Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Large time asymptotics of growth models on space-like paths I: PushASEP

Alexei Borodin[1], Patrik L. Ferrari[2]

submitted: September 10, 2007

[1] California Institute of Technology
Mathematics 253-37
Pasadena, CA 91125, USA
E-mail: borodin@caltech.edu

[2] Weierstraß-Institut
für Angewandte Analysis und Stochastik
Mohrenstraße 39
10117 Berlin, Germany
E-mail: ferrari@wias-berlin.de

No. 1251
Berlin 2007

2000 *Mathematics Subject Classification.* 82C22, 60K35, 15A52.

Key words and phrases. Simple exclusion process, KPZ class, Airy processes, space-like universality.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider a new interacting particle system on the one-dimensional lattice that interpolates between TASEP and Toom's model: A particle cannot jump to the right if the neighboring site is occupied, and when jumping to the left it simply pushes all the neighbors that block its way.

We prove that for flat and step initial conditions, the large time fluctuations of the height function of the associated growth model along any space-like path are described by the Airy_1 and Airy_2 processes. This includes fluctuations of the height profile for a fixed time and fluctuations of a tagged particle's trajectory as special cases.

1 Introduction

We consider a model of interacting particle systems, which is a generalization of the TASEP (totally asymmetric simple exclusion process) and the Toom model. Besides the extension of some universality results to a new model, the main feature of this paper is the extension of the range of analysis to any "space-like" paths in space-time, whose extreme cases are fixed time and fixed particle (tagged particle problem), see below for details.

Consider the system of N particles $x_1 > \cdots > x_N$ in $\mathbb{Z}$ that undergoes the following continuous time Markovian evolution: Each particle has two exponential clocks – one is responsible for its jumps to the left while the other one is responsible for its jumps to the right. All $2N$ clocks are independent, and the rates of all left clocks are equal to L while the rates of all right clocks are equal to R. When the ith left clock rings, the ith particle jumps to the nearest vacant site on its left. When the ith right clock rings, the ith particle jumps to the right by one provided that the site $x_i + 1$ is empty; otherwise it stays put. The main goal of the paper is to study the asymptotic properties of this system when the number of particles and the evolution time become large.

If $L = 0$ then the dynamics is known under the name of Totally Asymmetric Simple Exclusion Process (TASEP), and if $R = 0$ the dynamics is a special case of Toom's model studied in [9] (see references therein too). Both systems belong to the Kardar-Parisi-Zhang (KPZ) universality class of growth models in $1 + 1$ dimensions.

Particle's jump to the nearest vacant spot on its left can be also viewed as the particle pushing all its left neighbors by one if they prevent it from jumping to the left. This point of view is often beneficial because it remains meaningful for infinite systems, and also the order of particles is not being changed. Because of this pushing effect we call our system the Pushing Asymmetric Simple Exclusion Process or PushASEP.

Observe that for a N-particle PushASEP with particles $x_1(t) > \cdots > x_N(t)$, the evolution of $(x_1, \ldots, x_M)$ for any $M \leq N$ is the M-particle PushASEP not influenced by the presence of the remaining $N - M$ particles. This "triangularity property" seems to be a key feature of our model that allows our analysis to go through.

Our results split in two groups – algebraic and analytic.

Algebraically, we derive a determinantal formula for the distribution of the N-particle PushASEP with an arbitrary fixed initial condition, and we also represent this distribution as a gap probability for a (possibly, signed) determinantal point process (see [12, 16, 17, 21, 22] for information on determinantal processes). The result is obtained in greater generality with jump rates L and R being both time and particle-dependent (Proposition 3.1). The first part (the determinantal formula, see Proposition 2.1) is a generalization of similar results due to [2, 19, 20] obtained by the Bethe Ansatz techniques. Also, a closely related result have been obtained very recently in [10] using a version of the Robinson-Schensted-Knuth correspondence.

Analytically, we use the above-mentioned determinantal process to study the large time behavior of the infinite-particle PushASEP with two initial conditions:

1. Flat initial condition with particles occupying all even integers.

2. Step initial condition with particles occupying all negative integers.

It is not obvious that the infinite-particle PushASEP started from these initial configurations is correctly defined, and some work needs to be done to prove the existence of the Markovian dynamics. However, we take a simpler path here and consider our infinite-particle system as a limit of growing finite-particle systems. It turns out that for the above initial conditions, the distribution of any finite number of particles at any finitely many time moments stabilizes as the total number of particles in the system becomes large enough. It is this limiting distribution that we analyze.

We are able to control the asymptotic behavior of the joint distribution of $x_{n_1}(t_1), \ldots, x_{n_k}(t_k)$ with $x_{n_1}(0) \geq \cdots \geq x_{n_k}(0)$ and $t_1 \geq \cdots \geq t_k$. It is the second main novel feature of the present paper (the first one being the model itself) that we can handle joint distributions of different particles at different time moments. As special cases we find distributions of several particles at a given time moment and distribution of one particle at several time moments (a.k.a. the tagged particle).

In the growth model formulation of PushASEP (that we do not give here; it can be easily reconstructed from the growth models for TASEP and Toom's model described in [9] and references therein), this corresponds to joint distributions of values of the height function at a finite number of space-time points that lie on a space-like path; for that reason we use the term 'space-like path' below. The two extreme space-like paths were described above – they correspond to $t_1 = \cdots = t_k$ and $n_1 = \cdots = n_k$.

The algebraic techniques of handling space-like paths are used in the subsequent paper [5] to analyze two different models, namely the polynuclear growth (PNG) model on a flat substrate and TASEP in discrete time with parallel update.

Our main result states that large time fluctuations of the particle positions along any space-like path have exponents 1/3 and 2/3, and that the limiting process is the Airy_1 process for the flat initial condition and the Airy_2 process for the step initial condition (see the review [11] and Section 2.4 below for the definition of these processes).

In the PushASEP model, we have the fluctuation exponent 1/3 even in the case of zero drift. This is due to the asymmetry in the dynamical rules and it is consistent with the KPZ hypothesis. In fact, from KPZ we expect to have the 1/3 exponent when $j''(\rho) \neq 0$, where $j(\rho)$ is the current of particles as a function of their density ρ, and $j''(\rho) = -2(R + L/(1-\rho)^3)$ for PushASEP.

We find it remarkable that up to scaling factors, the fluctuations are independent of the space-like path we choose (this phenomenon was also observed in [7] for the polynuclear growth model (PNG) with step initial condition). It is natural to conjecture that this type of universality holds at least as broadly as KPZ-universality does.

Interestingly enough, so far it is unknown how to study the joint distribution of $x_{n_1}(t_1)$ and $x_{n_2}(t_2)$ with $x_{n_1}(0) > x_{n_2}(0)$ and $t_1 < t_2$ (two points on a time-like path); this question remains a major open problem of the subject.

Previous results. For the TASEP and PNG models, large time fluctuation results have already been obtained in the following cases: For the step initial condition the Airy_2 process has been shown to occur in the scaling limit for fixed time [14, 15, 18], and more recently for tagged particle [13]. For TASEP, the Airy_1 process occurs for flat initial conditions in continuous time [4] and in discrete time with sequential update [3] with generalization to the initial condition of one particle every $d \geq 2$ sites[1]. Also, a transition between the Airy_2 and Airy_1 processes was obtained in [6]. These are fixed time results; the only previous result concerning general space-like paths is to be found in [7] in the context of the PNG model, where the Airy_2 process was obtained as a limit for a directed percolation model.

Outline. The paper is organized as follows. In Section 2 we describe the model and the results. In Proposition 2.1 the transition probability of the model is given. Then, we define what we mean by space-like paths, and formulate the scaling limit results; the definitions of the Airy_1 and Airy_2 processes are recalled in Section 2.4. In Section 3 we state the general kernel for PushASEP (Proposition 3.1) and then particularize it to step and flat initial conditions (Proposition 3.4 and 3.6). In Section 4 we first prove Proposition 2.1 and then obtain the general kernel for a determinantal measure of a certain form (Theorem 4.2), which includes the one of

[1]Similar results for discrete time TASEP with parallel update and PNG model will follow from more general results of [5].

PushASEP. Finally, the asymptotic analysis is the content of Section 5.

Acknowledgments

A.Borodin was partially supported by the NSF grants DMS-0402047 and DMS-0707163.

2 The PushASEP model and limit results

2.1 The PushASEP

The model we consider is an extension of the well known totally asymmetric simple exclusion process (TASEP) on $\mathbb{Z}$. The allowed configuration are like in the TASEP, i.e., configurations consist of particles on $\mathbb{Z}$, with the constraint that at each site can be occupied by at most one particle (exclusion constraint). We consider a dynamics in continuous time, where particles are allowed to jump to the right and to the left as follows. A particle jumps to its right-neighbor site with some rate, provided the site is empty (TASEP dynamics). To the left, a particle jump to its left-neighbor site with some rate and, if the site is already occupied by another particle, this is pushed to its left-neighbor and so on (push dynamics).

To define precisely the jump rates, we need to introduce a few notations. The dynamics preserves the particle position, thus we can associate to each particle a label. Let $x_k(t)$ be the position of particle k at time t. We choose the right-left labelling, i.e., $x_k(t) > x_{k+1}(t)$ for all $k \in I \subseteq \mathbb{Z}$, $t \geq 0$. With this labelling, we consider $v_k > 0$, $k \in I$, and some smooth positive increasing functions $a(t), b(t)$ with $a(0) = b(0) = 0$. Then, the right jump rate of particle k is $\dot{a}(t)v_k$, while its left jump rate is $\dot{b}(t)/v_k$.

In Proposition 2.1 we derive the expression of the transition probability from time $t = 0$ to time t for N particles, proven in Section 4.

Proposition 2.1. *Consider N particles with initial conditions $x_i(0) = y_i$. Denote its transition probability until time t by*

$$G(x_N, \ldots, x_1; t) = \mathbb{P}(x_i(t) = x_i, 1 \leq i \leq N | x_i(0) = y_i, 1 \leq i \leq N). \tag{2.1}$$

Then

$$\begin{aligned} &G(x_N, \ldots, x_1; t) \\ &= \left(\prod_{n=1}^{N} v_n^{x_n - y_n} e^{-a(t)/v_n} e^{-b(t)v_n} \right) \det \left[F_{k,l}(x_{N+1-l} - y_{N+1-k}, a(t), b(t)) \right]_{1 \leq k,l \leq N}, \end{aligned} \tag{2.2}$$

where

$$F_{k,l}(x, a, b) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{x-1} \frac{\prod_{i=1}^{k-1}(1 - v_{N+1-i}z)}{\prod_{j=1}^{l-1}(1 - v_{N+1-j}z)} e^{bz} e^{a/z}, \tag{2.3}$$

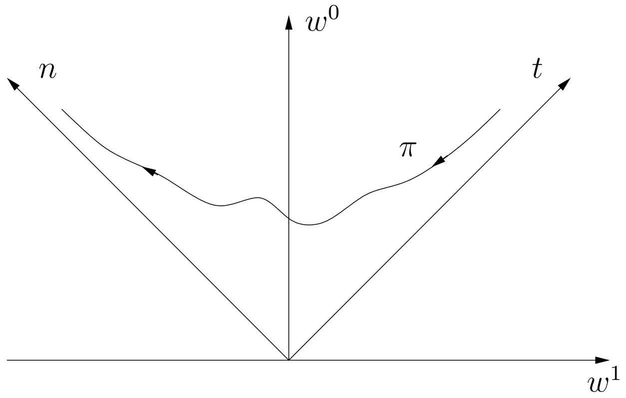

Figure 1: An example of a space-like path. Its slope is, in absolute value, at most 1.

where Γ_0 *is any anticlockwise oriented simple loop with including only the pole at* $z = 0$.

2.2 Space-like paths

From Proposition 2.1 one can compute the joint distribution of particle positions at a given time t, in a similar way of what we made in [4]. However, one of the main motivation for this work is to enlarge the spectrum of the situations which can be analyzed to what we call *space-like* paths. In this context, space-like paths are sequences of particle numbers and times in the ensemble

$$\mathcal{S} = \{(n_k, t_k), k \geq 1 | (n_k, t_k) \prec (n_{k+1}, t_{k+1})\}, \tag{2.4}$$

where, by definition,

$$(n_i, t_i) \prec (n_j, t_j) \text{ if } n_j \geq n_i, t_j \leq t_i, \text{ and the two couples are not identical.} \tag{2.5}$$

The two extreme cases are (1) fixed time, $t_k = t$ for all k, and (2) fixed particle number, $n_k = n$ for all k. This last situation is known as *tagged particle* problem. Since the analysis is of the same degree of difficulty for any space-like path, we will consider the general situation.

Consider any smooth function π, $w^0 = \pi(w^1)$, in the forward light cone of the origin that satisfies

$$|\pi'| \leq 1, \quad |w^1| \leq \pi(w^1). \tag{2.6}$$

These are space-like paths in $\mathbb{R} \times \mathbb{R}_+$, see Figure 1. The first condition (the space-like property) is related to the applicability of our result to sequences of particles in $\mathcal{S}$. The second condition just reflect the choice of having $t \geq 0$ and $n \geq 0$. Time and particle number are connected with the variables w^1 and w^0 by a rotation of 45 degrees. To avoid unnecessary $\sqrt{2}$'s, we set

$$\left\{ \begin{array}{lcl} w^1 & = & \frac{t-n}{2} \\ w^0 & = & \frac{t+n}{2} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{lcl} t & = & w^0 + w^1 \\ n & = & w^0 - w^1 \end{array} \right\} \tag{2.7}$$

For a large parameter T we consider the scaling

$$\begin{cases} w^1 = \theta T - uT^{2/3}, \\ w^0 = \pi(\theta)T - \pi'(\theta)uT^{2/3} + \frac{1}{2}\pi''(\theta)u^2T^{1/3}. \end{cases}$$

Then,

$$\begin{aligned} t(u) &= (\pi(\theta)+\theta)T - (\pi'(\theta)+1)uT^{2/3} + \tfrac{1}{2}\pi''(\theta)u^2T^{1/3}, \\ n(u) &= \left[(\pi(\theta)-\theta)T + (1-\pi'(\theta))uT^{2/3} + \tfrac{1}{2}\pi''(\theta)u^2T^{1/3}\right]. \end{aligned} \tag{2.8}$$

Setting $\pi(\theta) = 1-\theta$ we get the fixed time case with $t = T$, while setting $\pi(\theta) = \alpha+\theta$ we get the tagged particle situation with particle number $n = \alpha T$.

2.3 Scaling limits

Universality occurs in the large T limit. In Proposition 3.1 we will get an expression for the joint distribution in the general setting. For the asymptotic analysis we consider the case where all particles have the same jump rates, i.e., we set

$$v_k = 1 \text{ for all } k \in I. \tag{2.9}$$

Moreover, we consider time-homogeneous case, i.e., we set $a(t) = Rt$ and $b(t) = Lt$ for some $R, L \geq 0$ (for time non-homogeneous case, one would just replace R and L by some time-dependent functions). Two important initial conditions are

(a) *flat initial condition*: particles start from $2\mathbb{Z}$,

(b) *step initial condition*: particles start from $\mathbb{Z}_- = \{\ldots, -3, -2, -1\}$.

In the first case, the macroscopic limit shape is flat, while in the second case it is curved, see [11] for a review on universality in the TASEP. For TASEP with step initial conditions and particle-dependent rates v_k, the study of tagged particle has been carried out in [13].

Flat initial conditions

For the flat initial condition, it is not very difficult to get the proper scaling limit as $T \to \infty$. The initial position of particle $n(u)$ is $-2n(u)$ and during time $t(u)$ it will have travelled around $\mathbf{v}\, t(u)$, where $\mathbf{v}$ is the mean speed of particles, given by

$$\mathbf{v} = -2L + R/2. \tag{2.10}$$

The reason is that the density of particle is $1/2$ and the particles jumps to the right with rate R but the site on its right has a $1/2$ chance to be empty. Moreover, particles move (and push) to the left with rate L but typically every second move to the left is due to a push from another particle. Therefore, the rescaled process is given by

$$u \mapsto X_T(u) = \frac{x_{n(u)}(t(u)) - (-2n(u) + \mathbf{v}\, t(u))}{-T^{1/3}}, \tag{2.11}$$

where $n(u)$ and $t(u)$ are defined in (2.8). The rescaled process X_T has a limit for large T given in terms of the Airy$_1$ process, $\mathcal{A}_1$ (see [4, 6, 11] and Section 2.4 for details on $\mathcal{A}_1$).

