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Weak solutions to a time-dependent heat equation with nonlocal radiation boundary condition and right-hand side in L^p $(p \geq 1)$.

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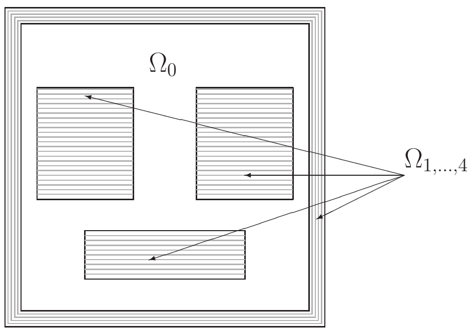
Abstract

We consider a model for transient conductive-radiative heat transfer in grey materials. Since the domain contains an enclosed cavity, nonlocal radiation boundary conditions for the conductive heat-flux are taken into account. We generalize known existence and uniqueness results to the practically relevant case of lower integrable heat-sources, and of nonsmooth interfaces. The purpose of the paper is to obtain energy estimates that involve only the L-p norm of the heat sources for some exponent p close to one. Such estimates are important for the investigation of models in which the heat equation is coupled to Maxwell's equations or to the Navier-Stokes equations (dissipative heating), with many applications such as crystal growth.

Introduction

Heat transfer processes that take place at high temperatures can neither be modeled nor simulated accurately without taking into account the phenomenon of heat radiation. A typical industrial field where radiation models are needed is crystal growth (see [Phi03], [KPS04], [KP05], [MPT06], [Mey06], [Voi01]).

In the present paper, we study from the analytical viewpoint a time-dependent heat transfer problem involving nonlocal radiation. The problem consists in computing the temperature distribution in several different opaque materials Ω_i $(i = 1, \ldots, m)$ that are separated from each other by an enclosed transparent medium Ω_0, as in the following picture:



Radiation occurs at the surface $\Sigma := \partial\Omega_0$, which is the boundary of a transparent cavity.

We denote by R the outgoing radiation (radiosity), and by J is the incoming radiation at each point of Σ. Then, R and J are connected by the simple relation

$$R = \epsilon\,\sigma\,|\theta|^3\,\theta + (1-\epsilon)\,J\,, \qquad \text{on } [0,T]\times\Sigma\,, \tag{1}$$

where the emissivity ϵ is a given material function that takes values in $[0,1]$, and σ denotes the Stefan-Boltzmann constant. The relation (1) simply states that the outgoing radiation has to be the sum of the radiation emitted according to Stefan-Boltzmann's law and of the reflected part of the incoming radiation.

A second constitutive relation between R and J is needed. The incoming radiation $J(z)$ at the point $z \in \Sigma$ is the weighted sum of the radiation outgoing at all points of the surface that are in the range of vision of z. One introduces for points pairs $(z,y) \in \Sigma \times \Sigma$ a *view factor* $w : \Sigma \times \Sigma \longrightarrow \mathbb{R}$ by setting

$$w(z,y) := \begin{cases} \frac{\vec{n}(z)\cdot(y-z)\ \vec{n}(y)\cdot(z-y)}{\pi|y-z|^4}\,\Theta(z,y) & \text{if } z \neq y\,, \\ 0 & \text{if } z = y\,, \end{cases} \tag{2}$$

where the Θ is a visibility factor that penalizes the presence of opaque obstacles

$$\Theta(z,y) = \begin{cases} 1 & \text{if }]z,y[\, \subset \Omega_0\,, \\ 0 & \text{else} \quad . \end{cases} \tag{3}$$

Here, we have used the notation $]z,y[:= \mathrm{conv}\{z,y\} \setminus \{z,y\}$, and $\vec{n}$ is the outward-pointing unit normal to Σ.

The second constitutive relation between R and J is then given by

$$J = K(R)\,, \qquad \text{on } [0,T] \times \Sigma\,, \tag{4}$$

where for diffuse grey materials (see [KPS04]), one can use the model

$$(K(R))(t,z) = \int_\Sigma w(z,y)\, R(t,y)\, dS_y \quad \text{for } (t,z) \in [0,T] \times \Sigma\,, \tag{5}$$

which we consider throughout the paper. Observe that the view factor w is obviously well defined if the surface Σ has $\mathcal{C}^1-$regularity. This can be generalized to the case of a Lipschitz continuous boundary (see for example [Dru07]).

We introduce a bounded domain, the computation domain $\Omega \subset \mathbb{R}^3$, with the representation $\overline{\Omega} = \bigcup_{i=1}^m \overline{\Omega_i}$. The set Ω is the union of the opaque materials. Note the important feature that Ω is disconnected. The equations that we consider are

$$(P) \begin{cases} \dfrac{\partial\theta}{\partial t} - \mathrm{div}(\kappa(\theta)\,\nabla\theta) = f & \text{in } [0,T] \times \Omega_i, \quad \text{for } i = 1,\dots,m\,, \\ -\kappa(\theta)\dfrac{\partial\theta}{\partial\vec{n}} = R - J & \text{on } [0,T] \times \Sigma\,, \end{cases}$$

where $\kappa = \kappa_i$ $(i = 1,\dots,m)$ denotes the temperature-dependent heat conductivity of the medium Ω_i.

We introduce the notation $\Gamma := \partial\Omega \setminus \Sigma$ for the part of the boundary where no radiative interactions take place. On the boundary part $[0,T] \times \Gamma$, one, or a combination, of the following conditions would make sense:

$$\theta = \theta_g\,, \qquad -\kappa(\theta)\frac{\partial\theta}{\partial\vec{n}} = \alpha\,(\theta - \theta_{\mathrm{Ext}})\,, \qquad -\kappa(\theta)\frac{\partial\theta}{\partial\vec{n}} = \sigma\,\epsilon\,(\theta^4 - \theta_{\mathrm{Ext}}^4)\,, \tag{6}$$

where the imposed temperature θ_g and the external temperature θ_{Ext} are given. In [Dru07], the decomposition $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ was assumed and the general boundary condition (4) was considered in the context of the stationary problem. However, one can readily see that the two last conditions in (6) lead to operators that are monotone, and do not complicate the analysis fundamentally. In order to keep the presentation as simple as possible, but at the same time still treating the essential difficulties, we will choose throughout this paper the Dirichlet condition

$$\theta = \theta_g \quad \text{on } [0, T] \times \Gamma\,.$$

The problem (P) has to be complemented by an initial condition.

Another similar problem arises if we do not neglect the heat conduction in the transparent medium Ω_0. In this case, we set $\overline{\Omega} := \bigcup_{i=0}^{m} \overline{\Omega_i}$, and consider the boundary condition

$$-\left[\kappa(\theta)\frac{\partial \theta}{\partial \vec{n}}\right] = R - J \qquad \text{on } \; [0, T] \times \Sigma\,,$$

where $[\cdot]$ denotes the jump of a quantity across Σ. The main difference to (P) is that the domain of computation Ω is connected, which also makes the problem simpler.

We now proceed to the weak formulation of the problem (P). It was shown in [Tii97b], [Tii97a], that one can eliminate R and J from (P) by introducing the operator

$$G \;:=\; (I - K)\,(I - (1-\epsilon)K)^{-1}\epsilon\,, \tag{7}$$

where the symbol I denotes the identity mapping, and the functions ϵ, $(1-\epsilon)$ in connection with integral operators simply imply multiplication. One can show that the operator $(I - (1-\epsilon)K)^{-1}$ is invertible in suitable Banach spaces, as will be made precise below. The condition (1), (4) on the boundary part $[0, T] \times \Sigma$ can then be rewritten as

$$-\kappa(\theta)\frac{\partial \theta}{\partial \vec{n}} = G(\sigma\,|\theta|^3\,\theta) \quad \text{on } [0, T] \times \Sigma\,. \tag{8}$$

Weak solution. We use the notations

$$Q_t :=]0\,,\, t[\times\Omega\,, \qquad \mathcal{S}_t :=]0\,,\, t[\times\Sigma \qquad \mathcal{C}_t :=]0\,,\, t[\times\Gamma\,.$$

We write Q instead of Q_T, $\mathcal{S}$ instead of $\mathcal{S}_T$, etc.

For $1 \le p\,,\, q < \infty$, we use the notation

$$L^{p,q}(Q) := \left\{ u \in L^1(Q) \,\middle|\, \left(\int_0^T \left(\int_\Omega |u|^q\,dx\right)^{\frac{p}{q}} dt\right) < \infty \right\},$$

and for $p = \infty$,

$$L^{\infty,q}(Q) := \left\{ u \in L^1(Q) \,\middle|\, \operatorname*{ess\,sup}_{t\in]0,T[} \left(\int_\Omega |u|^q\,dx\right)^{\frac{1}{q}} < \infty \right\}.$$

Analogously, one can define the spaces $L^{p,q}(\mathcal{S})$. We use also the notations $L^p(Q)$, $L^p(\mathcal{S})$ instead of $L^{p,p}(Q)$, $L^{p,p}(\mathcal{S})$.

For $1 \leq p < \infty$, we use the spaces

$$W_p^{1,0}(Q) := \left\{ u \in L^p(Q) \,\middle|\, \exists\, u_{x_i} \in L^p(Q) \text{ for } i = 1,\, 2,\, 3 \right\},$$

and

$$W_p^{1}(Q) := \left\{ u \in W_p^{1,0}(Q) \,\middle|\, \exists\, u_t \in L^p(Q) \right\},$$

where all partial derivatives are intended in the weak sense.

The space $V_2^{1,0}(Q)$ consists of all $u \in W_2^{1,0}(Q)$ such that $\operatorname*{ess\,sup}_{t\in]0,T[} \int_\Omega u^2(t,x)\,dx < \infty$.

Let $\Omega = \bigcup_{i=1}^m \Omega_i$, where Ω_i are disjoint domains, such that $\partial\Omega_i \in \mathcal{C}^{0,1}$. Taking into account that we expect that the set Σ will be at least Lipschitzian, we set

$$V^{p,q}(\Omega) \quad := \quad \left\{ u \in W^{1,p}(\Omega) \,\middle|\, \gamma(u) \in L^q(\Sigma) \right\},$$

where γ is the trace operator. The subscript Γ will denote subspaces of functions that vanish on the surface Γ. We set

$$\mathcal{V}^{p,q}(Q) := \left\{ u \in W_p^{1}(Q) \,\middle|\, \gamma(u) \in L^q(\mathcal{S}) \right\},$$

$$\mathcal{V}_0^{p,q}(Q) := \left\{ u \in W_p^{1,0}(Q) \,\middle|\, \gamma(u) \in L^q(\mathcal{S}) \right\}.$$

Using the subscript $\mathcal{C}$, we will denote subspaces of functions that vanish on the surface $]0\,,T[\times\Gamma$. Throughout the paper, we will assume that there exists positive constants κ_l, κ_u such that

$$0 < \kappa_l \leq \kappa_i(s) \leq \kappa_u < \infty \qquad \text{for all } s \in \mathbb{R}\,,\ \text{for } i = 1, \ldots, m\,. \tag{9}$$

We indicate that for a real number $s > 1$, we denote by s' the conjugated exponent $s/s-1$ to s. With these preliminaries, we can show that the following definition is meaningful:

Definition 0.1. A *weak solution* of (P) is a function $\theta \in \mathcal{V}_0^{s,4}(Q)$ such that $\theta = \theta_g$ on $\mathcal{C}$, and such that the integral relation

$$-\int_Q \theta\,\frac{\partial\psi}{\partial t} + \int_Q \kappa(\theta)\,\nabla\theta\cdot\nabla\psi + \int_{\mathcal{S}} G(\sigma\,|\theta|^3\,\theta)\,\psi = \int_\Omega \theta_0\,\psi(0) + \int_Q f\,\psi\,,$$

is valid for all $\psi \in \mathcal{V}_{\mathcal{C}}^{s',\infty}(Q)$, such that $\psi(T) = 0$ almost everywhere in Ω.

Situation of the paper. The papers [Tii97b], [Tii97a] were devoted to the stationary equations corresponding to the problem (P). The existence of weak solutions was proved for enclosure-free systems. In [Met99], a similar result was stated for the time-dependent case under the same geometrical restriction. The crucial point of the existence proof consists in ensuring coercivity for the nonlinear operator A

$$\langle A\theta,\, \psi\rangle := \int_\Omega \kappa\, \nabla\theta \cdot \nabla\psi + \int_\Sigma G(\sigma\, |\theta|^3\, \theta)\, \psi\,,$$

on a suitable Banach space. This point turns out to have an elementary solution in enclosure-free systems (see [Met99]).

In [LT01], new coercivity properties were established for the operator A, allowing to extend previous results concerning the stationary problem to enclosures. Since the coercivity inequality proved in [LT01] relies on smoothing properties (compactness) of the integral operator K, the surface Σ has at least to be of class $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$. In the same paper [LT01], a paragraph was also devoted to the time-dependent problem, and an existence result was stated for $f \in L^2(Q)$ in the case of a $\mathcal{C}^{1,\alpha}$ boundary.

In the present paper, using the tools developed in [Dru07], we will generalize these results in two directions. First, we prove the existence of weak solutions to (P), for $f \in L^p(Q)$ with arbitrary $1 \leq p \leq \infty$. Second, we propose a new method for proving the result in the case that $f \in L^2(Q)$, and that the surface Σ is only Lipschitzian.

The paper is organized as follows. The first section is devoted to existence results for the case that $f \in L^p(Q)$ with $p > 1$ arbitrary. We then briefly address the question of L^∞ regularity of weak solutions. The last section is devoted to the proof of existence in the case that $f \in L^1(Q)$. In the appendix, we have gathered some auxiliary results needed throughout the paper.

For the seek of completeness, we cite a uniqueness result.

Remark 0.2. For $i = 0, \ldots, m$, let $\kappa_i : \mathbb{R} \longrightarrow \mathbb{R}$ denote the heat conductivity of Ω_i, and assume that κ_i is a globally Lipschitz continuous function such that (9) is valid. Then, there exists at most one weak solution of (P) in the class $\mathcal{V}_0^{2,4}(Q) \cap C(0, T; L^1(\Omega))$.

Proof. The comparison method of the paper [LT01] can be extended to the case of temperature-dependent heat-conductivities. In the case that $\kappa_i \equiv$ *const* for $i = 1, \ldots, m$, the more elementary proof of [Dru07] may also be used in the time-dependent case. ☐

1 Existence of solutions

As in the stationary case, we can discuss the question of the existence of weak solutions with or without supposing $\mathcal{C}^{1,\alpha}$ regularity of the boundary Σ. In the latter case, we can only prove existence assuming a certain regularity for the data.

In the remainder of the paper, we suppose that the imposed temperature θ_g on $]0\,,\, T[\times\Gamma$ has an extension to Q. Still denoting by θ_g this extension, we suppose that $\theta_g(0)$ is a well-defined function and that it satisfies

$$\theta_g(0, x) = \theta_0(x) \ \text{ almost everywhere in } \Omega\,. \tag{10}$$

For real numbers $1 < r < \infty$, we will use the notation $r' := \frac{r}{r-1}$.

In this section, we prove the following results.

Theorem 1.1. Let $f \in L^{s_1}(0,T;L^{s_2}(\Omega))$ and $\theta_0 \in L^{s_1}(\Omega)$ with $s_1 > 3$ and $s_2 > \frac{9}{7}$. Let $\theta_g \in W^{1,0}_2(Q)$ satisfy (10), and let κ satisfy (9). Define

$$q := \begin{cases} \min\left\{\frac{3(s_2-1)}{3-2s_2}, s_1 - 1\right\} & \text{if } s_2 < \frac{3}{2}\,, \\ s_1 - 1 & \text{else}\,. \end{cases}$$

Then, there exists a weak solution $\theta \in \mathcal{V}_0^{2,\frac{4(q+1)}{3}}(Q)$ of (P) such that

$$\theta\,,\ |\theta|^{\frac{q+1}{2}} \in V^{1,0}_2(Q)\,, \qquad \theta' \in L^{\frac{q+1}{3}}(0,T;[V^{2,\frac{4(q+1)}{3}}(\Omega)]^*)\,.$$

Under the assumption that $\theta_g \in W^1_2(Q)$, we also have

$$\theta \in C(0,T;L^2(\Omega)) \ \text{ if } q \geq 2 + \frac{3}{4}\,.$$

We obtain more general results if we assume that $\Sigma \in \mathcal{C}^{1,\alpha}$.

