Error Estimates and Extrapolation for the Numerical Solution of Mellin Convolution Equations

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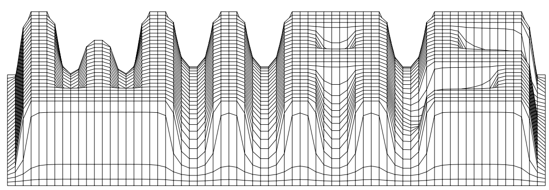

Abstract.
In this paper we consider a quadrature method for the numerical solution of a second kind integral equation over the interval, where the integral operator is a compact perturbation of a Mellin convolution operator. This quadrature method relies upon singularity subtraction and transformation technique. Stability and convergence order of the approximate solution are well known. We shall derive the first term in the asymptotics of the error which shows that, in the interior of the interval, the approximate solution converges with higher order than over the whole interval. This implies higher orders of convergence for the numerical calculation of smooth functionals to the exact solution. Moreover, the asymptotics allows us to define a new approximate solution extrapolated from the dilated solutions of the quadrature method over meshes with different mesh sizes. This extrapolated solution is designed to improve the low convergence order caused by the non-smoothness of the exact solution even when the transformation technique corresponds to slightly graded meshes. Finally, we discuss the application to the double layer integral equation over the boundary of polygonal domains and report numerical results.

Key words. Mellin convolution, potential equation, quadrature method, extrapolation

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0 INTRODUCTION

It is well known that the convergence of various numerical methods can be improved by extrapolation, i.e., the combination of approximate solutions obtained for different values of discretization parameters is closer to the exact solution than the approximate solutions themselves. A review of this topic is given e.g. in the works by Marchuk/Shaidurov [19] and Khoromski/Zhidkov [14]. The application of extrapolation to second kind integral equations with smooth kernel functions is described e.g. in the books of Baker [2] and Hackbusch [11] or in the papers by McLean [22] and Lin/Sloan/Xie [17]. The case of one-dimensional boundary integral operators over smooth curves is considered e.g. by Heise [12] and Saranen [28]. However, it seems to us that the theory of extrapolation techniques for boundary integral operators over non-smooth boundaries is rather incomplete. The only results in this direction we know about are those of Lin/Xie [18], Shi [29], and Graham/Lin/Xie [10], where Mellin convolution equations and double layer potential operators over polygonal curves are considered. The extrapolation in these papers improves the low order convergence which is caused by the low order of the implemented discretization scheme, i.e., caused by the low degree of the trial functions in the Galerkin scheme or by the low order quadrature rules used for discretization. The price for the faster convergence rate is that a stronger mesh grading near the points of singularity of the Mellin convolution kernels is required. Note that the use of strongly graded meshes is, in some sense, equivalent to the application of a transformation of variables with a large number of vanishing derivatives at the points of singularity. Finally, we remark that there are also other methods improving the numerical convergence in the case of these equations. We refer the reader to results on superconvergence and on p- and h-p-methods by Amini, Chandler, Elschner, Graham, Jeon, Kress, Mastroianni, Monegato, McLean, and Sloan [23, 1, 3, 16, 7, 13, 8, 21, 24].

In the present paper we also consider Mellin convolution equations. We shall establish the first term of an asymptotic error expansion and define an extrapolation method which is different from that in [18, 29, 10]. This extrapolation does not improve the convergence rate due to the low order discretization scheme. Instead, it improves the low order caused by the non-smoothness of the exact solution. In order to describe the nature of the asymptotics and the extrapolation, let us consider a Mellin convolution equation of the second kind on the interval $[0, 1]$ (cf. Equ. (1.1)), where the singularity of the kernel function is located at 0. Let $\tilde{x}$ denote the exact solution and suppose we solve our Mellin equation approximately using a quadrature method. Roughly speaking, for the approximate solution $\tilde{x}_h$ of the quadrature method, we shall derive an error expansion of the type (cf. Theorem 1.3)

$$\tilde{e}_h(\tilde{t}) := \tilde{x}(\tilde{t}) - \tilde{x}_h(\tilde{t}) = h^\gamma \tilde{f}(\tilde{t}/h) + O(h^{\gamma_1}), \quad 0 < \tilde{t} < 1, \tag{0.1}$$

where γ and γ_1 are positive reals with $\gamma < \gamma_1$ and h is the mesh size of the grid used for the quadratures. For the function $\tilde{f}$, we shall show $|\tilde{f}(\tau)| = O(\tau^{-\beta_1})$, $\tau \longrightarrow \infty$ with $\beta_1 > 0$. If $\tilde{f}$ admits the asymptotic expansion $\tilde{f}(\tau) \sim \sum_{i=1}^\infty c_i \tau^{-\beta_i}$, $\beta_1 < \beta_2 < \beta_3 \ldots$ for $\tau \longrightarrow \infty$, then we arrive at

$$\tilde{e}_h(\tilde{t}) \sim \sum_{i=1}^\infty h^{\gamma+\beta_i} \tilde{f}_i(\tilde{t}) + O(h^{\gamma_1}), \quad 0 < \tilde{t} < 1\,. \tag{0.2}$$

The functions $\tilde{f}_i$, however, are singular, i.e., we get $\tilde{f}_i(\tilde{t}) = c_i \tilde{t}^{-\beta_i}$ for $h \le \tilde{t} < 1$. Hence, the supremum norm error $\|\tilde{e}_h\|_{L^\infty[0,1]}$ is of order h^γ only. Fixing an $\epsilon > 0$ and considering the supremum norm error over the interval $[\epsilon, 1]$, we get the better estimate $\|\tilde{e}_h\|_{L^\infty[\epsilon,1]} \sim h^{\inf\{\gamma_1, \gamma+\beta_i : i=1,2,\dots\}}$. If we apply the usual Richardson extrapolation algorithm to our quadrature process and denote the extrapolated solution by $\tilde{x}_h^u$, then some of the first terms in (0.2) cancel out, i.e., the extrapolation error $\tilde{e}_h^u$ satisfies

$$\tilde{e}_h^u := \tilde{x}(\tilde{t}) - \tilde{x}_h^u(\tilde{t}) \sim \sum_{i=R}^{\infty} h^{\gamma+\beta_i} \tilde{f}_i(\tilde{t}) + O(h^{\gamma_1}), \quad 0 < \tilde{t} < 1 \ .$$

This gives the better convergence order $\gamma_* := \inf\{\gamma_1, \gamma + \beta_i : i = R, R+1 \dots\}$ for the supremum norm error $\|\tilde{e}_h^u\|_{L^\infty[\epsilon,1]} \sim h^{\gamma_*}$ but no improvement of the overall error.

However, one often seeks a good error estimate over the whole of $[0, 1]$. To get this, we observe

$$2^{-\gamma} \tilde{e}_{2h}(2\tilde{t}) = 2^{-\gamma}(2h)^\gamma \tilde{f}\Big((2\tilde{t})/(2h)\Big) + O(h^{\gamma_1}) = h^\gamma \tilde{f}(\tilde{t}/h) + O(h^{\gamma_1}) \ . \qquad (0.3)$$

Thus $\tilde{e}_h(\tilde{t})$ has the same error expansion as $2^{-\gamma}\tilde{e}_{2h}(2\tilde{t})$ and $\|\tilde{e}_h(\tilde{t}) - 2^{-\gamma}\tilde{e}_{2h}(2\tilde{t})\|_{L^\infty[0,1]} \sim h^{\gamma_1}$. Unfortunately, $\tilde{x}(\tilde{t})$ and $\tilde{e}_h(\tilde{t})$ are unknown. However, using the previous kind of argument for the expansion of the "discrete" error $\tilde{x}_{2h}(\tilde{t}) - \tilde{x}_h(\tilde{t})$ instead of $\tilde{e}_h = \tilde{x}(\tilde{t}) - \tilde{x}_h(\tilde{t})$, we shall observe (cf. Theorem 1.6) $\|\tilde{x}(\tilde{t}) - \tilde{x}_h^e(\tilde{t})\|_{L^\infty[0,1]} \sim h^{\gamma_1}$, where

$$\tilde{x}_h^e(\tilde{t}) := \tilde{x}_{2h}(\tilde{t}) + \sum_{l=1}^{L} 2^{-(l-1)\gamma} \left\{ \tilde{x}_h(2^{l-1}\tilde{t}) - \tilde{x}_{2h}(2^{l-1}\tilde{t}) \right\},$$

and L denotes the largest integer such that $2^{L-1}\tilde{t} \le 1/2$. Moreover, if we seek a linear functional $\int \tilde{x}\tilde{g}$ of the exact solution $\tilde{x}$ with a smooth function $\tilde{g}$, then the asymptotics (0.1) implies new orders for the convergence of the quadrature approximation $h \sum' \tilde{x}(t_i)\tilde{g}(t_i)$ to $\int \tilde{x}\tilde{g}$ without any extrapolation (cf. Corollary 1.5). These new orders improve those obtained by estimating the error for the functional by the L^p-errors for the function $\tilde{x}$.

The plan of this paper is as follows. In Sects. 1.1-1.2 we introduce the equation together with the necessary assumptions. The quadrature method including singularity subtraction and transformation technique will be derived in Sect. 1.3. A stability and convergence theorem follows. This theorem is perhaps new for the space $L^p[0,1]$, $(1 \le p < \infty)$ and for the special kind of singularity subtraction. However, it should also be possible to prove this result by extending the arguments of [3, 16, 7, 5, 21, 25] to the L^p setting. We shall present some details of the proof here only to prepare the derivation of an error expansion. In Sect. 1.4 we give this asymptotic error expansion and derive the corresponding extrapolation process. Sects. 2 and 3 are devoted to the proof of the error expansion. To this end, we first prove an error estimate for the quadrature rule analogous to the Euler-Maclaurin summation formula in Sect. 2.1. Then we show the stability of the quadrature method for the case of the half-axis in Sect.2.2. From this and well-known localization principles the stability for the equation over the interval follows. In Sect. 2.3 we analyse the solution of the quadrature method when the right-hand side is of the type $y(t) = y_h(t) := f(t/h)$. Since functions of the type $t \mapsto f(t/h)$ appear in the asymptotic expansion of the quadrature error (cf. Sect. 3.2), the structure of these special solutions is crucial for the proof of the asymptotic error expansion for the quadrature method. In Sect. 3.1 we split the error of the approximate solution into several terms, and we estimate these terms in Sect. 3.2. Finally, in Sect. 3.3 all previous results are combined to prove the error expansion of Theorem 1.3. The last section is devoted to the application of Theorems 1.1-1.6 to the special case of the double layer equation over polygonal boundaries. The presented numerical computations confirm our results or show even better results.

1 QUADRATURE METHOD AND EXTRAPOLATION

1.1 The equation

Let us consider an equation of the type

$$\tilde{x}(\tilde{t}) + \int_0^1 \tilde{k}(\tilde{t}, \tilde{s})\tilde{x}(\tilde{s})d\tilde{s} = \tilde{y}(\tilde{t}), \quad 0 < \tilde{t} < 1, \tag{1.1}$$

where

$$\tilde{k}(\tilde{t}, \tilde{s}) = \tilde{k}_M\left(\frac{\tilde{t}}{\tilde{s}}\right)\frac{1}{\tilde{s}} + \tilde{k}_S(\tilde{t}, \tilde{s}), \quad 0 < \tilde{t}, \tilde{s} < 1. \tag{1.2}$$

We shall formulate necessary assumptions on $\tilde{k}_M$, $\tilde{k}_S$ in Sect.1.2. Here we mention only that the double layer equation over polygonal boundaries is equivalent to a system of equations of the form (1.1). Namely, for the double layer kernel over a curve with angle θ, we get equations including the kernel functions (cf. e.g. [4, 3])

$$\tilde{k}(\tilde{t}, \tilde{s}) = \pm\frac{1}{\pi}\frac{\tilde{t}\sin\theta}{\tilde{t}^2 + \tilde{s}^2 - 2\tilde{t}\tilde{s}\cos\theta}, \quad \tilde{k}_M(\sigma) := \pm\frac{1}{\pi}\frac{\sigma\sin\theta}{\sigma^2 + 1 - 2\sigma\cos\theta}. \tag{1.3}$$

The application of our quadrature method to (1.1) requires first a singularity subtraction and then a substitution of variables corresponding to a transformation of the interval $[0, 1]$. Let us start with the **singularity subtraction** (cf.[26] and also the slightly different techniques in [3, 16, 5]). We introduce

$$\tilde{a}(\tilde{t}) := 1 + \int_0^1 \tilde{k}(\tilde{t}, \tilde{s})d\tilde{s}, \quad 0 < \tilde{t} < 1$$

and write (1.1) in the form

$$\tilde{a}(\tilde{t})\tilde{x}(\tilde{t}) + \int_0^1 \tilde{k}(\tilde{t}, \tilde{s})[\tilde{x}(\tilde{s}) - \tilde{x}(\tilde{t})]d\tilde{s} = \tilde{y}(\tilde{t}), \quad 0 < \tilde{t} < 1. \tag{1.4}$$

The new integrand $\tilde{k}(\tilde{t}, \tilde{s})[\tilde{x}(\tilde{s}) - \tilde{x}(\tilde{t})]$ of (1.4) is smoother than $\tilde{k}(\tilde{t}, \tilde{s})\tilde{x}(\tilde{s})$ in (1.1). Consequently, the quadrature for the integral in (1.4) converges faster.

