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Positivity and Time Behavior of a General Linear Evolution System, Non-local in Space and Time

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Abstract

We consider a general linear reaction-diffusion system in three dimensions and time, containing diffusion (local interaction), jumps (nonlocal interaction) and memory effects. We prove a maximum principle, and positivity of the solution, and investigate its asymptotic behavior. Moreover, we give an explicite expression of the limit of the solution for large times. In order to obtain these results we use the following method: We construct a Riemannian manifold with complicated microstructure depending on a small parameter. We study the asymptotic behavior of the solution of a simple diffusion equation on this manifold as the small parameter tends to zero. It turns out that the homogenized system coincides with the original reaction-diffusion system what allows us to investigate its properties.

1 Introduction

Linear reaction-diffusion systems play an important role in applied mathematics. They describe, for instance, the transport of particles of various species in a random medium and the transformation of the particles into each other, that means linear reactions. The transport can be forced by local (diffusion) and nonlocal interaction (jumps) of the particles with the medium. Moreover, the transport and the reactions can be nonlocal in time (memory effects).

Let Ω be a bounded domain in $\mathbb{R}^3$ and $[0,\infty)$ the time interval. We consider m kinds of species with concentrations $u_k = u_k(x,t)$, for $k = 1,...,m$, $x \in \Omega$ and $t \in [0,\infty)$. In the following, we give every species a special color and call transformations of the particles into each other "changing of color".

We consider a general linear reaction-diffusion system in $\Omega \times [0,\infty)$, describing the mentioned reactions and transport effects:

$$\begin{aligned}
0 &= \frac{\partial u_k}{\partial t} - \Delta u_k + \sum_{l=1|l\neq k}^{m} A_{kl}(x)\big(u_k(x,t) - u_l(x,t)\big) \\
&+ \sum_{l=1}^{m} \int_{\Omega} B_{kl}(x,y)\big(u_k(x,t) - u_l(y,t)\big)dy \\
&+ \frac{\partial}{\partial t}\int_0^t C_k(x)e^{-D_k(x)(t-\tau)}u_k(x,\tau)d\tau \\
&+ \sum_{l=1|l\neq k}^{m} \frac{\partial}{\partial t}\int_0^t E_{kl}(x)e^{-F_{kl}(x)(t-\tau)}\big(u_k(x,\tau) + u_l(x,\tau)\big)d\tau \\
&+ \sum_{l=1}^{m} \int_{\Omega} \frac{\partial}{\partial t}\int_0^t G_{kl}(x,y)e^{-H_{kl}(x,y)(t-\tau)}\big(u_k(x,\tau) + u_l(y,\tau)\big)d\tau dy \quad (1.1)
\end{aligned}$$

$$u_k(x,0) = f_k(x), \quad (1.2)$$

$$0 = \frac{\partial u_k}{\partial \vec{n}} + \sum_{l=1}^{m} U_{kl}\frac{\partial u_l}{\partial \vec{n}}, \ x \in \partial\Omega. \quad (1.3)$$

with $k = 1,...,m$, a smooth function $f_k(x)$, nonnegative smooth functions $A_{kl}(x)$, $B_{kl}(x,y)$, ..., $H_{kl}(x,y)$ satisfying the conditions

$$\begin{array}{rclrcl}
A_{kl}(x) &=& A_{lk}(x), & B_{kl}(x,y) &=& B_{lk}(y,x), \\
E_{kl}(x) &=& E_{lk}(x), & F_{kl}(x) &=& F_{lk}(x), \\
G_{kl}(x,y) &=& G_{lk}(y,x), & H_{kl}(x,y) &=& H_{lk}(y,x), \\
A_{kl}(x) &\geq& E_{kl}(x), & B_{kl}(x,y) &\geq& G_{kl}(x,y),
\end{array} \quad (1.4)$$

and a symmetric matrix $U = \{U_{kl},\ k,l = 1,m\}$ consisting of zeros and unities with one and only one unity in every line.

From a physical point of view our system can be understood in the following way. System (1.1) describe the diffusion of particles of m colors with concentrations u_k, which can change their coordinates and colors, in the following way in each point of Ω:

- Change their color (this is described by the terms with A_{kl}).
- Jump from one point of Ω to another one (described by the B_{kk}-terms).
- Jump and change its color simultaneously (described by the B_{kl}-terms with $k \neq l$).
- Disappear and appear after some period of time in the same place without change of color (this is described by the terms with C_k and D_k).
- Disappear and appear after some period of time in the same point of Ω, but with another color (described by the terms with E_{kl} and F_{kl}).
- Disappear and appear after some period of time without change of color, but in another point of Ω (described by the terms with G_{kk} and H_{kk}).
- Disappear and appear after some period of time with another color and in another point (described by the terms with G_{kl} and H_{kl} for $k \neq l$).

The last three processes influence the terms with A_{kl} and B_{kl}, too.

When a particle with the k-th color reaches the boundary of Ω, it is reflected and if $U_{kl} = 1$ for some $l \neq k$, it changes to the l-th color.

Important for the positivity of the solution to our system is the absents of differential operators on the off-diagonal of the main part. This problem was investigated in [19] for general linear drift-diffusion systems without memory effects.

We analyze the system (1.1)-(1.3), transforming the analytical difficulties into geometric ones but for a much simpler equation. The idea of this method comes from the article [5], where the authors consider the diffusion equation on a Riemannian manifold with a complicated microstructure. After homogenization they get a system of equations, which describes nonlocal spatial and time interactions of a systems of various species. In some sense we solve the inverse to this problem and that we construct a special Riemannian manifold and homogenize the diffusion equation (Theorem 3). As a result we obtain the desired system (1.1)-(1.3) and are able to prove some of its important properties (Theorems 1 and 2).

The homogenization of the diffusion equation was studied by many authors (see, for example, the monographs [2], [8], [10], [14], [16], [17], [21] and the references there). The effect of the appearance of memory terms in the homogenized equation also was investigated in many articles, see, in particular, [3], [4], [11], [13]. Homogenization problems on manifolds with complicated microstructure were studied, except [5], in [6], [7], [9], [12], [15].

2 The ideas and main results

It is well known, that the solution $u(x,t)$ of the initial-boundary value problem for the simple diffusion equation

$$\frac{\partial u}{\partial t} - \Delta u = 0, \tag{2.1}$$

$$u(x,0) = f(x), \tag{2.2}$$

$$\frac{\partial u}{\partial \vec{n}} = 0, \ x \in \partial\Omega. \tag{2.3}$$

satisfies the following properties

I. $\max\limits_{x\in\Omega}\max\limits_{t>0} u(x,t) = \max\limits_{x\in\Omega} f(x)$ (maximum principle),

II. If the function $f(x)$ is nonnegative then $u(x,t)$ is nonnegative for all $t > 0$ (conservation of positivity).

III. $u(x,t)$ converges to the constant $M = \frac{1}{|\Omega|}\int\limits_{\Omega} f(x)dx$ as $t \to \infty$, where $|\Omega|$ is the volume of the domain Ω. M is the solution to the stationary equation.

IV. $\int\limits_{\Omega} u(x,t)dx = \int\limits_{\Omega} f(x)dx$ for $t > 0$ (conservation of mass).

These properties are valid for general linear evolution problems, conserving positivity (see, e.g., [1, 18]).

The goal of the present paper is to prove analogous statements for the problem (1.1)-(1.3) except the statement IV and the tending of the solution to the stationary solution. This properties are not fulfilled because of the memory effects.

Theorem 1. *The system (1.1)-(1.3) has a unique solution* $u(x,t) = (u_1(x,t), ..., u_m(x,t))$ *with the following properties.*

1) If $M := \max\limits_{k}\max\limits_{x\in\Omega} f_k(x) \geq 0,$ *we have*

$$u_k(x,t) \leq M \text{ for almost all } (x,t) \in \Omega \times [0,\infty), \ \forall k.$$

2) If $m := \min\limits_{k}\min\limits_{x\in\Omega} f_k(x) \leq 0,$ *we have*

$$m \leq u_k(x,t) \text{ for almost all } (x,t) \in \Omega \times [0,\infty), \ \forall k.$$

Moreover, if $C_k(x) = D_k(x) = E_{kl}(x) = F_{kl}(x) = G_{kl}(x,y) = H_{kl}(x,y) \equiv 0, \ \forall k,l,$ *then the statements 1) and 2) are true even without the conditions* $M \geq 0$ *and* $m \leq 0$.

Corollary. *Let* $f_k(x) \geq 0, \ k = 1, ..., m.$ *Then* $u_k(x,t) \geq 0$ *for almost all* $(x,t) \in \Omega \times [0,\infty), \ \forall k.$

Theorem 2. *Let the functions $A_{kl}(x)...H_{kl}(x,y)$ be strictly positive and let $u(x,t) = (u_1(x,t),...,u_m(x,t))$ be the solution to (1.1)-(1.3). Then, $\forall k$ $u_k(x,t)$ converges in $L_2(\Omega)$ as $t \to \infty$ to the constant*

$$L = \frac{\sum\limits_{k=1}^{m} \int\limits_{\Omega} f_k(x)dx}{m \cdot |\Omega| + \sum\limits_{k=1}^{m} \int\limits_{\Omega} \frac{C_k}{D_k} dx + 2 \sum\limits_{k,l=1|k\neq l}^{m} \int\limits_{\Omega} \frac{E_{kl}}{F_{kl}} dx + 2 \sum\limits_{k,l=1}^{m} \int\limits_{\Omega} \int\limits_{\Omega} \frac{G_{kl}}{H_{kl}} dx dy}.$$

In order to prove these theorems we use the following method. We construct a special Riemannian manifold $\widetilde{M}^\varepsilon$, called the main manifold, depending on a small parameter ε. On $\widetilde{M}^\varepsilon$ we consider the initial-boundary problem for the usual diffusion equation

$$\frac{\partial u^\varepsilon}{\partial t} - \Delta^\varepsilon u^\varepsilon = 0,\ (\tilde{x}, t) \in \widetilde{M}^\varepsilon \times [0,T], \tag{2.4}$$

$$u^\varepsilon(\tilde{x}, 0) = f^\varepsilon(\tilde{x}), \tag{2.5}$$

$$\frac{\partial u^\varepsilon}{\partial \vec{n}} = 0,\ \tilde{x} \in \partial \widetilde{M}^\varepsilon, \tag{2.6}$$

where f^ε is a smooth function, and Δ^ε is the Laplace-Beltrami operator. We prove that it is possible to choose such manifolds M^ε and initial function f^ε that the solution of (2.4)-(2.6) $u^\varepsilon(\tilde{x}, t)$ converges (in a certain sense) to the solution of (1.1)-(1.3) $u(x,t) = (u_1(x,t),...,u_m(x,t))$ as $\varepsilon \to 0$. It is well known, that the statements I-III are still true for the problem (2.4)-(2.6). Using the convergence of $u^\varepsilon(\tilde{x}, t)$ to $u(x,t)$, we will extend the statements I-III to the problem (1.1)-(1.3).

Remark 1. It seems to be possible to prove Theorems 1, 2 directly analyzing (1.1)-(1.3). This is done for some particular cases. Our method gives a microscopic interpretation of the terms of the system as diffusing particles in different domains and allows us to calculate the constant L explicitely.

At first, we give an idea how to choose the manifold $\widetilde{M}^\varepsilon$ (see Figures 1 and 2). Note that all objects in the following are three dimensional. Because we cannot draw them, we will use two dimensional figures and use two dimensional notations for the objects like sheets, holes, tubes and bubbles.

Instead of particles of m colors moving in the domain Ω, we consider particles with **one** color moving on m copies (sheets) of the domain Ω which are connected between each other in a special manner. On the sheets are distributed holes $D_k^{\varepsilon i}$. All holes on all sheets are connected by special manifolds consisting of tubes, or bubbles and tubes.

All kinds of interaction between the particles and the medium and between different kinds of particles can be realized by a simple diffusion on explicitely constructed manifolds. We call this manifolds A, B, CD, EF and GH-manifolds to show the underlying connection with the term in the system (1.1)-(1.3), containing the functions

A_{kl}, B_{kl}, C_k and D_k, E_{kl} and F_{kl}, G_{kl} and H_{kl} resp. Note that the EF-manifolds give a contribution to the terms with A_{kl} and the GH-manifolds to the terms with B_{kl}, too.

