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On the inviscid limit of a model for crack propagation

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Abstract

We study the evolution of a single crack in an elastic body and assume that the crack path is known in advance. The motion of the crack tip is modeled as a rate-independent process on the basis of Griffith's local energy release rate criterion. According to this criterion, the system may stay in a local minimum before it performs a jump. The goal of this paper is to prove existence of such an evolution and to shed light on the discrepancy between the local energy release rate criterion and models which are based on a global stability criterion (as for example the Francfort/Marigo model). We construct solutions to the local model via the vanishing viscosity method and compare different notions of weak, local and global solutions.

1 Introduction

The prediction of the growth of cracks in brittle materials is of importance in many practical applications. However, mathematical models involving the full elastic interaction as well as the evolution of a freely growing crack are rare. Only within the last decade such models were developed based on the pioneering work in [FrM93, FrM98] that developed a quasistatic framework based on energy minimization. In a series of technical papers [DaT02, FrL03, DFT05, FrG06] the necessary analytical results have been developed to provide existence results for such solutions. In this setting the crack path may be an arbitrary set of finite Hausdorff dimension $d-1$ with the restriction that it is a non-decreasing family as a function of time. The displacements are allowed to lie in the function space GSBV (generalized special functions of bounded variations), where for each time instant the jump set of the deformation has to be contained in the corresponding crack set.

These solutions are in fact special cases of the so-called energetic solution for rate-independent processes as developed in [MiT99, MTL02, CHM02] for modeling the evolution of phase transformations in shape-memory materials or elastoplasticity. The energetic solutions can be considered as weak solutions of the flow laws usually posed in engineering. For the crack problem this relates to the Griffith criterion [Gri20] that states that a crack grows as soon as the energy release rate is bigger than the fracture toughness and it is stationary otherwise. The energetic concept is based on a global energetic stability principle that says that a crack grows if there is any bigger crack such that the total energy release is larger than the energy dissipated by creating the crack (surface). Otherwise the state is (globally) stable. A process is called an "irreversible quasistatic evolution" or, equivalently, an "energetic solution", if for each time instant the state is (globally) stable and the total energy balance holds.

In this work, we are interested in the discrepancy between the local energy-release-rate criterion (Griffith) and the global stability criterion. The problem is that energetic solutions tend to jump earlier because global minimizers are used. In many systems it is expected that physical systems will stay in local minimizers, and hence crack growth will occur later.

To generate solutions staying in local minimizers we will use the vanishing viscosity limit which again is close to the physical modeling. In fact, true physical systems are not strictly rate-independent but have some internal time scales (relaxation times) that are usually neglected when very slow loading is considered. However, if the rate-independent solutions are not continuous, then the corresponding solution with small viscosity develops very large rates that are governed by the viscosity. The aim is to understand the limits of viscous solutions when the viscosity is made smaller and smaller, see [EfM06, MRS07] for the general philosophy. For nontrivial PDE applications see also [DD*07, MiZ07].

The application of this idea to crack problems turns out to be technically very difficult. Hence, all of the rigorous results are restricted to problems where the crack path is prescribed in advance and either (i) the position of the crack tip is to be determined (cf. [NeO07, ToZ06]) or (ii) a function along the crack path, which measures the maximal opening of the crack, is to be calculated in so-called cohesive zone models or delamination problems, cf. [KMR06, Cag07].

In this work we mainly study the motion of one crack tip that is driven by stresses arising from elastic deformations. We fix an arbitrary crack path that is assumed to be twice continuously differentiable. We consider small strains and assume that the elastic energy is coercive and strictly convex, but not necessarily quadratic or uniformly convex. The external loading occurs through time-dependent displacement boundary conditions as well as volume and surface loading. Having given these data, we define the stored energy functional $\mathcal{E}$ on $[0,T]\times\mathcal{Q}$, for a suitable state space $\mathcal{Q}$, as the elastic energy minus the work of external loadings. The dissipative nature of the crack propagation is encoded in a fracture-toughness function $\kappa : [s_0, s_1] \to]0,\infty[$, which we assume to be continuous, and a positive viscosity parameter ν. The viscous crack-tip propagation problem for determining the displacement $u(t)$ and the crack-tip position $s(t)$ reads

$$\begin{aligned} &u(t) = \operatorname{argmin}\{\, \mathcal{E}(t, v, s(t)) \mid v \in \mathcal{Q}\,\}, \\ &0 \in \partial_{\dot{s}}\mathcal{R}_0(s(t), \dot{s}(t)) + \nu\dot{s}(t) - \mathcal{G}(t, u(t), s(t)), \end{aligned} \tag{1.1}$$

where $\mathcal{R}_0(s,\dot{s}) = \kappa(s)\dot{s}$ for $\dot{s} \geq 0$ and ∞ otherwise. The generalized energy-release rate $\mathcal{G}$ takes the form

$$\mathcal{G}(t,v,s) := -\lim_{\delta\to 0}\frac{1}{\delta}\Big(\mathcal{E}(t, v\circ T_{s,\delta}^{-1}, s{+}\delta) - \mathcal{E}(t,v,s)\Big),$$

where $T_{s,\delta}$ is a diffeomorphism between the domains with crack length s and $s+\delta$, respectively (see Section 3.2 for details).

In Section 2 we give the precise definitions and state the existence result that (1.1) has a solution $(u^\nu, s^\nu) \in \mathrm{L}^\infty([0,T];\mathrm{W}^{1,p}) \times \mathrm{H}^1([0,T])$ for each $\nu > 0$. The proof is done in Section 4 using a time-incremental minimization procedure.

The main goal of this work is to study the limiting behavior of (u^ν, s^ν) for the vanishing viscosity limit $\nu \to 0$ and to identify a rate-indepedent limit problem, which is satisfied by all possible limit solutions. For this purpose we use the convexity of $\mathcal{E}(t,\cdot,s)$, which guarantees that $u \mapsto \mathcal{E}(t,u,s)$ has a unique minimizer $\mathcal{U}(t,s)$. We define the reduced functional $\mathcal{I} : [0,T] \times [s_0, s_1] \to \mathbb{R}$ by minimizing out the displacements:

$$\mathcal{I}(t,s) := \mathcal{E}(t, \mathcal{U}(t,s), s).$$

The first major result (see Theorem 3.6) states that under fairly general conditions on the elastic energy $\mathcal{E}$ the reduced functional $\mathcal{I}$ is continuously differentiable and satisfies the relation

$$G(t,s) := -\partial_s \mathcal{I}(t,s) \;=\; \mathcal{G}(t, \mathcal{U}(t,s), s). \tag{1.2}$$

Moreover, we obtain an explicit formula for $G(t,s)$ in terms of the Eshelby tensor associated with $\mathcal{U}(t,s)$. Actually, we provide simplified proofs for more general situations and derive Theorem 3.6 from an abstract Theorem 3.2. In this theorem, we study the differentiability properties of reduced energies, which correspond to rather general (elastic) energy functionals depending on a finite number of parameters. Theorem 3.2 is also applicable to the case with interface cracks, non-interpenetration conditions and to finite-strain elasticity, where the energy density is no more convex, but polyconvex and may take the value $+\infty$. We refer to [DeD81, KhS00, Kne06, KnM07] for the discussion of representative special cases.

In Section 5 we study the limit behavior. Using suitable a priori estimates, we show that a subsequence converges pointwise on $[0,T]$ to a limiting process $s \in \mathrm{BV}([0,T])$. Moreover, defining the jump set $J(s) = \{\, t \in [0,T] \mid s(t+) \neq s(t-) \,\}$ and the set of differentiability $D(s) = \{\, t \in [0,T] \mid \dot{s}(t) \text{ exists } \}$, then any such limit has to satisfy the following rate-independent limit problem: $u(t) = \mathcal{U}(t, s(t))$ and

(a) $s : [0,T] \to [s_0, s_1]$ is nondecreasing;

(b) $\kappa(s(t)) - \mathcal{G}(t, u(t), s(t)) \geq 0$ for all $t \in [0,T] \backslash J(s)$;

(c) if $\kappa(s(t)) - \mathcal{G}(t, u(t), s(t)) > 0$ then $t \in D(s)$ and $\dot{s}(t) = 0$;

(d) for all $t \in J(s)$ and all $s_* \in [s(t-), s(t+)]$ we have $\kappa(s_*) - \mathcal{G}(t, \mathcal{U}(t, s_*), s_*) \leq 0$.

Here (a) provides the irreversibility saying that a crack can never heal. In (b) we see that the release rate $\mathcal{G}$ can never exceed the fracture toughness except in jumps, while (c) says that a crack cannot move if the release rate $\mathcal{G}$ is strictly less than the fracture toughness κ. Condition (d) states that along a jump path the release rate can never be smaller than the fracture toughness as then the crack would immediately stop, see (c).

Our formulation of the limit process via (a)–(d) is essentially the same as that given in [NeO07]. However, our approach using the vanishing-viscosity method is completely different from the monotonicity approach there. In fact, our approach can be generalized in several aspects. First we may allow healing of cracks by adding to the stored energy a suitable surface term and redefining $\mathcal{R}_0$ as $\kappa^+(s)\dot{s}$ for $\dot{s} \geq 0$, and as $\kappa^-(s)|\dot{s}|$ for $\dot{s} \leq 0$. Moreover, we are able to treat the case of several noninteracting cracks in one body, see Section 7 for details. In the latter case we rely on the theory developed in [EfM06].

2 Problem formulation and results

2.1 Setting of the problem

Throughout the paper we assume that the conditions described in this paragraph are satisfied.

Let $\Omega \subset \mathbb{R}^2$ be open, bounded with Lipschitz boundary $\partial\Omega$. We assume that $\partial\Omega$ is the union of two disjoint subsets Γ_D and Γ_N, with $\mathscr{H}^1(\Gamma_D) > 0$, where $\mathscr{H}^1$ denotes the one dimensional Hausdorff measure.

The prescribed crack path is a simple C^2-path $\mathcal{C} \subset \overline{\Omega}$ with $\mathscr{H}^1(\mathcal{C}) := L$ and let $\gamma : [0, L] \to \mathcal{C}$ be its arc-length parameterization. We assume that for every $s \in]0, L[$ we have $\gamma(s) \in \overline{\Omega} \setminus \partial\Omega$, while the endpoints of $\mathcal{C}$, that is $\gamma(0)$ and $\gamma(L)$, can meet the boundary $\partial\Omega$. Let us fix $0 < s_0 < s_1 < L$ and for each $s \in [s_0, s_1]$ we define the admissible crack set by $\mathcal{C}_s := \{\, \gamma(\sigma) \mid 0 \leq \sigma \leq s \,\}$. The cracked domain is then the set $\Omega_s := \Omega \setminus \mathcal{C}_s$.

We consider small strain elasticity and assume that the stored energy density $\widetilde{W} : \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}$ belongs to $\mathrm{C}^1(\mathbb{R}^{2\times 2}_{\mathrm{sym}}; \mathbb{R})$ and is strictly convex. Furthermore, there exist $p \in (1, \infty)$ and constants $c_i > 0$ such that for every $A \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ we have

$$c_1 \, |A|^p - c_2 \leq \widetilde{W}(A) \leq c_3(1 + |A|^p). \tag{2.1}$$

The convexity of $\widetilde{W}$ and (2.1) imply that there is a constant $c_4 > 0$ such that

$$\big|\mathrm{D}\widetilde{W}(A)\big| \leq c_4(1 + |A|^{p-1}) \tag{2.2}$$

for every $A \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$. Here, $\mathrm{D}\widetilde{W} : \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ denotes the derivative of $\widetilde{W}$. The given Dirichlet datum and the applied forces shall satisfy

$$\begin{gathered} u_{\mathrm{Dir}} \in \mathrm{C}^1([0, T]; \mathrm{W}^{1,p}(\Omega_{s_0/2}; \mathbb{R}^2)), \\ f \in \mathrm{C}^1([0, T]; \mathrm{W}^{1,q}(\Omega; \mathbb{R}^2)), \quad h \in \mathrm{C}^1([0, T]; L^q(\Gamma_N; \mathbb{R}^2)), \end{gathered} \tag{2.3}$$

where $p^{-1} + q^{-1} = 1$. The rather strong assumption $f(t) \in \mathrm{W}^{1,q}(\Omega)$ is made for technical reasons and could slightly be weakened, see Remark 3.7. For shortness, we put

$$\langle \ell(t), v \rangle := \int_\Omega f(t) \cdot v \, \mathrm{d}x + \int_{\Gamma_N} h(t) \cdot v \, \mathrm{d}\sigma$$

for every $v \in W^{1,p}(\Omega_{s_1}; \mathbb{R}^2)$. For given $t \in [0, T]$, $x \in \Omega$ and $A \in \mathbb{R}^{2\times 2}$ we define

$$W(t, x, A) := \widetilde{W}((A + \nabla u_{\mathrm{Dir}}(t))_{\mathrm{sym}}),$$

where $A_{\mathrm{sym}} = \frac{1}{2}(A + A^\top)$ is the symmetric part of A. Furthermore, we set

$$W^{1,p}_{\Gamma_D}(\Omega_s; \mathbb{R}^2) := \{\, w \in W^{1,p}(\Omega_s; \mathbb{R}^2) \mid w = 0 \text{ on } \Gamma_D \,\},$$

and the equality is understood in the sense of traces. We assume that the state space $\mathcal{Q}$ is the product

$$\mathcal{Q} := \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_{s_1}; \mathbb{R}^2) \times [s_0, s_1].$$

On this state space we define energy functional $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{E}(t,u,s) := \begin{cases} \int_{\Omega_s} W(t,x,\nabla u(x))\,\mathrm{d}x - \langle \ell(t), u\rangle & \text{if } u \in \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_s;\mathbb{R}^2) \\ \infty & \text{else.} \end{cases} \tag{2.4}$$

The assumption on $\widetilde{W}$ and the data guarantee that for every $t \in [0,T]$ and $s \in [s_0, s_1]$ there exists a unique element $\mathcal{U}(t,s) \in W^{1,p}_{\Gamma_D}(\Omega_s)$ with

$$\mathcal{U}(t,s) = \operatorname{argmin} \mathcal{E}(t,\cdot,s). \tag{2.5}$$

The reduced energy $\mathcal{I} : [0,T] \times [s_0,s_1] \to \mathbb{R}$ is defined as

$$\mathcal{I}(t,s) := \min\{\, \mathcal{E}(t,v,s) \mid v \in W^{1,p}_{\Gamma_D}(\Omega_{s_1};\mathbb{R}^2)\,\} = \mathcal{E}(t,\mathcal{U}(t,s),s). \tag{2.6}$$

We observe that for any $t \in [0,T]$ and any $s \in [s_0,s_1]$ we have

$$\mathcal{I}(t,s) = \mathcal{E}(t,\mathcal{U}(t,s),s) \le \mathcal{E}(t,0,s) < \infty.$$

By the definition of $\mathcal{E}$, our assumption (2.1), and Hölder's inequality we derive

$$\int_{\Omega_s} \big[c_1 |(\nabla\mathcal{U}(t,s)+\nabla u_{\mathrm{Dir}}(t))_{\mathrm{sym}}|^p - c_2\big]\,\mathrm{d}x \le \mathcal{E}(t,0,s) + \|\ell(t)\|_{(\mathrm{W}^{1,p}_{\Gamma_D}(\Omega_s;\mathbb{R}^2))'} \|\mathcal{U}(t,s)\|_{\mathrm{W}^{1,p}_{\Gamma_D}(\Omega_s;\mathbb{R}^2)}.$$

Applying then Korn's inequality to the left hand side and Young's inequality to the last term on the right hand side and using the assumptions on the data ℓ and u_{Dir}, we finally obtain that there exists a positive constant (independent of t and s) such that

$$\|\mathcal{U}(t,s)\|_{\mathrm{W}^{1,p}(\Omega_s;\mathbb{R}^2)} \le C.$$

We fix once and for all $u_0 := \mathcal{U}(0,s_0)$ and we are interested in finding an evolution starting from (u_0,s_0).

The energy release rate is defined by

$$G(t,s^*) := -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(t,\mathcal{U}(t,s),s)\Big|_{s=s^*} = -\frac{\partial}{\partial s}\mathcal{I}(t,s^*). \tag{2.7}$$

In Theorem 3.6 we show $\mathcal{I} \in \mathrm{C}^1([0,T]\times[s_0,s_1])$ and, hence, G is continuous. For the explicit formula and further properties of G, we refer to Theorem 3.6 again. In particular it holds that $G(t,s) = \mathcal{G}(t,\mathcal{U}(t,s),s)$ and

$$G_{\max} := \sup\{\, G(t,s) \mid (t,s) \in [0,T]\times[s_0,s_1]\,\} < \infty.$$

The motion of the crack tip is associated with the dissipation of energy via a dissipation potential $\mathcal{R}$. Let $\kappa \in \mathrm{C}^0([0,L])$ be positive and ν nonnegative, and define the dissipation potential

$$\mathcal{R}_\nu(s,\dot s) := \begin{cases} \kappa(s)\dot s + \frac{\nu}{2}\dot s^2 & \text{if } \dot s \ge 0 \\ \infty & \text{else.} \end{cases} \tag{2.8}$$

The function κ takes into account the toughness of the material. Throughout the paper we will assume

$$\kappa(s_1) > G_{\max}. \tag{2.9}$$

This condition will prevent the evolution $s(t)$ from reaching the endpoint s_1. On the other hand, in order to obtain a nontrivial evolution, we will assume

$$\kappa(s_0) < G_{\max}. \tag{2.10}$$

We are now ready to define the viscous crack evolution model (Section 2.2) and to formulate the rate-independent limit problem (Section 2.3). In the remainder of this section we formulate the different types of solutions (u, s) in terms of the elastic equilibrium condition and a crack-propagation law. To highlight the coupling between these two balance laws we use the full energy functional $\mathcal{E}$ and the generalized energy-release rate $\mathcal{G}$. Of course, using the elastic equilibrium $u(t) = \mathcal{U}(t, s(t))$ we have $\mathcal{I}(t, s(t)) = \mathcal{E}(t, u(t), s(t))$ and the crucial identity (1.2), namely $G(t, s(t)) = \mathcal{G}(t, u(t), s(t))$. In fact, in Section 4 and 5 the proofs depend essentially on this reduction to a problem in s alone.

2.2 Viscous problem

We start with our notion of viscous solution, depending on a (small) parameter ν.