For the **transformation of variables**, we choose a positive integer q and substitute $\tilde{t} = t^q$, $\tilde{s} = s^q$ in (1.4). Since there holds $\|\tilde{x}\|_{L^p} = \|x\|_{L^p}$, $1 \le p \le \infty$ for $x(t) := \tilde{x}(t^q)\sqrt[p]{qt^{q-1}}$, we multiply our equation (1.4) by $\sqrt[p]{qt^{q-1}}$ and get

$$a(t)x(t) + \int_0^1 k(t, s)\left[x(s) - x(t)\left(\frac{s}{t}\right)^{(q-1)/p}\right]ds = y(t), \quad 0 < t < 1, \tag{1.5}$$

where

$$a(t) := \tilde{a}(t^q), \quad x(t) := \tilde{x}(t^q)\sqrt[p]{qt^{q-1}}, \quad y(t) := \tilde{y}(t^q)\sqrt[p]{qt^{q-1}}, \tag{1.6}$$
$$k(t,s) := \sqrt[p]{qt^{q-1}}\tilde{k}(t^q,s^q)\sqrt[p']{qs^{q-1}}.$$

Here we define p' by $\frac{1}{p}+\frac{1}{p'}=1$ and set $1/p:=0$ for $p=\infty$. The new solution x of (1.5) is smoother in the neighbourhood of 0 than $\tilde{x}$. For instance, if $\tilde{x}(\tilde{t}) \sim \tilde{t}^\gamma$ for $\tilde{t} \longrightarrow 0$ with $\gamma > 0$, then $x(t) \sim t^{q\gamma+(q-1)/p}$ with $q\gamma+(q-1)/p \geq \gamma$.

1.2 Assumptions on the kernel, the right-hand side, and the solution

In view of (1.6) and (1.2) we get

$$k(t,s) = k_M\left(\frac{t}{s}\right)\frac{1}{s} + k_S(t,s), \tag{1.7}$$
$$k_M(\tau) := q\tilde{k}_M(\tau^q)\sqrt[p]{\tau^{q-1}}, \quad k_S(t,s) := \sqrt[p]{qt^{q-1}}\tilde{k}_S(t^q,s^q)\sqrt[p']{qs^{q-1}}.$$

However, we formulate the **assumptions in terms of the original kernels**. Here and in the following C stands for a generic constant the value of which varies from instance to instance. Even, if C appears twice at one line, the values may be different.

(A1) The kernel function $\tilde{k}_S : [0,1]\times[0,1] \longrightarrow \mathbb{R}$ is smooth, i.e., it is infinitely differentiable.

(A2) The kernel function $\tilde{k}_M : \mathbb{R}^+ := (0,\infty) \longrightarrow \mathbb{R}$ is infinitely differentiable and there exists constants C and real numbers α, α_1 with $1/p < \alpha < \alpha_1$ such that

$$(\sigma\partial/\partial\sigma)^m\tilde{k}_M(\sigma) = C\sigma^\alpha + O(\sigma^{\alpha_1}), \quad \sigma \longrightarrow 0,$$
$$(\sigma\partial/\partial\sigma)^m\tilde{k}_M(\sigma) = C\sigma^{-\alpha} + O(\sigma^{-\alpha_1}), \quad \sigma \longrightarrow \infty$$

for $m = 0,1,2,\ldots$.

Note that the exponents could have been different for $\sigma \longrightarrow \infty$ and for $\sigma \longrightarrow 0$. For simplicity we choose them to be equal. Furthermore, we note that the condition on the differentiability can be relaxed. A fourth order continuous derivative should be enough for our considerations.

Now we observe that, for $k_S \equiv 0$, the operator $\tilde{A}$ on the right-hand side of (1.1) is the restriction ("Wiener-Hopf" operator) of the Mellin convolution operator $I_H + \tilde{K}_H$, where $I_H \in \mathcal{L}(L^p(\mathbb{R}^+))$ is the identity and

$$\tilde{K}_H f(\tilde{t}) := \int_0^\infty \tilde{k}\left(\frac{\tilde{t}}{\tilde{s}}\right)\frac{1}{\tilde{s}}f(\tilde{s})d\tilde{s}, \quad \tilde{t} \in \mathbb{R}^+.$$

The operator $I_H + \tilde{K}_H$ is invertible if and only if its symbol does not vanish, i.e., if the Mellin transform of its kernel function

$$(\mathcal{M}\tilde{k}_M)(\xi) := \int_0^\infty \tilde{k}_M(\sigma)\sigma^{i\xi+1/p-1}d\sigma, \quad \xi \in I\!R$$

is different from -1 over $I\!R$. By (A2) the function $\xi \mapsto (\mathcal{M}\tilde{k}_M)(\xi)$ is continuous and vanishes at $\xi = \pm\infty$. Our next assumption (A3) is nothing else then the invertibility of the operator $\tilde{A}$. Using the theory of Wiener-Hopf operators, this assumption can be formulated as

(A3) i) For any $\xi \in I\!R$, there holds $1 + (\mathcal{M}\tilde{k}_M)(\xi) \neq 0$.
ii) The winding number of the curve $\{[1 + (\mathcal{M}\tilde{k}_M)(\xi)], \ \xi \in I\!R\}$ is zero.
iii) The homogeneous equation (1.1) (i.e., (1.1) with $\tilde{y} \equiv 0$) has only the trivial solution.

Next we need some **assumptions on the resolvent kernel**. It is a well-known fact that the inverse $(I_H + \tilde{K}_H)^{-1}$ takes the form $I_H + \tilde{L}_H$, where

$$\tilde{L}_H f(\tilde{t}) := \int_0^\infty \tilde{l}\left(\frac{\tilde{t}}{\tilde{s}}\right)\frac{1}{\tilde{s}}f(\tilde{s})d\tilde{s}, \quad \tilde{t} \in I\!R^+.$$

Note that the resolvent kernel $\tilde{l}_M$ is the solution of $(I_H + \tilde{K}_H)\tilde{l}_M = -\tilde{k}_M$. Thus the regularity theory of Mellin type equations implies the regularity of the resolvent kernel $\tilde{l}_M$. We require

(A4) The kernel function $\tilde{l}_M : I\!R^+ \longrightarrow I\!R$ is infinitely differentiable and there exists constants C and real numbers γ, γ_1 with $1/p < \gamma < \gamma_1$ such that

$$\begin{aligned}(\sigma\partial/\partial\sigma)^m\tilde{l}_M(\sigma) &= C\sigma^\gamma + O(\sigma^{\gamma_1}), \quad \sigma \longrightarrow 0,\\ (\sigma\partial/\partial\sigma)^m\tilde{l}_M(\sigma) &= C\sigma^{-\gamma} + O(\sigma^{-\gamma_1}), \quad \sigma \longrightarrow \infty\end{aligned}$$

for $m = 0, 1, 2, \ldots$.

For the **right-hand side and the solution**, we assume the following.

(A5) The function $\tilde{y} : [0,1] \longrightarrow I\!R$ is infinitely differentiable.

(A6) The solution $\tilde{x}$ of (1.1) is continuous on $[0,1]$ and infinitely differentiable on $(0,1)$ such that

$$\begin{aligned}\tilde{x}(\tilde{t}) &= C + C\tilde{t}^\gamma + O(\tilde{t}^{\gamma_1}), \quad \tilde{t} \longrightarrow 0,\\ (\tilde{t}\partial/\partial\tilde{t})^m\tilde{x}(\tilde{t}) &= C\tilde{t}^\gamma + O(\tilde{t}^{\gamma_1}), \quad \tilde{t} \longrightarrow 0\end{aligned}$$

for $m = 1, 2, \ldots$. Here γ, γ_1 are the same as in (A4).

Finally, we shall need the following **technical assumption**.

(A7) Suppose $q(\gamma + 1/p) < min\{4, q\alpha\}$.

1.3 The quadrature method

First we need a quadrature formula. For this purpose we take the **trapezoidal rule with end point correction**. Thus we set

$$\int_0^1 f(s)ds \sim h{\sum_j}' f(t_j), \tag{1.8}$$

where $h := 1/N,\ t_j := j/N$ for $j \in \mathbb{Z}$ and

$$\begin{aligned} {\sum_j}' T_j \ &:=\ \frac{127}{48}T_1 - \frac{59}{48}T_2 + \frac{39}{16}T_3 + \frac{31}{48}T_4 + \sum_{j=5}^{N-4} T_j \\ &\quad + \frac{49}{48}T_{N-3} + \frac{43}{48}T_{N-2} + \frac{59}{48}T_{N-1} + \frac{17}{48}T_N. \end{aligned}$$

Note that this quadrature will be derived in part a) of the proof to Lemma 2.1 and has convergence order four (cf. Corollary 2.2). Using (1.8) for the integral in (1.5), we arrive at the following **quadrature method**:

$$a(t_i)x_N(t_i) + h{\sum_j}' k(t_i,t_j)\left[x_N(t_j) - x_N(t_i)\left(\frac{t_j}{t_i}\right)^{(q-1)/p}\right] = y(t_i), \quad i = 1,\ldots,N. \tag{1.9}$$

To force stability we also introduce a slight modification (cf. [3, 16, 7, 26, 21]). We note, however, that in numerical computations this modification has often turned out to be not necessary (cf. also [5], where stability is proved without this modification under additional assumptions). So we only recommend to work with the modified method if a numerical instability has been observed in the unmodified version. Let us fix a non-negative integer j_* and set

$$\sum_{j>j_*}{}' T_j := {\sum_j}' T_j^*, \quad T_j^* := \begin{cases} T_j & \text{if } j > j_* \\ 0 & \text{if } j \le j_* \,. \end{cases}$$

Then the **modified method** looks like

$$a(t_i)x_N(t_i) + h\sum_{j>j_*}{}' k(t_i,t_j)\left[x_N(t_j) - x_N(t_i)\left(\frac{t_j}{t_i}\right)^{(q-1)/p}\right] = y(t_i), \quad i = 1,\ldots,N. \tag{1.10}$$

Clearly, (1.10) coincides with (1.9) if $j_* = 0$.

The solution x_N of (1.10) is given on a set of discrete points. To get an approximate function x_N, we introduce the **interpolation** $x_N := \sum_{j=1}^N x_N(t_j)\varphi_j^N$, where φ_j^N is a continuous and piecewise cubic function with minimal support such that $\varphi_j^N(t_m) = \delta_{j,m},\ m = 1,2,\ldots N$. More exactly, the basis function φ_j^N is defined as the unique function whose restriction to $(t_i,t_{i+1}),\ i \in \mathbb{Z}$ is the cubic polynomial $P_{j,i}$ which satisfies $P_{j,i}(t_m) = \delta_{m,j}$ for $m = m_i, m_i+1, m_i+2, m_i+3$ with

$$m_i := \begin{cases} 1 & \text{if } i = 1 \\ i-1 & \text{if } 2 \le i \le N-2 \\ N-3 & \text{if } i = N-1,\ N\ . \end{cases}$$

It is not hard to see that $supp\, \varphi_j^N \subseteq [t_{j-2}, t_{j+2}]$ and that

$$C^{-1}\sqrt[p]{h}\|\{x_N(t_i)\}_{i=1}^N\|_{l^p} \leq \|x_N\|_{L^p(0,1)} \leq C\sqrt[p]{h}\|\{x_N(t_i)\}_{i=1}^N\|_{l^p}, \tag{1.11}$$

where $\|\cdot\|_{l^p}$ denotes the norm in the N-dimensional discrete l^p-space $l^p(N)$.

Let $A \in \mathcal{L}(L^p(0,1))$ be the operator defined by the left-hand side of (1.5) and let A_N be the operator given by the left-hand side of (1.9) or (1.10). In view of (1.11), we consider A_N in $\mathcal{L}(l^p(N))$. Let us recall that the method (1.10) or the sequence $\{A_N\}_N$ is called **stable** if the A_N are invertible for sufficiently large N and if the norms of their inverses $\|A_N^{-1}\|_{\mathcal{L}(l^p(N))}$ are uniformly bounded with respect to N. The stability is an important prerequisite for the proof of error estimates and for the estimates of the condition numbers of the arising linear systems of equations.