- Color change: diffusion through a thin tube connecting two points with the same coordinate in Ω but on different sheets (see Figure 1: A-manifold),

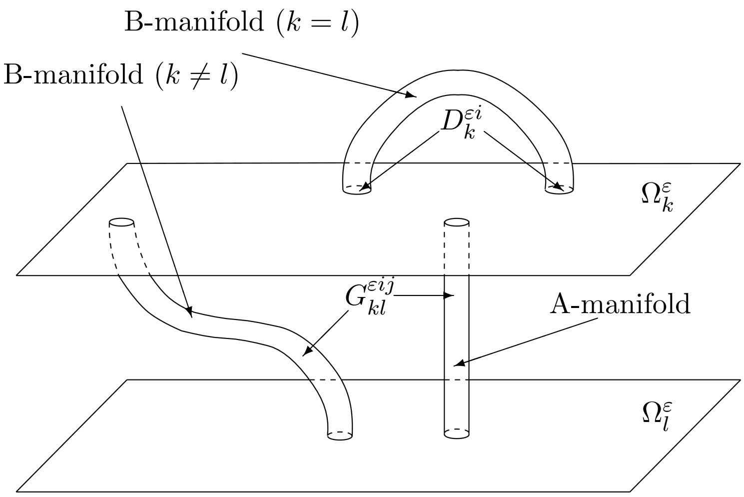

Figure 1: Manifolds without bubbles

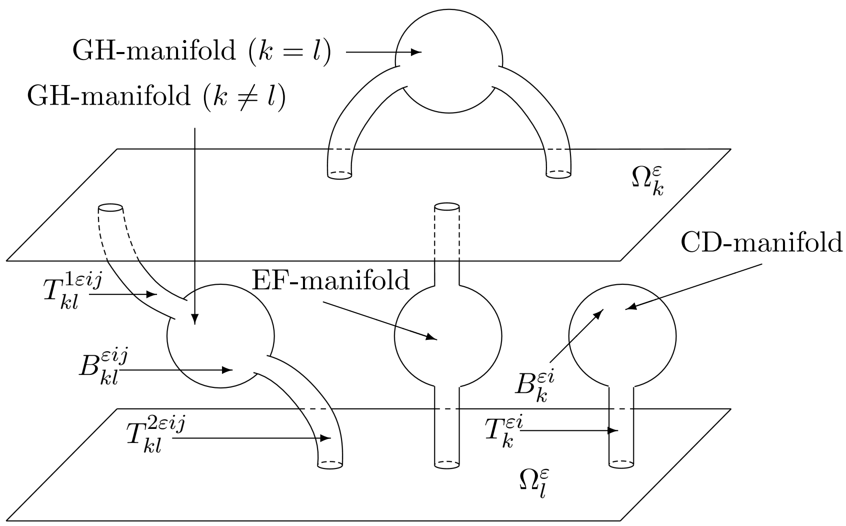

Figure 2: Manifolds with bubbles

- Jump from one point to another: diffusion through a thin tube connecting two points on the same sheet, but with different coordinates in Ω (see Figure 1: B-manifold ($k = l$)),

- Simultaneous change of color and jump from one point to another: diffusion through a thin tube connecting two points with different coordinates in Ω and on different sheets (see Figure 1: B-manifold ($k \neq l$)),
- Disappearance of a particle and appearance after some period of time: diffusion in a bubble which is joined to the sheet by a thin tube (see Figure 2: CD-manifold),
- Disappearance of a particle and appearance after some period of time with another color and/or in another place: diffusion through a manifold connecting two points with different coordinates in Ω and/or lying on the different sheets. This manifold consists of bubble and two thin tubes (see Figure 2: EF/GH-manifold).
- Behavior of particles on the boundary of the domain Ω can be realized by connecting the external boundaries of the k-th and the l-th sheets if $U_{kl} = 1, k \neq l$.

3 The construction of the main manifold

Let Ω be a bounded domain in $\mathbb{R}^3$, $\{D^{\varepsilon i} \subset \Omega, i = 1..N(\varepsilon)\}$ be a system of balls (holes) of radius d_i^ε and centers x_i^ε and

$$\Omega^\varepsilon = \Omega \backslash \bigcup_{i=1}^{N(\varepsilon)} D^{\varepsilon i}.$$

We consider m copies of the domain Ω^ε. We denote by Ω_k^ε the k-th copy and call it the k-th sheet. By $D_k^{\varepsilon i}$ we denote the copy of the i-th ball on the k-th sheet.

We associate with each hole $D_k^{\varepsilon i}$ at most one hole $D_l^{\varepsilon j}$ (possibly, $k = l, i = j$) and connect them via a manifold $G_{kl}^{\varepsilon ij}$ with boundary ${}^\alpha\Gamma_{kl}^{\varepsilon ij}$, where α counts the components of the boundary.

- If $k = l$ and $i = j$ we glue to $\partial D_k^{\varepsilon i}$ a 3-dimensional manifold $G_{kk}^{\varepsilon ii}$ with a boundary consisting of one component ${}^0\Gamma_k^{\varepsilon i}$. More exactly, we suppose that ${}^0\Gamma_k^{\varepsilon i}$ is diffeomorphic to $\partial D_k^{\varepsilon i}$, according to these diffeomorphism we glue the manifold $G_{kk}^{\varepsilon ii}$ to the sheet Ω_k^ε identifying ${}^0\Gamma_k^{\varepsilon i}$ and $\partial D_k^{\varepsilon i}$ (see Figure 2: CD-manifold).
- If $k \neq l$ or $i \neq j$ we connect $\partial D_k^{\varepsilon i}$ and $\partial D_l^{\varepsilon j}$ by a 3-dimensional manifold $G_{kl}^{\varepsilon ij}$ with boundary consisting of the components ${}^1\Gamma_{kl}^{\varepsilon ij}$ and ${}^2\Gamma_{kl}^{\varepsilon ij}$. More exactly, we suppose that ${}^1\Gamma_{kl}^{\varepsilon ij}$ is diffeomorphic to $\partial D_k^{\varepsilon i}$, ${}^2\Gamma_{kl}^{\varepsilon ij}$ is diffeomorphic to $\partial D_l^{\varepsilon j}$, according to these diffeomorphisms we glue the manifold $G_{kl}^{\varepsilon ij}$ to the sheets Ω_k^ε and Ω_l^ε identifying ${}^1\Gamma_{kl}^{\varepsilon ij}$ and $\partial D_k^{\varepsilon i}$, ${}^2\Gamma_{kl}^{\varepsilon ij}$ and $\partial D_l^{\varepsilon j}$ (see Figure 1,2: A, B, EF, GH -manifolds).

As a result we obtain a differentiable manifold M^ε

$$M^\varepsilon = \left(\bigcup_k \overline{\Omega_k^\varepsilon} \right) \cup \left(\bigcup_{k,l,i,j} G_{kl}^{\varepsilon ij} \right)$$

Let $U = \{U_{kl},\ k,l = \overline{1,m}\}$ be the symmetric matrix described in the previous section. If $U_{kl} = 1$ we identify the boundaries of the k-th and the l-th sheets. We denote the obtained manifold by $\widetilde{M^\varepsilon}$.

The boundary of $\widetilde{M^\varepsilon}$ consists of $\bigcup\limits_{k:U_{kk}=1} \partial\Omega_k$ and the boundaries of holes $D_k^{\varepsilon i}$ which do not have an associated hole $D_l^{\varepsilon j}$.

We denote the points of $\widetilde{M^\varepsilon}$ by $\tilde{x}$. If $\tilde{x} \in \Omega_k^\varepsilon$, then we assign the pair (x,k) to $\tilde{x}$, where x is the corresponding point in Ω.

We supposed that $\widetilde{M^\varepsilon}$ is equipped by the Riemannian metric $g_{\alpha\beta}^\varepsilon(\tilde{x})$, which coincides with the Euclidian metric on the domains $\bigcup\limits_k \Omega_k^\varepsilon$.

Now, we specify the size of the holes $D^{\varepsilon i}$ and the form of the manifold $G_{kl}^{\varepsilon ij}$. We consider two associated to each other holes $D_k^{\varepsilon i}$ and $D_l^{\varepsilon j}$. We set

$$d_i^\varepsilon = \begin{cases} a\varepsilon^3, & i = j, \\ a\varepsilon^6, & i \neq j, \end{cases} \quad a > 0. \tag{3.1}$$

Moreover, we suppose that

$$\exists c_1, c_2 > 0,\ \forall i:\ c_1 \cdot r_i^{\varepsilon 3} < d_i^\varepsilon < c_2 \cdot r_i^{\varepsilon 3},\ \varepsilon < \varepsilon_0, \tag{3.2}$$

where $r_i^\varepsilon = \min\limits_j \left(\operatorname{dist}(x_i^\varepsilon, x_j^\varepsilon)\right)$.

We introduce the set of smooth positive functions

$$q_{kl}^A(x),\ q_{kl}^B(x,y),\ q_k^C(x),\ b_k^D(x),\ q_{kl}^E(x),\ b_{kl}^F(x),\ q_{kl}^G(x,y),\ b_{kl}^H(x,y)$$

such that

$$\begin{gathered} q_{kl}^A(x) = q_{lk}^A(x),\ q_{kl}^B(x,y) = q_{lk}^B(y,x),\ q_{kl}^E(x) = q_{lk}^E(x),\ b_{kl}^F(x) = b_{lk}^F(x), \\ q_{kl}^G(x,y) = q_{lk}^G(y,x),\ b_{kl}^H(x,y) = b_{lk}^H(y,x). \end{gathered} \tag{3.3}$$

They will describe the metric on the manifolds $G_{kl}^{\varepsilon ij}$ and the coefficients of (1.1) will depend on these functions.

We describe the form of the manifolds $G_{kl}^{\varepsilon ij}$ and the metric on them.

- If $k = l, i = j$ then

$$G_{kl}^{\varepsilon ij} = B_k^{\varepsilon i} \cup T_k^{\varepsilon i} \quad \text{(CD-manifold)}, \tag{3.4}$$

where

$$B_k^{\varepsilon i} = \big\{(\varphi,\psi,\theta) : \varphi \in [0,2\pi], \psi \in [0,\pi], \theta \in [\theta_k^{\varepsilon i},\pi]\big\} \text{ (bubble)}, \tag{3.5}$$

$$T_k^{\varepsilon i} = \big\{(\varphi,\psi,z) : \varphi \in [0,2\pi], \psi \in [0,\pi], z \in [0,1]\big\} \text{ (tube)}, \tag{3.6}$$

so that ${}^0\Gamma_k^i = \big\{\tilde{x} \in T_k^{\varepsilon i} | z = 0\big\}$, $B_k^{\varepsilon i}$ and $T_k^{\varepsilon i}$ are connected in the points $\theta = \theta_k^{\varepsilon i}$ and $z = 1$, correspondingly. The metric is defined by the formula for the element of length

$$ds^2 = \begin{cases} (q_k^{\varepsilon i})^2 dz^2 + (d_i^\varepsilon)^2(\sin^2\psi d\varphi^2 + d\psi^2), \ \tilde{x} \in T_k^{\varepsilon i}, \\ (b_k^{\varepsilon i})^2(\sin^2\theta \sin^2\psi d\varphi^2 + \sin^2\theta d\psi^2 + d\theta^2), \ \tilde{x} \in B_k^{\varepsilon i}, \end{cases}$$

where $b_k^{\varepsilon i} = b_k^D(x_i^\varepsilon) \cdot \sqrt[3]{d_i^\varepsilon}$, $q_k^{\varepsilon i} = q_k^C(x_i^\varepsilon) \cdot d_i^\varepsilon$, $\sin\theta_k^{\varepsilon i} = \dfrac{d_i^\varepsilon}{b_k^{\varepsilon i}}$.[1]

- If $k \neq l$ or $i \neq j$ then two situations are possible

1.

$$G_{kl}^{\varepsilon ij} = \big\{(\varphi,\psi,z) : \varphi \in [0,2\pi], \psi \in [0,\pi], z \in [0,1]\big\}, \tag{3.7}$$

so that ${}^1\Gamma_{kl}^{ij} = \big\{\tilde{x} \in G_{kl}^{\varepsilon ij} | z = 0\big\}$, ${}^2\Gamma_{kl}^{ij} = \big\{\tilde{x} \in G_{kl}^{\varepsilon ij} | z = 1\big\}$. The metric is defined by the formula

$$ds^2 = (q_{kl}^{\varepsilon ij})^2 dz^2 + (d_i^\varepsilon)^2(\sin^2\psi d\varphi^2 + d\psi^2),$$

where $q_{kl}^{\varepsilon ij} = \begin{cases} q_{kl}^A(x_i^\varepsilon) \cdot d_i^\varepsilon, & i = j \quad \text{(A-manifold)}, \\ q_{kl}^B(x_i^\varepsilon, x_j^\varepsilon) \cdot d_i^\varepsilon, & i \neq j \quad \text{(B-manifold)}; \end{cases}$

2.

$$G_{kl}^{\varepsilon ij} = T_{kl}^{1\varepsilon ij} \cup B_{kl}^{\varepsilon ij} \cup T_{kl}^{2\varepsilon ij}, \tag{3.8}$$

where

$$B_{kl}^{\varepsilon ij} = \big\{(\varphi,\psi,\theta) : \varphi \in [0,2\pi], \psi \in [0,\pi], \theta \in [\theta_{kl}^{\varepsilon ij}, \pi - \theta_{kl}^{\varepsilon ij}]\big\},$$

$$T_{kl}^{1\varepsilon ij} = T_{kl}^{2\varepsilon ij} = \big\{(\varphi,\psi,z) : \varphi \in [0,2\pi], \psi \in [0,\pi], z \in [0,1]\big\},$$

so that ${}^1\Gamma_{kl}^{ij} = \big\{\tilde{x} \in T_{kl}^{1\varepsilon ij} | z = 0\big\}$, ${}^2\Gamma_{kl}^{ij} = \big\{\tilde{x} \in T_{kl}^{2\varepsilon ij} | z = 1\big\}$, $B_{kl}^{\varepsilon ij}$ and $T_{kl}^{1\varepsilon ij}$ are joined in the points with $\theta = \theta_{kl}^{\varepsilon ij}$ and $z = 1$, correspondingly, $B_{kl}^{\varepsilon ij}$ and $T_{kl}^{2\varepsilon ij}$ are joined in the points with $\theta = \pi - \theta_{kl}^{\varepsilon ij}$ and $z = 0$, correspondingly.