Definition 2.1 *For $\nu > 0$, a* **viscous solution** *associated with $\mathcal{E}$ and $\mathcal{R}_\nu$ is a map $t \mapsto (u^\nu(t), s^\nu(t))$ with $u^\nu \in \mathrm{L}^\infty([0,T]; \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_{s_1}; \mathbb{R}^2))$, $s^\nu \in \mathrm{H}^1([0,T]; [s_0, s_1])$ satisfying*

$$u^\nu(t) = \mathcal{U}(t, s^\nu(t)) := \operatorname{argmin} \mathcal{E}(t, \cdot, s^\nu(t)) \qquad \text{for every } t \in [0,T] \tag{2.11}$$

$$0 \in \partial_{\dot s} \mathcal{R}_\nu(s^\nu(t), \dot s^\nu(t)) - \mathcal{G}(t, u^\nu(t), s^\nu(t)) \qquad \text{for a.e. } t \in [0,T]. \tag{2.12}$$

We note that from the definition it follows that $u^\nu(t) \in \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_{s^\nu(t)}; \mathbb{R}^2)$ for every $t \in [0,T]$. Moreover, it is not difficult to prove that any viscous solution $(u^\nu(\cdot), s^\nu(\cdot))$ associated with $\mathcal{E}$ and $\mathcal{R}_\nu$ guarantees that the map $t \mapsto \partial_t \mathcal{E}(t, u^\nu(t), s^\nu(t)) \in \mathrm{L}^1(0,T)$ and that the following energy balance condition is satisfied for every $0 \le t_1 < t_2 \le T$ (for a proof see Lemma 4.5 below):

$$\begin{aligned} &\mathcal{E}(t_2, u^\nu(t_2), s^\nu(t_2)) + \int_{t_1}^{t_2} \big(\kappa(s^\nu(t))\dot s^\nu(t) + \nu|\dot s^\nu(t)|^2\big)\,\mathrm{d}t \\ &= \mathcal{E}(t_1, u^\nu(t_1), s^\nu(t_1)) + \int_{t_1}^{t_2} \partial_t \mathcal{E}(t, u^\nu(t), s^\nu(t))\,\mathrm{d}t. \end{aligned} \tag{2.13}$$

The main result of this section is the following one, the proof is given in Section 4.2 after Theorem 4.2.

Theorem 2.2 *There exists a viscous solution $t \mapsto (u^\nu(t), s^\nu(t))$ associated with $\mathcal{E}$ and $\mathcal{R}_\nu$ such that $(u^\nu(0), s^\nu(0)) = (u_0, s_0)$.*

2.3 Rate-independent limit

We are now interested in the limit of the solutions (u^ν, s^ν) in the case of vanishing viscosity, i.e., $\nu \to 0$. The limit $s : [0,T] \to [s_0, s_1]$ will in general not stay continuous but will lie in $\mathrm{BV}([0,T])$ only. We want to make precise what can be said about the limits and define a limit problem that contains as much information about the limits as possible, in particular at jump points.

We recall some basic properties of general functions in $\mathrm{BV}([0,T])$ and introduce some notations to formulate the limit problem. For a function $s \in \mathrm{BV}([0,T])$ the limit from the right $s(t+)$ and the limit $s(t-)$ from the left exist for all $t \in [0,T]$, if we let $s(0-) = s(0)$ and $s(T+) = s(T)$. As common in rate-independent evolution problems we consider the function s to be defined everywhere such that the three values $s(t-)$, $s(t)$, and $s(t+)$ may be different. We define the jump set $J(s) \subset [0,T]$ to be the set of points where s is not continuous.

The distributional derivative $\mathrm{D}s$ of s is a bounded, signed measure that can be decomposed into three parts, namely $\mathrm{D}s = \mathrm{D}^j s + \dot{s}\mathrm{d}t + \mathrm{D}^c s = \mathrm{D}^j s + \widetilde{\mathrm{D}}s$. Here $\widetilde{\mathrm{D}}s = \dot{s}\mathrm{d}t + \mathrm{D}^c s$ is the diffuse part of the derivative $\mathrm{D}s$, while $\mathrm{D}^j s$ is the discrete part associated with the jumps, namely $\mathrm{D}^j s = \sum_{t\in J(s)}(s(t+)-s(t-))\delta_t$. Let $D(s) \subset [0,T]$ denote the set of points where s is differentiable, $\dot{s}(t) = \lim_{h\to 0}(s(t+h) - s(t))/h$, then $D(s)$ has full measure and $\dot{s} \in \mathrm{L}^1([0,T])$.

Note that in general the fundamental theorem of calculus $s(t_2) - s(t_1) = \int_{t_1}^{t_2} \dot{s}(t)\,\mathrm{d}t$ does not hold because of jumps and because of the singular part. However, we have

$$s(t_2) - s(t_1) = \int_{]t_1,t_2[} \mathrm{D}s(\mathrm{d}t) + \big(s(t_2) - s(t_2-)\big) - \big(s(t_1) - s(t_1+)\big) \tag{2.14}$$

because we did not enforce continuity from the left or from the right, and there is a suitable generalization for the chain rule (see (5.9)). To avoid all these complications the following formulation does not make usage of derivatives like in the global energetic formulation (GES) given in Definition 2.5.

Definition 2.3 *A* **local energetic solution** *to the rate-independent problem associated with $\mathcal{E}$ and $\mathcal{R}_0$ is a map $t \mapsto (u(t), s(t))$ with $u \in \mathrm{L}^\infty([0,T]; \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_{s_1};\mathbb{R}^2))$ and $s \in \mathrm{BV}([0,T];[s_0,s_1])$ such that*

$$u(t) = \mathcal{U}(t, s(t)) := \operatorname{argmin} \mathcal{E}(t,\cdot,s(t)) \quad \textit{for every } t \in [0,T] \tag{2.15}$$

and the following four conditions hold true

(a) $s : [0,T] \to [s_0, s_1]$ *is nondecreasing;*

(b) $\kappa(s(t)) - \mathcal{G}(t, u(t), s(t)) \geq 0$ *for all* $t \in [0,T]\backslash J(s)$*;*

(c) if $\kappa(s(t)) - \mathcal{G}(t, u(t), s(t)) > 0$ *then* $t \in D(s)$ *and* $\dot{s}(t) = 0$*;*

(d) for all $t \in J(s)$ *and all* $s_* \in [s(t-), s(t+)]$ *we have* $\kappa(s_*) - \mathcal{G}(t, \mathcal{U}(t, s_*), s_*) \leq 0$.

Condition (b) states that the energy-release rate has to be smaller than the fracture toughness everywhere except at the jump times. However, assuming continuity from the left or from the right and continuity of κ and $\mathcal{G}$ would even prove this estimate at the jump times $t \in J(s)$. Condition (c) states that the crack cannot move if the energy-release rate is strictly less than the fracture toughness. Thus, so far the evolution is in full accordance with the Griffith criterion. Finally, condition (d), which is the essential new feature of the present formulation, states that during a jump the energy-release rate is not allowed to go below the fracture toughness. It is clear that this formulation is local in the sense that the evolution of s is determined solely by local properties of κ and $\mathcal{G}$.

We observe that if $(u(\cdot), s(\cdot))$ is a local energetic solution, then since $\mathcal{E}(t, u(t), s(t)) < \infty$ we have $u(t) \in \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_{s(t)}; \mathbb{R}^2)$.

As a consequence of Definition 2.3, we deduce that any local energetic solution $t \mapsto (u(t), s(t))$ associated with $\mathcal{E}$ and $\mathcal{R}_0$ satisfies $t \mapsto \partial_t \mathcal{E}(t, u(t), s(t)) \in \mathrm{L}^1([0,T])$ and the following energy inequality

$$\mathcal{E}(t_2, u(t_2), s(t_2)) + \int_{s(t_1)}^{s(t_2)} \kappa(s)\,\mathrm{d}s \le \mathcal{E}(t_1, u(t_1), s(t_1)) + \int_{t_1}^{t_2} \partial_t \mathcal{E}(t, u(t), s(t))\,\mathrm{d}t, \quad (2.16)$$

holds true for every $0 \le t_1 \le t_2 \le T$ (for a proof see Corollary 5.6).

For each jump time $t \in J(s)$ we define the nonnegative quantities $\Delta^+(t)$ and $\Delta^-(t)$ by

$$\begin{aligned} \Delta^+(t) &:= \int_{s(t)}^{s(t+)} [\mathcal{G}(t, \mathcal{U}(t,s), s) - \kappa(s)]\,\mathrm{d}s \ge 0, \\ \Delta^-(t) &:= \int_{s(t-)}^{s(t)} [\mathcal{G}(t, \mathcal{U}(t,s), s) - \kappa(s)]\,\mathrm{d}s \ge 0. \end{aligned} \quad (2.17)$$

Through them, we can define a nonnegative function μ on closed subintervals of $[0,T]$ as follows:

$$\mu([t_1,t_2]) := \Delta^+(t_1) + \Delta^-(t_2) + \sum_{t \in]t_1,t_2[\cap J(s)} (\Delta^+(t) + \Delta^-(t)). \quad (2.18)$$

Note that μ is finite, since $\mathcal{G}$ and κ are bounded and the sum of all jumps does not exceed $s_1 - s_0$. Using a chain rule for BV functions, (see, e.g.,[AFP00, Theorem 3.96] and (5.9)), it is then possible to derive an exact energy balance, i.e., we are able to characterize the energy missing in (2.16) via the function μ (see Lemma 5.5). For all $0 \le t_1 < t_2 \le T$ we have

$$\begin{aligned} &\mathcal{E}(t_2, u(t_2), s(t_2)) + \int_{s(t_1)}^{s(t_2)} \kappa(\sigma)\,\mathrm{d}\sigma + \mu([t_1,t_2]) \\ &= \mathcal{E}(t_1, u(t_1), s(t_1)) + \int_{t_1}^{t_2} \partial_t \mathcal{E}(\tau, u(\tau), s(\tau))\,\mathrm{d}\tau. \end{aligned} \quad (2.19)$$

We are now in a position to state the main result of this section.

Theorem 2.4 *There exists a local energetic solution $t \mapsto (u(t), s(t))$ to the rate-independent problem associated with $\mathcal{E}$ and $\mathcal{R}_0$ such that $(u(0), s(0)) = (u_0, s_0)$. In particular, every limit point of a subsequence of viscous solutions $t \mapsto (u^\nu(t), s^\nu(t))$ starting from (u_0, s_0) is a local energetic solution.*

2.4 Discussion and comparison with other types of solutions

We give now three different notions of solutions. For this reason we need some preliminary additional notations. Via the dissipation metric $\mathcal{R}_0$ we introduce the dissipation distance $\mathcal{D} : [s_0, s_1] \times [s_0, s_1] \to [0, \infty]$ defined by

$$\mathcal{D}(s_*^0, s_*^1) := \begin{cases} \int_{s_*^0}^{s_*^1} \mathcal{R}_0(s, \mathrm{d}s) & \text{for } s_*^1 \geq s_*^0, \\ \infty & \text{otherwise.} \end{cases}$$

Obviously, $\mathcal{D}$ satisfies $\mathcal{D}(s_*, s_*) = 0$ and the triangle inequality, but we put in evidence that due to the definition of $\mathcal{R}_0$, it turns out that $\mathcal{D}$ is a non-symmetric distance, since $\mathcal{D}(s, \widetilde{s}) = \infty$ for $\widetilde{s} < s$.

The $\mathcal{D}$-dissipation of a curve s is defined by

$$\mathrm{Diss}_{\mathcal{D}}(s; [t_1, t_2]) := \sup\{ \sum_{j=1}^{M} \mathcal{D}(s(r_{j-1}), s(r_j)) \mid M \in \mathbb{N}, t_1 \leq r_0 < \cdots < r_M \leq t_2 \}.$$

We observe that $\mathrm{Diss}_{\mathcal{D}}(s; [t_1, t_2]) < \infty$ implies that $s : [t_1, t_2] \to [0, L]$ is nondecreasing and then

$$\mathrm{Diss}_{\mathcal{D}}(s; [t_1, t_2]) = \mathcal{D}(s(t_1), s(t_2)).$$

Definition 2.5 *(LS) A* **local solution** *to the rate-independent problem associated with $\mathcal{E}$ and $\mathcal{R}_0$ is a map $t \mapsto (u(t), s(t))$ with $u \in \mathrm{L}^\infty([0,T]; \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_{s_1}; \mathbb{R}^2))$ and $s \in \mathrm{BV}([0,T]; [s_0, s_1])$ satisfying the following three conditions:*

(1) local stability:

$$u(t) = \mathcal{U}(t, s(t)) := \operatorname{argmin} \mathcal{E}(t, \cdot, s(t)) \quad \text{for every } t \in [0,T], \tag{2.20}$$
$$\kappa(s(t)) - \mathcal{G}(t, u(t), s(t)) \geq 0 \quad \text{for a.e. } t \in [0,T], \tag{2.21}$$

(2) irreversibility: the map $t \mapsto s(t)$ is nondecreasing,

(3) energy inequality: the map $t \mapsto \partial_t \mathcal{E}(t, u(t), s(t))$ lies in $\mathrm{L}^1([0,T])$ and

$$\mathcal{E}(t_2, u(t_2), s(t_2)) + \int_{s(t_1)}^{s(t_2)} \kappa(\sigma)\, \mathrm{d}\sigma \leq \mathcal{E}(t_1, u(t_1), s(t_1)) + \int_{t_1}^{t_2} \partial_t \mathcal{E}(t, u(t), s(t))\, \mathrm{d}t, \tag{2.22}$$

for every $0 \leq t_1 \leq t_2 \leq T$;

(GES) a **global energetic solution** *associated with $\mathcal{E}$ and $\mathcal{D}$ is a map $t \mapsto (u(t), s(t))$ with $t \mapsto \partial_t \mathcal{E}(t, u(t), s(t)) \in \mathrm{L}^1([0,T])$ satisfying for every $t \in [0,T]$ stability (S) and energy balance (E):*

$$(S) \quad \mathcal{E}(t, u(t), s(t)) \leq \mathcal{E}(t, \widetilde{u}, \widetilde{s}) + \mathcal{D}(s(t), \widetilde{s}) \quad \forall (\widetilde{u}, \widetilde{s}) \in \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_{s_1}; \mathbb{R}^2) \times [s_0, s_1],$$
$$(E) \quad \mathcal{E}(t, u(t), s(t)) + \mathrm{Diss}_{\mathcal{D}}(s; [0,t]) = \mathcal{E}(0, u(0), s(0)) + \int_0^t \partial_t \mathcal{E}(t, u(t), s(t))\, \mathrm{d}t;$$

(AS) *an* **approximable solution** *associated with the energy functional $\mathcal{E}$ and the dissipation metric $\mathcal{R}_0$ is a local solution $t \mapsto (u(t), s(t))$ which is the point wise limit of a subsequence of some viscous solution $t \mapsto (u^\nu(t), s^\nu(t))$ associated with $\mathcal{E}$ and $\mathcal{R}_\nu$.*

Remark 2.6 We note that if $t \mapsto (u(t), s(t))$ is a local solution to the rate-independent problem associated with $\mathcal{E}$ and $\mathcal{R}_0$, then

$$0 \in \partial_{\dot{s}} \mathcal{R}_0(s(t), \dot{s}(t)) - \mathcal{G}(t, u(t), s(t)) \quad \text{for a.e. } t \in [0, T].$$

Indeed, from the energy inequality we derive

$$\big(\kappa(s(t)) - \mathcal{G}(t, u(t), s(t))\big)\dot{s}(t) \leq 0 \quad \text{for a.e. } t \in [0, T],$$

but actually we can substitute inequality by equality due to the irreversibility condition and to stability (2.21).

By the previous definition it follows that the weakest notion of solution is the local one (LS) and therefore any other solution (among those defined in this work, including the local energetic one) is in particular a local solution. Its left-continuous version corresponds to the notion of irreversible quasistatic evolution given in [ToZ06, Definition 3.1].

The study of global energetic solutions (GES) is well developed in the literature, see, e.g., [MaM05, Mie05, FrM06] (and references therein). Moreover, the notion of global energetic solution in the case of a non-symmetric dissipation distance (like in this work) corresponds to the definition of irreversible quasistatic evolution considered in [FrM98, FrL03, DFT05] (see also references therein).

We note that the left-continuous version of an approximable solution (AS) fits the definition of approximable irreversible quasistatic evolution given in [ToZ06, Definition 3.7]. Anyway, in that paper, the authors considered a different viscous approximation, taking into account viscosity also for the bulk energy in the dissipation metric, and confined themselves to the case $\widetilde{W}(\nabla u) = |\nabla u|^2$ and $\kappa(s) \equiv 1$.

In general, we expect that a global energetic solution (GES) is different from a local energetic one. On the other hand, as stated in Theorem 2.4, we will prove that any approximable solution (AS) is a local energetic solution. On the contrary, maybe not any local energetic solution is approximable. For a specific situation comparing global energetic solution (GES), local energetic solution and approximable solution (AS) with each other, see Example 6.3.

The more general concept of **BV-solution** has been recently introduced in [MRS07]. This notion works on general metric spaces, but in the context of the present work it coincides with the local energetic solution.

We would like to mention also another notion of evolution which was recently introduced in the work [NeO07]:

(WS) *a* **weak solution** *associated with $\mathcal{E}$ and $\mathcal{R}_0$ is a mapping $t \mapsto (u(t), s(t))$ with $u \in \mathrm{L}^\infty([0,T]; \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_{s_1}; \mathbb{R}^2))$ and $s \in \mathrm{BV}([0,T]; [s_0, s_1])$ such that $(u(0), s(0)) = (\mathcal{U}(0, s_0), s_0)$ and the following three conditions are satisfied:*

(1) local stability condition: for every $t \in [0,T]$

$$u(t) = \mathcal{U}(t, s(t)) := \operatorname{argmin} \mathcal{E}(t, \cdot, s(t)),$$
$$\kappa(s(t)) - \mathcal{G}(t, u(t), s(t)) \geq 0,$$

(2) irreversibility: the map $t \mapsto s(t)$ is nondecreasing,

(3) weak activation criterion:

$$s(\cdot) \text{ not constant in }]t-\eta, t+\eta[\qquad \Rightarrow$$
$$\mathcal{G}(t, \mathcal{U}(t,\hat{s}), \hat{s}) \geq \kappa(\hat{s}) \quad \forall \hat{s} \in [s(t-), s(t+)] \setminus \{s_1\}.$$

The weak solution (WS) is defined according to [NeO07, Definition 2.2]. In that work the authors consider the case of bulk energy $\widetilde{W}(\nabla u) = |\nabla u|^2$ and fracture toughness $\kappa(s) \equiv G_c > 0$. This notion is very close to our definition of local energetic solution, and the main difference with the previous definitions is that they both do not require any condition on energies and that on the contrary they are given in terms of "slopes", involving energy release rate and toughness. Moreover both notions satisfy the extended energy balance (2.19), which easily implies the usual energy inequality (2.22). In general, a weak solution is a local energetic solution and vice versa, any local energetic solution $s(t)$ can be modified to be a weak solution, with $s(t) \in \{s(t-), s(t+)\}$ for every t.