THEOREM 1.1 *Suppose that the assumptions (A1)-(A6) together with the technical condition (A8) of Sect.2.2 are satisfied. Then the quadrature method (1.10) is stable. In particular, (1.9) is stable if condition (A8) holds for $j_* = 0$. If x is the exact solution of (1.5) and x_N that of the quadrature method, then we get*

$$\|x - x_N\|_{L^p(0,1)} \leq C h^{min\{q(\gamma+1/p),4,q\alpha\}} \begin{cases} 1 & \text{if } min\{4, q\alpha\} > q(\gamma+1/p) \\ (\log h^{-1})^{\varrho_{22}} & \text{else}, \end{cases}$$

where ϱ_{22} will be defined in Sect. 3.3. Especially, the number ϱ_{22} is zero if

$$\begin{array}{rclrclrcl} q\alpha &\neq& 4, & q\gamma &\neq& 2, & \gamma - \frac{1}{p} &\neq& \alpha, \\ q(1+\gamma) &\neq& 4, & q\left(\gamma - \frac{1}{p}\right) &\neq& 4, & q\left(\gamma_1 + \frac{1}{p}\right) &\neq& 4, \\ \alpha - \frac{1}{p} &\neq& \gamma_1, & \pm q(\alpha \pm \gamma) &\neq& 4, & 2\gamma &\neq& \alpha, \\ \alpha &\neq& \gamma, & q(\gamma_1 - \alpha) &\neq& 4, & \gamma &\neq& 1, \\ & & & \alpha - \gamma &\neq& 1. & & & \end{array} \tag{1.12}$$

REMARK 1.2 *If the operator A of (1.5) corresponds to the double layer operator or $\|\tilde{K}_H\|_{\mathcal{L}(L^\infty(\mathbb{R}^+))} < 1$ and if j_* is large enough, then condition (A8) is always satisfied for (1.10) (cf. [26]). For an arbitrary but invertible Mellin convolution operator, we do not know whether (A8) holds even for large j_*. However, if another singularity subtraction step is performed (cf. [3, 16]), then the corresponding condition (A8) for (1.10) with large j_* can be derived from the theory of finite section methods (cf. [27]). In particular, this condition (A8) holds even for (1.9) in the case of the double layer operator and for $\|\tilde{K}_H\|_{\mathcal{L}(L^\infty(\mathbb{R}^+))} < 1$ (cf. [5]). We feel that, similarly to the assumption of a second kind Fredholm integral operator to be invertible, it would be a rare accident if (A8) is not satisfied.*

The proof of the stability will be given in Sect.2. The error estimate follows from the stability and from (3.22). Results like that of Theorem 1.1 have already been proved in [3, 16, 7, 25, 26] (cf. also [1, 23, 21, 5, 13]).

1.4 Asymptotic expansion of the numerical error and extrapolation

In view of (1.11), we shall consider the discrete L^p-error $\sqrt[p]{h\sum_{j=1}^N |x(t_j) - x_N(t_j)|^p}$. To demonstrate the usefulness of this norm, let us suppose that we have to compute a **linear functional of the solution** $\tilde{x}$, i.e., we seek $\int_0^1 \tilde{x}\tilde{g}$, where $\tilde{g} \in L^{p'}(0,1)$. We get

$$\int_0^1 \tilde{x}(\tilde{s})\tilde{g}(\tilde{s})d\tilde{s} = \int_0^1 x(s)g(s)ds, \quad g(s) := \tilde{g}(t^q)\sqrt[p']{qt^{q-1}}. \tag{1.13}$$

Note that $\|\tilde{g}\|_{L^{p'}} = \|g\|_{L^{p'}}$. Replacing x by the approximate solution x_N and the integration by the quadrature, we get an approximation for $\int_0^1 \tilde{x}\tilde{g}$:

$$\int_0^1 \tilde{x}\tilde{g} \sim h{\sum_j}' x_N(t_j)g(t_j).$$

We remark that this approximate value for $\int_0^1 \tilde{x}\tilde{g}$ is independent of the numbers p and p' used in (1.5), (1.10) and (1.13). For the error of this approximation, we obtain

$$\begin{aligned}\left|\int_0^1 \tilde{x}\tilde{g} - h{\sum_j}' x_N(t_j)g(t_j)\right| &\leq \left|\int_0^1 x(s)g(s)ds - h{\sum_j}' x(t_j)g(t_j)\right| \\ &\quad + h{\sum_j}' |x(t_j) - x_N(t_j)|\,|g(t_j)|. \end{aligned} \tag{1.14}$$

If g is sufficiently smooth (e.g. if g is infinitely differentiable), then assumption (A6) for $\tilde{x}$ implies

$$\begin{aligned}(t\partial/\partial t)^m(xg)(t) &= Ct^{q\gamma+q-1} + O(t^{q(\gamma+1)+q-1}) + O(t^{q\gamma_1+q-1}) + \begin{cases} 0 & \text{if } m \geq 1 \\ g_0(t) & \text{if } m = 0 \end{cases}, \\ & \quad m = 0, 1, \ldots, \end{aligned}$$

where g_0 is smooth. Hence, we get (cf. Corollaries 2.2 and 2.3)

$$\left|\int_0^1 x(s)g(s)ds - h{\sum_j}' x(t_j)g(t_j)\right| \leq C \begin{cases} h^4 & \text{if } q(\gamma+1) > 4 \\ h^4 \log h^{-1} & \text{if } q(\gamma+1) = 4 \\ h^{q(\gamma+1)} & \text{if } q(\gamma+1) < 4\,. \end{cases} \tag{1.15}$$

Consequently, using the Cauchy-Schwarz inequality as well as the boundedness of the discrete $L^{p'}$-norm of our smooth test functional g, we arrive at

$$\begin{aligned}\left|\int_0^1 \tilde{x}\tilde{g} - h{\sum_j}' x(t_j)g(t_j)\right| &\leq C\sqrt[p]{h\sum_{j=1}^N |x(t_j) - x_N(t_j)|^p} \\ &\quad + Ch^{min\{4,q(\gamma+1)\}} \begin{cases} 1 & \text{if } q(\gamma+1) \neq 4 \\ \log h^{-1} & \text{if } q(\gamma+1) = 4\,. \end{cases} \end{aligned} \tag{1.16}$$

In other words we have to estimate the error $x(t_j) - x_N(t_j)$ in the discrete L^p-norm. Moreover, we seek an extrapolation x_N^e of the approximate solutions x_N and x_{2N} such that the discrete L^p-norm of $x(t_j) - x_N^e(t_j)$ is smaller than that of $x(t_j) - x_N(t_j)$. Let us express the dependence of t_i on N by writing $t_i^N := t_i$ in the rest of this section. The **error expansion** which is fundamental for our extrapolation looks as follows.

THEOREM 1.3 *Suppose that the assumptions (A1)-(A7) hold, that the quadrature method (1.9) or (1.10) is stable (cf. Theorem 1.1), and that x is the exact solution of (1.5). Then there exists a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ with*

$$|f(i)| \leq C i^{-min\{4-q(\gamma+1/p),q(\gamma-1/p),q(\alpha-\gamma-1/p)\}-1/p} (\log i)^{\varrho_{29}} \tag{1.17}$$

such that, for the solution x_N of the quadrature method, we get

$$\begin{aligned} x(t_i^N) - x_N(t_i^N) &= h^{q(\gamma+1/p)-1/p} f(i) + r_{N,i} \ , \\ \sqrt[p]{h \sum_{i=1}^{N} |r_{N,i}|^p} &\leq C h^{\varrho} (\log\, h^{-1})^{\varrho_{27}} \end{aligned} \tag{1.18}$$

with $\varrho := min\{q(\gamma_1 + 1/p), q2\gamma, 4, q, q\alpha\}$. The non-negative integers ϱ_{29} and ϱ_{27} will be defined in Sect. 3.3. In particular, $\varrho_{29} = \varrho_{27} = 0$ if (1.12) holds.

REMARK 1.4 *For many applications the convergence order $min\{q(\gamma + 1/p), 4, q\alpha\}$ of Theorem 1.1 is equal to $q(\gamma + 1/p)$. In this case the error term $r_{N,i}$ is of higher order and $h^{q(\gamma+1/p)-1/p} f(i)$ is the main part of the error $x(t_i^N) - x_N(t_i^N)$.*

The proof of this theorem will be given in Sect.3.

COROLLARY 1.5 *Let $p = \infty$, fix an $\epsilon > 0$ and a smooth function $\tilde{g} : [0,1] \longrightarrow \mathbb{R}$, and suppose the assumptions of Theorem 1.3 are satisfied. Then there holds*

$$\begin{aligned} \sup_{\epsilon \leq t \leq 1} |x(t) - x_N(t)| &\leq C h^{\varrho} (\log\, h^{-1})^{\varrho_{36}} \ , \\ \left| \int_0^1 \tilde{x}\tilde{g} - h {\sum_j}' x_N(t_j) g(t_j) \right| &\leq C h^{\varrho} (\log\, h^{-1})^{\varrho_{35}} \ , \end{aligned} \tag{1.19}$$

where ϱ is as in Theorem 1.3 and the integers ϱ_{35} and ϱ_{36} will be defined below. In particular, $\varrho_{35} = \varrho_{36} = 0$ if (1.12) holds.

PROOF: Setting

$$\varrho_{36} := \begin{cases} \varrho_{27} & \text{if } \ \varrho < min\{4,\ q2\gamma,\ q\alpha\} \\ max\{\varrho_{27},\ \varrho_{29}\} & \text{else} \ , \end{cases}$$

the first assertion is obvious. Let us turn to the second. In view of (1.14) and (1.15), we have to estimate $h{\sum_j}' |x(t_j) - x_N(t_j)|\,|g(t_j)|$. Let us set $\beta := min\{4 - q(\gamma + 1/p),\ q(\gamma - 1/p),\ q(\alpha - \gamma - 1/p)\} + 1/p$. Using Theorem 1.3 and the smoothness of $\tilde{g}$, we conclude $|g(t_j)| = |\tilde{g}(t_j^q) \sqrt[p']{q t_j^{q-1}}| \leq C \sqrt[p']{t_j^{q-1}}$ as well as

$$\begin{aligned} h {\sum_j}' |x(t_j) \ &- \ x_N(t_j)|\,|g(t_j)| \leq C h \sum_j h^{q(\gamma+1/p)-1/p} j^{-\beta} (jh)^{(q-1)/p'} (\log\, h^{-1})^{\varrho_{29}} \\ &\qquad\qquad + C h^{\varrho} (\log\, h^{-1})^{\varrho_{27}} \\ &\leq \ C h^{min\{q(\gamma+1),q(\gamma+1/p)-1/p+\beta\}} (\log\, h^{-1})^{\varrho_{33}} + C h^{\varrho} (\log\, h^{-1})^{\varrho_{27}} \\ &\leq \ C h^{min\{4,q(\gamma+1),q2\gamma,q\alpha\}} (\log\, h^{-1})^{\varrho_{33}} + C h^{\varrho} (\log\, h^{-1})^{\varrho_{27}} \ , \\ \varrho_{33} \ &:= \ \begin{cases} \varrho_{29} & \text{if } \ \beta - 1 \neq (q-1)/p' \\ 1 + \varrho_{29} & \text{if } \ \beta - 1 = (q-1)/p' \ . \end{cases} \end{aligned}$$

In other words,

$$h{\sum_j}' |x(t_j) \quad - \quad x_N(t_j)|\,|g(t_j)| \le C h^{\varrho} (\log\, h^{-1})^{\varrho_{34}}\ ,$$

$$\varrho_{34} \quad := \quad \begin{cases} \varrho_{27} & \text{if } \ min\{4,\, q(\gamma+1),\, q2\gamma,\, q\alpha\} > \varrho \\ max\{\varrho_{27},\, \varrho_{33}\} & \text{else .} \end{cases}$$

This and (1.14),(1.15) lead to (1.19), where

$$\varrho_{35} \quad := \quad \begin{cases} max\{1,\, \varrho_{34}\} & \text{if } \ 4 = q(\gamma+1) \le \varrho \\ \varrho_{34} & \text{else .} \end{cases}$$

∎

Theorem 1.3 allows us to derive an extrapolation result. We conclude from (1.18)

$$\begin{aligned} x(t_i^N) - x_{2^l N}(t_i^N) &= x(t_{2^l i}^{2^l N}) - x_{2^l N}(t_{2^l i}^{2^l N}) && (1.20) \\ &= (2^l N)^{-q\gamma-(q-1)/p} f(2^l i) + r_{2^l N, 2^l i}, \\ x_{2^l N}(t_i^N) - x_{2^{l-1} N}(t_i^N) &= (2^{l-1} N)^{-q\gamma-(q-1)/p} g_+(2^{l-1} i) && (1.21) \\ &\quad + r_{2^{l-1} N, 2^{l-1} i} - r_{2^l N, 2^l i}, \\ g_+(i) &:= \{f(i) - 2^{-q\gamma-(q-1)/p} f(2i)\}. \end{aligned}$$

