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Chaotic soliton walk in periodically modulated media

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Abstract

We show that a weak transverse spatial modulation in (2+1) nonlinear Schrödinger equation with saturable nonlinearity can result in nontrivial dynamics of radially symmetric solitons. In particular, in the case of hexagonal profile of the modulation the soliton moves chaotically.

The nonlinear Schrödinger (NLS) equation plays a central role in understanding various physical phenomena in plasma physics, hydrodynamics, Bose-Einstein condensation, nonlinear optics. In particular, NLS describes pulse propagation in nonlinear fibers and self-focusing of paraxial beams of light in a homogeneous Kerr medium [1]. In the case of purely cubic nonlinearity, (2+1)-dimensional NLS possesses a localized solution, known as Townes mode [2]. However, this solution is always unstable: small perturbations lead to a collapse, i.e. to an unbounded growth of the field amplitude within a finite time interval. A suppression of the collapse can be achieved by various means. In particular, replacing the cubic nonlinearity with a saturable one achieves an arrest of the collapse and a stable self-collimated propagation of a light beam [3]. In a spatially homogeneous medium the paraxial beam propagates with a constant velocity along a straight line. However, because of recent developments in fabrication of microstructured wave-guiding materials known as photonic crystals [4], there is a growing interest to the study of nonlinear beam propagation in various inhomogeneous settings [1, 5, 6, 7, 8].

In our paper we study mobility properties of the stable solitons of (2+1)-dimensional NLS equation with a saturable nonlinearity in the situation where the refractive index of the medium is modulated periodically in the transverse directions. It is known that the effective particle approach is very efficient in such type of problems [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. In particular, in the case of weak modulation amplitude we apply this approach to derive an equation, which describes the soliton as a Newtonian particle in the external potential created by the refractive index profile. This remains valid independently of the ratio of the soliton transverse size to the modulation period, even when the soliton is quite wide.

We show that like a particle in a two-dimensional potential, the soliton in the medium with a transversely modulated refractive index can move both in a regular and chaotic manner, and the choice between these two types of motion is foremost determined by the geometry of the refractive index profile. Thus, when the refractive index forms a rectangular lattice, the effective potential is integrable, and the soliton transverse motion is very close to integrable one for long time intervals. In this case there are two typical dynamical regimes: the first corresponds to low-energy quasi-periodic oscillations around local maximum of the refractive index (minimum of the effective potential), the second corresponds to quasi-periodic oscillations superimposed on a constant velocity drift. In the case of hexagonal lattice the situation is drastically different. Here, with the increase of energy the oscillations near a local maximum of refractive index become chaotic and transform into a random walk – an unbounded transverse motion of the soliton wandering chaotically between different cells of the refractive index profile. Thus, our results show that even in simple periodic media a soliton can exhibit very complicated motion patterns.

Consider the equation

$$\partial_t A = i\Delta A + Af\left(|A|^2\right) + i\varepsilon^2 g\left(\mathbf{r}\right) A, \tag{1}$$

where $\mathbf{r} = (x, y)$, $\Delta = \partial_{xx} + \partial_{yy}$, and $A(\mathbf{r}, t)$ is a complex field amplitude. Note that when f is purely imaginary, Eq. (1) is Hamiltonian, with the energy functional given by

$$H = \frac{1}{2}\int\left[|\partial_x A|^2 + |\partial_y A|^2 + \Phi\left(|A|^2\right) - \varepsilon^2 g(x,y)|A|^2\right] dxdy,$$

where $\Phi' \equiv if$. The conservation of energy H means that the purely imaginary f corresponds to the light propagation in a transparent medium. In order to ensure the stability of the soliton we use the saturable nonlinearity [3]

$$f\left(|A|^2\right) = \frac{-i}{1 + |A|^2}. \tag{2}$$

An important feature of Eq. (1) is that at $\varepsilon = 0$ it is invariant with respect to the Galilean transformation to a moving coordinate frame:

$$A(\mathbf{r}, t) \to A\left(\mathbf{r} - \mathbf{v}t, t\right)\exp\left(i\mathbf{r}\cdot\mathbf{v}/2 - i|\mathbf{v}|^2 t/4\right). \tag{3}$$

It follows that for $\varepsilon = 0$ any stationary solution of Eq. (1) coexists at with a family of uniformly moving solutions parameterized by the velocity vector $\mathbf{v}$ (here and below we use bold-face letters to denote spatial 2-component vectors, while the central dot denotes a scalar product of such vectors.