3 Reduced energy and energy release rate

In the proofs of Theorems 2.2 and 2.4 we use frequently that the energy release rate G is continuous. We will therefore investigate in this section the continuity and differentiability properties of the reduced energy $\mathcal{I}$ and derive a formula for G. We treat first a rather general case, where the energy $\mathcal{E}$ may depend on several parameters and have nonunique minimizers. Afterwards, the results are applied to the situation with a crack as described above.

3.1 Variation of reduced energies with respect to a finite number of parameters

Let V be a topological Hausdorff space and $\Sigma = [\sigma_1^1, \sigma_2^1] \times \ldots \times [\sigma_1^m, \sigma_2^m] \subset \mathbb{R}^m$ a set of parameters. For the energy functional $\mathcal{E}_0 : \Sigma \times V \to \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ we define

$$\mathcal{I}(\sigma) = \inf\{\, \mathcal{E}_0(\sigma, v) \mid v \in V \,\},$$
$$\mathcal{U}(\sigma) = \operatorname{Argmin} \mathcal{E}_0(\sigma, \cdot) = \{\, v \in V \mid \mathcal{E}_0(\sigma, v) = \mathcal{I}(\sigma) \,\}.$$

The following assumptions are imposed on $\mathcal{E}_0$, cf. [FrM06].

Compactness of energy sublevels:

$\forall\, \sigma \in \Sigma\ \exists\, E \in \mathbb{R} :\ L_{\sigma,E} := \{\, u \in V \mid \mathcal{E}_0(\sigma, u) \leq E \,\}$ is not empty. (E1)

Furthermore, $L_{\sigma,E}$ is compact for every $\sigma \in \Sigma$ and every $E \in \mathbb{R}$.

This assumption implies that for every $\sigma \in \Sigma$ the set $\mathcal{U}(\sigma)$ is not empty and that $\mathcal{I} : \Sigma \to \mathbb{R}$ is well defined.

$$\begin{aligned}&\textit{Uniform control of } \partial_\sigma \mathcal{E}_0\textit{:}\\ &\exists\, c_0 \in \mathbb{R}\; \exists\, c_1 > 0\; \forall (\widetilde{\sigma}, u) \in \Sigma \times V \text{ with } \mathcal{E}_0(\widetilde{\sigma}, u) < \infty :\\ &\mathcal{E}_0(\cdot, u) \in \mathrm{C}^1(\Sigma) \text{ and } |\partial_\sigma \mathcal{E}_0(\sigma, u)| \le c_1(c_0 + \mathcal{E}_0(\sigma, u)) \forall\, \sigma \in \Sigma.\end{aligned} \tag{E2}$$

Using Gronwall's inequality, the following fundamental estimate can be deduced from assumption (E2), see e.g. [FrM06]: For every $\sigma_1, \sigma_2 \in \Sigma$ and $u \in V$ with $\mathcal{E}_0(\sigma_1, u) < \infty$ it holds

$$\mathcal{E}_0(\sigma_1, u) \le \big(c_0 + \mathcal{E}_0(\sigma_2, u)\big) \mathrm{e}^{c_1 |\sigma_1 - \sigma_2|} - c_0.$$

This inequality implies in particular that for every $\sigma_1, \sigma_2 \in \Sigma$ and $u \in \mathcal{U}(\sigma_2)$, we have

$$\mathcal{I}(\sigma_1) \le \mathcal{E}_0(\sigma_1, u) \le \big(c_0 + \mathcal{I}(\sigma_2)\big) \mathrm{e}^{c_1 |\sigma_1 - \sigma_2|} - c_0,$$

and therefore,

$$\sup_{\sigma \in \Sigma} \mathcal{I}(\sigma) < \infty, \quad \sup\{\, \mathcal{E}_0(\sigma, u) \mid \sigma \in \Sigma,\, u \in \cup_{\tau \in \Sigma} \mathcal{U}(\tau) \,\} < \infty. \tag{3.1}$$

Proposition 3.1 *Assume that* (E1) *and* (E2) *are satisfied. Then the mapping* $\mathcal{I} : \Sigma \to \mathbb{R}$ *is Lipschitz continuous. Moreover, for every sequence* $\sigma_n \to \sigma$ *and every sequence* $(u_n)_{n \in \mathbb{N}}$ *with* $u_n \in \mathcal{U}(\sigma_n)$ *we have* $\lim_{n \to \infty} \mathcal{E}_0(\sigma, u_n) = \mathcal{I}(\sigma)$.

Proof: Let $\sigma_1, \sigma_2 \in \Sigma$ and $u_2 \in \mathcal{U}(\sigma_2)$. By condition (E2) and estimate (3.1) we obtain

$$\begin{aligned}\mathcal{I}(\sigma_1) - \mathcal{I}(\sigma_2) &\le \mathcal{E}_0(\sigma_1, u_2) - \mathcal{E}_0(\sigma_2, u_2)\\ &\le |\sigma_1 - \sigma_2| \int_0^1 |\partial_\sigma \mathcal{E}_0(\sigma_2 + s(\sigma_1 - \sigma_2), u_2)| \,\mathrm{d}s \le c\, |\sigma_1 - \sigma_2|\,,\end{aligned}$$

and the constant c is independent of σ_1 and σ_2. Interchanging σ_1 and σ_2 in the previous inequality shows that $\mathcal{I}$ is Lipschitz continuous.

Let $(\sigma_n, u_n)_{n \in \mathbb{N}}$ be a sequence as described in the second statement of Proposition 3.1. Again by property (E2) and estimate (3.1) we see that

$$|\mathcal{E}_0(\sigma_n, u_n) - \mathcal{E}_0(\sigma, u_n)| \le |\sigma_n - \sigma| \int_0^1 |\partial_\sigma \mathcal{E}_0(\sigma + s(\sigma_n - \sigma_0), u_n)| \,\mathrm{d}s \le c\, |\sigma_n - \sigma|\,.$$

Together with continuity of $\mathcal{I}$ it follows that $\mathcal{E}_0(\sigma, u_n) \to \mathcal{I}(\sigma)$ for $n \to \infty$. ■

For the proof of differentiability properties of $\mathcal{I}$, we need also a continuity assumption for $\partial_\sigma \mathcal{E}_0$ along sequences $(\sigma_n, u_n)_n$, where $u_n \in \mathcal{U}(\tau_n)$ for some τ_n.

$$\begin{aligned}&\textit{Continuity of } \partial_\sigma \mathcal{E}_0 \textit{ along sequences } (\sigma_n, u_n)_n\textit{:}\\ &\text{For } n \in \mathbb{N} \text{ let } \tau_n, \sigma_n \in \Sigma,\ u_n \in \mathcal{U}(\tau_n). \text{ Then the following implication holds:}\\ &(\sigma_n, \tau_n, u_n) \to (\sigma, \sigma, u) \text{ with } u \in \mathcal{U}(\sigma) \implies \partial_\sigma \mathcal{E}_0(\sigma_n, u_n) \to \partial_\sigma \mathcal{E}_0(\sigma, u).\end{aligned} \tag{E3}$$

If V is identified with a Banach space, which is equipped with the weak topology, then there are at least two cases such that assumption (E3) is satisfied. In the case, where $\mathcal{E}_0$ has nonunique minimizers (like in finite–strain elasticity), a sufficient condition for (E3) to hold is: For every $E \in \mathbb{R}$ there exists a modulus of continuity $\omega_E : [0,\infty) \to [0,\infty)$ such that $|\partial_\sigma \mathcal{E}_0(\sigma_1, u) - \partial_\sigma \mathcal{E}_0(\sigma_2, u)| \le \omega_E(|\sigma_1 - \sigma_2|)$ for every $u \in V$ with $\mathcal{E}_0(\sigma_1, u) \le E$. Property (E3) is then an immediate consequence of the fundamental convergence theorem in [FrM06], where it is proved that the convergence of a sequence $(u_n)_{n\in\mathbb{N}}$ together with the convergence of the corresponding energies implies the convergence of $\partial_\sigma \mathcal{E}_0(\sigma_n, u_n)$. The case, where $\mathcal{E}_0(\sigma, \cdot)$ is strictly convex, is discussed in detail in the next section.

For $\tau \in \mathbb{R}^m \backslash \{0\}$ and $\sigma \in \Sigma$ the right and left directional derivatives of $\mathcal{I}$ are denoted by

$$\partial_\tau^+ \mathcal{I}(\sigma) = \lim_{h \searrow 0} \tfrac{1}{h}\big(\mathcal{I}(\sigma + h\tau) - \mathcal{I}(\sigma)\big), \tag{3.2}$$

$$\partial_\tau^- \mathcal{I}(\sigma) = \lim_{h \searrow 0} \tfrac{1}{h}\big(\mathcal{I}(\sigma) - \mathcal{I}(\sigma - h\tau)\big). \tag{3.3}$$

Theorem 3.2 *Let* (E1)–(E3) *be satisfied. For every $\sigma \in \Sigma$ and $\tau \in \mathbb{R}^m \backslash \{0\}$ with $\sigma + h\tau \in \Sigma$ for small $h > 0$, the right and left directional derivatives with respect to τ exist and are given by*

$$\partial_\tau^+ \mathcal{I}(\sigma) = \min\{\, \partial_\sigma \mathcal{E}_0(\sigma, u) \cdot \tau \mid u \in \mathcal{U}(\sigma) \,\},$$
$$\partial_\tau^- \mathcal{I}(\sigma) = -\partial_{-\tau}^+ \mathcal{I}(\sigma) = \max\{\, \partial_\sigma \mathcal{E}_0(\sigma, v) \cdot \tau \mid v \in \mathcal{U}(\sigma) \,\}.$$

Moreover, $\partial_\tau^+ \mathcal{I}$ and $\partial_\tau^- \mathcal{I}$ are measurable and $\partial_\tau^+ \mathcal{I}(\sigma) = \partial_\tau^- \mathcal{I}(\sigma)$ for a.e. $\sigma \in \Sigma$. Finally, if $h_n > 0$ with $\lim_{n\to\infty} h_n = 0$, then $\partial_\tau^\pm \mathcal{I}(\sigma \pm h_n \tau) \to \partial_\tau^\pm \mathcal{I}(\sigma)$.

Remark 3.3 *From the last assertion we may conclude the following, using Theorem 2.5.1 of [Cla83]: Let $\Sigma = [\sigma_0, \sigma_1] \subset \mathbb{R}$. Then, under assumptions of Theorem 3.2, the Clarke generalized gradient of $\mathcal{I}$ is given by $\partial^{Cl}\mathcal{I}(\sigma) = [\partial^+ \mathcal{I}(\sigma), \partial^- \mathcal{I}(\sigma)]$, $\sigma \in (\sigma_0, \sigma_1)$. This fact will be used in a forthcoming paper.*

Proof: Let $\sigma \in \Sigma, \tau \in \mathbb{R}^m \backslash \{0\}$ such that $\sigma + h\tau \in \Sigma$ for $0 < h < h_0$, where h_0 is chosen small enough. The goal is to calculate the limit in (3.2).
Upper estimate: Let $u \in \mathcal{U}(\sigma)$ be arbitrary. Then

$$\tfrac{1}{h}\big(\mathcal{I}(\sigma + h\tau) - \mathcal{I}(\sigma)\big) \le \tfrac{1}{h}\big(\mathcal{E}_0(\sigma + h\tau, u) - \mathcal{E}_0(\sigma, u)\big) = \int_0^1 \partial_\sigma \mathcal{E}_0(\sigma + rh\tau, u) \cdot \tau \, \mathrm{d}r.$$

By assumption (E2) and inequality (3.1), the integrand is bounded by a constant, which is independent of s and r. Therefore, Lebesgue's Theorem of dominated convergence implies

$$\limsup_{h \searrow 0} \tfrac{1}{h}\big(\mathcal{I}(\sigma + h\tau) - \mathcal{I}(\sigma)\big) \le \lim_{h \searrow 0} \int_0^1 \partial_\sigma \mathcal{E}_0(\sigma + rh\tau, u) \cdot \tau \, \mathrm{d}r = \partial_\sigma \mathcal{E}_0(\sigma, u) \cdot \tau.$$

Since $u \in \mathcal{U}(\sigma)$ is arbitrary, we can take the infimum on the right hand side. In fact, the infimum is a minimum, which can be seen as follows. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{U}(\sigma)$ be an infimizing sequence for $\partial_\sigma \mathcal{E}_0(\sigma, \cdot) \cdot \tau$ with respect to $\mathcal{U}(\sigma)$. By assumption (E1) the set $\mathcal{U}(\sigma)$ is compact and therefore, there exists an element $u \in \mathcal{U}(\sigma)$ and a subsequence $(u_{n'})_{n' \in \mathbb{N}}$, which converges to u. Assumption (E3) implies that $\partial_\sigma \mathcal{E}_0(\sigma, u_{n'}) \cdot \tau \to \partial_\sigma \mathcal{E}_0(\sigma, u) \cdot \tau$. Thus, u is a minimizer of $\partial_\sigma \mathcal{E}_0(\sigma, \cdot) \cdot \tau$ on $\mathcal{U}(\sigma)$ and we have proved that

$$\limsup_{h \searrow 0} \tfrac{1}{h} \left(\mathcal{I}(\sigma + h\tau) - \mathcal{I}(\sigma) \right) \leq \min\{ \, \partial_\sigma \mathcal{E}_0(\sigma, u) \cdot \tau \mid u \in \mathcal{U}(\sigma) \, \}.$$

Lower estimate: For every $h \in [0, h_0]$ let $u_{\sigma+h\tau} \in \mathcal{U}(\sigma+h\tau)$. The lower semicontinuity of $\mathcal{E}_0$ (assumption (E1)) and Proposition 3.1 imply that there exists a sequence $h_n \to 0$ and an element $u_* \in \mathcal{U}(\sigma)$ such that $u_{\sigma+h_n\tau} \to u_*$. By assumption (E3) and Lebesgue's Theorem we obtain therefore

$$\begin{aligned} &\liminf_{n \to \infty} \tfrac{1}{h_n} \big(\mathcal{I}(\sigma + h_n\tau) - \mathcal{I}(\sigma) \big) \geq \lim_{n \to \infty} \tfrac{1}{h_n} \big(\mathcal{E}_0(\sigma + h_n\tau, u_{\sigma+h_n\tau}) - \mathcal{E}_0(\sigma, u_{\sigma+h_n\tau}) \big) \\ &= \lim_{n \to \infty} \int_0^1 \partial_\sigma \mathcal{E}_0(\sigma + r h_n \tau, u_{\sigma+h_n\tau}) \cdot \tau \, \mathrm{d}r = \partial_\sigma \mathcal{E}_0(\sigma, u_*) \cdot \tau. \end{aligned}$$

A proof by contradiction shows finally that

$$\liminf_{h \searrow 0} \tfrac{1}{h} \big(\mathcal{I}(\sigma + h\tau) - \mathcal{I}(\sigma) \big) \geq \min\{ \, \partial_\sigma \mathcal{E}_0(\sigma, v) \cdot \tau \mid v \in \mathcal{U}(\sigma) \, \}.$$

This finishes the proof of the first part of Theorem 3.2.

For the proof of the second part we extend $\mathcal{I}$ by reflection to a Lipschitz continuous and bounded function $\widetilde{\mathcal{I}} : \mathbb{R}^m \to \mathbb{R}$. For $\tau \in \mathbb{R}^m \backslash \{0\}$, $h > 0$ and $\sigma \in \Sigma$ we define $\mathcal{I}^+_{\tau,h}(\sigma) = h^{-1}(\widetilde{\mathcal{I}}(\sigma+h\tau) - \widetilde{\mathcal{I}}(\sigma))$ and $\mathcal{I}^-_{\tau,h}(\sigma) = h^{-1}(\widetilde{\mathcal{I}}(\sigma) - \widetilde{\mathcal{I}}(\sigma - h\tau))$. Obviously, the functions $\mathcal{I}^+_{\tau,h}$ and $\mathcal{I}^-_{\tau,h}$ are measurable with respect to Σ and we have due to the first part of Theorem 3.2 that $\mathcal{I}^\pm_{\tau,h}(\sigma) \to \partial^\pm_\tau \mathcal{I}(\sigma)$ for every $\sigma \in \Sigma$. Therefore, $\partial^\pm_\tau \mathcal{I}$ is measurable. Let $\varphi \in \mathrm{C}^\infty_0(\operatorname{int} \Sigma)$ be arbitrary. Lebesgue's Theorem and a change of coordinates imply that

$$\begin{aligned} \int_\Sigma \partial^+_\tau \mathcal{I}(\sigma) \varphi(\sigma) \mathrm{d}\sigma &= \lim_{h \to 0} h^{-1} \int_\Sigma (\widetilde{\mathcal{I}}(\sigma+h\tau) - \widetilde{\mathcal{I}}(\sigma)) \varphi(\sigma) \mathrm{d}\sigma \\ &= \lim_{h \to 0} h^{-1} \int_\Sigma \widetilde{\mathcal{I}}(\sigma) (\varphi(\sigma - h\tau) - \varphi(\sigma)) \mathrm{d}\sigma = - \int_\Sigma \mathcal{I}(\sigma) (\nabla \varphi \cdot \tau) \mathrm{d}\sigma. \end{aligned}$$

And similarly

$$\int_\Sigma \partial^-_\tau \mathcal{I}(\sigma) \varphi(\sigma) \mathrm{d}\sigma = - \int_\Sigma \mathcal{I}(\sigma) (\nabla \varphi \cdot \tau) \mathrm{d}\sigma.$$

Since $\varphi \in \mathrm{C}^\infty_0(\operatorname{int} \Sigma)$ is arbitrary, we finally obtain $\partial^-_\tau \mathcal{I}(\sigma) = \partial^+_\tau \mathcal{I}(\sigma)$ for a.e. $\sigma \in \Sigma$.

