \mathrm{e}^{-(V(k+1)-V(b_i))}$ for $k \in [a_i + 1, b_i - 1]$. Since the walk is reflected at a_i and $\bar{c}_i$, we have $\pi_i(a_i) = \mathrm{e}^{-(V(a_i+1)-V(b_i))}$ and $\pi_i(\bar{c}_i) = \mathrm{e}^{-(V(\bar{c}_i-1)-V(b_i))}$. Hence on the event $A_5(t)$ we have

$$(\pi_i)_{|[a_i,\bar{c}_i]\setminus(b_i - \eta\log t, b_i + \eta \log t)} \le C\mathrm{e}^{-C'''\eta \log t} = Ct^{-C'''\eta}.$$

Moreover, since π_i is an invariant measure and since $\pi_i(b_i) = 1$, we have, for all $k \ge 0$,

$$P^{b_i}_{\omega,|a_i,\bar{c}_i|}(X_k = x) \le \pi_i(x).$$

Hence, on the event $A(t) \cap A_5(t)$ we have, for all $k \ge 0$,

$$P^{b_i}_{\omega,|a_i,\bar{c}_i|}(|X_k - b_i| > \eta \log t) \le C(\log t) t^{-C'''\eta}. \tag{6.1}$$

Let ξ be a positive real, $0 < \xi < 1$. Then, let us write

$$\begin{aligned} & \liminf_{t\to\infty} \mathbb{P}(|X_t - b_{\ell_t}| \le \eta \log t) \\ \ge\ & \liminf_{t\to\infty} \mathbb{P}(|X_t - b_{\ell_t}| \le \eta\log t\,;\, \ell_t = \ell_{t(1+\xi)}) \\ \ge\ & \liminf_{t\to\infty} \mathbb{P}(\ell_t = \ell_{t(1+\xi)}) - \limsup_{t\to\infty} \mathbb{P}(|X_t - b_{\ell_t}| > \eta\log t\,;\, \ell_t = \ell_{t(1+\xi)}). \end{aligned}$$

Considering the first term, we get by using Proposition 5.4,

$$\begin{aligned}\liminf_{t\to\infty} \mathbb{P}(\ell_t = \ell_{t(1+\xi)}) &= \liminf_{t\to\infty} \mathbb{P}(T_{\ell_t+1} > t(1+\xi)) \\ &= \frac{\sin(\kappa\pi)}{\pi}\int_\xi^\infty \frac{\mathrm{d}x}{x^\kappa(1+x)}. \end{aligned} \tag{6.2}$$

In order to estimate the second term, let us introduce the event

$$TT(t) := A(t) \cap A_5(t) \cap DT(t) \cap DT^*(t) \cap A^*(t) \cap IA(t) \cap LT(t) \cap IT(t),$$

where $IT(t) := \{T_{\ell_t} \le t < T_{\ell_t} + \tau_{\ell_t}\}$. Observe that the preliminary results obtained in Section 4 together with Proposition 5.4 imply that $\mathbb{P}(TT(t)) \to 1$, when $t \to \infty$. Then, we have

$$\begin{aligned} & \limsup_{t\to\infty} \mathbb{P}(|X_t - b_{\ell_t}| > \eta \log t\,;\, \ell_t = \ell_{t(1+\xi)}) \\ \le\ & \limsup_{t\to\infty} \mathbb{P}(TT(t)\,;\, |X_t - b_{\ell_t}| > \eta \log t\,;\, \ell_t = \ell_{t(1+\xi)}) \\ \le\ & \limsup_{t\to\infty} \mathbb{E}\Big[\mathbf{1}_{TT(t)} \sum_{i=1}^{K_t} \mathbf{1}_{\{|X_t-b_i|>\eta\log t\,;\,\ell_t=\ell_{t(1+\xi)}=i\}}\Big]. \end{aligned}$$