The metric is defined by the formula

$$ds^2 = \begin{cases} (q_{kl}^{\varepsilon ij})^2 dz^2 + (d_i^\varepsilon)^2(\sin^2\psi d\varphi^2 + d\psi^2), \ \tilde{x} \in T_{kl}^{1\varepsilon ij} \cup T_{kl}^{2\varepsilon ij}, \\ (b_{kl}^{\varepsilon ij})^2(\sin^2\theta \sin^2\psi d\varphi^2 + \sin^2\theta d\psi^2 + d\theta^2), \ \tilde{x} \in B_{kl}^{\varepsilon ij}, \end{cases}$$

[1] The metric on $B_k^{\varepsilon i}$ is the usual metric on the sphere $S^3 \subset \mathbb{R}^4$ with radius $b_k^{\varepsilon i}$. The metric on $T_k^{\varepsilon i}$ is the usual metric on the cylinder $S^2 \times [0,1]$ with radius d_i^ε and length $q_k^{\varepsilon i}$.

where $b^{\varepsilon ij}_{kl} = \begin{cases} b^F_{kl}(x^\varepsilon_i)\cdot\sqrt[3]{d^\varepsilon_i},\ i=j,\\ b^H_{kl}(x^\varepsilon_i,x^\varepsilon_j)\cdot\sqrt[3]{d^\varepsilon_i},\ i\neq j,\end{cases}$

$q^{\varepsilon ij}_{kl} = \begin{cases} q^E_{kl}(x^\varepsilon_i)\cdot d^\varepsilon_i, & i=j \quad \text{(EF-manifold)},\\ q^G_{kl}(x^\varepsilon_i,x^\varepsilon_j)\cdot d^\varepsilon_i, & i\neq j \quad \text{(GH-manifold)},\end{cases}$ $\sin\theta^{\varepsilon ij}_{kl} = \dfrac{d^\varepsilon_i}{b^{\varepsilon ij}_{kl}}.$

4 Homogenization of the diffusion equation

We consider the following Cauchy problem on $\widetilde{M}^\varepsilon$

$$\frac{\partial u^\varepsilon}{\partial t} - \Delta^\varepsilon u^\varepsilon = 0,\ (\tilde{x},t)\in\widetilde{M}^\varepsilon\times[0,T], \tag{4.1}$$

$$u^\varepsilon(\tilde{x},0) = f^\varepsilon(\tilde{x}), \tag{4.2}$$

$$\frac{\partial u^\varepsilon}{\partial\vec{n}} = 0,\ \tilde{x}\in\partial\widetilde{M}^\varepsilon, \tag{4.3}$$

where Δ^ε is the Laplace-Beltrami operator which has the following form in local coordinates

$$\Delta^\varepsilon = \frac{1}{\sqrt{G^\varepsilon}}\sum_{\alpha,\beta=1}^{3}\frac{\partial}{\partial x_\alpha}\left(\sqrt{G^\varepsilon}g^{\alpha\beta}_\varepsilon\frac{\partial}{\partial x_\beta}\right)$$

where $G^\varepsilon = \det g^\varepsilon_{\alpha\beta}$, $g^{\alpha\beta}_\varepsilon$ are the components of the tensor inverse to $g^\varepsilon_{\alpha\beta}$. f^ε is a smooth function, which coincides with $f_k(x)$ on the sheets and is equal to zero on $G^{\varepsilon ij}_{kl}$ (outside some small neighborhood of $\partial G^{\varepsilon ij}_{kl}$). More exactly

$$f^\varepsilon(\tilde{x}) = \begin{cases} f_k(x), \tilde{x}=(x,k)\in\Omega^\varepsilon_k,\\ 0, \tilde{x}\in G^{\varepsilon ij}_{kl}\backslash U^{\varepsilon ij}_{kl}(\delta),\end{cases}$$

where

$$U^{\varepsilon ij}_{kl}(\delta) = \begin{cases} \left\{\tilde{x}\in G^{\varepsilon ij}_{kl} | \tilde{x}=(\varphi,\psi,z)\in T^{\varepsilon i}_k : |z|<\delta\right\},\ \text{if } G^{\varepsilon ij}_{kl} \text{ is of type (3.4)},\\ \left\{\tilde{x}\in G^{\varepsilon ij}_{kl} | \tilde{x}=(\varphi,\psi,z)\in G^{\varepsilon ij}_{kl} : |z|<\delta \vee |1-z|<\delta\right\},\\ \quad \text{if } G^{\varepsilon ij}_{kl} \text{ is of type (3.7)},\\ \left\{\tilde{x}\in G^{\varepsilon ij}_{kl} | \tilde{x}=(\varphi,\psi,z)\in T^{1\varepsilon ij}_{kl} : |z|<\delta \vee \right.\\ \quad \left. \vee\ \tilde{x}=(\varphi,\psi,z)\in T^{2\varepsilon ij}_{kl} : |1-z|<\delta\right\},\ \text{if } G^{\varepsilon ij}_{kl} \text{ is of type (3.8)}\end{cases}$$

is a δ-neighborhood of $G^{\varepsilon ij}_{kl}$. We also require $f^\varepsilon(\tilde{x}) \leq \max\limits_{k=1\dots m}\max\limits_{x\in\Omega} f_k(x), \tilde{x}\in \bigcup\limits_{k,l,i,j} U^{\varepsilon ij}_{kl}(\delta)$. We set $\delta = \bar{o}(d^\varepsilon_i)$.

Let $L_2(\widetilde{M}^\varepsilon)$ be the Hilbert space of real-valued functions with the norm

$$\|u^\varepsilon\|_{0\varepsilon} = \left\{\int\limits_{\widetilde{M}^\varepsilon}(u^\varepsilon)^2 d\tilde{x}\right\}^{1/2},$$

where $d\tilde{x} = \sqrt{G^\varepsilon} dx_1 dx_2 dx_3$ is a volume element on $\widetilde{M}^\varepsilon$; let $H^1(\widetilde{M}^\varepsilon)$ be the Hilbert space of real-valued functions with the norm

$$\|u^\varepsilon\|_{H^1(M^\varepsilon)} = \|u^\varepsilon\|_{0\varepsilon} + \|\nabla^\varepsilon u^\varepsilon\|_{0\varepsilon};$$

let $L_2(\Omega)^m$ be the Hilbert space of the real-valued m vector-functions with the norm

$$\|u\|_0 = \left\{ \int_\Omega \sum_{k=1}^m (u_k)^2 dx \right\}^{1/2}.$$

It is well known that the system (4.1)-(4.3) has a unique generalized solution in $L_2(0,T;H^1(M^\varepsilon))$ (see, e.g., [20]).

We say, that $f^\varepsilon \in L_2(\widetilde{M}^\varepsilon)$ converges to the function $f \in L_2(\Omega)^m$ if

$$\lim_{\varepsilon \to 0} \|Q^\varepsilon f^\varepsilon - f\|_{L_2(\Omega)^m} = 0, \tag{4.4}$$

where the operator $Q^\varepsilon : L_2(\widetilde{M}^\varepsilon) \to L_2(\Omega)^m$ is defined by the equality

$$\left(Q^\varepsilon u^\varepsilon\right)_k(x) = \begin{cases} u^\varepsilon(\tilde{x}), \tilde{x} = (x,k), \ \text{if } x \in \Omega^\varepsilon, \\ 0, \ \text{if } x \in \bigcup_i D^{\varepsilon i}. \end{cases}$$

Similarly, we say that $u^\varepsilon \in L_2(\widetilde{M}^\varepsilon \times [0,T])$ converges to the function $u \in L_2(\Omega \times [0,T])^m$ if

$$\lim_{\varepsilon \to 0} \int_0^T \|Q^\varepsilon u^\varepsilon(\cdot,t) - u(\cdot,t)\|^2_{L_2(\Omega)^m} dt = 0, \tag{4.5}$$

Theorem 3. *Let $A_{kl}(x), B_{kl}(x,y)...H_{kl}(x,y)$ be an arbitrary set of smooth non-negative functions which satisfy the conditions (1.4). Then there exist a number $a > 0$, a distribution of the points x_i^ε, which satisfy (3.2) and a set of functions $q_{kl}^A(x), ..., b_{kl}^H(x,y)$, which satisfies the condition (3.3), such that the solution $u^\varepsilon(\tilde{x},t)$ of (4.1)-(4.3) converges in the sense (4.5) to the solution $u(x,t)$ of the initial-boundary value problem (1.1)-(1.3). Moreover, the following equalities are valid*

$$\begin{aligned}
A_{kl}(x) &= E_{kl}(x) + \frac{4a\pi}{q_{kl}^A(x)+2}, & B_{kl}(x) &= G_{kl} + \frac{4a\pi}{q_{kl}^B(x,y)+2}, \\
C_k(x) &= \frac{4a\pi}{q_k^C(x)+2}, & D_k(x) &= \frac{2}{\pi(b_k^D(x))^3(q_k^C(x)+2)}, \\
E_{kl}(x) &= \frac{2a\pi}{q_{kl}^E(x)+2}, & F_{kl}(x) &= \frac{4}{\pi(b_{kl}^F(x))^3(q_{kl}^E(x)+2)}, \\
G_{kl}(x,y) &= \frac{2a\pi}{q_{kl}^G(x,y)+2}, & F_{kl}(x,y) &= \frac{4}{\pi(b_{kl}^H(x,y))^3(q_{kl}^G(x,y)+2)}.
\end{aligned} \tag{4.6}$$

5 Proof of Theorem 3

The asymptotic behavior of the solution of the diffusion equation on Riemannian manifolds with the same form as in Section 3 (without additional assumptions about the form of the manifolds $G_{kl}^{\varepsilon ij}$) was investigated in [5]. We will use the results obtained there.

We denote:

$$\begin{aligned} R_k^{\varepsilon i} &= \{\tilde{x} = (x,k) \in \Omega_k^\varepsilon : d_i^\varepsilon \le |x - x_i^\varepsilon| \le r_i^\varepsilon/2\}, \\ S_k^{\varepsilon i} &= \{\tilde{x} = (x,k) \in \Omega_k^\varepsilon : |x - x_i^\varepsilon| = r_i^\varepsilon/2\}, \\ \widetilde{G_{kl}^{\varepsilon ij}} &= \begin{cases} R_k^{\varepsilon i} \cup G_{kk}^{\varepsilon ii}, \ k = l \wedge i = j, \\ R_k^{\varepsilon i} \cup G_{kl}^{\varepsilon ij} \cup R_l^{\varepsilon j}, \ k \ne l \vee i \ne j, \end{cases} \end{aligned}$$

We consider the following boundary value problem in the domain $\widetilde{G_{kl}^{\varepsilon ij}}$:

$$-\Delta^\varepsilon v + \lambda \chi_{kl}^{\varepsilon ij} v = 0, \quad \tilde{x} \in \widetilde{G_{kl}^{\varepsilon ij}}, \ \lambda > 0, \tag{5.1}$$

$$v = 1, \tilde{x} \in S_k^{\varepsilon i}, \tag{5.2}$$

$$v = 0, \tilde{x} \in S_l^{\varepsilon j} \ (\text{if } k \ne l \vee i \ne j). \tag{5.3}$$

where $\chi_{kl}^{\varepsilon ij}$ is the characteristic function of $G_{kl}^{\varepsilon ij}$.

Let $v_{kl}^{\varepsilon ij}$ be the solution of the problem (5.1)-(5.3). We set

$$\begin{aligned} V_{kl}^{\varepsilon ij} &= \int\limits_{\widetilde{G_{kl}^{\varepsilon ij}}} \sum_{a\,\beta=1}^{3} \left\{ g_\varepsilon^{\alpha\beta} \frac{\partial v_{kl}^{\varepsilon ij}}{\partial x_\alpha} \frac{\partial v_{kl}^{\varepsilon ij}}{\partial x_\beta} + \lambda \chi_{kl}^{\varepsilon ij} \left(v_{kl}^{\varepsilon ij}\right)^2 \right\} d\tilde{x}, \\ W_{kl}^{\varepsilon ij} &= \int\limits_{\widetilde{G_{kl}^{\varepsilon ij}}} \sum_{a\,\beta=1}^{3} \left\{ g_\varepsilon^{\alpha\beta} \frac{\partial v_{kl}^{\varepsilon ij}}{\partial x_\alpha} \frac{\partial v_{lk}^{\varepsilon ji}}{\partial x_\beta} + \lambda \chi_{kl}^{\varepsilon ij} v_{kl}^{\varepsilon ij} v_{lk}^{\varepsilon ji} \right\} d\tilde{x}, \end{aligned} \tag{5.4}$$

and introduce the following $m \times m$ matrix-valued generalized functions

$$\begin{aligned} V^\varepsilon(x,\lambda) &= \left\{ \sum_{i=1}^{N(\varepsilon)} W_{kl}^{\varepsilon ii} \delta(x - x_i^\varepsilon); k,l = 1,..,m, \ k \ne l \right\} \\ &+ \operatorname{diag} \left\{ \sum_{i,j=1}^{N(\varepsilon)} V_{kl}^{\varepsilon ij} \delta(x - x_i^\varepsilon); k = 1,..,m \right\}, \\ W^\varepsilon(x,\lambda) &= \left\{ \sum_{i,j=1, i\ne j}^{N(\varepsilon)} W_{kl}^{\varepsilon ij} \delta(x - x_i^\varepsilon)\delta(y - x_j^\varepsilon); k,l = 1,..,m \right\}. \end{aligned} \tag{5.5}$$

We suppose that $\forall \lambda > 0$ the following limits exist (in $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$, respectively)

$$\lim_{\varepsilon \to 0} V^\varepsilon(x,\lambda) = V(x,\lambda), \quad \lim_{\varepsilon \to 0} W^\varepsilon(x,y,\lambda) = W(x,y,\lambda), \tag{5.6}$$

where $V(x,\lambda), W(x,y,\lambda)$ are continuous matrix-valued functions in Ω and $\Omega \times \Omega$, respectively.

