Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

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submitted: 2nd December 1994

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Preprint No. 126
Berlin 1994

1991 Mathematics Subject Classification. 62F05, 62F12, 62G20.
Key words and phrases. Moderate large deviations, Cramer type large deviations, asymptotic efficiency, asymptotically minimax estimation, asymptotically minimax hypothesis testing, Bahadur efficiency, Chernoff efficiency.

This work is partially supported by Weierstrass Institute for Applied Analysis and Stochastics.

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ON LOWER BOUNDS OF THE MODERATE AND CRAMER TYPE LARGE DEVIATION PROBABILITIES IN STATISTICAL INFERENCE

Mikhail S. Ermakov

Summary. We indicate new simple assignments of the lower bounds for the probabilities of the moderate and Cramer type large deviations of type I and type II errors of statistical tests. These assignments are based on a one natural property of the normal distribution. Using these results we deduce easily the lower bounds for the probabilities of the moderate and Cramer type large deviations of estimators. The lower bounds were obtained under the more weak assumptions then in the previous papers. The lower bound for the probabilities of the Cramer type large deviations of estimators has not been proved earlier. The results are also extended on the problems of asymptotically minimax statistical inference about a value of functional.

Key words and phrases: moderate large deviations, Cramer type large deviations, asymptotic efficiency, asymptotically minimax estimation, asymptotically minimax hypothesis testing, Bahadur efficiency, Chernoff efficiency.

AMS 1980 *subject classifications*: 62F05, 62F12, 62G20

1. Introduction

Nowadays the theory of the large deviations of tests and estimators has been obtained the comprehensive development. A wide range of problems of statistical inference has been studied on the base of the Bahadur (1960), Chernoff (1952) and Hodges-Lehmann (1956) efficiencies. The moderate and Cramer type large deviations of tests and estimators were also considered in many papers. However for such types of large deviations the problem of asymptotic efficiency has been treated by the unit authors. Up to the last time there existed only the Kallenberg (1983) intermediate efficiency for the characterization of the moderate large deviations and did not exist any asymptotic efficiency for the Cramer type large deviations.

In the last years the essential progress has been made by Borovkov and Mogulskii (1992a),(1992b), Ermakov (1990), (1993) and Radavichius (1991). In Ermakov (1990),(1993) a new natural lower bound has been proposed for the probabilities of the moderate large deviations in the problems of hypothesis testing. This lower bound allowed to introduce a new type of efficiency called in this paper by the moderate large deviation or MLD efficiency. Radavichius (1993) has obtained the lower bound for the moderate large deviations of estimators. This lower bound can be interpreted as intermediate between the Hajek–Le Cam (1972) and Bahadur (1960) lower bounds. At the same time it is easy to show that the Radavichius (1991) lower bound is a particular case of the lower bound for the MLD-efficiency in hypothesis testing.

The comprehensive analysis of the moderate and Cramer type large deviations of the Bayes and maximum likelihood ratio tests has been made by Borovkov and Mogulskii (1992a),(1992b). On the base of these results they extended the lower bound for the MLD efficiency on the multivariate case and proposed a similar version of this lower bound for the Cramer type large deviations. Note that their version of the last lower bound has another assignment than in this paper and has been proved under the strong assumptions.

Our results are based on the following property of the normal distribution.

Let $X_1, \ldots, X_n$ be Gaussian i.i.d.r.v.'s, $EX_1 = \theta$, $DX_1 = 1$. Suppose a hypothesis $\theta = 0$ has to be tested versus alternatives $\theta = u_n$, $u_n \to 0$, $nu_n^2 \to \infty$ as $n \to \infty$. For any test $K_n = K_n(X_1, \ldots, X_n)$ let $\alpha_n(K_n) = E_0 K_n$ be its level and let $\beta_n(K_n) = \beta(K_n, u_n) = E_{u_n}(1 - K_n)$ be its type II error probability. Put $\bar{X}_n = n^{-1}(X_1 + \ldots + X_n)$ and define the tests

$L_n = \chi(\bar{X}_n > c_n)$, $0 < c_n < u_n$. Here $\chi(A)$ denotes the indicator of event A. Denote by $\Phi(s) = (2\pi)^{-1/2} \int_{-\infty}^{s} \exp\{-x^2/2\}dx$, $s \in R^1$, the standard normal distribution function. Define the inverse function $s = \Phi^{-1}(y)$ by the equation $y = \Phi(s)$. Since $\Phi^{-1}(\alpha_n(L_n)) = -n^{1/2}c_n$, $\Phi^{-1}(\beta_n(L_n)) = -n^{1/2}(u_n - c_n)$ then

$$\Phi^{-1}(\alpha_n(L_n)) + \Phi^{-1}(\beta_n(L_n)) = -n^{1/2}u_n. \qquad (1.1)$$

Hence for any sequence of tests K_n

$$\liminf_{n\to\infty} (nu_n^2)^{-1/2}(\Phi^{-1}(\alpha_n(K_n)) + \Phi^{-1}(\beta_n(K_n))) \ge -1 \qquad (1.2)$$

and if $\alpha_n(K_n) < c < 1$, $\beta_n(K_n) < c < 1$ additionally then we obtain the lower bound for the LD efficiency proved in Ermakov [5]

$$\limsup_{n\to\infty} n^{-1/2}u_n^{-1}(|2\log\alpha_n(K_n)|^{1/2} + |2\log\beta_n(K_n)|^{1/2}) \le 1. \qquad (1.3)$$

The more exact version of (1.2) is as follows

$$\liminf_{n\to\infty} n^{1/2}u_n\{n^{1/2}u_n + \Phi^{-1}(\alpha_n(K_n)) + \Phi^{-1}(\beta_n(K_n)\} \ge 0. \qquad (1.4)$$

The similar results are also valid for the problem of hypothesis testing $\theta = 0$ against the twosided alternatives $|\theta| = u_n$. For any test K_n let $\beta(K_n, u_n)$, $\beta(K_n, -u_n)$ be respectively its type II error probabilities for the alternatives $\theta = u_n$ and $\theta = -u_n$. Denote

$$\beta_n(K_n) = \sup\{\beta(K_n, u_n), \beta(K_n, -u_n)\}.$$

Then the relation (1.3) remains valid. Define the function $s = \bar{\Phi}^{-1}(y)$ by the equation $y = 2\Phi(s)$. Put $L_n = \chi(|X_n| > c_n)$. Then

$$2n^{1/2}u_n + 2\bar{\Phi}^{-1}(\alpha_n(L_n)) + \Phi^{-1}(\beta(L_n, u_n) + \Phi^{-1}(\beta(L_n, -u_n) = 0$$

and for any sequence of tests K_n

$$\liminf_{n\to\infty} n^{1/2}u_n(n^{1/2}u_n + \bar{\Phi}^{-1}(\alpha_n(K_n)) + \Phi^{-1}(\beta_n(K_n)) \ge 0. \qquad (1.5)$$

It is easy to see that the equality in (1.2) or (1.3) takes place if and only if for any sequence of tests N_n, $\alpha_n(N_n) \le \alpha_n(K_n)$,

$$\limsup_{n\to\infty} \log\beta_n(K_n)/\log\beta_n(N_n) \le 1. \qquad (1.6)$$

At the same time if we assume additionally that the $nu_n^3 \to 0$ as $n \to \infty$ then the equality in (1.4) or (1.5) implies that for any sequence of tests $N_n, \alpha_n(N_n) \le \alpha_n(K_n)$,

$$\limsup_{n \to \infty} \beta_n(K_n)/\beta_n(N_n) \le 1. \qquad (1.7)$$

Thus the inequalities (1.2)–(1.5) can be considered as the lower bounds for the moderate and Cramer type large deviation probabilities of sequences of tests.

The purpose of the paper is to show that these lower bounds are valid for arbitrary regular families of distribution and to deduce easily on the base of these results the lower bounds for the moderate and Cramer type large deviations of estimators. After that the similar lower bounds will be obtained for the problem of asymptotically minimax statistical inference about a value of functional. For the estimation problems the lower bound of the Cramer type large deviations is proved in the traditional local asymptotic minimax setting. At the same time we show that the lower bound for the moderate large deviations can be obtained as a lower bound in essentially more simple problem of estimation of one from two possible values of parameter. In the paper we do not indicate directly the lower bounds for the problems of testing composite parametric hypothesis. As wellknown such results follow easily from the lower bounds for the simple hypothesis. The asymptotic minimax lower bounds for the problem of hypothesis testing about a value of functional confirm excelently this assertion.

On the base of the proposed lower bounds we introduce two new asymptotic efficiencies. A sequence of tests or estimators is called moderate large deviation or MLD asymptotic efficient if a lower bound for its probabilities of the moderate large deviations is achieved. If a lower bound for the probabilities of the Cramer type type large deviations is achieved we say that the corresponding sequence of tests or estimators is strong large deviation or SLD asymptotic efficient.

As abovementioned in the problems of hypothesis testing there exists a wide range of different asymptotic efficiencies. In this connection it is natural to discuss the advantages of the new asymptotic efficiencies in comparison with the asymptotic efficiencies proposed earlier. As it follows from the lower bounds (1.2)–(1.5) in the case of the MLD and SLD asymptotic efficiencies the type I and type II error probabilities can tend to zero simultaneously with increasing sample size. Thus, the MLD and SLD asymptotic efficiencies allow us to study the widespread problems of hypothesis testing with the small probabilities of the type I and type II errors. Earlier the asymptotic behaviour of test statistics in this situation has been characterised only on the base of the Chernoff efficiency. For the Pitman, Kallenberg intermediate, Bahadur, Hodges-Lehmann efficiencies at least one of the error probabilities is supposed to be fixed. The MLD-efficiency has also another merit. The Kallenberg intermediate and local Bahadur, Chernoff, Hodges-Lehmann efficiencies turn out to be particular cases of the MLD efficiency. From our point of view these arguments justify such general names of the new efficiencies.

The results of the paper show that the MLD and SLD efficiencies are closely connected with the Pitman efficiency in hypothesis testing and locally asymptotically minimax efficiency in estimation. Although these efficiencies correspond to different situations, they have the Fisher information as a common lower bound. Thus for the essentially more wide spectrum of the problems of statistical inference there exists the common measure of efficiencies and this measure equals to that of the traditional efficiencies in estimation and hypothesis testing.

In the paper a large number of different positive absolute constants will be used. All these constants will be denoted by the letters c, C.

2. Lower bounds of the MLD and SLD-asymptotic efficiencies

2.1. General setting. In the most general form the results will be obtained in the terms of the Hellinger metric. The Hellinger metric gives the best approximation of the Kullback-Leibler information than the Fisher one and this turns out essential in the problems of the moderate and Cramer type large deviations.

Let Λ be the set of all probability measures on a measurable space (S,B) and let $X_1,\dots X_n$ be i.i.d.r.v.'s with p.m. $P_\theta \in \Lambda$, $\theta \in R^1$. Suppose the p.m.'s P_θ, $\theta \in R^1$, are absolutely continuous with respect to measure $\nu \in \Lambda$ and have the densities $f(x,\theta) = dP_\theta/d\nu(x)$, $x \in S$. The Hellinger distance of p.m.'s $P_{\theta_1}, P_{\theta_2}$, $\theta_1, \theta_2 \in R^1$, equals

$$\rho(P_{\theta_1}, P_{\theta_2}) = \rho(\theta_1, \theta_2) = (\int_S (f^{1/2}(x,\theta_1) - f^{1/2}(x,\theta_2))^2 d\nu)^{1/2}$$

Fix $\theta = t$ and denote by P_θ^a, P_θ^s the absolutely continuous and singular components of p.m. P_θ, $\theta \in R^1$ w.r.t. p.m. P_t. Put $g(x,t+u) = ((f(x,t+u)/f(x,t))^{1/2} - 1)f^{-1/2}(x,t)$ for all $x \in S$, $u \in R^1$.