Theorem 1.2. Let Σ in $\mathcal{C}^{1,\alpha}$. Let $f \in L^{s_1}(0,T;L^{s_2}(\Omega))$ and $\theta_0 \in L^{s_1}(\Omega)$ for any $1 < s_1,\, s_2 \leq \infty$. Define

$$q := \begin{cases} \min\left\{\frac{3(s_2-1)}{3-2s_2}, s_1 - 1\right\} & \text{if } s_2 < \frac{3}{2}\,, \\ s_1 - 1 & \text{if } s_2 \geq \frac{3}{2}\,, \end{cases} \qquad s := \min\left\{2, \frac{5\,(q+1)}{q+4}\right\}\,.$$

If $\theta_g \in W^{1,0}_{s'}(Q)$ satisfies (10), and if κ satisfies (9), then there exists a weak solution $\theta \in \mathcal{V}_0^{s,q+4}(Q)$ of (P) such that

$$|\theta|^{\frac{q+1}{2}} \in V^{1,0}_2(Q)\,, \qquad \theta' \in L^{\frac{q+4}{4}}(0,T;[V^{2,q+4}(\Omega)]^*)\,.$$

Under the assumption that $\theta_g \in W^1_2(Q)$, we also have

$$\theta \in C(0,T;L^2(\Omega))\,, \ \text{ if } q \geq 1\,.$$

Remark 1.3. Making a systematical use of the embedding relations of Lemma 4.5, we can optimize the statements of Theorem 1.1, resp. 1.2, as follows. Suppose that $f \in L^{s_1}(0,T;L^{s_2}(\Omega))$, where $s_1,\, s_2 \in [1,+\infty]$ are such that

$$s_1 \in \begin{cases} \left]\frac{2\,s_2}{3\,(s_2-1)}, \infty\right] & \text{if } s_2 \leq \frac{3}{2}\,, \\ \left]\frac{2\,s_2}{3\,(s_2-1)}, \frac{2\,s_2}{2\,s_2-3}\right] & \text{if } s_2 > \frac{3}{2}\,. \end{cases}$$

Define $\bar{q} := \frac{5\,s_1\,s_2-(3\,s_1+2\,s_2)}{3\,s_1+2\,s_2-2\,s_1\,s_2}$, and assume that $\theta_0 \in L^{\bar{q}+1}(\Omega)$. Then, under the assumptions of Theorem 1.1, one can prove the existence of a weak solution θ such that $|\theta|^{\frac{\bar{q}+1}{2}} \in V^{1,0}_2(Q)$. Note that our choice of s_1, s_2 ensures that $\frac{(\bar{q}+1)}{3} \geq 1$. For example, in the case that $f \in L^2(Q)$ and $\theta_0 \in L^6(\Omega)$, we immediately get that the weak solution θ belongs to $L^8(\mathcal{S})$. The proof of the remark involving a lot of extensive algebraic computations of suitable exponents, we will restrict ourselves to proving the two theorems stated. □

We start the proof of the theorems 1.1 and 1.2 by constructing suitable approximate solutions. We first introduce some notations. For $p \geq 5$ fixed, we define

$$\mathcal{V} := L^p(0,T; W^{1,p}_\Gamma(\Omega))\,, \qquad L\,\theta := \theta'\,,$$

$$D(L) := \Big\{\theta \in L^p(0,T; W^{1,p}_\Gamma(\Omega)) \,\Big|\, \exists\, \theta' \in L^{p'}(0,T; [W^{1,p}_\Gamma(\Omega)]^*)\,;\, \theta(0) = 0\Big\}\,,$$

The symbol θ' denotes the distributional time derivative of θ. By classical results that can be found, for example, in [Lio69] (see Ch. 3, Lem. 1.1), the operator L is a densely defined, maximal monotone linear operator from the linear subspace $D(L)$ of $\mathcal{V}$ into the dual $\mathcal{V}^*$.

For $\delta > 0$ arbitrary and for s defined as in Theorem 1.2, we choose some sequence $\{\theta_{g,\delta}\} \subset W^1_p(Q)$ such that

$$\theta_{g,\delta} \longrightarrow \theta_g \ \text{ in } W^{1,0}_{s'}(Q)\,, \qquad \theta_{0,\delta} := \theta_{g,\delta}(0) \longrightarrow \theta_0 \ \text{ in } L^{s_1}(\Omega)\,,$$

as $\delta \to 0$. Defining for $\theta \in L^p(0,T; W^{1,p}_\Gamma(\Omega))$ $\hat{\theta} := \theta + \theta_{g,\delta}$, we introduce an operator

$$\big\langle \mathcal{A}\,\theta\,, \psi\big\rangle := \delta \int_0^T \int_\Omega \Big(|\nabla\hat{\theta}|^{p-2}\nabla\hat{\theta}\cdot\nabla\psi + |\hat{\theta}|^{p-2}\,\hat{\theta}\,\psi\Big) + \int_0^T \int_\Omega \kappa(\hat{\theta})\,\nabla\hat{\theta}\cdot\nabla\psi$$
$$+ \int_0^T \int_\Sigma G(\sigma\,|\hat{\theta}|^3\,\hat{\theta})\,\psi\,,$$

where the symbol $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathcal{V}$ and its dual $\mathcal{V}^*$.

We easily see that $\mathcal{A}$ is a well-defined, bounded operator from $\mathcal{V}$ into $\mathcal{V}^*$. As a matter of fact, for $p \geq 5$ the embedding $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ is continuous. Using also the property $\|G\|_{\mathcal{L}(L^\infty(\Sigma),L^\infty(\Sigma))} \leq 1$, we can estimate

$$\left|\int_0^T \int_\Sigma G(\sigma\,|\hat{\theta}|^3\,\hat{\theta})\,\psi\right| \leq \sigma\ \text{meas}(\Sigma) \int_0^T \max_\Sigma |\hat{\theta}(t)|^4\,\max_\Sigma |\psi(t)|$$
$$\leq \bar{c} \int_0^T \|\,\hat{\theta}(t)\,\|^4_{W^{1,p}(\Omega)}\,\|\,\psi(t)\,\|_{W^{1,p}_\Gamma(\Omega)} \leq \bar{c}\,\left(\int_0^T \|\,\hat{\theta}(t)\,\|^{4p'}_{W^{1,p}(\Omega)}\right)^{\frac{1}{p'}}\ \|\,\psi\,\|_{\mathcal{V}}$$
$$\leq \bar{c}\,T^{\frac{p-5}{p}}\,(\|\,\theta\,\|^4_{\mathcal{V}} + \|\,\theta_{g,\delta}\,\|^4_{W^{1,0}_p(Q)})\ \|\,\psi\,\|_{\mathcal{V}}\ .$$

We estimate the other terms in $\mathcal{A}$ by the Hölder inequality to obtain that

$$\|\,\mathcal{A}\,\theta\,\|_{\mathcal{V}^*} \ \leq\ c_\delta\,(1 + \|\,\theta\,\|^{p-1}_{\mathcal{V}})\,.$$

Now, we can construct approximate solutions.

Proposition 1.4. Let the assumptions of Theorem 1.1 be satisfied. If $p \geq \max\{s_1', 5\}$, then for all $\delta > 0$, there exists a $\theta \in D(L)$ such that for all ψ in $\mathcal{V}$

$$\big\langle \theta'\,,\,\psi\big\rangle + \big\langle \mathcal{A}\,\theta\,,\,\psi\big\rangle \ = \ \int_0^T \int_\Omega f\,\psi - \int_0^T \int_\Omega \frac{\partial\theta_{g,\delta}}{\partial t}\,\psi\,. \tag{11}$$

Proof. Consider the estimate

$$\left|\int_0^T \int_\Omega f\,\psi\right| \le \int_0^T \max_{\overline{\Omega}} |\psi(t)| \ \| f(t) \|_{L^1(\Omega)} \ \le \ c \int_0^T \| \psi(t) \|_{W^{1,p}_\Gamma(\Omega)} \| f(t) \|_{L^1(\Omega)}$$
$$\le \ c \ \| f \|_{L^{s_1}(0,T;L^1(\Omega))} \| \psi \|_{L^p(0,T;W^{1,p}_\Gamma(\Omega))} \ .$$

We have also that

$$\left|\int_0^T \int_\Omega \frac{\partial \theta_{g,\delta}}{\partial t}\, \psi\right| \le c_\delta \ \| \psi \|_{L^p(0,T;W^{1,p}_\Gamma(\Omega))} \ .$$

Therefore, the mapping $\mathcal{F}$ given by

$$\big\langle \mathcal{F}\,,\,\psi\big\rangle := \int_0^T \int_\Omega f\,\psi - \int_0^T \int_\Omega \frac{\partial \theta_{g,\delta}}{\partial t}\, \psi\,,$$

is a well-defined element of $\mathcal{V}^*$. We observe that $\theta \in D(L)$ satisfies the statement of Proposition 1.4 if and only if the equation $\big(L+\mathcal{A}\big)\theta = \mathcal{F}$ takes place in $\mathcal{V}^*$. In order to establish the existence of the solution θ, it is therefore sufficient to prove that the operator $L+\mathcal{A}$ is surjective from $D(L)$ into $\mathcal{V}^*$. In turn, if we can prove that $\mathcal{A}$ is coercive and pseudomonotone with respect to $D(L)$, then the theory of elliptic regularization (see [Lio69], Ch. 3, Th. 1.2) ensures the surjectivity of the operator $L+\mathcal{A}$.

We at first discuss the coercivity. Making use of Lemma 4.1, (4), we can easily prove that

$$\int_{\mathcal{S}} G(\sigma\,|\hat{\theta}|^3\,\hat{\theta})\,\hat{\theta} \ge (1-\|H\|_{\mathcal{L}(L^{5/4}(\mathcal{S}),L^{5/4}(\mathcal{S}))}) \int_{\mathcal{S}} |\hat{\theta}|^5 \ge 0\,.$$

Therefore, we can write that

$$\big\langle \mathcal{A}\,\theta\,,\theta\big\rangle = \big\langle \mathcal{A}\,\theta\,,\hat{\theta}-\theta_{g,\delta}\big\rangle = \delta \ \| \ \hat{\theta} \ \|^p_{L^p(0,T;W^{1,p}(\Omega))} + \int_Q \kappa(\hat{\theta})|\nabla\hat{\theta}|^2 + \int_{\mathcal{S}} G(\sigma\,|\hat{\theta}|^3\,\hat{\theta})\,\hat{\theta}$$
$$-\big\langle \mathcal{A}\,\theta\ ,\theta_{g,\delta}\big\rangle \ge \delta \ \| \ \hat{\theta} \ \|^p_{L^p(0,T;W^{1,p}(\Omega))} - \Big|\big\langle \mathcal{A}\,\theta\ ,\theta_{g,\delta}\big\rangle\Big|\,.$$

Using Hölder's and Young's inequality, it follows that

$$\big\langle \mathcal{A}\,\theta\,,\theta\big\rangle \ge \delta \ \| \ \hat{\theta} \ \|^p_{L^p(0,T;W^{1,p}(\Omega))} - c \ \| \ \theta_{g,\delta} \ \|_{L^p(0,T;W^{1,p}(\Omega))}$$
$$\times \left(\int_0^T \| \ \hat{\theta}(t) \ \|^{p-1}_{W^{1,p}(\Omega)} + \| \ \hat{\theta}(t) \ \|^4_{W^{1,p}(\Omega)} + \| \ \hat{\theta}(t) \ \|_{W^{1,p'}(\Omega)}\right) \ge \frac{\delta}{2} \ \| \ \hat{\theta} \ \|^p_{L^p(0,T;W^{1,p}(\Omega))} - C_\delta\,,$$

with a constant C_δ that depends on δ, but whose precise value is not needed.

We now prove that $\mathcal{A}$ is pseudomonotone. Let $\theta_k \rightharpoonup \theta$ in $D(L)$. We assume that $\limsup_{k\to\infty} \big\langle \mathcal{A}\,\theta_k\,,\,\theta_k-\theta\big\rangle \le 0$. The weak convergence in $D(L)$ means that

$$\theta_k \rightharpoonup \theta \text{ in } \mathcal{V}\,, \quad \theta_k' \rightharpoonup \theta' \text{ in } \mathcal{V}^* \ . \tag{12}$$

Applying the well-known compactness result of [Lio69], Ch. 1, Th. 5.1, we can find a subsequence, still denoted by $\{\theta_k\}$, such that

$$\theta_k \longrightarrow \theta \text{ in } L^p(0,T;L^p(\Omega))\,. \tag{13}$$

Using the inequality

$$\|u\|_{L^p(\Sigma)} \leq \gamma\, \|u\|_{W^{1,p}(\Omega)} + c_\gamma\, \|u\|_{L^p(\Omega)}\,, \tag{14}$$

which holds for any u in $W^{1,p}(\Omega)$ and arbitrary small $\gamma > 0$, we obtain from (12) and (13) the existence of a (not relabelled) subsequence such that

$$\theta_k \longrightarrow \theta \quad \text{ in } L^p(0,T;L^p(\Sigma))\,.$$

Using the monotonicity of the $p-$Laplace terms, it is then easy to verify that

$$\liminf_{k\to\infty} \left\langle \mathcal{A}\,\theta_k\;,\,\theta_k - \phi\right\rangle \;\geq\; \left\langle \mathcal{A}\,\theta\,,\,\theta-\phi\right\rangle,$$

for all ϕ in $\mathcal{V}$. The pseudomonotonicity, and with it the proposition, is proved. □

The next point consists in obtaining uniform estimates for the sequence of approximate solutions.

Proposition 1.5. Let $f \in L^{s_1}(0,T;L^{s_2}(\Omega))$ and $\theta_0 \in L^{s_1}(\Omega)$. Define

$$q := \begin{cases} \min\left\{\frac{3(s_2-1)}{3-2s_2},\, s_1-1\right\} & \text{if } \frac{3}{2} < s_2\,,\\ s_1 - 1 & \text{else}\,, \end{cases} \qquad s := \min\left\{2,\, \frac{5(q+1)}{q+4}\right\}.$$

Let $\theta_g \in W^{1,0}_{s'}(Q)$, and let κ satisfy (9). For the sequence of solutions $\{\theta_\delta\}$ of Proposition 1.4, we have the following *a priori* estimates:

(1) If $s_1 \geq 3$ and $s_2 \geq \frac{9}{7}$ we have

$$\|\,\theta_\delta\,\|_{V^{1,0}_2(Q)} + \|\,|\theta_\delta|^{\frac{q+1}{2}}\,\|_{V^{1,0}_2(Q)} + \|\,\theta_\delta\,\|_{L^{\frac{4(q+1)}{3}}(\mathcal{S})} \leq C + C_\delta\,. \tag{15}$$

(2) Let $\Sigma \in \mathcal{C}^{1,\alpha}$. Then, for any $1 < s_1, s_2 \leq \infty$, we have

$$\|\,|\theta_\delta|^{\frac{q+1}{2}}\,\|_{V^{1,0}_2(Q)} + \|\,\theta_\delta\,\|_{W^{1,0}_s(Q)} + \|\,\theta_\delta\,\|_{L^{q+4}(\mathcal{S})} \leq C + C_\delta\,. \tag{16}$$

The constant C depends continuously on $\|\,f\,\|_{L^{s_1}(0,T;L^{s_2}(\Omega))}$, on $\|\,u_0\,\|_{L^{s_1}(\Omega)}$, and on $\|\theta_g\|_{W^{1,0}_{s'}(Q)}$. The sequence $\{C_\delta\}$ depends on our approximation method, and we have $C_\delta \to 0$ as $\delta \to 0$.

Proof. In the following we write for convenience θ instead of θ_δ. We will prove the claim in the homogeneous case $\theta_g = 0$ on $\mathcal{C}$. The result in the general case can be obtained by making only slight modifications to this proof.

The method that we use is quite straightforward. For the family of parameters $0 < q < \infty$, we would like to test the approximate equation (11) with the signed powers $|\theta|^{q-1}\,\theta$, and obtain for $t_1 \in]0,T[$ the inequality

$$\frac{1}{q+1}\int_\Omega |\theta(t_1)|^{q+1} + \int_{Q_{t_1}} \frac{4q}{(q+1)^2}\,\kappa(\theta)\left|\nabla|\theta|^{\frac{q+1}{2}}\right|^2 + \int_{\mathcal{S}_{t_1}} \sigma\, G(|\theta|^3\,\theta)\,|\theta|^{q-1}\,\theta$$
$$\leq \int_{Q_{t_1}} f\,|\theta|^{q-1}\theta + \frac{1}{q+1}\int_\Omega |\theta_{0,\delta}|^{q+1}\,, \tag{17}$$

with the notation $\theta_{0,\delta} := \theta_{g,\delta}(0)$. However, testing with $|\theta|^{q-1}\,\theta$ is not directly possible since this function may not belong to $L^p(0,T;W^{1,p}(\Omega))$. Therefore, we have to construct diverse regularizations of the test functions. That is the only reason why our estimates may look somewhat technical.