The term $i\varepsilon^2 g$ in the right-hand side of Eq. (1) with small ε and real $g\left(\mathbf{r}\right)$ describes the spatial variation of the refractive index profile. If g is not spatially homogeneous, the translational and, hence, Galilean symmetries are broken at non-zero ε, which results in a non-trivial motion of the soliton in the (x, y)-plane.

The propagation of paraxial light beams in a dissipative media can be described by the same equation (1), where the function f in the right-hand side is no longer purely imaginary. In our simulations we take f real:

$$f\left(|A|^2\right) = -1 + \frac{G}{1 + |A|^2} - \frac{Q}{1 + s|A|^2}, \tag{4}$$

where G and Q are linear gain and, respectively, absorption coefficients, and $s > 1$ is the ratio of the saturation intensities of the gain and absorber media [19].

In this case, the Hamiltonian structure of the equation is lost, while the Galilean symmetry is preserved at $\varepsilon = 0$. As we show below, the presence of this symmetry results in a great similarity between the character of soliton motion in the conservative and dissipative cases, in spite of the difference between the physical mechanisms of the soliton formation.

Let Eq. (1) at $\varepsilon = 0$ have a radially symmetric stationary soliton $A(\mathbf{r}, t) = A_0(r)e^{i\omega_0 t}$, where $A_0 \to 0$ exponentially fast as $r \to \infty$ (we denote $r = |\mathbf{r}|$). Since the equation at $\varepsilon = 0$ is symmetric with respect to spatial translations, the vector-function

$$\mathbf{U} = \nabla A = \frac{\mathbf{r}}{r} A_0'(r)$$

satisfies $L\mathbf{U} = 0$, where the operator

$$L : X \mapsto [i(\Delta - \omega_0) + f(E_0) + E_0 f'(E_0)] X + A_0^2 f'(E_0) X^*$$

yields the linearization of the right-hand side of Eq. (1) at the soliton solution. Here the star denotes complex conjugation, and $E_0 = |A_0|^2$.

Note that the Galilean symmetry of Eq. (1) implies the existence of the vector-function $\mathbf{Z}$ such that $L\mathbf{Z} = \mathbf{U}$. By differentiating formula (3) with respect to $\mathbf{v}$, it is easy to find that $\mathbf{Z} = -i\mathbf{r}A_0(r)/2$.

Let us define the following inner product of the functions X and Y:

$$\langle X, Y \rangle = \int (XY + X^* Y^*) dx dy.$$

According to this definition, the adjoint to L operator $L^\dagger$ reads as

$$L^\dagger : X \mapsto [i(\Delta - \omega_0) + f(E_0) + E_0 f'(E_0)] X + \left[A_0^2 f'(E_0)\right]^* X^*.$$

Like L, the operator $L^\dagger$ has a non-trivial odd solution to $L^\dagger \mathbf{U}^\dagger = 0$. Due to the rotational symmetry we can write this solution in the form $\mathbf{U}^\dagger = \frac{\mathbf{r}}{r} U^\dagger(r)$, where $U^\dagger(r)$ is a scalar function. An easy computation gives

$$\left\langle Z_x, U_x^\dagger \right\rangle = \left\langle Z_y, U_y^\dagger \right\rangle = \int \Psi(r) dx dy,$$

where $\Psi(r) = \int_r^{+\infty} \mathrm{Im}\left[U^\dagger(r') A_0(r')\right] dr'$, while the functions $Z_{x,y}$ and $U_{x,y}$ denote the components of the vector-functions $\mathbf{Z}$ and $\mathbf{U}^\dagger$.