The main results are proved under the following assumption.

A. There exists a function $\omega(u)$, $\omega(u) < r\omega(u/r)$ for all $r \geq 1$, $\omega(u) \to 0$ as $u \to 0$, such that $P_{t+u}^s(S) = O(u^2\omega(u))$ as $u \to 0$ and

$$\int_S g^2(x,t+u)\chi(|g(x,t+u)| > r^{-1})dP_t < C\rho^2(t,t+u)\omega(ur) \qquad (2.1)$$

for any $r > 1$ and all $u \in R^1$.

Note that A does not imply the existence of the Fisher information. At the same time the existence of the Fisher information implies (2.1) with some function $\omega(u)$, $\omega(u) \to 0$ as $u \to 0$.

The statistical experiment $E = \{(B,S), P_\theta, \theta \in R^1\}$ has the finite Fisher information at the point $\theta = t$ if there exists the function $\varphi(x) = 1/2 f_\theta(x,t)f^{-1}(x,t) \in L_2(P_t)$ such that

A0. $\int_S (g(x,t+u) - u\varphi(x))^2 dP_t = o(u^2), \qquad P_{t+u}^s(S) = o(u^2) \qquad (2.2)$

as $u \to 0$.

The Fisher information equals $I = I(t) = 4\int_S \varphi^2(x) dP_t$.

Let a finite Fisher information exist at the point $\theta = t$. Then A fulfilled under the following sufficient conditions.

A1. For all $r \geq 1$ and all $u \in R^1$

$$\int_S (g(x,t+u)-u\varphi(x))^2 \chi(|g(x,t+u)-u\varphi(x)| > r^{-1}) dP_t \leq Cu^2\omega(ur) \qquad (2.3)$$

and $P^s_{t+u}(S) = O(u^2\omega(u))$ as $u \to 0$.

A2. $\quad \int_S \varphi^2(x)\chi(|\varphi(x)| > u^{-1})dP_t < C\omega(u), \qquad u \in R^1. \qquad (2.4)$

The results can be expressed in the terms of the Fisher information if one of the following additional assumptions is valid.

A3. $\quad |\int_S g^2(x,t+u)dP_t - Iu^2| < Cu^2\omega(u), \qquad u \in R^1. \qquad (2.5)$

A4. $\quad \int_S (g(x,t+u) - u\varphi(x))^2 dP_t < Cu^2\omega^2(u), \qquad u \in R^1 \qquad (2.6)$

and $P^s_{t+u}(S) = O(u^2\omega(u))$ as $u \to 0$.

In section 4 we show that A1–A3 follows from A2,A4.

Let $\omega(u) = u^\gamma$, $0 < \gamma \leq 1$, and let

$$g_\gamma(x,t+u) = (f(x,t+u)/f(x,t))^{1/(2+\gamma)} - 1.$$

Then A holds if

$$\int_S g_\gamma^{2+\gamma}(x,t+u)dP_t < Cu^{2+\gamma}, \qquad u \in R^1 \qquad (2.7)$$

and $P^s_{t+u}(S) = O(u^{2+\gamma})$ as $u \to 0$.

The assumptions A1,A2 are valid if

$$\int_S \varphi^{2+\gamma}(x)dP_t < C, \qquad (2.8)$$

$$\int_S (g_\gamma(x,t+u) - 2/(2+\gamma)u\varphi(x))^{2+\gamma}dP_t < Cu^{2+\gamma}, \qquad u \in R^1 \qquad (2.9)$$

and $P^s_{t+u}(S) = O(u^{2+\gamma})$ as $u \to 0$.

2.2. MLD and SLD asymptotic efficiencies in hypothesis testing. First we indicate the lower bounds of the MLD efficiency and discuss the relation of the MLD efficiency and the Pitman, Kallenberg intermediate, local Bahadur, Chernoff, Hodges–Lehmann efficiencies. Then Theorems 2.2,2.3 about the lower bounds of the SLD efficiency will be given.

Suppose the hypothesis $H_{t_{n1}} : \theta = t_{n1} = t + v_n$ has to be tested versus the alternatives $H_{t_{n2}} : \theta = t_{n2} = t + v_n + u_n$ with $v_n \to 0, u_n \to 0$, $nu_n^2 \to \infty$ as $n \to \infty$. For any test K_n let $\alpha_n(K_n) = \alpha(K_n, P_{t_{n1}})$ and $\beta_n(K_n) = \beta(K_n, P_{t_{n2}})$ be respectively its probabilities of the type I and type II errors. Denote $\rho_n = \rho(t_{n1}, t_{n2})$.

Theorem 2.1. *Assume* A. *Let K_n be a sequence of tests for testing a hypothesis $H_{t_{n1}}$ against the alternatives $H_{t_{n2}}$ such that*

$$|(n^{1/2}\rho_n)^{-1}\Phi^{-1}(\alpha_n(K_n))| < C.. \qquad (2.10)$$

Then

$$\limsup_{n\to\infty}(2n^{1/2}\rho_n)^{-1}|\Phi^{-1}(\alpha_n(K_n)) + \Phi^{-1}(\beta_n(K_n))| \le 1. \qquad (2.11)$$

The equality in (2.11) *is achieved on the sequences of the likelihood ratio tests L_n satisfiing the same assumption*

$$|(n^{1/2}\rho_n)^{-1}\Phi^{-1}(\alpha_n(L_n))| < C, \qquad (2.12)$$

that is,

$$\limsup_{n\to\infty}(2n^{1/2}\rho_n)^{-1}|\Phi^{-1}(\alpha_n(L_n)) + \Phi^{-1}(\beta_n(L_n))| = 1. \qquad (2.13)$$

This implies that if $\alpha(K_n) < C < 1$ and $\beta_n(K_n) < C < 1$ then

$$\limsup_{n\to\infty}(2n^{1/2}\rho_n)^{-1}(|\,2\log\alpha_n(K_n)\,|^{1/2} + |\,2\log\beta_n(K_n)\,|^{1/2}) \le 1 \qquad (2.14)$$

and if $\alpha_n(L_n) < c < 1$ and (2.12) *is valid then*

$$\lim_{n\to\infty}(2n^{1/2}\rho_n)^{-1}(|2\log\alpha_n(L_n)|^{1/2} + |2\log\beta_n(L_n)|^{1/2}) = 1. \qquad (2.15)$$

Assume A0. *Then in all relations of the Theorem we can take $2\rho_n = u_n I^{1/2}$.*

Remark 2.1. A similar result is valid for the problem of testing a simple hypothesis against twosided alternatives. Suppose a hypothesis $\theta = t$ has to be tested versus the alternatives $\theta = \theta_{n1} = t + u_n$ or $\theta = t - u_n$. For any test K_n denote $\beta_n(K_n) = \sup\{\beta(K_n, P_{\theta_{n1}}), \beta(K_n, P_{\theta_{n1}})\}$ and put $\rho_n = \hat{\rho_n} = \min\{\rho(t, \theta_{n1}), \rho(t, \theta_{n2})\}$. Then (2.14), (2.15) are valid under the same assumptions. It is easy to see that this assertion is a direct consequence of Theorem 2.1.

Remark 2.2. The lower bound for the Kallenberg intermediate efficiency follows easily from (2.14),(2.15). To obtain this bound it suffices to put $\beta(K_n, P_{nu_n})=$ const < 1.

Similarly to section 1 we say that a sequence of tests K_n satisfiing (2.10) is moderate large deviation or MLD-asymptotically efficient if the right hand-side of (2.11) equals one.

Indicate two important corollaries of Theorem 2.1. These corollaries are given in the terms of the Fisher information to emphasize their connection with the traditional results.

Corollary 2.1. *Assume* A0. *Let* $K_n(u), u \in R^1$, *be a sequence of tests for testing a hypothesis* $\theta = t$ *against the alternatives* $\theta = t + un^{-1/2}$ *such that* $\alpha_n(K_n(u)) < C < 1$ *and* $(Iu^2)^{-1} \mid 2\log\alpha_n(K_n(u)) \mid < C < 1$. *Then*

$$\limsup_{u\to\infty} \limsup_{n\to\infty} (uI^{1/2})^{-1}(\mid 2\log\alpha_n(K_n(u)) \mid^{1/2} +$$

$$\mid 2\log\beta_n(K_n(u)) \mid^{1/2}) \le 1. \qquad (2.16)$$

Remark 2.3. Corollary 2.1 shows that, under the natural assumptions, the LD efficiency turns into the Pitman efficiency. It is clear that (2.16) also follows directly from the lower bound for the Pitman efficiency.

Corollary 2.2. *Assume* A0. *Let* $K_n(v,u)$, $v, u \in R^1$, *be a sequence of tests for testing a hypothesis* $\theta = t + v$ *against the alternative* $\theta = t + v + u$. *Let* $\alpha_n(K_n(v,u)) < C < 1$ *and* $\beta_n(K_n(v,u)) < C < 1$. *Then*

$$\limsup_{r\to 0} \limsup_{n\to\infty} (nu^2I)^{-1/2}(\mid 2\log\alpha_n(K_n(v,u)) \mid^{1/2} +$$

$$\mid 2\log\beta_n(K_n(v,u)) \mid^{1/2}) \le 1 \qquad (2.17)$$

with $r = |v| + |w|$.

If $K_n(v,u)$ *is a sequence of the likelihood ratio tests then the equality takes place in* (2.17). .

Remark 2.4. The lower bounds of the local Bahadur, Chernoff and Hodges-Lehmann efficiencies are the particular cases of (2.17). To obtain these lower bounds it suffices to put in (2.17) $\beta_n(K_n(v,u)) = \beta$=const, $\alpha_n(K_n(v,u)) = \beta_n(K_n(v,u))$, $\alpha_n(K_n(v,u)) = \alpha =$ const respectively. In such a way the MLD

efficiency implies the local Bahadur, Chernoff and Hodges-Lehmann ones. Corollaries 2.1, 2.2 indicate the direct relation of the local Bahadur, Chernoff, Hodges-Lehmann and Pitman efficiencies. Such relations have been studied by Wieand (1976) and Kourauklis (1989),(1990) under the more restricted assumptions.
Remark 2.5. Assume A. Then (2.17) is valid if $u^2 I$ is replaced by $4\rho^2(t+v, t+v+u)$. Note that under the Assumption A0 and $v = 0$ Theorem 2.1 and Corollary 2.2 has been proved in Ermakov (1990),(1993).