It is possible to show, that $V(x, \lambda)$ and $W(x, \lambda)$ have analytic continuation with respect to the parameter λ to the domain $\mathbb{C}\backslash\{\arg\lambda = \pi\}$, where the matrix-valued functions $\lambda^{-1}V(x, \lambda)$ and $\lambda^{-1}W(x, \lambda)$ are the Laplace transforms of the matrix-valued functions $\mathrm{V}(x, t)$ and $\mathrm{W}(x, t)$:

$$\begin{aligned} \mathrm{V}(x,t) &= \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{V(x,\lambda)}{\lambda}e^{\lambda t}d\lambda, \\ \mathrm{W}(x,t) &= \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{W(x,y,\lambda)}{\lambda}e^{\lambda t}d\lambda,\ \sigma > 0. \end{aligned}$$

Now, we formulate the main result of [5].

Theorem 4. *Let*

(i) the condition (3.2) be fulfilled,

(ii) the limits (5.6) exist,

(iii) the function $f^\varepsilon(\tilde{x})$ *converges in the sense (4.4) to the vector-function* $f(x) = (f_1(x), .., f_m(x))$,

(iv) $\lim\limits_{\varepsilon\to 0}\sum\limits_{k,l=1}^{m}\sum\limits_{i,j=1}^{N(\varepsilon)}\int\limits_{G_{kl}^{\varepsilon ij}}(f^\varepsilon(\tilde{x}))^2 d\tilde{x} = 0,$

Then the solution $u^\varepsilon(\tilde{x}, t)$ *of the problem (4.1)-(4.3) converges in the sense (4.5) to the solution of the following problem*

$$\frac{\partial u_k}{\partial t} - \Delta u_k + \sum_{l=1}^{m}\frac{\partial}{\partial t}\int\limits_0^t \mathrm{V}_{kl}(x, t-\tau)u_l(x,\tau)d\tau +$$

$$+\sum_{l=1}^{m}\int\limits_\Omega\frac{\partial}{\partial t}\int\limits_0^t \mathrm{W}_{kl}(x,y,t-\tau)u_l(y,\tau)d\tau dy = 0, k = 1...m, \tag{5.7}$$

$$u_k(x,0) = f_k(x), \tag{5.8}$$

$$\frac{\partial u_k}{\partial \vec{n}} + \sum_{l=1}^{m}U_{kl}\frac{\partial u_l}{\partial \vec{n}} = 0,\ x \in \partial\Omega. \tag{5.9}$$

In the case, when $G_{kl}^{\varepsilon ij}$ have a structure like (3.4), (3.7) or (3.8), then (with a suitable distribution of the points x_i^ε) the limits (5.6) exist and it is possible to find the functions $\mathrm{V}(x, t)$ and $\mathrm{W}(x, t)$ explicitely.

At first we consider some typical cases of the manifold $\widetilde{M}^\varepsilon$ with different types of $G_{kl}^{\varepsilon ij}$. We restrict ourself to the case of a manifold $\widetilde{M}^\varepsilon$ which consist of $m = 2$ sheets. For the case $m > 2$ the Theorem is proved in a similar way.

Since we use the same method as in [5], we do not replay all details in every case. We explain in more detail the Case 3.

Case 1. We divide the domain Ω in cubes $K^{\varepsilon i}$ in such a way that they form a periodic cubic lattice with side length ε. The number i counts the cubes, x_i^ε are the centers of the cubes. Further, in the center of each cube $K^{\varepsilon i}$, fully lying in Ω, we cut out a ball $D^{\varepsilon i}$ with radius $d_i^\varepsilon = a\varepsilon^3$ and center x_i^ε (Figure 3). As before, $\Omega^\varepsilon = \Omega \setminus \bigcup_{i=1}^{N(\varepsilon)} D^{\varepsilon i}$, Ω_1^ε and Ω_2^ε are the two copies of the domain Ω, $D_k^{\varepsilon i}$ is the copy of the i-th ball on the k-th sheet, $k = 1, 2$. We connect the boundaries of $D_1^{\varepsilon i}$ and $D_2^{\varepsilon i}$ by the manifold $G_{12}^{\varepsilon ij}$ which has the form (3.7). Finally, we obtain the manifold $\widetilde{M^\varepsilon} = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \left(\bigcup_{i=1}^{N(\varepsilon)} G_{12}^{ii}\right)$. It is easy to see that the conditions (i),(iii),(iv) of Theorem 4 hold.

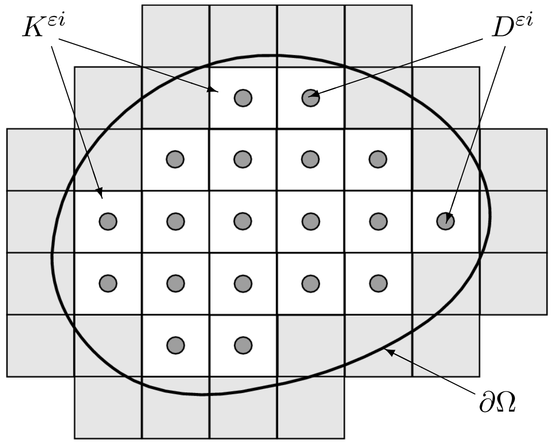

Figure 3

We obtain

$$V_{12}^{\varepsilon ii} = V_{21}^{\varepsilon ii} = -W_{12}^{\varepsilon ii} = -W_{21}^{\varepsilon ii} = \frac{4a\pi\varepsilon^3}{q_{12}^A(x_i^\varepsilon) + 2}(1 + \bar{o}(1))$$

Let $\varphi(x) \in C^\infty(\Omega)$, then

$$\left\langle \sum_i V_{12}^{\varepsilon ii}\delta(x - x_i^\varepsilon); \varphi(x) \right\rangle = \sum_i V_{12}^{\varepsilon ii}\varphi(x_i^\varepsilon)(1 + \bar{o}(1)) =$$
$$= \sum_i \frac{4a\pi\varphi(x_i^\varepsilon)}{q_{12}^A(x_i^\varepsilon) + 2}|K^{\varepsilon i}|(1 + \bar{o}(1)) \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \frac{4a\pi\varphi(x)dx}{q_{12}^A(x) + 2},$$

i.e.

$$V_{12}(x,\lambda) = V_{21}(x,\lambda) = -W_{12}(x,\lambda) = -W_{21}(x,\lambda) = \frac{4a\pi}{q_{12}^A(x) + 2} \Longrightarrow$$
$$\Longrightarrow \mathrm{V}_{12}(x,t) = \mathrm{V}_{21}(x,t) = -\mathrm{W}_{12}(x,t) = -\mathrm{W}_{21}(x,t) = \frac{4a\pi}{q_{12}^A(x) + 2}.$$

Thus, the homogenized system has the form

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \Delta u_1 + \dfrac{4a\pi}{q_{12}^A(x) + 2}\big(u_1(x,t) - u_2(x,t)\big) = 0, \\ \dfrac{\partial u_2}{\partial t} - \Delta u_2 + \dfrac{4a\pi}{q_{12}^A(x) + 2}\big(u_2(x,t) - u_1(x,t)\big) = 0. \end{cases}$$

This is a two species diffusion-reaction system.

Case 2. We divide the domain Ω in cubes $K^{\varepsilon i}$ in such a way that they form a periodic cubic lattice with side length ε. Let $n(\varepsilon)$ be the number of cubes which fully lie in Ω. In each of such cubes we cut out $n(\varepsilon)$ balls $D^{\varepsilon i}$ with radiuses $d_i^\varepsilon = a\varepsilon^6$ and centers x_i^ε. Thus, the total number of holes is equal to $n^2(\varepsilon)$. Moreover, we distribute the balls in such a way that condition (3.2) holds (e.g., choosing the centers of the balls in the knots of the periodic lattice with period $\sim c\varepsilon^2$).

Again $\Omega^\varepsilon = \Omega \backslash \bigcup_{i=1}^{N(\varepsilon)} D^{\varepsilon i}$, Ω_1^ε and Ω_2^ε are the two copies of Ω, $D_k^{\varepsilon i}$ is the copy of the ball $D^{\varepsilon i}$ on the k-th sheet, $k = 1, 2$.

Within each cube $K^{\varepsilon\alpha}$ we renumber the out-cut balls from 1 to $n(\varepsilon)$. For each ball $D^{\varepsilon i}$ we denote by $\alpha(i)$ the number of the cube containing the ball and by $\beta(i)$ the number of the balls inside this cube. If $\alpha(i) = \beta(j), a(j) = \beta(i)$ (and only in this case) we join the boundaries of the balls $D_1^{\varepsilon i}$ and $D_2^{\varepsilon j}$ by manifolds $G_{12}^{\varepsilon ij}$ having form (3.7). Figure 4 shows an example of two copies of Ω where some holes are connected by tubes.

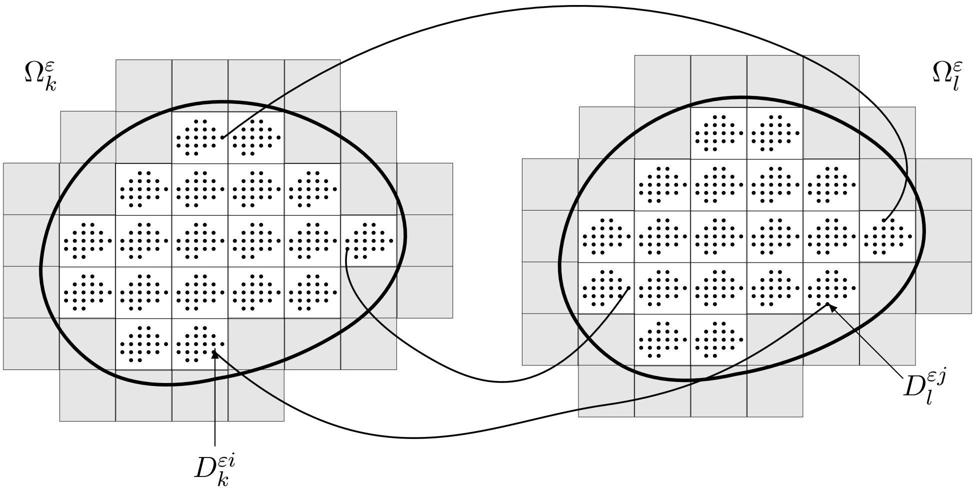

Figure 4

We obtain the manifold $\widetilde{M}^\varepsilon = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \left(\bigcup_{i,j} G_{12}^{\varepsilon ij} \right)$. Conditions (i),(iii),(iv) of Theorem 4 hold.

In this case we have

$$V_{12}^{\varepsilon ij} = V_{21}^{\varepsilon ji} = -W_{12}^{\varepsilon ij} = -W_{21}^{\varepsilon ji} = \frac{4a\pi\varepsilon^6}{q_{12}^B(x_i^\varepsilon, x_j^\varepsilon) + 2}(1 + \bar{o}(1)).$$

Let $\varphi(x,y) \in C^\infty(\Omega \times \Omega)$, then

$$\left\langle \sum_{i,j} W_{12}^{\varepsilon ij} \delta(x - x_i^\varepsilon)\delta(x - x_j^\varepsilon); \varphi(x) \right\rangle = \sum_{i,j} W_{12}^{\varepsilon ij} \varphi(x_i^\varepsilon, x_j^\varepsilon)(1 + \bar{o}(1)) =$$

$$= -\sum_{i,j} \frac{4a\pi\varphi(x_i^\varepsilon, x_j^\varepsilon)}{q_{12}^B(x_i^\varepsilon, x_j^\varepsilon) + 2} |K^{\varepsilon\alpha(i)}| \cdot |K^{\varepsilon\alpha(j)}|(1 + \bar{o}(1)), \tag{5.10}$$

where the sum contains only the terms with pairs (i,j), which are connected by the tube $G_{kl}^{\varepsilon ij}$. Since $\forall K^{\varepsilon\alpha_1}, K^{\varepsilon\alpha_2}$ there exists a joining pair of the holes $D_1^{\varepsilon i}$ and $D_2^{\varepsilon j}$ such, that $D_1^{\varepsilon i} \subset K^{\varepsilon\alpha_1}$, $D_2^{\varepsilon j} \subset K^{\varepsilon\alpha_2}$ (by the construction of the $\widetilde{M}^\varepsilon$), the sum (5.10) is an integral sum for the function $\dfrac{4\pi\varphi(x,y)}{q_{12}^B(x,y) + 2}$, i.e.,

$$W_{12}(x,y,\lambda) = -\frac{4a\pi}{q_{12}^B(x,y) + 2},\ W_{21}(x,y,\lambda) = -\frac{4a\pi}{q_{21}^B(x,y) + 2}$$

$$V_{12}(x,\lambda) = -\int_\Omega W_{12}(x,y)dy,\ V_{21}(x,\lambda) = -\int_\Omega W_{21}(x,y)dy.$$

Finally, the homogenized system has the form

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \Delta u_1 + \displaystyle\int_\Omega \frac{4a\pi}{q_{12}^B(x,y) + 2}\big(u_1(x,t) - u_2(y,t)\big)dy = 0, \\ \dfrac{\partial u_2}{\partial t} - \Delta u_2 + \displaystyle\int_\Omega \frac{4a\pi}{q_{21}^B(x,y) + 2}\big(u_2(x,t) - u_1(y,t)\big)dy = 0. \end{cases}$$

This is a two species diffusion-reaction system with nonlocal spatial interaction.