First step: For a number $q \geq 1$, and a parameter $k > 0$, we consider the function $g = g_{q,k}$, $F = F_{q,k} \in C(\mathbb{R})$ given by

$$g(s) := \begin{cases} k^q & \text{if } s > k\,,\\ |s|^{q-1}\,s & \text{if } |s| \leq k\,,\\ -k^q & \text{if } s < -k\,.\end{cases} \qquad F(s) = \begin{cases} k^q\,s + \left(\frac{1}{q+1} - 1\right)\,k^{q+1} & \text{if } s > k\,,\\ \frac{1}{q+1}\,|s|^{q+1} & \text{if } -k \leq s \leq k\,,\\ -k^q\,s + \left(\frac{1}{q+1} - 1\right)\,k^{q+1} & \text{if } s < -k\,.\end{cases}$$

The function F is the primitive function of g that vanishes at zero. We introduce the notation $s^{(k)} := \text{sign}(s)\,\min\{|s|,\,k\}$, and observe that $F(s) \geq \frac{1}{q+1}\,|s^{(k)}|^{q+1}$. Applying Lemma 4.6, we can for all $t_1 < T$ produce the inequality

$$\frac{1}{q+1}\int_\Omega \left|\theta^{(k)}(t_1)\right|^{q+1} + \int_{Q_{t_1}} \delta\left[|\nabla\theta|^{p-2}\,\nabla\theta\cdot\nabla(|\theta^{(k)}|^{q-1}\,\theta^{(k)}) + |\theta|^{p-2}\,\theta\,|\theta^{(k)}|^{q-1}\,\theta^{(k)}\right]$$
$$+ \int_{Q_{t_1}} \kappa(\theta)\,\nabla\theta\cdot\nabla(|\theta^{(k)}(t)|^{q-1}\,\theta^{(k)}) + \int_{\mathcal{S}_{t_1}} G(\sigma\,|\theta|^3\,\theta)\,|\theta^{(k)}|^{q-1}\,\theta^{(k)}$$
$$\leq \int_{Q_{t_1}} f\,|\theta^{(k)}|^{q-1}\,\theta^{(k)} + \int_\Omega F\Big(\theta^{(k)}(0)\Big)\,.$$

Now we can use the following facts

$$\nabla\theta\cdot\nabla(|\theta^{(k)}|^{q-1}\,\theta^{(k)}) = \frac{4q}{(q+1)^2}\left|\nabla|\theta^{(k)}|^{\frac{q+1}{2}}\right|^2, \qquad |\theta|^{p-2}\,\theta\,\Big(|\theta^{(k)}|^{q-1}\,\theta^{(k)}\Big) \geq 0\,. \tag{18}$$

By the fact that G is selfadjoint (see Lemma 4.1, (4) below), we also have that

$$\int_{\mathcal{S}_{t_1}} G(\sigma\,|\theta|^3\,\theta)\;|\theta^{(k)}|^{q-1}\,\theta^{(k)} = \int_{\mathcal{S}_{t_1}} \sigma\,|\theta|^3\,\theta\;G(|\theta^{(k)}|^{q-1}\,\theta^{(k)})$$
$$= \int_{\mathcal{S}_{t_1}} \sigma\,\Big(|\theta|^3\,\theta - |\theta^{(k)}|^3\,\theta^{(k)}\Big)\,G(|\theta^{(k)}|^{q-1}\,\theta^{(k)}) + \int_{\mathcal{S}_{t_1}} \sigma\,|\theta^{(k)}|^3\,\theta^{(k)}\;G(|\theta^{(k)}|^{q-1}\,\theta^{(k)})\,.$$

Now, using Lemma 4.1, (4), we have in the sets $\{(t,z) \in \mathcal{S} \,:\, \theta(t,z) > k\}$ the inequality

$$G(|\theta^{(k)}|^{q-1}\,\theta^{(k)}) = k^q - H(|\theta^{(k)}|^{q-1}\,\theta^{(k)}) \geq (1- \parallel H \parallel_{\mathcal{L}(\infty,\infty)})\, k^q \geq 0\,.$$

By an analogous consideration concerning the sets $\{(t,z) \in \mathcal{S} \,:\, \theta < -k\}$, we obtain that

$$\int_{\mathcal{S}_{t_1}} \sigma \left(|\theta|^3\,\theta - |\theta^{(k)}|^3\,\theta^{(k)} \right) G(|\theta^{(k)}|^{q-1}\,\theta^{(k)}) \geq 0\,. \tag{19}$$

The facts (18) and (19) yield the relation

$$\frac{1}{q+1} \int_\Omega \left|\theta^{(k)}(t_1)\right|^{q+1} + \int_{Q_{t_1}} \frac{4q}{(q+1)^2}\,\kappa(\theta)\left|\nabla|\theta^{(k)}|^{\frac{q+1}{2}}\right|^2 + \int_{\mathcal{S}_{t_1}} \sigma\,|\theta^{(k)}|^3\,\theta^{(k)}\,G(|\theta^{(k)}|^{q-1}\,\theta^{(k)})$$
$$\leq \int_{Q_{t_1}} f\,|\theta^{(k)}|^{q-1}\,\theta^{(k)} + \int_\Omega F\left(\theta^{(k)}(0)\right).$$

We easily can see that for $k \longrightarrow \infty$, $\nabla|\theta^{(k)}|^{\frac{q+1}{2}} \rightharpoonup \nabla|\theta|^{\frac{q+1}{2}}$ in $W_2^{1,0}(Q)$. Hence, by using the lower semicontinuity of the norm and monotone convergence, we can pass to the limit $k \to \infty$ in the last relation and finally obtain the inequality (17). Now, define

$$w := |\theta|^{\frac{q+1}{2}}\,. \tag{20}$$

Since $G = I - H$ with a positive operator H, we have

$$\int_{\mathcal{S}_{t_1}} G(\sigma\,|\theta|^3\,\theta)\,|\theta|^{q-1}\,\theta \geq \int_{\mathcal{S}_{t_1}} G(\sigma\,|\theta|^4)\,|\theta|^q = \int_{\mathcal{S}_{t_1}} G(\sigma\,w^{\frac{8}{q+1}})\,w^{\frac{2q}{q+1}}\,.$$

Rewriting (17), we obtain that

$$\int_\Omega w^2(t_1) + \int_{Q_{t_1}} |\nabla w|^2 + \int_{\mathcal{S}_{t_1}} G(\sigma\,w^{\frac{8}{q+1}})\,w^{\frac{2q}{q+1}} \leq c_q \left(\int_\Omega |\theta_{0,\delta}|^{q+1} + \int_{Q_{t_1}} |f|\,w^{\frac{2q}{q+1}} \right).$$

Using Sobolev's embedding relations and Young's inequality, we can write

$$\begin{aligned}
\int_0^{t_1}\int_\Omega |f|\,w^{\frac{2q}{q+1}} &\leq \int_0^{t_1} \parallel f(t) \parallel_{L^{\frac{3(q+1)}{2q+3}}(\Omega)} \parallel w(t) \parallel_{L^6(\Omega)}^{\frac{2q}{q+1}} \\
&\leq c \int_0^{t_1} \parallel f(t) \parallel_{L^{\frac{3(q+1)}{2q+3}}(\Omega)} \parallel w(t) \parallel_{W^{1,2}_\Gamma(\Omega)}^{\frac{2q}{q+1}} \\
&\leq \gamma \int_0^{t_1} \parallel w(t) \parallel_{W^{1,2}_\Gamma(\Omega)}^2 + c_\gamma \int_0^{t_1} \parallel f(t) \parallel_{L^{\frac{3(q+1)}{2q+3}}(\Omega)}^{q+1}\,.
\end{aligned}$$

Choosing $\gamma > 0$ sufficiently small, we can achieve

$$\int_\Omega w(t_1)^2 + \int_{Q_{t_1}} |\nabla w|^2 + \int_{\mathcal{S}_{t_1}} G(\sigma\,w^{\frac{8}{q+1}})\,w^{\frac{2q}{q+1}}$$
$$\leq c \left(\int_\Omega |\theta_{0,\delta}|^{q+1} + \int_{Q_{t_1}} w^2 + \int_0^{t_1} \parallel f(t) \parallel_{L^{\frac{3(q+1)}{2q+3}}(\Omega)}^{q+1} \right). \tag{21}$$

By the Gronwall inequality, we first obtain that $w = |\theta_\delta|^{\frac{q+1}{2}}$ remains bounded in $L^{\infty,2}(Q)$. This implies immediately a bound in the norm of $W^{1,0}_{2,\mathcal{C}}(Q)$. More precisely, we find positive constants C_1, C_2 such that

$$\| \, |\theta|^{\frac{q+1}{2}} \, \|_{L^{\infty,2}(Q)} \leq C_1 \, , \qquad \| \, |\theta|^{\frac{q+1}{2}} \, \|_{W^{1,0}_{2,\mathcal{C}}(Q)} \leq C_2 \, , \tag{22}$$

where C_1, C_2 depend continuously on q, $\| \, f \, \|_{L^{q+1}(0,T;L^{\frac{3(q+1)}{2q+3}}(\Omega))}$, $\| \, \theta_0 \, \|_{L^{q+1}(\Omega)}$. Note that this provides already an estimate of θ on the boundary $\mathcal{S}$. In view of Lemma 4.5, the embedding $V^{1,0}_2(Q) \hookrightarrow L^{8/3}(\mathcal{S})$ is continuous. By (20), we can write $\int_{\mathcal{S}} |\theta|^{\frac{4\,(q+1)}{3}} = \int_{\mathcal{S}} w^{\frac{8}{3}}$. With the help of estimate (22), we obtain that

$$\|\theta\|^{\frac{4\,(q+1)}{3}}_{L^{\frac{4\,(q+1)}{3}}(\mathcal{S})} = \|w\|^{\frac{8}{3}}_{L^{\frac{8}{3}}(\mathcal{S})} \leq c\, \|w\|^{8/3}_{V^{1,0}_2(Q)} \leq C_3 \, . \tag{23}$$

If $\Sigma \in \mathcal{C}^{1,\alpha}$, we can apply Lemma 4.3, and we have in addition the inequality

$$\int_0^{t_1} \int_\Sigma G(\sigma\, w^{\frac{8}{q+1}})\, w^{\frac{2q}{q+1}} \geq c_{1,q} \int_0^{t_1} \int_\Sigma w^{\frac{2(q+4)}{q+1}} - c_{2,q} \int_0^{t_1} \left(\int_\Sigma w \right)^{\frac{2(q+4)}{q+1}} \, ,$$

where we chose $\psi := w^{\frac{2q}{q+1}}$, $r := \frac{4}{q}$, $s := \frac{q+1}{2q}$. By Lemma 4.5, the embedding $V^{1,0}_2(Q) \hookrightarrow L^{\infty,4/3}(\mathcal{S})$ is continuous. Taking (22) into account, we obtain that

$$\int_0^T \left(\int_\Sigma w \right)^{\frac{2(q+4)}{q+1}} \leq T \, \| \, w \, \|^{\frac{2(q+4)}{q+1}}_{L^{\infty,1}(\mathcal{S})} \leq T\, c \, \| \, w \, \|^{\frac{2(q+4)}{q+1}}_{V^{1,0}_2(Q)} \leq C_3 \, .$$

Reconsidering (21), we now obtain for arbitrary $t_1 < T$ that

$$\int_0^{t_1} \int_\Sigma w^{\frac{2(q+4)}{q+1}} \leq C_4 \quad \Longrightarrow \quad \| \, \theta_\delta \, \|^{q+4}_{L^{q+4}(\mathcal{S})} \leq C_4 \, . \tag{24}$$

Second step: An easy calculation shows us that the estimates obtained in the first step involve at least the norms $\| \, f \, \|_{L^2(0,T;L^{6/5}(\Omega))}$ and $\| \, \theta_0 \, \|_{L^2(\Omega)}$. Next we search for estimates in the case that f is less regular. For an arbitrary small parameter $\alpha > 0$ and $0 < q < 1$, we consider functions $g = g_{\alpha,q}$, $F = F_{\alpha,q} \in C(\mathbb{R})$ given by

$$g(s) := (|s| + \alpha)^{q-1}\, s, \quad F(t) := \begin{cases} -\frac{(\alpha-t)^q\, t}{q} - \frac{(\alpha-t)^{q+1}}{q\,(q+1)} + \frac{\alpha^{q+1}}{q\,(q+1)} & \text{if } t \leq 0 \, , \\ \frac{(\alpha+t)^q\, t}{q} - \frac{(\alpha+t)^{q+1}}{q\,(q+1)} + \frac{\alpha^{q+1}}{q\,(q+1)} & \text{if } t > 0 \, . \end{cases} .$$

Observe that g vanishes at zero and that

$$g'(s) = (|s| + \alpha)^{q-2}\, (q\, |s| + \alpha) \, .$$

Clearly, F is the primitive function of g that vanishes at zero. Observe that the function F is positive and that for all $t \in \mathbb{R}$, $F_{\alpha,q}(t) \longrightarrow \frac{|t|^{q+1}}{q+1}$ for $\alpha \to 0$. Using again Lemma 4.6, we obtain that

$$\int_\Omega F\Big(\theta(t_1)\Big) + \int_{Q_{t_1}} \delta\Big[|\nabla\theta|^{p-2}\,\nabla\theta\cdot\Big((\alpha+|\theta|)^{q-2}\,(\alpha+q|\theta|)\nabla\theta\Big)+|\theta|^p(\alpha+|\theta|)^{q-1}\Big]$$

$$+\int_{Q_{t_1}}\kappa(\theta)\,\nabla\theta\cdot\Big((\alpha+|\theta|)^{q-2}\,(\alpha+q|\theta|)\nabla\theta\Big)+\int_{\mathbb{S}_{t_1}} G(\sigma\,|\theta|^3\,\theta)\,\theta(\alpha+|\theta|)^{q-1}$$

$$=\int_{Q_{t_1}} f\,\theta(\alpha+|\theta|)^{q-1}+\int_\Omega F\Big(\theta_{0,\delta}\Big)\,.$$

Since the terms involving the $p-$power are positive, we can drop them without changing the sense of the inequality. Letting $\alpha\to 0$ in the last relation, we obtain by Fatou's lemma the relation (17), this time with $0<q<1$. Repeating the argumentation that follows (17), we obtain estimate (22) also in this case; we obtain (24) if $\Sigma\in\mathcal{C}^{1,\alpha}$.

Third step:

The next step consist in obtaining estimates on $\nabla\theta$. By the separability of $W^{1,p}_\Gamma(\Omega)$, we can conclude from Proposition 1.4 that

$$\Big\langle\theta'_\delta(t)\,,\,\psi\Big\rangle+\int_\Omega\delta\Big[|\nabla\theta_\delta(t)|^{p-2}\,\nabla\theta_\delta(t)\cdot\nabla\psi+|\theta_\delta(t)|^{p-2}\,\theta(t)\,\psi\Big]+\int_\Omega\kappa(\theta_\delta(t))\,\nabla\theta_\delta(t)\cdot\nabla\psi$$

$$+\int_\Sigma G(\sigma|\theta_\delta(t)|^3\,\theta_\delta(t))\,\psi=\int_\Omega f(t)\,\psi\,, \tag{25}$$

for all ψ in $W^{1,p}_\Gamma(\Omega)$, and for almost all $t\in[0,T]$. The null set where this relation fails depends neither on δ, nor on ψ. Here, the symbol $\langle\cdot,\cdot\rangle$ stands for the duality product of $W^{1,p}(\Omega)$.