Theorem 2.2. *Assume* A. *Let* $u_n > Cn^{-1/2}$, $|v_n| < Cu_n$, $nu_n^2\omega(u_n) \to 0$ *as* $n \to \infty$. *Then for any sequence of likelihood ratio tests* L_n *satisfiing* (2.12)

$$\lim_{n\to\infty} (2n^{1/2}\rho_n)^{-1}(2n^{1/2}\rho_n + \Phi^{-1}(\alpha_n(L_n)) + \Phi^{-1}(\beta_n(L_n))) = 0. \qquad (2.18)$$

This implies that for any sequence of tests K_n *satisfiing* (2.10)

$$\lim\inf_{n\to\infty} (2n^{1/2}\rho_n)^{-1}(2n^{1/2}\rho_n + \Phi^{-1}(\alpha_n(K_n)) + \Phi^{-1}(\beta_n(K_n))) \geq 0. \qquad (2.19)$$

There exists a sequence of events U_n, $P_t(U_n) \to 1$ *as* $n \to \infty$, *such that if the equality takes place in* (2.19) *then*

$$\lim_{n\to\infty} E_{t_{n1}}\{|K_n - \chi(\sum_{i=1}^{n}(g(X_i, t+v_n) + 1/2\rho_n^2) > (n\rho_n^2)^{-1/2}\Phi^{-1}(\alpha_n(K_n)))|\times$$
$$\chi(B_n)\}(\alpha_n(K_n))^{-1} = 0, \qquad (2.20)$$

$$\lim_{n\to\infty} E_{t_{n2}}\{|K_n - \chi(\sum_{i=1}^{n}(g(X_i, t+v_n) + 1/2\rho_n^2) > (n\rho_n^2)^{-1/2}\Phi^{-1}(\alpha(K_n)))|\times$$
$$\chi(B_n)\}(\beta_n(K_n))^{-1} = 0, \qquad (2.21)$$

$$\lim_{n\to\infty} E_{t_{n1}}\{K_n(1-\chi(U_n))\}/\alpha_n(K_n) = 0, \qquad (2.22)$$

$$\lim_{n\to\infty} E_t\{(1-K_n)(1-\chi(U_n))\}/\beta_n(K_n) = 0. \qquad (2.23)$$

If A1-A3 *are also valid then* (2.18),(2.19) *can be replaced by*

$$\lim_{n\to\infty} (nu_n^2 I)^{1/2}((nu_n^2 I)^{1/2} + \Phi^{-1}(\alpha_n(L_n)) + \Phi^{-1}(\beta_n(L_n))) = 0, \qquad (2.24)$$

$$\lim\inf_{n\to\infty} (nu_n^2 I)^{1/2}((nu_n^2 I)^{1/2} + \Phi^{-1}(\alpha_n(K_n)) + \Phi^{-1}(\beta_n(K_n))) \geq 0 \qquad (2.25)$$

respectively.

If A2,A4 *holds then* (2.20),(2.21) *can be replaced by*

$$\lim_{n\to\infty} E_{t_{n1}}\{|K_n - \chi(\sum_{i=1}^{k}\varphi(X_i)(f(X_i,t)/f(X_i,t+v_n))^{1/2} >$$

$$(nI/4)^{1/2}\Phi^{-1}(\alpha_n(K_n)))|\chi(B_n)\}(\alpha_n(K_n))^{-1} = 0, \qquad (2.26)$$

$$\lim_{n\to\infty} E_{t_{n2}}\{|K_n - \chi(\sum_{i=1}^{k} g(X_i,t) > (nI/4)^{1/2}\Phi^{-1}(\alpha_n(K_n)))|\times$$

$$\chi(B_n)\}(\beta_n(K_n))^{-1} = 0 \qquad (2.27)$$

respectively.

Theorems 2.1,2.2 will be proved in section 4.

Remark 2.6. It follows from the proof of Theorem 2.2 that the set U_n can be defined as follows

$$U_n = \{X_1, ..., X_n : |f^{1/2}(X_i,t_{n1}) - f^{1/2}(X_i,t_{n2})| < \epsilon f^{1/2}(X_i,t_{n1}), 1 \le i \le n\}$$

Remark 2.7. Since we do not assume in (2.18)–(2.23) the existence of the Fisher information these results are also of interest for the standard problem of hypothesis testing $\theta = t+vn^{-1/2}$ versus the alternatives $\theta = t+(v+u)n^{-1/2}$.
Remark 2.8. It follows from the Hajek (1970) Theorem that if the regular sequence of estimators $\hat{\theta}_n$ is locally asymptotically minimax then

$$nI^{1/2}(\hat{\theta}_n - t) - 2(nI)^{-1/2}\sum_{i=1}^{n}\varphi(X_i) \to 0$$

in probability as $n \to \infty$. The relations (2.19)–(2.23),(2.26),(2.27) and the further relations (2.29)–(2.32),(2.37),(2.38) are the versions of this assertion for the strong large deviation efficiency.
Example 2.1. Let $X_1, \ldots, X_n$ be i.i.d.r.v.'s with p.m. $P \in \Lambda$ and let the function φ satisfy (2.4) with $P_t = P_0, P_0 \in \Lambda$. Denote

$$\zeta_n(x) = \varphi(x)\chi(|\,\varphi(x)\,| < \epsilon_n u_n^{-1}), \qquad \xi_n(x) = \zeta_n(x) - E(\zeta_n(X_1))$$

where $\epsilon_n \to 0$ as $n \to \infty$. Define the probability measures P_n with the densities $f_n(x) = dP_n/dP_0(x) = 1 + 2u_n\xi_n(x)$. Suppose the problem is to test a hypothesis $P = P_0$ against alternatives $P = P_n$. It is easy to test that for this problem the assumptions A1,A3 are fulfilled and (2.18) is valid with $\rho_n^2 = 1/4\ u_n^2 I$.

Theorem 2.3. *Assume* A. *Let* $u_n > Cn^{-1/2}$, $nu_n^2\omega(u_n) \to 0$ *as* $n \to \infty$. *Suppose the problem is to test a hypothesis* $H_0 : \theta = t$ *against the alternatives* $H_n : \theta = \theta_{n1} = t + u_n$ *or* $\theta = \theta_{n2} = t - u_n$. *Then for any sequence of tests* K_n *such that* $\alpha_n(K_n) < c < 1$ *and* $(n\hat\rho_n^2)^{-1}|2\log\alpha_n(K_n)| < 1$ *it holds*

$$\liminf_{n\to\infty}(4n\hat\rho_n^2)^{1/2}(4n^{1/2}\hat\rho_n + 2\Phi^{-1}(\alpha_n(K_n)) + \Phi^{-1}(\beta(K_n, P_{n\theta_{n1}})+$$

$$\Phi^{-1}(\beta(K_n, P_{n\theta_{n2}})) \geq 0. \qquad (2.28)$$

For any sequence $\alpha_n < C < 1$, $(4n\hat\rho_n^2)^{-1}|2\log\alpha_n| < 1$ *there exists a sequence of the likelihood ratio tests* L_n, $\alpha_n = \alpha_n(L_n)$, *such that the left handside of* (2.28) *equals zero.*

Assume A1-A3. *Then in* (2.28) *we can put* $2\hat\rho_n = u_n I^{1/2}$.

Assume $A2, A4$. *Then there exists a sequence of sets* U_n, $P_t(U_n) \to 1$ *as* $n \to \infty$, *such that if in* (2.28) *the equality takes place then*

$$\lim_{n\to\infty} E_t\{|K_n - \chi(|\sum_{i=1}^n \varphi(X_i)| > (1/4nI)^{1/2}\bar\Phi^{-1}(\alpha_n(K_n)))|\times$$

$$\chi(U_n)\}/\alpha_n(K_n) = 0, \qquad (2.29)$$

$$\lim_{n\to\infty} E_{\theta_{nj}}\{|1 - K_n - \chi(|\sum_{i=1}^n \varphi(X_i)| < (1/4nI)^{1/2}\bar\Phi^{-1}(\alpha_n(K_n)))|\times$$

$$\chi(U_n)\}/\beta(K_n, P_{\theta_{nj}}) = 0, \qquad (2.30)$$

$$\lim_{n\to\infty} E_t\{K_n(1 - \chi(U_n))\}/\alpha_n(K_n) = 0, \qquad (2.31)$$

$$\lim_{n\to\infty} E_{t_{nj}}\{(1 - K_n)(1 - \chi(U_n))\}/\beta(K_n, P_{\theta_{nj}}) = 0 \qquad (2.32)$$

for $j = 1, 2$.
Here the assignment of $\hat\rho_n$ *is the same as in Remark* 2.2.

The proof of Theorem 2.3 is similar to that of Theorem 2.2 and is omitted.

We call the sequence of tests K_n satisfiing (2.12) strong large deviation or SLD-asymptotically efficient in the problem of hypothesis testing $\theta = t + v_n$ against alternatives $\theta = t + v_n + w_n$ (respectively twosided alternatives

$\mid \theta - t - v_n \mid = w_n)$ if the left hand-side of (2.18) (respectively (2.28)) equals zero.

2.3. MLD and SLD asymptotic efficiencies in estimation problems. The lower bound of the MLD and SLD-asymptotic efficiencies in estimation problems are easy consequences of the similar lower bounds in hypothesis testing. Such a direct connection of the lower bounds of probabilities of large deviations in hypothesis testing and estimation has been discovered by Bahadur (1960).

Consider the problem of hypothesis testing $\theta = t$ against the alternatives $\theta = t_n = t + 2u_n$. On a sequence of estimators $\hat{\theta}_n = \hat{\theta}_n(X_1, \ldots, X_n)$ define the sequence of tests $K_n = \chi(\hat{\theta}_n - t > u_n)$. Let $\alpha_n(K_n) = P_t(\hat{\theta}_n - t > u_n) < C < 1$ and $\beta_n(K_n) = P_{t_n}(\hat{\theta}_n - t_n < u_n) < C < 1$ then by (2.14) we have

$$(4n\rho^2(t, t_n))^{-1/2}(|2\log P_t(\hat{\theta}_n - t > u_n)|^{1/2} + |2\log P_{t_n}(\hat{\theta}_n - t_n < u_n)|^{1/2}) \leq 1 + o(1)$$

as $n \to \infty$. Since $P_t(\hat{\theta}_n - t > u_n) \leq P_t(|\hat{\theta}_n - t| > u_n)$ and $P_{t+2u_n}(\hat{\theta}_n - 2u_n < u_n) \leq P_{t+2u_n}(|\hat{\theta}_n - 2u_n| > u_n)$ this implies as follows.

Theorem 2.4. *Assume* A. *Let* $u_n > 0$, $u_n \to 0$, $nu_n^2 \to \infty$ *as* $n \to \infty$. *Then for any sequence of estimators* $\hat{\theta}_n$ *of parameter* θ

$$\liminf_{n\to\infty} \sup_{\theta = t, t+2u_n} (2n\rho^2(t, t+2u_n))^{-1} \log P_\theta(|\hat{\theta}_n - \theta| > u_n) \geq -1. \qquad (2.33)$$

It is easy to see that the traditional local asymptotical minimax lower bound (see Radavichius (1991)) follows from Theorem 2.4.

Corollary 2.3.(*see Radavichius* (1991)) *Assume* A0. *Let* $u_n > 0$, $u_n \to 0$, $nu_n^2 \to \infty$ *as* $n \to \infty$. *Then for any* $\delta > 0$ for any sequence of estimators $\hat{\theta}_n$

$$\liminf_{n\to\infty} \sup_{|\theta - t| < \delta} (nu_n^2 I/2)^{-1} \log P_\theta(|\hat{\theta}_n - \theta| > u_n) \geq -1. \qquad (2.34)$$

We say that a sequence of estimators $\hat{\theta}_n$ is u_n–consistent if $P_\theta(|\hat{\theta}_n - \theta| > \delta u_n) \to 0$ as $n \to \infty$ for any $\delta > 0$ and all θ , $|\theta - t| < \epsilon$, $\epsilon > 0$.

Arguing similarly to the proof of Theorem 2.4 we obtain the following assertion.

Theorem 2.5. *Assume* A. *Let a sequence of estimators* $\hat{\theta}_n$ *is* u_n*–consistent. Then*

$$\limsup_{n\to\infty} (4n\rho_n^2)^{-1/2}(|2\log P_t(|\hat{\theta}_n - t| > cu_n)|^{1/2} +$$

$$|2 \log P_{t+u_n}(|\hat{\theta}_n - t - u_n| > (1-c)u_n)|^{1/2}) \le 1 \qquad (2.35)$$

for any $0 < c < 1$.