But on the event $TT(t)\cap\{\ell_t = \ell_{t(1+\xi)} = i\}$ we know that for all $k \in [T_i, t]$ the walk X_k is in the interval $[a_i, \overline{c}_i - 1]$. (Indeed, on the event $LT(t)\cap DT(t)\cap IA(t)$ we know that once the position $\overline{c}_i$ is reached then within a time $t/\log t$ the position b_{i+1} is reached which would contradict the fact that $\ell_{t(1+\xi)} = i$. Hence, we obtain, for all $i \in \mathbb{N}$,

$$\begin{aligned} & \mathbb{P}\left(TT(t)\,;\, i \le K_t\,;\, |X_t - b_i| > \eta\log t\,;\, \ell_t = \ell_{t(1+\xi)} = i\right) \\ \le\ & \mathbb{E}\Big[\mathbf{1}_{\{i\le K_t\}}\mathbf{1}_{A(t)\cap A_5(t)} \sup_{k\in[0,t]} P^{b_i}_{\omega,|a_i,\overline{c}_i|}\left(|X_k - b_i| > \eta\log t\right)\Big] \\ \le\ & C(\log t)t^{-C'''\eta}, \end{aligned}$$

where we used the estimate (6.1) on the event $A(t)\cap A_5(t)$. Considering now that, on the event $A(t)$, the number $K(t)$ of deep valleys is smaller than $(\log t)^{\frac{\kappa+1}{2}}$ we get

$$\begin{aligned}\limsup_{t\to\infty} \mathbb{P}(|X_t - b_{\ell_t}| > \eta\log t\,;\, \ell_t = \ell_{t(1+\xi)}) &\le \limsup_{t\to\infty} C(\log t)^{\frac{3+\kappa}{2}} t^{-C'''\eta} \\ &= 0. \end{aligned}$$

Then, letting ξ tends to 0 in (6.2) concludes the proof of part *(i)* of Theorem 2.3. □

7 Part *(ii)* of Theorem 2.3: the quenched law of the last visited valley

In order to prove the proximity of the distributions of ℓ_t and $\ell^{(\mathbf{e})}_{t,\omega}$, we go through $\ell^*_t = \sup\{n \ge 0,\quad \tau^*_1 + \cdots + \tau^*_n \le t\}$ whose advantage is to involve independent random variables whose laws are clearly identified.

Proposition 7.1. *Under assumptions (a)-(b) of Theorem 2.1, we have, for all $\delta > 0$,*

$$\lim_{t\to\infty} P\Big(d_{TV}(\ell_t^*, \ell_{t,\omega}^{(\mathbf{e})}) > \delta\Big) = 0,$$

where d_{TV} denotes the distance in total variation.

Proof. The strategy is to build a coupling between ℓ_t^* and $\ell_{t,\omega}^{(\mathbf{e})}$ such that

$$\lim_{t\to\infty} P(P_{0,\omega}(\ell_t^* \neq \ell_{t,\omega}^{(\mathbf{e})}) > \delta) \to 0.$$

Let us first associate to the exponential variable $\mathbf{e}_i$ the following geometric random variable

$$N_i := \Big\lfloor \big(-\frac{1}{\log(p_i(\omega))}\big)\mathbf{e}_i \Big\rfloor,$$

where $1 - p_i(\omega)$ denotes the probability for the random walk starting at b_i to go to d_i before returning to b_i, which is equal to $\omega_b \frac{e^{V(b_i)}}{\sum_{x=b_i}^{d_i-1} e^{V(x)}}$. The parameter of this geometric law is now clearly equal to $1 - p_i(\omega)$.

Now one can introduce like in [15] two random variables $F^{(i)}$ (resp. $S^{(i)}$) whose law are given by the time it takes for the random walk reflected at a_i, starting at b_i, to hit b_i (resp. d_i) conditional on the event that d_i (resp. b_i) is not hitten in between.

We introduce now a sequence of independent copies of $F^{(i)}$ we denote by $(F_n^{(i)})_{n\geq 0}$. The law of τ_i^* is clearly the same as $F_1^{(i)} + \cdots + F_{N_i}^{(i)} + S^{(i)}$ which is going now to be compared with $E_\omega[\tau_i^*]\mathbf{e}_i$.