Equation (1.21) with $l = 1$ yields

$$N^{-q\gamma-(q-1)/p} g_+(i) = x_{2N}(t_i^N) - x_N(t_i^N) + r_{2N,2i} - r_{N,i}.$$

Hence, we obtain

$$\begin{aligned} x(t_i^N) &= x_{2^L N}(t_i^N) + r_{2^L N, 2^L i} + (2^L N)^{-q\gamma-(q-1)/p} f(2^L i) \\ &= x_N(t_i^N) + \sum_{l=1}^{L} [x_{2^l N}(t_i^N) - x_{2^{l-1} N}(t_i^N)] + r_{2^L N, 2^L i} + (2^L N)^{-q\gamma-(q-1)/p} f(2^L i) \\ &= x_N(t_i^N) + \sum_{l=1}^{L} \left\{ (2^{l-1} N)^{-q\gamma-(q-1)/p} g_+(2^{l-1} i) + r_{2^{l-1} N, 2^{l-1} i} - r_{2^l N, 2^l i} \right\} \\ &\quad + r_{2^L N, 2^L i} + (2^L N)^{-q\gamma-(q-1)/p} f(2^L i) \\ &= x_N(t_i^N) + \sum_{l=1}^{L} 2^{-(l-1)[q\gamma+(q-1)/p]} \left\{ x_{2N}(t_{2^{l-1} i}^N) - x_N(t_{2^{l-1} i}^N) + r_{2N, 2^l i} - r_{N, 2^{l-1} i} \right\} \\ &\quad + r_{N,i} + (2^L N)^{-q\gamma-(q-1)/p} f(2^L i) \\ &= x_N(t_i^N) + \sum_{l=1}^{L} 2^{-(l-1)[q\gamma+(q-1)/p]} \left\{ x_{2N}(t_{2^{l-1} i}^N) - x_N(t_{2^{l-1} i}^N) \right\} \\ &\quad + \sum_{l=1}^{L} 2^{-(l-1)[q\gamma+(q-1)/p]} \left\{ r_{2N, 2^l i} - r_{N, 2^{l-1} i} \right\} \\ &\quad + r_{N,i} + (2^L N)^{-q\gamma-(q-1)/p} f(2^L i). && (1.22) \end{aligned}$$

We define the **extrapolated solution** x_N^e by

$$x_N^e(t_i^N) := x_N(t_i^N) + \sum_{l=1}^{L_i} 2^{-(l-1)[q\gamma+(q-1)/p]} \left\{ x_{2N}(t_{2^{l-1}i}^N) - x_N(t_{2^{l-1}i}^N) \right\}, \tag{1.23}$$

where L_i is the largest non-negative integer such that $t_{2^{L_i-1}i}^N \leq 1/2$, i.e., $i \leq 2^{-L_i}N$.

THEOREM 1.6 *Suppose that the assumptions (A1)-(A7) hold, that the quadrature method (1.9) or (1.10) is stable (cf. Theorem 1.1), that x is the exact solution of (1.5), and that x_N^e is the approximate solution extrapolated from x_N and x_{2N} by (1.23). Then there holds*

$$\sqrt[p]{h \sum_{i=1}^{N} |x(t_i) - x_N^e(t_i)|^p} \leq C h^{\varrho} (\log\ h^{-1})^{\varrho_{36}} \tag{1.24}$$

with ϱ and ϱ_{36} as in Theorem 1.3 and Corollary 1.5.

PROOF: Let us assume, for simplicity, that $N = 2^n$. From (1.22) we conclude

$$\begin{aligned} \sqrt[p]{h} \| \{x(t_i^N) - x_N^e(t_i^N)\}_{i=1}^N \|_{l^p} &\leq C\sqrt[p]{h} \left\{ \|\{r_{2N,i}\}_{i=1}^N\|_{l^p} + \|\{r_{N,i}\}_{i=1}^N\|_{l^p} \right\} \sum_{l=0}^{n} 2^{-[q\gamma+(q-1)/p]l} \\ &\quad + \sqrt[p]{h \sum_{i=1}^{N} [(2^{L_i}N)^{-q\gamma-(q-1)/p} f(2^{L_i}i)]^p}\ . \end{aligned} \tag{1.25}$$

First we estimate the last term. We set $\xi := q\gamma + (q-1)/p$ and $\zeta := min\{4 - q(\gamma + 1/p), q(\gamma - 1/p), q(\alpha - \gamma - 1/p)\} + 1/p$. Then (1.17) implies $|f(i)| \leq i^{-\zeta} (\log\ i)^{\rho_{29}}$. We get $L_i = n - L$ for $i = 2^{L-1}+1, 2^{L-1}+2, \ldots, 2^L$ and

$$\begin{aligned} \sqrt[p]{h \sum_{i=1}^{N} [(2^{L_i}N)^{-\xi} f(2^{L_i}i)]^p} &\leq C \sqrt[p]{h \sum_{L=1}^{n} \sum_{i=2^{L-1}+1}^{2^L} [(2^{n-L}N)^{-\xi} (2^{n-L}i)^{-\zeta} (\log(2^{n-L}i))^{\rho_{29}}]^p} \\ &\leq C\sqrt[p]{h} (\log\ N)^{\rho_{29}} N^{-2\xi-\zeta} \sqrt[p]{\sum_{L=1}^{n} 2^{Lp(\xi+\zeta)} \sum_{i=2^{L-1}+1}^{2^L} i^{-\zeta p}} \\ &\leq C\sqrt[p]{h} (\log\ N)^{\rho_{29}} N^{-2\xi-\zeta} \sqrt[p]{\sum_{L=1}^{n} 2^{Lp(\xi+\zeta)-L(\zeta p-1)}} \\ &\leq C\sqrt[p]{h} (\log\ N)^{\rho_{29}} N^{-\xi-\zeta+1/p} \leq h^{\xi+\zeta} (\log h^{-1})^{\rho_{29}}\ . \end{aligned}$$

Together with (1.18) and (1.25), this implies (1.24). ∎

2 PROOF OF STABILITY AND SOME CONSEQUENCES

2.1 Euler-Maclaurin formula for the quadrature

Analogously to the Euler-Maclaurin summation formula we get the following.

LEMMA 2.1 *For any function* $f : (0,N) \longrightarrow \mathbb{R}$ *which has an integrable fourth order derivative, we get*

$$\int_0^N f(\sigma)d\sigma - {\sum_j}' f(j) = \int_0^N H(\sigma) f^{(4)}(\sigma)d\sigma \ ,$$

where $H = H_N : (0,N) \longrightarrow \mathbb{R}$ *is uniformly bounded with respect to* σ *and* N*. Moreover,* $H(\sigma) = \frac{1}{24}\sigma^4$ *if* $0 < \sigma < 1$.

PROOF: a) Let us first derive our fourth order rule (1.8) as a sum of quadratures over the subintervals. Clearly, Simpson's rule

$$\int_{j-1}^{j+1} f(\sigma)d\sigma \sim 2\left\{\frac{1}{6}f(j-1) + \frac{2}{3}f(j) + \frac{1}{6}f(j+1)\right\} \tag{2.1}$$

is of order four. To get fourth order interpolatory rules over $(0,1)$ and $(1,2)$, we denote the unique cubic polynomial with $P(i) = f(i),\ i = 1,2,3,4$ by P and get

$$\int_0^1 f(\sigma)d\sigma \ \sim \ \int_0^1 P(\sigma)d\sigma = \frac{55}{24}f(1) - \frac{59}{24}f(2) + \frac{37}{24}f(3) - \frac{9}{24}f(4) \ , \tag{2.2}$$

$$\int_1^2 f(\sigma)d\sigma \ \sim \ \int_1^2 P(\sigma)d\sigma = \frac{9}{24}f(1) + \frac{19}{24}f(2) - \frac{5}{24}f(3) + \frac{1}{24}f(4) \ . \tag{2.3}$$

Similarly, we obtain

$$\int_{N-1}^N f(\sigma)d\sigma \ \sim \ \frac{9}{24}f(N) + \frac{19}{24}f(N-1) - \frac{5}{24}f(N-2) + \frac{1}{24}f(N-3) \ . \tag{2.4}$$

Using (2.1)-(2.4) as well as

$$\int_0^N f(\sigma)d\sigma = \int_0^1 f(\sigma)d\sigma + \frac{1}{2}\left\{\int_1^2 f(\sigma)d\sigma + \sum_{j=2}^{N-1}\int_{j-1}^{j+1} f(\sigma)d\sigma + \int_{N-1}^N f(\sigma)d\sigma\right\} \tag{2.5}$$

we get the rule $\int_0^N f(\sigma)d\sigma \sim {\sum_j}' f(j)$. The substitution $s = \sigma h$ yields (1.8).

b) Now let P be as in part a) of the present proof and let T stand for the cubic Taylor polynomial of f at 1. For the quadrature error, we obtain

$$\begin{aligned} E \ &:= \ \int_0^1 f(\sigma)d\sigma - \left\{\frac{55}{24}f(1) - \frac{59}{24}f(2) + \frac{37}{24}f(3) - \frac{9}{24}f(4)\right\} \\ &= \ \int_0^1 \{f-P\}(\sigma)d\sigma = \int_0^1 \{(f-T)-(P-T)\}(\sigma)d\sigma \\ &= \ \int_0^1 (f-T)(\sigma)d\sigma - \\ & \quad \left\{\frac{55}{24}(f-T)(1) - \frac{59}{24}(f-T)(2) + \frac{37}{24}(f-T)(3) - \frac{9}{24}(f-T)(4)\right\} \ . \end{aligned}$$

Now the formula for the remainder of the Taylor series expansion implies

$$\int_0^1 (f-T)(\sigma)d\sigma \ = \ \int_0^1\int_1^\sigma \frac{(\sigma-\tau)^3}{3!} f^{(4)}(\tau)d\tau d\sigma = -\int_0^1\int_0^\tau \frac{(\sigma-\tau)^3}{3!}d\sigma f^{(4)}(\tau)d\tau$$

$$= \int_0^1 \frac{\tau^4}{4!} f^{(4)}(\tau) d\tau \ .$$

Hence, we get

$$\begin{aligned} E &= \int_0^1 \frac{\tau^4}{4!} f^{(4)}(\tau) d\tau - \left\{ -\frac{59}{24} \int_1^2 \frac{(2-\sigma)^3}{3!} f^{(4)}(\sigma) d\sigma + \frac{37}{24} \int_1^3 \frac{(3-\sigma)^3}{3!} f^{(4)}(\sigma) d\sigma \right. \\ & \qquad \left. -\frac{9}{24} \int_1^4 \frac{(4-\sigma)^3}{3!} f^{(4)}(\sigma) d\sigma \right\} \end{aligned}$$

Similar formulas hold for the quadrature errors of the other integrals in (2.5). Summing up these errors, we get the assertion of the lemma. ∎

COROLLARY 2.2 *For any f which has an integrable fourth order derivative over $(0,1)$, we get*

$$\left| \int_0^1 f(s) ds - h {\sum_j}' f(t_j) \right| \le C h^4 \int_0^1 |f^{(4)}(s)| ds,$$

where C is independent of f and N.

The proof is straightforward.