Below we assume that $\int \Psi(r) dx dy \neq 0$, i.e. $\left\langle \mathbf{Z}, \mathbf{U}^\dagger \right\rangle \neq 0$, which means that there is no solution to $L\mathbf{X} = \mathbf{Z}$. Note that in the Hamiltonian case where f is purely imaginary and A_0 is real, it is easy to see that $L^\dagger(iX) = iLX$, which implies that

$$\mathbf{U}^\dagger = i\mathbf{U} = i\frac{\mathbf{r}}{r} A_0'(r), \quad \text{and} \quad \Psi(r) = -\frac{1}{2} A_0^2(r).$$

In the non-Hamiltonian case these relations are no longer true, and we do not have explicit formulas for $\mathbf{U}^\dagger$ and Ψ.

At non-zero ε we will be looking for a slowly moving soliton solution in the form of series expansion

$$\left\{A_0\left[\left|\mathbf{r}-\mathbf{R}\left(\varepsilon t\right)\right|\right]+\varepsilon A_1\left[\mathbf{r}-\mathbf{R}\left(\varepsilon t\right),\varepsilon t\right]+\varepsilon^2 A_2\left[\mathbf{r}-\mathbf{R}\left(\varepsilon t\right),\varepsilon t\right]+\ldots\right\}e^{i\omega_0 t}, \tag{5}$$

where $\mathbf{R}$ is the soliton center position and $A_{1,2,\ldots}$ describe a small correction to the soliton shape. Substituting expansion (5) into Eq. (1) and collecting first order terms in ε we find that

$$LA_1 = -\dot{\mathbf{R}}\cdot\nabla A_0 \equiv -\dot{\mathbf{R}}\cdot\mathbf{U}$$

(the dot over $\mathbf{R}$ denotes the derivative with respect to the slow time εt). Since $\mathbf{U}=L\mathbf{Z}$, we can take $A_1=-\dot{\mathbf{R}}\cdot\mathbf{Z}$.

Now, collecting the second order terms in ε we obtain

$$LA_2 \;=\; -\ddot{\mathbf{R}}\cdot\mathbf{Z}(\mathbf{r}-\mathbf{R})-ig(\mathbf{r})A_0(|\mathbf{r}-\mathbf{R}|)-\mathcal{F}(\mathbf{r}-\mathbf{R}), \quad \text{with} \quad \mathcal{F}(-\mathbf{r})=\mathcal{F}(\mathbf{r}).$$

According to the Fredholm alternative, the solvability of this equation with respect to A_2 requires the orthogonality of its right-hand side to the solutions of the homogeneous equation $L^\dagger X=0$. So, by taking the inner product of the right-hand side to $U_x^\dagger(\mathbf{r}-\mathbf{R})$ and $U_y^\dagger(\mathbf{r}-\mathbf{R})$, and noticing that $\langle\mathcal{F},U_{x,y}^\dagger\rangle=0$, we get the following necessary solvability condition:

$$\ddot{\mathbf{R}}\int\Psi(r)dxdy-2\int g\left(\mathbf{r}+\mathbf{R}\right)\mathrm{Im}\left[\mathbf{U}^\dagger(\mathbf{r})A_0(r)\right]dxdy=0.$$

Integrating by parts, we obtain finally the following equation for the soliton motion:

$$\ddot{\mathbf{R}}=-\nabla V(\mathbf{R}), \tag{6}$$

where

$$V(\mathbf{R})=-2\frac{\int g\left(\mathbf{r}+\mathbf{R}\right)\Psi(r)dxdy}{\int\Psi(r)dxdy}. \tag{7}$$

As we mentioned, $\Psi=-A_0^2/2$ in the Hamiltonian case. Formula (7) generalizes the expression for the effective potential obtained for the integrable case of 1D NLS in Refs. [11, 17].

As we see, both in transparent and active-dissipative media, the transverse soliton motion is described, to the leading order, by the Hamiltonian equation (6). This is the equation of a unit mass particle moving in the external potential. Up to the factor of (-2) the potential is obtained by averaging the refractive index g with a weight determined by the soliton intensity. Note that Eqs. (6) and (7) are valid for arbitrary ratio of the soliton width to the characteristic period of the refractive index modulation. When this ratio is small, we obtain $V\left(\mathbf{R}\right)=-2g\left(\mathbf{R}\right)$. As the ratio grows, the averaging smooths the inhomogeneity of the refractive index. Therefore, when the soliton is sufficiently wide it moves, essentially, like a free particle.