For the Cramer type large deviations of estimators we could not prove the direct analogues of (2.18),(2.19). Here the proof of the lower bound is also reduced to the problems of hypothesis testing . However the number of problems of hypothesis testing under consideration increases with increasing sample size. The Hellinger distance $\rho(\theta_1, \theta_2)$ can satisfy the assumption A and have as the function of two variables a rather irregular character. To take into account such a variation of the Hellinger distance the lower bound of the SLD efficiensy are given in the terms of the function $R(t, C, u) = \sup\{\rho(\theta, \theta + u) : |\theta - u| < Cu\}$. The values of $R(t, C, u)$ used in the lower bound can be considered as the least favourable values of the Hellinger distance.

Theorem 2.6. *Assume* A. *Let* $u_n > 0$, $nu_n^2\omega(u_n) \to 0$, $nu_n^2 \to \infty$ *as* $n \to \infty$. *Then for any sequence of estimators* $\hat{\theta}_n$

$$\liminf_{n\to\infty} \sup_{|\theta - t| < C_n u_n} P_\theta(|\hat{\theta}_n - \theta| > u_n)(2\Phi(-2n^{1/2}R(t, C_n, u_n)))^{-1} \ge 1 \qquad (2.36)$$

for any sequence $C_n \to \infty$ *as* $n \to \infty$.

If A1–A3 *is valid then* (2.36) *can be replaced by*

$$\liminf_{n\to\infty} \sup_{|\theta - t| < C_n u_n} P_\theta(|\hat{\theta}_n - \theta| > u_n)(2\Phi(-(nI)^{1/2}u_n)))^{-1} \ge 1. \qquad (2.37)$$

Assume $A2, A4$. *Then there exists a sequence events* U_n, $P(U_n) \to 1$ as $n \to \infty$, *such that if in* (2.37) *the equality takes place for a sequence* C_n, $C_n^2 u_n^2 \omega(u_n C_n) \to 0$ *as* $n \to \infty$ *then*

$$\lim_{n\to\infty} E_{\theta_n}\{|\chi(|\hat{\theta}_n - \theta_n| > u_n) -$$

$$\chi(|\sum_{i=1}^n g_n(X_{ni}, \theta_n)| > 1/2nu_n^2 I)|\chi(U_n)\}(2\Phi(-(nI)^{1/2}u_n))^{-1} = 0, \qquad (2.38)$$

$$E_{\theta_n}\chi(|\hat{\theta}_n - \theta_n| > u_n)(1 - \chi(U_n))(2\Phi(-(nI)^{1/2}u_n))^{-1} = 0 \qquad (2.39)$$

for any sequence θ_n, $|\theta_n - t| < C_n u_n$.

Similarly to the problems of hypothesis testing we say that a sequence of estimators $\hat{\theta}_n$ is MLD (SLD-respectively) asymptotically efficient if in (2.34) ((2.36) respectively) the equality takes place.

Remark 2.9. Theorems 2.1–2.6 are easily generalized on the k -sample case. Let $X_{j1},\ldots,X_{jn_j}$, $1 \le j \le k$, be i.i.d.r.v.'s with p.m.'s $P_{j\theta}$, $\theta \in R^1$, $n = n_1 + \ldots + n_k$ and let $n_j/n \to \nu_j$ as $n \to \infty$. Let the p.m.'s $P_{j\theta}$, $1 \le j \le k$, satisfy the assumption A. Then (2.11),(2.13),(2.33),(2.35) are valid with

$$\rho_n^2 = \sum_{j=1}^{k} \nu_j \rho^2(P_{jt_{n1}}, P_{jt_{n2}})$$

Let the p.m.'s $P_{j\theta}$, $1 \le j \le k$, satisfy A1–A3 and let $I_j = I_{jt}$ be the Fisher information of $P_{j\theta}$ at the point $\theta = t$. Let $\nu_j, 1 \le j \le k$ satisfy the following assumption.

A5. $u_n^2 \max\{|n_j - n\nu_j|, 1 \le j \le k\} \to 0$ as $n \to \infty$.

Then under this additional assumption (2.24),(2.25),(2.37) hold with $I = \sum_{j=1}^k \nu_j I_j$. For the brevity we omit here the versions of Theorem 2.3 and (2.20)–(2.23),(2.26), (2.27),(2.38),(2.39).

Remark 2.10. Let $\Psi(P_t)$ be the set of all maps $\Phi : u \to P_{t+u}$, $u \in (-\delta, \delta)$, $\delta > 0$, satisfiing (2.1) with a fixed constant C. Then it is easy to see from the proofs of Theorems 2.1,2.2,2.6 that the lower bounds given by (2.11),(2.14),(2.18),(2.25),(2.28),(2.33)–(2.37) are uniform on $\Phi \in \Psi(P_t)$. This implies that if we put $\inf_{\Phi\in\Psi(P_t)}$ after $\liminf_{n\to\infty}$ in (2.19),(2.25), (2.28), (2.33),(2.34),(2.36), (2.37) and $\sup_{\Phi\in\Psi(P_t)}$ after $\limsup_{n\to\infty}$ in (2.11),(2.14), (2.35) then these inequalities remain valid. It is easy to see that the convergence in (2.13),(2.15) is also uniform on $\Phi \in \Psi(P_t)$.

Remark 2.11. The proofs of Theorems 2.4 and 2.6 show the direct connection of the MLD and SLD asymptotic efficiencies in the problems of estimation and hypothesis testing. It follows from the proof of Theorem 2.6 that the SLD asymptotic efficiency of the sequence of estimates $\hat\theta_n$ is equivalent to the SLD asymptotic efficiency of the sequence of tests statistics $\hat\theta_n$. At the same time the MLD asymptotic efficiency of tests statistics $\hat\theta_n$ for all sequencies u_n, $nu_n^2 \to \infty$ as $n \to \infty$, implies that the sequence of estimators $\hat\theta_n$ is MLD asymptotically efficient.

Proof of Theorem 2.6. The proof will be given under the assumptions A2,A4. In the other cases the differences in the proof of (2.36) are insignificant. For a sequence of parameters t_n whose values will be defined later consider the l–problems of hypothesis testing $H_{nj} : \theta = t_{nj} = t_n + 2ju_n$

versus the alternatives $H_{n,j+1} : \theta = t_{n,j+1}$, $0 \le j \le l-1$. Define the sequences of tests $K_{nj} = \chi(\hat{\theta}_n - t_{nj} > u_n)$. Denote by α_{nj}, β_{nj} respectively the type I and type II error probabilities of the tests K_{nj}, $0 \le j \le l-1$. Put $r_{nj} = \alpha_{nj}/\Phi(-(nI)^{1/2}u_n)) - 1$, $s_{nj} = 1 - \beta_{nj}/\Phi(-(nI)^{1/2}u_n))$.

Assume that (2.37) is not valid. Then there exist a sequence t_n, $|t_n - t| < C_n u_n$ and $\epsilon > 0$ such that

$$r_{n,l-2} - s_{n,l-1} < -\epsilon \quad \text{and} \quad r_{n,j-1} - s_{nj} < o(1) \qquad 1 \le j \le l-2. \tag{2.40}$$

Note that $r_{nj} < 1 + o(1)$ since otherwise implies

$$P(|\hat{\theta}_n - t| > u_n) \ge P(\hat{\theta}_n - t > u_n) = 2\Phi(-(nI)^{1/2}u_n)(1 + o(1)).$$

By (2.18) we also have $\log(1 + r_{nj}) + \log(1 - s_{nj}) > o(1)$

Using (2.40), $r_{n0} < 1 + o(1)$ and $(1 + r_{nj})(1 - s_{nj}) > 1 + o(1)$ we obtain inductively $s_{n0} < 1/2 + o(1)$, $r_{n1} < 1/2 + o(1), s_{n1} < 1/3 + o(1), \ldots, s_{nj} < 1/(j+2) + o(1)$, $r_{n,j+1} < 1/(j+2) + o(1)$ and so on. Since $o(1)$ in all estimates does not depend on l the last inequalities contradict $r_{n,l-2} - s_{n,l-1} < -\epsilon$ as $l \to \infty$.

Assume that in (2.37) the equality takes place. Let $l = l_n = o(C_n)$, $l_n \to \infty$ as $n \to \infty$. Then, arguing similarly, we obtain that $s_{nj} < 1/j + o(1)$, $r_{nj} < 1/j + o(1)$ as $n \to \infty$. By (2.20)–(2.23) this implies (2.38),(2.39) for $\theta_n = t_{nj_n}$, $j_n \to \infty$ as $n \to \infty$. Since here the choice of t_n, $|t_n - t| < C_n u_n/2$, is arbitrary we can take $t_{nj_n} = \theta_n$, $|\theta_n - t| < C_n u_n/2$. This completes the proof of Theorem 2.6.

3. MLD and SLD asymptotically minimax lower bounds.
In section we consider k–sample case in contrast to section 2. This caused the applications considered in the further papers.

Section treats the following setting. Let (S, B) be a measurable space, let Λ be the set of all probability measures on (S, B), let $X_{j1}, \ldots, X_{jn_j}$ be i.i.d.r.v.'s with probability measure $P_j \in \Lambda$, $1 \le j \le k$, and let the functional $T : \Lambda^k \to R^1$ be defined. Suppose the a priori information is given that $P = P_1 \times \ldots \times P_k \in \Gamma \subseteq \Lambda^k$. Denote $n = n_1 + \ldots + n_k$ and assume $n_j/n \to \nu_j$, $1 \le j \le k$, as $n \to \infty$. The problems under consideration are as follows. The estimation problem is to estimate a value of functional $T(P)$ on the set Γ. The hypothesis testing problem is to test a hypothesis $P \in \Gamma_t = \Gamma_t(T) = \{P : T(P) = t, P \in \Gamma\}$ against the alternatives $P \in \Gamma_t(T, u_n) = \{P : T(P) - t > u_n \ , P \in \Gamma\}, u_n > 0$.

Such an approach to the theory of nonparametric statistical inference has been proposed by C.Stein (1956) and developed by Levit (1974), Koshevnick and Levit (1976), Millar (1983), Phanzagl (1982), van der Vaart (1991), Ibragimov and Khasminskii (1991) and others.

3.1. MLD asymptotic minimaxity. Introduce the standard terminology arising in the problems of asymptotic statistical inference on a value of functional (see Phanzagl (1982), van der Vaart (1991) and Ermakov (1992)).

For a fixed $P = P_1 \times \ldots \times P_k \in \Gamma$ let $\Pi(\Gamma, P)$ be the set of all maps $\Phi : u \to P_u$ from some interval $(0, \delta)$ into Γ satisfiing for some function $\varphi_\Phi(x_1, ..., x_k) = (\varphi_1(x_1), \ldots, \varphi_k(x_k))$, $\varphi_j \in L_2(P_j)$, $1 \le j \le k$,

$$\sum_{j=1}^{k} \int_S [(dP^c_{ju}/dP_j)^{1/2} - 1 - u\varphi_j]^2 dP_j = o(u^2), \;\; P^s_{ju}(S) = o(u^2) \qquad (3.1)$$

as $u \to 0$. Here P^c_{ju} and P^s_{ju} , $1 \le j \le k$, denote the absolutely continuous and singular components of p.m. P_{ju} w.r.t. p.m. P_j. Let $\Delta(\Gamma, P)$ be the set of all functions φ_Φ, $\Phi \in \Pi(\Gamma, P)$, thus defined and let $\bar{\Delta}(\Gamma, P)$ be the closure in $L_2(P)$ of $\Delta(\Gamma, P)$. Define the linear space $L(\Gamma, P)$ as the closure in $L_2(P)$ of the linear space generated by the functions $\varphi_\Phi \in \Delta(\Gamma, P)$. The linear space $L(\Gamma, P)$ can be interpreted as a tangent space of Γ at a point P in the Hellinger metric. In the problems of asymptotic statistical inference the derivative of functional T is natural to define as an element of the linear space $L(\Gamma, P)$.