For the proof of the last part of Theorem 3.2 let $\sigma_n := \sigma + h_n \tau$ and $v_n \in \mathcal{U}(\sigma_n)$ such that $\partial^+_\tau \mathcal{I}(\sigma_n) = \partial_\sigma \mathcal{E}(\sigma_n, v_n) \cdot \tau$. In view of (E1) and Proposition 3.1 we may assume that $v_n \to v$ with $v \in \mathcal{U}(\sigma)$. Thus, by (E3) and formula (3.2) we have

$$\partial^+_\tau \mathcal{I}(\sigma_n) = \partial_\sigma \mathcal{E}_0(\sigma_n, v_n) \cdot \tau \to \partial_\sigma \mathcal{E}_0(\sigma, v) \cdot \tau \geq \partial^+_\tau \mathcal{I}(\sigma). \tag{3.4}$$

Moreover,

$$\partial^+_\tau \mathcal{I}(\sigma) \geq \lim_{n \to \infty} \tfrac{1}{h_n} \big(\mathcal{E}_0(\sigma_n, v_n) - \mathcal{E}_0(\sigma, v_n) \big) = \partial_\sigma \mathcal{E}_0(\sigma, v) \cdot \tau. \tag{3.5}$$

Combining (3.4) and (3.5) finishes the proof. ∎

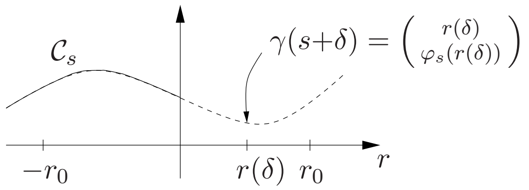

Figure 3.1: Local description of the crack $\mathcal{C}$ via φ_s

Corollary 3.4 *Let* (E1)–(E3) *be satisfied. If for every $\sigma \in \Sigma$ the corresponding minimizer of $\mathcal{E}_0(\sigma, \cdot)$ is unique, then $\mathcal{I} \in \mathrm{C}^1(\Sigma)$. Moreover, $\mathrm{D}\mathcal{I}(\sigma) = \partial_\sigma \mathcal{E}_0(\sigma, v)$, where $v = v_\sigma$ is the minimizer of $\mathcal{E}_0(\sigma, \cdot)$.*

Proof: Note first that for every $\tau \in \mathbb{R}^m \backslash \{0\}$ and every $\sigma \in \Sigma$ it holds

$$\partial_\tau \mathcal{I}(\sigma) \equiv \partial_\tau^+ \mathcal{I}(\sigma) = \partial_\tau^- \mathcal{I}(\sigma) = \partial_\sigma \mathcal{E}_0(\sigma, v_\sigma) \cdot \tau,$$

where v_σ is the unique minimizer of $\mathcal{E}_0(\sigma, \cdot)$. It remains to prove the continuity of $\partial_\tau \mathcal{I}(\sigma)$. Let $(\sigma_n)_{n\in\mathbb{N}} \subset \Sigma$ be a sequence with $\sigma_n \to \sigma$ and let $(u_n)_{n\in\mathbb{N}} \subset V$ be the corresponding minimizers. The uniqueness assumption and Proposition 3.1 imply that $u_n \to u$, where $u \in V$ is the minimizer of $\mathcal{E}_0(\sigma, \cdot)$. Assumption (E3) now guarantees that $\partial_\sigma \mathcal{E}_0(\sigma_n, u_n) \to \partial_\sigma \mathcal{E}_0(\sigma, u)$ and the proof is finished. ∎

3.2 Application to the problem with prescribed crack path

The scope of this section is to show that the reduced energy $\mathcal{I} : [0, T] \times [s_0, s_1] \to \mathbb{R}$, which is defined in (2.6), is well defined and belongs to $\mathrm{C}^1([0, T] \times [s_0, s_1])$. Moreover, we provide a formula for the energy release rate $G(t, s) = -\partial_s \mathcal{I}(t, s)$.

In order to study the differentiability properties of $\mathcal{I}$ with respect to s we introduce a family of diffeomorphisms $T_{s,\delta} : \Omega_s \to \Omega_{s+\delta}$ for $s \in [s_0, s_1]$ and $|\delta| \le \delta_0$, where $\delta_0 > 0$ is some small enough constant. Due to the smoothness assumptions on the crack path $\mathcal{C}$, the subsequent considerations can be carried out uniformly with respect to $s \in [s_0, s_1]$.

Since the crack path $\mathcal{C}$ is a simple C^2–curve, after a suitable rotation, it can locally be described as the graph of a C^2–function. Let $s \in [s_0, s_1]$, $r_0, \delta_l, \delta_r > 0$ and $\varphi_s \in \mathrm{C}^2([-r_0, r_0], \mathbb{R})$ such that for $\delta \in [-\delta_l, \delta_r]$ we have (for simplicity, we neglect the rotation):

$$\gamma(s{+}\delta) = \left(\begin{smallmatrix} r(\delta) \\ \varphi_s(r(\delta)) \end{smallmatrix}\right), \quad \mathcal{C}_{s+\delta} \backslash \mathcal{C}_{s-\delta_l} = \{\, (r, \varphi_s(r)) \mid r \in \,]{-r_0}, r(\delta)] \,\} \quad \text{and} \quad r(0) = 0,$$

see figure 3.1. Choose $\theta \in \mathrm{C}_0^\infty(B_{r_0}(0))$ with $\theta\big|_{B_{r_0/3}(0)} = 1$. Similar to [Kov03] we define the mapping $T_{s,\delta} : \mathbb{R}^2 \to \mathbb{R}^2$ via

$$T_{s,\delta}(x) = x + \begin{pmatrix} (\gamma_1(s+\delta) - \gamma_1(s))\theta(\gamma(s) - x) \\ \varphi_s(x_1 + (\gamma_1(s+\delta) - \gamma_1(s))\theta(\gamma(s) - x)) - \varphi_s(x_1) \end{pmatrix}.$$

Lemma 3.5 (Properties of $T_{s,\delta}$) *There exists a constant $\delta_0 > 0$ such that we have*

(a) $T_{s,\cdot} \in \mathrm{C}^2([-\delta_0, \delta_0] \times \mathbb{R}^2, \mathbb{R}^2)$ *and for every* $|\delta| \le \delta_0$ *the mapping* $T_{s,\delta}$ *is a* C^2–*diffeomorphism. Moreover,* $T_{s,\delta}(\Omega_s) = \Omega_{s+\delta}$, $T_{s,\delta}(\gamma(s)) = \gamma(s+\delta)$, $T_{s,\delta}(\mathcal{C}_s) = \mathcal{C}_{s+\delta}$ *and* $T_{s,\delta}(x) = x$ *for every* $x \in \mathbb{R}^2 \backslash B_{r_0}(\gamma(s))$.

(b) The norms $\|T_{s,\delta}\|_{\mathrm{C}^2(\mathbb{R}^2)}$ *and* $\left\|T_{s,\delta}^{-1}\right\|_{\mathrm{C}^2(\mathbb{R}^2)}$ *are uniformly bounded with respect to* δ. *There exist constants* $c_3, c_4 > 0$ *such that for every* $|\delta| \le \delta_0$ *and* $x \in \mathbb{R}^2$ *we have* $c_3 \le \det \nabla T_{s,\delta}(x) \le c_4$.

(c) Some derivatives:

$$\varrho_s(x) := \partial_\delta (T_{s,\delta}(x))\big|_{\delta=0} = \gamma_1'(s)\theta(\gamma(s)-x)\left(\begin{smallmatrix} 1 \\ \varphi_s'(x_1)\end{smallmatrix}\right), \tag{3.6}$$

$$\partial_\delta(\det \nabla T_{s,\delta})\big|_{\delta=0} = \operatorname{div} \varrho_s, \qquad \partial_\delta \left(\nabla T_{s,\delta}\right)^{-1}\big|_{\delta=0} = -\nabla \varrho_s. \tag{3.7}$$

(d) There is a constant $c > 0$ such that for every $u \in \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_s)$ *and* $|\delta| \le \delta_0$ *we have*

$$\|u\|_{\mathrm{W}^{1,p}(\Omega_s)} \le c \left\|(\nabla u (\nabla T_{s,\delta})^{-1})_{sym}\right\|_{\mathrm{L}^p(\Omega_s)}. \tag{3.8}$$

Proof: The proofs of parts (a)–(c) of Lemma 3.5 are carried out in [GiH96] for C^∞–diffeomorphisms. Without any changes, the arguments are also applicable to C^2 mappings $T_{s,\delta}$. Part (d) follows by a perturbation argument. ■

We make use of the following abbreviations

$$x_\delta(y) = T_{s,\delta}(y), \quad q_\delta(y) = \det \nabla T_{s,\delta}(y), \quad B_\delta(y) = (\nabla T_{s,\delta}(y))^{-1}.$$

For elements $v \in \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_s)$ and $(t,\delta) \in [0,T] \times [-\delta_0, \delta_0]$ we define

$$\begin{aligned}\mathcal{E}_0(t,\delta,v) = & \int_{\Omega_s} q_\delta(y) W(t,y,\nabla v(y) B_\delta(y))\, \mathrm{d}y \\ & - \int_{\Omega_s} q_\delta(y) f(t, x_\delta(y)) \cdot v(y)\, \mathrm{d}y - \int_{\Gamma_N} h(t) \cdot v \, \mathrm{d}\sigma. \end{aligned} \tag{3.9}$$

The definition of $\mathcal{E}_0$ is chosen in such a way that for every $v \in W^{1,p}_{\Gamma_D}(\Omega_{s+\delta})$ we have

$$\mathcal{E}(t, v, s+\delta) = \mathcal{E}_0(t, \delta, v \circ T_{s,\delta}). \tag{3.10}$$

Note that $T_{s,\delta}$ induces an isomorphism between the spaces $W^{1,p}_{\Gamma_D}(\Omega_s)$ and $W^{1,p}_{\Gamma_D}(\Omega_{s+\delta})$ through $u \mapsto u \circ T_{s,\delta}^{-1}$. Therefore, for every $|\delta| \le \delta_0$ the following identity is valid with $\mathcal{I}$ as in (2.6):

$$\mathcal{I}(t, s+\delta) = \min\{\, \mathcal{E}_0(t,\delta,v) \mid v \in W^{1,p}_{\Gamma_D}(\Omega_s)\,\},$$

and $\operatorname{argmin} \mathcal{E}_0(t,\delta,\cdot) = u \circ T_{s,\delta}$, where u is the unique minimizer of $\mathcal{E}(t,\cdot,s+\delta)$.

Theorem 3.6 *Assumptions* (2.1) *and* (2.3) *imply that* $\mathcal{I} \in \mathrm{C}^1([0,T] \times [s_0, s_1])$ *and the following formulas are valid with* ϱ_s *from* (3.6)

$$\partial_t \mathcal{I}(t,s) = \int_{\Omega_s} \partial_t W(t,y,\nabla u(y))\,\mathrm{d}y - \int_{\Omega_s} \dot{f}(t) \cdot u\,\mathrm{d}y - \int_{\Gamma_N} \dot{h}(t) \cdot u\,\mathrm{d}s, \tag{3.11}$$

$$\begin{aligned} -G(t,s) = \partial_s \mathcal{I}(t,s) = &\int_{\Omega_s} \big(W(t,y,\nabla u)\mathbb{I} - \nabla u^\top \mathrm{D}_A W(t,y,\nabla u)\big) : \nabla \varrho_s\,\mathrm{d}y \\ &- \int_{\Omega_s} u \cdot \mathrm{div}(f(t) \otimes \varrho_s)\,\mathrm{d}y. \end{aligned} \tag{3.12}$$

In both formulas, u *is the unique minimizer of* $\mathcal{E}(t,\cdot,s)$.

The quantity $\nabla u^\top \mathrm{D}_A W(t,y,\nabla u) - W(t,y,\nabla u)\mathbb{I}$ is the Eshelby or Hamilton tensor. It follows from the proof of Theorem 3.6 that

$$G(t,s) = \mathcal{G}(t, \mathcal{U}(t,s), s). \tag{3.13}$$

Moreover, we observe that $G_{\max}$ which appears in (2.9) is in fact a maximum.

Remark 3.7 Integration by parts shows that $\int_{\Omega_s} \mathrm{div}(f \otimes \varrho_s) \cdot v\ \mathrm{d}x = -\int_{\Omega_s} f \cdot (\nabla v \varrho_s)\,\mathrm{d}x$. This indicates that it would be sufficient to assume $f(t) \in L^q(\Omega;\mathbb{R}^2)$ instead of $f(t) \in W^{1,q}(\Omega;\mathbb{R}^2)$. In [KnM07], we deduced a formula for the energy release rate in the stationary case with this weaker assumption on f.

Proof: In order to prove Theorem 3.6, we apply Corollary 3.4 to the energy density $\mathcal{E}_0$. Thus, we only have to show that $\mathcal{E}_0$ satisfies conditions (E1)–(E3) from the previous section. The formula for the energy release rate can then be calculated using $\partial_\delta \mathcal{I}(t, s+\delta) = \partial_\delta \mathcal{E}_0(t,\delta,u)$, where $u = u_\delta$ is the minimizer of $\mathcal{E}_0(t,\delta,\cdot)$. We choose $V = \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_s)$ together with the weak topology and $\Sigma = [0,T] \times [-\delta_0, \delta_0]$.

Condition (E1) is an immediate consequence of the growth and convexity properties of the energy density W and relies on identity (3.10). Moreover, for every $(t,\delta) \in \Sigma$ and $v \in W^{1,p}(\Omega_s)$ the partial derivatives $\partial_t \mathcal{E}_0$ and $\partial_\delta \mathcal{E}_0$ exist and are given by

$$\begin{aligned} \partial_t \mathcal{E}_0(t,\delta,v) = &\int_{\Omega_s} q_\delta(y) \partial_t W(t,y,\nabla v(y) B_\delta(y))\,\mathrm{d}y \\ &- \int_{\Omega_s} q_\delta(y) \dot{f}(t, x_\delta(y)) \cdot v(y)\,\mathrm{d}y - \int_{\Gamma_N} \dot{h}(t) \cdot v\,\mathrm{d}s, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \partial_\delta \mathcal{E}_0(t,\delta,v) = &\int_{\Omega_s} \partial_\delta q_\delta(y)\, W(t,y,\nabla v(y) B_\delta(y))\,\mathrm{d}y \\ &+ \int_{\Omega_s} q_\delta(y) \big(\nabla v(y)^\top \mathrm{D}_A W(t,y,\nabla v(y) B_\delta(y))\big) : \partial_\delta B_\delta(y)\,\mathrm{d}y \\ &- \int_{\Omega_s} \partial_\delta q_\delta(y)\, f(t, x_\delta(y)) \cdot v(y)\,\mathrm{d}y \\ &- \int_{\Omega_s} q_\delta(y) \big(\nabla f(t, x_\delta(y)) \partial_\delta x_\delta(y)\big) \cdot v(y)\,\mathrm{d}y. \end{aligned} \tag{3.15}$$

These formulas can be verified using Lebesgue's Theorem, see also [Els05, Satz IV.5.7], and by applying a generalized variant of Lemma 4.1 from [KnM07]. There, for a straight crack it is shown that $f(t, x_{\delta_n}) \to f(t, x_\delta)$ strongly in $\mathrm{L}^q(\Omega_s)$ for $\delta_n \to \delta$ and that $\delta_n^{-1}(f(t, x_{\delta_n}) - f(t, x_\delta)) \to \nabla f(t, x_\delta)\partial_\delta x_\delta$ strongly in $\mathrm{L}^q(\Omega_s)$. The generalization of this lemma to a smooth, curved crack is straightforward.

Furthermore, $\partial_t \mathcal{E}_0$, $\partial_\delta \mathcal{E}_0 : \Sigma \times \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_s) \to \mathbb{R}$ are strongly continuous. This is again a consequence of Lemma 4.1 from [KnM07] together with properties of Nemytskij operators [Zei86] (for the terms with W) and the Lebesgue Theorem.

It remains to verify the estimate in (E2) and property (E3). Taking into account the uniform bounds of the family $T_{s,\delta}$ and assumptions (2.1) and (2.3), we obtain, based on the generalized Korn's inequality (3.8) and relation (3.10), the following estimate for elements $v \in \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_s)$:

$$\begin{aligned} \mathcal{E}_0(t, \delta, v) \geq\ & c_2 \|v\|^p_{\mathrm{W}^{1,p}(\Omega_s)} \\ & - c_3(1 + \|u_{\mathrm{Dir}}\|^p_{\mathrm{C}^1([0,T];\mathrm{W}^{1,p}(\Omega_s))} + \|f\|^q_{\mathrm{C}^1([0,T];\mathrm{W}^{1,q}(\Omega_s))} + \|h\|^q_{\mathrm{C}^1([0,T];\mathrm{L}^q(\Gamma_N))}). \end{aligned} \tag{3.16}$$

The constants $c_i > 0$ are independent of v, δ and t. On the other hand, from (3.14) and (3.15) by Hölder's inequality we obtain the estimate

$$\begin{aligned} & |\partial_t \mathcal{E}_0(t, \delta, v)| + |\partial_\delta \mathcal{E}_0(t, \delta, v)| \\ & \quad \leq c\big(\|v\|^p_{\mathrm{W}^{1,p}(\Omega_s)} + \|f\|^q_{\mathrm{C}^1([0,T];\mathrm{W}^{1,q}(\Omega_s))} + \|u_{\mathrm{Dir}}\|^p_{\mathrm{C}^1([0,T];\mathrm{W}^{1,p}(\Omega_s))} + \|h\|^q_{\mathrm{C}^1([0,T];\mathrm{L}^q(\Gamma_N))} \big) \end{aligned}$$

and $c > 0$ is independent of v, t and δ. Together with (3.16) this proves (E2).