COROLLARY 2.3 *Suppose the function $\tilde{x}$ satisfies (A6). Then*

$$\left| \int_0^1 \tilde{x}(s) ds - h {\sum_j}' \tilde{x}(t_j) \right| \le C \begin{cases} h^{\gamma+1} & \text{if } \gamma < 3 \\ h^4 \log h^{-1} & \text{if } \gamma = 3 \\ h^4 & \text{if } \gamma > 3 \end{cases} .$$

PROOF: We get

$$\begin{aligned} \int_0^1 \tilde{x}(s) ds - h {\sum_j}' \tilde{x}(t_j) &= h \left\{ \int_0^N \tilde{x}(\sigma h) d\sigma - {\sum_j}' \tilde{x}(jh) \right\} \\ &= h \int_0^N H(\sigma) (\partial/\partial\sigma)^4 \{\tilde{x}(h\sigma)\} d\sigma \ , \\ \left| \int_0^1 \tilde{x}(s) ds - h {\sum_j}' \tilde{x}(t_j) \right| &\le Ch \left\{ \int_0^1 \sigma^4 |\tilde{x}^{(4)}(h\sigma)| h^4 d\sigma + \int_1^N |\tilde{x}^{(4)}(h\sigma)| h^4 d\sigma \right\} \\ &\le Ch \left\{ \int_0^1 (h\sigma)^4 |\tilde{x}^{(4)}(h\sigma)| d\sigma + \int_1^N (h\sigma)^4 |\tilde{x}^{(4)}(h\sigma)| \sigma^{-4} d\sigma \right\} \ . \end{aligned}$$

Using (A6), we arrive at

$$\begin{aligned} \left| \int_0^1 \tilde{x}(s) ds - h {\sum_j}' \tilde{x}(t_j) \right| &\le Ch \left\{ \int_0^1 (h\sigma)^\gamma d\sigma + \int_1^N (h\sigma)^\gamma \sigma^{-4} d\sigma \right\} \\ &\le C \begin{cases} h^{\gamma+1} & \text{if } \gamma < 3 \\ h^4 \log h^{-1} & \text{if } \gamma = 3 \\ h^4 & \text{if } \gamma > 3 \ . \end{cases} \end{aligned}$$

∎

Analogously to the quadrature (1.8) over the finite interval, we introduce the corresponding rule over the half-axis.

$$\int_0^\infty f(s)ds \sim h{\sum_j}'' f(t_j), \tag{2.6}$$
$$\sum_j{}'' T_j := \frac{127}{48}T_1 - \frac{59}{48}T_2 + \frac{39}{16}T_3 + \frac{31}{48}T_4 + \sum_{j=5}^{\infty} T_j \ .$$

Similarly to Lemma 2.1 we get

LEMMA 2.4 *For any function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ which has an integrable fourth order derivative, we get*

$$\int_0^\infty f(\sigma)d\sigma - {\sum_j}'' f(j) = \int_0^\infty G(\sigma)f^{(4)}(\sigma)d\sigma \ ,$$

where $G : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is bounded and $G(\sigma) = \frac{1}{24}\sigma^4$ if $0 < \sigma < 1$.

2.2 The quadrature method over the half-axis and the stability proof

It is well known that localization principles apply to the stability analysis of numerical methods for Mellin convolution operators (cf. [24, 25, 26]). In other words, the quadrature method (1.10) is stable if and only if the corresponding methods for the "locally equivalent" operators over the "tangent spaces" are stable. Since the kernel is smooth for $t \neq 0$ and $s \neq 0$, the only non-trivial localized method is that over the half-axis which we shall introduce next (cf. (2.7)). If the stability of this method is proved (cf. Theorem 2.5), then the localization technique implies Theorem 1.1. For the details, we refer to [15, 30, 24, 25, 26].

The equation (1.5) over $(0,1)$ is "locally equivalent" at $t = 0$ to the equation $(I_H + K_H)x_H = y_H$ over the half-axis $\mathbb{R}^+$, where I_H is the identity and

$$K_H f(t) := \int_0^\infty k_M\left(\frac{t}{s}\right)\frac{1}{s}f(s)ds, \quad 0 < t < \infty \ .$$

Writing equation $(I_H + K_H)x_H = y_H$ with singularity subtraction, we get

$$(1+\kappa)x_H(t) + \int_0^\infty k_M\left(\frac{t}{s}\right)\frac{1}{s}\left[x_H(s) - x_H(t)\left(\frac{s}{t}\right)^{(q-1)/p}\right]ds = y_H(t), \quad 0 < t < \infty \ ,$$

where

$$\kappa := \int_0^\infty k_M\left(\frac{t}{s}\right)\frac{1}{s}\left(\frac{s}{t}\right)^{(q-1)/p} ds = \int_0^\infty k_M(\sigma)\,\sigma^{-(q-1)/p-1}d\sigma = (\mathcal{M}\tilde{k}_M)(0) \ .$$

The corresponding modified quadrature method for the equation over the axis is defined as follows (cf. (2.6)).

$$(1+\kappa)x_{H,N}(t_i) + \sum_{j>j_*}{}'' k_M\left(\frac{t_i}{t_j}\right)\frac{1}{t_j}\left[x_{H,N}(t_j) - x_{H,N}(t_i)\left(\frac{t_j}{t_i}\right)^{(q-1)/p}\right] = y_H(t_i), \quad i = 1,2,\dots . \tag{2.7}$$

In particular, we set $j_* = 0$ in (2.7) if we consider the quadrature method (1.9) without modification. The operator defined by the left-hand side of (2.7) will be denoted by $A_{H,N}$. In view of (1.11) we consider $A_{H,N}$ in $\mathcal{L}(l^p)$. Having defined $A_{H,N}$, we are in the position to formulate our last technical assumption of Theorems 1.1 and 2.5.

(A8) The null space of $A_{H,1} \in \mathcal{L}(l^p)$ (i.e., of $A_{H,N}$ for $N = 1$) is trivial.

For the validity of (A8), we refer to Remark 1.2.

THEOREM 2.5 *Suppose the assumptions (A1)-(A6) and (A8) are satisfied. Then the quadrature method (2.7) is stable.*

PROOF: The approximate operator $A_{H,N}$ takes the form

$$\begin{aligned}
A_{H,N} &:= Id + \mu_{H,N} Id + K_{H,N} \qquad (2.8)\\
Id\{f(t_i)\}_{i=1}^{\infty} &:= \{f(t_i)\}_{i=1}^{\infty}, \quad \mu_{H,N} Id\{f(t_i)\}_{i=1}^{\infty} := \{\mu_{H,N}(t_i) f(t_i)\}_{i=1}^{\infty},\\
K_{H,N}\{f(t_i)\}_{i=1}^{\infty} &:= \left\{h\sum_{j>j_*}{}'' k_M\left(\frac{t_i}{t_j}\right)\frac{1}{t_j} f(t_j)\right\}_{i=1}^{\infty} = \left\{\sum_{j>j_*}{}'' k_M\left(\frac{i}{j}\right)\frac{1}{j} f(t_j)\right\}_{i=1}^{\infty},\\
\mu_{H,N}(t_i) &:= \kappa - h\sum_{j>j_*}{}'' k_M\left(\frac{t_i}{t_j}\right)\frac{1}{t_j}\left(\frac{t_j}{t_i}\right)^{(q-1)/p} .
\end{aligned}$$

Of course, we have

$$\mu_{H,N}(t_i) = \mu(t_i/h) = \mu(i), \quad \mu(\tau) := \kappa - \sum_{j>j_*}{}'' k_M\left(\frac{\tau}{j}\right)\frac{1}{j}\left(\frac{j}{\tau}\right)^{(q-1)/p} . \tag{2.9}$$

We observe that the matrix operator $A_{H,N}$ is independent of N. Hence, the $A_{H,N}$ are invertible and their inverses are uniformly bounded with respect to N if and only if $A_{H,1}$ is invertible. The null space of this operator is trivial by assumption. To finish our proof it is sufficient to show the next lemma. ∎

LEMMA 2.6 *The operator $A_{H,1}$ is Fredholm and its index is zero.*

PROOF: a) First we shall estimate the function μ for large values of τ. By Lemma 2.4 we get

$$\begin{aligned}
\mu(\tau) &= \kappa - \sum_{j}{}'' k_M\left(\frac{\tau}{j}\right)\frac{1}{j}\left(\frac{j}{\tau}\right)^{(q-1)/p} + \sum_{j\le j_*}{}' k_M\left(\frac{\tau}{j}\right)\frac{1}{j}\left(\frac{j}{\tau}\right)^{(q-1)/p}\\
&= \int_0^{\infty} k_M\left(\frac{\tau}{\sigma}\right)\frac{1}{\sigma}\left(\frac{\sigma}{\tau}\right)^{(q-1)/p} d\sigma - \sum_{j}{}'' k_M\left(\frac{\tau}{j}\right)\frac{1}{j}\left(\frac{j}{\tau}\right)^{(q-1)/p}\\
&\quad + \sum_{j\le j_*}{}' k_M\left(\frac{\tau}{j}\right)\frac{1}{j}\left(\frac{j}{\tau}\right)^{(q-1)/p}\\
&= \int_0^{\infty} G(\sigma)(\partial/\partial\sigma)^4\left\{k_M\left(\frac{\tau}{\sigma}\right)\frac{1}{\sigma}\left(\frac{\sigma}{\tau}\right)^{(q-1)/p}\right\} d\sigma + \sum_{j\le j_*}{}' k_M\left(\frac{\tau}{j}\right)\frac{1}{j}\left(\frac{j}{\tau}\right)^{(q-1)/p}
\end{aligned}$$

Using

$$(\partial/\partial\sigma)^4 \left\{ k_M \left(\frac{\tau}{\sigma}\right) \frac{1}{\sigma} \left(\frac{\sigma}{\tau}\right)^{(q-1)/p} \right\} = (\partial/\partial\sigma)^4 \left\{ q\tilde{k}_M \left(\left(\frac{\tau}{\sigma}\right)^q \right) \frac{1}{\sigma} \right\}$$
$$= \sum_{m=0}^{4} C\tilde{k}_M^{(m)} \left(\left(\frac{\tau}{\sigma}\right)^q \right) \left(\frac{\tau}{\sigma}\right)^{qm} \frac{1}{\sigma}\sigma^{-4}$$

as well as (A2), we arrive at

$$|\mu(\tau)| \leq C\int_0^1 \left(\frac{\tau}{\sigma}\right)^{-\alpha q} \sigma^{-1} d\sigma + C\int_1^\tau \left(\frac{\tau}{\sigma}\right)^{-\alpha q} \sigma^{-5} d\sigma + C\int_\tau^\infty \left(\frac{\tau}{\sigma}\right)^{\alpha q} \sigma^{-5} d\sigma$$
$$+C\tau^{-\alpha q} \leq C\tau^{-min\{\alpha q, 4\}} \begin{cases} \log\tau & \text{if } \alpha q = 4 \\ 1 & \text{else} \end{cases} \quad (2.10)$$

for $\tau \longrightarrow \infty$. This estimate implies that $\mu_{H,N} Id \in \mathcal{L}(l^p)$ is a compact operator and we are left with showing the Fredholmness of $(Id + K_{H,N})$. Moreover, we may suppose $j_* = 0$ since the difference between $K_{H,N}$ for $j_* > 0$ and $K_{H,N}$ for $j_* = 0$ is a finite rank operator.

b) In order to prove that $(Id + K_{H,N})$ is Fredholm, we shall construct a left regularizer. This will be done in such a manner that with the same technique the existence of a left regularizer for $(\lambda Id + K_{H,N})$, $\lambda \notin \{\mathcal{M}\tilde{k}_M(\xi),\ \xi \in I\!R\}$ can be shown. Hence, $(\lambda Id + K_{H,N}) \in \mathcal{L}(l^p)$ is a Φ_+-operator (semi Fredholm operator) for any $\lambda \in \mathbb{C} \setminus \{\mathcal{M}\tilde{k}_M(\xi),\ \xi \in I\!R\}$. Since 1 is contained in the unbounded component of $\mathbb{C} \setminus \{\mathcal{M}\tilde{k}_M(\xi),\ \xi \in I\!R\}$ (cf. assumptions (A3)i) and ii)) and since $(\lambda Id + K_{H,N})$ is invertible for large $|\lambda|$, we conclude that $(Id + K_{H,N})$ is a Fredholm operator with index zero.

To construct a left regularizer of $(Id + K_{H,N})$, we consider the Mellin convolution operator with the resolvent kernel l_M. More precisely, analogously to $(I_H + \tilde{K}_H)^{-1} = (I_H + \tilde{L}_H)$ (cf. Sect.1.2) we get $(I_H + K_H)^{-1} = (I_H + L_H)$ with

$$L_H f(t) := \int_0^\infty l_M\left(\frac{t}{s}\right)\frac{1}{s} f(s) ds, \quad 0 < t < \infty$$
$$l_M(\tau) := q\tilde{l}_M(\tau^q)\sqrt[p]{\tau^{q-1}}\,. \quad (2.11)$$

It is natural to seek the regularizer of $(Id + K_{H,N})$ in the form $(Id + L_{H,N})$ with

$$L_{H,N}\{f(t_i)\}_{i=1}^\infty := \left\{ h{\sum_j}'' l_M\left(\frac{t_i}{t_j}\right)\frac{1}{t_j} f(t_j) \right\}_{i=1}^\infty = \left\{ {\sum_j}'' l_M\left(\frac{i}{j}\right)\frac{1}{j} f(t_j) \right\}_{i=1}^\infty .$$

Indeed, we get $(Id + L_{H,N})(Id + K_{H,N}) = Id + R_{H,N}$ with

$$R_{H,N}\{f(t_i)\}_{i=1}^\infty := \left\{ {\sum_j}'' r_{i,j} f(t_j) \right\}_{i=1}^\infty ,$$
$$r_{i,j} := hl_M\left(\frac{t_i}{t_j}\right)\frac{1}{t_j} + hk_M\left(\frac{t_i}{t_j}\right)\frac{1}{t_j} + h^2{\sum_m}'' l_M\left(\frac{t_i}{t_m}\right)\frac{1}{t_m} k_M\left(\frac{t_m}{t_j}\right)\frac{1}{t_j} .$$