Case 3. Let Ω^ε be the domain constructed in Case 1. To the boundary of the i-th hole we glue a manifold G_{11}^{ii} of the form (3.4). So we obtain the manifold $\widetilde{M}^\varepsilon = \Omega^\varepsilon \cup \left(\bigcup_i G_{11}^{ii} \right)$.

Now, we calculate the function $V_{11}(x,\lambda)$. Let $v^\varepsilon \equiv v_{11}^{\varepsilon ii}(\tilde{x},\lambda)$ be the solution to (5.1)-(5.3). We represent v^ε in the form $v^\varepsilon = \widehat{v}^\varepsilon + w^\varepsilon$, where

$$\widehat{v}^\varepsilon = \begin{cases} 1 - A_2^{\varepsilon i} \dfrac{d^\varepsilon}{|x - x_i^\varepsilon|} \Phi_1(|x - x_i^\varepsilon|\varepsilon^{-1}), & \tilde{x} = (x,1) \in \Omega^\varepsilon, \\ A^{\varepsilon i} z + B^{\varepsilon i}, & \tilde{x} = (\varphi,\psi,z) \in T_1^{\varepsilon i}, \\ A_0^{\varepsilon i} + A_1^{\varepsilon i} \dfrac{\cot\theta}{\cot\theta^\varepsilon} \Phi_2(\theta), & \tilde{x} = (\varphi,\psi,\theta) \in B_1^{\varepsilon i}, \end{cases} \tag{5.11}$$

$\Phi_1(r)$ is a smooth function equal to 1 when $r < 1/4$ and equal to 0 when $r \geq 1/2$ and $\Phi_2(\theta)$ is a smooth function equal to 1 when $\theta < \pi/3$ and equal to 0 when $\theta \geq 2\pi/3$.

We choose the constants $A_0^{\varepsilon i}, A_1^{\varepsilon i}, A_2^{\varepsilon i}, A^{\varepsilon i}, B^{\varepsilon i}$ in such a way that the function $\widehat{v}^\varepsilon \in H^2(\widetilde{G_{kl}^{\varepsilon ij}})$ satisfies the following condition

$$\int\limits_{G_{11}^{\varepsilon ii}} \big(- \Delta^\varepsilon \widehat{v}^\varepsilon + \lambda \widehat{v}^\varepsilon \big) d\widetilde{x} = 0. \tag{5.12}$$

As a result we obtain for $\varepsilon \to 0$ the asymptotic

$$\begin{aligned}
A_0^{\varepsilon i} &= \frac{2}{2 + (q_1^C(x_i^\varepsilon) + 2)\lambda\pi(b_1^D(x_i^\varepsilon))^3}(1 + \bar{o}(1)), \\
A_1^{\varepsilon i} &= \frac{\lambda\pi(b_1^D(x_i^\varepsilon))^3}{2 + (q_1^C(x_i^\varepsilon) + 2)\lambda\pi(b_1^D(x_i^\varepsilon))^3}(1 + \bar{o}(1)), \\
A_2^{\varepsilon i} &= \frac{\lambda\pi(b_1^D(x_i^\varepsilon))^3}{2 + (q_1^C(x_i^\varepsilon) + 2)\lambda\pi(b_1^D(x_i^\varepsilon))^3}(1 + \bar{o}(1)), \\
A^{\varepsilon i} &= -\frac{\lambda\pi(b_1^D(x_i^\varepsilon))^3 q_1^C(x_i^\varepsilon)}{2 + (q_1^C(x_i^\varepsilon) + 2)\lambda\pi(b_1^D(x_i^\varepsilon))^3}(1 + \bar{o}(1)), \\
B^{\varepsilon i} &= A_0^{\varepsilon i} + A_1^{\varepsilon i} - A^{\varepsilon i}.
\end{aligned} \tag{5.13}$$

Estimating w^ε, we set

$$I^\varepsilon[v^\varepsilon] = \int\limits_{\widetilde{G_{kl}^{\varepsilon ij}}} \sum_{\alpha,\beta=1}^{3} \left\{ g_\varepsilon^{\alpha\beta} \frac{\partial v^\varepsilon}{\partial x_\alpha} \frac{\partial v^\varepsilon}{\partial x_\beta} + \lambda \chi^\varepsilon \cdot (v^\varepsilon)^2 \right\} d\tilde{x},$$

where χ^ε is the characteristic function of $G_{11}^{\varepsilon ii}$.

Since v^ε minimizes the functional I^ε in the class of functions in $H^1(\widetilde{G_{kl}^{\varepsilon ij}})$ equal to 1 on $\partial \widetilde{G_{kl}^{\varepsilon ij}} \equiv S_1^{\varepsilon i}$, w^ε minimizes the functional

$$J^\varepsilon[w^\varepsilon] = I^\varepsilon[w^\varepsilon] - 2 \int\limits_{\widetilde{G_{kl}^{\varepsilon ij}}} \big(\Delta^\varepsilon \widehat{v}^\varepsilon - \lambda \chi^\varepsilon \widehat{v}^\varepsilon \big) w^\varepsilon d\tilde{x}$$

in the class $H_0^1(\widetilde{G_{kl}^{\varepsilon ij}})$. Therefore $J[w^\varepsilon] \le J[0] \equiv 0$ and using (5.12) we have

$$I^\varepsilon[w^\varepsilon] \le 2 \left| \int\limits_{R_1^{\varepsilon i}} \Delta \widehat{v}^\varepsilon w^\varepsilon dx \right| + 2 \left| \int\limits_{G_{11}^{\varepsilon ii}} (\Delta \widehat{v}^\varepsilon - \lambda \widehat{v}^\varepsilon) \cdot (w^\varepsilon - \bar{w}^\varepsilon) dx \right|, \tag{5.14}$$

where $\bar{w}^\varepsilon$ is the average value of w^ε in $G_{11}^{\varepsilon ii}$.

It follows from Friedrich's and Poincare's inequalities that

$$\int\limits_{R_1^{\varepsilon i}} |w^\varepsilon|^2 d\tilde{x} \le C\varepsilon^2 I^\varepsilon[w^\varepsilon],$$
$$\int\limits_{G_{11}^{\varepsilon ii}} |w^\varepsilon - \bar{w}^\varepsilon|^2 d\tilde{x} \le C\varepsilon^2 I^\varepsilon[w^\varepsilon]. \tag{5.15}$$

Moreover, from (5.11) and (5.13) we have

$$\int\limits_{R_1^{\varepsilon i}} |\Delta^\varepsilon \widehat{v}^\varepsilon|^2 d\tilde{x} + \int\limits_{G_{11}^{\varepsilon ii}} \left(|\Delta^\varepsilon \widehat{v}^\varepsilon|^2 + \lambda |\widehat{v}^\varepsilon|^2\right) d\tilde{x} \le C \cdot \varepsilon^3. \tag{5.16}$$

Taking into account the inequalities (5.15), (5.16) and using Cauchy's inequality, from (5.14) we obtain the estimate

$$I^\varepsilon[w^\varepsilon] \le C \cdot \varepsilon^5. \tag{5.17}$$

On the other hand we get from (5.11) and (5.13)

$$I^\varepsilon[\widehat{v}^\varepsilon] = \frac{4a\pi^2 \left(b^D(x_i^\varepsilon)\right)^3 \lambda}{2 + \lambda\pi \left(b_1^D(x_i^\varepsilon)\right)^3 \left(q_1^C(x_i^\varepsilon) + 2\right)} \varepsilon^3 \cdot (1 + \bar{o}(1)). \tag{5.18}$$

Therefore from (5.17)-(5.18) we have

$$V_{11}^{\varepsilon ii} = \frac{4a\pi^2 \left(b^D(x_i^\varepsilon)\right)^3 \lambda}{2 + \lambda\pi \left(b_1^D(x_i^\varepsilon)\right)^3 \left(q_1^C(x_i^\varepsilon) + 2\right)} \varepsilon^3 \cdot (1 + \bar{o}(1)).$$

In the same way as in Cases 1 we obtain

$$V_{11}(x, \lambda) = \frac{4a\pi^2 \left(b^D(x)\right)^3 \lambda}{2 + \lambda\pi \left(b_1^D(x)\right)^3 \left(q_1^C(x) + 2\right)},$$

hence

$$\mathrm{V}_{11}(x, t) = \frac{4a\pi}{q_1^C(x) + 2} \exp\left(\frac{-2t}{\pi \left(b_1^D(x)\right)^3 \left(q_1^C(x) + 2\right)}\right).$$

The homogenized equation has the form

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\partial}{\partial t} \int\limits_0^t \frac{4a\pi}{q_1^C(x) + 2} \exp\left(\frac{-2(t-\tau)}{\pi \left(b_1^D(x)\right)^3 \left(q_1^C(x) + 2\right)}\right) u(x, \tau) d\tau = 0.$$

This is a one-species diffusion equation with memory.

Case 4. We construct the manifold $\widetilde{M}^\varepsilon$ in the same way as in Case 1, but $G_{12}^{\varepsilon ii}$ we choose in the form (3.8). In this case we obtain

$$V_{12}(x,\lambda) = V_{21}(x,\lambda) = 4a\pi \frac{2 + \lambda\pi\big(b_{12}^F(x)\big)^3\big(q_{12}^E(x)+2\big)}{\big(q_{12}^E(x)+2\big)\cdot\big(4+\lambda\pi\big(b_{12}^F(x)\big)^3\big(q_{12}^E(x)+2\big)\big)},$$

$$W_{12}(x,\lambda) = W_{21}(x,\lambda) = -4a\pi \frac{2}{\big(q_{12}^E(x)+2\big)\cdot\big(4+\lambda\pi\big(b_{12}^F(x)\big)^3\big(q_{12}^E(x)+2\big)\big)}.$$

Further we get

$$\mathrm{V}_{12}(x,t) = \mathrm{V}_{21}(x,t) = \frac{2a\pi}{q_{12}^E(x)+2}\left\{1+\exp\left(\frac{-4t}{\lambda\pi\big(b_{12}^F(x)\big)^3\big(q_{12}^E(x)+2\big)}\right)\right\},$$

$$\mathrm{W}_{12}(x,t) = \mathrm{W}_{21}(x,t) = -\frac{2a\pi}{q_{12}^E(x)+2}\left\{1-\exp\left(\frac{-4t}{\lambda\pi\big(b_{12}^F(x)\big)^3\big(q_{12}^E(x)+2\big)}\right)\right\}.$$

The homogenized system has the form

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &- \Delta u_1 + \frac{2a\pi}{q_{12}^E(x)+2}\big(u_1(x,t)-u_2(x,t)\big) + \\
&+ \frac{\partial}{\partial t}\int_0^t \frac{2a\pi}{q_{12}^E(x)+2}\exp\left(\frac{-4(t-\tau)}{\pi\big(b_{12}^F(x)\big)^3\big(q_{12}^E(x)+2\big)}\right)\big(u_1(x,\tau)+u_2(x,\tau)\big)d\tau = 0, \\
\frac{\partial u_2}{\partial t} &- \Delta u_2 + \frac{2a\pi}{q_{12}^E(x)+2}\big(u_2(x,t)-u_1(x,t)\big) + \\
&+ \frac{\partial}{\partial t}\int_0^t \frac{2a\pi}{q_{12}^E(x)+2}\exp\left(\frac{-4(t-\tau)}{\pi\big(b_{12}^F(x)\big)^3\big(q_{12}^E(x)+2\big)}\right)\big(u_2(x,\tau)+u_1(x,\tau)\big)d\tau = 0.
\end{aligned}$$

Case 5. Finally, we construct the manifold $\widetilde{M}^\varepsilon$ in the same way as in Case 2, but $G_{12}^{\varepsilon ij}$ we choose in the form (3.8). In this case the homogenized system has the form