Theorem 2.2 (Convergence to the Airy$_1$ process)**.** *Let us set the vertical and horizontal rescaling*

$$S_v = ((8L+R)(\pi(\theta)+\theta))^{1/3}, \quad S_h = \frac{4((8L+R)(\pi(\theta)+\theta))^{2/3}}{(R+4L)(\pi'(\theta)+1)+4(1-\pi'(\theta))}. \tag{2.12}$$

Then

$$\lim_{T\to\infty} X_T(u) = S_v \mathcal{A}_1(u/S_h) \tag{2.13}$$

in the sense of finite dimensional distributions.

The proof of this theorem is in Section 5. The specialization for fixed time $t = T$ is

$$S_v = (8L+R)^{1/3}, \quad S_h = \frac{(8L+R)^{2/3}}{2}, \tag{2.14}$$

and the one for tagged particle $n = \alpha T$ at times $t(u) = T - 2uT^{2/3}$, obtained by setting $\theta = (1-\alpha)/2$, is

$$S_v = (8L+R)^{1/3}, \quad S_h = \frac{2(8L+R)^{2/3}}{4L+R}. \tag{2.15}$$

Step initial condition

The proper rescaled process for step initial condition is quite intricate. Denote by βt the typical position of particle with number around αt at time t. In the situations previously studied in the literature, there was a nice function $\beta = \beta(\alpha)$. In the present situation this is not anymore true, but we can still describe the limit shape. More precisely, α and β are parametrized by a $\mu \in (0,1)$ via

$$\alpha(\mu) = (1-\mu)^2(R + L/\mu^2), \quad \beta(\mu) = -((1-2\mu)R + L/\mu^2). \tag{2.16}$$

In particular, we have

$$\alpha(\mu) = \frac{\pi(\theta)-\theta}{\pi(\theta)+\theta}. \tag{2.17}$$

For any given θ, there exists only one μ such that (2.17) holds, because α is strictly monotone in μ. Some computations are needed, but finally we get the rescaling of the position x as a function of u, namely,

$$x(u) = \sigma_0 T - \sigma_1 u T^{2/3} + \sigma_2 u^2 T^{1/3}, \tag{2.18}$$

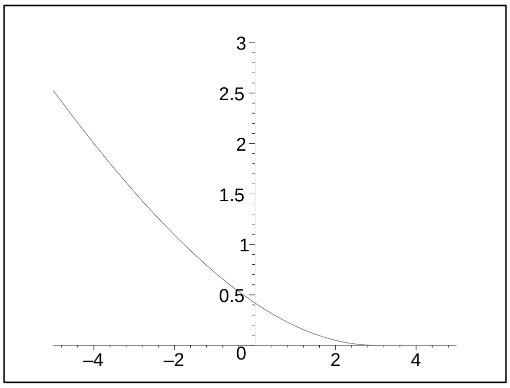

Figure 2: Parametric plot of $(\beta(\mu), \alpha(\mu))$, for $L = 1, R = 4$.

where

$$\begin{aligned} \sigma_0 &= (\pi(\theta)+\theta)\beta(\mu) \\ \sigma_1 &= 1 + (\pi'(\theta)+1)\left(\mu R - \tfrac{L}{\mu}\right) + (1-\pi'(\theta))\frac{1}{1-\mu} \\ \sigma_2 &= \tfrac{1}{2}\pi''(\theta)\left(\mu R + \tfrac{L}{\mu} - \tfrac{1}{1-\mu}\right) + \frac{(\pi'(\theta)(1-\alpha(\mu)) - (1+\alpha(\mu)))^2}{4(1-\mu)^3(\pi(\theta)+\theta)(R+L/\mu^3)}. \end{aligned} \tag{2.19}$$

The rescaled process is then given by

$$u \mapsto X_T(u) = \frac{x_{n(u)}(t(u)) - (\sigma_0 T - \sigma_1 u T^{2/3} + \sigma_2 u^2 T^{1/3})}{-T^{1/3}}, \tag{2.20}$$

with $n(u)$ as in (2.8). Let us define the constants

$$\begin{aligned} \kappa_0 &= \frac{(\pi(\theta)+\theta)(R+L/\mu^3)}{\mu(1-\mu)}, \\ \kappa_1 &= \frac{(\pi'(\theta)+1)(R+L/\mu^2)}{2\mu} - \frac{\pi'(\theta)-1}{2\mu(1-\mu)^2}. \end{aligned} \tag{2.21}$$

Then, a detailed asymptotic analysis would lead to,

$$\lim_{T\to\infty} X_T(u) = \mu \kappa_0^{1/3} \mathcal{A}_2(\kappa_1 \kappa_0^{-2/3} u), \tag{2.22}$$

in the sense of finite dimensional distributions, where $\mathcal{A}_2$ is the Airy$_2$ process (see [11, 14, 18] and Section 2.4 for details on $\mathcal{A}_2$). As for the flat PNG, special cases are tagged particle and fixed time. In Section 5.2 we obtain (2.22) by looking at the contribution coming from the series expansion around a double critical point. To get (2.22) rigorously, one has to control (1) the error terms in the convergence on bounded sets and (2) get some bounds to get convergence of the Fredholm determinants. This is what we actually do the in the flat initial condition setting.

2.4 Limit processes

For completeness, we shortly recall the definitions of the limit process $\mathcal{A}_1$ and $\mathcal{A}_2$ appearing above. The notation $\mathrm{Ai}(x)$ below stands for the classical Airy function [1].

Definition 2.3 (The Airy$_1$ process)**.** *The* Airy$_1$ process $\mathcal{A}_1$ *is the process with m-point joint distributions at* $u_1 < u_2 < \ldots < u_m$ *given by the Fredholm determinant*

$$\mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathcal{A}_1(u_k) \leq s_k\}\Big) = \det(\mathbb{1} - \chi_s K_{\mathcal{A}_1}\chi_s)_{L^2(\{u_1,\ldots,u_m\}\times\mathbb{R})}, \tag{2.23}$$

where $\chi_s(u_k, x) = \mathbb{1}(x > s_k)$ *and the kernel* $K_{\mathcal{A}_1}$ *is given by*

$$\begin{aligned} K_{\mathcal{A}_1}(u_1, s_1; u_2, s_2) &= -\frac{1}{\sqrt{4\pi(u_2-u_1)}}\exp\left(-\frac{(s_2-s_1)^2}{4(u_2-u_1)}\right)\mathbb{1}(u_2 > u_1) \\ &+\mathrm{Ai}(s_1+s_2+(u_2-u_1)^2)\exp\left((u_2-u_1)(s_1+s_2)+\frac{2}{3}(u_2-u_1)^3\right). \end{aligned} \tag{2.24}$$

Definition 2.4 (The Airy$_2$ process)**.** *The* Airy$_2$ process $\mathcal{A}_2$ *is the process with m-point joint distributions at* $u_1 < u_2 < \ldots < u_m$ *given by the Fredholm determinant*

$$\mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathcal{A}_2(u_k) \leq s_k\}\Big) = \det(\mathbb{1} - \chi_s K_{\mathcal{A}_1}\chi_s)_{L^2(\{u_1,\ldots,u_m\}\times\mathbb{R})}, \tag{2.25}$$

where $\chi_s(u_k, x) = \mathbb{1}(x > s_k)$ *and the kernel* $K_{\mathcal{A}_2}$ *is given by*

$$K_{\mathcal{A}_2}(u_1, s_1; u_2, s_2) = \begin{cases} \int_{\mathbb{R}_+} e^{-\lambda(u_2-u_1)}\mathrm{Ai}(s_1+\lambda)\mathrm{Ai}(s_2+\lambda), & u_2 \geq u_1, \\ -\int_{\mathbb{R}_-} e^{-\lambda(u_2-u_1)}\mathrm{Ai}(s_1+\lambda)\mathrm{Ai}(s_2+\lambda), & u_2 < u_1. \end{cases} \tag{2.26}$$

3 Finite time kernel

3.1 General kernel for PushASEP

In Theorem 4.2 we will derive a general expression for joint distributions of a determinantal measure. In particular, it follows that the joint distribution of particle positions is given by a Fredholm determinant of the form

$$\mathbb{P}\Big(\bigcap_{k=1}^{m}\{x_{n_k}(t_k) \geq a_k\}\Big) = \det\left(\mathbb{1} - \tilde{\chi}_a K \tilde{\chi}_a\right)_{\ell^2(\{(n_1,t_1),\ldots,(n_m,t_m)\}\times\mathbb{Z})} \tag{3.1}$$

with $((n_1, t_1), \ldots, (n_m, t_m)) \in \mathcal{S}$, and $\tilde{\chi}_a((n_k, t_k))(x) = \mathbb{1}(x < a_k)$.

Before stating the result, proven in Section 4, we introduce a space of functions V_n. Consider the set of numbers $\{v_1, \ldots, v_n\}$ and let $\{u_1 < u_2 < \ldots < u_\nu\}$ be their different values, with α_k being the multiplicity of u_k (v_k is the jump rate of particle with label k). Then we define the space

$$V_n = \mathrm{span}\{x^l u_k^x, 1 \leq k \leq \nu, 0 \leq l \leq \alpha_k - 1\}. \tag{3.2}$$

Proposition 3.1 (PushASEP kernel)**.** *The kernel K for the PushASEP is given by*

$$K((n_1,t_1),x_1;(n_2,t_2),x_2) = -\phi^{((n_1,t_1),(n_2,t_2))}(x_1,x_2) + \sum_{k=1}^{n_2} \Psi^{n_1,t_1}_{n_1-k}(x_1)\Phi^{n_2,t_2}_{n_2-k}(x_2) \tag{3.3}$$

where

$$\Psi^{n,t}_{n-l}(x) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{x-y_l-1} e^{a(t)/z+b(t)z} \frac{(1-v_1z)\cdots(1-v_nz)}{(1-v_1z)\cdots(1-v_lz)}, \tag{3.4}$$

the functions $\{\Phi^{n,t}_{n-l}, l=1,\ldots,n\}$, are obtained by the orthogonality relation

$$\sum_{x\in\mathbb{Z}} \Psi^{n,t}_{n-l}(x)\Phi^{n,t}_{n-k}(x) = \delta_{k,l}, \tag{3.5}$$

and by the requirement $\mathrm{span}\{\Phi^{n,t}_{n-l}(x), 1\leq l\leq n\} = V_n$*. Finally, the first term has the form*

$$\phi^{((n_1,t_1),(n_2,t_2))}(x,y) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}z}{z^{y-x+1}} \frac{e^{(a(t_1)-a(t_2))/z} e^{(b(t_1)-b(t_2))z}}{(1-v_{n_1+1}z)\cdots(1-v_{n_2}z)} \mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]}. \tag{3.6}$$

3.2 Kernel for step initial condition

We set all the jump rates to 1: $v_1 = v_2 = \cdots = 1$. The transition function (3.6) does not depend on initial conditions. It is useful to rewrite it in a slightly different form.

Lemma 3.2. *The transition function can be rewritten as*

$$\begin{aligned} &\phi^{((n_1,t_1),(n_2,t_2))}(x,y) \\ &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{0,1}} \mathrm{d}w \frac{1}{w^{x-y+1}} \left(\frac{w}{w-1}\right)^{n_2-n_1} \frac{e^{a(t_1)w+b(t_1)/w}}{e^{a(t_2)w+b(t_2)/w}} \mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]}. \end{aligned} \tag{3.7}$$

Proof of Lemma 3.2. The proof follows by the change of variable $z=1/w$ in (3.6). $\square$

Lemma 3.3. *Let $y_i=-i, i\geq 1$. Then, the functions Φ and Ψ are given by*

$$\begin{aligned} \Psi^{n,t}_k(x) &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{0,1}} \mathrm{d}w \frac{(w-1)^k}{w^{x+n+1}} e^{a(t)w+b(t)/w}, \\ \Phi^{n,t}_j(x) &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_1} \mathrm{d}z \frac{z^{x+n}}{(z-1)^{j+1}} e^{-a(t)z-b(t)/z}. \end{aligned} \tag{3.8}$$

Proof of Lemma 3.3. $\Psi^{n,t}_k(x)$ comes by the change of variable $z=1/w$ in (3.4). For $k\geq 0$, the pole at $w=1$ is irrelevant, but in the kernel $\Psi^{n,t}_k$ enters also for negatives values of k. Let us compute $\sum_{x\in\mathbb{Z}} \Phi^{n,t}_j(x)\Psi^{n,t}_k(x)$. The x-dependent terms give

$$\sum_{x\in\mathbb{Z}} (z/w)^x = \frac{w}{w-z}\mathbb{1}_{\{|z|<|w|\}} - \frac{w}{w-z}\mathbb{1}_{\{|w|<|z|\}}. \tag{3.9}$$

Thus, before taking inside the sum in the integral we divide it into $\{x \geq 0\}$ and $\{x < 0\}$ which gives the two contributions in the r.h.s. of (3.9). The difference between the two terms is that the integration paths satisfy $|z| < |w|$ for the first term and $|w| < |z|$ for the second term. At $w = z$ there is a simple pole, therefore by deforming the integration paths to make them coinciding, the net result is the residue at $w = z$. The terms in the exponential and the terms like $z^{(\cdots)}$ simplify, leading to

$$\sum_{x\in\mathbb{Z}} \Phi_j^{n,t}(x)\Psi_k^{n,t}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_1} \mathrm{d}z (z-1)^{k-j-1} = \delta_{j,k}. \tag{3.10}$$

□

Proposition 3.4 (Step initial conditions, finite time kernel)**.** *The kernel for $y_i = -i$, $i \geq 1$, is given by*

$$\begin{aligned} &K((n_1,t_1),x_1;(n_2,t_2),x_2) \qquad (3.11)\\ &= -\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \frac{1}{w^{x_1-x_2+1}} \left(\frac{w}{1-w}\right)^{n_2-n_1} \frac{e^{a(t_1)w+b(t_1)/w}}{e^{a(t_2)w+b(t_2)/w}} \mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]}\\ &+ \frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}w \oint_{\Gamma_1} \mathrm{d}z \frac{e^{b(t_1)/w+a(t_1)w}}{e^{b(t_2)/z+a(t_2)z}} \frac{(1-w)^{n_1+1}}{w^{x_1+n_1+1}} \frac{z^{x_2+n_2}}{(1-z)^{n_2+1}} \frac{1}{w-z}. \end{aligned}$$

The contours Γ_0 and Γ_1 include the poles $w = 0$ and $z = 1$, respectively, and no other poles.

Proof of Proposition 3.4. Consider the main term of the kernel, namely

$$\begin{aligned} \sum_{k=1}^{n_2} \Psi_{n_1-k}^{n_1,t_1}(x_1)\Phi_{n_2-k}^{n_2,t_2}(x_2) &= \sum_{k=1}^{n_2} \Bigg(\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{0,1}} \mathrm{d}w \frac{(w-1)^{n_1-k}}{w^{x_1+n_1+1}} e^{a(t_1)w+b(t_1)/w}\Bigg)\\ &\quad \times \Bigg(\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_1} \mathrm{d}z \frac{z^{x_2+n_2}}{(z-1)^{n_2-k+1}} e^{-a(t_2)z-b(t_2)/z}\Bigg). \end{aligned}$$

First take the sum inside and then we extend it to $+\infty$, since the second term is identically equal to zero for $k > n_2$. The integration paths are taken so that $|z-1| < |w-1|$. The k-dependent terms are

$$\sum_{k\geq 1} \left(\frac{z-1}{w-1}\right)^k = \frac{w-1}{w-z}. \tag{3.12}$$

Notice now we have a new pole at $w = z$, but at $w = 1$ the pole vanished. Therefore the main part of the kernel equals

$$\frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_1} \mathrm{d}z \oint_{\Gamma_{0,z}} \mathrm{d}w \frac{e^{a(t_1)w+b(t_1)/w}}{e^{a(t_2)z+b(t_2)/z}} \frac{(w-1)^{n_1+1}}{w^{x_1+n_1+1}} \frac{z^{x_2+n_2}}{(z-1)^{n_2+1}} \frac{1}{w-z}. \tag{3.13}$$

The contribution of the pole at $w = z$ is exactly equal to the contribution of the pole at $z = 1$ in the transition function (3.7). Therefore in the final result the first term coming from (3.7) has the integral only around $z=0$, and the second term is (3.13) but with the integral over w only around the pole at $w=0$. Finally, a conjugation by a factor $(-1)^{n_1-n_2}$ leads to the result. □

3.3 Kernel for flat initial condition

We again consider the case $v_1=v_2=\cdots=1$.