It remains to show that $R_{H,N} \in \mathcal{L}(l^p)$ is compact. From $(I_H + L_H)(I_H + K_H) = I_H$ we conclude $L_H + K_H + L_H K_H = 0$, i.e.,

$$l_M\left(\frac{t}{s}\right)\frac{1}{s}+k_M\left(\frac{t}{s}\right)\frac{1}{s}+\int_0^\infty l_M\left(\frac{t}{u}\right)\frac{1}{u}k_M\left(\frac{u}{s}\right)\frac{1}{s}du=0.$$

Consequently, we obtain

$$\begin{aligned} r_{i,j} &:= -h\left\{\int_0^\infty l_M\left(\frac{t_i}{u}\right)\frac{1}{u}k_M\left(\frac{u}{t_j}\right)du - h{\sum_m}'' l_M\left(\frac{t_i}{t_m}\right)\frac{1}{t_m}k_M\left(\frac{t_m}{t_j}\right)\right\}\frac{1}{t_j} \\ &= -\left\{\int_0^\infty l_M\left(\frac{i}{\sigma}\right)\frac{1}{\sigma}k_M\left(\frac{\sigma}{j}\right)d\sigma - {\sum_m}'' l_M\left(\frac{i}{m}\right)\frac{1}{m}k_M\left(\frac{m}{j}\right)\right\}\frac{1}{j}\,. \end{aligned}$$

In view of Lemma 2.4, we arrive at

$$r_{i,j}=-\int_0^\infty G(\sigma)(\partial/\partial\sigma)^4\left\{l_M\left(\frac{i}{\sigma}\right)\frac{1}{\sigma}k_M\left(\frac{\sigma}{j}\right)\right\}d\sigma\frac{1}{j}\,,$$

where (cf. (1.7) and (2.11))

$$\begin{aligned} &(\partial/\partial\sigma)^4\left\{\ l_M\left(\frac{i}{\sigma}\right)\ \ \frac{1}{\sigma}\ \ k_M\left(\frac{\sigma}{j}\right)\ \right\} \\ &\quad= (\partial/\partial\sigma)^4\left\{q\tilde{l}_M\left(\left(\frac{i}{\sigma}\right)^q\right)\left(\frac{i}{\sigma}\right)^{(q-1)/p}\frac{1}{\sigma}q\tilde{k}_M\left(\left(\frac{\sigma}{j}\right)^q\right)\left(\frac{\sigma}{j}\right)^{(q-1)/p}\right\} \\ &\quad= q^2 i^{(q-1)/p}(\partial/\partial\sigma)^4\left\{\tilde{l}_M\left(\left(\frac{i}{\sigma}\right)^q\right)\frac{1}{\sigma}\tilde{k}_M\left(\left(\frac{\sigma}{j}\right)^q\right)\right\}j^{-(q-1)/p} \\ &\quad= q^2 i^{(q-1)/p}\left\{\sum_{m_1+m_2+m_3=4} C\tilde{l}_M^{(m_1)}\left(\left(\frac{i}{\sigma}\right)^q\right)\left(\frac{i}{\sigma}\right)^{qm_1}\sigma^{-m_1}\frac{1}{\sigma}\right. \\ &\qquad\qquad \left.\cdot\,\sigma^{-m_2}\tilde{k}_M^{(m_3)}\left(\left(\frac{\sigma}{j}\right)^q\right)\left(\frac{\sigma}{j}\right)^{qm_3}\sigma^{-m_3}\right\}j^{-(q-1)/p} \\ &\quad= q^2 i^{(q-1)/p}\left\{\sum_{m_1+m_2+m_3=4} C\tilde{l}_M^{(m_1)}\left(\left(\frac{i}{\sigma}\right)^q\right)\left(\frac{i}{\sigma}\right)^{qm_1}\frac{1}{\sigma}\right. \\ &\qquad\qquad \left.\cdot\,\tilde{k}_M^{(m_3)}\left(\left(\frac{\sigma}{j}\right)^q\right)\left(\frac{\sigma}{j}\right)^{qm_3}\sigma^{-4}\right\}j^{-(q-1)/p}\,. \end{aligned}$$

It remains to apply the properties of G in Lemma 2.4 and the assumptions (A2), (A4). For $i \ge j$, this yields

$$\begin{aligned} |r_{i,j}| &\le Ci^{(q-1)/p}j^{-(q-1)/p-1}\left\{\int_0^1\left(\frac{i}{\sigma}\right)^{-q\gamma}\frac{1}{\sigma}\left(\frac{\sigma}{j}\right)^{q\alpha}d\sigma+\int_1^j\left(\frac{i}{\sigma}\right)^{-q\gamma}\frac{1}{\sigma}\left(\frac{\sigma}{j}\right)^{q\alpha}\sigma^{-4}d\sigma+\right. \\ &\qquad \left.\int_j^i\left(\frac{i}{\sigma}\right)^{-q\gamma}\frac{1}{\sigma}\left(\frac{\sigma}{j}\right)^{-q\alpha}\sigma^{-4}d\sigma+\int_i^\infty\left(\frac{i}{\sigma}\right)^{q\gamma}\frac{1}{\sigma}\left(\frac{\sigma}{j}\right)^{-q\alpha}\sigma^{-4}d\sigma\right\} \\ &\le Ci^{-q\gamma+(q-1)/p}j^{-q\alpha-(q-1)/p-1} \\ &\quad +Ci^{-q\gamma+(q-1)/p}j^{q\gamma-(q-1)/p-5}\begin{cases}\log j & \text{if } q(\gamma\pm\alpha)=4\\ 1 & \text{else}\end{cases} \end{aligned}$$

$$+Ci^{-q\alpha+(q-1)/p-4}j^{q\alpha-(q-1)/p-1}\begin{cases}\log i & \text{if } q(\gamma-\alpha)=4\\ 1 & \text{else .}\end{cases} \tag{2.12}$$

For $i \le j$, we arrive at

$$\begin{aligned}|r_{i,j}| &\le Ci^{(q-1)/p}j^{-(q-1)/p-1}\left\{\int_0^1\left(\frac{i}{\sigma}\right)^{-q\gamma}\frac{1}{\sigma}\left(\frac{\sigma}{j}\right)^{q\alpha}d\sigma+\int_1^i\left(\frac{i}{\sigma}\right)^{-q\gamma}\frac{1}{\sigma}\left(\frac{\sigma}{j}\right)^{q\alpha}\sigma^{-4}d\sigma+\right.\\ &\qquad\left.\int_i^j\left(\frac{i}{\sigma}\right)^{q\gamma}\frac{1}{\sigma}\left(\frac{\sigma}{j}\right)^{q\alpha}\sigma^{-4}d\sigma+\int_j^\infty\left(\frac{i}{\sigma}\right)^{q\gamma}\frac{1}{\sigma}\left(\frac{\sigma}{j}\right)^{-q\alpha}\sigma^{-4}d\sigma\right\}\\ &\le Ci^{-q\gamma+(q-1)/p}j^{-q\alpha-(q-1)/p-1}+\\ &\quad Ci^{q\alpha-4+(q-1)/p}j^{-q\alpha-(q-1)/p-1}\begin{cases}\log i & \text{if } q(\alpha\pm\gamma)=4\\ 1 & \text{else}\end{cases}+\\ &\quad Ci^{q\gamma+(q-1)/p}j^{-q\gamma-(q-1)/p-5}\begin{cases}\log j & \text{if } q(\alpha-\gamma)=4\\ 1 & \text{else .}\end{cases}\end{aligned} \tag{2.13}$$

From these estimates and the inequalities $1/p < \gamma$ and $1/p - 4/q < \alpha$ (cf. assumptions (A2) and (A4)) we conclude $\{\sum_{i=1}^\infty[\sum_{j=1}^\infty |r_{i,j}|^{p'}]^{p/p'}\}^{1/p} < \infty$. Hence, $R_{H,N}$ is compact.
∎

2.3 Special solutions of the quadrature equation

For the derivation of the asymptotic expansion in Theorem 1.3, we need the following property of the quadrature method.

THEOREM 2.7 *Suppose we are given a real number $\beta > 1/p$, a real $\omega \ge 0$, and a function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that $|g(\tau)| \le C\tau^{-\beta}(\log\tau)^\omega$ for $\tau \longrightarrow \infty$. Let us consider the right-hand side $y(t) := y_N(t) := g(t/h)$, $0 < t < 1$ and let $\{x_N(t_i)\}_{i=1}^N$ stand for the solution of (1.10), i.e., of $A_N\{x_N(t_i)\}_{i=1}^N = \{y_N(t_i)\}_{i=1}^N := \{g(i)\}_{i=1}^N$. Then there exists a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that*

$$\begin{aligned}|f(i)| &\le Ci^{-\varrho_1}(\log i)^{\varrho_2},\ i\longrightarrow\infty\ , & (2.14)\\ \varrho_1 &:= min\{q(\alpha-1/p)+1/p,\ q(\gamma-1/p)+1/p,\ \beta,\ 4+1/p\}\ ,\\ \varrho_2 &:= \begin{cases}\varrho_2' & \text{if } \beta\le q(\alpha-1/p)+1/p\\ \varrho_2'' & \text{if } \beta> q(\alpha-1/p)+1/p\ ,\end{cases}\\ \|\{x_N(t_i)\}_{i=1}^N &- \{f(t_i/h)\}_{i=1}^N\|_{l^p}\le Ch^{\varrho_3}(\log h^{-1})^{\varrho_4} & (2.15)\\ \varrho_3 &:= min\{q(\gamma-1/p),\ q(\alpha-1/p),\ \beta-1/p,\ 4,\ q(1-1/p)\}\ ,\end{aligned}$$

where the integers ϱ_2', ϱ_2'', and ϱ_4 are defined as in the following proof.

PROOF: a) Consider $y_N(t) := g(t/h)$. First we shall show that there is a function f such that the solution $\{x_N(t_i)\}_{i=1}^\infty$ of $A_{H,N}\{x_N(t_i)\}_{i=1}^\infty = \{y_N(t_i)\}_{i=1}^\infty$ satisfies $\{x_N(t_i)\}_{i=1}^\infty = \{f(t_i/h)\}_{i=1}^\infty = \{f(i)\}_{i=1}^\infty$ and that f satisfies (2.14). Since the matrix of the operator $A_{H,N}$ is independent of N and the right-hand side $\{y_N(t_i)\}_{i=1}^\infty := \{g(i)\}_{i=1}^\infty$ is independent of N too, it is clear that $\{x_N(t_i)\}_{i=1}^\infty$ takes the form $\{x_N(t_i)\}_{i=1}^\infty = \{f(i)\}_{i=1}^\infty$. It remains to prove (2.14). Let us start with the special case $j_* = 0$. Using the notation of the proofs to Theorem 2.5 and Lemma 2.6, we get

$$
\begin{aligned}
(Id + \mu_{H,N} Id + K_{H,N}) \{f(i)\}_{i=1}^\infty &= \{g(i)\}_{i=1}^\infty, \\
(Id + K_{H,N}) \{f(i)\}_{i=1}^\infty &= \{g(i) - \mu(i) f(i)\}_{i=1}^\infty, \\
(Id + R_{H,N}) \{f(i)\}_{i=1}^\infty &= (Id + L_{H,N}) \{g(i) - \mu(i) f(i)\}_{i=1}^\infty, \\
\{f(i)\}_{i=1}^\infty &= \{g(i) - \mu(i) f(i)\}_{i=1}^\infty - R_{H,N}\{f(i)\}_{i=1}^\infty + \\
& \quad L_{H,N}\{g(i) - \mu(i) f(i)\}_{i=1}^\infty, \\
\{[1+\mu(i)] f(i)\}_{i=1}^\infty &= \{g(i)\}_{i=1}^\infty + L_{H,N}\{g(i)\}_{i=1}^\infty - R_{H,N}\{f(i)\}_{i=1}^\infty \\
& \quad - L_{H,N}\{\mu(i) f(i)\}_{i=1}^\infty \, . \qquad (2.16)
\end{aligned}
$$

Using (A4), for the i-th component $[L_{H,N}\{g(j)\}_{j=1}^\infty]_i$ of the sequence $L_{H,N}\{g(j)\}_{j=1}^\infty$, we get

$$
\begin{aligned}
[L_{H,N}\{g(j)\}_{j=1}^\infty]_i &= \sum_j{}'' l_M\left(\frac{i}{j}\right) \frac{1}{j} g(j) \\
\left|[L_{H,N}\{g(j)\}_{j=1}^\infty]_i\right| &\le C \sum_{j:j\le i} \left(\frac{i}{j}\right)^{-q\gamma+(q-1)/p} j^{-\beta-1} (\log\, j)^\omega + \\
& \quad C \sum_{j:j>i} \left(\frac{i}{j}\right)^{q\gamma+(q-1)/p} j^{-\beta-1} (\log\, j)^\omega \\
&\le C \begin{cases} i^{-\beta}(\log\, i)^\omega & \text{if } \; q(\gamma-1/p)+1/p > \beta \\ i^{-\beta}(\log\, i)^{1+\omega} & \text{if } \; q(\gamma-1/p)+1/p = \beta \\ i^{-q(\gamma-1/p)-1/p}(\log\, i)^\omega & \text{if } \; q(\gamma-1/p)+1/p < \beta. \end{cases} \qquad (2.17)
\end{aligned}
$$