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &- \Delta u_1 + \int_\Omega \frac{2a\pi}{q_{12}^G(x,y)+2}\big(u_1(x,t)-u_2(y,t)\big)dy + \\
&+ \int_\Omega \frac{\partial}{\partial t}\int_0^t \frac{2a\pi}{q_{12}^G(x,y)+2}\exp\left(\frac{-4(t-\tau)}{\pi\big(b_{12}^H(x,y)\big)^3\big(q_{12}^G(x,y)+2\big)}\right)\cdot \\
&\cdot \big(u_1(x,\tau)+u_2(y,\tau)\big)d\tau dy = 0, \\
\frac{\partial u_2}{\partial t} &- \Delta u_2 + \int_\Omega \frac{2a\pi}{q_{21}^G(x,y)+2}\big(u_2(x,t)-u_1(y,t)\big)dy + \\
&+ \int_\Omega \frac{\partial}{\partial t}\int_0^t \frac{2a\pi}{q_{21}^G(x,y)+2}\exp\left(\frac{-4(t-\tau)}{\pi\big(b_{21}^H(x,y)\big)^3\big(q_{21}^G(x,y)+2\big)}\right)\cdot \\
&\cdot \big(u_2(x,\tau)+u_1(y,\tau)\big)d\tau dy = 0.
\end{aligned}$$

Let us combine the results of Cases 1-5. We divide the domain Ω in cubes $K^{\varepsilon i}$ in such a way that they form a periodic cubic lattice with side length ε. In each cube we pick out 7 disjoint cubes $K^{S\varepsilon\alpha}, s = 1..7$, such that $\mathrm{diam} K^{s\varepsilon\alpha} \sim c\varepsilon$. We call them sub-cubes. In the sub-cubes $K^{1\varepsilon}, K^{2\varepsilon}, K^{3\varepsilon}$ we cut out a single hole - a ball with radius $a\varepsilon^3$, whereas in the sub-cubes $K^{4\varepsilon}, K^{5\varepsilon}, K^{6\varepsilon}, K^{7\varepsilon}$ we cut out $n(\varepsilon)$ holes - balls with radius $a\varepsilon^6$ (we require the condition (3.2) holds - see Case 2). We obtain a system of balls $D^{\varepsilon i}, i = 1..N(\varepsilon) = 3n(\varepsilon)+4n^2(\varepsilon)$. As before, $\Omega^\varepsilon = \Omega \setminus \bigcup_{i=1}^{N(\varepsilon)} D^{\varepsilon i}$. Now, we consider two copies (sheets) of the domain Ω^ε - Ω_1^ε and Ω_2^ε. We denote by $D_k^{\varepsilon i}$ the copy of the i-th ball on the k-th sheet ($k = 1, 2$). We can write the index i in the form $i = i_{\alpha,s,\beta}$, where α is the number of the cube containing the ball, s is the number of the sub-cube, and index β appears only in the case $s = 4, 5, 6, 7$ and denote the number of the ball within the sub-cube.

Now, we connect the manifolds $G_{kl}^{\varepsilon ij}$ with the sheets by the following rules:

1. Via the manifold $G_{12}^{\varepsilon i_{\alpha,1} i_{\alpha,1}}$ of the form (3.7) (see Case 1) we join the boundaries of the holes $D_1^{\varepsilon i_{\alpha,1}}$ and $D_2^{\varepsilon i_{\alpha,1}}$.
2. Via the manifold $G_{12}^{\varepsilon i_{\alpha,2} i_{\alpha,2}}$ of the form (3.8) (see. Case 4) we join the boundaries of the holes $D_1^{\varepsilon i_{\alpha,2}}$ and $D_2^{\varepsilon i_{\alpha,2}}$.
3. We glue the manifold $G_{11}^{\varepsilon i_{\alpha,3} i_{\alpha,3}}$ of the form (3.4) to the boundary of the hole $D_1^{\varepsilon i_{\alpha,3}}$ and glue the manifold $G_{22}^{\varepsilon i_{\alpha,3} i_{\alpha,3}}$ of the form (3.4) to the boundary of the hole $D_2^{\varepsilon i_{\alpha,3}}$ (see. Case 3).
4. Via the manifold $G_{12}^{\varepsilon i_{\alpha,4,\beta} i_{\beta,4,\alpha}}$ of the form (3.7) (see. Case 2) we join the boundaries of the holes $D_1^{\varepsilon i_{\alpha,4,\beta}}$ and $D_2^{\varepsilon i_{\beta,4,\alpha}}$.

5. Via the manifold $G_{12}^{\varepsilon i_{\alpha,5,\beta} i_{\beta,5,\alpha}}$ of the form (3.8) (see. Case 5) we join the boundaries of the holes $D_1^{\varepsilon i_{\alpha,5,\beta}}$ and $D_2^{\varepsilon i_{\beta,5,\alpha}}$.

6. Via the manifold $G_{11}^{\varepsilon i_{\alpha,6,\beta} i_{\beta,6,\alpha}}$ of the form (3.7) we join the boundaries of the hole $D_1^{\varepsilon i_{\alpha,6,\beta}}$ and $D_1^{\varepsilon i_{\beta,6,\alpha}}$ and via the manifold $G_{22}^{\varepsilon i_{\alpha,6,\beta} i_{\beta,6,\alpha}}$ of the form (3.7) we join the boundaries of the holes $D_2^{\varepsilon i_{\alpha,6,\beta}}$ and $D_2^{\varepsilon i_{\beta,6,\alpha}}$. This is analogously to Case 2, but here the tube starts and ends on the same sheet.

7. Via the manifold $G_{11}^{\varepsilon i_{\alpha,7,\beta} i_{\beta,7,\alpha}}$ of the form (3.8) we join the boundaries of the holes $D_1^{\varepsilon i_{\alpha,7,\beta}}$ and $D_1^{\varepsilon i_{\beta,7,\alpha}}$ and via the manifold $G_{22}^{\varepsilon i_{\alpha,7,\beta} i_{\beta,7,\alpha}}$ of the form (3.8) we join the boundaries of the holes $D_2^{\varepsilon i_{\alpha,7,\beta}}$ and $D_2^{\varepsilon i_{\beta,7,\alpha}}$. This is analogously to Case 5, but as before the tube starts and ends on the same sheet.

As a result we obtain the manifold $\widetilde{M}^\varepsilon$ as a combination of the Cases 1-5. In this case the homogenized system has the form

$$\begin{aligned}
\frac{\partial u_k}{\partial t} &- \Delta u_k + \sum_{l\neq k}\left(\frac{4a\pi}{q_{kl}^A(x)+2}+\frac{2a\pi}{q_{kl}^E(x)+2}\right)\big(u_k(x,t)-u_l(x,t)\big) + \\
&+ \sum_{l=1}^{2}\int_\Omega\left(\frac{4a\pi}{q_{kl}^B(x,y)+2}+\frac{2a\pi}{q_{kl}^G(x,y)+2}\right)\big(u_k(x,t)-u_l(y,t)\big)dy + \\
&+ \frac{\partial}{\partial t}\int_0^t \frac{4a\pi}{q_k^C(x)+2}\exp\left(\frac{-2(t-\tau)}{\pi\big(b_k^D(x)\big)^3\big(q_k^C(x)+2\big)}\right)u_k(x,\tau)d\tau + \\
&+ \sum_{l\neq k}\frac{\partial}{\partial t}\int_0^t \frac{2a\pi}{q_{kl}^E(x)+2}\exp\left(\frac{-4(t-\tau)}{\pi\big(b_{kl}^F(x)\big)^3\big(q_{kl}^E(x)+2\big)}\right)\big(u_k(x,\tau)+u_l(x,\tau)\big)d\tau + \\
&+ \sum_{l=1}^{2}\int_\Omega\frac{\partial}{\partial t}\int_0^t \frac{2a\pi}{q_{kl}^G(x,y)+2}\exp\left(\frac{-4(t-\tau)}{\pi\big(b_{kl}^H(x,y)\big)^3\big(q_{kl}^G(x,y)+2\big)}\right)\cdot \\
&\cdot \big(u_k(x,\tau)+u_l(y,\tau)\big)d\tau dy = 0,\ k=1,2. \qquad (5.19)
\end{aligned}$$

Note, that the introduced metric g_{ij}^ε has a discontinuous part on the boundary of the manifolds $G_{kl}^{\varepsilon ij}$, but it is possible to approximate it by a smooth metric $g_{ij}^\varepsilon(\delta)$. This metric differ from g_{ij}^ε in a small $\delta=\delta(\varepsilon)$-neighborhood of $\partial G_{kl}^{\varepsilon ij}$. If $\delta=\delta(\varepsilon)$ tends to zero sufficiently fast as $\varepsilon\to 0$, then the functions $V_{kl}(x,\lambda)=V_{kl}(x,\lambda,\delta)$ and $W_{kl}(x,y,\lambda)=W_{kl}(x,y,\lambda,\delta)$, calculated by the formulas (5.4)-(5.6) with metric $g_{kl}^{\varepsilon ij}(\delta)$, are equal to the functions $V_{kl}(x,\lambda)$ and $W_{kl}(x,y,\lambda)$ calculated with the metric $g_{kl}^{\varepsilon ij}$.

We set

$$
\begin{aligned}
q^A_{kl}(x) &= \frac{4a\pi}{A_{kl}(x)-E_{kl}(x)}-2, & q^B_{kl}(x) &= \frac{4a\pi}{B_{kl}(x,y)-G_{kl}(x,y)}-2,\\
q^C_k(x) &= \frac{4a\pi}{C_k(x)}-2, & b^D_k(x) &= \sqrt[3]{\frac{C_k(x)}{2a\pi^2 D_k(x)}},\\
q^E_{kl}(x) &= \frac{2a\pi}{E_{kl}(x)}-2, & b^F_{kl}(x) &= \sqrt[3]{\frac{2E_{kl}(x)}{a\pi^2 F_{kl}(x)}},\\
q^G_{kl}(x,y) &= \frac{2a\pi}{G_{kl}(x,y)}-2, & b^H_{kl}(x,y) &= \sqrt[3]{\frac{2G_{kl}(x,y)}{a\pi^2 H_{kl}(x,y)}}.
\end{aligned}
\tag{5.20}
$$

The functions (5.20) satisfy the conditions (3.3) and are positive, if a is sufficiently large. Then the system (5.19) has the form (1.1). In the same way the proof can be done for $m > 2$ sheets . Theorem 3 is proved.

6 Proofs of the main theorems

P r o o f of Theorem 1. Using the same methods as in [7] it is easy to show that (1.1)-(1.3) has the unique solution $u(\cdot,t) \in C(0,T;H^1(\Omega)^m)$ with $\frac{d}{dt}u(\cdot,t) \in L_2(0,T;L_2(\Omega)^m)$, $\forall T > 0$.

We construct the manifold $\widetilde{M}^\varepsilon$ in the same way as in Theorem 3. Moreover, we require that the point $x_{\max}$, providing the maximum to $\max\limits_k \max\limits_{x\in\Omega} f_k(x)$, does not lie in any out-cut ball $D^{\varepsilon i}$. This can be done, because of the construction of the manifold $\widetilde{M}^\varepsilon$. Let the function $f^\varepsilon(\tilde{x})$ be the same as in the proof of Theorem 3, i.e., it coincides with $f_k(x)$ if $\tilde{x} = (x,k) \in \Omega^\varepsilon_k$ and is equal to zero in $G^{\varepsilon ij}_{kl}$, except of a small neighborhoods of ${}^0\Gamma^{\varepsilon i}_k$, ${}^1\Gamma^{\varepsilon ij}_{kl}$ and ${}^2\Gamma^{\varepsilon ij}_{kl}$. In these neighborhoods we construct f^ε in such a way that $f^\varepsilon(\tilde{x}) \le \max\limits_{k=1\dots m} \max\limits_{x\in\Omega} f_k(x)$. Then, $\max\limits_{\widetilde{M}^\varepsilon} f^\varepsilon(\tilde{x})$ is reached on some sheet and, therefore, $\max\limits_{\widetilde{M}^\varepsilon} f^\varepsilon(\tilde{x}) = \max\limits_l \max\limits_{x\in\Omega} f_l(x) \equiv M$.

In view of the maximum principle (see, e.g., [20]) we have

$$
u^\varepsilon(\tilde{x},t) \le \max_{\widetilde{M}^\varepsilon} f^\varepsilon(\tilde{x}) = M,
$$

where u^ε is a solution of the problem (4.1)-(4.3). We get

$$
\left(Q^\varepsilon u^\varepsilon\right)_k(x,t) \le M,\ x \in \Omega,\ t > 0,\ \forall k.
$$

By Theorem 3 $\left(Q^\varepsilon u^\varepsilon\right)_k$ converges to u_k in $L_2(\Omega \times [0,T])$, $\forall T > 0$. Therefore, there exists a sequence $\varepsilon = \varepsilon_n$ such that for almost all $x \in \Omega,\ t > 0$

$$
\left(Q^\varepsilon u^\varepsilon\right)_k(x,t) \to u_k(x,t),\ \varepsilon = \varepsilon_n \to 0.
$$

Then, for almost all $x \in \Omega,\ t > 0, \forall k : u_k(x,t) \le M$.

In the same way the minimum principle can be proved.