Lemma 3.5. *Let $y_i=-2i$, $i\geq 1$. Then, the functions Φ and Ψ are given by*

$$\begin{aligned}\Psi_k^{n,t}(x) &= \frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{0,1}} \mathrm{d}w \frac{(w(w-1))^k}{w^{x+2n+1}} e^{a(t)w+b(t)/w},\\ \Phi_j^{n,t}(x) &= \frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_1} \mathrm{d}z \frac{(2z-1)z^{x+2n}}{(z(z-1))^{j+1}} e^{-a(t)z-b(t)/z}.\end{aligned} \tag{3.14}$$

Proof of Lemma 3.5. The proof is like in Lemma 3.3, but the residue terms lead this time to

$$\sum_{x\in\mathbb{Z}} \Phi_j^{n,t}(x)\Psi_k^{n,t}(x) = \frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_1} \mathrm{d}z (2z-1)(z(z-1))^{k-j-1} = \delta_{j,k} \tag{3.15}$$

by the change of variable $w=z(z-1)$. □

Proposition 3.6 (Flat initial conditions, finite time kernel)**.** *The kernel for $y_i=-2i$, $i\in\mathbb{Z}$, is given by*

$$\begin{aligned}&K((n_1,t_1),x_1;(n_2,t_2),x_2)\\ &= -\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_0} \mathrm{d}w \frac{1}{w^{x_1-x_2+1}}\left(\frac{w}{1-w}\right)^{n_2-n_1} \frac{e^{a(t_1)w+b(t_1)/w}}{e^{a(t_2)w+b(t_2)/w}} \mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]}\\ &+ \frac{-1}{2\pi \mathrm{i}}\oint_{\Gamma_1} \mathrm{d}z \frac{e^{a(t_1)(1-z)+b(t_1)/(1-z)}}{e^{a(t_2)z+b(t_2)/z}} \frac{z^{n_1+n_2+x_2}}{(1-z)^{n_1+n_2+x_1+1}}.\end{aligned} \tag{3.16}$$

Proof of Proposition 3.6. The strategy is similar to the one of Proposition 3.4. This time, the sum in k is

$$\sum_{k\geq 1}\left(\frac{z(z-1)}{w(w-1)}\right)^k = \frac{w(w-1)}{(w-z)(w-1+z)}. \tag{3.17}$$

So, the pole for $w=1$ is now replaced by two simple poles, one at $w=z$ and one at $w=1-z$. The pole at $w=z$ cancels with the one at $z=1$ of (3.7). Thus we are left with

$$\frac{1}{(2\pi \mathrm{i})^2}\oint_{\Gamma_1} \mathrm{d}z \oint_{\Gamma_{0,1-z}} \mathrm{d}w \frac{e^{a(t_1)w+b(t_1)/w}}{e^{a(t_2)z+b(t_2)/z}} \frac{(w-1)^{n_1+1}}{(z-1)^{n_2+1}} \frac{z^{x_2+n_2-1}}{w^{x_1+n_1}} \frac{2z-1}{(w-z)(w-1+z)}. \tag{3.18}$$

This is the main part of the kernel for the initial condition $y_i = -2i$, $i \geq 1$. To obtain the kernel for $y_i = -2i$, $i \in \mathbb{Z}$, we just have to look far enough into the bulk of our system, until when the influence of the fact that there are only a finite number of particles on the right vanishes. For the kernel, this means that the pole at $w = 0$ vanishes. Therefore, we are left with the contribution of the simple pole at $w = 1-z$, and computing the corresponding residue leads to the result of the Proposition, up to a factor $(-1)^{n_1-n_2}$, which however have no impact on the Fredholm determinant in question. $\square$

4 Determinantal measures

In this section we first prove Proposition 2.1. Then, we use it to extend the measure to space-like paths. More precisely, we first obtain a general determinantal formula in Theorem 4.1. Then, in Theorem 4.2, we prove that the measure has determinantal correlations and obtain an expression of the associated kernel.

Proof of Proposition 2.1. We first prove that the initial condition is satisfied. We have

$$F_{k,l}(x,0) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{x-1} \frac{\prod_{i=1}^{k-1}(1-v_{N+1-i}z)}{\prod_{j=1}^{l-1}(1-v_{N+1-j}z)}. \tag{4.1}$$

(a) $F_{k,l}(x,0) = 0$ for $x \geq 1$ because the pole at $z = 0$ vanishes.
(b) $F_{k,l}(x,0) = 0$ for $k \geq l$ and $x < l-k$, because then

$$F_{k,l}(x,0) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{x-1}(1-v_l z)\cdots(1-v_{k-1}z) \tag{4.2}$$

and the residue at infinity equals to zero for $x < l-k$.

Assume that $x_N < \cdots < x_1$. If $x_N > y_N$, also $x_l > y_N$ for $l = 1,\ldots,N-1$. Thus $F_{1,l}(x_{N+1-l} - y_N, 0) = 0$ using (a). Therefore $G(x_N,\ldots,x_1;0) = 0$. On the other hand, it $x_N < y_N$, then $x_N < y_k - N + k$, $k = 1,\ldots,N-1$. Thus $F_{1,k}(x_N - y_{N+1-k}, 0) = 0$ using (b) and the fact that $x_N - y_{N+1-k} < 1-k$. Therefore we conclude that $G(x_N,\ldots,x_1;0) = 0$ if $x_N \neq y_N$. For $x_N = y_N$, $F_{1,1}(0,0) = 1$ and by (a) $F_{1,l}(x_{N+1-l} - y_N, 0) = 0$ for $l = 2,\ldots,N$. This means that

$$G(x_N,\ldots,x_1;0) = \delta_{x_N,y_N} G(x_{N-1},\ldots,x_1;0). \tag{4.3}$$

By iterating the procedure we obtain

$$G(x_N,\ldots,x_1;0) = \prod_{k=1}^{N} \delta_{x_k,y_k}. \tag{4.4}$$

Notice that the prefactor in (2.2) is equal to one at $t = 0$.

The initial condition being settled, we need to prove that (2.2) satisfies the PushASEP dynamics. For that purpose, let us first compute $\frac{\mathrm{d}F_{k,l}(x,t)}{\mathrm{d}t}$.

$$\frac{\mathrm{d}F_{k,l}(x,t)}{\mathrm{d}t} = \dot{a}(t)F_{k,l}(x-1,t) + \dot{b}(t)F_{k,l}(x+1,t), \tag{4.5}$$

from which it follows, by differentiating the prefactor and the determinant column by column,

$$\begin{aligned}\frac{\mathrm{d}G(x_N,\ldots,x_1;t)}{\mathrm{d}t} &= -\Big(\dot{a}(t)\sum_{k=1}^N v_k + \dot{b}(t)\sum_{k=1}^N \frac{1}{v_k}\Big)G(x_N,\ldots,x_1;t) \\ &\quad +\dot{a}(t)\sum_{k=1}^N v_k G(\ldots,x_k-1,\ldots;t) \\ &\quad +\dot{b}(t)\sum_{l=1}^N \frac{1}{v_l}G(\ldots,x_l+1,\ldots;t).\end{aligned} \tag{4.6}$$

To proceed, we need an identity. Using

$$\frac{z^x}{1-v_{N+1-l}z} = \frac{v_{N+1-l}z^{x+1}}{1-v_{N+1-l}z} + z^x \tag{4.7}$$

it follows that

$$F_{k,l+1}(x,t) = F_{k,l}(x,t) + v_{N+1-l}F_{k,l+1}(x+1,t). \tag{4.8}$$

Therefore, for $j=2,\ldots,N$, by setting $\tilde{y}_k = y_{N+1-k}$,

$$\begin{aligned}&G(\ldots,x_j,x_{j-1}=x_j,\ldots;t) = \frac{1}{Z_N}\det\Big[v_{N+1-l}^{x_{N+1-l}}F_{k,l}(x_{N+1-l}-\tilde{y}_k,t)\Big]_{1\le k,l\le N} \\ &= \frac{1}{Z_N}\det\Big[\ldots \quad v_j^{x_j}F_{k,N+1-j}(x_j-\tilde{y}_k,t) \quad v_{j-1}^{x_j}F_{k,N+2-j}(x_{j-1}-\tilde{y}_k,t)\cdots\Big].\end{aligned}$$

Here Z_N does not depend on the x_j's. Using (4.8) we have

$$\begin{aligned}&v_{j-1}^{x_j}F_{k,N+2-j}(x_j-\tilde{y}_k,t) \\ &= v_{j-1}^{x_j}F_{k,N+1-j}(x_j-\tilde{y}_k,t) + v_{j-1}^{x_j+1}F_{k,N+2-j}(x_j+1-\tilde{y}_k,t)\frac{v_j}{v_{j-1}}.\end{aligned} \tag{4.9}$$

Using this identity in the previous formula, the first term cancels being proportional to its left column, and the second term yields

$$G(\ldots,x_j,x_{j-1}=x_j,\ldots;t) = \frac{v_j}{v_{j-1}}G(\ldots,x_j,x_{j-1}=x_j+1,\ldots;t). \tag{4.10}$$

With (4.10) we can go back to (4.6). First, consider all the terms in (4.6) which are proportional to $\dot{a}(t)$. They have the form

$$
\begin{aligned}
& -\sum_{k=1}^{N} v_k G(\ldots;t) + \sum_{k=1}^{N} v_k G(\ldots,x_k-1,\ldots;t) && (4.11)\\
= \;& -v_1 G(\ldots;t) - \sum_{k=2}^{N} v_k (1-\delta_{x_{k-1},x_k+1}) G(\ldots;t) && (4.12)\\
& +v_N G(x_N-1,\ldots;t) + \sum_{k=1}^{N-1} v_k (1-\delta_{x_{k+1},x_k}) G(\ldots,x_k-1,\ldots;t) && (4.13)\\
& -\sum_{k=2}^{N} v_k G(\ldots,x_k,x_{k-1}=x_k+1,\ldots;t) && (4.14)\\
& +\sum_{k=1}^{N-1} v_k G(\ldots,x_{k+1}=x_k,x_k,\ldots;t). && (4.15)
\end{aligned}
$$

By using (4.10) and shifting the summation index by one, we get that (4.15) equals

$$
\sum_{k=2}^{N} v_{k-1} G(\ldots,x_k,x_{k-1}=x_k+1,\ldots;t)\frac{v_k}{v_{k-1}}, \tag{4.16}
$$

which cancels (4.14). The expression (4.12) is the contribution in the master equation of the particles jumping to the right and leaving the state $(x_N,\ldots,x_1)$ with jump rate $\dot{a}(t)v_k$, while (4.13) is the contribution of the particles arriving to the state $(x_N,\ldots,x_1)$. Therefore, the jumps to the right satisfy the exclusion constraint.

Secondly, consider all the terms in (4.6) which are proportional to $\dot{b}(t)$. They are

$$
-\sum_{k=1}^{N} \frac{1}{v_k} G(\ldots;t) + \sum_{k=1}^{N} \frac{1}{v_k} G(\ldots,x_k+1,\ldots;t). \tag{4.17}
$$

Let us denote by $m(k)$ the index of the last particle to the right of particle k such that particle $m(k)$ belongs to the same block of particles as particle k (we say that two particles are in the same block if between them all sites are occupied). Then, (4.17) takes the form

$$
(4.17) = -\sum_{k=1}^{N} \frac{1}{v_k} G(\ldots;t) + \sum_{k=1}^{N} \frac{1}{v_k} G(\ldots,x_k+1,x_k+1,\ldots,x_k+k-m(k),\ldots;t). \tag{4.18}
$$

Using (4.10) we get

$$
\begin{aligned}
& \frac{1}{v_k} G(\ldots,x_k+1,x_k+1,\ldots,x_k+k-m(k),\ldots;t) && \\
= \;& \frac{1}{v_k}\frac{v_k}{v_{k-1}} G(\ldots,x_k+1,x_k+2,\ldots,x_k+k-m(k),\ldots;t) && (4.19)\\
= \;& \frac{1}{v_{k-1}} G(\ldots,x_k+1,x_{k-1}+1,\ldots,x_k+k-m(k),\ldots;t). && (4.20)
\end{aligned}
$$

By iterations we finally obtain

$$(4.17) = -\sum_{k=1}^{N} \frac{1}{v_k} G(\ldots;t) + \sum_{k=1}^{N} \frac{1}{v_{m(k)}} G(\ldots, x_k+1, x_{k-1}+1, \ldots, x_{m(k)}+1, \ldots; t). \tag{4.21}$$

The first term in (4.21) is the contribution of particles pushing to the left and leaving the state $(x_N, \ldots, x_1)$, while the second term is the contribution of particles arriving at the state $(x_N, \ldots, x_1)$ because they were pushed, and the particle number k pushes to the left with rate $\dot{b}(t)/v_k$. □

We would like to obtain the joint distribution of particle N_k at time t_k for $N_1 \geq N_2 \geq \ldots \geq N_m \geq 1$ and $0 \leq t_1 \leq t_2 \leq \ldots \leq t_m$. By Proposition 2.1, this can be written as an appropriate marginal of a product of m determinants.

Notational remark: Below there is an abuse of notation. For example, $x_l^n(t_i)$ and $x_l^n(t_{i+1})$ are considered different variables even if $t_i = t_{i+1}$. One could call them simply $x_l^n(i)$ and $x_l^n(i+1)$, but then one loses the connection with the times t_i's. In this sense, t_i is considered as a symbol, not as a number.

Theorem 4.1. *Let us set $t_0 = 0$, $a(t_0) = b(t_0) = 0$, and $N_{m+1} = 0$. The joint distribution of PushASEP particles is a marginal of a determinantal measure, obtained by summation of the variables in the set*

$$D = \{x_k^l(t_i), 1 \leq k \leq l, 1 \leq l \leq N_i, 0 \leq i \leq m\} \setminus \{x_1^{N_i}(t_i), 1 \leq i \leq m\}; \tag{4.22}$$

the range of summation for any variable in this set in $\mathbb{Z}$. Precisely,

$$\begin{aligned} &\mathbb{P}(x_{N_i}(t_i) = x_1^{N_i}(t_i), 1 \leq i \leq m | x_k(0) = y_k(0), 1 \leq k \leq N_1) \\ &= \mathrm{const} \times \sum_D \det\left[\Psi_{N_1-l}^{N_1}(x_k^{N_1}(t_0))\right]_{1 \leq k,l \leq N_1} \\ &\times \prod_{i=1}^{m} \Bigg[\det[\mathcal{T}_{t_i,t_{i-1}}(x_l^{N_i}(t_i), x_k^{N_i}(t_{i-1}))]_{1\leq k,l \leq N_i} \\ &\qquad \times \prod_{n=N_{i+1}+1}^{N_i} \det[\phi_n(x_k^{n-1}(t_i), x_l^n(t_i))]_{1 \leq k,l \leq n} \Bigg] \end{aligned} \tag{4.23}$$

where

$$\mathcal{T}_{t_j,t_i}(x,y) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{x-y-1} e^{(a(t_j)-a(t_i))/z} e^{(b(t_j)-b(t_i))z}, \tag{4.24}$$

$$\Psi_{N_1-l}^{N_1}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{x-y_l-1}(1-v_{l+1}z)\cdots(1-v_{N_1}z), \tag{4.25}$$

$$\phi_n(x,y) = v_n^{y-x} \mathbb{1}_{[y \geq x]} \quad \textit{and} \quad \phi_n(x_n^{n-1}, y) = v_n^y. \tag{4.26}$$

Remark: the variables x_n^{n-1} participating in the last factor of (4.23) are fictitious, cf. (4.26), and are used for convenience of notation only.