Being a Hamiltonian system with two degrees of freedom, equation (6) may exhibit both regular and chaotic dynamical regimes, depending on the shape of the potential and on the value of energy. Below we show that the transverse dynamics of the soliton depends strongly indeed on the structure of the refractive index profile. We consider square ($g = g_4$) and hexagonal ($g = g_3$) lattices, defined by

$$g_j\left(\mathbf{r}\right) = -\sum_{l=0}^{l=j-1} \cos(\vec{k}_{jl} \cdot \vec{r}), \quad \vec{k}_{sl} = k \left[\begin{array}{c} \cos(\pi l/j) \\ \sin(\pi l/j) \end{array} \right].$$

Here $j = 4$ and $k = \sqrt{2}\pi/d$ for the square lattice, $j = 3$ and $k = 4\pi/\left(3d\right)$ for the hexagonal lattice, with d being the size of the lattice cell defined as the distance between a local maximum of g and its nearest local minimum. According to Eq. (7), the effective potential that governs the soliton transverse motion is given by

$$V\left(x,y\right) = -g(x,y)S\left(k\right),$$

where the response coefficient $S\left(k\right)$ is defined by

$$S\left(k\right) = 2\frac{\int \cos\left(kx\right)\Psi(r)dxdy}{\int \Psi(r)dxdy}.$$

In the limit of a narrow soliton ($k \rightarrow 0$), we have $S\left(k\right) \rightarrow 2$, while in the opposite limit $k \rightarrow \infty$ the response function decays exponentially. In Fig. 1 we plot the graph of amplitude profile and the response coefficient $S(k)$ for the solitons whose dynamics we studied in our numeric simulations. As it can be seen from the figure, $S(k)$ is not negligibly small for the soliton sizes up to roughly twice the size of the lattice cell.

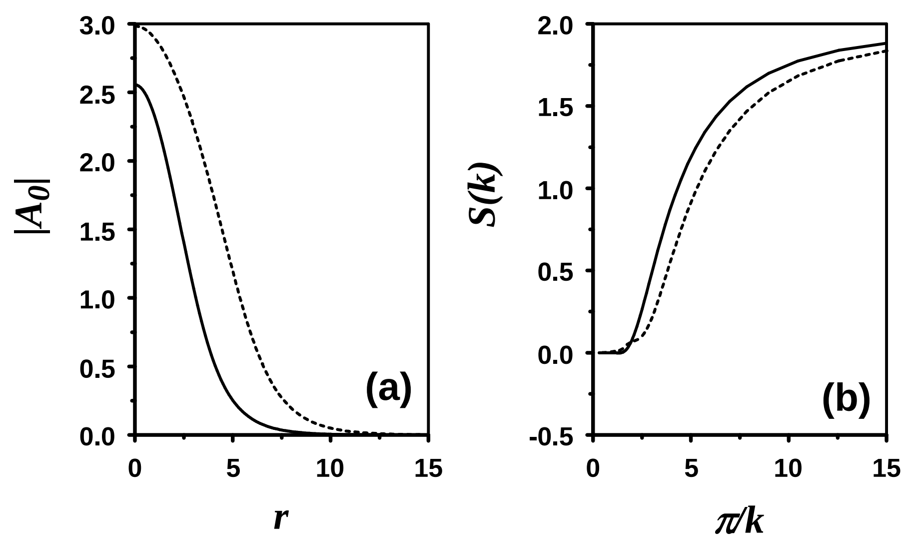

Figure 1: Considered soliton amplitude profiles (a) and the response function $S(k)$ (b). Solid and dotted lines correspond, respectively, to conservative nonlinearity (2) and dissipative nonlinearity (4) with $G = 2.11$, $Q = 2.0$, and $s = 10$.