Finally, we consider the case $C_k(x) = D_k(x) = E_{kl}(x) = F_{kl}(x) = G_{kl}(x,y) = H_{kl}(x,y) \equiv 0$, $\forall k,l$. Again, we construct the manifold $\widetilde{M}^\varepsilon$ so, that the point $x_{\max}$ does not lie in any ball $D^{\varepsilon i}$. Let the function $f^\varepsilon(\tilde{x})$ coincides with $f_k(x)$ if $\tilde{x} = (x,k) \in \Omega_k^\varepsilon$, while in $G_{kl}^{\varepsilon ij}$ we set $f^\varepsilon(\tilde{x})$ equal to the constant C (except some small neighborhood of $\partial G_{kl}^{\varepsilon ij}$), such that $C < M$ (Theorem 3 is still true, because $\sum\limits_{i,j=1}^{N(\varepsilon)} \sum\limits_{k,l=1}^{m} |G_{kl}^{\varepsilon ij}| \to 0$ for $\varepsilon \to 0$). Then again $\max\limits_{\widetilde{M}^\varepsilon} f^\varepsilon(\tilde{x}) = M$ (i.e. the maximum of $f^\varepsilon(\tilde{x})$ is reached on the sheets). Further, the proof is done in the same way. Theorem 1 is proved.

Remark 2. We give an example which shows that the condition $\max\limits_k \max\limits_{x\in\Omega} f_k(x) \geq 0$ is necessary for the maximum principle.

We consider a particular case of the problem (1.1)-(1.3):

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\partial}{\partial t}\int\limits_0^t Ce^{-D(t-\tau)}u(x,\tau)d\tau = 0, \tag{6.1}$$

$$u(x,0) = f, \ \frac{\partial u}{\partial \vec{n}} = 0, \tag{6.2}$$

where C, D and f are constant.

It is easy to see that the function $u(x,t) = \dfrac{f}{C+D}\left(D + C\cdot e^{-(C+D)t}\right)$ is a solution of (6.1)-(6.2). If $f<0$ then, obviously, the maximum principle is not fulfilled.

P r o o f of Theorem 2. We construct the manifold $\widetilde{M}^\varepsilon$ in the same way as in Theorem 3. Let $u^\varepsilon(\tilde{x},t)$ be the solution of the problem (4.1)-(4.3). In order to estimate $u^\varepsilon(\tilde{x},t)$ we prove the following uniform Poincare inequality

Lemma. *For all $u^\varepsilon \in H^1(\widetilde{M}^\varepsilon)$ such that $\bar{u}^\varepsilon \equiv \dfrac{1}{|\widetilde{M}^\varepsilon|}\int\limits_{\widetilde{M}^\varepsilon} u^\varepsilon(\tilde{x})d\tilde{x} = 0$ the following inequality holds*

$$\int\limits_{\widetilde{M}^\varepsilon} \left(u^\varepsilon(\tilde{x})\right)^2 d\tilde{x} \leq c_p \int\limits_{\widetilde{M}^\varepsilon} \sum_{\alpha,\beta=1}^{3} g_\varepsilon^{\alpha\beta} \frac{\partial u^\varepsilon}{\partial x_\alpha}\frac{\partial u^\varepsilon}{\partial x_\beta} d\tilde{x}, \tag{6.3}$$

where the constant c_p does not depend on ε.

P r o o f. We prove the Lemma for two special cases of the manifold $\widetilde{M}^\varepsilon$. For the general case it can be proved in a similar way.

1^0. Suppose that our manifold $\widetilde{M}^\varepsilon$ has the same form as in Case 3 in the proof of Theorem 3.

$$\widetilde{M}^\varepsilon = \Omega_1^\varepsilon \cup \left(\bigcup_{i=1}^{N(\varepsilon)} G_{11}^{\varepsilon ii}\right)$$

Remind, that $G_{11}^{\varepsilon ii} = B_1^{\varepsilon i} \cup T_1^{\varepsilon i}$, $B_1^{\varepsilon i}$ and $T_1^{\varepsilon i}$ are defined by the formulas (3.5), (3.6). Proving the lemma indirectly, we assume the opposite, i.e., (6.3) does not holds. Then a sequence (still denoted by ε) and functions $u^\varepsilon(\tilde{x}) \in H^1(\widetilde{M^\varepsilon})$ exist such that

$$\begin{gathered} \int\limits_{\widetilde{M^\varepsilon}} \big(u^\varepsilon(\tilde{x})\big)^2 d\tilde{x} = 1, \\ \bar{u}^\varepsilon \equiv \frac{1}{|\widetilde{M^\varepsilon}|} \int\limits_{\widetilde{M^\varepsilon}} u^\varepsilon(\tilde{x}) d\tilde{x} = 0, \\ \int\limits_{\widetilde{M^\varepsilon}} \sum_{\alpha,\beta=1}^{3} g_\varepsilon^{\alpha\beta} \frac{\partial u^\varepsilon}{\partial x_\alpha} \frac{\partial u^\varepsilon}{\partial x_\beta} d\tilde{x} \to 0, \ \varepsilon \to 0. \end{gathered} \tag{6.4}$$

Then, it follows from (6.4) that $Q^\varepsilon u^\varepsilon$ converges in $L_2(\Omega)$ to some constant C_0. Denote by C_i^ε the average value of u^ε in the domain $B_1^{\varepsilon i}$, i.e.,

$$C_i^\varepsilon = \frac{1}{|B_1^{\varepsilon i}|} \int\limits_{B_1^{\varepsilon i}} u^\varepsilon(\tilde{x}) d\tilde{x}.$$

We represent u^ε in the form $u^\varepsilon = v^\varepsilon + w^\varepsilon$, where

$$v^\varepsilon = \begin{cases} C_0 + \sum\limits_{i=1}^{N(\varepsilon)} \alpha_i^\varepsilon \dfrac{d_i^\varepsilon}{|x - x_i^\varepsilon|} \Phi_1\left(\dfrac{|x - x_i^\varepsilon|}{\varepsilon}\right), \ \tilde{x} = (x, 1) \in \Omega_1^\varepsilon \\ A_i^\varepsilon z + B_i^\varepsilon, \ \tilde{x} = (\varphi, \psi, z) \in T_1^{\varepsilon i} \\ C_i^\varepsilon + \beta_i^\varepsilon \dfrac{\cot\theta}{\cot\theta^\varepsilon} \Phi_2(\theta), \ \tilde{x} = (\varphi, \psi, \theta) \in B_1^{\varepsilon i} \end{cases}$$

and

$$\alpha_i^\varepsilon = \frac{C_i^\varepsilon - C_0}{1 + \cos\theta_1^{\varepsilon i} + q_1^C(x_i^\varepsilon)}, \ \beta_i^\varepsilon = -\alpha_i^\varepsilon \cdot \cos\theta_1^{\varepsilon i}, \ A_i^\varepsilon = \alpha_i^\varepsilon \cdot q_1^C(x_i^\varepsilon), \ B_i^\varepsilon = C_0 + \alpha_i^\varepsilon,$$

$\Phi_1(r)$ is a smooth function equal to 1 when $r < 1/4$ and equal to 0 when $r \geq 1/2$, $\Phi_2(\theta)$ is a smooth function equal to 1 when $\theta < \pi/3$ and equal to 0 when $\theta \geq 2\pi/3$. The coefficients $\alpha_i^\varepsilon, \ \beta_i^\varepsilon, \ A_i^\varepsilon, \ B_i^\varepsilon$ are taken in such a way that $v^\varepsilon \in H^1(\widetilde{M^\varepsilon})$ and

$$\Delta^\varepsilon v^\varepsilon = 0, \ \tilde{x} \in \bigcup_i \Big[B_1^{\varepsilon i} \cup \big\{\tilde{x} = (x, 1) \in \Omega^\varepsilon : \ d_i^\varepsilon < |x - x_i^\varepsilon| < d_i^\varepsilon + r_i^\varepsilon/4\big\}\Big].$$

We have

$$\|\nabla^\varepsilon u^\varepsilon\|_{0\varepsilon}^2 \geq \|\nabla^\varepsilon v^\varepsilon\|_{0\varepsilon}^2 + 2(\nabla^\varepsilon v^\varepsilon, \nabla^\varepsilon w^\varepsilon)_{0\varepsilon} = \|\nabla^\varepsilon v^\varepsilon\|_{0\varepsilon}^2 - 2(\Delta^\varepsilon v^\varepsilon, w^\varepsilon)_{0\varepsilon} \tag{6.5}$$

From the explicit form of the function v^ε and Poincare's inequality for the domain $B_1^{\varepsilon i}$, we obtain the following inequalities

$$
\begin{gathered}
M_1 \sum_{i=1}^{N(\varepsilon)} (C_0 - C_i^\varepsilon)^2 |B_1^{\varepsilon i}| \le \|\nabla^\varepsilon v^\varepsilon\|^2_{0\varepsilon} \le M_2 \sum_{i=1}^{N(\varepsilon)} (C_0 - C_i^\varepsilon)^2 |B_1^{\varepsilon i}|, \\
\|\Delta^\varepsilon v^\varepsilon\|^2_{0\varepsilon} \le M_3 \sum_{i=1}^{N(\varepsilon)} (C_0 - C_i^\varepsilon)^2 |B_1^{\varepsilon i}|, \\
\|w^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} \le 2\|u^\varepsilon - C_0\|^2_{L_2(\Omega^\varepsilon)} + M_4 \varepsilon^4 \sum_{i=1}^{N(\varepsilon)} (C_0 - C_i^\varepsilon)^2 |B_1^{\varepsilon i}|, \\
\sum_{i=1}^{N(\varepsilon)} \|w^\varepsilon\|^2_{L_2(B_1^{\varepsilon i})} \le \varepsilon^2 M_5 \sum_{i=1}^{N(\varepsilon)} \|\nabla^\varepsilon w^\varepsilon\|^2_{L_2(B_1^{\varepsilon i})} + \sum_{i=1}^{N(\varepsilon)} \frac{1}{|B_1^{\varepsilon i}|} \left(\int\limits_{B_1^{i\varepsilon}} (C_i^\varepsilon - v^\varepsilon) d\tilde{x} \right)^2 \le \\
\le 2\varepsilon^2 M_5 (\|\nabla^\varepsilon v^\varepsilon\|^2_{0\varepsilon} + \|\nabla^\varepsilon u^\varepsilon\|^2_{0\varepsilon}) + \varepsilon^4 M_6 \sum_{i=1}^{N(\varepsilon)} (C_0 - C_i^\varepsilon)^2 |B_1^{\varepsilon i}|,
\end{gathered}
\tag{6.6}
$$

where $M_i, i = 1..6$ are positive constants.

Further, we prove that

$$
\exists c_1, c_2, > 0 : \; c_1 < \sum_{i=1}^{N(\varepsilon)} (C_0 - C_i^\varepsilon)^2 |B_1^{\varepsilon i}| < c_2. \tag{6.7}
$$

From the inequalities (6.6), (6.7), using Cauchy's inequality, we have

$$
\begin{aligned}
\|\nabla^\varepsilon v^\varepsilon\|^2_{0\varepsilon} &\ge c_1 \cdot M_1 > 0, \\
|(\Delta^\varepsilon v^\varepsilon, w^\varepsilon)_{0\varepsilon}| &\le \|\Delta^\varepsilon v^\varepsilon\|_{L_2(\Omega^\varepsilon)} \cdot \|\Delta^\varepsilon w^\varepsilon\|_{L_2(\Omega^\varepsilon)} \\
&+ \left[\sum_{i=1}^{N(\varepsilon)} \|\Delta^\varepsilon v^\varepsilon\|^2_{L_2(B_1^{\varepsilon i})} \right]^{\frac{1}{2}} \cdot \left[\sum_{i=1}^{N(\varepsilon)} \|w^\varepsilon\|^2_{L_2(B_1^{\varepsilon i})} \right]^{\frac{1}{2}} \underset{\varepsilon \to 0}{\longrightarrow} 0.
\end{aligned}
$$

Then, from (6.5) we have $\lim\limits_{\varepsilon \to 0} \|\nabla u^\varepsilon\|_{0\varepsilon} > 0$ – a contradiction.