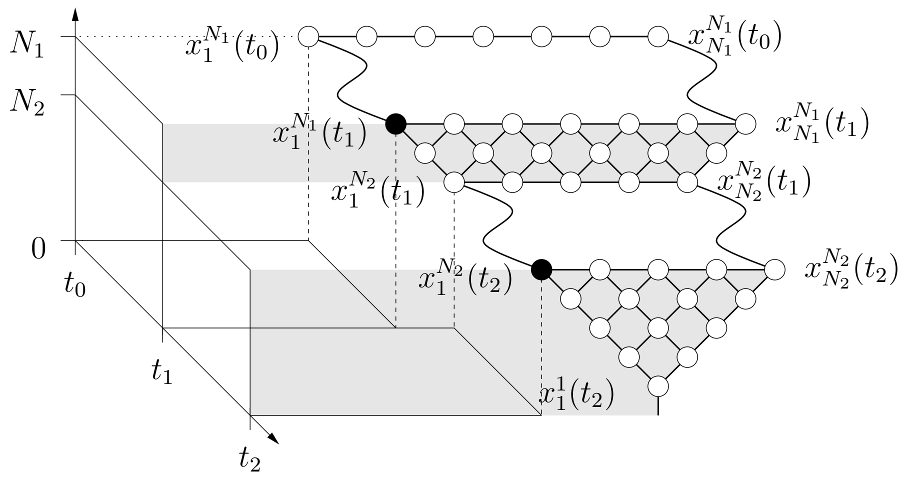

Figure 3: A graphical representation of variables entering in the determinantal structure, illustrated for $m = 2$. The wavy lines represents the time evolution between t_0 and t_1 and from t_1 to t_2. The rest is the interlacing structure on the variables induced by the $\det[\phi_n(\cdots)]$. The black dots are the only variables which are not in the summation set $D = D(0) \cup D^*(t_1) \cup \cdots \cup D^*(t_m)$ (see Figure 4 too). The variables of the border of the interlacing structures are explicitly indicated.

We illustrate the determinantal structure in Figure 3.

Proof of Theorem 4.1. Since the evolution is Markovian, we have

$$\begin{aligned}
&\mathbb{P}(x_{N_i}(t_i) = x_1^{N_i}(t_i), 1 \leq i \leq m | x_k(0) = x_1^k, 1 \leq k \leq N_1) \\
= &\sum \mathbb{P}(x_k(0) = x_1^k(0), 1 \leq k \leq N_1 | x_k(0) = y_k, 1 \leq k \leq N_1) \qquad (4.27)\\
\times &\prod_{i=1}^{m} \mathbb{P}(x_k(t_i) = x_1^k(t_i), 1 \leq k \leq N_i | x_k(t_{i-1}) = x_1^k(t_{i-1}), 1 \leq k \leq N_i)
\end{aligned}$$

where the sum is over $x_1^k(0)$, $1 \leq k \leq N_1$, and $x_1^k(t_i)$, $1 \leq k \leq N_i - 1$, $i = 1, \ldots, m$. Note that so far the lower index of all variables x_l^k is identically equal to 1.

The continuation of the proof requires a series of Lemmas collected at the end of this section, see Section 4.1. We apply Proposition 2.1 to the $m+1$ factors in (4.27), namely,

$$\begin{aligned}
&\mathbb{P}(x_k(t_i) = x_1^k(t_i), 1 \leq k \leq N_i | x_k(t_{i-1}) = x_1^k(t_{i-1}), 1 \leq k \leq N_i) \qquad (4.28)\\
&= \text{const} \times \Big(\prod_{n=1}^{N_i} v_n^{x_1^n(t_i) - x_1^n(t_{i-1})} \Big) \det \big[F_{k,l}(x_1^{N_i+1-l}(t_i) - x_1^{N_i+1-k}(t_{i-1}) \big]_{1 \leq k,l \leq N_i}.
\end{aligned}$$

First we collect all the factors coming from the $\prod_{n=1}^{N_i} v_n^{x_1^n(t_i) - x_1^n(t_{i-1})}$. We have the factor

$$
\begin{aligned}
&\Big(\prod_{n=1}^{N_1} v_n^{x_1^n(0)-y_k}\Big)\prod_{k=1}^{m}\prod_{n=1}^{N_n} v_n^{x_1^n(t_k)-x_1^n(t_{k-1})} \\
=\ &\Big(\prod_{n=1}^{N_1} v_n^{-y_n}\Big)\Big(\prod_{i=1}^{m-1}\prod_{n=N_{i+1}+1}^{N_i} v_n^{x_1^n(t_i)}\Big)\prod_{n=1}^{N_m} v_n^{x_1^n(t_m)}. \qquad (4.29)
\end{aligned}
$$

Then we apply Lemma 4.4 to all the factors $\det[F_{k,l}(\cdots)]$. For the initial condition we have

$$
\sum_{\widetilde{D}(0)} \det\left[F_{N_1+1-l,1}(x_k^{N_1}(0)-y_l,0,0)\right]_{1\le k,l\le N_1}\prod_{n=2}^{N_1}\det\left[\varphi_n(x_k^{n-1}(0),x_l^n(0))\right]_{1\le k,l\le n}. \qquad (4.30)
$$

For the other terms, $i=1,\ldots,m$, we get

$$
\begin{aligned}
&\sum_{\widetilde{D}(t_i)} \det\left[F_{N_i+1-l,1}(x_k^{N_i}(t_i)-x_1^l(t_{i-1}),a_i,b_i)\right]_{1\le k,l\le N_i} \\
&\quad\times\prod_{n=2}^{N_i}\det\left[\varphi_n(x_k^{n-1}(t_i),x_l^n(t_i))\right]_{1\le k,l\le n}. \qquad (4.31)
\end{aligned}
$$

Thus, the probability we want to compute in (4.27) is obtained by a marginal of a measure on $m+1$ interlacing triangles, when we sum over all the variables in $D(0),D^*(t_1),\ldots,D^*(t_m)$, see Figure 4 for the definitions of these sets. At this point we apply Lemma 4.5 as follows. For $i=1,\ldots,m-1$ we do the sum over the variables in $\widehat{D}(t_i)$. Notice that the remaining variables in (4.29) do not belong to the $\widehat{D}(t_i)$, thus we factorize them out. So, r.h.s. of (4.27) is, up to a constant, equal to

$$
\begin{aligned}
&\sum (4.29)\times\det\left[F_{N_1+1-l,1}(x_k^{N_1}(0)-y_l,0,0)\right]_{1\le k,l\le N_1} \\
\times\ &\Bigg[\prod_{i=0}^{m-1}\Big(\prod_{n=2}^{N_i}\det\left[\varphi_n(x_k^{n-1}(t_i),x_l^n(t_i))\right]_{1\le k,l\le n}\Big) \\
\times\ &\det\left[F_{N_{i+1}+1-l,1}(x_k^{N_{i+1}}(t_{i+1})-x_1^l(t_i),a_{i+1},b_{i+1})\right]_{1\le k,l\le N_{i+1}}\Bigg] \\
\times\ &\prod_{n=2}^{N_m}\det\left[\varphi_n(x_k^{n-1}(t_m),x_l^n(t_m))\right]_{1\le k,l\le n} \qquad (4.32)
\end{aligned}
$$

with the sum is over the variables described just above. By summing over the $\widehat{D}(t_i)$, the determinant with $F_{N_{i+1}+1-l,1}$ becomes a determinant with $F_{1,1}$ and the product of the $\det[\varphi_n(\cdots)]$ is restricted to $n = N_{i+1}+1, \ldots, N_i$. Thus,

$$\begin{aligned}
(4.27) &= \text{const} \times \sum (4.29) \times \det\left[F_{N_1+1-l,1}(x_k^{N_1}(0) - y_l, 0, 0)\right]_{1\le k,l\le N_1} \\
&\times \prod_{i=1}^{m} \Bigg(\det\left[F_{1,1}(x_k^{N_i}(t_i) - x_l^{N_i}(t_{i-1}), a_i, b_i)\right]_{1\le k,l\le N_i} \\
&\times \prod_{n=N_{i+1}+1}^{N_i} \det\left[\varphi_n(x_k^{n-1}(t_i), x_l^n(t_i))\right]_{1\le k,l\le n} \Bigg)
\end{aligned} \tag{4.33}$$

where we set $N_{m+1} = 0$ (the contribution from $n = 1$ is 1). Finally, by using Lemma 4.6 we can include the terms in (4.29) into the φ_n's by modifying the last row, i.e., by setting it equal to v_n^y. Thus,

$$\begin{aligned}
(4.27) &= \text{const} \times \det\left[F_{N_1+1-l,1}(x_k^{N_1}(0) - y_l, 0, 0)\right]_{1\le k,l\le N_1} \\
&\times \prod_{i=1}^{m} \Bigg(\det\left[F_{1,1}(x_k^{N_i}(t_i) - x_l^{N_i}(t_{i-1}), a_i, b_i)\right]_{1\le k,l\le N_i} \\
&\times \prod_{n=N_{i+1}+1}^{N_i} \det\left[\phi_n(x_k^{n-1}(t_i), x_l^n(t_i))\right]_{1\le k,l\le n} \Bigg).
\end{aligned} \tag{4.34}$$

The identification to the expressions in Theorem 4.1 uses the representations (2.3) and (3.4). ☐

The first line represent the initial condition at $t_0 = 0$, the term with $\Psi_{N_1-l}^{N_1}$ in Theorem 4.1. These N_1 variables evolves until time t_1 and this is represented by the first line (term $\mathcal{T}_{t_1,t_0}$). After that, there is a reduction of the number of variables from N_1 to N_2 by the interlacing structure, which is followed by the time evolution from t_1 to t_2. This is repeated $m-1$ times. Finally it ends with an interlacing structure. If $N_1 = N_2$, then the first interlacing structure is trivial (not present), while if for example $t_2 = t_1$, then the time evolution is just the identity.

In what follows, the picture to keep in mind is that, starting from bottom to top in Figure 3, it corresponds to having a sort of vicious walkers with increasing number of walkers when the transition is made by the ϕ's, and with constant number of walkers if the transition is the temporal one made by $\mathcal{T}$.

The determinantal measure in (4.23) is written with outer product over time moments but it can be rewritten by taking the outer product over the index n in the variables x_k^n's. Let us introduce the following notations. For any level n there is a number $c(n) \in \{0, \ldots, m+1\}$ of products of terms $\mathcal{T}$ which are the time evolution of n particles between consecutive times in the set $\{t_1, \ldots, t_m\}$ (in other words $c(n)$ is $\#\{i | N_i = n\}$). Let us denote them by $t_0^n < \ldots < t_{c(n)}^n$. Notice that $t_0^n = t_{c(n+1)}^{n+1}$,

$t_0^{N_1} = t_0$, $t_1^{N_1} = t_1$, and $t_0^0 = t_{c(0)}^0 = t_m$. Then, the measure in (4.23) takes the form

$$\text{const} \times \prod_{n=1}^{N_1} \Bigg[\det[\phi_n(x_k^{n-1}(t_0^{n-1}), x_l^n(t_{c(n)}^n))]_{1\le k,l\le n} \tag{4.35}$$
$$\times \prod_{a=1}^{c(n)} \det[\mathcal{T}_{t_a^n, t_{a-1}^n}(x_k^n(t_a^n), x_l^n(t_{a-1}^n))]_{1\le k,l\le n} \Bigg] \det[\Psi_{N_1-l}^{N_1}(x_k^{N_1}(t_0^{N_1}))]_{1\le k,l\le N_1}.$$

In Theorem 4.2 we show that a measure on the $x_k^n(t_a^n)$ of the form (4.35) is determinantal and we give the expression for the kernel. Then we particularize it in case of the PushASEP with particle dependent jump rates. For this purpose, we introduce a couple of notations. For any two time moments $t_{a_1}^{n_1}, t_{a_2}^{n_2}$, we define the convolution over all the transitions between them by $\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})}$ (backwards in time, since forward in the n's), i.e.,

$$\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})} = \mathcal{T}_{t_{a_1}^{n_1}, t_0^{n_1}} * \phi_{n_1} * \mathcal{T}^{n_1+1} * \cdots * \phi_{n_2-1} * \mathcal{T}_{t_{c(n_2)}^{n_2}, t_{a_2}^{n_2}} \tag{4.36}$$

where

$$\mathcal{T}^n = \mathcal{T}_{t_{c(n)}^n, t_0^n}. \tag{4.37}$$

If no such factor exists, then we set $\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})} = 0$. Above we used

$$\mathcal{T}_{t_3,t_2} * \mathcal{T}_{t_2,t_1} = \mathcal{T}_{t_3,t_1}, \tag{4.38}$$

which is an immediate corollary of (4.24). In a more general case considered in Theorem 4.2 below, if (4.38) does not holds, then $\mathcal{T}^n$ is just the convolution of the transitions between $t_{c(n)}^n$ and t_0^n by definition. Moreover, define the matrix M with entries $M_{k,l}$, $1 \le k, l \le N_1$,

$$M_{k,l} = \big(\phi_k * \mathcal{T}^k * \cdots * \phi_{N_1} * \mathcal{T}^{N_1} * \Psi_{N_1-l}^{N_1}\big)(x_k^{k-1}) \tag{4.39}$$

and the vector

$$\Psi_{n-l}^{n,t_a^n} = \phi^{(t_a^n, t_0^{N_1})} * \Psi_{N_1-l}^{N_1}. \tag{4.40}$$

Theorem 4.2. *Assume that the matrix M is invertible. Then, the probability measure of the form (4.35) viewed as $(N_1 + \ldots + N_m)$-point process is determinantal, and the correlation kernel can be computed as follows*

$$\begin{aligned} K(t_{a_1}^{n_1}, x_1; t_{a_2}^{n_2}, x_2) &= -\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})}(x_1, x_2) \\ &+ \sum_{k=1}^{N_1} \sum_{l=1}^{n_2} \Psi_{n_1-k}^{n_1, t_{a_1}^{n_1}}(x_1) [M^{-1}]_{k,l} (\phi_l * \phi^{(t_{c(l)}^l, t_{a_2}^{n_2})})(x_l^{l-1}, x_2). \end{aligned} \tag{4.41}$$

In the case when the matrix M is upper triangular, there is a simpler way to write the kernel. Set

$$\Phi_{n-k}^{n,t_a^n}(x) = \sum_{l=1}^{n} [M^{-1}]_{k,l} \big(\phi_l * \phi^{(t_{c(l)}^l, t_a^n)}\big)(x_l^{l-1}, x) \tag{4.42}$$

for all $n = 1, \dots, N_1$ *and* $k = 1, \dots, n$*. Then,* $\{\Phi^{n,t^n_a}_{n-k}\}_{k=1,\dots,n}$ *is the unique basis of the linear span of*

$$\Big\{(\phi_1 * \phi^{(t^1_{c(1)},t^n_a)})(x^0_1, x), \dots, (\phi_n * \phi^{(t^n_{c(n)},t^n_a)})(x^{n-1}_n, x)\Big\} \tag{4.43}$$

that is different from (4.43) by a triangular matrix (as in (4.42)), and that is biorthogonal to $\{\Psi^{n,t^n_a}_{n-k}\}$*:*

$$\sum_{x\in\mathbb{Z}} \Phi^{n,t^n_a}_i(x)\Psi^{n,t^n_a}_j(x) = \delta_{i,j}, \quad i,j = 0,\dots,n-1. \tag{4.44}$$

The correlation kernel can then be written as

$$K(t^{n_1}_{a_1}, x_1; t^{n_2}_{a_2}, x_2) = -\phi^{(t^{n_1}_{a_1},t^{n_2}_{a_2})}(x_1,x_2) + \sum_{k=1}^{n_2} \Psi^{n_1,t^{n_1}_{a_1}}_{n_1-k}(x_1)\Phi^{n_2,t^{n_2}_{a_2}}_{n_2-k}(x_2). \tag{4.45}$$

Moreover, one has the identity

$$\phi^{(t^{n_1}_{a_1},t^{n_2}_{a_2})} * \Phi^{n_2,t^{n_2}_{a_2}}_{n_2-l} = \Phi^{n_1,t^{n_1}_{a_1}}_{n_1-l} \tag{4.46}$$

for $n_1 \geq n_2$ *and* $a_1 \leq a_2$ *for* $n_1 = n_2$*.*

Proof of Theorem 4.2. The proof is similar to the one of Lemma 3.4 in [4], which is in its turn based on the formalism of [8]. The only place where the argument changes substantially is the definition of the matrix L, see [4], formula (3.32). The variables of interest are in the space $\mathfrak{Y} = \mathfrak{X}^{(1)} \cup \dots \cup \mathfrak{X}^{(N_1)}$, with $\mathfrak{X}^{(n)} = \mathfrak{X}^{(n)}_0 \cup \dots \cup \mathfrak{X}^{(n)}_{c(n)}$, where $\mathfrak{X}^{(n)}_a = \mathbb{Z}$ is the space where the n variables at time t^n_a live. Let us also denote $I = \{1,\dots,N_1\}$. Then, the matrix L written with the order given by the entries in the set of all variables $\mathfrak{X} = I \cup \mathfrak{Y}$ becomes

$$L = \begin{pmatrix}
0 & E_0 & 0 & E_1 & 0 & E_2 & 0 & \cdots & E_{N_1-1} & 0 \\
0 & 0 & -T_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & -W_{[1,2)} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & -T_2 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -W_{[2,3)} & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -T_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -W_{[N_1-1,N_1)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -T_{N_1} \\
\Psi^{(N_1)} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix} \tag{4.47}$$

with the matrix blocks in L have the following entries:

$$[\Psi^{(N_1)}]_{x,j} = \Psi^{N_1}_{N_1-j}(x), \quad x \in \mathfrak{X}^{(N_1)}_0, j \in I, \tag{4.48}$$

$$[E_n]_{i,y} = \begin{cases} \phi_{n+1}(x^n_{n+1}, y), & i = n+1, y \in \mathfrak{X}^{(n+1)}_{c(n+1)}, \\ 0, & i \in I \setminus \{n+1\}, y \in \mathfrak{X}^{(n+1)}_{c(n+1)}, \end{cases} \tag{4.49}$$

$$[W_{[n,n+1)}]_{x,y} = \phi_{n+1}(x,y), \quad x \in \mathfrak{X}^{(n)}_0, y \in \mathfrak{X}^{(n+1)}_{c(n+1)}, \tag{4.50}$$

and T_n is the matrix made of blocks

$$T_n = \begin{pmatrix} T_{n,1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & T_{n,c(n)} \end{pmatrix}, \tag{4.51}$$

where

$$[T_{n,a}]_{x,y} = \mathcal{T}_{t_a^n, t_{a-1}^n}(x,y), \quad x \in \mathfrak{X}_a^{(n)}, y \in \mathfrak{X}_{a-1}^{(n)}. \tag{4.52}$$

The rest of the proof is along the same lines as that of Lemma 3.4 in [4].