We say that the function $\xi_P \in L(\Gamma, P)$ is a gradient T on Γ at a point $P \in \Gamma$ if it holds

$$T(P_u) - T(P) = uE\xi'_P\varphi_\Phi + o(u), \qquad u \downarrow 0 \qquad (3.2)$$

for all $\Phi \in \Pi(\Gamma, P)$. Here $\xi'_P\varphi_\Phi$ is the scalar product of $\xi_P = (\xi_1, \ldots, \xi_k)$ and φ_Φ.

The Fisher information of functional T on Γ at a point P equals $I(\Gamma, P) = (4\nu_1 E\xi_1^2(X_{11}) + \ldots + 4\nu_k E\xi_k^2(X_{k1}))^{-1}$.

Denote $I(\Gamma_t) = \sup\{I(\Gamma, P) : P \in \Gamma_t\}$.

Make the following assumption.

B. For all $P \in \Gamma_t$ there exists a gradient ξ_P of the functional T on Γ and $\xi_P \in \bar{\Delta}(\Gamma, P)$.

For any sequence of tests K_n denote $\alpha(K_n) = \sup\{\alpha(K_n, P) : P \in \Gamma_t\}$ and $\beta(K_n, u_n) = \sup\{\beta(K_n, P) : P \in \Gamma_t(T, u_n)\}$.

Theorem 3.1. *Assume* B. *Let* $u_n \to 0$, $nu_n^2 \to \infty$ *as* $n \to \infty$. *Then for any sequence of tests* K_n *such that*

$$(nu_n^2)^{-1/2} \mid \Phi^{-1}(\alpha(K_n)) \mid \leq C \qquad (3.3)$$

it holds

$$\limsup_{n\to\infty} (nu_n^2 I(\Gamma_t))^{-1/2} \mid \Phi^{-1}(\alpha(K_n)) + \Phi^{-1}(\beta(K_n, u_n)) \mid \leq 1. \qquad (3.4)$$

Remark 3.1. It easy to see that a similar result holds for the sets of alternatives $\Gamma_t^s(T, u_n) = \{P : |T(P) - t| > u_n,\ P \in \Gamma\}$ under the additional assumptions on the sequences of tests K_n

$$\alpha(K_n) < C < 1, \qquad (nu_n^2 I(\Gamma_t))_{-1} |2 \log \alpha(K_n)|^{1/2} < C < 1.$$

Corollary 3.1. *Assume* B *and let* $u_n = u$. *Then for any sequences of tests* K_{nu} *such that* $\alpha(K_{nu}) < C < 1$, $(nu^2 I(\Gamma_t))^{-1/2} \mid 2\log\alpha(K_n) \mid^{1/2} < C < 1$ *it holds*

$$\limsup_{u\to 0} \limsup_{n\to\infty} (nu^2 I(\Gamma_t))^{-1/2} (\mid 2\log\alpha(K_{nu}) \mid^{1/2} +$$

$$\mid 2\log\beta(K_{nu}, u) \mid^{1/2}) \leq 1. \qquad (3.5)$$

The asymptotically minimax lower bounds for the local Bahadur, Hodges-Lehmann and Chernoff efficiencies are the particular cases of (3.5).

Indicate the similar lower bounds for the probabilities of the moderate large deviations of estimators.

Theorem 3.2. *Assume* B. *Let* $u_n \to 0$, $nu_n^2 \to \infty$ *as* $n \to \infty$. *Then for any sequence of estimators* $\hat{\theta}_n$ *for all* $\delta > 0$

$$\liminf_{n\to\infty} \inf_{P\in U(\delta,t)} (nu_n^2 I(\Gamma_t))^{-1/2} \log P(|\hat{\theta}_n - T(P)| > u_n) \le -1. \qquad (3.6)$$

Here $U(\delta, t) = \{P : |T(P) - t| < \delta,\ P \in \Gamma\}$.

The proofs of Theorems 3.1,3.2 are omitted. These proofs follow directly from Theorems 2.1,2.4 using the same arguments as the proof of the traditional asymptotically minimax lower bound for the problem of functional estimation (see Koshevnick and Levit (1976) Theorem 2). Note that the proof of asymptotically minimax lower bound for the Pitman efficiency is also based on the same ideas (see Ermakov (1992)).

We say that a sequence of tests K_n satisfiing (3.3) (estimators $\hat{\theta}_n$) is moderate large deviation or MLD asymptotically minimax if the left hand-side of (3.4) ((3.6) respectively) equals one. Similarly the sequence of test statistics V_n is called MLD asymptotically minimax if any sequence of tests K_n generated by the test statistics V_n and satisfiing (3.3) is MLD asymptotically minimax.

3.2. SLD asymptotic minimaxity. Section treats the two settings. The first setting is the standard approach of the section 3.1. The second setting has the geometrical character and uses the Hellinger distance as a measure of asymptotic efficiency. The results are given in the case of $\omega(u) = u^\gamma$, $0 < \gamma \le 1$.

For the SLD asymptotically minimax inference the approach of section 3.1 has the more complicated form. We were compelled to make the assumptions of uniform convergence similar to that used in Theorems 2.2,2.6.

Define the space $L_{2+\gamma}(P), P = P_1 \times \ldots \times P_k$, with the norm

$$||\zeta||_{2+\gamma}^{2+\gamma} = \sum_{j=1}^{k} \int_S |\zeta_j|^{2+\gamma} dP_j$$

where $\zeta(x_1, \ldots, x_k) = (\zeta_1(x_1), \ldots, \zeta_k(x_k))$.

Fix $C > 0$ and introduce the set $\Pi_C(\Gamma, P)$, $P \in \Gamma$, of all maps $\Phi : u \to P_u$ from some interval $(0, \delta)$ into Γ satisfiing for some function $\varphi_\Phi(x_1, \ldots, x_k) =$

$(\varphi_1(x_1), \ldots, \varphi_k(x_k)) \in L_{2+\gamma}(P)$, $||\varphi_\Phi||_{2+\gamma} < C$,

$$\sum_{j=1}^{k} \int_S (P_{ju}^c/dP_j - 1 - u\varphi_j)^2 dP_j < Cu^{2+2\gamma}, \qquad P_{ju}^s(S) < Cu^{2+\gamma} \qquad (3.7)$$

for $1 \le j \le k$. Let $\Delta_C(\Gamma, P)$ be the set of all functions φ_Φ, $\Phi \in \Pi_C(\Gamma, P)$, thus defined and $\bar{\Delta}_C(\Gamma, P)$ denote the closure in $L_{2+\gamma}(P)$ of $\Delta_C(\Gamma, P)$.

Make the following assumptions

B1. There exists C such that $\xi_P \in \bar{\Delta}_C(\Gamma, P)$.

Here ξ_P is a gradient T on Γ at the point P.

B2. There exists C_1 such that

$$|T(P_u) - T(P) - uE\xi'_P\varphi| < C_1 u^{1+\gamma}$$

for all $\Phi \in \Pi_C(\Gamma, P)$.

Theorem 3.3. *Assume* B1,B2,A5. *Let* $u_n > Cn^{-1/2}$, $nu_n^{2+\gamma} \to 0$ *as* $n \to \infty$. *Then for any sequence of tests* K_n satisfiing (3.3)

$$\liminf_{n\to\infty} (nu_n^2 I(\Gamma_t))^{1/2}((nu_n^2 I(\Gamma_t))^{1/2} + \Phi^{-1}(\alpha(K_n)) +$$

$$\Phi^{-1}(\beta(K_n, u_n))) \ge 0. \qquad (3.8)$$

Remark 3.2. Under the same assumptions the similar results are valid for the sets of alternatives $\Gamma_t^s(T, u_n)$. For any sequence of tests K_n, $\alpha(K_n) < C < 1$, $(nu_n^2 I(\Gamma_t))^{-1}|2\log\alpha(K_n)| < 1$ it holds

$$\liminf_{n\to\infty} (nu_n^2 I(\Gamma_t))^{1/2}((nu_n^2 I(\Gamma_t))^{1/2} + \bar{\Phi}^{-1}(\alpha(K_n)) +$$

$$\Phi^{-1}(\beta(K_n, u_n))) \ge 0. \qquad (3.9)$$

Theorem 3.4. *Assume* B1,B2,A5 *and let* $nu_n^2 \to \infty$, $nu_n^2\omega(u_n) \to 0$ *as* $n \to \infty$. *Then for any sequence of estimators* $\hat{\theta}_n$

$$\limsup_{n\to\infty} \sup_{P\in U(C_n, u_n, t)} P(|\hat{\theta}_n - T(P)| > u_n)/(2\Phi(-n^{-1/2}u_n I^{1/2}(\Gamma_t))) \ge 1. \qquad (3.10)$$

for all sequences $C_n \to \infty$ *as* $n \to \infty$.

The main idea of the proofs of Theorems 3.3,3.4 is the same as in the proofs of Theorems 3.1,3.2. Let $\Phi_n : u \to P_{nu}$, $\Phi_n \in \Pi_C(\Gamma, P)$ be a sequence of maps such that $||\varphi_n - \xi_P||_{2+\gamma} < Cu_n^\gamma$, $\varphi_n = \varphi_{\Phi_n}$. Since $\Pi_c(\Gamma, P) \subseteq \Psi(P)$ then by Remark 2.10 we have

$$\sup_{|\theta|<C_n u_n} P_{n\theta}(|\hat\theta_n - \theta| > u_n)/(2\Phi(-2n^{1/2}u_n E^{1/2}\varphi_n^2(X_1))) \ge 1+\delta_n(C_n) \qquad (3.11)$$

where $\delta_n(C_n) \to 0$ as $n \to \infty$ for any sequence $C_n \to \infty$ as $n \to \infty$ and $\delta_n(C_n)$ does not depend on the sequence of maps $\Phi_n \in \Pi_c(\Gamma, P)$. This implies (3.10). The proof of Theorem 3.3 is similar and is omitted.

The second setting is as follows. For any p.m.'s $P_1 = P_{11} \times \ldots \times P_{1k}$, $P_2 = P_{21} \times \ldots \times P_{2k} \in \Lambda^k$ denote

$$\rho_{\gamma\nu}(P_1, P_2) = (\sum_{j=1}^{k} \int_S ((dP_{1j}/dQ_j)^{1/(2+\gamma)} - (dP_{2j}/dQ_j)^{1/(2+\gamma)})^{2+\gamma} dQ)^{1/(2+\gamma)}$$

with $Q_j = 1/2\,(P_{1j} + P_{2j})$. Put $\rho_\gamma(P_1, P_2) = \rho_{\gamma 0}(P_1, P_2)$. For any $C > 0$ define the functions

$$R(t, u, C) = \inf\{\rho(P_1, P_2) : P_1 \in \Gamma_t, P_2 \in \Gamma_{t+u}, \rho_\gamma(P_1, P_2) < Cu\}. \qquad (3.12)$$

If the p.m.'s $P_1 \in \Gamma_t, P_2 \in \Gamma_{t+u}$ such that $\rho_\gamma(P_1, P_2) < Cu$ do not exist then put $R(t, u, C) = 0$. For any C,C_n denote $R_{C_n}(t, u, C) = \sup\{R(\theta, u, C) : |\theta - t| < C_n u\}$.