We first suppose that $f\in L^2(0,T;L^{\frac{6}{5}}(\Omega))$ and $\theta_0\in L^2(\Omega)$. Choosing $\psi=\theta(t)$ in (25), and integrating on $[0,t_1]$, we obtain for almost all $t_1\in[0,T]$ that

$$\frac{1}{2}\int_\Omega|\theta(t_1)|^2+\int_0^{t_1}\int_\Omega\kappa(\theta(t))\,|\nabla\theta(t)|^2\le\frac{1}{2}\int_\Omega|\theta_{0,\delta}|^2+\int_0^{t_1}\int_\Omega f(t)\,\theta(t)$$

$$\le\int_\Omega|\theta_{0,\delta}|^2+c\int_0^{t_1}\parallel f(t)\parallel_{L^{\frac{6}{5}}(\Omega)}\parallel\theta(t)\parallel_{W^{1,2}(\Omega)}$$

$$\le\frac{1}{2}\int_\Omega|\theta_{0,\delta}|^2+\int_0^{t_1}c_\gamma\parallel f(t)\parallel^2_{L^{\frac{6}{5}}(\Omega)}+\gamma\parallel\theta(t)\parallel^2_{W^{1,2}(\Omega)}\,. \tag{26}$$

The Gronwall inequality implies that

$$\operatorname*{ess\,sup}_{t_1\in]0,T[}\int_\Omega|\theta(t_1)|^2\le C_5\,,\qquad\parallel\nabla\theta\parallel_{L^2(0,T;L^2(\Omega))}\le C_6\,, \tag{27}$$

with constants C_5, C_6 independent of δ. If the regularity of θ_0 and f does not allow to argue in this way, we write

$$\int_Q |\nabla\theta|^r = \int_Q \frac{|\nabla\theta|^r}{|\theta|^{\frac{(1-q)r}{2}}}\ |\theta|^{\frac{(1-q)r}{2}} \leq \Big(\int_Q \frac{|\nabla\theta|^2}{|\theta|^{1-q}}\Big)^{\frac{r}{2}}\ \Big(\int_Q |\theta|^{\frac{(1-q)r}{2-r}}\Big)^{\frac{2-r}{2}}$$

$$\leq c\ \|\ \nabla|\theta|^{\frac{q+1}{2}}\ \|^r_{L^2(Q)}\ \|\ |\theta|^{\frac{q+1}{2}}\ \|^{\frac{(1-q)r}{q+1}}_{L^{\frac{2(1-q)r}{(2-r)(q+1)}}(Q)}\ .$$

Now, in view of estimate (22) and of Lemma 4.5, we see that if the relation

$$2(1-q)r \leq \frac{10}{3}(2-r)\,(q+1)\,,$$

is satisfied, then $\nabla\theta$ will be uniformely bounded in $L^r(Q)$. This is true exactly for the range $1 \leq r \leq \frac{5\,(q+1)}{q+4}$. In order to finish the proof of the proposition, we must determine, according to the regularity of f, θ_0, for which range of q we will obtain estimates with this method. In other words, we look at the values of q for which $q+1 \leq s_1$ and $\frac{3\,(q+1)}{2q+3} \leq s_2$. The result of this elementary calculation exactly proves the statement of the proposition. □

In order to pass to the limit, we state in the following lemma some technical estimates.

Lemma 1.6. Let the hypothesis of Proposition 1.5 be satisfied, and define the number q as in this proposition.

(1) If $s_1 \geq 3$ and $s_2 \geq \frac{9}{7}$, we have $\|\ \theta'_\delta\ \|_{L^{\frac{q+1}{3}}(0,T;[W^{1,p}_\Gamma(\Omega)]^*)} \leq C$.

(2) Let $\Sigma \in \mathcal{C}^{1,\alpha}$. Then, for any $1 < s_1, s_2 \leq \infty$, we have $\|\ \theta'_\delta\ \|_{L^{\frac{q+4}{4}}(0,T;[W^{1,p}_\Gamma(\Omega)]^*)} \leq C$.

Proof. For the sake of notational simplicity, we write θ instead of θ_δ. We prove the claim in the case of homogeneous boundary conditions. In (25), we test with $\theta(t)$, and by usual considerations, we obtain the inequality

$$\big\langle \theta'(t)\,,\,\theta(t)\big\rangle + \delta\,\|\theta(t)\|^p_{W^{1,p}(\Omega)} \leq \int_\Omega f(t)\,\theta(t)\,.$$

We integrate this inequality on $]0,t_1[$, and since $p > s'_1$, we get

$$\|\theta(t_1)\|^2_{L^2(\Omega)} + \delta\int_0^{t_1}\|\theta(t)\|^p_{W^{1,p}(\Omega)} \leq \|\theta_{g,\delta}(0)\|^2_{L^2(\Omega)} + c\,\|f\|_{L^{s_1}(0,T;L^1(\Omega))}\,\|\theta\|_{L^p(0,T;W^{1,p}(\Omega))}\,.$$

Therefore,

$$\delta\,\|\theta\|^p_{L^p(0,T;W^{1,p}(\Omega))} \leq \|\theta_{g,\delta}(0)\|^2_{L^2(\Omega)} + c\,\|f\|_{L^{s_1}(0,T;L^1(\Omega))}\,\|\theta\|_{L^p(0,T;W^{1,p}(\Omega))}\,.$$

If $\|\theta\|_{L^p(0,T;W^{1,p}(\Omega))} \geq 1$, it then follows that

$$\delta\,\|\theta\|^{p-1}_{L^p(0,T;W^{1,p}(\Omega))} \leq \|\theta_{g,\delta}(0)\|^2_{L^2(\Omega)} + c\,\|f\|_{L^{s_1}(0,T;L^1(\Omega))}\,.$$

Thus, we get that

$$\left\|\delta^{\frac{1}{p-1}}\,\theta\right\|^{p-1}_{L^p(0,T;W^{1,p}(\Omega))} \le \max\left\{\delta\,,\ \|\theta_{g,\delta}(0)\|^2_{L^2(\Omega)} + c\,\|f\|_{L^{s_1}(0,T;L^1(\Omega))}\right\}\,. \tag{28}$$

Starting again from (25), we can write for $\psi \in W^{1,p}_\Gamma(\Omega)$,

$$\begin{aligned}\left|\langle\theta'(t)\,,\,\psi\rangle\right| \le &\int_\Omega \left[\delta\,|\nabla\theta(t)|^{p-1} + \kappa(\theta(t))\,|\nabla\theta(t)|\right]|\nabla\psi| \\ &+ \int_\Omega \left(\delta\,|\theta(t)|^{p-1} + |f(t)|\right)|\psi| + \int_\Sigma \left|G(\sigma|\theta(t)|^3\,\theta(t))\right|\,|\psi| \\ \le\ & \delta\ \|\ \theta(t)\ \|^{\frac{p}{p'}}_{W^{1,p}(\Omega)}\ \|\ \psi\ \|_{W^{1,p}_\Gamma(\Omega)} + \kappa_u\ \|\ \nabla\theta(t)\ \|_{L^{p'}(\Omega)}\ \|\ \nabla\psi\ \|_{L^p(\Omega)} \\ &+ \|\ f(t)\ \|_{L^1(\Omega)}\ \max_{\overline{\Omega}}|\psi| + c\ \|\ \theta(t)\ \|^4_{L^4(\Sigma)}\ \max_{\overline{\Omega}}|\psi|\,.\end{aligned}$$

Using one more time the continuity of the embedding $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$, we get

$$\|\ \theta'(t)\ \|_{[W^{1,p}_\Gamma(\Omega)]^*} \le c\left(\delta\ \|\ \theta(t)\ \|^{\frac{p}{p'}}_{W^{1,p}(\Omega)} + \|\ \nabla\theta(t)\ \|_{L^{p'}(\Omega)} + \|\ f(t)\ \|_{L^1(\Omega)} + \|\ \theta(t)\ \|^4_{L^4(\Sigma)}\right)\,. \tag{29}$$

We have $\delta\ \|\ \theta(t)\ \|^{\frac{p}{p'}}_{W^{1,p}(\Omega)} = \|\ \delta^{\frac{1}{p-1}}\,\theta(t)\ \|^{p-1}_{W^{1,p}(\Omega)}$, which, in view of (28), is uniformly bounded in the space $L^{p'}(0,T)$.

With the notations of Proposition 1.5, we find for $q \ge 2$ that the sequence $\{\|\ \theta_\delta\ \|^4_{L^4(\Sigma)}\}$ is bounded in the space $L^{\frac{q+1}{3}}(0,T)$. If $\Sigma \in \mathcal{C}^{1,\alpha}$, then $\{\|\ \theta_\delta\ \|^4_{L^4(\Sigma)}\}$ is even bounded in the space $L^{\frac{q+4}{4}}(0,T)$.

Thus, by means of (22), we get

$$\|\ \theta'_\delta\ \|_{L^{p_1}(0,T;[W^{1,p}_\Gamma(\Omega)]^*)} \le C\,, \tag{30}$$

with $p_1 = \min\left\{\frac{q+1}{3}\,,\,p'\,,\,s_1\right\} \ge 1$. If we assume that $\Sigma \in \mathcal{C}^{1,\alpha}$, we see in view of (24), that the right-hand side in (29) is bounded in the space $L^{p_1}(0,T)$ with $p_1 = \min\left\{p'\,,\,s_1\,,\,\frac{q+4}{4}\right\} > 1$. □

Proof of Theorem 1.1 and 1.2. We start from Propostion 1.5, (1), where we made no assumption on the regularity of the surface Σ.

Thanks to the *a priori* estimates (15) and the compactness theorems of [Lio69] generalized in [Sim86], we can find a sequence $\delta \to 0$ and a function θ such that

$$\theta_\delta \rightharpoonup \theta\ \text{ in } W^{1,0}_2(Q)\,,\quad \theta_\delta \longrightarrow \theta\ \text{ in } L^2(Q)\,,\quad \theta_\delta \longrightarrow \theta\ \text{ a. e. in } Q\,. \tag{31}$$

By means of the inequality (14), we also find subsequences such that

$$\theta_\delta \longrightarrow \theta\ \text{ in } L^2(\mathcal{S})\,,\quad \theta_\delta \longrightarrow \theta\ \text{ a. e. on } \mathcal{S}\,. \tag{32}$$

In addition, we see that there must exists some w , u such that

$$|\theta_\delta|^{\frac{q+1}{2}} \rightharpoonup w \;\text{ in } W_2^{1,0}(Q)\,, \quad \theta_\delta\,|\theta_\delta|^3 \rightharpoonup u \;\text{ in } L^{\frac{q+1}{3}}(\mathcal{S})\,. \tag{33}$$

But then, the convergence properties (31) and (32) imply that

$$w = |\theta|^{\frac{q+1}{2}}\,, \qquad u = \theta\,|\theta|^3\,.$$

If we start from Proposition 1.5, (2), which corresponds to the supposition that $\Sigma \in \mathcal{C}^{1,\alpha}$, we find, by the same arguments, sequences with the properties

$$\begin{aligned}
&\theta_\delta \rightharpoonup \theta \;\text{ in } W_s^{1,0}(Q)\,, \quad \theta_\delta \longrightarrow \theta \;\text{ in } L^2(Q) \text{ and in } L^2(\mathcal{S})\,,\\
&\theta_\delta \longrightarrow \theta \;\text{ a. e. in } Q \text{ and a. e. on } \mathcal{S}\,.
\end{aligned} \tag{34}$$

In addition, we see that there must exists some w, u such that

$$|\theta_\delta|^{\frac{q+1}{2}} \rightharpoonup w \;\text{ in } W_2^{1,0}(Q)\,, \quad \theta_\delta\,|\theta_\delta|^3 \rightharpoonup u \;\text{ in } L^{\frac{q+4}{4}}(\mathcal{S})\,. \tag{35}$$

But then, the convergence properties (34) imply that

$$w = |\theta|^{\frac{q+1}{2}}\,, \qquad u = \theta\,|\theta|^3\,.$$

Now, testing in (11) with an arbitrary ψ in $C^\infty(\overline{Q})$ which vanishes in $\{0\} \times \Omega$ and on $\mathcal{C}$, we can write

$$\begin{aligned}
&-\int_0^T \Big(\theta_\delta(t), \psi'(t)\Big)_{L^2(\Omega)} + \ldots + \int_0^T \int_\Omega \kappa(\theta_\delta)\,\nabla\theta_\delta \cdot \nabla\psi + \int_0^T \int_\Sigma G(\sigma\,|\theta_\delta|^3\,\theta_\delta)\,\psi\\
&\quad = (\theta_\delta(0), \psi(0))_{L^2} + \int_0^T \int_\Omega f\,\psi\,,
\end{aligned}$$

where the $(\ldots)$ represents the terms involving the $p-$power. Passing to the limit in the last relation, we easily can show that

$$\begin{aligned}
-\int_0^T \Big(\theta(t), \psi'(t)\Big)_{L^2(\Omega)} + \int_0^T \int_\Omega \kappa(\theta)\,\nabla\theta \cdot \nabla\psi + \int_0^T \int_\Sigma G(\sigma\,|\theta|^3\,\theta)\,\psi&\\
= (\theta_0, \psi(0))_{L^2} + \int_0^T \int_\Omega f\,\psi\,.&
\end{aligned} \tag{36}$$

In addition, we have, almost everywhere on $\mathcal{C}$, that

$$\theta(t,z) = \lim_{\delta\to 0} \theta_\delta(t,z) \;=\; \lim_{\delta\to 0} \theta_{g,\delta}(t,z) \;=\; \theta_g(t,z)\,.$$

The result stated on the existence on θ' follows easily from the estimates of Proposition 1.5. We now prove the continuity assumption. First we note that $\theta \in L^5(\mathcal{S})$, is true if $\frac{4(q+1)}{3} \geq 5$ in the case that $\Sigma \in \mathcal{C}^{0,1}$, and if $q + 4 \geq 5$ in the case that $\Sigma \in \mathcal{C}^{1,\alpha}$. We easily verify that these conditions are exactly satisfied when f has the regularity stated by the theorem. For any $\phi \in C^\infty(\overline{\Omega})$, that vanishes on Γ, and $\zeta \in C_c^\infty(0,t_1)$, where $t_1 < T-h$, we can consider the test function $(\phi(x)\,\zeta(t))_{(\underline{h})}$ and use it in (36). Observe that

$$-\int_0^T \Big(\theta(t)\,,\,\psi'(t)\Big)_{L^2(\Omega)} = -\int_0^T \Big(\theta_{(h)}(t)\,,\,\phi\Big)_{L^2(\Omega)}\,\zeta'(t)\,.$$

This implies the relation

$$\frac{\partial}{\partial t}\Big(\theta_{(h)}\,,\,\phi\Big)_{L^2(\Omega)} \;=\; -\int_\Omega \{\kappa(\theta)\,\nabla\theta\}_{(h)}\cdot\nabla\phi - \int_\Sigma \{G(\sigma\,|\theta|^3\,\theta)\}_{(h)}\,\phi + \int_\Omega f_{(h)}\,\phi\,.$$

But since $\theta_{(h)} \in W^{1,1}_2(Q)$, we can even write

$$\Big(\frac{\partial}{\partial t}\theta_{(h)}\,,\,\phi\Big)_{L^2(\Omega)} \;=\; -\int_\Omega \{\kappa(\theta)\,\nabla\theta\}_{(h)}\cdot\nabla\phi - \int_\Sigma \{G(\sigma\,|\theta|^3\,\theta)\}_{(h)}\,\phi + \int_\Omega f_{(h)}\,\phi\,.$$

We recall that $\tilde\theta := \theta - \theta_g$ vanishes on $\mathcal{C}$. For $0 < h_1 < h_2 < T - t_1$, it is possible to choose

$$\phi = (\theta-\theta_g)_{(h_1)} - (\theta-\theta_g)_{(h_2)}\,.$$

Subtracting the respective integral identities, we get

$$\frac{1}{2}\,\frac{\partial}{\partial t}\Big\|\tilde\theta_{(h_1)}(t) - \tilde\theta_{(h_2)}(t)\Big\|^2_{L^2(\Omega)} = -\int_\Omega \Big[\{\kappa(\theta)\,\nabla\theta\}_{(h_1)} - \{\kappa(\theta)\,\nabla\theta\}_{(h_2)}\Big]\cdot\nabla(\tilde\theta_{(h_1)} - \tilde\theta_{(h_2)})$$

$$-\int_\Sigma \Big[\{G(\sigma\,|\theta|^3\,\theta)\}_{(h_1)} - \{G(\sigma\,|\theta|^3\,\theta)\}_{(h_2)}\Big]\,(\tilde\theta_{(h_1)} - \tilde\theta_{(h_2)}) + \int_\Omega (f_{(h_1)} - f_{(h_2)})\,(\tilde\theta_{(h_1)} - \tilde\theta_{(h_2)})$$

$$-\left(\frac{\partial}{\partial t}\big((\theta_g)_{(h_1)} - (\theta_g)_{(h_2)}\big)\,,\,\tilde\theta_{(h_1)} - \tilde\theta_{(h_2)}\right)_{L^2(\Omega)}\,,$$

where we did not indicate the dependence of the right-hand side on t for the sake of notational commodity. Integrating this relation on $]0\,,t[$ for any $t \leq t_1$, we get by our assumptions on θ_g that

$$\max_{t\in[0,t_1]}\;\|\,\tilde\theta_{(h_1)}(t) - \tilde\theta_{(h_2)}(t)\,\|_{L^2(\Omega)} \;\longrightarrow 0\;\;\text{as } h_1,\,h_2 \to 0\,.$$