Although the argument gives a proof in the case when all variables $x_a^n(t_b^n)$ vary over finite sets, a simple limiting argument immediately extends the statement to any discrete sets, provided the series that defines $M_{k,l}$ are absolutely convergent, which is certainly true in our case. □

A special case of Theorem 4.2 is Proposition 3.1 stated in Section 3, which we prove below.

Proof of Proposition 3.1. This is a specialization of Theorem 4.2. The kernel depends only on the actual times and particle numbers, therefore we might drop the label a_i of $t_{a_i}^{n_i}$. Equivalently, we can use the notation (n_i, t_i) instead of $t_{a_i}^{n_i}$, to go back to the natural notations of the model. For PushASEP we have $\Psi_{N_1-l}^{N_1}(x) = F_{N_1+1-l,1}(x - y_l, 0, 0)$ and

$$\mathcal{T}_{t_j,t_i}(x,y) = F_{1,1}(x-y, a(t_j)-a(t_i), b(t_j)-b(t_i)). \tag{4.53}$$

First of all, we sum over the $\{x_k^{N_1}(0), 1 \leq k \leq N_1\}$ variables, since we are not interested in the initial conditions (being fixed). While applied to the $F_{k,l}(x, a(t_i), b(t_i))$, the time evolution $\mathcal{T}_{t_j,t_i}$ changes it into $F_{k,l}(x, a(t_j), b(t_j))$,

$$\sum_{y\in\mathbb{Z}} \mathcal{T}_{t_j,t_i}(x,y) F_{k,l}(y, a(t_i), b(t_i)) = F_{k,l}(x, a(t_j), b(t_j)). \tag{4.54}$$

This implies that Theorem 4.2 still holds but with $t_0^{N_1} = t_1$ and

$$\Psi_{N_1-l}^{N_1}(x) = F_{N_1+1-l,1}(x - y_l, a(t_1), b(t_1)). \tag{4.55}$$

We have, see (4.65), that

$$(\phi_k * F_{l,N_1+1-k})(x,a,b) = F_{l,N_1+2-k}(x,a,b). \tag{4.56}$$

Using (4.54) and (4.56) repeatedly one then gets

$$\Psi_{n-l}^{n,t_k^n}(x) = F_{N_1+1-l,N_1+1-n}(x - y_l, a(t_k^n), b(t_k^n)) \tag{4.57}$$

which can be rewritten as (3.4).

Next we show that the matrix M is upper triangular. Once again, (4.54) and (4.56) are applied several times, leading to

$$M_{k,l} = \sum_{y\in\mathbb{Z}} v_k^y F_{N_1+1-l,N_1+1-k}(y-y_l, a(t^k_{c(k)}), b(t^k_{c(k)})). \tag{4.58}$$

Set $a_k = a(t^k_{c(k)})$ and $b_k = b(t^k_{c(k)})$. Then, for $k<l$,

$$M_{k,l} = \sum_{y\in\mathbb{Z}} v_k^y \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{y-y_l-1} e^{a_k/z+b_k z} \frac{(1-v_{l+1}z)\cdots(1-v_{N_1}z)}{(1-v_{k+1}z)\cdots(1-v_{N_1}z)}. \tag{4.59}$$

We divide the sum over y in two regions, $\{y\geq 0\}$ and $\{y<0\}$, and then we take them inside the integral and use

$$\sum_{y\in\mathbb{Z}}(az)^y = \sum_{y\geq 0}(az)^y + \sum_{y<0}(az)^y = \frac{1}{1-az}\mathbb{1}_{\{|az|<1\}} - \frac{1}{1-az}\mathbb{1}_{\{|az|>1\}}. \tag{4.60}$$

For $k>l$ the new term in the denominator, $1-v_k z$, is cancelled so that this is not a pole and we can deform the contours to be the same. Thus for $k>l$ the net result is zero. This is not the case for $k\leq l$, since in that case the new pole at $1/v_k$ does not have to vanish. Moreover, the diagonal terms are not zero, thus the matrix M is invertible. In fact, $M_{k,k} = v_k^{y_l+1} e^{v_k a_k + b_k/v_k} \neq 0$.

Since M is upper triangular, we need to determine the space V_{N_1} where the orthogonalization has to be made. The k-th basis vector is

$$(\phi_k * \phi^{(t^k_{c(k)},t_1)})(x_k^{k-1},x) = \sum_{y\in\mathbb{Z}} v_{N_1}^y \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{y-x-1} \frac{e^{a_k/z+b_k z}}{(1-v_{k+1}z)\cdots(1-v_{N_1}z)}. \tag{4.61}$$

We apply (4.60) and obtain

$$f_k(x) \equiv \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{1/v_k}} \mathrm{d}z z^{-x-1} \frac{e^{a_k/z+b_k z}}{(1-v_k z)(1-v_{k+1}z)\cdots(1-v_{N_1}z)} \tag{4.62}$$

plus residue terms which are linear combinations of the $(\phi_n * \phi^{(t^n_{c(n)},t_1)})(x_n^{n-1},x)$ with $n>k$. Therefore the space V_{N_1} is generated by the functions f_k for $k=1,\ldots,N_1$. For $k=N_1$, the evaluation of the residue leads to $f_{N_1}(x) = \mathrm{const}\times v_{N_1}^x$. For $k = N_1-1$, if $v_{N_1-1}\neq v_{N_1}$, then $f_{N_1-1}(x) = \mathrm{const}\times v_{N_1-1}^x$, while if $v_{N_1-1} = v_{N_1}$, it gives $f_{N_1-1}(x) = \mathrm{const}\times x\, v_{N_1}^x$, since the pole is of order 2. In general, $f_k(x) = \mathrm{const}\times v_k^x$ if $v_k\neq v_l$ for all $l>k$ and $f_k(x) = \mathrm{const}\times \mathrm{Poly}_m(x) v_k^x$ if there are m values of $l\in\{k+1,\ldots,N_1\}$ such that $v_k = v_l$, where $\mathrm{Poly}_m(x)$ is a polynomial of order m in x. This is due to the fact that the pole is of order $m+1$. Therefore, the space where the orthogonalization has to be done is the one indicated in the Proposition.

Finally, we need an expression for the transition between two times, which is given by (4.36). Every time that we convolute a ϕ_k with $\mathcal{T}$, we get an extra factor $1/(1-v_k z)$ in the integral. Therefore, if $t^{n_2}_{a_2}\leq t^{n_1}_{a_1}$ and $n_2\geq n_1$, then

$$\phi^{(t^{n_1}_{a_1},t^{n_2}_{a_2})}(x,y) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{x-y-1} \frac{e^{(a(t^{n_1}_{a_1})-a(t^{n_2}_{a_2}))/z} e^{(b(t^{n_1}_{a_1})-b(t^{n_2}_{a_2}))z}}{(1-v_{n_1+1}z)\cdots(1-v_{n_2}z)}, \tag{4.63}$$

while $\phi^{(t_{a_1}^{n_1},t_{a_2}^{n_2})}(x,y)=0$ otherwise. ☐

4.1 Some lemmas

In this subsection we state and prove the Lemmas used in the proof of Theorem 4.2.

Lemma 4.3. *Let us define the function*

$$\varphi_n(x,y)=\left\{\begin{array}{ll} v_n^{y-x}, & y\geq x, \\ 0, & y<x. \end{array}\right. \tag{4.64}$$

Then the following recurrence relations holds

$$F_{k,l+1}(x,a,b)=(\varphi_{N+1-l}*F_{k,l})(x,a,b) \tag{4.65}$$

and

$$F_{k-1,l}(x,a,b)=(\varphi_{N+2-k}*F_{k,l})(x,a,b). \tag{4.66}$$

From (4.66) and $\varphi_n(x,y)=\varphi_n(0,y-x)=\varphi_n(-y,-x)$ *it follows*

$$F_{k-1,l}(-x,a,b)=\sum_{y\in\mathbb{Z}}F_{k,l}(-y,a,b)\varphi_{N+2-k}(y,x). \tag{4.67}$$

Proof of Lemma 4.3. We have

$$F_{k,l}(x,a,b)=\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}z z^{x-1}e^{bz}e^{a/z}\frac{(1-v_Nz)\cdots(1-v_{N+2-k}z)}{(1-v_Nz)\cdots(1-v_{N+2-l}z)}. \tag{4.68}$$

Then applying $\sum_{y\geq x}v_{N+1-l}^{y-x}z^y=z^x/(1-v_{N+1-l}z)$ (for $|z|\ll 1$), we get that in the denominator we have an extra factor, which corresponds to increasing l by one. Similarly, applying φ_{N+2-k}, the extra factor in the denominator cancels the last one in the numerator, thus this is equivalent to decreasing k by one. ☐

We define the following domains, which will occurs several times in the following. A graphical representation is in Figure 4. Let us denote the set of interlacing variables at time t_i by

$$D(t_i)=\{x_k^n(t_i),1\leq n\leq N_i,1\leq k\leq n|x_k^{n+1}(t_i)<x_k^n(t_i)\leq x_{k+1}^{n+1}(t_i)\}. \tag{4.69}$$

Then let

$$\widetilde{D}(t_i)=\{x_k^n(t_i)\in D(t_i)|k\geq 2\},\quad \widehat{D}(t_i)=\{x_k^n(t_i)\in D(t_i)|n\leq N_{i+1}-1\}, \tag{4.70}$$

and

$$D^*(t_i)=D(t_i)\setminus\{x_1^{N_i}(t_i)\},\quad \widehat{D}^*(t_i)=D^*(t_i)\setminus\widehat{D}(t_i). \tag{4.71}$$

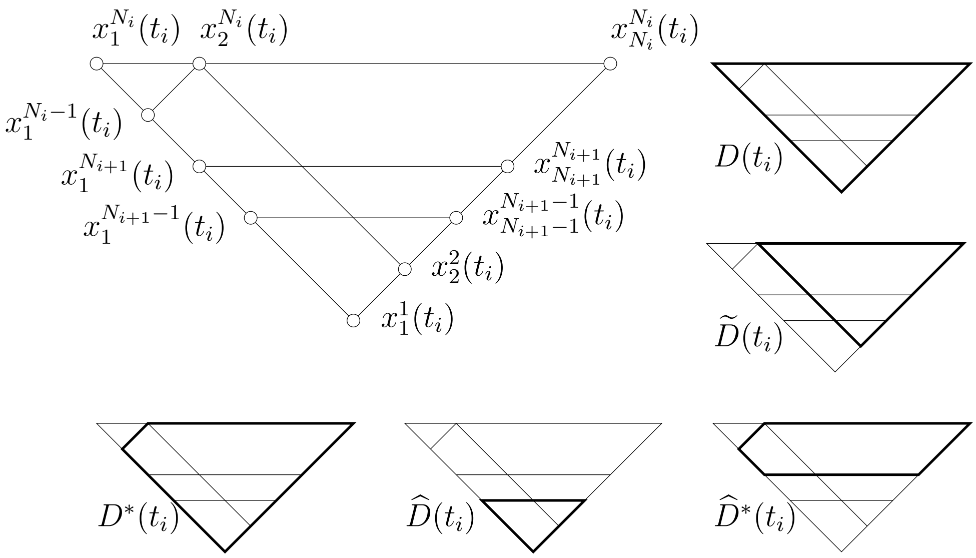

Figure 4: A graphical representation of the summation domains that occurs in the next lemmas and theorem. The bold lines passes through the border of the domains.

Lemma 4.4. *We have the identity*

$$\begin{aligned} &\det\left[F_{k,l}(x_1^{N_i+1-l}(t_i) - x_1^{N_i+1-k}(t_{i-1}), a, b)\right]_{1\le k,l\le N_i} \\ = \;& \text{const} \sum_{\widetilde{D}(t_i)} \left(\prod_{n=2}^{N_i} \det\left[\varphi_n(x_k^{n-1}(t_i), x_l^n(t_i))\right]_{1\le k,l\le n} \right) \\ \times \;& \det\left[F_{N_i+1-l,1}(x_k^{N_i}(t_i) - x_1^l(t_{i-1}), a, b)\right]_{1\le k,l\le N_i} \end{aligned} \tag{4.72}$$

where we set $\varphi_n(x_n^{n-1}, x) = 1$.

Proof of Lemma 4.4. By changing the indices we get that l.h.s. of (4.72) is, up to a sign, equal to

$$\det\left[F_{N_i+1-l,k}(x_1^{N_i+1-k}(t_i) - x_1^l(t_{i-1}), a, b)\right]_{1\le k,l\le N_i} \tag{4.73}$$

Using repeatedly the identity (4.65) we have

$$F_{n,k}(x, a, b) = (\varphi_{N_i+2-k} * \cdots * \varphi_{N_i} * F_{n,1})(x, a, b). \tag{4.74}$$

Therefore,

$$(4.73) = \det\left[(\varphi_{N_i+2-k} * \cdots * \varphi_{N_i} * F_{N_i+1-l,1})(x_1^{N_i+1-k} - x_1^l(t_{i-1}), a, b)\right]_{1\le i,j\le N_i} \tag{4.75}$$

We write explicitly the convolution by introducing explicit summation variables as follows

$$
\begin{aligned}
&(\varphi_{N_i+2-k} * \cdots * \varphi_{N_i} * F_{N_i+1-l,1})(x_1^{N_i+1-k} - x_1^l(t_{i-1}), a, b) \\
&= \sum_{\substack{x_n^{N_i+1-k+n}, \\ 1\leq n\leq k-1}} \Big(\prod_{n=1}^{k-1} \varphi_{N_i+1-k+n}(x_n^{N_i-k+n}, x_{n+1}^{N_i+1-k+n}) \Big) \\
&\qquad \times F_{N_i+1-l,1}(x_k^{N_i} - x_1^l(t_{i-1}), a, b),
\end{aligned} \tag{4.76}
$$

where we used the fact that $\varphi_m(x,y) = \varphi_m(x+c, y+c)$ for any $c \in \mathbb{Z}$. By multilinearity of the determinant, we can take the sums and the factors φ's out of the determinant with the result

$$
\begin{aligned}
(4.73) \;=\; & \sum_{\substack{x_k^n(t_i), \\ 2\leq n\leq N_i, \\ 2\leq k\leq n}} \Big(\prod_{n=2}^{N_i} \prod_{k=1}^{n-1} \varphi_n(x_k^{n-1}(t_i), x_{k+1}^n(t_i)) \Big) \\
& \times \; \det \left[F_{N_i+1-l,1}(x_k^{N_i} - x_1^l(t_{i-1}), a, b) \right]_{1\leq i,j\leq N_i}.
\end{aligned} \tag{4.77}
$$

The product of the φ's is non-zero only if $x_k^{n-1}(t_i) \leq x_{k+1}^n(t_i)$ is satisfied for all the variables. Moreover, in the symmetric part of the remaining summation domain, e.g., when $x_3^3(t_i) \geq x_2^2(t_i)$ *and* $x_2^3(t_i) \geq x_2^2(t_i)$, the product of the φ's is symmetric, while the last determinant is antisymmetric in the variables $\{x_k^{N_i}, k = 1, \ldots, N_i\}$. By iteration (a simple generalization of Lemma 3.3 in [4]) it follows that the result is unchanged if we restrict the sum to $\widetilde{D}(t_i)$, i.e., to the interlacing configurations.