Now, we prove the inequalities (6.7). The right-hand inequality follows from $\|u^\varepsilon\|_{0\varepsilon} = 1$. Now, suppose that the left-hand inequality does not hold. Then, there exists a sequence (again denoted by ε) such that

$$
\lim_{\varepsilon \to 0} \sum_{i=1}^{N(\varepsilon)} (C_0 - C_i^\varepsilon)^2 |B_1^{\varepsilon i}| = 0. \tag{6.8}
$$

From (6.8) we obtain the inequalities

$$\left(\sum_{i=1}^{N(\varepsilon)}(C_0 - C_i^\varepsilon)|B_1^{\varepsilon i}|\right)^2 \le N(\varepsilon)\sum_{i=1}^{N(\varepsilon)}(C_0 - C_i^\varepsilon)^2|B_1^{\varepsilon i}|^2 \le$$

$$\le c\sum_{i=1}^{N(\varepsilon)}(C_0 - C_i^\varepsilon)^2|B_1^{\varepsilon i}|, \ c > 0 \tag{6.9}$$

Using Poincare's inequality for the domain $B_1^{\varepsilon i}$ we have

$$0 \le \sum_{i=1}^{N(\varepsilon)}\left\{\int\limits_{B_1^{\varepsilon i}}(u^\varepsilon(\tilde{x}))^2 d\tilde{x} - (C_i^\varepsilon)^2|B_1^{\varepsilon i}|\right\} \le \varepsilon^2\sum_{i=1}^{N(\varepsilon)}\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(B_1^{\varepsilon i})} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Hence,

$$1 = \lim_{\varepsilon\to 0}\|u^\varepsilon\|^2_{0\varepsilon} = C_0^2|\Omega| + \lim_{\varepsilon\to 0}\sum_{i=1}^{N(\varepsilon)}(C_i^\varepsilon)^2|B_1^{\varepsilon i}| \tag{6.10}$$

Further, we get

$$0 = \lim_{\varepsilon\to 0}\int\limits_{\widetilde{M}^\varepsilon} u^\varepsilon(\tilde{x})d\tilde{x} = C_0|\Omega| + \lim_{\varepsilon\to 0}\sum_{i=1}^{N(\varepsilon)} C_i^\varepsilon|B_1^{\varepsilon i}| =$$

$$= C_0\lim_{\varepsilon\to 0}|\widetilde{M}^\varepsilon| + \lim_{\varepsilon\to 0}\sum_{i=1}^{N(\varepsilon)}(C_i^\varepsilon - C_0)|B_1^{\varepsilon i}|. \tag{6.11}$$

It follows from (6.9) and (6.11) that $C_0 = 0$. But this, together with (6.8) contradicts (6.10). Thus, the right-hand inequality of (6.7) is true.

2^0. Suppose that our manifold $\widetilde{M}^\varepsilon$ has the same form as in Case 1 in the proof of Theorem 3:

$$\widetilde{M}^\varepsilon = \Omega_1^\varepsilon \cup \left(\bigcup_{i=1}^{N(\varepsilon)} G_{12}^{\varepsilon ii}\right) \cup \Omega_2^\varepsilon$$

Remind that the manifolds $G_{12}^{\varepsilon ii}$ are defined by the formulas (3.7).

We suppose the opposite: (6.3) does not hold. Then, there exist functions $u^\varepsilon(\tilde{x}) \in H^1(\widetilde{M}^\varepsilon)$ such that

$$\begin{gathered}\int\limits_{\widetilde{M}^\varepsilon}\left(u^\varepsilon(\tilde{x})\right)^2 d\tilde{x} = 1,\\ \bar{u}^\varepsilon = \frac{1}{|\widetilde{M}^\varepsilon|}\int\limits_{\widetilde{M}^\varepsilon} u^\varepsilon(\tilde{x})d\tilde{x} = 0,\\ \int\limits_{\widetilde{M}^\varepsilon}\sum_{\alpha,\beta=1}^{3} g_\varepsilon^{\alpha\beta}\frac{\partial u^\varepsilon}{\partial x_\alpha}\frac{\partial u^\varepsilon}{\partial x_\beta}d\tilde{x} \to 0, \ \varepsilon \to 0.\end{gathered} \tag{6.12}$$

It follows from (6.12) that $(Q^\varepsilon u^\varepsilon)_1$ converges in $L_2(\Omega)$ to some constant C_1 and $(Q^\varepsilon u^\varepsilon)_2$ converges in $L_2(\Omega)$ to some constant C_2. Later we prove that $C_1 \neq C_2$.

Again, we represent u^ε in the form $u^\varepsilon = v^\varepsilon + w^\varepsilon$, where

$$v^\varepsilon = \begin{cases} C_1 + \sum\limits_{i=1}^{N(\varepsilon)} \alpha_i^\varepsilon \dfrac{d_i^\varepsilon}{|x - x_i^\varepsilon|} \Phi\left(\dfrac{|x - x_i^\varepsilon|}{\varepsilon}\right), \ \tilde{x} = (x, 1) \in \Omega_1^\varepsilon \\ A_i^\varepsilon z + B_i^\varepsilon, \ \tilde{x} = (\varphi, \psi, z) \in G_{12}^{\varepsilon ii} \\ C_2 + \sum\limits_{i=1}^{N(\varepsilon)} \beta_i^\varepsilon \dfrac{d_i^\varepsilon}{|x - x_i^\varepsilon|} \Phi\left(\dfrac{|x - x_i^\varepsilon|}{\varepsilon}\right), \ \tilde{x} = (x, 2) \in \Omega_2^\varepsilon \end{cases} \tag{6.13}$$

and

$$\alpha_i^\varepsilon = \frac{C_2 - C_1}{2 + q_{12}^A(x_i^\varepsilon)}, \ \beta_i^\varepsilon = -\alpha_i^\varepsilon, \ A_i^\varepsilon = \alpha_i^\varepsilon \cdot q_{12}^A(x_i^\varepsilon), \ B_i^\varepsilon = C_1 + \alpha_i^\varepsilon, \tag{6.14}$$

$\Phi(r)$ is a smooth function equal to 1 for $r < 1/4$ and equal to 0 for $r \geq 1/2$. The coefficients $\alpha_i^\varepsilon, \ \beta_i^\varepsilon, \ A_i^\varepsilon, \ B_i^\varepsilon$ are taken in such a way that $v^\varepsilon \in H^1(\widetilde{M}^\varepsilon)$.

We have

$$\|\nabla^\varepsilon u^\varepsilon\|_{0\varepsilon}^2 \geq \|\nabla^\varepsilon v^\varepsilon\|_{0\varepsilon}^2 + 2(\nabla^\varepsilon v^\varepsilon, \nabla^\varepsilon w^\varepsilon)_{0\varepsilon} = \|\nabla^\varepsilon v^\varepsilon\|_{0\varepsilon} - 2(\Delta^\varepsilon v^\varepsilon, w^\varepsilon)_{0\varepsilon} \tag{6.15}$$

From the explicit form of the function v^ε it is easy to obtain the following inequalities

$$\begin{gathered} M_1(C_1 - C_2)^2 \leq \|\nabla^\varepsilon v^\varepsilon\|_{0\varepsilon}^2 \leq M_2(C_1 - C_2)^2, \\ \|\Delta^\varepsilon v^\varepsilon\|_{0\varepsilon}^2 \leq M_3, \\ \|w^\varepsilon\|_{L_2(\Omega_k^\varepsilon)}^2 \leq 2\|u^\varepsilon - C_k\|_{L_2(\Omega_k^\varepsilon)}^2 + M_4 \varepsilon^4 \sum_{i=1}^{N(\varepsilon)} |B_1^{\varepsilon i}|, \ k = 1, 2, \end{gathered} \tag{6.16}$$

where $M_i, i = 1..4$ are positive constants.

From the inequalities (6.16) we have

$$\|\nabla^\varepsilon v^\varepsilon\|_{0\varepsilon} \geq (C_1 - C_2)^2 \cdot M_1 > 0,$$

$$|(\Delta v^\varepsilon, w^\varepsilon)_{0\varepsilon}| \leq \sum_{k=1}^{2} \|\Delta v^\varepsilon\|_{L_2(\Omega_k^\varepsilon)} \cdot \|w^\varepsilon\|_{L_2(\Omega_k^\varepsilon)} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

and from (6.15) we have $\lim\limits_{\varepsilon \to 0} \|\nabla u^\varepsilon\|_{0\varepsilon}^2 > 0$. We obtain a contradiction, i.e., (6.3)holds.

Now, let's prove that $C_1 \neq C_2$. It is easy to prove that $\lim\limits_{\varepsilon \to 0} \sum\limits_{i=1}^{N(\varepsilon)} \|u^\varepsilon\|_{L_2(G_{12}^{\varepsilon ii})}^2 = 0$.

Then we have

$$1 = \lim_{\varepsilon \to 0} \|u^\varepsilon\|_{0\varepsilon}^2 = (C_1^2 + C_2^2)|\Omega|, \tag{6.17}$$

$$0 = \lim_{\varepsilon \to 0} \int\limits_{\widetilde{M}^\varepsilon} u^\varepsilon(\tilde{x}) d\tilde{x} = (C_1 + C_2)|\Omega|. \tag{6.18}$$

(6.17) and (6.18) imply $C_1 \neq C_2$.

The lemma is proved.

We continue the proof of Theorem 2.

From Gronwall's lemma we get that the solution of (4.1)-(4.3) satisfies the inequality

$$\|u^\varepsilon - L^\varepsilon\|^2_{0\varepsilon} \leq \|f^\varepsilon - L^\varepsilon\|^2_{0\varepsilon} \cdot \exp\left[\frac{-2t}{c_p}\right], \; L^\varepsilon = \frac{1}{|\widetilde{M}^\varepsilon|}\int\limits_{\widetilde{M}^\varepsilon} f^\varepsilon(\tilde{x})d\tilde{x}. \tag{6.19}$$

It is clear, that the norms $\|f^\varepsilon - L^\varepsilon\|_{0\varepsilon}$ are uniformly bounded with respect to ε. Therefore $\exists \; c_1 > 0$:

$$\|u^\varepsilon - L^\varepsilon\|^2_{0\varepsilon} \leq c_1 \cdot \exp\left[\frac{-2t}{c_p}\right], \tag{6.20}$$

hence,

$$\|Q^\varepsilon u^\varepsilon - L^\varepsilon\|^2_0 \leq c_1 \cdot \exp\left[\frac{-2t}{c_p}\right] + (L^\varepsilon)^2 \sum_{k=1}^{m}\sum_{i=1}^{N(\varepsilon)} |D^\varepsilon_i|. \tag{6.21}$$

By the construction of $\widetilde{M}^\varepsilon$ we have

$$|\widetilde{M}^\varepsilon| = m \cdot |\Omega^\varepsilon| + \sum_{k=1}^{m}\sum_{i=1}^{N(\varepsilon)} |G^{\varepsilon ii}_{kk}| + \frac{1}{2}\sum_{k,l=1|k\neq l}^{m}\sum_{i=1}^{N(\varepsilon)} |G^{\varepsilon ii}_{kl}| + \frac{1}{2}\sum_{k,l=1}^{m}\sum_{i,j=1|i\neq j}^{N(\varepsilon)} |G^{\varepsilon ij}_{kl}|. \tag{6.22}$$

From the formulas (3.3), (4.6) and (6.22) it follows that

$$|\widetilde{M}^\varepsilon| \xrightarrow[\varepsilon\to 0]{} m \cdot |\Omega| + 2\pi^2 \left(\sum_{k=1}^{m} \int\limits_{\Omega} a\big(b^D_k(x)\big)^3 dx + \frac{1}{2}\sum_{k,l=1|k\neq l}^{m} \int\limits_{\Omega} a\big(b^F_{kl}(x)\big)^3 dx + \right.$$

$$\left. + \sum_{k,l=1}^{m} \frac{1}{2}\int\limits_{\Omega}\int\limits_{\Omega} a\big(b^H_{kl}(x,y)\big)^3 dxdy \right) =$$

$$= m \cdot |\Omega| + \sum_{k=1}^{m} \int\limits_{\Omega} \frac{C_k(x)}{D_k(x)} dx + 2\sum_{k,l=1|k\neq l}^{m} \int\limits_{\Omega} \frac{E_{kl}(x)}{F_{kl}(x)} dx + 2\sum_{k,l=1}^{m} \int\limits_{\Omega}\int\limits_{\Omega} \frac{G_{kl}(x,y)}{H_{kl}(x,y)} dxdy,$$

$$\int\limits_{\widetilde{M}^\varepsilon} f^\varepsilon(\tilde{x})d\tilde{x} \xrightarrow[\varepsilon\to 0]{} \sum_{k=1}^{m} \int\limits_{\Omega} f_k(x)dx,$$

i.e.,

$$\lim_{\varepsilon\to 0} L^\varepsilon = L. \tag{6.23}$$

The formula (6.22) give a heuristic idea to obtain the constant L.

Let $\delta > 0$ be an arbitrary number, let's fix t. We have

$$\|u(\cdot,t) - L\|_0^2 \leq 3 \cdot \left\{ \|Q^\varepsilon u^\varepsilon(\cdot,t) - u(\cdot,t)\|_0^2 + \|Q^\varepsilon u^\varepsilon(\cdot,t) - L^\varepsilon\|_0^2 + \|L^\varepsilon - L\|_0^2 \right\} \tag{6.24}$$

From Theorem 3 and (6.23) it follows, that there exists such $\varepsilon > 0$ that

$$\|Q^\varepsilon u^\varepsilon(\cdot,t) - u(\cdot,t)\|_0^2 + \|L^\varepsilon - L\|_0^2 + (L^\varepsilon)^2 \sum_{k=1}^{m} \sum_{i=1}^{N(\varepsilon)} |D_k^{\varepsilon i}| \leq \delta.$$

Then

$$\|u(\cdot,t) - L\|_0^2 \leq 3\delta + 3 \cdot c_1 \cdot \exp\left[\frac{-2t}{c_p}\right].$$

Passing to the limit as $\delta \to 0$ we have

$$\|u(\cdot,t) - L\|_0^2 \leq 3 \cdot c_1 \cdot \exp\left[\frac{-2t}{c_p}\right].$$

Theorem 2 is proved.

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