Make the following assumption.

B3. There exist $A > 0$, $a > 0$ such that $R(t, u, A) > au$, for all u, $0 < u < u_0$.

Note that in B3 we do not make any direct assumptions on the functional T. The essential role plays only the geometrical structure of the sets Γ_{t+u} defined on T.

Theorem 3.5. *Assume* B3,A5. *Let* $u_n > Cn^{-1/2}$, $nu_n^{2+\gamma} \to 0$ *as* $n \to \infty$. *Then for any sequence of tests* K_n *satisfiing* (3.3)

$$\liminf_{n\to\infty} (4nR^2(t, u_n, C))^{1/2}((nR^2(t, u_n, C))^{1/2} + \Phi^{-1}(\alpha_n(K_n))+$$

$$\Phi^{-1}(\beta(K_n, u_n)))) \ge 0. \qquad (3.13)$$

for all $C > 0$.

Theorem 3.6. *Assume* B1. *Let* $nu_n^2 \to \infty$, $nu_n^{2+\gamma} \to 0$ *as* $n \to \infty$. *Then for any sequence of estimators* $\hat{\theta}_n$

$$\liminf_{n\to\infty} \sup_{P\in U(C_n,u_n,t)} \frac{P(|\hat{\theta}_n - T(P)| > u_n)}{2\Phi(-2n^{1/2}R_{C_n}(T,u_n,C))} \geq 1 \qquad (3.14)$$

for all sequences $C_n \to \infty$ *as* $n \to \infty$.

The proofs of Theorems 3.5,3.6 are based on the same arguments as the proofs of Theorems 3.3,3.4 and are omitted. The uniform convergence of the reminder term in the corresponding version of (3.11) is ensured here by the condition $\rho_\gamma(P_1, P_2) < Cu$ in (3.12).

We say that a sequence of tests K_n satisfiing (3.3) (respectively estimators $\hat{\theta}_n$) is strong large deviation or SLD-asymptotically minimax if (3.8) or (3.13) (respectively (3.10) or (3.14)) is valid. Similarly we call SLD asymptotically minimax a sequence of test statistics generating the SLD asymptotically minimax sequences of tests.

4. Proofs of Theorems 2.1,2.2.

To simplify the notations we assume $t = 0$ and denote $E\zeta = E_t\zeta$, $P(D) = P_t(D)$ for any random variable ζ and event D. For any $v \in R^1$ put

$$\tau_i(v) = \varphi(X_i)f^{-1/2}(X_i, v), \quad \eta_{ni}(v) = (f(X_i, v+u_n)/f(X_i, v))^{1/2} - 1,$$

$\tau_i = \tau_i(0)$, $\eta_{ni} = \eta_{ni}(0)$, $1 \le i \le n$. Since the relations are proved usually for the arbitrary value of index i, $1 \le i \le n$, we shall omit in such cases the index i in notations assuming that the value $i = 1$ is considered, that is, $\tau(v) = \tau_1(v), \eta_n = \eta_{n1}$ and so on. For a fixed $\epsilon > 0$ and all $1 \le i \le n$ define the events

$$A_{ni} = \{X_i : \mid \eta_{ni} \mid > \epsilon\}, \quad B_{ni} = \{X_i : \mid \eta_{ni} \mid < \epsilon\}.$$

Denote $U_n = \bigcap_{i=1}^n B_{ni}$.

For fixed constants C_n , $|C_n| < Cn^{1/2}\rho_n$, define the sequence of the likelihood ratio tests

$$L_n = \chi\{\prod_{i=1}^n f(X_i, u_n + v_n)/f(X_i, v_n) > \exp\{-C_n\}\}$$

and the sequences of tests

$$N_{n1} = \chi\{u_n \sum_{i=1}^n \tau_i > n\rho_n^2 - C_n/2\}\chi(U_n),$$

$$N_{n2} = \chi\{\sum_{i=1}^n (\eta_{ni} + 1/2\ \rho_n^2) > n\rho_n^2 - C_n/2\}\chi(U_{n1})$$

where the sets U_{n1} will be defined later.

The plan of the proof is as follows. First we find the asymptotics of $\alpha_n(L_n), \beta_n(L_n)$ and as a result obtain (2.10)–(2.15),(2.18),(2.19),(2.24),(2.25). Then we prove that the asymptotics of the type I and type II error probabilities of the tests N_{n1} and N_{n2} coincide with the corresponding asymptotics of $\alpha_n(L_n), \beta_n(L_n)$. A simple analysis of the proof of the Neyman-Pearson Lemma (see Lehmann (1986)) shows that this implies (2.20)-(2.23),(2.26),(2.27).

For the sake of simplicity the estimates will be given under the assumption $P_\theta^s(S) = 0$, $|\theta| < \delta$, $\delta > 0$. The additional addendums arising without this assumption are easily estimated and have the smaller order.

For the proof of Theorems 2.1,2.2 we shall use the following representation of the likelihood ratio

$$\prod_{i=1}^{n} f(X_i, u_n + v_n)/f(X_i, v_n) = \exp\{2\sum_{i=1}^{n}(\eta_{ni}(v_n) - 1/2\ \rho_n^2) - 2n\rho_n^2 + \zeta_{n1}\}$$

with $\zeta_{n1} \to 0$ in probability as $n \to \infty$.

This representation allow us to prove the results under Assumption A instead of the traditional assumptions assuming the existence of the finite Fisher information. Thus we refuse from the traditional representation of the likelyhood ratio (see Hajek (1970),(1972))

$$\prod_{i=1}^{n} f(X_i, u_n + v_n)/f(X_i, v_n) = \exp\{2u_n \sum_{i=1}^{n} \tau_i - nu_n^2 I/2 + \zeta_{n2}\}$$

with $\zeta_{n2} \to 0$ in probability as $n \to \infty$.

The asymptotics of the type I and type II error probabilities of tests L_n, N_{n1} and N_{n2} will be obtained on the base of the following Theorem about the moderate and Cramer type large deviations of sums of independent identically distributed random variables. This Theorem is a version of Theorem 3.2 in Saulis and Statuliavichius (1990).

Let $Y_1, \ldots, Y_n$ be i.i.d.r.v.'s, $EY_1 = 0$, $EY_1^2 = \sigma^2$. Denote $S_n = Y_1 + \ldots + Y_n$ and put $F_n(x) = P(S_n < x\sigma n^{1/2})$.

Make the following assumption.

P. There exist a constant C and a sequence of constants D_n, $nD_n^2 \to \infty$ as $n \to \infty$, such that

$$|\log E\exp\{sY_1\}| < Cs^2, \qquad |s| < D_n. \qquad (4.1)$$

Theorem 4.1. *Assume* P. *Then there exists* $\delta > 0$ *such that*

$$(1 - F_n(x))/(1 - \Phi(x)) = 1 + o(1) \qquad (4.2)$$

for $0 < x < \min\{\delta n^{1/6}, n^{1/2}D_n\}$ *and*

$$\log(1 - F_n(x)) = \log(1 - \Phi(x))(1 + o(1)) \qquad (4.3)$$

for $0 < x < \min\{n^{1/2}D_n, \delta n^{1/2}\}$ *as* $n \to \infty$. *The convergence in* (4.2), (4.3) *is uniform on the sets of all distributions satisfiing* (4.1).

The next four Lemmas have the auxilliary character.

Lemma 4.1. *Assume* A *and let* $|v_n| < Cu_n$. *Then*

$$E_{v_n}\eta_n^2(v_n)\chi(|\eta_n(v_n)| > \epsilon/r) \le Cu_n^2\omega(ru_n) \qquad (4.4)$$

for any $r > 1$.

Proof. Denote $\bar{\eta_n}(v_n) = (f(X_1, v_n)/f(X_1, 0))^{1/2} - 1$. Since $\eta_n(v_n) = (\bar{\eta_n}(v_n + u_n) - \bar{\eta_n}(v_n))/(1 + \bar{\eta_n}(v_n))$ then $|\eta_n(v_n)| > \epsilon/r$ implies $|\bar{\eta_n}(v_n)| > \epsilon/(2r)$ or $|\bar{\eta_n}(v_n + u_n)| > \epsilon/(2r)$. Therefore

$$E_{v_n}\eta_n^2(v_n)\chi(|\eta_n(v_n)| > \epsilon/r) =$$

$$E(\bar{\eta_n}(v_n + u_n) - \bar{\eta_n}(v_n))^2\chi(|\eta_n(v_n)| > \epsilon/r) \le$$
$$4E\bar{\eta}_n^2(v_n + u_n)\chi(|\bar{\eta}_n(v_n + u_n)| > \epsilon/(2r)) +$$
$$4E\bar{\eta}_n^2(v_n)\chi(|\bar{\eta}_n(v_n)| > \epsilon/(2r)) \le Cu_n^2\omega(u_n/r).$$

This completes the proof of Lemma 4.1.

Lemma 4.2. A1,A2 *imply* A.

Proof. We have

$$E\eta_n^2\chi(|\eta_n| > \epsilon/r) \le 4E(\eta_n - u_n\tau)^2\chi(|\eta_n - u_n\tau| > \epsilon/(2r)) +$$

$$4u_n^2E\tau^2\chi(|\tau| > \epsilon/(2r)u_n^{-1}).$$

By A1,A2 this implies A.

Lemma 4.3. *Assume* A. *Then*

$$P(A_n) = O(u_n^2\omega(u_n)), \qquad (4.5)$$

$$P_{u_n}(A_n) = O(u_n^2\omega(u_n)), \qquad (4.6)$$
$$E|\eta_n|^3\chi(B_n) = O(u_n^2\omega(u_n)). \qquad (4.7)$$

Proof. By A and the Chebyshov inequality

$$P(A_n) \le \epsilon^{-2}E\eta_n^2\chi(|\eta_n| > \epsilon) = O(u_n^2\omega(u_n)). \qquad (4.8)$$

Denote $\hat{\eta}_n = (f(X_1,0)/f(X_1,u_n))^{1/2} - 1$, $1 \le i \le k$. Since $|\eta_n| > \epsilon$ implies $|\hat{\eta}_n| > \epsilon/2$ then, by Lemma 4.1,

$$P_{u_n}(A_n) \le P_{u_n}(|\hat{\eta}_n| > \epsilon/2) \le$$

$$\epsilon^{-2} E_{u_n} \hat{\eta}_n^2 \chi(|\hat{\eta}_n| > \epsilon/2) = O(u_n^2 \omega(u_n)). \qquad (4.9)$$

Prove (4.7). Let $k \ge 1/2|\log(u_n\omega(u_n))|/\log 2$. Then

$$E|\eta_n|^3 \chi(B_n) < C u_n^2 \omega(u_n) + C E \eta_n^2 \sum_{j=1}^{k} 2^{-j} \chi(\epsilon 2^{-j} < |\eta_n| < \epsilon 2^{1-j}) <$$

$$C u_n^2 \omega(u_n) + C 2^{-k} E \eta_n^2 \sum_{j=1}^{k} \chi(\epsilon 2^{-j} < |\eta_n| < \epsilon) <$$

$$C u_n^2 \omega(u_n) + C 2^{-k} u_n^2 \sum_{j=1}^{k} 2^{k-j} \omega(2^{-k}\epsilon) < C u_n^2 \omega(u_n) \qquad (4.10)$$

since $\omega(2^{-j}\epsilon) < 2^{k-j}\omega(2^{-k}\epsilon) < C 2^{k-j}\omega(u_n)$.