The product of the determinants of φ's in the right-hand side of (4.72) is either 1 or 0 depending on whether the variables interlace (belongs to $D(t_i)$) or not. This implies (4.72). ☐

Lemma 4.5. *We have the identity*

$$
\begin{aligned}
& \sum_{\widehat{D}(t_i)} \Big(\prod_{n=2}^{N_{i+1}} \det \left[\varphi_n(x_k^{n-1}(t_i), x_l^n(t_i)) \right]_{1\leq k,l\leq n} \Big) \\
\times \;& \det \left[F_{N_{i+1}+1-l,1}(x_k^{N_{i+1}}(t_{i+1}) - x_1^l(t_i), a, b) \right]_{1\leq k,l\leq N_{i+1}} \\
= \;& \det \left[F_{1,1}(x_k^{N_{i+1}}(t_{i+1}) - x_l^{N_{i+1}}(t_i), a, b) \right]_{1\leq k,l\leq N_{i+1}}.
\end{aligned} \tag{4.78}
$$

Proof of Lemma 4.5. By an analogue (essentially inverse) procedure as in the proof of Lemma 4.4, we first get

$$
\begin{aligned}
(4.78) \;=\; & \sum_{\substack{x_k^n(t_i), \\ 2\leq n\leq N_{i+1}-1, \\ 1\leq k\leq n}} \Big(\prod_{n=2}^{N_{i+1}} \prod_{k=1}^{n-1} \varphi_n(x_k^{n-1}(t_i), x_{k+1}^n(t_i)) \Big) \\
& \times \; \det \left[F_{N_{i+1}+1-l,1}(x_k^{N_{i+1}}(t_{i+1}) - x_1^l(t_i), a, b) \right]_{1\leq k,l\leq N_{i+1}}.
\end{aligned} \tag{4.79}
$$

Now we insert by linearity the factor $\prod_{n=l+1}^{N_{i+1}} \varphi_n(x_l^{n-1}(t_i), x_{l+1}^n(t_i))$ to terms $F_{N_{i+1}+1-l,1}(x_k^{N_{i+1}}(t_{i+1}) - x_1^l(t_i), a, b)$ as well as the sum over the corresponding variables. The sums are carried out by using (4.67), from which we get the r.h.s. of (4.78). □

Lemma 4.6. *Let us define*

$$\phi_n(x,y) = \varphi_n(x,y), \quad \phi_n(x_n^{n-1}, y) = v_n^y. \tag{4.80}$$

Then

$$v_n^{x_1^n} \det \left[\varphi_n(x_k^{n-1}, x_l^n)\right]_{1\le k,l\le n} = \det \left[\phi_n(x_k^{n-1}, x_l^n)\right]_{1\le k,l\le n} \tag{4.81}$$

Proof of Lemma 4.6. It is a consequence of the fact that both determinants are zero if the variables x_i^j do not interlace and when they do, the matrices are upper-triangular with diagonal equal to zero and with equal entries in the first $n-1$ rows. The only difference is for the last row, where the matrix in l.h.s. of (4.81) has entries 1 and r.h.s. of (4.81) has entries $v_n^{x_l^n}$. □

5 Asymptotic analysis

5.1 Flat initial conditions

To prove Theorem 2.2 we need the uniform convergence of the kernel in bounded sets as well as bounds uniform in T. These results are provided in the following Propositions 5.1, 5.2, 5.3.

Let us define the rescaled and conjugate kernel by

$$K_T^{\rm resc}(u_1,s_1;u_2,s_2) = K((n_1,t_1),x_1;(n_2,t_2),x_2) T^{1/3} \frac{e^{t_2(2L+R/2)} 2^{x_2}}{e^{t_1(2L+R/2)} 2^{x_1}} \tag{5.1}$$

where $n_i = n(u_i)$, $t_i = t(u_i)$, and

$$x_i = [-2n_i + \mathbf{v}\, t_i - s_i T^{1/3}]. \tag{5.2}$$

Proposition 5.1 (Uniform convergence in a bounded set)**.** *Fix u_1, u_2, then for any fixed $\ell > 0$, the rescaled kernel $K_T^{\rm resc}$ converges uniformly for $(s_1,s_2) \in [-\ell,\ell]^2$ as*

$$\lim_{T\to\infty} K_T^{\rm resc}(u_1,s_1;u_2,s_2) = S_v^{-1} K_{\mathcal{A}_1}(S_h^{-1}u_1, S_v^{-1}s_1; S_h^{-1}u_2, S_v^{-1}s_2), \tag{5.3}$$

with $K_{\mathcal{A}_1}$ the kernel of the Airy$_1$ process, see (2.24), and S_v, S_h are defined in (2.12).

Proof of Proposition 5.1. First we consider the term coming from the second integral in (3.16), namely

$$\frac{-T^{1/3}}{2\pi {\rm i}} \oint_{\Gamma_1} {\rm d}z \frac{e^{Rt_1(1-z)+Lt_1/(1-z)}}{e^{Rt_2 z + Lt_2/z}} \frac{z^{n_1+n_2+x_2}}{(1-z)^{n_1+n_2+x_1+1}} \frac{e^{t_2(2L+R/2)} 2^{x_2}}{e^{t_1(2L+R/2)} 2^{x_1}}. \tag{5.4}$$

Define the functions

$$\begin{aligned} H(z) &= Rz + L/z - (R/2 - 2L)\ln(z), \\ g_0(z) &= (\pi(\theta)+\theta)H(z), \\ g_1(z,u) &= -u(\pi'(\theta)+1)H(z) + u(1-\pi'(\theta))\ln(z(1-z)), \\ g_2(z,u,s) &= u^2\pi''(\theta)[H(z)+\ln(z(1-z))] + s\ln(z), \end{aligned} \tag{5.5}$$

from which we then set

$$\begin{aligned} f_0(z) &= g_0(1-z) - g_0(z), \\ f_1(z) &= g_1(1-z,u_1) - g_1(z,u_2) - g_1(1/2,u_1) + g_1(1/2,u_2), \\ f_2(z) &= g_2(1-z,u_1,s_1) - g_2(z,u_2,s_2) - g_2(1/2,u_1,s_1) + g_2(1/2,u_2,s_2), \\ f_3(z) &= -\ln(1-z). \end{aligned} \tag{5.6}$$

With these notations we get

$$(5.4) = \frac{-T^{1/3}}{2\pi \mathrm{i}} \oint_{\Gamma_1} \mathrm{d}z e^{Tf_0(z)+T^{2/3}f_1(z)+T^{1/3}f_2(z)+f_3(z)}. \tag{5.7}$$

The function $f_0(z)$ has a double critical point at $z = 1/2$ and the contribution for large T will be dominated by the one close $z = 1/2$. Thus we need to do series expansions around the critical point. Computations leads to

$$\begin{aligned} f_0(z) &= \tfrac{1}{3}\kappa_0(z-1/2)^3 + \mathcal{O}((z-1/2)^4), \\ f_1(z) &= -(u_1-u_2)\kappa_1(z-1/2)^2 + \mathcal{O}((z-1/2)^3), \\ f_2(z) &= -(s_1+s_2)(z-1/2) + \mathcal{O}((z-1/2)^2), \\ f_3(z) &= \ln(2) + \mathcal{O}((z-1/2)) \end{aligned} \tag{5.8}$$

with

$$\kappa_0 = 8(8L+R)(\pi(\theta)+\theta), \quad \kappa_1 = (R+4L)(\pi'(\theta)+1) + 4(1-\pi'(\theta)). \tag{5.9}$$

First we choose Γ_1 to be a steep descent path[2] for $f_0(z)$. We consider $\Gamma_1 = \gamma \vee \gamma_c \vee \bar{\gamma}$, where $\gamma = \{1/2 + e^{-I\pi/3}\xi, 0 \leq \xi \leq 1/2\}$, $\bar{\gamma}$ its image with respect to complex conjugation, and $\gamma_c = \{1 - 1/2e^{I\phi}, \pi/6 \leq \phi \leq 2\pi - \pi/6\}$. We also have $f_0(z) = S_R(z)R(\pi(\theta)+\theta) + S_L(z)L(\pi(\theta)+\theta)$, with

$$S_R(z) = 1 - 2z + \frac{1}{2}\ln(z/(1-z)), \quad S_L(z) = \frac{1}{1-z} - \frac{1}{z} - 2\ln(z/(1-z)). \tag{5.10}$$

On γ, simple computations leads to

$$\begin{aligned} \frac{\mathrm{dRe}(S_R(z))}{\mathrm{d}\xi} &= -\frac{8\xi^2(1+2\xi^2)}{((1+\xi^2)+2\xi^2)((1-\xi)^2+2\xi^2)}, \\ \frac{\mathrm{dRe}(S_L(z))}{\mathrm{d}\xi} &= -\frac{64\xi^2((1+2\xi^2)^2-12\xi^4)}{((1+\xi^2)+2\xi^2)^2((1-\xi)^2+2\xi^2)^2} \end{aligned} \tag{5.11}$$

[2]For an integral $I = \int_\gamma \mathrm{d}z e^{tf(z)}$, we say that γ is a steep descent path if (1) $\mathrm{Re}(f(z))$ is maximum at some $z_0 \in \gamma$: $\mathrm{Re}(f(z)) < \mathrm{Re}(f(z_0))$ for $z \in \gamma \setminus \{z_0\}$, and (2) $\mathrm{Re}(f(z))$ is monotone along γ except at its maximum point z_0 and, if γ is closed, at a point z_1 where the minimum of $\mathrm{Re}(f)$ is reached.

which are both strictly less than 0 for $\xi \in (0,1/2)$. Moreover, on γ_c,

$$\begin{aligned} \frac{\mathrm{dRe}(S_R(z))}{\mathrm{d}\phi} &= -\frac{4\sin(\phi)(1-\cos(\phi))}{5-4\cos(\phi)}, \\ \frac{\mathrm{dRe}(S_L(z))}{\mathrm{d}\phi} &= -\frac{32\sin(\phi)(1-\cos(\phi))(2-\cos(\phi))}{(5-4\cos(\phi))^2} \end{aligned} \tag{5.12}$$

which are both strictly less than 0 for $\cos(\phi) \in (-1,1)$. Therefore the chosen Γ_1 is a steep descent path for $f_0(z)$.

Take any $\delta > 0$ and set $\Gamma_1^\delta = \{z \in \Gamma_0 | |z-1/2| \leq \delta\}$. Then, if in (5.7) we integrate only along Γ_1^δ instead of integrating along Γ_1, the error made is just of order $\mathcal{O}(e^{-cT})$ for some $c > 0$ (more exactly, $c \sim \delta^3$ for δ small). Thus we now consider the integral on Γ_1^δ only. There, we can use the above series expansions to obtain

$$\begin{aligned} &\frac{-2T^{1/3}}{2\pi \mathrm{i}} \int_{\Gamma_1^\delta} \mathrm{d}z e^{\frac{1}{3}\kappa_0 T(z-1/2)^3 + (u_2-u_1)\kappa_1 T^{2/3}(z-1/2)^2 - 2(s_1+s_2)(z-1/2)} \\ &\times e^{\mathcal{O}\left(T(z-1/2)^4, T^{2/3}(z-1/2)^3, T^{1/3}(z-1/2), (z-1/2)\right)}. \end{aligned} \tag{5.13}$$

The difference between (5.13) and the same integral without the error term can be bounded by applying $|e^x - 1| \leq |x| e^{|x|}$ to $\mathcal{O}(\cdots)$. Thus, this error term can be bounded by

$$\begin{aligned} &\frac{2T^{1/3}}{2\pi} \int_{\Gamma_1^\delta} \mathrm{d}z \Big| e^{\frac{1}{3}c_0\kappa_0 T(z-1/2)^3 + (u_2-u_1)c_1\kappa_1 T^{2/3}(z-1/2)^2 - 2c_2(s_1+s_2)(z-1/2)} \\ &\times \mathcal{O}\big(T(z-1/2)^4, T^{2/3}(z-1/2)^3, T^{1/3}(z-1/2), (z-1/2)\big)\Big| \end{aligned} \tag{5.14}$$

for some c_0, c_1, c_2 which can be taken as close to 1 as needed by setting δ small enough. Then, by the change of variable $T^{1/3}(z-1/2) = w$ one gets that this error term is of order $\mathcal{O}(T^{-1/3})$ (what is needed is just $c_0 > 0$).

It remains to consider the leading term, namely (5.13) without the error terms. By extending the integral to infinity by continuing the two small straight segments forming Γ_1^δ, the error made is of order $\mathcal{O}(e^{-cT})$. Thus we obtained that (5.4) is, up to an error $\mathcal{O}(e^{-cT}, T^{-1/3})$ uniform for $s_1, s_2 \in [-\ell, \ell]^2$, equal to

$$\frac{-2T^{1/3}}{2\pi \mathrm{i}} \int_{\gamma_\infty} \mathrm{d}z e^{\frac{1}{3}\kappa_0 T(z-1/2)^3 + (u_2-u_1)\kappa_1 T^{2/3}(z-1/2)^2 - 2(s_1+s_2)(z-1/2)}, \tag{5.15}$$

where γ_∞ is a path going from $e^{\mathrm{i}\pi/3}\infty$ to $e^{-\mathrm{i}\pi/3}\infty$. By the change of variable $w = (\kappa_0 T)^{1/3}(z-1/2)$, we get

$$\begin{aligned} (5.15) &= \frac{-1}{2\pi \mathrm{i}} \int_{\gamma_\infty} \mathrm{d}w \frac{2}{\kappa_0^{1/3}} e^{\frac{1}{3}w^2 + (u_2-u_1)w^2\kappa_1/\kappa_0^{2/3} - 2(s_1+s_2)w/\kappa^{1/3}} \\ &= S_v^{-1} \mathrm{Ai}\big(S_h^{-1}(u_2-u_1)^2 + S_v^{-1}(s_1+s_2)\big) \\ &\quad \times e^{\frac{2}{3}S_h^{-1}(u_2-u_1)^3 + S_v^{-1}S_h^{-1/2}(u_2-u_1)(s_1+s_2)} \end{aligned} \tag{5.16}$$

with S_v and S_h defined in (2.12). Here we used the Airy function representation

$$\frac{-1}{2\pi \mathrm{i}}\int_{\gamma_\infty} \mathrm{d}v e^{v^3/3+av^2+bv} = \mathrm{Ai}(a^2-b)\exp(2a^3/3-ab). \tag{5.17}$$

To finish the proof, we need to consider the term coming from the first integral in (3.16), namely

$$-\frac{T^{1/3}}{2\pi \mathrm{i}}\oint_{\Gamma_0} \mathrm{d}w \frac{1}{w^{x_1-x_2+1}}\left(\frac{w}{1-w}\right)^{n_2-n_1} e^{(Rw+L/w)(t_1-t_2)}\frac{e^{t_2(2L+R/2)}2^{x_2}}{e^{t_1(2L+R/2)}2^{x_1}}. \tag{5.18}$$