Let now $t, t_n, \tilde{t}_n \in [0, T]$, $\delta, \delta_n, \tilde{\delta}_n \in [-\delta_0, \delta_0]$ with $(t_n, \tilde{t}_n) \to (t, t)$, $(\delta_n, \tilde{\delta}_n) \to (\delta, \delta)$ and assume that u_n is the unique minimizer of $\mathcal{E}_0(\tilde{t}_n, \tilde{\delta}_n, \cdot)$ with $u_n \rightharpoonup u$ weakly in $\mathrm{W}^{1,p}_{\Gamma_D}(\Omega_s)$, where u is the minimizer of $\mathcal{E}_0(t, \delta, \cdot)$. Proposition 3.1 implies that the sequence $(u_n)_{n\in\mathbb{N}}$ is a minimizing sequence for $\mathcal{E}_0(t, \delta, \cdot)$. Since the energy density $\widetilde{W}$ is assumed to be strictly convex, it follows from a result by Visintin [Vis84] that the minimizing sequence converges also strongly in $\mathrm{W}^{1,p}(\Omega_s)$. From the continuity properties of $\partial_t \mathcal{E}_0$ and $\partial_\delta \mathcal{E}_0$ we conclude therefore that

$$\partial_t \mathcal{E}_0(t_n, \delta_n, u_n) \to \partial_t \mathcal{E}_0(t, \delta, u), \qquad \partial_\delta \mathcal{E}_0(t_n, \delta_n, u_n) \to \partial_\delta \mathcal{E}_0(t, \delta, u).$$

This proves condition (E3). Corollary 3.4 now implies that $\mathcal{I} \in \mathrm{C}^1([0, T] \times [s_0, s_1])$. The formulas for the derivatives of $\mathcal{I}$ follow from (3.14) and (3.15) with $\delta = 0$ taking into account relations (3.6) and (3.7). ■

Remark 3.8 Non-interpenetration can be included in our model for both, straight and curved cracks. This means that we have to restrict the space $\mathrm{W}^{1,p}_{\Gamma_D}(\Omega_s; \mathbb{R}^2)$ to the convex cone $V_\geq(\Omega_s) = \{\, v \in \mathrm{W}^{1,p}_{\Gamma_D}(\Omega_s) \mid [v]\nu \geq 0 \,\}$, where $[v] = v^+ - v^-$ denotes the difference of the traces of v on the positive and negative side of $\mathcal{C}_s$, and ν is the unit normal to $\mathcal{C}_s$ pointing from the negative to the positive side. If the crack is straight, the proof of Theorem 3.6 is still valid, since $T_{s,\delta}$ induces an isomorphism between $V_\geq(\Omega_s)$ and $V_\geq(\Omega_{s+\delta})$. In the case of a curved crack we use the Piola transform $P_\delta : V_\geq(\Omega_{s+\delta}) \to V_\geq(\Omega_s)$ with $P_\delta\, v = (\mathrm{cof}\, \nabla T_{s,\delta})^\top\, v \circ T_{s,\delta}$, where cof denotes the cofactor matrix. The Piola transform generates an isomorphism between $V_{\geq}(\Omega_{s+\delta})$ and $V_{\geq}(\Omega_s)$. The energy $\mathcal{E}_0$ from (3.9) has to be replaced by $\mathcal{E}_{\geq}$ with

$$\begin{aligned}\mathcal{E}_{\geq}(t,\delta,v) = &\int_{\Omega_s} q_\delta\, W\big(t,y,\nabla\big((\operatorname{cof}\nabla T_{s,\delta})^{-\top}v\big)B_\delta\big)\,\mathrm{d}y\\ &-\int_{\Omega_s} q_\delta\, f(t,x_\delta)\cdot\big((\operatorname{cof}\nabla T_{s,\delta})^{-\top}v\big)\,\mathrm{d}y - \int_{\Gamma_N} h(t)\cdot v\,\mathrm{d}\sigma.\end{aligned}$$

Note that $\mathcal{E}(t,v,s+\delta)=\mathcal{E}_{\geq}(t,\delta,P_\delta v)$ for every $v\in V_{\geq}(\Omega_{s+\delta})$. Now the same arguments as in the proof of Theorem 3.6 can be applied to $\mathcal{E}_{\geq}$ under the additional assumption that the crack is C^3-smooth. The energy release rate is given by (with $u=\mathcal{U}(t,s)$):

$$\begin{aligned}-G(t,s)=\partial_\delta\mathcal{E}_{\geq}(t,0,u)=&\int_{\Omega_s}\big(W(t,y,\nabla u)\mathbb{I}-\nabla u^\top \mathrm{D}_A W(t,y,\nabla u)\big):\nabla\varrho_s\,\mathrm{d}y\\ &-\int_{\Omega_s} v\cdot\operatorname{div}(f\otimes\varrho_s)\,\mathrm{d}y-\int_{\Omega_s} f\cdot\big((\nabla\varrho_s-\operatorname{div}\varrho_s\mathbb{I})v\big)\,\mathrm{d}y\\ &+\int_{\Omega_s}\mathrm{D}_A W(t,y,\nabla u):\nabla\big((\nabla\varrho_s-\operatorname{div}\varrho_s\mathbb{I})v\big)\,\mathrm{d}y. \end{aligned}\tag{3.17}$$

If the crack is straight, then this formula reduces to (3.12). It remains open whether this is also true in the general case. This investigation will be continued in a subsequent paper.

4 Solutions for the viscous problem

In this section we deal with the reduced functional $\mathcal{I}(t,s)$ defined in (2.6) and with the corresponding energy release rate $G(t,s)$ defined in (2.7).

The existence of a viscous solution s^ν is obtained by minimizing a sequence defined through time-discretization, i.e., using the minimizing movements theory of De Giorgi [De 93] (see also [Amb95] and the recent book [AGS05]). In this section the viscosity parameter $\nu>0$ is fixed.

4.1 Time-incremental problems

For $N\in\mathbb{N}\setminus\{0\}$ we define the time-step $\tau=T/N$ and $t_k:=k\tau$ for $k=0,1,\ldots,N$. We define by induction s_k^ν as follows: $s_0^\nu:=s_0$ and for $k\geq1$ the value s_k^ν is defined by

$$s_k^\nu\in\operatorname{argmin}\Big\{\,\mathcal{I}(t_k,\widetilde{s})+\tau\mathcal{R}_\nu\Big(s_{k-1}^\nu,\frac{\widetilde{s}-s_{k-1}^\nu}{\tau}\Big)\;\Big|\;\widetilde{s}\in[s_0,s_1]\,\Big\}.\tag{4.1}$$

The existence of s_k^ν is an easy consequence of the direct method in the calculus of variations, since $s\mapsto\mathcal{I}(t,s)$ is continuous and $s\mapsto\mathcal{R}_\nu(s_{k-1}^\nu,\frac{s-s_{k-1}^\nu}{\tau})$ is lower semicontinuous. We observe that s_k^ν satisfies

$$0\in\partial_{\dot s}\mathcal{R}_\nu\Big(s_{k-1}^\nu,\frac{s_k^\nu-s_{k-1}^\nu}{\tau}\Big)-G(t_k,s_k^\nu)+\partial\chi_{[s_0,s_1]}(s_k^\nu),\tag{4.2}$$

for every $k=1,\ldots,N$.

If $s_k^\nu < s_1$, then by (4.2) we deduce that

$$\Big(\kappa(s_{k-1}^\nu) - G(t_k, s_k^\nu) + \nu\frac{s_k^\nu - s_{k-1}^\nu}{\tau})\Big)\frac{s_k^\nu - s_{k-1}^\nu}{\tau} = 0. \tag{4.3}$$

Indeed, let us first observe that (4.2) is equivalent to

$$\mathcal{R}_0\Big(s_{k-1}^\nu, \frac{\widetilde{s} - s_{k-1}^\nu}{\tau}\Big) - \mathcal{R}_0\Big(s_{k-1}^\nu, \frac{s_k^\nu - s_{k-1}^\nu}{\tau}\Big) + \Big(\nu\frac{s_k^\nu - s_{k-1}^\nu}{\tau} - G(t_k, s_k^\nu)\Big)\frac{\widetilde{s} - s_k^\nu}{\tau} \geq 0$$

for all $\widetilde{s} \in \mathbb{R}$. Using $\mathcal{R}_0(s, \dot{s}) = \infty$ for $\dot{s} < 0$, it is sufficient to consider $\widetilde{s} \geq s_{k-1}^\nu$ which gives

$$\Big(\kappa(s_{k-1}^\nu) + \nu\frac{s_k^\nu - s_{k-1}^\nu}{\tau} - G(t_k, s_k^\nu))\Big)\frac{\widetilde{s} - s_k^\nu}{\tau} \geq 0.$$

In particular, for any $\widetilde{s} > s_k^\nu$ we obtain $\kappa(s_{k-1}^\nu) + \nu\frac{s_k^\nu - s_{k-1}^\nu}{\tau} - G(t_k, s_k^\nu) \geq 0$. If we choose now $\widetilde{s} = s_{k-1}^\nu$ then we derive

$$\Big(\kappa(s_{k-1}^\nu) + \nu\frac{s_k^\nu - s_{k-1}^\nu}{\tau} - G(t_k, s_k^\nu)\Big)\frac{s_k^\nu - s_{k-1}^\nu}{\tau} \leq 0.$$

The last two inequalities together with the fact that $s_k^\nu \geq s_{k-1}^\nu$ give (4.3).

Let $\overline{s}_\tau^\nu$ and $\underline{s}_\tau^\nu$ be the left-continuous and right-continuous piecewise constant interpolants of s_k^ν such that $\overline{s}_\tau^\nu(t_k) = \underline{s}_\tau^\nu(t_k) = s_k^\nu$, i.e.,

$$\overline{s}_\tau^\nu(t) := s_k^\nu \quad \forall t \in \,]t_{k-1}, t_k], \quad \underline{s}_\tau^\nu(t) := s_{k-1}^\nu \quad \forall t \in [t_{k-1}, t_k[, \quad k = 1, \ldots, N. \tag{4.4}$$

Let $\overline{t}_k : [0, T] \to [0, T]$ be given by

$$\overline{t}_\tau(0) := 0, \quad \overline{t}_\tau(t) := t_k \text{ for } t \in \,]t_{k-1}, t_k].$$

Moreover, we define the piecewise affine interpolants

$$\hat{s}_\tau^\nu(t) := s_{k-1}^\nu + \frac{t - t_{k-1}}{\tau}(s_k^\nu - s_{k-1}^\nu) \quad \forall t \in \,]t_{k-1}, t_k]\,. \tag{4.5}$$

Hence, we can rewrite the time-incremental problem (4.2) by

$$0 \in \partial_{\dot{s}}\mathcal{R}_\nu(\underline{s}_\tau^\nu(t), \dot{\hat{s}}_\tau^\nu(t)) - G(\overline{t}_\tau(t), \overline{s}_\tau^\nu(t)) + \partial\chi_{[s_0, s_1]}(\overline{s}_\tau^\nu(t)). \tag{4.6}$$

We now prove that these interpolants satisfy suitable a priori bounds.

Lemma 4.1 *There exists a positive constant C such that for every $\nu > 0$ and every $\tau > 0$ the following estimates hold true.*

$$\|\underline{s}_\tau^\nu\|_{\mathrm{L}^\infty(0,T)}, \|\overline{s}_\tau^\nu\|_{\mathrm{L}^\infty(0,T)} \leq C \tag{4.7}$$

$$\int_0^T \mathcal{R}_\nu(\underline{s}_\tau^\nu(t), \dot{\hat{s}}_\tau^\nu(t))\,\mathrm{d}t \leq C \tag{4.8}$$

$$\|\dot{\hat{s}}_\tau^\nu\|_{\mathrm{L}^2(0,T)} \leq \frac{C}{\sqrt{\nu}} \tag{4.9}$$

$$\|\overline{s}_\tau^\nu - \hat{s}_\tau^\nu\|_{\mathrm{L}^\infty(0,T)}, \|\underline{s}_\tau^\nu - \hat{s}_\tau^\nu\|_{\mathrm{L}^\infty(0,T)} \leq C\frac{\sqrt{\tau}}{\sqrt{\nu}}. \tag{4.10}$$

Moreover, for every $\nu > 0$ there exists $\tau_0 = \tau_0(\nu)$ such that

$$\overline{s}_\tau^\nu(t) < s_1 \quad \forall \tau < \tau_0 \; \forall t \in [0, T]. \tag{4.11}$$

Proof: Since s_k^ν belongs to $[s_0, s_1]$ for every $k = 1, \dots, N$, estimate (4.7) is trivially satisfied by any constant $C \geq s_1$. By the minimality of s_k^ν and taking s_{k-1}^ν as test function we deduce

$$\begin{aligned}\mathcal{I}(t_k, s_k^\nu) + \tau\mathcal{R}_\nu\Big(s_{k-1}^\nu, \frac{s_k^\nu - s_{k-1}^\nu}{\tau}\Big) &\leq \mathcal{I}(t_k, s_{k-1}^\nu) + \tau\mathcal{R}_\nu(s_{k-1}^\nu, 0)\\ &= \mathcal{I}(t_{k-1}, s_{k-1}^\nu) + \int_{t_{k-1}}^{t_k} \partial_t\mathcal{I}(t, s_{k-1}^\nu)\,\mathrm{d}t,\end{aligned}$$

that is

$$\mathcal{I}(t_k, s_k^\nu) - \mathcal{I}(t_{k-1}, s_{k-1}^\nu) + \int_{t_{k-1}}^{t_k} \mathcal{R}_\nu\Big(s_{k-1}^\nu, \frac{s_k^\nu - s_{k-1}^\nu}{\tau}\Big)\,\mathrm{d}t \leq \int_{t_{k-1}}^{t_k} \partial_t\mathcal{I}(t, s_{k-1}^\nu)\,\mathrm{d}t.$$

By adding this inequality we obtain, for every $0 \leq i \leq k \leq N$,

$$\mathcal{I}(t_k, s_k^\nu) - \mathcal{I}(t_i, s_i^\nu) + \int_{t_i}^{t_k} \mathcal{R}_\nu(\underline{s}_\tau^\nu(t), \dot{\hat{s}}_\tau^\nu(t))\,\mathrm{d}t \leq \int_{t_i}^{t_k} \partial_t\mathcal{I}(t, \underline{s}_\tau^\nu(t))\,\mathrm{d}t.$$

Thus

$$\mathcal{I}(\overline{t}_\tau(t), \overline{s}_\tau^\nu(t)) + \int_0^{\overline{t}_\tau(t)} \mathcal{R}_\nu(\underline{s}_\tau^\nu(r), \dot{\hat{s}}_\tau^\nu(r))\,\mathrm{d}r \leq \mathcal{I}(0, s_0) + \int_0^{\overline{t}_\tau(t)} \partial_t\mathcal{I}(r, \underline{s}_\tau^\nu(r))\,\mathrm{d}r$$

holds true for every $t \in [0, T]$. Since now $\mathcal{I} \in \mathrm{C}^1([0, T] \times [s_0, s_1])$ by Theorem 3.6, we deduce the existence of a positive constant C independent of τ and ν such that

$$\int_0^T \mathcal{R}_\nu(\underline{s}_\tau^\nu(t), \dot{\hat{s}}_\tau^\nu(t))\,\mathrm{d}t \leq C(T+1) + \mathcal{I}(0, s_0),$$

which proves estimate (4.8). From the definition of $\mathcal{R}_\nu$ given by (2.8) it follows that (4.8) is equivalent to

$$\int_0^T \Big(\kappa(\underline{s}_\tau^\nu(t))\dot{\hat{s}}_\tau^\nu(t) + \frac{\nu}{2}|\dot{\hat{s}}_\tau^\nu(t)|^2\Big)\,\mathrm{d}t \leq C.$$

The non-negativity of the first term implies the estimate (4.9).

In order to prove (4.10), let now $t \in\,]t_{k-1}, t_k]$. Then by the definition of $\overline{s}_\tau^\nu(t)$ and $\hat{s}_\tau^\nu(t)$ given by (4.4) and (4.5), respectively, we derive

$$\overline{s}_\tau^\nu(t) - \hat{s}_\tau^\nu(t) = s_k^\nu - s_{k-1}^\nu - \frac{t - t_{k-1}}{\tau}(s_k^\nu - s_{k-1}^\nu) = (\tau - t + t_{k-1})\dot{\hat{s}}_\tau^\nu(t) \leq \tau|\dot{\hat{s}}_\tau^\nu(t)|.$$

Thus,

$$|\overline{s}_\tau^\nu(t) - \hat{s}_\tau^\nu(t)| \leq \int_{t_{k-1}}^{t_k} |\dot{\hat{s}}_\tau^\nu(t)|\,\mathrm{d}t \leq \sqrt{\tau}\Big(\int_{t_{k-1}}^{t_k} |\dot{\hat{s}}_\tau^\nu(t)|^2\,\mathrm{d}t\Big)^{1/2} \leq \sqrt{\tau}\,\|\dot{\hat{s}}_\tau^\nu\|_{\mathrm{L}^2(0,T)},$$

which, thanks to (4.9), gives the first estimate in (4.10). The second one is obtained in a similar way, since for every $t \in\,]t_{k-1}, t_k[$ we have

$$|\underline{s}_\tau^\nu(t) - \hat{s}_\tau^\nu(t)| = |\frac{t - t_{k-1}}{\tau}(s_k^\nu - s_{k-1}^\nu)| \leq \tau|\dot{\hat{s}}_\tau^\nu(t)|.$$

We observe that from (4.7), (4.9) and (4.10) it follows that $\hat{s}^\nu_\tau \in \mathrm{L}^\infty(0,T)$.

To conclude, we need to prove the existence of τ_0 such that (4.11) is satisfied. We start by defining the quantity $s_{\max}$ as

$$s_{\max} := \max\{\, s \in [s_0, s_1] \mid \kappa(s) \le G_{\max} \,\}.$$

By assumptions (2.10) and (2.9) it turns out that $s_{\max}$ is well defined and that $s_{\max} < s_1$. Moreover, $\kappa(s) > G_{\max}$ for every $s \in \,]s_{\max}, s_1]$.

Let $k^* \in \mathbb{N}$ be such that $s^\nu_k \le s_{\max}$ for all $k = 1, \ldots, k^*$ and (for $N > k^*$) let us assume $s^\nu_{k^*+1} > s_{\max}$. If $s^\nu_{k^*+1} < s_1$ then $\kappa(s^\nu_{k^*+1}) > G_{\max}$ and therefore by (4.3) $s^\nu_k = s^\nu_{k^*+1}$ for all $k = k^* + 1, \ldots, N$.