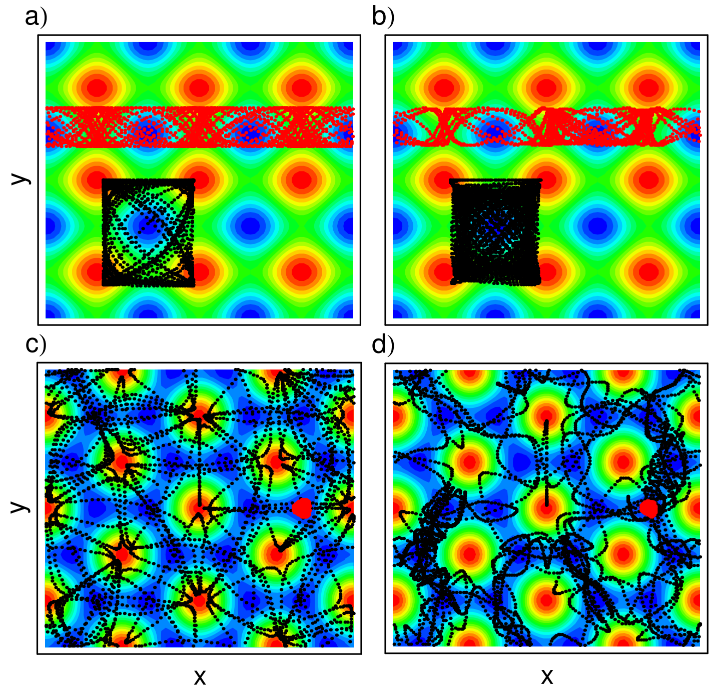

Figure 2: Regular and chaotic soliton motion (black and red curves) in the refractive index profile forming square (a,b) and hexagonal (c,d) lattice. Blue (red) color indicates higher (lower) values of the refractive index. The left two figures (a,c) correspond to conservative nonlinearity (2) and the right two figures (b,d) – to dissipative nonlinearity (4) with the parameter values given in the caption of Fig. 1.

In the case of square lattice, the effective potential

$$V = S(k)\left[\cos(kx) + \cos(ky)\right]$$

is separable, and therefore Eq. (6) is integrable, which means a quasiperiodic motion for the soliton. This result is confirmed by direct numerical integration of Eq. (1). Indeed, as we see in Figs. 2a,b, the soliton in the square lattice is either trapped in a lattice cell and oscillates quasiperiodically in it, or the quasiperiodic oscillations accompany a constant velocity drift. This picture is the same both for the conservative nonlinearity (2) and for the active-dissipative case (4). However, in the

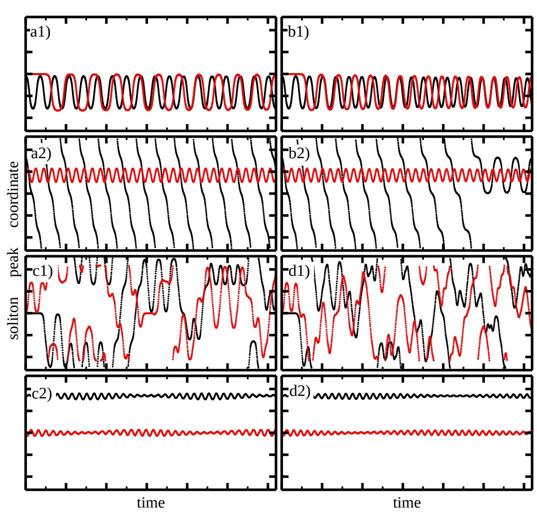

Figure 3: Time dependencies of the coordinates x (black) and y (red) of the soliton moving in square and hexagonal lattices. The figures (a1) and (a2), (b1) and (b2), (c1) and (c2), (d1) and (d2) present the timetraces corresponding to the trajectories shown in figures 2a, 2b, 2c, 2d, respectively.