This can be rewritten as

$$(5.18) = \frac{-T^{1/3}}{2\pi \mathrm{i}}\oint_{\Gamma_0}\frac{\mathrm{d}w}{w} e^{T^{2/3}(p_0(w)-p_0(1/2))+T^{1/3}(p_1(w)-p_1(1/2))} \tag{5.19}$$

with

$$\begin{aligned} p_0(w) &= (u_2-u_1)(\pi'(\theta)+1)H(w)-(u_2-u_1)(1-\pi'(\theta))\ln(w(1-w)),\\ p_1(w) &= -(u_2^2-u_1^2)\frac{\pi''(\theta)}{2}[H(w)+\ln(w(1-w))]-(s_2-s_1)\ln(w), \end{aligned} \tag{5.20}$$

where $H(w)$ is the function defined in (5.5). Remark that we need to do the analysis only for $u_2 > u_1$. The function p_0 has critical point at $w = 1/2$. The series expansions of p_0 and p_1 around $w = 1/2$ are

$$\begin{aligned} p_0(w) &= \kappa_1(u_2-u_1)(w-1/2)^2+\mathcal{O}((w-1/2)^3),\\ p_1(w) &= 2(s_1-s_2)(w-1/2)+\mathcal{O}((w-1/2)^2). \end{aligned} \tag{5.21}$$

We choose as path $\Gamma_0 = \{\frac{1}{2}e^{\mathrm{i}\phi}, \phi\in(-\pi,\pi]\}$. This is a steep descent path for p_0. In fact, for $w\in\Gamma_0$,

$$\begin{aligned} \mathrm{Re}(H(w)) &= (R/2+2L)\cos(\phi)+(R/2-2L)\ln(2), \qquad (5.22)\\ \mathrm{Re}(-\ln(w(1-w))) &= \ln(2)-\ln|1-w| = 2\ln(2)-\tfrac{1}{2}\ln(5-4\cos(\phi)), \end{aligned}$$

which are decreasing when $\cos(\phi)$ decreases. Thus, we can integrate only on $\Gamma_0^\delta = \{w\in\Gamma_0 \,|\, |w-1/2|\leq\delta\}$ and, for a small δ, the error term is just of order $\mathcal{O}(e^{-cT^{2/3}})$ with $c>0$ ($c\sim\delta^2$ as $\delta\ll 1$). The integral over Γ_0^δ is then given by

$$\begin{aligned} &\frac{-2T^{1/3}}{2\pi \mathrm{i}}\int_{\Gamma_0^\delta}\mathrm{d}w e^{\kappa_1(u_2-u_1)(w-1/2)^2T^{2/3}+2(s_1-s_2)(w-1/2)T^{1/3}}\\ &\times e^{\mathcal{O}((w-1/2)^3T^{2/3},(w-1/2)^2T^{1/3},(w-1/2))}. \end{aligned} \tag{5.23}$$

As above, we use $|e^x-1|\leq|x|e^{|x|}$, to control the difference between (5.23) and the same expression without the error terms. By taking $\delta\ll 1$ and the change of variable $(w-1/2)T^{1/3}=W$, we get that this difference is of order $\mathcal{O}(T^{-1/3})$ uniformly for s_1,s_2 in a bounded set. Once we have taken away the error terms in

(5.23), we extend the integral to $1/2 \pm \mathrm{i}\infty$. By this we make only an error of order $\mathcal{O}(e^{-cT^{2/3}})$. The integration path can be deformed to $1/2+\mathrm{i}\mathbb{R}$ without passing by any poles, therefore by setting $w = 1/2 + \mathrm{i}yT^{-1/3}$ we get

$$
\begin{aligned}
&-\frac{1}{\pi}\int_{\mathbb{R}} \mathrm{d}y e^{-\kappa_1(u_2-u_1)y^2+2(s_1-s_2)} = -\frac{1}{\sqrt{\pi\kappa_1(u_2-u_1)}}\exp\left(-\frac{(s_2-s_1)^2}{\kappa_1(u_2-u_1)}\right) \\
&= -\frac{S_v^{-1}}{\sqrt{4\pi(u_2-u_1)S_h^{-1}}}\exp\left(-\frac{(s_2-s_1)^2 S_v^{-1}}{4(u_2-u_1)S_h^{-1}}\right). \qquad (5.24)
\end{aligned}
$$

Since all the error terms in the series expansions are uniform for $(s_1,s_2)\in[-\ell,\ell]^2$, the result of the Proposition is proven. □

Proposition 5.2 (Bound for the diffusion term of the kernel)**.**
For any $s_1,s_2\in\mathbb{R}$ and $u_2-u_1>0$ fixed, the bound

$$
\begin{aligned}
&\left|\frac{e^{t_2(2L+R/2)}2^{x_2}}{e^{t_1(2L+R/2)}2^{x_1}}\frac{T^{1/3}}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}w\frac{1}{w^{x_1-x_2+1}}\left(\frac{w}{1-w}\right)^{n_2-n_1} e^{(Rw+L/w)(t_1-t_2)}\right| \\
&\leq \mathrm{const}\, e^{-|s_1-s_2|} \qquad (5.25)
\end{aligned}
$$

holds for T large enough and const *independent of T.*

Proof of Proposition 5.2. From the analysis in Proposition 5.1, we just need a bound for $|s_2-s_1|\geq\ell$, $\ell>0$ fixed. We start with (5.19) but to obtain a decaying bound for large $|s_2-s_1|$ we consider another path Γ_0.

Consider an ε with $0<\varepsilon\ll 1$ and set $\Gamma_0=\{w=\rho e^{\mathrm{i}\phi},\phi\in[-\pi,\pi)\}$, with

$$
\rho = \begin{cases} \frac{1}{2}+\frac{(s_2-s_1)T^{-1/3}}{(u_2-u_1)\kappa_1}, & \text{if } |s_2-s_1|\leq\varepsilon T^{1/3}, \\ \frac{1}{2}+\frac{\varepsilon}{(u_2-u_1)\kappa_1}, & \text{if } s_2-s_1\geq\varepsilon T^{1/3}, \\ \frac{1}{2}-\frac{\varepsilon}{(u_2-u_1)\kappa_1}, & \text{if } s_2-s_1\leq-\varepsilon T^{1/3}. \end{cases} \qquad (5.26)
$$

We have $\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{Re}(w-\frac{1}{2}\ln(w)) = -\rho\sin(\phi)$, $\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{Re}(1/w+2\ln(w)) = -\frac{4}{\rho}\sin(\phi)$, and $\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{Re}(-\ln(w(1-w))) = -\frac{\rho\sin(\phi)}{1-2\rho\cos(\phi)+\rho^2}$. Thus Γ_0 is a steep descent path for $p_0(z)$ but also for the term in p_1 proportional to s_2-s_1. Let, for a small $\delta>0$ fixed, $\Gamma_0^\delta=\{w=\rho e^{\mathrm{i}\phi},\phi\in(-\delta,\delta)\}$. Then

$$
\begin{aligned}
(5.19) &= e^{T^{2/3}(p_0(\rho)-p_0(1/2))+T^{1/3}(p_1(\rho)-p_1(1/2))} \qquad (5.27)\\
&\times \left(\mathcal{O}(e^{-cT^{2/3}})+\frac{-T^{1/3}}{2\pi\mathrm{i}}\int_{\Gamma_0^\delta}\frac{\mathrm{d}w}{w}e^{T^{2/3}(p_0(w)-p_0(\rho))+T^{1/3}(p_1(w)-p_1(\rho))}\right)
\end{aligned}
$$

for some $c>0$ (for small δ, $c\sim\delta^2$). On Γ_0^δ the s_i-dependent term in $\mathrm{Re}(p_1(w)-p_1(\rho))$ is equal to zero and the rest is of order $\mathcal{O}(\phi^2)$. Therefore the last integral can be bounded by

$$
\frac{T^{1/3}}{2\pi}\int_{-\delta}^{\delta}\frac{\mathrm{d}\phi}{\rho}e^{-\frac{1}{2}T^{2/3}(u_2-u_1)\left[(\pi'(\theta)+1)(R\rho+L/\rho)+(1-\pi'(\theta))\rho/(1-\rho)^2\right]\phi^2+\mathcal{O}(T^{2/3}\phi^4,T^{1/3}\phi^2)}. \qquad (5.28)
$$

For δ small enough, and T large enough, the terms $\mathcal{O}(T^{2/3}\phi^4)$ and $\mathcal{O}(T^{1/3}\phi^2)$ are both controlled by the first term in the exponential. Then, by the change of variable $T^{1/3}\phi = \psi$ one sees that r.h.s. of (5.28) is bounded by a constant, uniformly in T.

What remains is therefore to bound the first term in the r.h.s. of (5.27). By the choice in (5.26) of ρ, $|\rho - 1/2| \leq \varepsilon/((u_2-u_1)\kappa_1) \ll 1$ for ε small enough which can be still chosen. Series expansion for ρ close to $1/2$ leads to

$$\begin{aligned} p_0(\rho) - p_0(1/2) &= -2(s_2-s_1)(\rho-1/2)T^{1/3}(1+\mathcal{O}(\rho-1/2)) \\ &+ \kappa_1(u_2-u_1)(\rho-1/2)^2 T^{2/3}(1+\mathcal{O}(\rho-1/2)). \end{aligned} \tag{5.29}$$

By (5.26) we obtain the bounds

$$\begin{aligned} p_0(\rho) - p_0(1/2) &= -\frac{(s_2-s_1)^2}{(u_2-u_1)\kappa_1}(1+\mathcal{O}(\varepsilon)), \text{ if } |s_2-s_1| \leq \varepsilon T^{1/3}, \\ p_0(\rho) - p_0(1/2) &= -\frac{(s_2-s_1)\varepsilon T^{1/3}}{(u_2-u_1)\kappa_1}(1+\mathcal{O}(\varepsilon)), \text{ if } |s_2-s_1| \geq \varepsilon T^{1/3}. \end{aligned} \tag{5.30}$$

Combining the above result we have

$$|(5.25)| \leq \Big[\mathcal{O}(e^{-\mu T^{2/3}}) + \mathcal{O}(1)\Big]\Big[e^{-\frac{(s_2-s_1)^2}{(u_2-u_1)\kappa_1}(1+\mathcal{O}(\varepsilon))} + e^{-\frac{(s_2-s_1)\varepsilon T^{1/3}}{(u_2-u_1)\kappa_1}(1+\mathcal{O}(\varepsilon))}\Big]. \tag{5.31}$$

Thus by taking an ε small enough and then T large enough the bound (5.31) implies the statement to be proven, since for any $\alpha > 0$, there exists a $C_\alpha < \infty$ such that $e^{-\alpha(s_2-s_1)^2} \leq C_\alpha e^{-|s_2-s_1|}$. ☐

Proposition 5.3 (Bound on the main term of the kernel).
For any $(s_1,s_2) \in [-\ell,\infty)^2$, the bound

$$\begin{aligned} &\left|\frac{-T^{1/3}}{2\pi \mathrm{i}}\oint_{\Gamma_1} \mathrm{d}z \frac{e^{Rt_1(1-z)+Lt_1/(1-z)}}{e^{Rt_2 z+Lt_2/z}} \frac{z^{n_1+n_2+x_2}}{(1-z)^{n_1+n_2+x_1+1}} \frac{e^{t_2(2L+R/2)}2^{x_2}}{e^{t_1(2L+R/2)}2^{x_1}}\right| \\ &\leq \mathrm{const}\, e^{-(s_1+s_2)} \end{aligned} \tag{5.32}$$

holds for T large enough, where const *is a constant independent of T.*

Proof of Proposition 5.3. For $(s_1,s_2) \in [-\ell,\ell]^2$, this is a consequence of the estimates in the proof of Proposition 5.1. Therefore we can consider just $(s_1,s_2) \in [-\ell,\infty)^2 \setminus [-\ell,\ell]^2$. Let us introduce the notations $\tilde{s}_i = (s_i + 2\ell)T^{-2/3}$, which then belongs to $[\ell T^{-2/3},\infty)$. Then, the integral to be bounded is

$$\frac{-T^{1/3}}{2\pi \mathrm{i}}\oint_{\Gamma_1} \mathrm{d}z e^{Tf_0(z)+T^{2/3}f_1(z)+T^{1/3}f_2(z)+f_3(z)} \tag{5.33}$$

where $f_1(z)$ and $f_3(z)$ are given in (5.6), and $f_0(z)$ and $f_2(z)$ are just slight modifications of the functions in (5.6), namely

$$\begin{aligned} f_0(z) &= (\pi(\theta)+\theta)(H(1-z)-H(z)) + \tilde{s}_1\ln(2(1-z)) - \tilde{s}_2\ln(2z), \\ f_2(z) &= g_2(1-z,u_1,-2\ell) - g_2(z,u_2,-2\ell) - g_2(1/2,u_1,-2\ell) + g_2(1/2,u_2,-2\ell). \end{aligned}$$

We put $\tilde{s}_1$ and $\tilde{s}_2$ in $f_0(z)$, because they are not restricted to be of order $T^{-2/3}$ (as it was the case in Proposition 5.1).

First we need to find a steep descent path for $f_0(z)$. We choose it as $\Gamma_1 = \{1 - \rho e^{\mathrm{i}\phi}, \phi \in [-\pi, \pi)\}$ with $0 < \rho \leq 1/2$, choosen as follows,

$$\rho = \begin{cases} \frac{1}{2} - ((\tilde{s}_1 + \tilde{s}_2)/\kappa_0)^{1/2}, & |\tilde{s}_1 + \tilde{s}_2| \leq \varepsilon, \\ \frac{1}{2} - (\varepsilon/\kappa_0)^{1/2}, & |\tilde{s}_1 + \tilde{s}_2| \geq \varepsilon, \end{cases} \tag{5.34}$$

for some small $\varepsilon > 0$ to be fixed later.

To see that Γ_1 is a steep descent path, we consider $f_0(z)$ term by term. The term proportional to $R(\pi(\theta) + \theta)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\phi} \mathrm{Re}(1 - 2z + \tfrac{1}{2}\ln(z/(1-z))) = -\frac{\rho(3 - 8\rho\cos(\phi) + 4\rho^2)\sin(\phi)}{1 - 2\rho\cos(\phi) + \rho^2} \leq 0 \tag{5.35}$$

for all $0 < \rho \leq 1/2$, with equality only at $\phi = 0, \pm\pi$. The term proportional to $L(\pi(\theta) + \theta)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\phi} \mathrm{Re}(1/(1-z) - 1/z - 2\ln(z/(1-z))) = -\frac{((1 - 2\rho\cos(\phi) + 2\rho^2)^2 - \rho^2)\sin(\phi)}{(1 - 2\rho\cos(\phi) + \rho^2)^2\rho} \leq 0 \tag{5.36}$$

for all $0 < \rho \leq 1/2$, with equality only at $\phi = 0, \pm\pi$. Finally, $\mathrm{Re}(\ln(1-z))$ is constant on Γ_1 and $-\mathrm{Re}(\ln(2z)) = -\ln(2|z|)$ is strictly decreasing while moving on Γ_1 with $|\phi|$ increasing.