Thus, $\tilde\theta$ is equal to a continuous function from $[0,t_1]$ into $L^2(\Omega)$. Since t_1 was arbitrary, and $\tilde\theta \in L^{\infty,2}(Q)$, we can conclude that $\tilde\theta \in C([0,T];L^2(\Omega))$. By the regularity of θ_g, we see that also θ satisfies this continuity assumption. □

2 Boundedness of Solutions

Lemma 2.1. Let $f \in L^r(Q)$ for a $r > \frac{5}{2}$, and $\theta_0 \in L^\infty(\Omega)$, as well as $\theta_g \in L^\infty(\mathcal{C})$. Then the weak solution of (P) is bounded in Q, and we have

$$\|\theta\|_{L^\infty(Q)} \leq \max\Big\{\,\|\theta_g\|_{L^\infty(\mathcal{S})}\,,\,\|\theta_0\|_{L^\infty(\Omega)}\Big\} + C\,\|f\|_{L^r(Q)}\,.$$

Proof. We define $k_0 := \max\left\{\operatorname*{ess\,sup}_{\mathcal{C}} \theta_g\,,\ \operatorname*{ess\,sup}_{\Omega} \theta_0\right\}$. Observe that if $f \in L^r(Q)$, $r > 5/2$, then we obtain by Theorem 1.1 or by Theorem 1.2 in all cases that the weak solution θ belongs to $C([0,T]; L^2(\Omega)) \cap W_2^{1,0}(Q)$. For an arbitrary $k > k_0$, we can therefore apply Lemma 4.6 with $u = \theta$ and $g(\theta) = (\theta - k)^+$. Since our choice of k implies that $(\theta - k)^+(0) = (\theta_g(0) - k)^+ = 0$ almost everywhere in Ω, we obtain for all $t_1 < T$ the relation

$$\frac{1}{2} \int_\Omega (\theta - k)^{+^2}(t_1) + \int_{Q_{t_1}} \kappa(\theta) \left|\nabla([\theta - k]^+)\right|^2 + \int_{\mathcal{S}_{t_1}} G(\sigma\,|\theta|^3\,\theta)\,(\theta - k)^+ = \int_{Q_{t_1}} f\,(\theta - k)^+ \,.$$

In view of Lemma 4.4 we have $\int_{\mathcal{S}_{t_1}} G(\sigma\,|\theta|^3\,\theta)\,(\theta - k)^+ \geq 0$, which implies that

$$\max_{t_1 \in [0,T]} \int_\Omega [(\theta - k)^+(t_1)]^2 + \int_Q \kappa(\theta) \left|\nabla([\theta - k]^+)\right|^2 \leq \int_Q f\,(\theta - k)^+ \,.$$

We recall the continuity of the embedding $V_2^{1,0}(Q) \hookrightarrow L^{10/3}(Q)$, and we write

$$\left|\int_Q f\,(\theta - k)^+\right| \leq \| f \|_{L^r(Q)} \, \| (\theta - k)^+ \|_{L^{\frac{10}{3}}(Q)} \ |A(k)|^{\frac{7}{10} - \frac{1}{r}} \,,$$

where $A(k) := \{(t,x) \,:\, \theta(t,x) > k\}$. It follows that

$$\left(\max_{t_1 \in [0,T]} \int_\Omega [(\theta - k)^+(t_1)]^2 + \int_Q \kappa(\theta) \left|\nabla([\theta - k]^+)\right|^2\right)^{\frac{1}{2}} \leq C\,|A(k)|^{\frac{7}{10} - \frac{1}{r}} \,. \tag{37}$$

On the other hand we have, for $h > k > k_0$,

$$\begin{aligned}
(h - k)\,|A(h)|^{\frac{3}{10}} &\leq \|[\theta - k]^+\|_{L^{10/3}(Q)} \\
&\leq c \left(\max_{t_1 \in [0,T]} \int_\Omega [(\theta - k)^+(t_1)]^2 + \int_Q \kappa(\theta)\,|\nabla([\theta - k]^+)|^2\right)^{\frac{1}{2}} .
\end{aligned}$$

This combined with (37) yields

$$|A(h)| \leq \frac{K}{(h - k)^{\frac{10}{3}}}\,|A(k)|^{\frac{7}{3} - \frac{10}{3r}} \,.$$

Obviously, $\frac{7}{3} - \frac{10}{3r} > 1 \iff r > \frac{5}{2}$. We can repeat these considerations for the test functions $(\theta + k)^-$, where $k > -\inf\left\{\operatorname*{ess\,inf}_{\mathcal{C}} \theta_g\,,\ \operatorname*{ess\,inf}_{\Omega} \theta_0\right\}$. By the classical results of [Sta65], this proves the lemma. ☐

3 L^1-Estimates

We want to prove an existence result for the case that the heat-sources have poor regularity. Throughout this section, we require only that $\Sigma \in \mathcal{C}^{0,1}$.

Theorem 3.1. Let Ω satisfy (68). Let $f \in L^1(Q)$, $\theta_0 \in L^1(\Omega)$ and $\theta_g \in W_2^{1,0}(Q)$ satisfy (10). Then, there exists

$$\theta \in \bigcap_{1 \le p < \frac{5}{4}} W_p^{1,0}(Q) \cap L^{\infty,1}(Q)\,,$$

such that $\theta = \theta_g$ on $\mathcal{C}$ and

$$-\int_Q \theta\,\frac{\partial \psi}{\partial t} + \int_Q \kappa(\theta)\,\nabla\theta\cdot\nabla\psi + \int_{\mathcal{S}} \sigma\,|\theta|^3\,\theta\,G(\psi) = \int_\Omega \theta_0\,\psi(0,x) + \int_Q f\,\psi\,,$$

for all $\psi \in C^\infty(\overline{Q})$, such that $\psi = 0$ on $]\,0\,,\,T\,[\times\Gamma$, and $\psi(T) = 0$.

As to the proof of this theorem, we start by constructing approximate solutions. For $\delta > 0$, we define $f^{[\delta]} := \operatorname{sign}(f)\,\min\left\{|f|\,,\,\frac{1}{\delta}\right\}$. Then, by Theorem 1.2, we find a $\theta_\delta \in \mathcal{V}_0^{2,5}(Q)$ such that $\theta = \theta_g$ on $\mathcal{C}$, and

$$-\int_Q \theta_\delta\,\frac{\partial\,\psi}{\partial t} + \int_Q \kappa(\theta)\,\nabla\theta_\delta\cdot\nabla\psi + \int_{\mathcal{S}} G(\sigma|\theta_\delta|^3\,\theta_\delta)\,\psi = \int_\Omega \theta_0^{[\delta]}\,\psi(0) + \int_Q f^{[\delta]}\,\psi\,,$$

for all $\psi \in \mathcal{V}_{1,\mathcal{C}}^{2,5}(Q)$ such that $\psi(T) = 0$. From Theorem 1.2, we also obtain the existence of $\theta'_\delta \in L^2(0,T;[V_\Gamma^{2,5}(\Omega)]^*)$ such that the relation

$$\left\langle \theta'_\delta\,,\,\psi\right\rangle + \int_Q \kappa(\theta)\,\nabla\theta_\delta\cdot\nabla\psi + \int_{\mathcal{S}} G(\sigma|\theta_\delta|^3\,\theta_\delta)\,\psi = \int_Q f^{[\delta]}\,\psi\,. \tag{38}$$

is valid for the same class of test functions.

We cannot get *a priori*-estimates with the method of Proposition 1.5.

Proposition 3.2. For any sequence of approximate solutions $\{\theta_\delta\}$ that satisfy (38), the following uniform estimates are valid:

(1) There exists a positive constant C_1 independent of δ such that $\|\,\theta_\delta\,\|_{L^{\infty,1}(Q)} \le C_1$.

(2) For all $1 \le r < \frac{5}{4}$, there exists a positive constant $C_2 = C_2(r)$ independent of δ such that $\|\,\theta_\delta\,\|_{W_r^{1,0}(Q)} \le C_2$.

(3) There exists a positive constant C_3 and a number $1 < q < \infty$ such that for all $i = 0, \ldots, m$, $\|\,\theta'_\delta\,\|_{L^1(0,T;[W_0^{1,q}(\Omega_i)]^*)} \le C_3$.

The constants C_j ($j = 1, \ldots, 3$) depend continuously on $\|f\|_{L^1(Q)}$, on $\|\theta_0\|_{L^1(\Omega)}$ and on $\|\theta_g\|_{W_2^{1,0}(Q)}$.

Proof. For the sake of notational simplicity, we write θ instead of θ_δ. Again, we prove the proposition for the homogeneous case $\theta_g = 0$ on $]0\,,\,T[\times\Gamma$. The general case follows by similar arguments. For a parameter $\gamma > 0$, we consider functions $g = g_\gamma$, $F = F_\gamma \in C(\mathbb{R})$ given by

$$g_\gamma(s) := \frac{1}{\gamma}\,\mathrm{sign}(s)\,\min\{|s|,\gamma\}, \quad F_\gamma(s) = \begin{cases} -s + \frac{\gamma^2}{2} - \gamma & \text{if } s < -\gamma\,, \\ \frac{s^2}{2} & \text{if } -\gamma \le s \le \gamma\,, \\ s + \frac{\gamma^2}{2} - \gamma & \text{if } s > \gamma\,. \end{cases}.$$

Clearly, F is the primitive function of g that vanishes at zero. Applying Lemma 4.6, we get the relation

$$\int_\Omega F_\gamma(\theta)(t_1) + \int_{Q_{t_1}} \kappa(\theta)\,\nabla\theta\cdot\nabla g_\gamma(\theta) + \int_{\mathcal{S}_{t_1}} G(\sigma|\theta|^3\,\theta\,)\,g_\gamma(\theta) = \int_\Omega F_\gamma(\theta_0^{[\delta]}) + \int_{Q_{t_1}} f^{[\delta]}\,g_\gamma(\theta)\,.$$

Since g_γ is nondecreasing, we have $\nabla\theta\cdot\nabla g_\gamma(\theta) \ge 0$. Letting $\gamma \to 0$ in the previous relation, we obtain the inequality

$$\int_\Omega |\theta(t_1)| + \int_{\mathcal{S}_{t_1}} G(\sigma|\theta|^3\,\theta)\,\mathrm{sign}(\theta) \le \|\,\theta_0\,\|_{L^1(\Omega)} + \|\,f\,\|_{L^1(Q)}\;. \tag{39}$$

Now, use Lemma 4.1, (4) to show that

$$\int_{\mathcal{S}} G(\sigma|\theta|^3\,\theta)\,\mathrm{sign}(\theta) \ge (1 - \|H\|_{\mathcal{L}(\infty,\infty)}) \int_{\mathcal{S}} |\theta|^4 \ge 0\,.$$

This proves the estimate (1).

For the next estimate, we follow the techniques of [Lew97]. For $n \in \mathbb{N}$, we consider the functions

$$g_n(t) := \begin{cases} -1 & \text{for } t < -(n+1)\,, \\ t+n & \text{for } t \in [-(n+1), -n[\,, \\ 0 & \text{for } t \in [-n, n[\,, \\ t-n & \text{for } t \in [n, n+1[\,, \\ 1 & \text{for } t \ge n+1\,. \end{cases}, \quad F_n(t) = \begin{cases} 0 & \text{for } t \in [0,n[\,, \\ \frac{1}{2}t^2 - n\,t + \frac{n^2}{2} & \text{if } t \in [n, n+1[\,, \\ t - n - \frac{1}{2} & \text{if } t \ge n+1\,, \end{cases}$$

where F_n is extended to the negative axis such as to be an even function. Observe that g_n is continuous, nondecreasing and bounded and that F_n is the primitive function of g_n that vanishes at zero. Applying Lemma 4.6, we obtain that

$$\int_\Omega F_n(\theta(t_1)) + \int_0^{t_1}\int_\Omega \kappa(\theta)\,|\nabla\theta|^2\,g_n'(\theta) + \int_{\mathcal{S}_{t_1}} G(\sigma\,|\theta|^3\,\theta)\,g_n(\theta) = \int_\Omega F_n(\theta_0^{[\delta]}) + \int_{Q_{t_1}} f^{[\delta]}\,g_n(\theta)$$

Recalling Lemma 4.4, we know that $\int_{\mathcal{S}_{t_1}} G(\sigma\,|\theta|^3\,\theta)\,g_n(\theta) \ge 0$. Letting $t_1 \to T$ yields the inequality

$$\int_0^T\int_\Omega g_n'(\theta)\,\kappa(\theta)\,|\nabla\theta|^2 \le \int_\Omega F_n(\theta_{0,\delta}) + \int_Q f^{[\delta]}\,g_n(\theta) \le \int_\Omega |\theta_0| + \frac{1}{2}\,\mathrm{meas}(\Omega) + \|\,f\,\|_{L^1(Q)}\;. \tag{40}$$

As in Proposition 4.7, we introduce

$$B_n := \Big\{(t,x) \in Q \,\Big|\, n \leq |\theta(t,x)| < n+1\Big\}.$$

Relation (40) amounts to say that

$$\int_{B_n} \kappa(\theta)\,|\nabla\theta|^2 \leq \int_\Omega |\theta_0| + \frac{1}{2}\,\text{meas}(\Omega) + \| f \|_{L^1(Q)}$$

Now, Proposition 4.7 applies. Combined to (1), it gives (2).

Finally we want to estimate the time derivatives. The relation (38) is equivalent to

$$\big\langle \theta'(t)\,,\,\psi\big\rangle = -\int_\Omega \kappa(\theta(t))\,\nabla\theta(t)\cdot\nabla\psi - \int_\Sigma G(\sigma|\theta|^3\,\theta(t))\,\psi + \int_\Omega f^{[\delta]}(t)\,\psi\,. \tag{41}$$

for almost all $t \in]0,T[$ and all $\psi \in V^{2,5}_\Gamma(\Omega)$. Here, $\langle\cdot,\cdot\rangle$ is the duality pairing in $V^{2,5}(\Omega)$. We recall that $\Omega = \bigcup_{i=1}^m \Omega_i$. In (41), we can choose any test function $\psi \in W^{1,q}_0(\Omega_i)$ ($q \geq 3$) if we extend it by zero to the rest of Ω. For such a ψ, it follows that

$$\big\langle \theta'(t)\,,\,\psi\big\rangle = -\int_{\Omega_i} \kappa(\theta(t))\,\nabla\theta(t)\cdot\nabla\psi + \int_{\Omega_i} f^{[\delta]}(t)\,\psi\,.$$

We obtain that

$$\Big|\big\langle \theta'(t)\,,\,\psi\big\rangle\Big| \leq c\,\Big(\,\| \nabla\theta(t) \|_{L^{q'}(\Omega_i)} + \| f(t) \|_{L^1(\Omega_i)}\Big)\,\| \psi \|_{W^{1,q}_0(\Omega_i)}\,.$$

Thus,

$$\| \theta'(t) \|_{[W^{1,q}_0(\Omega_i)]^*} \leq c\,\Big(\,\| \nabla\theta(t) \|_{L^{q'}(\Omega_i)} + \| f(t) \|_{L^1(\Omega_i)}\Big)\,,$$

the right-hand side being bounded uniformly in $L^1(0,T)$ by the previous results. This was the last claim that we had to prove. □

Proof of Theorem 3.1. Applying Proposition 3.2, we first find a sequence such that

$$\theta_\delta \rightharpoonup \theta \text{ in } W^{1,0}_r(Q) \quad \text{for } 1 \leq r < \frac{5}{4}\,. \tag{42}$$

We now want to prove additional convergence properties. We have the situation

$$W^{1,r}(\Omega_i) \hookrightarrow L^r(\Omega_i) \hookrightarrow [W^{1,q}_0(\Omega_i)]^*\,.$$

We introduce the notations $Q_i :=]0,T[\times\Omega_i$, and $S_i :=]0,T[\times\partial\Omega_i$. From Proposition 3.2, (3), and the generalized Lemma of Aubin-Lions, we get for all $i = 0,\ldots,m$ that $\theta_\delta \longrightarrow \theta$ in $L^r(Q_i)$. By the inequality (14), we now find a subsequence such that $\theta_\delta \longrightarrow \theta$ in $L^r(S_i)$ and, after extracting subsequences, even

$$\theta_\delta \longrightarrow \theta \text{ in } L^r(Q)\,, \quad \theta_\delta \longrightarrow \theta \text{ in } L^r(\mathcal{S})\,, \quad \theta_\delta \longrightarrow \theta \text{ pointwise a. e. in } Q \text{ and on } \mathcal{S}\,. \tag{43}$$

Taking the limit $\delta \to 0$ in (38), we get

$$- \int_Q \theta \, \frac{\partial \psi}{\partial t} + \int_Q \kappa(\theta) \, \nabla\theta \cdot \nabla\psi + \lim_{\delta \to 0} \int_{\mathbb{S}} G(\sigma \, |\theta_\delta|^3 \, \theta_\delta) \, \psi = \int_\Omega \theta_0 \, \psi + \int_Q f \, \psi \,, \tag{44}$$

for all $\psi \in C^\infty(\overline{Q})$ such that $\psi = 0$ on $]0\,,\,T[\times\Gamma$ and $\psi(T) = 0$.