Let us now estimate, for a given $\xi > 0$ (small enough),

$$\begin{aligned}
&\mathbb{P}\Big((1-\xi)(F_1^{(i)} + \cdots + F_{N_i}^{(i)} + S^{(i)}) \leq E_\omega[\tau_i^*]\mathbf{e}_i < (1+\xi)(F_1^{(i)} + \cdots + F_{N_i}^{(i)} + S^{(i)})\Big) \\
\geq\ &\mathbb{P}\left((1-\tfrac{\xi}{2})(F_1^{(i)} + \cdots + F_{N_i}^{(i)}) \leq E_\omega[\tau_i^*]\mathbf{e}_i < (1+\tfrac{\xi}{2})(F_1^{(i)} + \cdots + F_{N_i}^{(i)})\right) \\
&\qquad -\mathbb{P}\Big(S^{(i)} > \frac{\xi}{3}(F_1^{(i)} + \cdots + F_{N_i}^{(i)})\Big).
\end{aligned} \tag{7.1}$$

Let us first treat the second quantity of the rhs of (7.1). For this purpose, we need an upper bound for $E_\omega[S^{(i)}]$ which is obtained exactly like in Lemma 13 of [15] and can be estimated by controlling the size of the falls (resp. rises) of the potential during its rises from $V(b_i)$ to $V(c_i)$ (resp. falls from $V(c_i)$ to $V(d_i)$), see Lemma 4.6. Indeed, the random variable $S^{(i)}$ concerns actually the random walk which is *conditioned* to hit d_i before b_i. Therefore, this involves an h-process which can be viewed as a random walk in a modified potential between b_i and d_i. This potential has a decreasing trend (which encourages the particle to go to the right), and the main contribution to $S^{(i)}$ comes from the small risings of this potential along its global fall which are similar to the fluctuations of the original potential during its fall and similar to their opposite during its rise.

This reasoning yields for δ small enough (one easily observes that the smaller δ, the stronger the result)

$$\forall 0 < \varepsilon < \delta, \quad P(E_\omega[S^{(i)}] \leq t^\delta) = 1 - o(t^{-\kappa\varepsilon}).$$

This implies, by Markov inequality, that

$$\forall\delta>0,\quad P\Big(P_\omega(S^{(i)}>t^{2\delta})<\frac{1}{t^\delta}\Big)=1-o\Big(\frac{1}{(\log t)^2}\Big).$$

On the other hand, $P_\omega(F_1^{(i)}+\dots+F_{N_i}^{(i)}<t^{2\delta})\le P_\omega(N_i<t^{2\delta})=1-p_i(\omega)^{\lfloor t^{2\delta}\rfloor}$. But, obviously,

$$P\Big(p_i(\omega)<1-\frac{1}{\sqrt{t}}\Big)=o\Big(\frac{1}{(\log t)^2}\Big). \tag{7.2}$$

Hence, we have

$$P\Big(P_\omega(F_1^{(i)}+\dots+F_{N_i}^{(i)}<t^{2\delta})\le\frac{1}{t^{\frac12-2\delta}})\Big)=1-o\Big(\frac{1}{(\log t)^2}\Big).$$

Gathering these two informations on $S^{(i)}$ and $F_1^{(i)}+\dots+F_{N_i}^{(i)}$, we obtain that, for all $\xi>0$,

$$\mathbb{P}\Big(S^{(i)}>\frac{\xi}{3}(F_1^{(i)}+\dots+F_{N_i}^{(i)})\Big)=o\Big(\frac{1}{(\log t)^2}\Big).$$