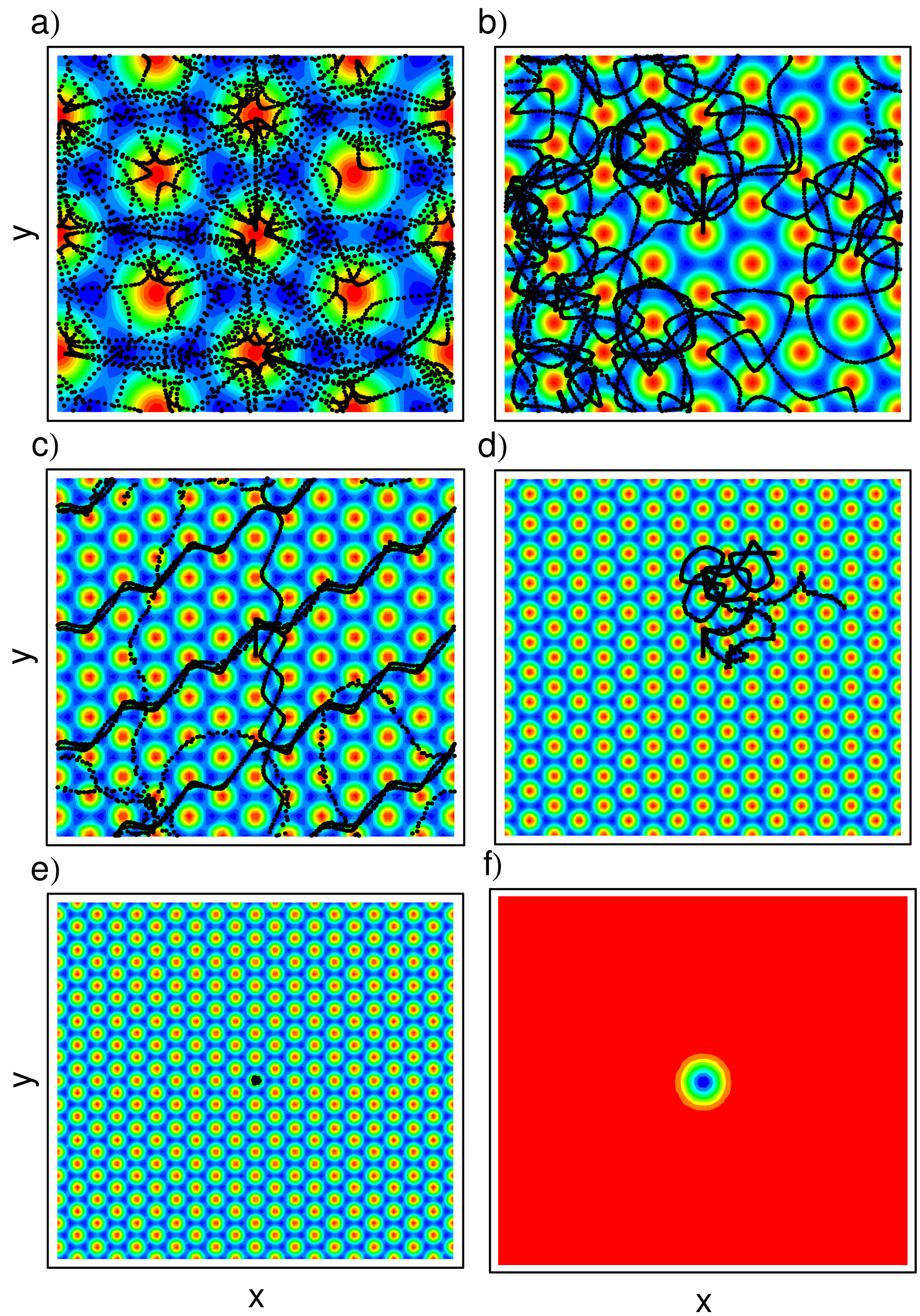

Figure 4: Soliton motion in hexagonal refractive index profiles with different size $d = 8\pi/(3n)$ of an elementary cell. Here, $n = 1$ (panel a), $n = 2$ (b), $n = 3$ (c) $n = 4$ (d), and $n = 5$ (e). The soliton size is approximately visible in panel f).