Now, we turn our attention to the main challenge of the proof, which consists in computing the value of $\lim_{\delta\to 0} \int_{\mathbb{S}} G(\sigma \, |\theta_\delta|^3 \, \theta_\delta) \, \psi$. For $\gamma > 0$, consider the function

$$g_\gamma(s) := \begin{cases} 1 & \text{if } s < 0\,, \\ \frac{1}{1+\gamma \, s^4} & \text{if } s \geq 0\,. \end{cases}$$

As usual, we denote by F_γ the primitive of g_γ that vanishes at zero. In (38), we use Steklov averagings and can prove, with the same method as in the proof of Lemma 4.6, that

$$\int_0^{t_1} \left(\frac{\partial \theta_{(h)}}{\partial t}, \psi \right)_{L^2(\Omega)} + \int_{Q_{t_1}} \{\delta \, |\nabla\theta|^{p-2} \, \nabla\theta\}_{(h)} . \nabla\psi$$

$$+ \int_{Q_{t_1}} \{\kappa(\theta) \, \nabla\theta\}_{(h)} \cdot \nabla\psi + \int_{\mathbb{S}_{t_1}} \{G(\sigma|\theta|^3 \, \theta\,)\}_{(h)} \, \psi = \int_{Q_{t_1}} \{f^{[\delta]}\}_{(h)} \, \psi \,, \tag{45}$$

for all $t_1 \in]0\,,\,T[$. We now consider an arbitrary $\tilde{\psi} \in C^\infty(0, t_1; C^\infty(\overline{\Omega}))$, such that $\tilde{\psi} = 0$ on $]0\,,\,t_1[\times\Gamma$, $\tilde{\psi} \geq 0$ in $\overline{Q}_{t_1}$, and $\tilde{\psi}(t_1) = 0$.

We extend $\tilde{\psi}$ by zero, and we choose in (45) the test function $\psi := g_\gamma(\theta_{(h)}) \, \tilde{\psi}$. Using integration by parts and observing that $\psi(t_1) = 0$, we find that

$$- \int_{Q_{t_1}} F_\gamma(\theta_{(h)}) \, \frac{\partial \tilde{\psi}}{\partial t} + \int_{Q_{t_1}} \{\kappa(\theta) \, \nabla\theta\}_h \cdot \nabla\tilde{\psi} \, g_\gamma(\theta_{(h)}) + \int_{\mathbb{S}_{t_1}} \{G(\sigma \, |\theta_\delta|^3 \, \theta_\delta\,)\}_{(h)} \, \tilde{\psi} \, g_\gamma(\theta_{(h)}) + R_h$$

$$= \int_\Omega F_\gamma(\theta_{(h)}(0)) \, \tilde{\psi}(0) + \int_{Q_{t_1}} \{f^{[\delta]}\}_{(h)} \, \tilde{\psi} \, g_\gamma(\theta_{(h)}) \,, \tag{46}$$

with the notation

$$R_h := \int_{Q_{t_1}} \{\kappa(\theta) \, \nabla\theta\}_h \cdot \nabla\theta_{(h)} \, \tilde{\psi} \, g'_\gamma(\theta_{(h)}) \,,$$

Note that, as $h \to 0$,

$$R_h \longrightarrow \int_{Q_{t_1}} \kappa(\theta) \, |\nabla\theta|^2 \tilde{\psi} \, g'_\gamma(\theta) \, \leq \, 0 \,,$$

because g_γ is decreasing for every fixed γ. Thus, taking the limes $h \to 0$ in (46), and writing again the indices δ, we find the inequality

$$- \int_{Q_{t_1}} F_\gamma(\theta_\delta) \, \frac{\partial \tilde{\psi}}{\partial t} + \int_{Q_{t_1}} \kappa(\theta_\delta) \, \nabla\theta_\delta \cdot \nabla\tilde{\psi} \, g_\gamma(\theta_\delta) + \int_{\mathbb{S}_{t_1}} G(\sigma \, |\theta_\delta|^3 \, \theta_\delta\,) \, \tilde{\psi} \, g_\gamma(\theta_\delta)$$

$$\geq \int_\Omega F_\gamma(\theta_0^{[\delta]}) \, \tilde{\psi}(0) + \int_{Q_{t_1}} f^{[\delta]} \, \tilde{\psi} \, g_\gamma(\theta_\delta) \,. \tag{47}$$

By the absolute continuity of the integral, we can also write $t_1 = T$ in the last inequality. By an approximation argument, we can, as well, suppose that $\tilde{\psi}$ is an arbitrary C^∞-function, which vanishes on $]0\,,\,T[\times\Gamma$ and in $\{T\}\times\Omega$, and is positive in $\overline{Q}$.

We now want to pass to the limit $\delta \to 0$ in (47). This is easily done, apart from terms on the boundary $\mathbb{S}$. We observe that

$$\int_{\mathbb{S}} G(\sigma\,|\theta_\delta|^3\,\theta_\delta)\,\tilde{\psi}\,g_\gamma(\theta_\delta) = \int_{\mathbb{S}} \sigma\,|\theta_\delta|^3\,\theta_\delta\,\tilde{\psi}\,g_\gamma(\theta_\delta) - \int_{\mathbb{S}} \sigma\,|\theta_\delta|^3\,\theta_\delta\,H(\tilde{\psi}\,g_\gamma(\theta_\delta))$$

$$= \int_{\mathbb{S}} \sigma\,\frac{\theta_\delta^{+^4}}{1+\gamma\,\theta_\delta^{+^4}}\,\tilde{\psi} - \int_{\mathbb{S}} \sigma\,\theta_\delta^{+^4}\,H(\tilde{\psi}\,g_\gamma(\theta_\delta)) + \int_{\mathbb{S}} \sigma\,|\theta_\delta|^3\,\theta_\delta^-\,\tilde{\psi} - \int_{\mathbb{S}} \sigma|\theta_\delta|^3\,\theta_\delta^-\,H(\tilde{\psi}\,g_\gamma(\theta_\delta))$$

$$\leq \int_{\mathbb{S}} \sigma\,\frac{\theta_\delta^{+^4}}{1+\gamma\,\theta_\delta^{+^4}}\,\tilde{\psi} - \int_{\mathbb{S}} \sigma\,\theta_\delta^{+^4}\,H(\tilde{\psi}\,g_\gamma(\theta_\delta)) + \int_{\mathbb{S}} \sigma\,|\theta_\delta|^3\,\theta_\delta^-\,\tilde{\psi} - \int_{\mathbb{S}} \sigma|\theta_\delta|^3\,\theta_\delta^-\,H(\tilde{\psi}\,)\,.$$

We justify the last inequality by the fact that $g_\gamma \leq 1$, and by the positivity of the operator H. This implies that

$$\lim_{\delta\to 0} \int_{\mathbb{S}} G(\sigma\,|\theta_\delta|^3\,\theta_\delta)\,\tilde{\psi}\,g_\gamma(\theta_\delta)$$

$$\leq \int_{\mathbb{S}} \frac{\sigma\,\theta^{+^4}}{1+\gamma\,\theta^{+^4}}\,\tilde{\psi} - \liminf_{\delta\to 0} \int_{\mathbb{S}} \sigma\,\theta_\delta^{+^4}\,H(\tilde{\psi}\,g_\gamma(\theta_\delta)) + \lim_{\delta\to 0} \int_{\mathbb{S}} \sigma\,G(|\theta_\delta|^3\,\theta_\delta^-\,)\,\tilde{\psi}\,, \tag{48}$$

where we made use of the facts that G is selfadjoint (see Lemma 4.1, and of the dominated convergence theorem.

On the other hand, by Fatou's lemma, and by the fact that $g_\gamma(\theta_\delta) \to g(\theta)$ in $L^1(\mathbb{S})$, we have

$$\liminf_{\delta\to 0} \int_{\mathbb{S}} \sigma\,\theta_\delta^{+^4}\,H(\tilde{\psi}\,g_\gamma(\theta_\delta)) \geq \int_{\mathbb{S}} \sigma\,\liminf_{\delta\to 0}\{\theta_\delta^{+^4}\,H(\tilde{\psi}\,g_\gamma(\theta_\delta))\}$$

$$\geq \int_{\mathbb{S}} \sigma\,\theta^{+^4}\,\liminf_{\delta\to 0} H(\tilde{\psi}\,g_\gamma(\theta_\delta)) = \int_{\mathbb{S}} \sigma\,\theta^{+^4}\,H(\tilde{\psi}\,g_\gamma(\theta))\,. \tag{49}$$

Returning to (48), we can write

$$\lim_{\delta\to 0} \int_{\mathbb{S}} G(\sigma\,\theta_\delta^{+^4})\,\tilde{\psi}\,g_\gamma(\theta_\delta) \leq \int_{\mathbb{S}} \sigma\,\theta^{+^4}\,g_\gamma(\theta)\,\tilde{\psi} - \int_{\mathbb{S}} \sigma\theta^{+^4}\,H(\tilde{\psi}\,g_\gamma(\theta)) + \lim_{\delta\to 0} \int_{\mathbb{S}} \sigma\,G(|\theta_\delta|^3\,\theta_\delta^-\,)\,\tilde{\psi}\,.$$

Passing to the limit $\delta \to 0$ in (47), the inequality is preserved, and we obtain

$$- \int_Q F_\gamma(\theta)\,\frac{\partial\tilde{\psi}}{\partial t} + \int_Q \kappa(\theta)\,\nabla\theta\cdot\nabla\tilde{\psi}\,g_\gamma(\theta) + \int_{\mathbb{S}} \sigma\,\theta^{+^4}\,g_\gamma(\theta)\,\tilde{\psi} - \int_{\mathbb{S}} \sigma\,\theta^{+^4}\,H(\tilde{\psi}\,g_\gamma(\theta))$$

$$+ \lim_{\delta\to 0} \int_{\mathbb{S}} \sigma\,G(|\theta_\delta|^3\,\theta_\delta^-\,)\,\tilde{\psi} \geq \int_\Omega F_\gamma(\theta_0)\,\tilde{\psi}(0) + \int_Q f\,\tilde{\psi}\,g_\gamma(\theta)\,. \tag{50}$$

We easily verify that F_γ monotonely increases to the identity, and that g_γ monotonely increases to 1. This last property makes it possible to write

$$\int_{\mathcal{S}} \sigma\, \theta^{+^4}\, g_\gamma(\theta)\, \tilde{\psi} \longrightarrow \int_{\mathcal{S}} \sigma\, \theta^{+^4}\, \tilde{\psi}\,,$$

for $\gamma \to 0$. By the same argument that led to (49), we also have

$$\liminf_{\gamma\to 0} \int_{\mathcal{S}} \sigma\, \theta^{+^4}\, H(\tilde{\psi}\, g_\gamma(\theta)) \geq \int_{\mathcal{S}} \sigma\theta^{+^4}\, H(\tilde{\psi})\,.$$

In the limit $\gamma \to 0$ of (50), we then obtain the inequality

$$\begin{aligned} -\int_Q \theta\, \frac{\partial\tilde{\psi}}{\partial t} + \int_Q \kappa(\theta)\, \nabla\theta\cdot\nabla\tilde{\psi} + \int_{\mathcal{S}} \sigma\, \theta^{+^4}\, G(\tilde{\psi}) + \lim_{\delta\to 0}\int_{\mathcal{S}} \sigma\, G(|\theta_\delta|^3\, \theta_\delta^-\,)\, \tilde{\psi} \\ \geq \int_\Omega \theta_0\, \tilde{\psi}(0) + \int_Q f\, \tilde{\psi}\,. \qquad (51) \end{aligned}$$

We compare (44), where we choose $\psi = \tilde{\psi}$, with (51), and we get

$$\lim_{\delta\to 0} \int_{\mathcal{S}} \sigma\, \theta_\delta^{+^4}\, G(\psi) = \lim_{\delta\to 0}\int_{\mathcal{S}} G(\sigma\, \theta_\delta^{+^4})\, \psi \leq \int_{\mathcal{S}} \sigma\, \theta^{+^4}\, G(\psi)\,, \qquad (52)$$

for all ψ in $C^\infty(\overline{Q})$ such that $\psi \geq 0$ in $[0\,,T]\times\Omega$, $\psi = 0$ on $]0\,,T[\times\Gamma$, and $\psi(T) = 0$. Elementarily, the inequality

$$\lim_{\delta\to 0} \int_{\mathcal{S}} \sigma\, \theta_\delta^{+^4}\, G(\psi) \geq \int_{\mathcal{S}} \sigma\, \theta^{+^4}\, G(\psi)\,, \qquad (53)$$

is valid for all ψ in $C^\infty(\overline{Q})$ such that $\psi \leq 0$ in $[0\,,T]\times\Omega$, $\psi = 0$ on $]0\,,T[\times\Gamma$, and $\psi(T) = 0$.

Now, we use the construction of Lemma 4.8. For an arbitrary ψ in $C^\infty(\overline{Q})$ such that $\psi \geq 0$ in $[0\,,T]\times\Omega$, $\psi = 0$ on $]0\,,T[\times\Gamma$, and $\psi(T) = 0$, we denote by $\bar{\psi}$ the negative function given by Lemma 4.8.

With the help of (52) and (53), we get

$$\int_{\mathcal{S}} \sigma\, \theta^{+^4}\, G(\psi) = \int_{\mathcal{S}} \sigma\, \theta^{+^4}\, G(\bar{\psi}) \leq \lim_{\delta\to 0}\int_{\mathcal{S}} \sigma\, \theta_\delta^{+^4}\, G(\bar{\psi}) = \lim_{\delta\to 0}\int_{\mathcal{S}} \sigma\, \theta_\delta^{+^4}\, G(\psi) \leq \int_{\mathcal{S}} \sigma\, \theta^{+^4}\, G(\psi)\,,$$

which shows that

$$\lim_{\delta\to 0} \int_{\mathcal{S}} \sigma\, \theta_\delta^{+^4}\, G(\psi) = \int_{\mathcal{S}} \sigma\, \theta^{+^4}\, G(\psi) \qquad (54)$$

for all ψ in $C^\infty(\overline{Q})$ such that $\psi = 0$ on $]0\,,T[\times\Gamma$, and $\psi(T) = 0$.

If for $\gamma > 0$, we consider the function

$$g_\gamma(s) := \begin{cases} \frac{-1}{1+\gamma\, s^4} & \text{if } s < 0\,, \\ -1 & \text{if } s \geq 0\,, \end{cases}$$

and test with $g_\gamma(\theta_{(h)})\,\psi$, we will find by similar considerations that

$$\lim_{\delta\to 0}\int_{\mathcal{S}} \sigma\,\theta_\delta^{-4}\,G(\psi) = \int_{\mathcal{S}} \sigma\,\theta^{-4}\,G(\psi)\,. \tag{55}$$

Therefore, we finally have found that

$$\lim_{\delta\to 0}\int_{\mathcal{S}} G(\sigma\,|\theta_\delta|^3\,\theta_\delta)\,\psi = \int_{\mathcal{S}} \sigma\,|\theta|^3\,\theta\,G(\psi)\,.$$

Returning to (44) with these informations, we find the relation

$$-\int_Q \theta\,\frac{\partial\psi}{\partial t} + \int_Q \kappa(\theta)\,\nabla\theta\cdot\nabla\psi + \int_{\mathcal{S}} \sigma\,|\theta|^3\,\theta\,G(\psi) = \int_\Omega \theta_0\,\psi + \int_Q f\,\psi\,, \tag{56}$$

for all ψ in $C^\infty(\overline{Q})$ such that $\psi = 0$ on $]0\,,\,T[\times\Gamma$, and $\psi(T) = 0$. This is what we wanted to prove. □

Remark 3.3. In comparison to the results obtained in [Dru07] for the stationary problem, we do not obtain a continuous estimate for the radiation energy $\|\theta^4\|_{L^1(\mathcal{S})}$ in terms of the heating power $\|f\|_{L^1(Q)}$. We finish this section by proving that in the case of a $\mathcal{C}^{1,\alpha}$ boundary, we can in some sense control θ^4 if we pass to a weaker norm.