On the other hand, if $s^\nu_{k^*+1} = s_1$, then, by the definition (4.1) we get

$$-G(t_{k^*+1}, s_1) + \kappa(s^\nu_{k^*}) + \frac{\nu}{\tau}(s_1 - s^\nu_{k^*}) \le 0$$

or, equivalently, since $s_1 - s^\nu_{k^*} > 0$,

$$\frac{\nu}{\tau} \le \frac{G(t_{k^*+1}, s_1) - \kappa(s^\nu_{k^*})}{s_1 - s^\nu_{k^*}} \le \frac{G_{\max} - \kappa(s^\nu_{k^*})}{s_1 - s^\nu_{k^*}} < \frac{\kappa(s_1) - \kappa(s^\nu_{k^*})}{s_1 - s^\nu_{k^*}} \le \frac{\kappa(s_1)}{s_1 - s_{\max}} =: L_*$$

where the second inequality is derived from the definition of $G_{\max}$ and the third one comes from our assumption (2.9).

Therefore, by taking $\tau_0 < \nu/L_*$ we deduce that this second case cannot occur and hence $s^\nu_{k^*+1} < s_1$ for every k^* and the proof is complete. ∎

From now on we will consider $\tau < \tau_0$ so that, thanks to (4.11) the time-incremental problem (4.6) becomes

$$0 \in \partial_{\dot{s}} \mathcal{R}_\nu(\underline{s}^\nu_\tau(t), \dot{\hat{s}}^\nu_\tau(t)) - G(\overline{t}_\tau(t), \overline{s}^\nu_\tau(t)). \tag{4.12}$$

4.2 Existence of a viscous solution

We consider now the limit in τ and prove that it is a viscous solution.

Theorem 4.2 *There exist a function $s^\nu \in \mathrm{H}^1([0,T]; [s_0, s_1])$ and a subsequence of τ (not labeled) such that*

$$\underline{s}^\nu_\tau, \overline{s}^\nu_\tau, \hat{s}^\nu_\tau \;\to\; s^\nu \qquad \text{in } \mathrm{L}^\infty([0,T]\,; [s_0, s_1]) \tag{4.13}$$

$$\dot{\hat{s}}^\nu_\tau \rightharpoonup \dot{s}^\nu \qquad \text{in } \mathrm{L}^2([0,T]\,; \mathbb{R}). \tag{4.14}$$

Moreover, for a.e. $t \in [0,T]$

$$0 \in \partial_{\dot{s}} \mathcal{R}_\nu(s^\nu(t), \dot{s}^\nu(t)) - G(t, s^\nu(t)). \tag{4.15}$$

Proof: We essentially use the continuous embedding $\mathrm{H}^1([0,T]) \subset \mathrm{C}^{0,1/2}([0,T])$ and the compact embedding of $\mathrm{C}^{0,1/2}([0,T]) \subset \mathrm{C}^0([0,T])$ (via the Arzela-Ascoli theorem).

Using estimates (4.9) the sequence $(\hat{s}^\nu_\tau)_\tau$ is bounded in $\mathrm{H}^1([0,T]; [s_0, s_1])$ and we find a weakly convergent subsequence (not renamed). In particular, (4.14) holds.

By the compact embedding into $\mathrm{C}^0([0,T])$ it also converges uniformly on $[0,T]$. Employing (4.10) we have also proved (4.13).

To establish the differential inclusion (4.15) we pass to the limit in (4.12). First note that G is continuous, hence we have

$$g^\tau(t) := G(\bar{t}_\tau(t), \overline{s}^\nu_\tau(t)) \to g^0(t) := G(t, s^\nu(t)) \text{ for all } t \in [0,T].$$

Equation (4.12) is equivalent to

$$\int_0^T \mathcal{R}_\nu(\underline{s}^\nu_\tau(t), w(t)) - \mathcal{R}_\nu(\underline{s}^\nu_\tau(t), \dot{\hat{s}}^\nu_\tau(t)) - g^\tau(t)(w(t) - \dot{\hat{s}}^\nu_\tau(t))\,\mathrm{d}t \geq 0 \tag{4.16}$$

for all $w \in \mathrm{L}^2([0,T])$. In fact, it suffices to consider w with $w \geq 0$ a.e. in $[0,T]$. For passing to the limit $\tau \to 0$ note that the first term converges pointwise with a majorant $\kappa_{\max} w + \frac{\nu}{2} w^2$, hence its limit is $\int_0^T \mathcal{R}_\nu(s^\nu, w)\,\mathrm{d}t$. The third term converges because it is a scalar product of a strongly and a weakly convergent sequence. For the second term, using the fact that $\dot{\hat{s}}^\nu_\tau(t) \geq 0$, we estimate

$$\int_0^T |\mathcal{R}_\nu(\underline{s}^\nu_\tau(t), \dot{\hat{s}}^\nu_\tau(t)) - \mathcal{R}_\nu(\hat{s}^\nu_\tau(t), \dot{\hat{s}}^\nu_\tau(t))|\,\mathrm{d}t \leq \omega_\kappa\big(\|\underline{s}^\nu_\tau - \hat{s}^\nu_\tau\|_\infty\big) \int_0^T \dot{\hat{s}}^\nu_\tau(t)\,\mathrm{d}t,$$

where ω_κ is a modulus of continuity of $\kappa \in \mathrm{C}^0([0,L])$. As the last integral equals $\overline{s}^\nu_\tau(T) - \underline{s}^\nu_\tau(0) \leq s_1 - s_0$ and by (4.10), the difference tends to 0 for $\tau \to 0$. Thus it remains to show the convergence of $\int_0^T \mathcal{R}_\nu(\hat{s}^\nu_\tau(t), \dot{\hat{s}}^\nu_\tau(t))\,\mathrm{d}t$ but this equals again $\int_{s_0}^{\overline{s}^\nu_\tau(T)} \kappa(s)\,\mathrm{d}s + \frac{\nu}{2}\int_0^T |\dot{\hat{s}}^\nu_\tau(t)|^2\,\mathrm{d}t$. The convergence of the first term follows with (4.13), while according to (4.14) lower semicontinuity can be applied to the second term. In particular, taking the lim inf as $\tau \to 0$ in (4.16) we find

$$\int_0^T \mathcal{R}_\nu(s^\nu(t), w(t)) - \mathcal{R}_\nu(s^\nu(t), \dot{s}^\nu(t)) - g^0(t)(w(t) - \dot{s}^\nu(t))\,\mathrm{d}t \geq 0$$

for all $w \in \mathrm{L}^2([0,T])$, which is equivalent to the desired equation (4.15). This concludes the proof. ∎

Now we are in a position to prove the main result of Section 2.2, which turns out to be an easy consequence of the previous Theorem 4.2.

Proof of Theorem 2.2: For the given $s^\nu : [0,T] \to [s_0, s_1]$ we choose $u^\nu(t) \equiv \mathcal{U}(t, s^\nu(t))$ for every $t \in [0,T]$, then (2.11) of Definition 2.1 is satisfied. Moreover, (4.15) together with (3.13) provides (2.12). ∎

Lemma 4.3 *The subdifferential formulation* (4.15) *is equivalent to the following three properties which hold true for a.e.* $t \in [0,T]$*:*

(a^ν) $\dot{s}^\nu(t) \geq 0$;

(b^ν) $\kappa(s^\nu(t)) + \nu\dot{s}^\nu(t) - G(t, s^\nu(t)) \geq 0$;

(c^ν) $(\kappa(s^\nu(t)) + \nu\dot{s}^\nu(t) - G(t, s^\nu(t)))\dot{s}^\nu(t) = 0$.

We note that by Lemma 4.3 it turns out that the viscous solution $t \mapsto s^\nu(t)$ satisfies the Karush-Kuhn-Tucker conditions ([Kar39, KuT51]).

Proof: It is sufficient to prove that conditions (a^ν)–(c^ν) are equivalent to the following evolutionary variational inequality

$$\mathcal{R}_0(s^\nu(t), \dot\sigma) - \mathcal{R}_0(s^\nu(t), \dot s^\nu(t)) + \big[\nu \dot s^\nu(t) - G(t, s^\nu(t))\big](\dot\sigma - \dot s^\nu(t)) \geq 0 \quad \forall \dot\sigma \in \mathbb{R}. \quad (4.17)$$

The direction (a^ν)–$(c^\nu) \Rightarrow (4.17)$ is immediate, while the opposite direction is obtained by an argument very similar to the one proving (4.3) and therefore it is omitted. ∎

We define

$$s_{\min} := \min\{\, s > s_0 \mid \kappa(s) \geq G_{\max} \,\}$$

and note that by our assumption (2.9) we have $s_{\min} < s_1$. It turns out that

$$s^\nu(t) \leq s_{\min} < s_1 \text{ for every } t \in [0,T]. \quad (4.18)$$

Indeed, if there is $t_* \in\,]0,T]$ with $s^\nu(t_*) > s_{\min}$ then there exist $t_1 < t_2 \leq t_*$ such that

$$s^\nu(t_1) = s_{\min},\ s^\nu(t) > s_{\min}\ \forall t \in\,]t_1, T] \text{ and } \kappa(s^\nu(t)) > G_{\max}\ \forall t \in\,]t_1, t_2]$$

(see also Figure 4.1).

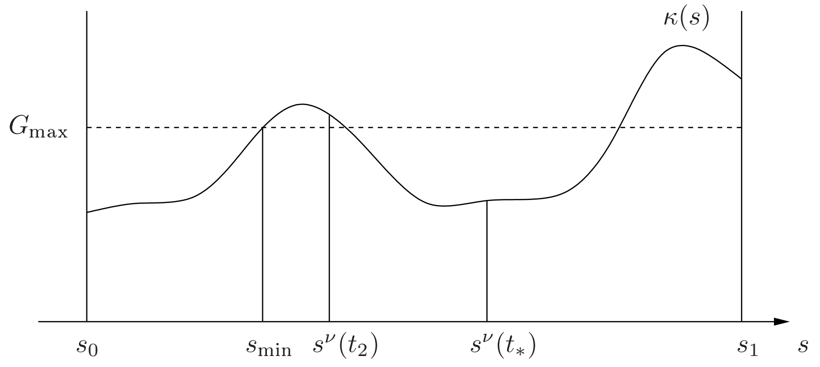

Figure 4.1: A possible situation for the graph of κ and the quantities $s_{\min} = s^\nu(t_1)$, $s^\nu(t_2)$ and $s^\nu(t_*)$.

Therefore $\kappa(s^\nu(t)) - G(t, s^\nu(t)) > 0$ for every $t \in\,]t_1, t_2]$. Condition (c^ν) in Lemma 4.3 implies then $\dot s^\nu(t) = 0$ a.e. on $]t_1, t_2]$. By the continuity of s^ν we derive $s_{\min} = s^\nu(t_1) = s^\nu(t_2) > s_{\min}$, a contradiction and (4.18) is proven.

Let us note that the same argument can be used to prove the following lemma.

Lemma 4.4 *Let s^ν be a viscous solution for $\mathcal{I}$ and $\mathcal{R}_\nu$ and let $t \in [0,T]$ be such that $\kappa(s^\nu(t)) - G(t, s^\nu(t)) > 0$. Then there exists $\delta > 0$ such that the map s^ν is constant in $[t-\delta, t+\delta] \cap [0,T]$.*

Proof: By continuity, there exists $\delta > 0$ such that $\kappa(s^\nu(\hat t)) - G(\hat t, s^\nu(\hat t)) > 0$ for every $\hat t \in [t-\delta, t+\delta] \cap [0,T]$.

Now we can conclude following the same argument as above (replacing the interval $]t_1, t_2]$ with $[t-\delta, t+\delta] \cap [0,T]$). ∎

We end this section by proving the energy balance condition.

Lemma 4.5 *Let $s^\nu \in \mathrm{H}^1([0,T];[s_0,s_1])$ be a function satisfying (4.15). Then the following energy balance condition holds true*

$$\mathcal{I}(t_2, s^\nu(t_2))+\int_{t_1}^{t_2} \left(\kappa(s^\nu(t))\dot{s}^\nu(t)+\nu|\dot{s}^\nu(t)|^2\right) \mathrm{d}t = \mathcal{I}(t_1, s^\nu(t_1))+\int_{t_1}^{t_2} \partial_t\mathcal{I}(t, s^\nu(t))\,\mathrm{d}t, \quad (4.19)$$

for every $0 \le t_1 < t_2 \le T$.

Proof: By Lemma 4.3 it follows that the map s^ν satisfies conditions (a^ν)–(c^ν). Moreover, for a.e. $t \in [0,T]$ we have, via the chain rule for $s^\nu \in \mathrm{H}^1([0,T])$,

$$-G(t, s^\nu(t))\dot{s}^\nu(t) = \partial_s\mathcal{I}(t, s^\nu(t))\dot{s}^\nu(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t, s^\nu(t)) - \partial_t\mathcal{I}(t, s^\nu(t)).$$

Now (4.19) is an immediate consequence of the integral version of condition (c^ν). ∎

We observe that, since $u^\nu(t) = \mathcal{U}(t, s^\nu(t))$, condition (4.19) turns out to be equivalent to energy balance (2.13).

5 Rate-independent limit

In this section we want to pass to the limit in ν, in order to prove the existence of a local energetic solution associated with $\mathcal{I}$ and $\mathcal{R}_0$. This procedure is usually called vanishing viscosity method. We begin by stating some a priori estimates.

Lemma 5.1 *Let $s^\nu \in \mathrm{H}^1([0,T];[s_0,s_1])$ be a solution of (4.15). Then there exists a positive constant (independent of ν) such that the following estimates hold true:*

$$\|s^\nu\|_{L^\infty(0,T)} \le C \quad (5.1)$$

$$\int_0^T \mathcal{R}_\nu(s^\nu(t), \dot{s}^\nu(t))\,\mathrm{d}t \le C \quad (5.2)$$

$$\nu\int_0^T |\dot{s}^\nu(t)|^2\,\mathrm{d}t \le C. \quad (5.3)$$

Proof: Since $s^\nu(t) \in [s_0, s_1]$ for every $t \in [0,T]$, we deduce that estimate (5.1) is satisfied by any constant $C \ge s_1$. Lemma 4.3 guarantees that condition (c^ν) holds true. Thus we derive

$$\begin{aligned}\int_0^T \left[\kappa(s^\nu(t))\dot{s}^\nu(t) + \nu|\dot{s}^\nu(t)|^2\right] \mathrm{d}t &= \int_0^T G(t, s^\nu(t))\dot{s}^\nu(t)\,\mathrm{d}t \\ &= -\mathcal{I}(T, s^\nu(T)) + \mathcal{I}(0, s_0) + \int_0^T \partial_t\mathcal{I}(t, s^\nu(t))\,\mathrm{d}t\end{aligned}$$

and the last right-hand side is bounded since $\mathcal{I} \in \mathrm{C}^1([0,T] \times [s_0,s_1])$, by Theorem 3.6. By condition (a^ν) and (2.8) we get that (5.2) and then (5.3) hold true, and therefore the proof is complete. ∎

Theorem 5.2 *There exist a function $s \in \mathrm{BV}(0,T;[s_0,s_1])$ and a subsequence of ν (not labeled) such that*

$$s^\nu \overset{*}{\rightharpoonup} s \text{ in } \mathrm{BV}(0,T;[s_0,s_1]) \tag{5.4}$$

$$s^\nu(t) \to s(t) \text{ for all } t \in [0,T]. \tag{5.5}$$

Moreover, the limit function s is a local energetic solution for $\mathcal{R}_0$ and $\mathcal{I}$ as defined in Definition 2.3, namely

- *(a) s is nondecreasing;*
- *(b) $\kappa(s(t)) - G(t,s(t)) \geq 0$ for all $t \in [0,T]\backslash J(s)$;*
- *(c) if $\kappa(s(t)) - G(t,s(t)) > 0$, then $t \in D(s)$ and $\dot{s}(t) = 0$;*
- *(d) for all $t \in J(s)$ and all $s_* \in [s(t-), s(t+)]$ we have $\kappa(s_*) - G(t,s_*) \leq 0$,*

where $J(s)$ and $D(s)$ denote the jump set and the set of differentiability, respectively.

Proof: An application of the classical Helly selection theorem (see, e.g., [Rud76]) together with the a priori estimates of Lemma 5.1 provide the existence of a subsequence of ν and of a function $s \in \mathrm{BV}([0,T];[s_0,s_1])$ satisfying (5.4)–(5.5).

Taking into account these convergences, we want to derive the limit problem solved by the map $t \mapsto s(t)$. The idea is to consider the limit in the formulation (a^ν)–(c^ν) which is equivalent to (4.15) as shown in Lemma 4.3.

First of all, let us note that condition (a) is an immediate consequence of Helly's Theorem. It follows that $t \mapsto s(t)$ is continuous at a.e. $t \in [0,T]$, and the jump set $J(s)$ is at most countable, since the sum of jumps is bounded by $s_1 - s_0$.

Further, we observe that a priori bound (5.3) implies

$$\nu \dot{s}^\nu \to 0 \quad \text{in } \mathrm{L}^2([0,T]). \tag{5.6}$$

Moreover, by condition (b^ν)

$$\int_0^T \psi(t)\big[\kappa(s^\nu(t)) - G(t,s^\nu(t)) + \nu\dot{s}^\nu(t)\big]\,\mathrm{d}t \geq 0$$

for every $\psi \in \mathrm{L}^2([0,T])$ with $\psi \geq 0$. Thanks to (5.6) we can pass to the limit and obtain an integral version of condition (b), namely

$$\int_0^T \psi(t)\big(\kappa(s(t)) - G(t,s(t))\big)\,\mathrm{d}t \geq 0 \qquad \forall \psi \in \mathrm{L}^2([0,T]),\ \psi \geq 0$$

because of convergence (5.5), and continuity of κ and of G. Then, $\kappa(s(t)) - G(t,s(t)) \geq 0$ for a.e. $t \in [0,T]$. In particular, the inequality is true for every t in which the map s is continuous, and therefore condition (b) is proven.