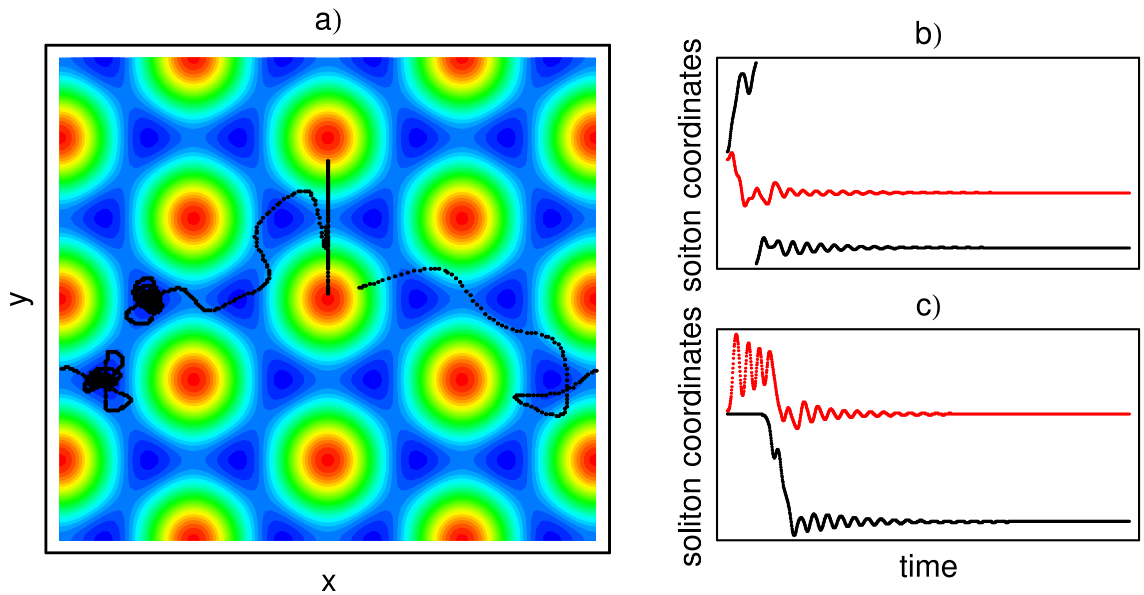

Figure 5: Soliton motion in a medium with nonzero spatial spectral filtering coefficient $\delta = 0.01$. Other parameter values are the same as in Figs. 2d and 3d1,d2.

non-Hamiltonian case we may see (Fig. 3b) a slow decay in the oscillation amplitude, due to higher order corrections which we neglected in our derivation of Eq. (6).

The hexagonal refractive index lattice induces a different type of soliton motion, as the hexagonal potential is known to create chaotic dynamics [20]. Namely, while low-energy oscillations near the minimum of the potential remain typically quasiperiodic (by KAM-theorem), the increase of the energy leads to a random walk between the cells. The numerical simulations both for conservative (2) and active-dissipative (4) nonlinearities in Eq. (1) confirms this conclusion, showing either low-amplitude regular oscillations or chaotic wanderings of the soliton center, see Figs. 2c,d and 3c,d.

The effect of the decrease of the lattice cell size d on the soliton motion is illustrated in Fig. 4. For soliton diameters up to $\sim 4d$ we see a behavior resembling that of a particle in the potential. In Fig. 5 we present the results of numerical integration of Eqs. (1), (4) with the term $\delta \Delta A$ added into the right-hand side. This term with real and small $\delta > 0$ corresponds to a spatial spectral filtering, which breaks the Galilean symmetry of Eq. (1) at $\varepsilon = 0$. One can show that taking it into account results in adding the dissipation term $\gamma \dot{\mathbf{R}}$ in the left-hand side of Eq. (6), with $\gamma = O(\delta/\varepsilon)$. Indeed, as we see in Fig. 5, we have at $\delta \neq 0$ a steady decrease of the amplitude of oscillations, regular and chaotic alike. In this case the soliton transverse motion finally halts at some position corresponding to a local maximum of the refractive index.

To conclude: we have established a chaotic character of motion of a soliton on hexagonal lattice engages itself in chaotic motion, even for a weak amplitude of the spatial modulation. The effect occurs both in the conservative case and in the case of dissipative solitons.

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