Lemma 4.4. *Assume* $A2, A4$. *Then*

$$|\rho_n^2 - u_n^2 I| < C u_n^2 \omega(u_n), \qquad (4.11)$$

$$|E_{v_n}\tau(v_n)(\eta_n(v_n) - u_n\tau(v_n))| < C u_n \omega(u_n). \qquad (4.12)$$

Proof. By the Schwartz inequality we have

$$E_{v_n}\tau(v_n)(\eta_n(v_n) - u_n\tau(v_n)) \le$$

$$(E_{v_n}\tau^2(v_n))^{1/2}(E_{v_n}(\eta_n(v_n) - u_n\tau(v_n))^2)^{1/2} = O(u_n\omega(u_n)). \qquad (4.13)$$

Hence we obtain

$$\rho_n^2 = u_n^2 E_{v_n}\tau^2(v_n) + 2u_n E_{v_n}\tau(v_n)(\eta_n(v_n) - u_n\tau(v_n)) +$$

$$E_{v_n}(\eta_n(v_n) - u_n\tau(v_n))^2 = u_n^2 I + O(u_n^2\omega(u_n))). \qquad (4.14)$$

This completes the proof of Lemma 4.4.

In that follows the proofs will be given in the case $v_n = 0$. The estimates in the general case are obtained similarly using (4.4) instead of (2.1).

Lemma 4.5.1. *Assume* A. *Let* $u_n \to 0, nu_n^2 \to \infty$ *as* $n \to \infty$. *Then*

$$\alpha(L_n) > \exp\{-(2n\rho_n^2 - C_n)^2/(8n\rho_n^2)(1+o(1)) + o(n\rho_n^2)\}, \qquad (4.15)$$

$$\beta_n(L_n) > \exp\{-(2n\rho_n^2 + C_n)/(8n\rho_n^2)(1+o(1)) + o(n\rho_n^2)\} \qquad (4.16)$$

as $n \to \infty$.

Now (2.14) follows from (4.15),(4.16).

Lemma 4.5.2. *Assume* A. *Let* $u_n > Cn^{-1/2}$, $nu_n^2\omega(u_n) \to 0$ as $n \to \infty$. *Then*

$$\alpha_n(L_n) \geq \Phi(-(2n\rho_n^2 - C_n)/(2n^{1/2}\rho_n))(1+o(1)), \qquad (4.17)$$

$$\beta_n(L_n) \geq \Phi(-(2n\rho_n^2 + C_n)/(2n^{1/2}\rho_n))(1+o(1)) \qquad (4.18)$$

as $n \to \infty$.

Now (4.17),(4.18) imply (2.18).

Proof of Lemmas 4.5.1, 4.5.2. By the simmetry of the problem it suffices to prove only (4.15),(4.17).

Denote $Y_{ni} = \log(f(X_i, u_n)/f(X_i, 0))$, $1 \leq i \leq n$. Define the random variables $Y_{ni}(\epsilon)$, having the conditional distribution Y_{ni} under the condition B_{ni}. Put

$$S_n = \sum_{i=1}^{n} Y_{ni}, \qquad S_n(\epsilon) = \sum_{i=1}^{n} Y_{ni}(\epsilon).$$

By (4.5) we have

$$P(B_n) = \prod_{i=1}^{n}(1-P(A_{ni})) \leq \exp\{-\sum_{i=1}^{n} P(A_{ni})) \leq \exp\{O(nu_n^2\omega(u_n))\}. \qquad (4.19)$$

Hence we obtain

$$\alpha_n(L_n) \geq P(S_n > -C_n) \geq P(S_n > -C_n \mid B_n)\exp\{O(nu_n^2\omega(u_n))\} \qquad (4.20)$$

as $n \to \infty$.

Thus, for the proof of (4.15),(4.17) it suffices to apply Theorem 4.1 to $S_n(\epsilon)$. We have

$$E(Y_n \mid B_n) = 2J_{n1} - J_{n2} + J_{n3} \qquad (4.21)$$

where

$$J_{n1} = E(\eta_n \mid B_n), \; J_{n2} = E(\eta_n^2 | B_n), \; J_{n3} = 1/3 E(\eta_n^3/(1+\gamma\eta_n)^3 \mid B_n)$$

with $0 \le \gamma \le 1$.

By (4.7) we obtain

$$J_{n3} < CE(|\eta_n|^3 \mid B_n) = O(u_n^2 \omega(u_n)). \qquad (4.22)$$

By A1 and (4.5),(4.6) we obtain

$$J_{n2} = \rho_n^2 + O(u_n^2\omega(u_n)), \qquad (4.23)$$

$$J_{n1} = -1/2 \; J_{n2} + O(u_n^2\omega(u_n)). \qquad (4.24)$$

Now (4.21)–(4.24) together imply

$$E(Y_n \mid B_n) = -2\rho_n^2 + O(u_n^2\omega(u_n)). \qquad (4.25)$$

Arguing similarly we obtain

$$E(Y_n^2 \mid B_n) = 4\rho_n^2 + O(u_n^2\omega(u_n)). \qquad (4.26)$$

Thus it remains to test (4.1). By the Taylor formular we have

$$E(\exp\{sY_n\}|B_n) = 1 + sE(Y_n|B_n) + s^2/2 \; E(Y_n^2|B_n) + M_n \qquad (4.27)$$

with $M_n = s^3/6 \; E(Y_n^3 \exp\{\gamma s Y_n\}|B_n)$ and $0 \le \gamma \le 1$.

By (4.7) we obtain

$$M_n = 4/3 \; s^3 E(\log^3(1+\eta_n)(1+\eta_n)^{2\gamma s}|B_n) <$$

$$Cs^3 E(\log^3(1+\eta_n)|B_n) \le Cs^3 E(|\eta_n|^3|B_n) \le Cs^3 u_n^2 \omega(u_n). \qquad (4.28)$$

Now (4.26)–(4.28) together imply that

$$\log E(\exp\{s(Y_n - E(Y_n|B_n))\}|B_n) = 2s^2\rho_n^2 + O(s^3u_n^2\omega(u_n) + s^2u_n^2\omega(u_n)).$$

Therefore the Assumption P of Theorem 4.1 is satisfied with $D_n = u_n/\omega(u_n)$. This completes the proof of Lemmas 4.5.1, 4.5.2.

Lemma 4.6.1. *Assume* A. *Let* $nu_n^2 \to \infty$ *as* $n \to \infty$. *Then*

$$\alpha_n(L_n) < \exp\{-(2n\rho_n^2 - C_n)^2/(8n\rho_n^2)(1+o(1)) + o(n\rho_n^2)\}, \qquad (4.29)$$

$$\beta_n(L_n) < \exp\{-(2n\rho_n^2 + C_n)^2/(8n\rho_n^2)(1+o(1)) + o(n\rho_n^2)\} \qquad (4.30)$$

as $n \to \infty$.

Lemma 4.6.2. *Assume* A. *Let* $u_n > Cn^{-1/2}$, $nu_n^2\omega(u_n) \to 0$ *as* $n \to \infty$. *Then*

$$\alpha_n(L_n) < \Phi(-(2n\rho_n^2 - C_n)/(2n^{1/2}\rho_n))(1+o(1)), \qquad (4.31)$$

$$\beta_n(L_n) < \Phi(-(2n\rho_n^2 + C_n)/(2n^{1/2}\rho_n))(1+o(1)) \qquad (4.32)$$

as $n \to \infty$.

Lemmas 4.5.1, 4.5.2 and 4.6.1, 4.6.2 together imply that the left and right handsides of (4.29)–(4.32) equal. Thus we obtain the exact asymptotics of $\alpha_n(L_n), \beta_n(L_n)$ that implies (2.13),(2.18).

Proof. Show that the asymptotics of $\alpha_n(N_{n1}), \beta_n(N_{n1})$ are given by the right handsides of (4.29)–(4.32). Then (4.29)–(4.31) will follow from Lemmas 4.5.1,4.5.2 and the Neyman-Pearson Lemma. We find only the asymptotics of $\alpha_n(N_{n1})$. The estimates in the case of $\beta_n(N_{n1})$ are similar and are omitted.

Similarly to (4.23),(4.24),(4.27) we have

$$E\eta_n\chi(B_n) = -1/2\ \rho_n^2 + O(u_n^2\omega(u_n)), \qquad (4.33)$$

$$E\eta_n^2\chi(B_n) = \rho_n^2 + O(u_n^2\omega(u_n)), \qquad (4.34)$$

$$E\exp\{s\eta_n\chi(B_n)\} = 1 + sE\eta_n\chi(B_n) + s^2/2\ E\eta_n^2\chi(B_n) + M_{n1} \qquad (4.35)$$

with $M_{n1} = s^3E\eta_n^3\exp\{s\gamma\eta_n\}\chi(B_n)$ and $0 \le \gamma \le 1$.

By (4.7) we have

$$M_{n1} < Cs^3E|\eta_n|^3\chi(B_n) < Cs^3u_n^2\omega(u_n). \qquad (4.36)$$

Now (4.33)–(4.36) together imply that Assumption P of Theorem 4.1 is satisfied with $D_n = u_n/\omega(u_n)$. Therefore $\alpha_n(N_{n1})$ has the required asymptotics. This completes the proof of Lemmas 4.6.1, 4.6.2.

Since $\alpha_n(N_{n1}), \beta_n(N_{n1})$ have the same asymptotics as $\alpha_n(L_n), \beta_n(L_n)$ then by the Neyman-Pearson Lemma this implies (2.20)–(2.23).

Prove (2.26),(2.27) and the corresponding versions of (2.22), (2.23). In this case we change the definitions of the sets $A_{ni}, B_{ni}, 1 \le i \le n$, and the set $U_n = U_{n1}$. Put

$$A_{ni} = \{X_i : |\eta_{ni}| > \epsilon \text{ or } |\tau_i| > \epsilon u_n^{-1}\},$$

$$B_{ni} = \{X_i : |\eta_{ni}| < \epsilon \text{ and } |\tau_i| < \epsilon u_n^{-1}\}.$$

Denote $U_n = U_{n1} = \bigcap_{i=1}^n B_{ni}$.

Lemma 4.7. *Assume* A1,A2. *Then*

$$E_{v_n}(\eta_n(v_n+v) - u_n\tau(v_n+v))^2 < Cu_n^2\omega(u_n) \qquad (4.37)$$

for all $v, |v - v_n| < Cu_n$. *It also holds*

$$E_{v_n}\tau^2(v_n)\chi(|\tau(v_n)| > \epsilon u_n^{-1}) < C\omega(u_n), \qquad (4.38)$$

$$P_{v_n}(|\tau(v_n)| > Cu_n^{-1}) < Cu_n^2\omega(u_n), \qquad (4.39)$$

$$u_nE_{v_n}|\tau(v_n)|^3\chi(B_n) < C\omega(u_n), \qquad (4.40)$$

$$u_nE_{w_n}|\tau(v_n)|^3\chi(B_n) < C\omega(u_n), \qquad (4.41)$$

$$u_nE_{w_n}\tau^2(v_n)\chi(|\tau(v_n)| > \epsilon u_n^{-1}) < C\omega(u_n), \qquad (4.42)$$

$$P_{w_n}(|\tau(v_n)| > \epsilon u_n^{-1}) < Cu_n^2\omega(u_n). \qquad (4.43)$$

Here $w_n = u_n + v_n$.