Furthermore, by the Cauchy-Schwarz inequality, by (A4), and by (2.10) we conclude

$$
\begin{aligned}
\left|[L_{H,N}\{\mu(j) f(j)\}_{j=1}^\infty]_i\right| &\le C \sum_{j:j\le i} \left(\frac{i}{j}\right)^{-q\gamma+(q-1)/p} f(j)\, j^{-min\{q\alpha,4\}-1} \begin{cases} \log\, j & \text{if } \; q\alpha = 4 \\ 1 & \text{if } \; q\alpha \ne 4 \end{cases} \\
&+ C \sum_{j:j>i} \left(\frac{i}{j}\right)^{q\gamma+(q-1)/p} f(j)\, j^{-min\{q\alpha,4\}-1} \begin{cases} \log\, j & \text{if } \; q\alpha = 4 \\ 1 & \text{if } \; q\alpha \ne 4 \end{cases} \\
&\le C \|\{f(j)\}_{j=1}^\infty\|_{l^p} \cdot \qquad (2.18)
\end{aligned}
$$

$$
\begin{aligned}
\cdot \Bigg\{ & i^{-q\gamma+(q-1)/p} \left[\sum_{j:j\le i} j^{[q\gamma-(q-1)/p-1-min\{q\alpha,4\}]p'} \begin{cases} (\log\, j)^{p'} & \text{if } \; q\alpha = 4 \\ 1 & \text{if } \; q\alpha \ne 4 \end{cases} \right]^{1/p'} \\
& + i^{q\gamma+(q-1)/p} \left[\sum_{j:j>i} j^{[-q\gamma-(q-1)/p-1-min\{q\alpha,4\}]p'} \begin{cases} (\log\, j)^{p'} & \text{if } \; q\alpha = 4 \\ 1 & \text{if } \; q\alpha \ne 4 \end{cases} \right]^{1/p'} \Bigg\} \\
& \le C i^{-\varrho_9} (\log\, i)^{\varrho_{10}} \, ,
\end{aligned}
$$

$$
\varrho_{10} := \begin{cases} 1 & \text{if } \; q\alpha = 4 \;\text{ and }\; q(\gamma-1/p) > 4 \\ 1/p' & \text{if } \; q\alpha \ne 4 \;\text{ and }\; q(\gamma-1/p) = min\{q\alpha, 4\} \\ 1+1/p' & \text{if } \; q\alpha = 4 = q(\gamma-1/p) \\ 0 & \text{else} \, , \end{cases}
$$

$$
\varrho_9 := min\{q\alpha + 1/p,\; q(\gamma-1/p)+1/p,\; 4+1/p\} \, .
$$

Using (2.12),(2.13) and $p > 1/\alpha$ from (A2) as well as $p > 1/\gamma$ from (A4), we conclude

$$\begin{aligned}
\left|[R_{H,N}\{f(j)\}_{j=1}^{\infty}]_i\right| &\leq C i^{(q-1)/p-q\gamma}\left\{\sum_j j^{[-(q-1)/p-q\alpha-1]p'}\right\}^{1/p'} \\
&+C\begin{cases} \log i & \text{if } \ q(\gamma-\alpha)=4 \\ 1 & \text{else}\end{cases} \quad i^{(q-1)/p-q\alpha-4}\{\sum_{j:j\leq i} j^{[-(q-1)/p+q\alpha-1]p'}\}^{1/p'} \\
&+C\begin{cases} \log i & \text{if } \ q(\alpha\pm\gamma)=4 \\ 1 & \text{else}\end{cases} \quad i^{(q-1)/p+q\alpha-4}\{\sum_{j:j>i} j^{[-(q-1)/p-q\alpha-1]p'}\}^{1/p'} \\
&+C i^{(q-1)/p-q\gamma}\begin{cases} \log i & \text{if } \ q(\gamma\pm\alpha)=4 \\ 1 & \text{else}\end{cases} \quad \{\sum_{j:j\leq i} j^{[-(q-1)/p+q\gamma-5]p'}\}^{1/p'} \\
&+C i^{(q-1)/p+q\gamma}\left\{\sum_{j:j>i} j^{[-(q-1)/p-q\gamma-5]p'}\begin{cases} (\log i)^{p'} & \text{if } \ q(\alpha-\gamma)=4 \\ 1 & \text{else}\end{cases}\right\}^{1/p'} \\
&\leq C i^{-min\{4,q(\gamma-1/p)\}-1/p}\begin{cases} 1 & \text{if } \ q(\gamma-1/p)\neq 4 \ \text{ and } \ \pm q(\gamma\pm\alpha)\neq 4 \\ (\log i)^{1/p'} & \text{if } \ q(\gamma-1/p)=4 \ \text{ and } \ \pm q(\gamma\pm\alpha)\neq 4 \\ \log i & \text{else .}\end{cases}
\end{aligned} \tag{2.19}$$

Let us set $\varrho_5 := min\{q\alpha, 4\}$ and $\varrho_6 := 1$ if $q\alpha = 4$ and $\varrho_6 := 0$ if $q\alpha \neq 4$. Then (2.10) yields $|\mu(i)| \leq C i^{-\varrho_5}(\log i)^{\varrho_6}$. Applying the assumptions on the right-hand side $\{g(i)\}_{i=1}^{\infty}$ and (2.16)-(2.19), we get that

$$\begin{aligned}
|f(i)| &\leq C i^{-\varrho_7}(\log i)^{\varrho_8} + C i^{-\varrho_9}(\log i)^{\varrho_{10}} , \qquad (2.20)\\
\varrho_7 &:= min\{\beta,\, q(\gamma-1/p)+1/p,\, 4+1/p\} , \\
\varrho_8 &:= \begin{cases}
0 & \text{if } \ 4+1/p < min\{q(\gamma-1/p)+1/p,\, \beta\} \\ & \quad \text{and } \ \pm q(\alpha\pm\gamma)\neq 4 \\
1 & \text{if } \ 4+1/p < min\{q(\gamma-1/p)+1/p,\, \beta\} \\ & \quad \text{and } \ q(\alpha-\gamma)=4 \ \text{ or } \ q(\gamma-\alpha)=4 \\
\omega & \text{if } \ \beta < min\{4+1/p,\, q(\gamma-1/p)+1/p\} \\
\omega & \text{if } \ q(\gamma-1/p)+1/p < min\{\beta,\, 4+1/p\} \\ & \quad \text{and } \ \pm q(\alpha\pm\gamma)\neq 4 \\
\omega & \text{if } \ 4 < q(\gamma-1/p) \ , \ \beta = 4+1/p \ , \\ & \quad \text{and } \ \pm q(\alpha\pm\gamma)\neq 4 \\
max\{\omega,\, 1/p'\} & \text{if } \ 4+1/p = q(\gamma-1/p)+1/p < \beta \\ & \quad \text{and } \ \pm q(\alpha\pm\gamma)\neq 4 \\
max\{\omega,\, 1\} & \text{if } \ \beta > q(\gamma-1/p)+1/p \ , \ 4 \geq q(\gamma-1/p) \\ & \quad \text{and } \ q(\alpha-\gamma)=4 \ \text{ or } \ q(\alpha+\gamma)=4 \\
max\{\omega,\, 1\} & \text{if } \ 4 < q(\gamma-1/p) \ , \ \beta = 4+1/p \ , \ \text{and} \\ & \quad q(\alpha-\gamma)=4 \ \text{ or } \ q(\alpha+\gamma)=4 \\
\omega+1 & \text{if } \ q(\gamma-1/p)+1/p = \beta \leq 4+1/p
\end{cases}
\end{aligned}$$

This proves the estimate (2.14) for $j_* = 0$ and for ϱ_2 replaced by

$$\varrho_2' := \begin{cases} \varrho_8 & \text{if } \ \beta < \varrho_9 \\ \varrho_{10} & \text{if } \ q\alpha + 1/p < \varrho_7 \\ max\{\varrho_8,\, \varrho_{10}\} & \text{else .}\end{cases}$$

To treat the case $j_* > 0$, we express the dependence on j_* in the notation $K_{H,N} =: K^{j_*}_{H,N}$. We get

$$\begin{aligned}
\left(Id + \mu_{H,N} Id + K^{j_*}_{H,N}\right) \{f(i)\}_{i=1}^\infty &= \{g(i)\}_{i=1}^\infty , \\
\left(Id + \mu_{H,N} Id + K^{0}_{H,N}\right) \{f(i)\}_{i=1}^\infty &= \{\tilde g(i)\}_{i=1}^\infty , \qquad (2.21)\\
\{\tilde g(i)\}_{i=1}^\infty &:= \{g(i)\}_{i=1}^\infty + (K^0_{H,N} - K^{j_*}_{H,N})\{f(i)\}_{i=1}^\infty .
\end{aligned}$$

Note that $\mu_{H,N}$ depends also on j_*. However, μ fulfills (2.10) for any choice of j_* and, therefore, (2.14) with ϱ_2 replaced by ϱ_2' holds for $\{f(i)\}_{i=1}^\infty = (Id + \mu_{H,N} Id + K^0_{H,N})^{-1}\{g(i)\}_{i=1}^\infty$ including $\mu_{H,N}$ with $j_* > 0$. Using the definition of $K_{H,N}$ and (A2), we get the estimate

$$\begin{aligned}
[(K^0_{H,N} - K^{j_*}_{H,N})\{f(j)\}_{j=1}^\infty]_i &= \sum_{j\le j_*}{}' k_M\left(\frac{i}{j}\right)\frac{1}{j} f(j) , \\
|[(K^0_{H,N} - K^{j_*}_{H,N})\{f(j)\}_{j=1}^\infty]_i| &\le C\|\{f(j)\}_{j=1}^\infty\|_{l^p} \sup_{j\le j_*} |k_M\left(\frac{i}{j}\right)\frac{1}{j}| \\
&\le C\|\{f(j)\}_{j=1}^\infty\|_{l^p}\, i^{-q(\alpha-1/p)-1/p} , \\
|\tilde g(i)| &\le C i^{-min\{\beta, q(\alpha-1/p)+1/p\}} (\log\, i)^{\varrho_*} , \\
\varrho_* &:= \begin{cases} 0 & \text{if } q(\alpha-1/p)+1/p < \beta \\ \omega & \text{else .}\end{cases}
\end{aligned}$$

If $q(\alpha-1/p)+1/p \ge \beta$, then $\tilde g$ satisfies the same estimate as g and the just proved estimate (2.14) with ϱ_2 replaced by ϱ_2' applies to the solution of (2.21). This yields (2.14) for $q(\alpha-1/p)+1/p \ge \beta$ and $j_* > 0$. If $q(\alpha-1/p)+1/p < \beta$, then $\tilde g$ satisfies a similar estimate as g, where β is to be replaced by $\beta'' := q(\alpha-1/p)+1/p$ and ω by $\omega'' := 0$. Let us define ϱ_8'' analogously to ϱ_8 but with β'', ω'' instead of β, ω, i.e., we set

$$\varrho_8'' := \begin{cases}
0 & \text{if } 4 < q(\gamma-1/p)\ , 4 \le q(\alpha-1/p)\ , \\
 & \quad \text{and } \pm\, q(\alpha-\gamma) \ne 4 \\
0 & \text{if } q(\alpha-1/p) < min\{4,\, q(\gamma-1/p)\} \\
0 & \text{if } q(\gamma-1/p) < min\{q(\alpha-1/p),\, 4\} \text{ and} \\
 & \quad q(\alpha\pm\gamma) \ne 4 \\
1/p' & \text{if } 4 = q(\gamma-1/p) < q(\alpha-1/p) \\
 & \quad \text{and } q(\alpha-\gamma) \ne 4\ , \\
1 & \text{if } 4 < q(\gamma-1/p)\ ,\ 4 \le q(\alpha-1/p)\ ,\ \text{and} \\
 & \quad q(\alpha-\gamma) = 4 \text{ or } q(\gamma-\alpha) = 4 \\
1 & \text{if } \alpha > \gamma\ ,\ 4 \ge q(\gamma-1/p)\ ,\ \text{and} \\
 & \quad q(\alpha-\gamma) = 4 \text{ or } q(\alpha+\gamma) = 4 \\
1 & \text{if } 4 \ge q(\gamma-1/p) = q(\alpha-1/p)\ .
\end{cases}$$