For the remainder of the section, we need to assume that the approximate solutions $\{\theta_\delta\}$ according to (38) satisfy

$$\operatorname*{ess\,inf}_Q \theta_\delta \geq k_0 > 0\,. \tag{57}$$

This condition is for example satisfied when the right-hand side f is a positive function. By the maximum principle, we then choose $k_0 := \inf\{\operatorname*{ess\,inf}_{\mathcal{C}} \theta_{g,\delta},\ \operatorname*{ess\,inf}_{\Omega} \theta_{0,\delta}\}$. In addition to the estimates of Proposition 3.2, we have the following lemma.

Lemma 3.4. We consider any sequence $\{\theta_\delta\}$ of approximations according to Proposition 3.2. Assume that (57) is satisfied. Then for all $0 < \gamma < \frac{1}{2}$, the uniform estimate

$$\left\|\nabla\theta_\delta^\gamma\right\|_{[L^2(Q)]^3} \leq C_4\,,$$

is valid, with a constant $C_4 = C_4(\gamma)$ that depends continuously on $\|\theta_{0,\delta}\|_{L^1(\Omega)}$, on $\|f\|_{L^1(Q)}$ and on $\|\theta_{g,\delta}\|_{W_2^{1,0}(Q)}$.

Proof. For the sake of simplicity, we prove the claim in the case that $\theta_{g,\delta} = k_0$ on $\mathcal{C}$. The general claim follows by similar ideas. In this proof, we do not indicate the indices δ. Under the assumptions of the lemma, we first see that the integral

$$I_\gamma := \int_{k_0}^{+\infty} \frac{1}{\tau^{2\,(1-\gamma)}}\,d\,\tau \tag{58}$$

is finite. For $t \in \mathbb{R}$, we then define

$$g(t) = g_\gamma(t) := \begin{cases} \int_{k_0}^t \frac{1}{\tau^{2\,(1-\gamma)}}\, d\tau & \text{if } t \geq k_0\,, \\ 0 & \text{otherwise,} \end{cases}$$

and we denote by $F = F_\gamma$ the primitive function of g that vanishes at zero.

Observe that the function g is globally bounded by the number I_γ given in (58) and that the function $g(\theta)$ vanishes on the boundary $\mathcal{C}$.

With the help of (38) and of Lemma 4.6, we can prove, for all $t_1 \in [0, T]$, the relation

$$\int_\Omega F(\theta(t_1)) + \int_{Q_{t_1}} \kappa(\theta)\, g'(\theta)\, |\nabla\theta|^2 + \int_{\mathcal{S}_{t_1}} G(\sigma\,\theta^4)\, g(\theta) = \int_\Omega F(\theta_{0,\delta}) + \int_{Q_{t_1}} f^{[\delta]}\, g(\theta)\,.$$

Clearly, we have

$$\left| \int_\Omega F(\theta_{0,\delta}) + \int_{Q_{t_1}} f^{[\delta]}\, g(\theta) \right| \leq I_\gamma \left(\|\theta_{0,\delta}\|_{L^1(\Omega)} + \|f\|_{L^1(Q)} \right).$$

On the other hand, $g'(\theta)\, |\nabla\theta|^2 = |\nabla\theta|^2/\theta^{2\,(1-\gamma)} = 1/\gamma^2\, |\nabla\theta^\gamma|^2$. Since in view of Lemma 4.4, the integrals $\int_{\mathcal{S}_{t_1}} G(\sigma\,\theta^4)\, g(\theta)$ are positive, we therefore obtain for all $t_1 \in]0, T$ that

$$\int_{Q_{t_1}} \kappa(\theta)\, |\nabla\theta^\gamma|^2 \leq \gamma^2\, I_\gamma \left(\|\theta_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q)} \right),$$

and the claim follows. □

We now recall the definition of the weak L^p spaces.

Definition 3.5. Let $(X, \mathcal{A}, \mu)$ denote a measurable space. Let $1 \leq p < \infty$. Then, the space $L^p_w(X, \mathcal{A}, \mu)$ consists of all $\mu-$measurable functions $u : X \to \mathbb{R}$ such that

$$\sup_{\lambda \geq 0} \left\{ \lambda\, \mu \left(\left\{ x \in X \,\Big|\, |u(x)| > \lambda \right\}^{1/p} \right) \right\} < \infty\,.$$

Proposition 3.6. Assume that $\Sigma \in \mathcal{C}^{1,\alpha}$. Under the assumptions of Lemma 3.4, the estimate

$$\|\theta_\delta\|_{L^4_w(\mathcal{S})} \leq \mathcal{P}\Big(\|\theta_0\|_{L^1(\Omega)},\, \|f\|_{L^1(Q)},\, \|\theta_g\|_{W^{1,0}_2(Q)} \Big)\,,$$

is valid with a continuous function $\mathcal{P}$ of the data.

Proof. For simplicity, we skip the indices δ. For arbitrary $\lambda > k_0$, we consider the function $g_\lambda : \mathbb{R} \to \mathbb{R}$ given by $g(t) = g_\lambda(t) := \text{sign}(t)\, \min\{|t|, \lambda\}$. We denote by $F = F_\lambda$ the primitive function of g that vanishes at k_0.

We write for the sake of commodity $\theta^{(\lambda)}$ instead of $g_\lambda(\theta)$. Testing in (38) with $\theta^{(\lambda)} - k_0$, we obtain with the help of Lemma 4.6 that

$$\int_\Omega (F(\theta(t_1)) - k_0\,\theta(t_1)) + \int_{Q_{t_1}} \kappa(\theta)\,|\nabla\theta^{(\lambda)}|^2 + \int_{\mathcal{S}_{t_1}} G(\sigma\,\theta^4)\,\theta^{(\lambda)}$$
$$= \int_\Omega (F(\theta_{0,\delta}) - k_0\,\theta_{0,\delta}) + \int_{Q_{t_1}} f^{[\delta]}\,(\theta^{(\lambda)} - k_0)\,.$$

Obviously, since $\lambda > k_0$ we have

$$\left|\int_\Omega (F(\theta_{0,\delta}) - k_0\,\theta_{0,\delta}) + \int_{Q_{t_1}} f^{[\delta]}\,(\theta^{(\lambda)} - k_0)\right| \leq 2\,\lambda\left(\|\theta_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q)}\right).$$

On the other hand, we observe that by Lemma 4.1 $\int_{\mathcal{S}_{t_1}} G(\sigma\,\theta^4)\,\theta^{(\lambda)} = \int_{\mathcal{S}_{t_1}} \sigma\,\theta^4\,G(\theta^{(\lambda)})$.

In the set $\{(t,\,z) \in \mathcal{S}_{t_1}\,:\,\theta > \lambda\}$, we have $G(\theta^{(\lambda)}) = \lambda - H(\theta^{(\lambda)}) \geq 0$. We therefore can write that

$$\int_{\mathcal{S}_{t_1}} \sigma\,\theta^4\,G(\theta^{(\lambda)}) \geq \int_{\mathcal{S}_{t_1}} \sigma\,\left(\theta^{(\lambda)}\right)^4 G(\theta^{(\lambda)}) = \int_{\mathcal{S}_{t_1}} \sigma\,G\big(\left(\theta^{(\lambda)}\right)^4\big)\,\theta^{(\lambda)}\,,$$

where we again made use of Lemma 4.1, (4). Now, using the inequality of Lemma 4.3 with $r = 4$, we obtain for arbitrary $0 < s \leq 5$ that

$$\int_{\mathcal{S}_{t_1}} \sigma\,G\big(\left(\theta^{(\lambda)}\right)^4\big)\,\theta^{(\lambda)} \geq c_s \int_{\mathcal{S}_{t_1}} \left|\theta^{(\lambda)}\right|^5 - \int_0^{t_1}\left(\int_\Sigma \left|\theta^{(\lambda)}\right|^s\right)^{5/s}.$$

The latter inequalities now give the estimate

$$\int_{\mathcal{S}_{t_1}} \left|\theta^{(\lambda)}\right|^5 \leq C_s\,\lambda\left(\|\theta_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q)} + \int_0^{t_1}\left(\int_\Sigma \left|\theta^{(\lambda)}\right|^{\frac{4s}{5}}\right)^{5/s}\right). \tag{59}$$

Now, define $\beta := \frac{4s}{5}$. In view of inequality of Lemma 4.9 below, we can write for q in the range $]4/3,\,4[$ that

$$\int_\Sigma \left|\theta^{(\lambda)}\right|^\beta = \left\|\left|\theta^{(\lambda)}\right|^{\beta/q}\right\|^q_{L^q(\Sigma)} \leq c_q\left(\left\|\left|\theta^{(\lambda)}\right|^{\beta/q}\right\|^{1-\alpha}_{L^2(\Omega)}\left\|\nabla\left(\theta^{(\lambda)}\right)^{\beta/q}\right\|^\alpha_{[L^2(\Omega)]^3}\right)^q, \tag{60}$$

with $\alpha = 3/2 - 2/q$. Under the condition $\beta/q \leq 1/2$, we have

$$\left\|\left|\theta^{(\lambda)}\right|^{\beta/q}\right\|^{q\,(1-\alpha)}_{L^2(\Omega)} = \left\|\left|\theta^{(\lambda)}\right|\right\|^{\beta\,(1-\alpha)}_{L^{2\,\beta/q}(\Omega)} \leq c\left\|\theta^{(\lambda)}\right\|^{\beta\,(1-\alpha)}_{L^1(\Omega)}.$$

Now, we recall the estimate of Proposition 3.2, (1), and obtain

$$\left\|\left|\theta^{(\lambda)}\right|^{\beta/q}\right\|^{q\,(1-\alpha)}_{L^2(\Omega)} \leq c\,\|\theta\|^{\beta\,(1-\alpha)}_{L^{\infty,1}(Q)} \leq c\,C_1^{\beta\,(1-\alpha)}\,.$$

Therefore, by (60), we will obtain that

$$\int_{\Sigma} \left|\theta^{(\lambda)}\right|^{\beta} \leq \tilde{c}_q\, C_1^{\beta\,(1-\alpha)} \left\| \nabla \left(\theta^{(\lambda)}\right)^{\beta/q} \right\|_{[L^2(\Omega)]^3}^{q\,\alpha} ,$$

which leads to

$$\int_0^{t_1} \left(\int_{\Sigma} \left|\theta^{(\lambda)}\right|^{\beta} \right)^{5/s} \leq \bar{c}_q\, C_1^{\frac{5\,\beta\,(1-\alpha)}{s}} \int_0^{t_1} \left\| \nabla \left(\theta^{(\lambda)}\right)^{\beta/q} \right\|_{[L^2(\Omega)]^3}^{\frac{5\,q\,\alpha}{s}} . \tag{61}$$

Now, assume that the condition

$$\gamma := \frac{\beta}{q} < \frac{1}{2} , \qquad \frac{5\,q\,\alpha}{s} \leq 2 , \tag{62}$$

are satisfied. Then, by Hölder's inequality

$$\int_0^{t_1} \left\| \nabla \theta^{\beta/q} \right\|_{[L^2(\Omega)]^3}^{\frac{5\,q\,\alpha}{s}} \leq c \int_0^{t_1} \left\| \nabla \left(\theta^{(\lambda)}\right)^{\gamma} \right\|_{[L^2(\Omega)]^3}^{2} ,$$

with some $\gamma < 1/2$. Now Lemma 3.4 gives that

$$\int_0^{t_1} \left\| \nabla \theta^{\beta/q} \right\|_{[L^2(\Omega)]^3}^{\frac{5\,q\,\alpha}{s}} \leq c\, C_4^2 .$$

Observing that $5\beta/s = 4$ and getting back to (61), we thus have

$$\int_0^{t_1} \left(\int_{\Sigma} \left|\theta^{(\lambda)}\right|^{\beta} \right)^{5/s} \leq c_q\, C_1^{4\,(1-\alpha)}\, C_4^2 . \tag{63}$$

Now, we have to ensure that a choice of parameters s, q with the property (62) is possible. In order to satisfy the first condition of (62), we choose $s = 5/8\,q - \delta$, where δ is an arbitrary small positive parameter.

The second condition of (62) will be satisfied if

$$5\,q \left(\frac{3}{2} - \frac{2}{q} \right) \leq 2 \left(\frac{5\,q}{8} - \delta \right) = \frac{5\,q}{4} - \tilde{\delta} .$$

We easily satisfy the last relation by choosing the number q in the range $q \in]4/3,\, 8/5 - \bar{\delta}]$, where $\bar{\delta}$ is arbitrary small. Since (63) is valid, the relation (59) now reads

$$\int_{\mathcal{S}_{t_1}} \left|\theta^{(\lambda)}\right|^{5} \leq \tilde{C}_q\, \lambda \left(\|\theta_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q)} + C_1^{4\,(1-\alpha)}\, C_4^2 \right) . \tag{64}$$

It follows that

$$\lambda^4 \text{ meas} \left(\left\{ (t,\, z) \in \mathcal{S} \,\middle|\, \theta > \lambda \right\} \right) \leq \tilde{C}_q \left(\|\theta_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q)} + C_1^{4\,(1-\alpha)}\, C_4^2 \right) ,$$

Thus,

$$\sup_{\lambda > 0} \left\{ \lambda \text{ meas} \left(\left\{ (t,\, z) \in \mathcal{S} \,\middle|\, \theta > \lambda \right\} \right)^{\frac{1}{4}} \right\} \leq C \left(\|\theta_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q)} + C_1^{4\,(1-\alpha)}\, C_4^2 \right)^{\frac{1}{4}} ,$$

proving the claim.

□

4 Appendix: auxiliary results

For $1 \le p, q \le \infty$, we introduce

$$\mathcal{L}(p,q) := \mathcal{L}\big(L^{p,q}(\mathcal{S})\,,\,L^{p,q}(\mathcal{S})\big)\,,$$

the space of all linear continuous maps from $L^{p,q}(\mathcal{S})$ into itself. For working with the operators K and G in the time-dependent case, we need the following properties:

Lemma 4.1. (1) For each $1 \le p, q \le \infty$ the operator K extends to a bounded linear operator from $L^{p,q}(\mathcal{S})$ into itself, and we have the norm estimate $\| K \|_{\mathcal{L}(p,q)} \le 1$.

(2) The operator K is positive, in the sense that $K(f) \ge 0$ almost everywhere on $\mathcal{S}$, whenever $f \ge 0$ almost everywhere on $\mathcal{S}$. Moreover, K is positive semi-definite and selfadjoint in $L^2(\mathcal{S})$.

(3) If $\epsilon : \mathcal{S} \to \mathbb{R}$ is such that

$$0 < \epsilon_l \le \epsilon(t,z) \le 1 \quad \text{on }]0,T[\times\Sigma\,,$$

then the operator $[I - (1-\epsilon)K]^{-1}$ has an inverse in $\mathcal{L}(L^{p,q}(\mathcal{S}), L^{p,q}(\mathcal{S}))$ having the representation

$$[I - (1-\epsilon)K]^{-1} \;=\; \sum_{i=0}^{\infty} (1-\epsilon)^i\, K^i\,.$$

(4) The operator G is positive semi-definite and selfadjoint in $L^2(\mathcal{S})$. It has the representation $G = I - H$, where the operator H is positive, selfadjoint in $L^2(\mathcal{S})$, and satisfies for all $1 \le p,\, q \le \infty$ the norm estimate $\| H \|_{\mathcal{L}(p,q)} \le 1$.