The first quantity of (7.1) is treated by going through

$$\mathbb{P}\Big((1-\frac{\xi}{4})N_iE_\omega[F^{(i)}]\le F_1^{(i)}+\dots+F_{N_i}^{(i)}\le(1+\frac{\xi}{4})N_iE_\omega[F^{(i)}]\Big),$$

which, for all $\delta>0$, is larger than

$$1-\mathbb{P}\left(\left\{\left|\frac{F_1^{(i)}+\dots+F_{N_i}^{(i)}}{N_i}-E_\omega[F^{(i)}]\right|>\frac{\xi}{4}E_\omega[F^{(i)}]\right\}\cap\{N_i\neq0\}\cap\{E_\omega[(F^{(i)})^2]\le t^\delta\}\right)$$
$$-P(E_\omega[(F^{(i)})^2]\ge t^\delta),$$

which is in turn, using Bienaimé-Chebicheff inequality, larger than

$$1-E\Big[E(\frac{t^\delta}{N_i}\mathbf{1}_{(N_i\neq0)}\,|\,N_i)\frac{16}{\xi^2E_\omega\,[F]^2}\Big]-P(E_\omega[(F^{(i)})^2]\ge t^\delta)$$

$$\ge1-\frac{16t^\delta}{\xi^2}E\Big[\frac{1}{N_i}\mathbf{1}_{(N_i\neq0)}\Big]-P(E_\omega[(F^{(i)})^2]\ge t^\delta).$$

Now, we use again the reasoning of [15] involving h-processes to get an upper bound for $E_\omega[(F^{(i)})^2]$ (see Lemma 11 of [15]), which is, like for $E_\omega[S^{(i)}]$, estimated by controlling the small fluctuations of the potential inside the valleys, see Lemma 4.6. We are even in a more favorable setting than in [15], since the number of valleys we have to control is much smaller (see Remark 3.1). So, we get

$$\forall\delta>0,\quad P(E_\omega[(F^{(i)})^2]\ge t^\delta)=o\Big(\frac{1}{(\log t)^2}\Big).$$

Moreover, using (7.2), we get

$$E\Big[\frac{1}{N_i}\mathbf{1}_{(N_i\neq0)}\Big]=E\Big[-\frac{1-p_i(\omega)}{p_i(\omega)}\log(1-p_i(\omega))\Big]=o\Big(\frac{1}{t^{1/3}}\Big).$$

As a result,

$$\mathbb{P}\Big((1-\frac{\xi}{4})N_iE_\omega[F^{(i)}] \le F_1^{(i)}+\cdots+F_{N_i}^{(i)} \le (1+\frac{\xi}{4})N_iE_\omega[F^{(i)}]\Big) = 1-o\Big(\frac{1}{(\log t)^2}\Big).$$

Now, the second step in the estimation of the first quantity of the rhs of (7.1) is the examination, for $\xi>0$, of

$$\mathbb{P}\Big((1-\frac{\xi}{4})N_iE_\omega[F^{(i)}] \le E_\omega[\tau_i]\mathbf{e}_i \le (1+\frac{\xi}{4})N_iE_\omega[F^{(i)}]\Big)$$

i.e.

$$\mathbb{P}\Big((1-\frac{\xi}{4})N_iE_\omega[F^{(i)}] \le (E_\omega[N_i]E_\omega[F^{(i)}]+E_\omega[S^{(i)}])\mathbf{e}_i \le (1+\frac{\xi}{4})N_iE_\omega[F^{(i)}]\Big).$$

Neglecting again, like above, the contribution of $S^{(i)}$ we are back to prove that

$$\mathbb{P}\Big((1-\frac{\xi}{4})\Big\lfloor(-\frac{1}{\log(p_i(\omega))})\mathbf{e}_i\Big\rfloor \le \frac{p_i(\omega)}{1-p_i(\omega)}\mathbf{e}_i \le (1+\frac{\xi}{4})\Big\lfloor(-\frac{1}{\log(p_i(\omega))})\mathbf{e}_i\Big\rfloor\Big) = 1-o\Big(\frac{1}{(\log t)^2}\Big),$$

which is a direct consequence of (7.2) and the fact that, for all $\varepsilon>0$,

$$P^{(\mathbf{e})}\Big(\mathbf{e}_i > \frac{1}{t^{1/2-\varepsilon}}\Big) = 1-o\Big(\frac{1}{(\log t)^2}\Big).$$

This concludes the proof that the lhs of (7.1) is $1-o(\frac{1}{(\log t)^2})$.