In order to obtain condition (d), let us fix $\hat{t} \in J(s)$ and $s(\hat{t}-) \leq s^a < s^b \leq s(\hat{t}+)$. From the continuity of the map $t \mapsto s^\nu(t)$ we deduce that for every ν there exist $\hat{t}^\nu_-$ and $\hat{t}^\nu_+$ such that

$$\hat{t}^\nu_- < \hat{t}^\nu_+, \quad \hat{t}^\nu_- \to \hat{t}, \quad \hat{t}^\nu_+ \to \hat{t}, \quad s^\nu(\hat{t}^\nu_-) \equiv s^a, \quad s^\nu(\hat{t}^\nu_+) \equiv s^b.$$

Condition (c^ν) of Lemma 4.3 implies

$$\int_{\hat t_-^\nu}^{\hat t_+^\nu} \varphi(s^\nu(t))\big(\kappa(s^\nu(t)) - G(t, s^\nu(t))\big)\dot s^\nu(t)\,\mathrm{d}t \le 0 \tag{5.7}$$

for every $\varphi \in \mathrm{L}^2([s_0, s_1])$ with $\varphi \ge 0$. Now we change variables, putting $\sigma := s^\nu(t)$ and defining $t^\nu(\sigma) := \min\{\, t \in [\hat t_-^\nu, \hat t_+^\nu] \mid s^\nu(t) = \sigma \,\}$ so that inequality (5.7) becomes

$$\int_{s^a}^{s^b} \varphi(\sigma)\big(\kappa(\sigma) - G(t^\nu(\sigma), \sigma)\big)\,\mathrm{d}\sigma \le 0$$

for every $\varphi \in \mathrm{L}^2([s_0, s_1])$, $\varphi \ge 0$. Passing now to the limit as $\nu \to 0$, since $t^\nu(\sigma) \to \hat t$, for every $\sigma \in [s(\hat t-), s(\hat t+)]$, and since G is continuous thanks to Theorem 3.6, we get

$$\int_{s^a}^{s^b} \varphi(\sigma)\big(\kappa(\sigma) - G(\hat t, \sigma)\big)\,\mathrm{d}\sigma \le 0.$$

Therefore, $\kappa(s_*) - G(\hat t, s_*) \le 0$ for every $s_* \in [s^a, s^b]$ and by the fact that s^a and s^b were arbitrarily chosen in $[s(\hat t-), s(\hat t+)]$ we obtain finally condition (d).

We are left with condition (c). Let t be such that $\kappa(s(t)) - G(t, s(t)) > 0$. Then by condition (d) $t \notin J(s)$ so that the map s is continuous in t. By continuity of κ and G and the pointwise convergence (5.5) we derive the existence of $\nu_0 > 0$ and of $\delta > 0$ such that for every $\nu \in [0, \nu_0]$ and every $\hat t \in [t - \delta, t + \delta] \cap [0, T]$ we have $\kappa(s^\nu(\hat t)) - G(\hat t, s^\nu(\hat t)) > 0$. Applying now Lemma 4.4 we deduce that (for some possibly smaller $\delta > 0$) the map s^ν is constant on $[t-\delta, t+\delta] \cap [0, T]$ for every $\nu \in [0, \nu_0]$. Therefore, the limit map s is constant on $[t - \delta, t + \delta] \cap [0, T]$, so that $t \in D(s)$ and $\dot s(t) = 0$.

This concludes the proof of condition (c) and the theorem is proven. ∎

We observe that from (4.18) and (5.5) it follows that

$$s(t) \le s_{\min} < s_1 \ \forall t \in [0, T]$$

and therefore $s(t)$ does not reach the point s_1 during the time interval $[0, T]$.

With the help of the monotone inverse $\hat t : [s(0), s(T)] \to [0, T]$ of $s : [0, T] \to [s_0, s_1]$, we can distinguish between the following three different regimes:

Regime I (sticking crack tip, i.e., no motion of crack tip):
$\dot s(t) = 0$, $\hat t(s)$ jumps, $\kappa(s(t)) - G(t, s(t)) \ge 0$;

Regime II (crack grows slowly):
both $\hat t$ and s are continuous and $\kappa(s(t)) - G(t, s(t)) = 0$;

Regime III (crack tip jumps):
s jumps at t, $\hat t'(s) = 0$ and $\kappa(s(t)) - G(t, s(t)) \le 0$.

When $\kappa(s(t)) - G(t, s(t)) = 0$ the three different situations are all admissible. On the other hand, the case $\kappa(s(t)) - G(t, s(t)) > 0$ will always express Regime I (i.e., no crack growth), while the case $\kappa(s(t)) - G(t, s(t)) < 0$ shall correspond only to Regime III. However, by additionally assuming $s(t) \in \{s(t-), s(t+)\}$ this last case disappears even though jumps occur along which $\kappa(\widetilde s) < G(t, \widetilde s)$ for $\widetilde s \in \,]s(t-), s(t+)[$ is possible.

Remark 5.3 Under the additional assumption that the map $s \mapsto G(t,s)$ is Lipschitz continuous, uniqueness of the viscous solution s^ν is guaranteed, and then $s^\nu(t)$ converges monotonically to the limit $s(t)$ (personal communication by Negri). Therefore in this situation $t \mapsto s(t)$ turns out to be continuous from the left (i.e., $s(t) = s(t-)$), and the situation $\kappa(s(t)) - G(t, s(t)) < 0$ cannot occur.

Proof of Theorem 2.4: Arguing in the same manner as in the proof of Theorem 2.2, for the map $s : [0,T] \to [s_0, s_1]$ obtained from Theorem 5.2 we choose $u(t) := \mathcal{U}(t, s(t)) = \operatorname{argmin} \mathcal{E}(t, \cdot, s(t))$ so that (2.15) in Definition 2.3 is satisfied. Theorem 5.2 together with equality (3.13) provides conditions (a)-(d). ∎

The following lemma implies that any local energetic solution is a local solution (LS), which was defined in Definition 2.5.

Lemma 5.4 *Conditions (a)–(c) of Theorem 5.2 are equivalent to the subdifferential formulation*

$$0 \in \partial_{\dot{s}} \mathcal{R}_0(s(t), \dot{s}(t)) - G(t, s(t)) \quad \textit{for every } t \in D(s).$$

The proof is very similar to the proof of Lemma 4.3 and therefore it is omitted.

Now we are in a position to prove our extended energy balance. For the sake of clarity, we recall first the definition of the function μ given in (2.18):

$$\mu([t_1, t_2]) := \Delta^+(t_1) + \Delta^-(t_2) + \sum_{t \in \,]t_1, t_2[\, \cap J(s)} (\Delta^+(t) + \Delta^-(t)),$$

where for each jump time $t \in J(s)$ we defined in (2.17) the nonnegative quantities $\Delta^+(t)$ and $\Delta^-(t)$ by

$$\Delta^+(t) := \int_{s(t)}^{s(t+)} [G(t,\sigma) - \kappa(\sigma)] \, \mathrm{d}\sigma \text{ and } \Delta^-(t) := \int_{s(t-)}^{s(t)} [G(t,\sigma) - \kappa(\sigma)] \, \mathrm{d}\sigma$$

respectively. We observe that μ is a nonnegative set function defined on closed subintervals of $[0,T]$, that it is finite and additive, so that

$$\mu([t_1, t_2]) = \mu([t_1, t_3]) + \mu([t_3, t_2]) \quad \forall 0 \le t_1 < t_3 < t_2 \le T.$$

Lemma 5.5 *Every local energetic solution $t \mapsto s(t)$ associated with $\mathcal{I}$ and $\mathcal{R}_0$ satisfies the following extended energy balance: for every $0 \le t_1 < t_2 \le T$ we have*

$$\mathcal{I}(t_2, s(t_2)) + \int_{s(t_1)}^{s(t_2)} \kappa(\sigma) \, \mathrm{d}\sigma + \mu([t_1, t_2]) = \mathcal{I}(t_1, s(t_1)) + \int_{t_1}^{t_2} \partial_t \mathcal{I}(\tau, s(\tau)) \, \mathrm{d}\tau, \tag{5.8}$$

where the function μ is given by (2.18).

Proof: The proof is essentially an application of the chain rule in BV. Indeed, we have that $\mathcal{I}(\cdot, s(\cdot)) \in \mathrm{BV}([0,T])$ and

$$\mathrm{D}\mathcal{I}(\cdot, s(\cdot)) = \partial_t \mathcal{I}(\cdot, s(\cdot)) \mathrm{d}t + \partial_s \mathcal{I}(\cdot, s(\cdot)) \widetilde{\mathrm{D}} s + \sum_{t \in J(s)} \big[\mathcal{I}(t, s(t+)) - \mathcal{I}(t, s(t-))\big] \delta_t \tag{5.9}$$

where $J(s) \subset [0,T]$ is the set of discontinuity points of s, and $\widetilde{\mathrm{D}}s = \dot{s}\mathrm{d}t + \mathrm{D}^c s$ is the diffuse part of the derivative $\mathrm{D}s$ (for a proof see, e.g., [AFP00, Theorem 3.96]).

We note that

$$\partial_s \mathcal{I}(\cdot, s(\cdot))\widetilde{\mathrm{D}}s = -\kappa(s(\cdot))\widetilde{\mathrm{D}}s$$

since by conditions (b) and (c) we have $(\kappa(s(\cdot)) - G(\cdot, s(\cdot)))\widetilde{\mathrm{D}}s = 0$. On the other hand, for the jump part of the derivative we have

$$\mathrm{D}^j \mathcal{I}(\cdot, s(\cdot)) = -\sum_{t \in J(s)} \int_{s(t-)}^{s(t+)} G(t,\sigma)\,\mathrm{d}\sigma\,\delta_t.$$

Now, by (2.14) we derive

$$\mathcal{I}(t_2, s(t_2)) - \mathcal{I}(t_1, s(t_1)) = \int_{]t_1,t_2[} \mathrm{D}\mathcal{I}(\cdot, s(\cdot)) - \int_{s(t_2-)}^{s(t_2)} G(t_2,\sigma)\,\mathrm{d}\sigma - \int_{s(t_1)}^{s(t_1+)} G(t_1,\sigma)\,\mathrm{d}\sigma.$$

Therefore, (5.9) and

$$\int_{s(t-)}^{s(t+)} G(t,\sigma)\,\mathrm{d}\sigma = \Delta^+(t) + \Delta^-(t) + \int_{s(t-)}^{s(t+)} \kappa(\sigma)\,\mathrm{d}\sigma$$

yield

$$\begin{aligned}\mathcal{I}(t_2, s(t_2)) - \mathcal{I}(t_1, s(t_1)) &= \int_{t_1}^{t_2} \partial_t \mathcal{I}(t, s(t))\,\mathrm{d}t - \mu([t_1,t_2]) - \int_{]t_1,t_2[} \kappa(s(\cdot))\widetilde{\mathrm{D}}s \\ &- \int_{s(t_2-)}^{s(t_2)} \kappa(\sigma)\,\mathrm{d}\sigma - \int_{s(t_1)}^{s(t_1+)} \kappa(\sigma)\,\mathrm{d}\sigma - \sum_{t \in J(s)\cap]t_1,t_2[} \int_{s(t-)}^{s(t+)} \kappa(\sigma)\,\mathrm{d}\sigma\,\delta_t,\end{aligned}$$

which is equal to (5.8), and the proof is complete. ■

This proves also (2.19).

Now the usual energy inequality turns out to be a direct consequence of the previous result, (simply by using the fact that $\mu([t_1,t_2]) \geq 0$).

Corollary 5.6 *Every local energetic solution $t \mapsto s(t)$ associated with $\mathcal{I}$ and $\mathcal{R}_0$ satisfies the following simplified energy inequality: for every $0 \leq t_1 \leq t_2 \leq T$ we have*

$$\mathcal{I}(t_2, s(t_2)) + \int_{s(t_1)}^{s(t_2)} \kappa(s)\,\mathrm{d}s \leq \mathcal{I}(t_1, s(t_1)) + \int_{t_1}^{t_2} \partial_t \mathcal{I}(t, s(t))\,\mathrm{d}t.$$

This gives energy inequality (2.16).

6 Examples

Here we present a few examples, which highlight the features of the functionals and solutions constructed above. Throughout we restrict to the case of linearized elasticity, such that the energy $\mathcal{E}$ is quadratic in u. For Dirichlet boundary conditions and loading of the form $(u_{\mathrm{Dir}}(t,\cdot), \ell(t)) = a(t)(u^0_{\mathrm{Dir}}, \ell^0)$ the reduced energy $\mathcal{I}$ takes the form $\mathcal{I}(t,s) = a(t)^2 \hat{\mathcal{I}}(s)$. Moreover, we will assume that the crack path is the straight line $\mathcal{C} := [0,L]\times\{0\}$ and that the fracture toughness is constant, i.e., $\kappa(s) \equiv \kappa$.

Example 6.1 In this example we treat a toy problem, which can be considered as a singular limit of a very thin body $\Omega =]0, L[\times]{-h}, h[$ with $0 < h \ll 1$. The prescribed crack path is $\gamma(s) = (s, 0)$ and the displacement u is restricted to be symmetric with respect to the x-axis, i.e., $u(t, x, y) = \mathrm{diag}(1, -1)\, u(t, x, -y)$. Moreover, for very small h, it is reasonable to assume that the displacement has the form $u(t, x, y) = (0, v(t, x)\mathrm{sign}(y))$, where $v(t, x) = 0$ for $x > s(t)$ (ahead of the crack). The purpose of the resulting toy model is to show that we are able to generate a large class of possible release rate functionals G in the form $G(t, s) = a(t)^2 \hat{G}(s)$. Moreover, we find the asymptotics for crack length going to 0. In our toy problem we have $\hat{G}(s) \sim s^2$ which indicates that we are not able to describe crack initiation.

Under these assumptions the PDE problem reduces to the following ODE problem. For any $s \in]0, L[$ we set

$$V_s := \{\, v \in \mathrm{H}^1_0([0, L]) \mid \mathrm{spt}(v) \subset [0, s] \,\} \quad \text{and}\; \mathcal{E}(t, v) := \int_0^L [\tfrac{1}{2} v'(x)^2 + a(t)\hat{f}(x) v(x)]\, \mathrm{d}x.$$

With $\hat{\mathcal{I}}(s) := \min\{\, \mathcal{E}(1, v) \mid v \in V_s \,\}/a(1)^2$ we find

$$\mathcal{I}(t, s) = a(t)^2 \hat{\mathcal{I}}(s) \quad \text{and}\; G(t, s) = a(t)^2 \hat{G}(s),$$

where $\hat{G}(s) = -\hat{\mathcal{I}}'(s) \geq 0$. In fact, $\hat{\mathcal{I}}$ can be determined explicitly using $F(x) = \int_0^x (x - \xi)\hat{f}(\xi)\, \mathrm{d}\xi$, i.e., we have $F(0) = 0 = F'(0)$, $F'' = \hat{f}$. The unique minimizer $v = \mathcal{V}_s \in V_s$ of $\mathcal{E}(1, \cdot)$ reads

$$\mathcal{V}_s(x) = \begin{cases} F(x) - \dfrac{F(s)}{s} x & \text{for } x \in [0, s], \\ 0 & \text{otherwise.} \end{cases}$$

Some explicit calculations yield

$$\hat{\mathcal{I}}(s) = \frac{F(s)^2}{2s} - \frac{1}{2}\int_0^s F'(x)^2\, \mathrm{d}x \quad \text{and} \quad \hat{G}(s) = -\hat{\mathcal{I}}'(s) = \frac{1}{2}\big(F'(s) - \frac{F(s)}{s}\big)^2 \geq 0.$$

From the last expression we see that every $\hat{G}$ in the form $\hat{G}(s) = \frac{s^2}{2}\gamma(s)^2$ with $\gamma \in \mathrm{W}^{1,1}([0, L])$ can be realized as a release rate by taking $F(s) = s \int_0^s \gamma(x)\, \mathrm{d}x$, i.e., $\hat{f}(s) = s\gamma'(s) + 2\gamma(s)$.

Example 6.2 Let us recall the example proposed in [ToZ06, Section 7]: there, $a(t) = t$, $\ell(t) = 0$, and $\mathcal{E}(t, u, s) = \int_{\Omega_s} |\nabla u(x)|^2\, \mathrm{d}x$ for $u = t u^0_{\mathrm{Dir}}$ on Γ_D. The reduced energy functional then takes the form $\mathcal{I}(t, s) = t^2 \hat{\mathcal{I}}(s)$. The authors focus on the shape of the graph of $\hat{\mathcal{I}}$ (instead of $\hat{G}$) and provide an explicit example in which $\hat{\mathcal{I}}$ is concave on some subinterval of $[s_0, s_1]$. Actually the construction goes by approximation, and a domain $\Omega_\varepsilon \subset \mathbb{R}^2$ and a prescribed boundary displacement $u^\varepsilon_{\mathrm{Dir}}$, both dependent on a small parameter ε, are provided.

Specifically Ω_ε consists of two discs B_{-2} and B_2 of radius 1 centered at $(-2, 0)$ and $(2, 0)$ respectively, connected by a rectangle T_ε of height $\sim 2\varepsilon$. The boundary displacement $u^\varepsilon_{\mathrm{Dir}}$ is such that on half part of T_ε (the left one) the body experiences some "closing" force,

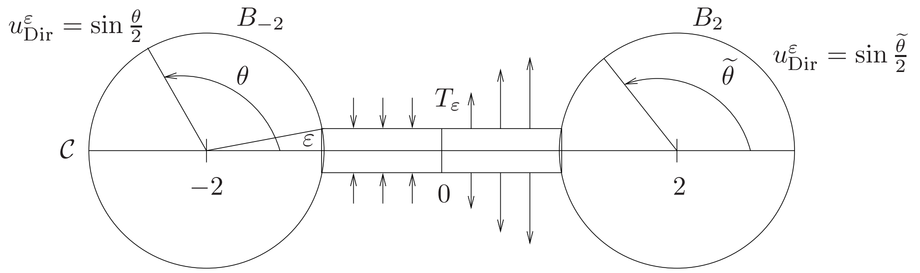

Figure 6.1: The set Ω_ε and the effect of the prescribed boundary displacement.

while on the other half part (the right one), the body experiences some "opening" force (see Figure 6.1).

Then, the limit of the energy

$$\hat{\mathcal{I}}_\varepsilon(s) := \min\{ \int_{\Omega_\varepsilon \setminus \mathcal{C}_s} |\nabla u|^2 \,\mathrm{d}x \mid u \in \mathrm{H}^1(\Omega_\varepsilon \setminus \mathcal{C}_s; \mathbb{R}),\ u = u^{\varepsilon}_{\mathrm{Dir}} \text{ on } \partial\Omega_\varepsilon \setminus \mathcal{C}_s \}$$

as $\varepsilon \to 0$ is considered. Note that here the prescribed crack path is $\mathcal{C} = [-3, 3] \times \{0\}$.

As $s \mapsto \hat{\mathcal{I}}_\varepsilon(s)$ is a C^2-function, in order to obtain that the map is not convex on the whole interval $[-2, 2]$ the following three facts are established: $\limsup_{\varepsilon\to 0+} \hat{\mathcal{I}}_\varepsilon(2)$ is finite, $\liminf_{\varepsilon\to 0+} \hat{\mathcal{I}}_\varepsilon(-2) = +\infty$; while $\limsup_{\varepsilon\to 0+} \hat{\mathcal{I}}'_\varepsilon(-2)$ is finite.