Note that (4.37),(4.38) coincide with (2.3),(2.4) in the case $v_n = 0, v = 0$. Thus if we prove (4.37),(4.38) then in the furher arguments it suffices to consider the case $v_n = 0$.

Proof. Since

$$(f^{1/2}(x, v_n+v) - f^{1/2}(x,v_n))^2 \le 2(f^{1/2}(x,v_n+v) - f^{1/2}(x,0))^2 +$$

$$2(f^{1/2}(x,v_n) - f^{1/2}(x,0))^2$$

then (4.37) follows from Lemma 4.2.

By A2 and Lemma 4.2 we have

$$E_{v_n}\tau^2(v_n)\chi(|\tau(v_n)| > \epsilon u_n^{-1}) =$$

$$E\tau^2\chi(|\tau(v_n)| > \epsilon u_n^{-1}, |\tau| < \epsilon/2\, u_n^{-1}) + O(\omega(u_n)) \le$$
$$Cu_n^{-2}P(|\tau(v_n)| > \epsilon u_n^{-1}, |\tau| < \epsilon/2\, u_n^{-1}) + O(\omega(u_n)) =$$
$$Cu_n^{-2}P(|\tau(v_n)| > \epsilon u_n^{-1}, |\tau| < \epsilon/2\, u_n^{-1}, |\eta_n| < \epsilon) + O(\omega(u_n)). \qquad (4.44)$$

Since $\tau(v_n) = \tau(1 + 2\eta_n)/(1 + \eta_n)$ then the first addendum in the right handside of (4.44) equals zero that implies (4.38).

The proof of (4.40) is similar to that of (4.7). Let $k > |\log u_n/\log 2|$. Then we have

$$u_n E|\tau|^3\chi(B_n) < Cu_n + E\tau^2 \sum_{j=1}^{k} 2^{-j}\chi(2^{-j}u_n^{-1} < |\tau| < 2^{1-j}u_n^{-1}) <$$

$$Cu_n + C2^{-k}E\tau^2 \sum_{j=1}^{k} \chi(2^{-j}u_n^{-1} < |\tau| < u_n^{-1}) <$$

$$Cu_n + C2^{-k} \sum_{j=1}^{k} \omega(2^j u_n) < C\omega(u_n)$$

since $\omega(2^j u_n) < 2^j\omega(u_n)$.

We have

$$u_n E_{u_n}|\tau|^3\chi(B_n) = u_n E|\tau|^3(1+\eta_n)^{-2}\chi(B_n) <$$

$$Cu_n E|\tau|^3\chi(B_n) < C\omega(u_n).$$

This implies (4.41).

The proof of (4.42) is similar to that of (4.41) and is omitted.

By the Chebyshov inequality we obtain (4.39),(4.43) from (4.38),(4.42). This completes the proof of Lemma 4.7

Lemma 4.8. *The asymptotics of* $\alpha_n(N_{n1}), \beta_n(N_{n1})$ *are given by the right handsides of* (4.31),(4.32) *respectively.*

Proof. Using A2,A4,(4.39), $E\tau = 0$ and the Chebyshov and Schwartz inequalities we obtain

$$E\tau\chi(B_n) = E\tau\chi(A_n) <$$

$$E|\tau|\chi(|\eta_n - u_n\tau| > \epsilon/2 \text{ or } |\tau| > \epsilon/2\, u_n^{-1}) <$$

$$E|\tau|\chi(|\eta_n - u_n\tau| > \epsilon/2) + E|\tau|\chi(|\tau| > \epsilon/2\, u_n^{-1}) <$$
$$2(E\tau^2)^{1/2}P^{1/2}(|\eta_n - u_n\tau| > \epsilon/2)+$$
$$E^{1/2}\tau^2\chi(|\tau| > \epsilon/2\, u_n^{-1})\times$$
$$P^{1/2}(|\tau| > \epsilon/2\, u_n^{-1}) < Cu_n\omega(u_n). \qquad (4.45)$$

Estimate $E\tau^2\chi(B_n)$. By A2,A4 we obtain

$$|E\tau^2\chi(B_n) - E\tau^2| < E\tau^2\chi(|\tau| > \epsilon/4\, u_n^{-1})+$$
$$E\tau^2\chi(|\eta_n - u_n\tau| > \epsilon/4, |\tau| < \epsilon/4\, u_n^{-1}) <$$
$$C\omega(u_n) + u_n^{-2}P(|\eta_n - u_n\tau| > \epsilon/4) < C\omega(u_n). \qquad (4.46)$$

We have

$$E_{u_n}\tau\chi(B_n) = E\tau(1+\eta_n)^2\chi(B_n) =$$
$$E\tau(1 + (\eta_n - u_n\tau) + u_n\tau)^2\chi(B_n) =$$
$$E\tau\chi(B_n) + 2u_nE\tau^2\chi(B_n) + u_n^2E\tau^3\chi(B_n) + R_{n1} + R_{n2} + R_{n3} \qquad (4.47)$$

where, by A4,

$$R_{n1} = 2E\tau(\eta_n - u_n\tau)\chi(B_n) \leq$$
$$2E^{1/2}\tau^2E^{1/2}(\eta_n - u_n\tau)^2 < Cu_n\omega(u_n), \qquad (4.48)$$
$$R_{n2} = 2u_nE\tau_n^2(\eta_n - u_n\tau)\chi(B_n) <$$
$$CE|\tau(\eta_n - u_n\tau)| < Cu_n\omega(u_n), \qquad (4.49)$$
$$R_{n3} = E\tau(\eta_n - u_n\tau)^2\chi(B_n) <$$
$$CE|\tau(\eta_n - u_n\tau_n)| < Cu_n\omega(u_n). \qquad (4.50)$$

Now (4.47)-(4.50) together imply that

$$E_{u_n}\tau\chi(B_n) = 1/2\, u_nI + O(u_n\omega(u_n)). \qquad (4.51)$$

By (4.40),(4.49) we have

$$|E_{u_n}\tau^2\chi(B_n) - E\tau^2\chi(B_n)| = E\tau^2((1+\eta_n)^2 - 1)\chi(B_n) <$$
$$3E\tau^2|\eta_n|\chi(B_n) < 3u_nE\tau^3\chi(B_n) + 3u_n^{-1}R_{n2} < C\omega(u_n). \qquad (4.52)$$

Thus it remains to test (4.1) for the random variables $Y_i = u_n \tau_i \chi(B_{ni})$. We have

$$E\exp\{su_n\tau\chi(B_n)\} = 1 + su_n E\tau\chi(B_n) + \\ 1/2\, s^2 u_n^2 E\tau^2\chi(B_n) + M_{n2} \qquad (4.53)$$

where

$$M_{n2} = \sum_{k=1} 1/k!\ s^k u_n^k E\tau^k \chi(B_n)$$

By (4.40) we obtain

$$|M_{n2}| \le \sum_{k=3}^{\infty} 1/k!\ s^k u_n^3 \epsilon^{k-3} E|\tau|^3 \chi(B_n) \le$$

$$C\sum_{k=3}^{\infty} 1/k!\ s^k u_n^2 \epsilon^{k-3} \omega(u_n).$$

This implies that

$$|\log E\exp\{su_n(\tau\chi(B_n) - E\tau\chi(B_n))\}| < Cs^2u_n^2 E\tau^2\chi(B_n)$$

for all $|s| \le A_n = u_n/\omega(u_n)$. The estimates of $E_{u_n}\exp\{su_n\tau\chi(B_n)\}$ are similar. This implies (2.26),(2.27).

The proof of (2.22),(2.23) follows easily from (4.39),(4.43) and Lemmas 4.2,4.8. This completes the proof of Theorem 2.2.

References

Bahadur,R.R.(1960). Asymptotic efficiency of tests and estimates. *Sankhya* **22** 229–252.

Borovkov,A.A. and Mogulskii,A.A.(1992a). Large deviations and testing of statistical hypothesis. *Proc. Inst. Math. Siberian Department RAN* **19** Novosibirsk, Science.

Borovkov,A.A. and Mogulskii,A.A.(1992b). Large deviations and statistical invariance principle. *Theory Probab. Appl.* **37** 11–18.

Chernooff,H.(1952). A measure of asymptotic efficiency for tests of a hypothesis based on sums of observations. *Ann. Math. Statist.* **23** 493–507.

Ermakov,M.S.(1990). Asymptotic minimaxity of usual goodness of fit tests. *Probability Theory and Mathematical Statistics. Proc. of the 5th Vilnius Conf.* Grigelionis et al. Vilnius, VSP/Mokslas **1** 323–331.

Ermakov,M.S.(1992). On asymptotic minimaxity of rank tests *Statist. Probab. Letters* **15** 191–196.

Ermakov,M.S.(1993). Large deviations of empirical measures and hypothesis testing. *Zap. Nauchn. Semin. LOMI RAN* **207** 37–60.

Hajek,J.(1970). A characterization of limiting distributions of regular estimates. *Z. Wahrsch. Verw. Gebiete.* **14** 323–330.

Hajek,J.(1972). Local asymptotic minimax and admisibility in estimation. *Proc.Sixth Berkeley Symp. on Math. Statist. and Probab.* Los Angelos, California Univ. Press. **1** 175–194.

Hodges,J.L. and Lehman,E.L.(1956). On some nonparametric competitors of the t-test. *Ann.Math.Statist.* **27** 324–335.

Ibragimov,I.A. and Khasminskii,R.Z.(1991). Asymptotically normal families of distributions and efficient estimation. *Ann. Statist.* **19** 1681–1721.

Kallenberg,W.C.M.(1983). Intermediate efficiency, theory and examples. *Ann.Statist.* **11**, 170–182.

Koshevnik,Yu,A. and Levit,B.Ya.(1976). On a nonparametric analogue of the information matrix. *Theory Probab.Appl.* **21** 738–753.

Kourauklis,S.(1989). On the relation between Hodges-Lehman efficiency and Pitman efficiency. *Canad. J. Statist.* **17** 311–318.

Kourauklis,S.(1990). A relation between the Chernoff index and the Pitman efficiency. *Statist.Probab.Lett.* **9** 391–395.

Lehmann,E.L.(1986). *Testing Statistical Hypothesis* New York, Wiley.

Levit,B.Ya.(1974). On optimality of some statistical estimates. *Proc.Prague Sympos. on Asymptotic Statist.* Prague **2** 215–238.

Millar,P.W.(1983). The minimax principle in asymptotic statistical theory. *Ecole d'Ete de Probabilites de Saint-Flour* XI. *Lecture Notes in Math.* **976** 76–265. New York, Springer.

Phanzagl,J.(1982). *Contribution to a General Asymptotic Statistical Theory. Lecture Notes in Statistics.* **13** New York, Springer.

Radavichius,M.(1991). From asymptotic efficiency in minimax sense to Bahadur efficiency *New Trends in Probab. and Statist.* V.Sazonov and T.Shervashidze (Eds) Vilnius, VSP/Mokslas **1** 629–635.

Saulis,L.,Statuliavicius,V.(1990). *Limit Theorems for Large Deviations.* Dordrecht, Boston,London: Kluver Academic Publishers.

Stein,Ch.(1956). Efficient nonparametric testing and estimation. *Third Berkeley Symp. Math. Statist. and Probab.* Berkeley, Univ. California Press **1** 187– 195.

van der Vaart,A.W.(1991). On differentiable functionals. *Ann.Statist.* **19** 178-204.

Wieand,H.S.(1976). A condition under which the Pitman and Bahadur approaches to efficiency coincide. *Ann. Statist.* **4** 1003-1011.

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