Now, since $P(K_t \le (\log t)^{\frac{1+\kappa}{2}}) \to 1$, when $t\to\infty$, we deduce,

$$\mathbb{P}\Big(\forall i\le K_t,\ (1-\xi)(F_1^{(i)}+\cdots+F_{N_i}^{(i)}+S^{(i)}) \le E_\omega[\tau_i^*]\mathbf{e}_i < (1+\xi)(F_1^{(i)}+\cdots+F_{N_i}^{(i)}+S^{(i)})\Big) \to 1.$$

Hence,

$$\mathbb{P}\Big(\forall i\le K_t,\ (1-\xi)(\tau_1^*+\cdots+\tau_i^*) \le \sum_{k=1}^{i} E_\omega[\tau_k^*]\mathbf{e}_k < (1+\xi)(\tau_1^*+\cdots+\tau_i^*)\Big) \to 1.$$

Applying this, for $i=\ell_t^*$ and $i=\ell_{t,\omega}^{(\mathbf{e})}$ we get respectively that, for all $\xi>0$,

$$\mathbb{P}\Big(\ell_t^* \le \ell^{(\mathbf{e})}_{\frac{t}{1-\xi},\omega}\Big) \to 1 \quad \text{and} \quad \mathbb{P}(\ell_{t,\omega}^{(\mathbf{e})} \le \ell^*_{t(1+\xi)}) \to 1.$$

We conclude now the proof by reminding that $\lim_{\xi\to0}\mathbb{P}(\ell_t^*=\ell^*_{(1+\xi)t})=1$ as well as $\lim_{\xi\to0}\mathbb{P}(\ell_{t,\omega}^{(\mathbf{e})}=\ell^{(\mathbf{e})}_{(1+\xi)t,\omega})=1$. □

Proof of part (ii) of Theorem 2.3. The passage from Proposition 7.1 to part (*ii*) of Theorem 2.3 is of the same kind as the passage from Proposition 5.1 to Proposition 5.4. □

8 Proof of Theorem 2.1

We fix $h > 1$ and $\eta > 0$ (η was used to define the event $A_5(t)$). Let us introduce the event

$$TT(t,h) := TT(t) \cap \{X_t - b_{\ell_t} \leq \frac{\eta}{2}\log t\} \cap \{X_{th} - b_{\ell_{th}} \leq \frac{\eta}{2}\log t\},$$

whose probability tends to 1, when t tends to infinity (it is a consequence of Section 4 together with part (ii) of Theorem 2.3). Then, we easily have

$$(\{\ell_{th} = \ell_t\} \cap TT(t,h)) \subset (\{|X_{th} - X_t| \leq \eta \log t\} \cap TT(t,h)).$$

Moreover, observe that on $TT(t)$, $\ell_{th} > \ell_t$ implies that $|b_{\ell_{th}} - b_{\ell_t}| \geq t^{\kappa/2}$ (by definition of $A_3(t)$). Therefore, we get

$$(\{|X_{th} - X_t| \leq \eta \log t\} \cap TT(t,h)) \subset (\{\ell_{th} = \ell_t\} \cap TT(t,h)),$$

for all large t. Thus, since Proposition 5.4 implies that $\lim_{t\to\infty} \mathbb{P}(\ell_{th} = \ell_t)$ exists, we obtain

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{P}(|X_{th} - X_t| \leq \eta \log t) &= \lim_{t\to\infty} \mathbb{P}(\ell_{th} = \ell_t) \\
&= \lim_{t\to\infty} \mathbb{P}(T_{\ell_t+1} \geq th) \\
&= \frac{\sin(\kappa\pi)}{\pi} \int_{h-1}^{\infty} \frac{\mathrm{d}x}{x^\kappa(1+x)} \\
&= \frac{\sin(\kappa\pi)}{\pi} \int_0^{1/h} y^{\kappa-1}(1-y)^{-\kappa}\,\mathrm{d}y,
\end{aligned}$$

which concludes the proof of Theorem 2.1. □

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