For a small $\delta > 0$, $\Gamma_1^\delta = \{1 - \rho e^{\mathrm{i}\phi}, \phi \in (-\delta, \delta)\}$. We also define

$$Q(\rho) = e^{\mathrm{Re}\left(T(f_0(1-\rho) - f_0(1/2)) + T^{2/3}(f_1(1-\rho) - f_1(1/2)) + T^{1/3}(f_2(1-\rho) - f_2(1/2))\right)} \tag{5.37}$$

Since Γ_1 is a steep descent path of $f_0(z)$, the integral over $\Gamma_1 \setminus \Gamma_1^\delta$ is bounded by

$$Q(\rho)\mathcal{O}(e^{-cT}) \tag{5.38}$$

for some $c > 0$ independent of T. The contribution of the integral over Γ_1^δ is bounded by

$$Q(\rho)\left|\frac{-T^{1/3}}{2\pi\mathrm{i}} \int_{\Gamma_1^\delta} \mathrm{d}z e^{T(f_0(z) - f_0(1-\rho)) + T^{2/3}(f_1(z) - f_1(1-\rho)) + T^{1/3}(f_2(z) - f_2(1-\rho)) + f_3(z)}\right| \tag{5.39}$$

The series expansion around $\phi = 0$ is

$$\mathrm{Re}(f_0(1 - \rho e^{\mathrm{i}\phi}) - f_0(1-\rho)) = -\gamma_1\phi^2(1 + \mathcal{O}(\phi)) \tag{5.40}$$

with

$$\gamma_1 = \frac{\tilde{s}_2\rho}{2(1-\rho)^2} + \frac{(\pi(\theta) + \theta)(1 - 2\rho)}{(1-\rho)^2}\left(\frac{R\rho(3 - 2\rho)}{4} + \frac{L(1 - \rho + 2\rho^2)}{3\rho(1-\rho)}\right), \tag{5.41}$$

and

$$\mathrm{Re}(f_1(1-\rho e^{\mathrm{i}\phi})-f_1(1-\rho))=\gamma_2\phi^2(1+\mathcal{O}(\phi)),\tag{5.42}$$

with

$$\gamma_2=(u_2-u_1)\kappa_1+\mathcal{O}(\rho-1/2).\tag{5.43}$$

Finally, $\mathrm{Re}(f_2(1-\rho e^{\mathrm{i}\phi})-f_2(1-\rho))=\mathcal{O}(\phi^2)$. Thus, by the change of variable $z=1-\rho e^{\mathrm{i}\phi}$, the above estimates, and by setting $\gamma=\gamma_1+\gamma_2 T^{-1/3}$, we get

$$(5.39)=Q(\rho)\frac{T^{1/3}\rho}{2\pi(1-\rho)}\int_{-\delta}^{\delta}\mathrm{d}\phi e^{-\gamma\phi^2 T(1+\mathcal{O}(\phi))(1+\mathcal{O}(T^{-1/3}))}.\tag{5.44}$$

By choosing δ small enough (independent of T) and then T large enough, the factors with the error terms can be replaced by $1/2$, thus

$$(5.39)\leq Q(\rho)\frac{T^{1/3}\rho}{2\pi(1-\rho)}\int_{-\delta}^{\delta}\mathrm{d}\phi e^{-\gamma\phi^2 T/2}\leq Q(\rho)\frac{1}{\sqrt{2\pi\gamma T^{1/3}}}.\tag{5.45}$$

Remark that, the worse case is when γ becomes small, and this happens when $\rho\to 1/2$, i.e., it is the case of small values of $\tilde{s}_1+\tilde{s}_2$. But even in this case, $\gamma_1 T^{1/3}\sim(s_1+s_2+4\ell)^{1/2}\geq(2\ell)^{1/2}$, which dominates $\gamma_2\sim\mathcal{O}(1)$ for ℓ large. Thus by setting ℓ large enough, $(5.39)\leq Q(\rho)\mathcal{O}(1)$. This estimate, combined with (5.38), implies that the Proposition will be proven by showing that $Q(\rho)\leq\mathrm{const}\, e^{-(s_1+s_2)}$. Since $1-\rho$ is close to $1/2$, we can apply the series expansion of f_i around $z=1/2$. The expansion of f_1 is in (5.8), while the one of f_2 is the same as in (5.8) with $s_1+s_2=-4\ell$. Finally,

$$f_0(z)=\tfrac{1}{3}\kappa_0(z-1/2)^3(1+\mathcal{O}(z-1/2)^2)-(\tilde{s}_1+\tilde{s}_2)(z-1/2)(1+\mathcal{O}(z-1/2)).\tag{5.46}$$

First consider $\tilde{s}_1+\tilde{s}_2\leq\varepsilon$. Then, with ρ chosen as in (5.34), we get

$$\begin{aligned}Q(\rho)&=e^{-\frac{2}{3}T(\tilde{s}_1+\tilde{s}_2)^{3/2}\kappa_0^{-1/2}T(1+\mathcal{O}(\sqrt{\varepsilon}))}e^{(u_2-u_1)\kappa_1(\tilde{s}_1+\tilde{s}_2)T^{2/3}\kappa_0^{-1}(1+\mathcal{O}(\sqrt{\varepsilon}))}\\&\quad\times e^{-4\ell(\tilde{s}_1+\tilde{s}_2)\kappa_0^{-1/2}T^{1/3}(1+\mathcal{O}(\sqrt{\varepsilon}))}\\&=e^{-\frac{2}{3}(s_1+s_2+4\ell)^{3/2}\kappa_0^{-1/2}(1+\mathcal{O}(\sqrt{\varepsilon}))}e^{(u_2-u_1)\kappa_1(s_1+s_2+4\ell)\kappa_0^{-1}(1+\mathcal{O}(\sqrt{\varepsilon}))}\\&\quad\times e^{-4\ell(s_1+s_2+4\ell)\kappa_0^{-1/2}T^{-1/3}(1+\mathcal{O}(\sqrt{\varepsilon}))}.\end{aligned}\tag{5.47}$$

Recall that $s_1+s_2+4\ell\geq 2\ell\gg 1$ for $\ell\gg 1$. Therefore by choosing ℓ large enough (depending only on the coefficients $\kappa_0,\kappa_1,u_1,u_2$ which are however fixed), all the terms are controlled by the first one, i.e.,

$$Q(\rho)\leq e^{-\frac{1}{3}(s_1+s_2+4\ell)^{3/2}\kappa_0^{-1/2}}\leq e^{-\frac{1}{3}(s_1+s_2)^{3/2}\kappa_0^{-1/2}}.\tag{5.48}$$

Since this decays more rapidly that $\exp(-(s_1+s_2))$, the Proposition holds for $\tilde{s}_1+\tilde{s}_2\leq\varepsilon$.

The last case is $\tilde{s}_1+\tilde{s}_2\geq\varepsilon$. In this case, with ρ chosen as in (5.34), we obtain

$$\begin{aligned}Q(\rho)&=e^{T\kappa_0^{-1/2}(1+\mathcal{O}(\sqrt{\varepsilon}))\sqrt{\varepsilon}(\varepsilon/3-(\tilde{s}_1+\tilde{s}_2))}e^{(u_2-u_1)\kappa_1\kappa_0^{-1}\varepsilon T^{2/3}(1+\mathcal{O}(\sqrt{\varepsilon}))}\\&\quad\times e^{-8\ell\kappa_0^{-1/2}\varepsilon T^{1/3}(1+\mathcal{O}(\sqrt{\varepsilon}))}.\end{aligned}\tag{5.49}$$

But now, $\varepsilon/3-(\tilde{s}_1+\tilde{s}_2)\leq -\frac{2}{3}(\tilde{s}_1+\tilde{s}_2)$, thus the first term in the exponential is, up to a positive constant, $-\sqrt{\varepsilon}T^{1/3}(s_1+s_2+4\ell)$, which dominates the second term $\sim \varepsilon T^{2/3}\leq s_1+s_2+4\ell$, and it also dominates the third term. Therefore, for any choice of ε and ℓ made before, we can take T large enough such that

$$Q(\rho)\leq e^{-\frac{1}{3}\sqrt{\varepsilon}T^{1/3}(s_1+s_2)}, \tag{5.50}$$

which ends the proof of the Proposition. □

Proof of Theorem 2.2. The proof of Theorem 2.2 is the complete analogue of Theorem 2.5 in [3]. The results in Propositions 5.1,5.3,5.4, and 5.5 in [3] are replaced by the ones in Proposition 5.1, 5.2, 5.3. The strategy is to write the Fredholm series of the expression for finite T and, by using the bounds in Propositions 5.2 and 5.3, see that it is bounded by a T-independent and integrable function. Once this is proven, one can exchange the sums/integrals and the $T\to\infty$ limit by the theorem of dominated convergence. For details, see Theorem 2.5 in [3]. □

5.2 Sketch of the result (2.22)

With the rescaling (2.8) and (2.18), the rescaled kernel writes

$$K^{\mathrm{resc}}(u_1,s_1;u_2,s_2)=K((n_1,t_1),x_1;(n_2,t_2),x_2)T^{1/3}. \tag{5.51}$$

The main part of the kernel (the second term in (3.11)) writes

$$\frac{T^{1/3}}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}w\oint_{\Gamma_1}\mathrm{d}z\frac{e^{Tf_0(w)+T^{2/3}f_1(w;u_1)+T^{1/3}f_2(w;u_1,s_1)}}{e^{Tf_0(z)+T^{2/3}f_1(z;u_2)+T^{1/3}f_2(z;u_2,s_2)}}\frac{w-1}{(z-1)w}\frac{1}{w-z} \tag{5.52}$$

with

$$\begin{aligned}
&f_0(w)=(\pi(\theta)+\theta)\left(Rw+\tfrac{L}{w}\right)+(\pi(\theta)-\theta)\ln\left(\tfrac{1-w}{w}\right)-\sigma_0\ln(w),\\
&f_1(w;u_i)=-\left[(\pi'(\theta)+1)\left(Rw+\tfrac{L}{w}\right)+(\pi'(\theta)-1)\ln\left(\tfrac{1-w}{w}\right)-\sigma_1\ln(w)\right]u_i,\\
&f_2(w;u_i,s_i)=\left[\tfrac{1}{2}\pi''(\theta)\left(Rw+\tfrac{L}{w}+\ln\left(\tfrac{1-w}{w}\right)\right)-\sigma_2\right]u_i^2+s_i\ln(w).
\end{aligned} \tag{5.53}$$

The parameter μ is actually the position of the double critical point of $f_0(w)$. Series expansions gives

$$\begin{aligned}
f_0(w)&=f_0(\mu)-\frac{\kappa_0}{3}(w-\mu)^3+\mathcal{O}((w-\mu)^4),\\
f_1(w;u_1)&=f_1(\mu;u_1)-u_1\kappa_1(w-\mu)^2+\mathcal{O}((w-\mu)^3),\\
f_2(w;u_1,s_1)&=f_2(\mu;u_1,s_1)-\left(\frac{\kappa_1^2u_1^2}{\kappa_0}-\frac{s_1}{\mu}\right)(w-\mu)+\mathcal{O}((w-\mu)^2).
\end{aligned} \tag{5.54}$$

The terms $f_1(\mu;u_i)$ and $f_2(\mu;u_i,s_i)$ cancel out by an appropriate conjugation of the kernel (5.52). We denote by $\simeq$ an equality up to conjugation. Thus, asymptotically, (5.52) goes to

$$\frac{T^{1/3}}{\mu(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}w\oint_{\Gamma_1}\frac{\mathrm{d}z}{w-z}\frac{e^{-\kappa_0(w-\mu)^3T/3-u_1\kappa_1(w-\mu)^2T^{2/3}+T^{1/3}(w-\mu)(s_1/\mu-\kappa_1^2u_1^2/\kappa_0)}}{e^{-\kappa_0(z-\mu)^3T/3-u_2\kappa_1(z-\mu)^2T^{2/3}+T^{1/3}(z-\mu)(s_2/\mu-\kappa_1^2u_2^2/\kappa_0)}} \tag{5.55}$$

With the change of variable $(w-\mu)(\kappa_0 T)^{1/3}=W$, $(z-\mu)(\kappa_0 T)^{1/3}=Z$, we then obtain

$$(5.55)=\frac{\kappa_0^{-1/3}}{\mu(2\pi\mathrm{i})^2}\int \mathrm{d}W\int \mathrm{d}Z\frac{1}{W-Z}\frac{e^{\frac{1}{3}Z^3+u_2Z^2\kappa_1/\kappa_0^{2/3}-Z(s_2/\mu-\kappa_1^2u_2^2/\kappa_0)/\kappa_0^{1/3}}}{e^{\frac{1}{3}W^3+u_1W^2\kappa_1/\kappa_0^{2/3}-W(s_1/\mu-\kappa_1^2u_1^2/\kappa_0)/\kappa_0^{1/3}}}. \quad (5.56)$$

Let us denote by $\tilde{S}_v=\mu\kappa_0^{1/3}$ and $\tilde{S}_h=\kappa_1^{-1}\kappa_0^{2/3}$ the vertical and horizontal scaling. Then

$$(5.56)=\tilde{S}_v^{-1}K_{\mathcal{A}_2}(\tilde{S}_h^{-1}u_1,\tilde{S}_v^{-1}s_1;\tilde{S}_h^{-1}u_2,\tilde{S}_v^{-1}s_2) \quad (5.57)$$

where $K_{\mathcal{A}_2}$ is the extended Airy kernel associated to the Airy_2 process. An asymptotic analysis of large deviations similar to Propositons 5.2 and 5.3 above would then lead to the result of (2.22).

References

[1] M. Abramowitz and I.A. Stegun, *Pocketbook of mathematical functions*, Verlag Harri Deutsch, Thun-Frankfurt am Main, 1984.

[2] M. Alimohammadi, V. Karimipour, and M. Khorrami, *A two-parametric family of asymmetric exclusion processes and its exact solution*, J. Stat. Phys. **97** (1999), 373–394.

[3] A. Borodin, P.L. Ferrari, and M. Prähofer, *Fluctuations in the discrete TASEP with periodic initial configurations and the Airy_1 process*, Int. Math. Res. Papers **2007** (2007), rpm002.

[4] A. Borodin, P.L. Ferrari, M. Prähofer, and T. Sasamoto, *Fluctuation properties of the TASEP with periodic initial configuration*, preprint: arXiv:math-ph/0608056; To appear in J. Stat. Phys. (2006).

[5] A. Borodin, P.L. Ferrari, and T. Sasamoto, *Large time asymptotics of growth models on space-like paths II: PNG and parallel TASEP*, in preparation (2007).

[6] A. Borodin, P.L. Ferrari, and T. Sasamoto, *Transition between Airy_1 and Airy_2 processes and TASEP fluctuations*, preprint: arXiv:math-ph/0703023; To appear in CPAM (2007).

[7] A. Borodin and G. Olshanski, *Stochastic dynamics related to Plancherel measure*, AMS Transl.: Representation Theory, Dynamical Systems, and Asymptotic Combinatorics (V. Kaimanovich and A. Lodkin, eds.), 2006, pp. 9–22.

[8] A. Borodin and E.M. Rains, *Eynard-Mehta theorem, Schur process, and their Pfaffian analogs*, J. Stat. Phys. **121** (2006), 291–317.

[9] B. Derrida, J.L. Lebowitz, E.R. Speer, and H. Spohn, *Dynamics of an anchored toom interface*, J. Phys. A: Math. Gen. **24** (1991), 4805–4834.

[10] A. B. Dieker and J. Warren, *Determinantal transition kernels for some interacting particles on the line*, arXiv:0707.1843 (2007).

[11] P.L. Ferrari, *The universal Airy$_1$ and Airy$_2$ processes in the Totally Asymmetric Simple Exclusion Process*, preprint: arXiv:math-ph/0701021 (2007).

[12] J. Ben Hough, M. Krishnapur, Y. Peres, and B. Virag, *Determinantal processes and independence*, Probability Surveys **3** (2006), 206–229.

[13] T. Imamura and T. Sasamoto, *Dynamical properties of a tagged particle in the totally asymmetric simple exclusion process with the step-type initial condition*, preprint: arXiv:math-ph/0702009; To appear in J. Stat. Phys. (2007).

[14] K. Johansson, *Discrete polynuclear growth and determinantal processes*, Comm. Math. Phys. **242** (2003), 277–329.

[15] K. Johansson, *The arctic circle boundary and the Airy process*, Ann. Probab. **33** (2005), 1–30.

[16] K. Johansson, *Random matrices and determinantal processes*, Mathematical Statistical Physics, Session LXXXIII: Lecture Notes of the Les Houches Summer School 2005 (A. Bovier, F. Dunlop, A. van Enter, F. den Hollander, and J. Dalibard, eds.), Elsevier Science, 2006, pp. 1–56.

[17] R. Lyons, *Determinantal probability measures*, Publ. Math. Inst. Hautes Etudes Sci. **98** (2003), 167–212.

[18] M. Prähofer and H. Spohn, *Scale invariance of the PNG droplet and the Airy process*, J. Stat. Phys. **108** (2002), 1071–1106.

[19] A. Rákos and G. Schütz, *Bethe Ansatz and current distribution for the TASEP with particle-dependent hopping rates*, Markov Process. Related Fields **12** (2006), 323–334.

[20] G.M. Schütz, *Exact solution of the master equation for the asymmetric exclusion process*, J. Stat. Phys. **88** (1997), 427–445.

[21] A. Soshnikov, *Determinantal random fields*, Encyclopedia of Mathematical Physics (J.-P. Francoise, G. Naber, and T. S. Tsun, eds.), Elsevier, Oxford, 2006, pp. 47–53.

[22] H. Spohn, *Exact solutions for KPZ-type growth processes, random matrices, and equilibrium shapes of crystals*, Physica A **369** (2006), 71–99.