Proof. Denote by S the surface measure on Σ. We can prove that the mapping $(z,y) \longmapsto w(z,y)$ is $S \times S-$measurable on $\Sigma \times \Sigma$, provided that Σ is a Lipschitz surface. This will ensure, for $f \in L^1(\mathcal{S})$, that the mapping

$$(t,z,y) \longmapsto w(z,y)\, f(t,y)\,,$$

is $\lambda_1 \times S \times S-$measurable on $[0,T] \times \Sigma \times \Sigma$. Thus, by Fubini's theorem, we can easily derive the assertions of the lemma from the properties that were established in [Tii97b], [Met99], [Dru07] for the stationary operators. ☐

We recall the definition

Definition 4.2. (1) We say that two points $z, y \in \Sigma$ *see each other* if and only if $w(z,y) \neq 0$.

(2) We call Ω an *enclosure* if and only if for $S-$almost all $z \in \Sigma$ we have $\int_\Sigma w(z,y)\, dS_y = 1$.

We now need to recall a few auxiliary lemmas that we will use in the following.

Lemma 4.3. Let $\Sigma \in \mathcal{C}^{1,\alpha}$. Let $r, s > 0$ be two real numbers such that $s \le r+1$. There exists a positive constant $c_{r,s}$ such that for all $\psi \in L^{r+1}(\Sigma)$,

$$\int_\Sigma G(|\psi|^{r-1}\,\psi)\,\psi + \left(\int_\Sigma |\psi|^s\right)^{\frac{r+1}{s}} \ge c \parallel \psi \parallel^{r+1}_{L^{r+1}(\Sigma)} \ .$$

Proof. See [Dru07], Lemma 2.4. □

Lemma 4.4. Let Ω be an enclosure. Let $F : \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuous function such that $F(0) = 0$ and $|F(t)| \le C_0\,(1 + |t|^s)$ as $|t| \to \infty$ $(0 \le s < \infty)$. Let $0 \le r < \infty$ be an arbitrary number. Then, for all $\psi \in L^{r+s}(\Sigma)$,

$$\int_\Sigma G(|\psi|^{r-1}\,\psi)\,F(\psi) \ge 0\,.$$

Proof. See [Dru07], Lemma 2.6. □

The following embedding result is well known.

Lemma 4.5. Let $\Omega \subset \mathbb{R}^3$ be such that $\partial\Omega \in \mathcal{C}^{0,1}$. For $T > 0$, let $Q :=]0, T[\times\Omega$.
If r, q satisfy

$$r \in [2, \infty]\,, \qquad q \in [2, 6]\,, \qquad \frac{1}{r} + \frac{3}{2q} = \frac{3}{4}\,,$$

then there exists a positive constant $c_{r,q}$ such that

$$\parallel u \parallel_{L^{r,q}(Q)} \ \le \ c \parallel u \parallel_{V_2^{1,0}(Q)} \ .$$

If $\tilde{r}, \tilde{q}$ satisfy

$$\tilde{r} \in [2\,, \infty]\,, \qquad \tilde{q} \in \left[\frac{4}{3}\,, 4\right]\,, \qquad \frac{1}{\tilde{r}} + \frac{1}{\tilde{q}} = \frac{3}{4}\,,$$

then there exists a positive constant $\tilde{c}_{\tilde{r},\tilde{q}}$ such that

$$\parallel u \parallel_{L^{\tilde{r},\tilde{q}}(]0\,,T[\times\partial\Omega)} \ \le \ \tilde{c} \parallel u \parallel_{V_2^{1,0}(Q)} \ .$$

Proof. See [LSU68], Chapter II, paragraph 3. □

Finally we recall that for functions defined on Q, we can introduce for all $h \in]0, T[$

$$u_{(h)}(x,t) \ := \ \frac{1}{h}\int_t^{t+h} u(x,\tau)\,d\tau\,.$$

The function $u_{(h)}$ is called the Steklov averaging of u and it belongs to $W_2^{1,1}(Q_{T-h})$, whenever u belongs to $W_2^{1,0}(Q)$. Its fundamental properties are listed in [LSU68], Chapter II, paragraph 4.

The notation

$$u_{(\underline{h})}(x,t) \;:=\; \frac{1}{h}\int_{t-h}^{t} u(x,\tau)\,d\tau\,,$$

will make sense if we extend u, for instance by zero, to the interval $[-h,0]$.

For functions $u,\eta : Q \to \mathbb{R}$, such that η vanishes in the intervals $[-h,0]$ and $[T-h,T]$, and such that $\int_Q u\,\eta\,dx\,dt < \infty$, it holds that

$$\int_Q u\,\eta_{(\underline{h})}\,dx\,dt \;=\; \int_Q u_{(h)}\,\eta\,dx\,dt\,. \tag{65}$$

We now prove a lemma that will help us to shorten some technical arguments.

Lemma 4.6. Let ξ_1, $\xi_2 \in L^1(Q)$, and suppose that $\xi_3 \in [L^p(Q)]^3$ for some $p > 1$.

Denoting as usual by p' the conjugated exponent to p, suppose that $u \in W^{1,0}_{p',\mathcal{C}}(Q) \cap C(0,T;L^1(\Omega))$ satisfies

$$-\int_Q u\,\frac{\partial\psi}{\partial t} = \int_Q \xi_1\,\psi + \xi_3\cdot\nabla\psi + \int_{\mathcal{S}} \xi_2\,\psi \tag{66}$$

for all $\psi \in C_c^\infty(0,T;C^\infty(\Omega))$ that vanish on $\mathcal{C}$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a globally Lipschitz continuous and bounded function that satisfies $g(0) = 0$, and let F denote the primitive function of g that vanishes at zero.

Then for all $t_1 < T$, it holds that

$$\int_\Omega F(u(t_1)) = \int_\Omega F(u(0)) + \int_{Q_{t_1}} \xi_1\,g(u) + \int_{Q_{t_1}} \xi_3\cdot\nabla g(u) + \int_{\mathcal{S}_{t_1}} \xi_2\,g(u)\,.$$

Proof. We denote by $C^\infty_\Gamma(\Omega)$ the set of all smooth functions in Ω that vanish on Γ.

We consider $t_1 < T$ arbitrary, and choose some positive number $h < T - t_1$.

For an arbitrary $\tilde\psi \in C_c^\infty(0,t_1;C^\infty_\Gamma(\Omega))$ that we extend by zero to $[t_1,T]$ and $[-h,0]$, the test function $\psi := \tilde\psi_{(\underline{h})}$ can be used in (66). Observe that by definition

$$\begin{aligned}
-\int_Q u\,\frac{\partial\tilde\psi_{(\underline{h})}}{\partial t} &= -\int_Q u(t,\cdot)\,\frac{1}{h}\Big(\tilde\psi(t,\cdot) - \tilde\psi(t-h,\cdot)\Big)\,d\,t\\
&= \frac{1}{h}\int_Q u(t,\cdot)\,\tilde\psi(t-h,\cdot)\,d\,t - \frac{1}{h}\int_Q u(t,\cdot)\,\tilde\psi(t,\cdot)\,d\,t\,.
\end{aligned}$$

Using the translation $\tau := t-h$, we obtain that

$$\frac{1}{h}\int_Q u(t,\cdot)\,\tilde\psi(t-h,\cdot)\,d\,t = \frac{1}{h}\int_Q u(\tau+h,\cdot)\,\tilde\psi(\tau,\cdot)\,d\,\tau\,,$$

so that

$$-\int_Q u\,\frac{\partial \tilde\psi_{(\underline{h})}}{\partial t} = \frac{1}{h}\int_Q \Big(u(\tau+h,\cdot)\,\tilde\psi(\tau,\cdot) - \int_Q u(\tau,\cdot)\,\tilde\psi(\tau,\cdot)\Big)\,d\,\tau = \int_Q \frac{\partial u_{(h)}}{\partial t}\,\tilde\psi\,.$$

Using also the fact that the Steklov averaging operator commutes with derivation with respect to space, we transfer for each integral the Steklov averaging according to (65), and we obtain that

$$\int_Q \frac{\partial u_{(h)}}{\partial t}\,\tilde\psi = \int_Q (\xi_1)_{(h)}\,\tilde\psi + \int_Q (\xi_3)_{(h)}\cdot\nabla\tilde\psi + \int_{\mathcal{S}} (\xi_2)_{(h)}\,\tilde\psi\,, \tag{67}$$

for all $\tilde\psi \in C_c^\infty(0,t_1;C_\Gamma^\infty(\Omega))$.

Now, note that by assumption, the function $g(u_{(h)})$ belongs to the space $W^{1,1}_{p',\mathcal{C}}(Q_{t_1})$. Therefore, it is possible to approximate the function $g(u_{(h)})$ in the norm of $W^{1,0}_{p',\mathcal{C}}(Q_{t_1})$ by a sequence $\{\tilde\psi_k\} \subset C_c^\infty(0,t_1;C_\Gamma^\infty(\Omega))$. We insert $\tilde\psi_k$ in (67).

Passing to the limit $k\to\infty$, we obtain that

$$\int_{Q_{t_1}} \frac{\partial u_{(h)}}{\partial t}\,g(u_{(h)}) = \int_{Q_{t_1}} (\xi_1)_{(h)}\,g(u_{(h)}) + \int_{Q_{t_1}} (\xi_3)_{(h)}\cdot\nabla g(u_{(h)}) + \int_{\mathcal{S}_{t_1}} (\xi_2)_{(h)}\,g(u_{(h)})\,.$$

Now, we observe that $\frac{\partial u_{(h)}}{\partial t}\,g(u_{(h)}) = \frac{\partial}{\partial t}F\big(u_{(h)}\big)$, so that the last relation is equivalent to the equation

$$\int_\Omega F(u_{(h)}(t_1))$$
$$= \int_\Omega F(u_{(h)}(0)) + \int_{Q_{t_1}} (\xi_1)_{(h)}\,g(u_{(h)}) + \int_{Q_{t_1}} (\xi_3)_{(h)}\cdot\nabla g(u_{(h)}) + \int_{\mathcal{S}_{t_1}} (\xi_2)_{(h)}\,g(u_{(h)})\,.$$

Since $u\in C([0,T];L^1(\Omega))$, we have for all $t\in[0,T]$ and $h\to 0$ that $u_{(h)}(t)\longrightarrow u(t)$ in $L^1(\Omega)$. Since the function g is globally bounded, its primitive F has at most linear growth at infinity, which implies that

$$F(u_{(h)}(t)) \longrightarrow F(u(t)) \quad \text{in } L^1(\Omega)\,,$$

for all $t\in[0,T]$.

Now, we check the convergence of the other integral terms. We know that

$$u_{(h)} \longrightarrow u \quad \text{in } L^1(Q_{t_1}) \text{ and in } L^1(\mathcal{S}_{t_1})\,.$$

Therefore, we can extract a subsequence such that

$$u_{(h)} \longrightarrow u \text{ almost everywhere in } Q_{t_1} \text{ and on } \mathcal{S}_{t_1}\,.$$

Since $(\xi_1)_{(h)} \longrightarrow \xi_1$ in $L^1(Q_{t_1})$, and $(\xi_2)_{(h)} \longrightarrow \xi_2$ in $L^1(\mathcal{S}_{t_1})$, we easily verify that, as $h\to 0$,

$$\int_{Q_{t_1}} (\xi_1)_{(h)}\,g(u_{(h)}) \longrightarrow \int_{Q_{t_1}} \xi_1\,g(u)\,, \qquad \int_{\mathcal{S}_{t_1}} (\xi_2)_{(h)}\,g(u_{(h)}) \longrightarrow \int_{\mathcal{S}_{t_1}} \xi_2\,g(u)\,.$$

By similar arguments, $\int_{Q_{t_1}} (\xi_3)_{(h)}\cdot\nabla g(u_{(h)}) \longrightarrow \int_{Q_{t_1}} \xi_3\cdot\nabla g(u)$ for $h\to 0$. This proves the claim. □

For obtaining *a-priori* estimates in the L^1-case, we will need two further auxiliary results.

Proposition 4.7. For $n \in \mathbb{N}$ and $u \in W_p^{1,0}(Q) \cap L^{\infty,1}(Q)$, define

$$B_n := \left\{ (t,x) \in [0,T] \times \Omega \,\middle|\, n \leq |u(t,x)| < n+1 \right\}.$$

Suppose that there exists a positive constant C_* such that $\sup_{n\in\mathbb{N}} \int_{B_n} |\nabla u|^p \, dx \, dt \leq C_*$. If $p < \frac{15}{4}$, then for all $1 \leq q < p - \frac{3}{4}$, we can find positive constants c_1, c_2 that depends only on Ω, q, p, such that for $s = p - q/3\,q$

$$\| \nabla u \|_{L^q(Q)} \leq c_1 + c_2 \, \| u \|^s_{L^{\infty,1}(Q)} \; C_*^{1/q} \,.$$

Proof. Similar results were proved in [BG92]. We can also follow the argumentation of [Lew97]. □

Lemma 4.8. Let Ω be an enclosure with the property

$$\operatorname{dist}(\Gamma\,,\,\Sigma) > 0\,. \tag{68}$$

Then for an arbitrary ψ in $C^\infty(\overline{Q})$ such that $\psi \geq 0$ in $[\,0\,,\,T\,] \times \Omega$, $\psi = 0$ on $[\,0\,,\,T\,] \times \Gamma$, and $\psi(T) = 0$, there exists

$$\bar{\psi} \in C^\infty(\overline{Q})\,, \begin{cases} \bar{\psi} \leq 0 & \text{in } [\,0\,,\,T\,] \times \Omega\,, \\ \bar{\psi} = 0 & \text{on } [\,0\,,\,T\,] \times \Gamma\,, \\ \bar{\psi} = 0 & \text{on } \{T\} \times \Omega\,, \end{cases}$$

such that $G(\psi) = G(\bar{\psi})$ on $[0,T\,] \times \Sigma$. In addition, $\|\bar{\psi}\|_{L^\infty(Q)} \leq \|\psi\|_{L^\infty(Q)}$.

Proof. Consider an arbitrary ψ in $C^\infty(\overline{Q})$ such that $\psi \geq 0$ in $[\,0\,,\,T\,] \times \Omega$, $\psi = 0$ on $[\,0\,,\,T\,] \times \Gamma$, and $\psi(T) = 0$. Defining

$$\phi(x) := \max_{t\in[0,T]} \frac{\psi(x,t)}{\max\limits_{y\in\overline{\Omega}} \psi(t,y)}\,,$$

we see that ϕ is continuous, that $0 \leq \phi \leq 1$ in $\overline{\Omega}$ and that $\phi = 0$ on Γ. Now, in view of assumption (68), we can choose a function $\bar{\phi} \in C^\infty(\overline{\Omega})$ that vanishes on Γ and such that

$$\bar{\phi} \geq \phi \;\text{ in } \overline{\Omega}\,, \qquad \bar{\phi} \equiv 1 \;\text{ on } \Sigma\,.$$

Note that the function $t \mapsto \max_{y\in\overline{\Omega}} \psi(t,y)$ is continuous on $[0,T]$ and vanishes at time T. Therefore, we can choose $\zeta \in C^\infty([0,T])$ such that $\zeta(T) = 0$ and

$$\zeta(t) \geq \max_{y\in\overline{\Omega}} \psi(t,y) \quad \text{ on } [0,T]\,.$$

Let $\bar{\psi}(t,x) := \psi(t,x) - \zeta(t)\,\bar{\phi}(x)$. Then, $\bar{\psi}$ belongs to $C^\infty(\overline{Q})$, it is nonpositive in $[\,0\,,\,T\,] \times \Omega$, and it vanishes at time T, as well as on $[0\,,\,T] \times \Gamma$. Obviously, $\bar{\psi}(t,\,x) \geq -\|\psi\|_{L^\infty(Q)}$ for all $(t,x) \in Q$.

Since Ω is an enclosure in the sense of Definition 4.2, observe that $G(1) \equiv 0$. Thus, we can write for all $t \in [\,0\,,\,T\,]$ that

$$G(\bar{\psi}(t)) = G(\psi(t)) - \zeta(t)\,G(\bar{\phi}) = G(\psi(t)) - \zeta(t)\,G(1) = G(\psi(t))\,.$$

□

Lemma 4.9. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then for each $q \in]4/3, 4[$ there exists a positive constant $c = c(q)$ such that

$$\|u\|_{L^q(\partial\Omega)} \leq c\,\|u\|_{L^2(\Omega)}^{1-\alpha}\,\|\nabla u\|_{[L^2(\Omega)]^3}^{\alpha}\,,$$

for all $u \in W^{1,2}(\Omega)$. Here, the number $0 < \alpha < 1$ is given by

$$\alpha = \frac{3}{2} - \frac{2}{q}\,.$$

Proof. See [LSU68],Ch. II, paragraph 2, equation 2.21. □

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