We note here in addition that it is possible to prove that $\limsup_{\varepsilon\to 0+} \hat{\mathcal{I}}'_\varepsilon(2)$ is finite, too. The proof follows the lines of the one proving that $\limsup_{\varepsilon\to 0+} \hat{\mathcal{I}}'_\varepsilon(-2)$ is finite, for which we refer to [ToZ06, Section 7].

Thus, we can conclude that the profile of $\hat{\mathcal{I}}(s)$ is concave in a first subinterval of $[-2, 2]$ and it is convex in the last part.

Example 6.3 We discuss here the different behavior of our local energetic solution defined in Definition 2.3, the global energetic solution (GES), and a "generic" local solution (LS) defined in Definition 2.5, in the particular case of $a(t) = t$, $\ell(t) = 0$, and

$$\hat{G}(s) = \begin{cases} s - s_0 + 1 & \text{if } s_0 \le s \le 2s_0 \\ 3s_0 + 1 - s & \text{if } 2s_0 \le s \le L. \end{cases}$$

Thanks to our Example 6.1 such a choice for $\hat{G}(s)$ is admissible.

In general, we have to compare the position of $\hat{G}(s(t))$ with the line $\frac{\kappa}{t^2}$, which is moving down as time increases. According to Griffith, we distinguish between three different situations (see also Figure 6.2):

(1) Regime I: no crack growth in the region strictly above the graph of $\hat{G}$, since there we have $\frac{\kappa}{t^2} - \hat{G}(s(t)) > 0$;

(2) Regime III: jumps in the region strictly below the graph of $\hat{G}$, where $\frac{\kappa}{t^2} - \hat{G}(s(t)) < 0$;

(3) Regime II: slow crack propagation when $\frac{\kappa}{t^2} - \hat{G}(s(t)) = 0$.

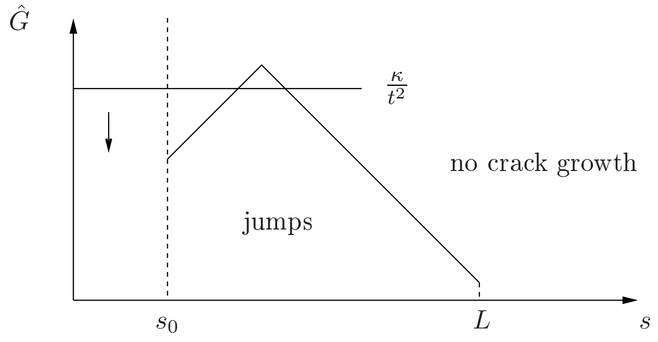

Figure 6.2: Interplay between $\hat{G}(s(t))$ and κ/t^2.

Let us start with the global energetic solution (GES), that we denote here by $s_{\mathrm{G}}(t)$. According to the stability condition (S), we have

$$t^2\hat{\mathcal{I}}(s_{\mathrm{G}}(t)) \leq t^2\hat{\mathcal{I}}(\hat{s}) + \kappa(\hat{s} - s_{\mathrm{G}}(t)) \quad \forall \hat{s} \geq s_{\mathrm{G}}(t),\, \forall t \in [0,T]$$

which is equivalent to

$$\int_{s_{\mathrm{G}}(t)}^{\hat{s}} \big(\hat{G}(\sigma) - \frac{\kappa}{t^2}\big)\,\mathrm{d}\sigma \leq 0 \quad \forall \hat{s} \geq s_{\mathrm{G}}(t),\, \forall t \in [0,T].$$

On the other hand, energy balance condition (E) gives

$$\big(- \hat{G}(s_{\mathrm{G}}(t)) + \frac{\kappa}{t^2}\big)\dot{s}_{\mathrm{G}}(t) = 0.$$

Therefore, assuming $s_{\mathrm{G}}(0) = s_0$, we expect that s_{G} will start to propagate (with a jump) at the first time $t = t_1$ such that the following equal-area rule is satisfied:

$$\int_{s_0}^{s_{\mathrm{G}}(t)} \big(\hat{G}(\sigma) - \frac{\kappa}{t^2}\big)\,\mathrm{d}\sigma = 0. \tag{6.1}$$

This behavior is represented in picture (GES) of Figure 6.3. At time t_1 we have (6.1) satisfied, so that the two triangles denoted by Λ in the picture have the same area. For $t \in \,]t_1, T]$, the global energetic solution will grow continuously.

In our specific example, we get

$$s_{\mathrm{G}}(t) = \begin{cases} s_0 & \text{if } 0 \leq t < \sqrt{\frac{\kappa(1+\sqrt{2})}{1+\sqrt{2}(1+s_0)}} =: t_1 \\ 3s_0 + 1 - \frac{\kappa}{t^2} & \text{if } t_1 < t \leq T. \end{cases}$$

We continue now with the local energetic solution $s(t)$. As already discussed, according to the Definition 2.3, we expect that for any time t, $s(t)$ will belong to the epigraph of $\hat{G}$. By condition (c) $s(t)$ will remain constantly equal to s_0 until some time $t_2 \geq t_1$ for which $\frac{\kappa}{t_2^2} - \hat{G}(s_0) = 0$. The local energetic solution has then to move, and according to condition (d) it will jump to the next point at which $\frac{\kappa}{t_2^2} - \hat{G}(s(t_2+)) = 0$. From this time on, the solution will grow continuously. See picture (LES) of Figure 6.3.

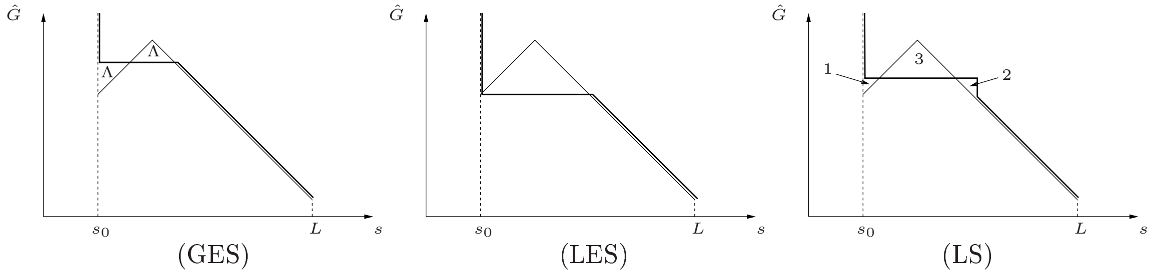

Figure 6.3: Different behavior of three notions of evolutions. Picture (GES) corresponds to the global energetic solution, picture (LES) to the local energetic solution, and picture (LS) to a possible local solution.

In this specific example, it turns out that approximable solution (AS), weak solution (WS) and BV-solution introduced in subsection 2.4 coincide with the local energetic solution. To be precise, here we get:

$$s(t) = \begin{cases} s_0 & \text{if } 0 \leq t < \sqrt{\kappa} =: t_2 > t_1 \\ 3s_0 + 1 - \frac{\kappa}{t^2} & \text{if } t_2 < t \leq T. \end{cases}$$

Finally, concerning the local solution (LS), denoted here by $s_L(t)$, from the energy inequality (2.22), we derive

$$t^2\big(\hat{\mathcal{I}}(s_L(t+)) - \hat{\mathcal{I}}(s_L(t+))\big) + \kappa(s_L(t+) - s_L(t-)) \leq 0$$

or, equivalently,

$$\int_{s_L(t-)}^{s_L(t+)} \big(- \hat{G}(\sigma) + \frac{\kappa}{t^2}\big)\,\mathrm{d}\sigma \leq 0.$$

Hence, a local solution can also jump at some time $t_3 \in\,]t_1, t_2]$ in the region above the graph of $\hat{G}$, then remaining constant up to time t_4 at which equality $\frac{\kappa}{t_4^2} - \hat{G}(s(t_4)) = 0$ holds, and hence growing continuously.

A possible local solution is represented in picture (LS) of Figure 6.3. Starting from s_0, a local solution $s_L(t)$ can jump at any time t_3 in the interval $[t_1, t_2]$. The maximal reachable position of $s_L(t_3+)$ is the one such that the sum of the areas of the triangles 1 and 2 is equal to the area of the triangle 3.

In our specific case, for any $y \in [0, \frac{\sqrt{2}}{1+\sqrt{2}}s_0]$ and any $\widetilde{s} \in [3s_0 - y, 3s_0 - y + \sqrt{2(s_0-y)^2 - y^2}]$ we obtain the following local solution

$$s(t) = \begin{cases} s_0 & \text{if } 0 \leq t \leq \sqrt{\frac{\kappa}{1+y}} =: t_3 \\ \widetilde{s} & \text{if } t_3 < t < \sqrt{\frac{\kappa}{1+3s_0-\widetilde{s}}} =: t_4 \\ 3s_0 + 1 - \frac{\kappa}{t^2} & \text{if } t_4 < t \leq T. \end{cases}$$

An example using full two dimensional elasticity and showing the different behavior of the global energetic solution (GES) and the approximable solution (AS) is already present in [ToZ06, Section 4]. However, our Example 6.3, which is constructed following a completely different approach, provides some geometrical characterization of the two solutions, and, additionally, it also gives a description of the general behavior of the local solution (LS), which was not discussed in [ToZ06, Section 4].

7 Discussion and outlook

We have shown that the rate-independent limit problem and its solutions are quite different from other solutions suggested in the literature. However, they essentially coincide with the "weak solutions" of [NeO07] and the BV-solutions in [MRS07]. This coincidence may be lost if we generalize the model.

First consider a situation where the crack tip may move backward and forward. This may model the delamination of a tape that is originally glued to a glass plate. After pulling it off it is possible to glue it again by pushing hard onto the plate again. In this case, the surface energy is not totally dissipated and part of it is stored. Hence, to model this situation we need to consider a new (reduced) energy functional obtained by adding to the previous one a nonnegative term representing the created surface energy:

$$\mathcal{I}(t,s) := \mathcal{E}(t,\mathcal{U}(t,s),s) + \int_{s_0}^{s} a(\sigma)\,\mathrm{d}\sigma$$

where $\mathcal{E}$ and $\mathcal{U}$ are defined in (2.4) and (2.5), respectively, and $a \in \mathrm{C}^0([0,L])$ is positive. The dissipation metric takes the form

$$\mathcal{R}_0(s,\dot{s}) = \begin{cases} \kappa^+(s)\dot{s} & \text{for } \dot{s} \geq 0, \\ \kappa^-(s)|\dot{s}| & \text{for } \dot{s} \leq 0, \end{cases}$$

with $\kappa^\pm \in \mathrm{C}^0([0,L])$ positive. Note that the case of nondecreasing crack tip studied in this paper corresponds to the choice $\kappa = \kappa^+ + a$ and $\kappa^- = \infty$. Then, the viscous problem $0 \in \partial_{\dot{s}}\mathcal{R}_0(s,\dot{s}) + \nu\dot{s} + \partial_s\mathcal{I}(t,s)$ can be still solved by the same incremental method developed in Section 4 and the extraction of a limit process still works. To have global existence of solutions, we make the following assumptions on κ^+ and κ^-. Let us denote $J_{\max} = \max_{(t,s)} \partial_s\mathcal{I}(t,s)$ and $J_{\min} = \min_{(t,s)} \partial_s\mathcal{I}(t,s)$. To prevent the crack tip from reaching the endpoint s_1 we assume $\kappa^+(s_1) > -J_{\min}$ which corresponds to (2.9), while to do not returning to the starting point s_0 we assume $\kappa^-(s_0) > J_{\max}$. Moreover, in order to obtain a nontrivial solution we assume $\kappa^+(s_0) < -\partial_s\mathcal{I}(t,s_0)$ for some $t \in [0,T]$ (which corresponds to (2.10)), while for allowing the crack tip to move backward we assume that there exists $(t,s) \in [0,T] \times [s_0,s_1]$ such that $\kappa^-(s) < \partial_s\mathcal{I}(t,s)$. The corresponding limit problem then reads

(a) $s \in \mathrm{BV}([0,T];[s_0,s_1])$;

(b) for all $t \in [0,T]\backslash J(s)$ we have $\partial_s\mathcal{I}(t,s(t)) \in [-\kappa^+(s(t)), \kappa^-(s(t))]$;

(c) if $\partial_s\mathcal{I}(t,s(t)) \in \,]-\kappa^+(s(t)), \kappa^-(s(t))[$, then $t \in D(s)$ and $\dot{s}(t) = 0$;

(d) for $t \in J(s)$ and s_* between $s(t-)$ and $s(t+)$ we have $\partial_s\mathcal{I}(t,s_*) \notin \,]-\kappa^+(s_*), \kappa^-(s_*)[$.

A second generalization concerns the modeling of several, noninteracting crack paths $\mathcal{C}_1, \ldots, \mathcal{C}_N$. Let $\boldsymbol{s} = (s_1, \ldots, s_N) \in \Sigma \subset \mathbb{R}^N$ denote the N-tuple containing the position of each crack tip. As above we obtain a reduced energy functional $\mathcal{I} : [0,T] \times \Sigma \to \mathbb{R}$, such that $G_j(t, \boldsymbol{s}) = -\partial_{s_j} \mathcal{I}(t, \boldsymbol{s})$ denotes the energy release rate for the j-th crack tip if all the others stay fixed. Moreover, we define the dissipation functional

$$\mathcal{R}_0(\boldsymbol{s}, \dot{\boldsymbol{s}}) = \begin{cases} \sum_{j=1}^N \kappa_j(s_j)\dot{s}_j & \text{for } \dot{\boldsymbol{s}} \in [0,\infty[^N, \\ \infty & \text{otherwise.} \end{cases}$$

Introducing the vector $\boldsymbol{G}(t, \boldsymbol{s}) = (G_1(t, \boldsymbol{s}), \ldots, G_N(t, \boldsymbol{s}))$ of all release rates, the viscous approximation takes the form

$$\mathbb{R}^N \ni 0 \in \partial_{\dot{\boldsymbol{s}}} \mathcal{R}_0(\boldsymbol{s}, \dot{\boldsymbol{s}}) + \nu \dot{\boldsymbol{s}} - \boldsymbol{G}(t, \boldsymbol{s}).$$

Again the methods in Sections 4 and 5 provide viscous solutions $\boldsymbol{s}^\nu \in \mathrm{H}^1([0,T]; \mathbb{R}^N)$ which are bounded in $\mathrm{BV}([0,T]; \mathbb{R}^N)$, independently of ν. Hence, Helly's selection principle still allows us to select a subsequence that converges pointwise to a limit function $\boldsymbol{s} \in \mathrm{BV}([0,T]; \mathbb{R}^N)$.

However, it is not so easy to see what problem the limit solutions have to satisfy. The problem is that some cracks may behave well while others jump. In particular, one should expect that a jump in one crack path changes the other release rates significantly and hence generates jumps at these cracks as well. One way of obtaining a limit problem is to use the arclength parameterization introduced in [EfM06]. We will not give the details here but just state the result if we transform back the limiting equation from there into the original time setting. For this we introduce the dissipation potential

$$\mathcal{R}^\infty(\boldsymbol{s}, \dot{\boldsymbol{s}}) = \begin{cases} \mathcal{R}_0(\boldsymbol{s}, \dot{\boldsymbol{s}}) & \text{for } |\dot{\boldsymbol{s}}|_2 \le 1, \\ \infty & \text{otherwise.} \end{cases}$$

Here the Euclidian norm $|v|_2 = (v \cdot v)^{1/2}$ corresponds to the viscous dissipation potential $\mathcal{R}_{\text{visc}}(\boldsymbol{s}, \dot{\boldsymbol{s}}) = \frac{\nu}{2} |\dot{\boldsymbol{s}}|_2^2$.

Now the limit functions $\boldsymbol{s}$ satisfy

(a) $\boldsymbol{s} \in \mathrm{BV}([0,T]; \mathbb{R}^N)$ with $\boldsymbol{s}(t) \in \Sigma$;

(b) for $t \in D(\boldsymbol{s})$ we have $0 \in \partial_{\dot{\boldsymbol{s}}} \mathcal{R}_0(\boldsymbol{s}(t), \dot{\boldsymbol{s}}(t)) - \boldsymbol{G}(t, \boldsymbol{s}(t))$;

(c) for each $t_* \in J(\boldsymbol{s})$ there exists $\sigma_* \in \mathrm{W}^{1,\infty}([0,1]; \mathbb{R}^N)$ with

   (c1) $\sigma_*(0) = \boldsymbol{s}(t_*-), \quad \sigma_*(1) = \boldsymbol{s}(t_*+), \quad$ and

   (c2) $\sigma_*'(\tau) \neq 0$ and $0 \in \partial_{\dot{\boldsymbol{s}}} \mathcal{R}^\infty\Big(\sigma_*(\tau), \dfrac{\sigma_*'(\tau)}{|\sigma_*'(\tau)|_2}\Big) - \boldsymbol{G}(t, \sigma_*(\tau))$ for a.e. $\tau \in [0,1]$.

Note that $\boldsymbol{s}$ has at most a countable number of jump points in $J(\boldsymbol{s})$. The function σ_* may be considered as connecting the point $\boldsymbol{s}(t_*-)$, where the jump starts, with the point $\boldsymbol{s}(t_*+)$ where the jump ends. Condition (c2) says that along the whole curve σ_* at least for one of the crack tips the energy release rate has to reach the corresponding fracture toughness.

Thus, this type of solution is close to the notion of BV-solutions in [MRS07], but there the viscosity norm $\mathcal{R}_{\text{visc}}(\boldsymbol{s}, \dot{\boldsymbol{s}}) = \frac{\nu}{2}\mathcal{R}_0(\boldsymbol{s}, \dot{\boldsymbol{s}})^2$ is used instead of $\frac{\nu}{2}|\dot{\boldsymbol{s}}|_2^2$. This simplifies the theory but seems less physical.

To conclude, we point out that our model is not suitable to describe crack initiation unless the crack toughness vanishes in a similar fashion at the proposed crack tip or stress concentrations make the release rate much bigger. Example 6.1 shows that the release rate $\hat{G}$ tends to 0 for the crack length s tending to 0. In that toy problem the decay is like $O(s^2)$, while for the full crack problem we expect $O(s)$ only. This behavior is justified in [DTV07].

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