Using this ϱ_8'' instead of ϱ_8 and $\varrho_7'' := min\{\beta'',\, q(\gamma-1/p)+1/p,\, 4+1/p\}$ instead of ϱ_7, we define ϱ_2'' analogously to ϱ_2'.

$$\varrho_2'' := \begin{cases}
\varrho_8'' & \text{if } \beta'' < \varrho_9 \\
\varrho_{10} & \text{if } q\alpha + 1/p < \varrho_7'' \\
max\{\varrho_8'',\, \varrho_{10}\} & \text{else}
\end{cases}$$

$$= \begin{cases} \varrho_8'' & \text{if } q(\alpha - 1/p) < min\{4,\, q(\gamma - 1/p)\} \\ max\{\varrho_8'',\, \varrho_{10}\} & \text{if } q(\alpha - 1/p) \geq min\{4,\, q(\gamma - 1/p)\}\ . \end{cases}$$

Now, for the solution of (2.21), the estimate (2.14) holds with ϱ_2, β replaced by ϱ_2'', β''. This yields (2.14) for $q(\alpha - 1/p) + 1/p < \beta$ and $j_* > 0$.

b) Let us suppose that $k_S \equiv 0$. We observe that the equation $A_N\{x_N(t_i)\}_{i=1}^N = \{g(i)\}_{i=1}^N$ is just the finite section of $A_{H,N}\{f(i)\}_{i=1}^\infty = \{g(i)\}_{i=1}^\infty$. Let Π_N denote the restriction operator

$$\Pi_N\{h(i)\}_{i=1}^\infty = \{\tilde{h}_N(i)\}_{i=1}^\infty, \ \ \tilde{h}_N(i) := \begin{cases} h(i) & \text{if } i \leq N \\ 0 & \text{else}\ . \end{cases}$$

Since our finite section method is stable (cf. Theorem 1.1), we get

$$\|\{x_N(t_i)\}_{i=1}^N - \{f(i)\}_{i=1}^N\|_{l^p} \leq C\|(I - \Pi_N)\{f(i)\}_{i=1}^\infty\|_{l^p}.$$

This and the estimate (2.14) together with $h = N^{-1}$ yield (2.15) with ϱ_4 replaced by ϱ_2.

c) Now consider $k_S \not\equiv 0$, set

$$T_N\{f(i)\}_{i=1}^N = \left\{ h \sum_{j>j_*}{}' k_S(t_i, t_j) f(t_j) \right\}_{i=1}^N, \quad I_N\{f(i)\}_{i=1}^N = \{f(i)\}_{i=1}^N\ ,$$

and let A_N^0 denote the approximate operator on the left-hand side of (1.10) for $k_S \equiv 0$. Then $A_N = A_N^0 + T_N$ and $A_N^{-1} = (A_N^0)^{-1} - A_N^{-1}T_N(A_N^0)^{-1}$. To estimate $A_N^{-1}\{y_N(t_i)\}_{i=1}^N$, we first apply T_N to the solution $\{f_N(t_i)\}_{i=1}^N$ of $A_N^0\{f_N(t_i)\}_{i=1}^N = \{y_N(t_i)\}_{i=1}^N$ which satisfies (2.15) with ϱ_4 replaced by ϱ_2 (cf. part b) of the present proof). Moreover, since $\|A_N^{-1}T_N\|$ is bounded and since (2.15) is true for $\{f_N(t_i)\}_{i=1}^N$, without loss of generality we may assume that $f_N(t_i) = f(t_i/h)$ and that f satisfies the estimate (2.14). We obtain

$$\begin{aligned} [T_N\{f(j)\}_{j=1}^N]_i &= h \sum_{j>j_*}{}' k_S(t_i, t_j) f(j)\ , \\ |[T_N\{f(j)\}_{j=1}^N]_i| &\leq C h t_i^{(q-1)/p} \sum_{j=j_*+1}^N t_j^{(q-1)/p'} j^{-\varrho_1} (\log\, j)^{\varrho_2}\ , \\ &\leq C i^{(q-1)/p} h^{min\{\varrho_1 + (q-1)/p,\, q\}} (\log\, h^{-1})^{\varrho_4}\ , \\ \varrho_4 &:= \begin{cases} \varrho_2 + 1 & \text{if } \varrho_1 = (q-1)/p' + 1 \\ \varrho_2 & \text{else}\ . \end{cases} \end{aligned}$$

Hence, we get

$$\|A_N^{-1}T_N\{f(j)\}_{j=1}^N\|_{l^p} \leq C\|T_N\{f(j)\}_{j=1}^N\|_{l^p} \ \leq \ C h^{min\{\varrho_1 - 1/p,\, q(1-1/p)\}} (\log\, h^{-1})^{\varrho_4}\ .$$

In other words, $A_N^{-1}\{y_N(t_j)\}_{j=1}^N = (A_N^0)^{-1}\{y_N(t_j)\}_{j=1}^N + A_N^{-1}T_N(A_N^0)^{-1}\{y_N(t_j)\}_{j=1}^N$, where $(A_N^0)^{-1}\{y_N(t_j)\}_{j=1}^N$ satisfies (2.15) with ϱ_4 replaced by ϱ_2 (cf. part b) of the present proof) and $A_N^{-1}T_N(A_N^0)^{-1}\{y_N(t_j)\}_{j=1}^N$ is bounded by the right-hand side of (2.15) plus $C h^{min\{\varrho_1 - 1/p,\, q(1-1/p)\}} (\log\, h^{-1})^{\varrho_4}$. This proves (2.15) for $k_S \not\equiv 0$. ∎

3 PROOF OF THE ASYMPTOTIC ERROR EXPANSION

3.1 Structure of the approximate operator and the corresponding splitting of the error

Let us introduce the restriction operator $R_N y := \{y(t_i)\}_{i=1}^N$. Then the error $e_N := \{x(t_i) - x_N(t_i)\}_{i=1}^N$ is equal to

$$e_N = R_N x - A_N^{-1} R_N y = A_N^{-1}[A_N R_N x - R_N A x]. \tag{3.1}$$

Since we have defined A_N by singularity subtraction, we get $A_N R_N x_0 - R_N A x_0 = 0$ for $x_0(t) := \sqrt[p]{q t^{q-1}}$. In other words, without loss of generality we may suppose that the exact solution of $Ax = y$ admits the asymptotic expansion (cf. (A6) and (1.6))

$$x(t) = C t^{q\gamma + (q-1)/p} + O(t^{q\gamma_1 + (q-1)/p}) , \quad t \longrightarrow 0 \tag{3.2}$$

Analogously to (2.8) the approximate operator A_N takes the form

$$A_N = I_N + \mu_N I_N + K_N + T_N, \tag{3.3}$$

where

$$\begin{aligned}
I_N\{f(t_i)\}_{i=1}^N &:= \{f(t_i)\}_{i=1}^N , \\
\mu_N I_N\{f(t_i)\}_{i=1}^N &:= \{\mu_N(t_i) f(t_i)\}_{i=1}^N , \\
K_N\{f(t_i)\}_{i=1}^N &:= \left\{ h \sum_{j>j_*}{}' k_M\left(\frac{t_i}{t_j}\right) \frac{1}{t_j} f(t_j) \right\}_{i=1}^N , \\
T_N\{f(t_i)\}_{i=1}^N &:= \left\{ h \sum_{j>j_*}{}' k_S(t_i, t_j) f(t_j) \right\}_{i=1}^N , \\
\mu_N(t_i) &:= \int_0^1 k(t_i, s) \left(\frac{s}{t_i}\right)^{(q-1)/p} ds - h \sum_{j>j_*}{}' k(t_i, t_j) \left(\frac{t_j}{t_i}\right)^{(q-1)/p} .
\end{aligned}$$

Clearly, the operators K_N and T_N are approximate operators for the operators K and T defined by

$$K f(t) := \int_0^1 k_M\left(\frac{t}{s}\right) \frac{1}{s} f(s) ds \quad \text{and} \quad T f(t) := \int_0^1 k_S(t, s) f(s) ds ,$$

respectively.

In accordance to (3.3), we arrive at the following splitting of the error e_N:

$$\begin{aligned}
e_N &:= A_N^{-1}[T_3 - T_1 - T_2] , \\
T_1 &:= \{(R_N K - K_N R_N) x(t_i)\}_{i=1}^N , \quad T_2 := \{(R_N T - T_N R_N) x(t_i)\}_{i=1}^N , \\
T_3 &:= \{\mu_N I_N x(t_i)\}_{i=1}^N .
\end{aligned} \tag{3.4}$$

3.2 Consistency estimates

To estimate the consistency error $(A_N R_N - R_N A)x$ we have to consider the terms T_j, $j = 1, 2, 3$.

LEMMA 3.1 *i) There exists a function* $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ *such that*

$$\begin{aligned} |f(i)| &\le Ci^{-\varrho_{12}} \begin{cases} \log i & \text{if } q(\alpha+\gamma)=4 \\ 1 & \text{else ,} \end{cases} \\ \varrho_{12} &:= min\{q(\alpha-1/p)+1/p,\, 4-q(\gamma+1/p)+1/p\} \,, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \|T_1 &- h^{q(\gamma+1/p)-1/p}\{f(i)\}_{i=1}^N\|_{l^p} \le Ch^{\varrho_{13}}(\log\, h^{-1})^{\varrho_{14}} \,, \\ \varrho_{13} &:= min\{q(\gamma_1+1/p)-1/p,\, 4-1/p\} \,, \\ \varrho_{14} &:= \begin{cases} 1 & \text{if } q(\gamma_1-\alpha)=4 \\ 1 & \text{if } q(\gamma-\alpha)=4 \\ 1/p & \text{if } q(\gamma_1+1/p)=4 \\ 0 & \text{else .} \end{cases} \end{aligned} \tag{3.6}$$

The function f *is equal to zero if* $q(\gamma-\alpha) \ge 4$.

ii) There holds

$$\|T_2\|_{l^p} \le Ch^{min\{4-1/p,q(\gamma+1)-1/p\}} \begin{cases} \log\, h^{-1} & \text{if } q(\gamma+1)=4 \\ 1 & \text{else .} \end{cases} \tag{3.7}$$

iii) There exists a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ *such that*

$$\begin{aligned} |f(i)| &\le Ci^{-\varrho_{15}} \begin{cases} \log i & \text{if } q\alpha=4 \\ 1 & \text{else} \end{cases} \,, \\ \varrho_{15} &:= min\{q(\alpha-\gamma-1/p)+1/p,\, 4-q(\gamma+1/p)+1/p\} \,, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \|T_3 &- h^{q(\gamma+1/p)-1/p}\{f(i)\}_{i=1}^N\|_{l^p} \le Ch^{\varrho_{16}}(\log\, h^{-1})^{\varrho_{17}} \,, \\ \varrho_{16} &:= min\{q-1/p,\, q\alpha-1/p, 4-1/p,\, q(\gamma_1+1/p)-1/p\} \,, \\ \varrho_{17} &:= \begin{cases} 1 & \text{if } q\alpha=4 \text{ and } \gamma_1 \ne \alpha-1/p \\ 1/p & \text{if } min\{\alpha, 4/q\}-1/p=\gamma_1 \text{ and } q\alpha \ne 4 \\ 1+1/p & \text{if } \alpha-1/p=\gamma_1 \text{ and } q\alpha=4 \\ 0 & \text{else .} \end{cases} \end{aligned} \tag{3.9}$$

PROOF OF LEMMA 3.1 i): Set $\sum_j{}'''g(j) := \sum_j{}''g(j) - \sum_j{}'g(j)$ and $x(t) = x_+(t) + Ct^{q(\gamma+1/p)-1/p}$ with $|x_+(t)| \le Ct^{q(\gamma_1+1/p)-1/p}$ (cf.(3.2)). Then we get $T_1 = T_{11} + T_{12} - T_{13} - T_{14}$, where $T_{1j} := \{T_{1j,i}\}_{i=1}^N$, $l = 1, 2, 3$ and

$$T_{11,i} := \int_0^\infty k_M\left(\frac{t_i}{s}\right)\frac{1}{s}Cs^{q(\gamma+1/p)-1/p}ds - h\sum_j{}''k_M\left(\frac{t_i}{t_j}\right)\frac{1}{t_j}Ct_j^{q(\gamma+1/p)-1/p} \,,$$

$$
\begin{aligned}
T_{12,i} &:= \int_0^1 k_M\left(\frac{t_i}{s}\right)\frac{1}{s}x_+(s)ds - h{\sum_j}' k_M\left(\frac{t_i}{t_j}\right)\frac{1}{t_j}x_+(t_j)\,,\\
T_{13,i} &:= \int_1^\infty k_M\left(\frac{t_i}{s}\right)\frac{1}{s}Cs^{q(\gamma+1/p)-1/p}ds - h{\sum_j}''' k_M\left(\frac{t_i}{t_j}\right)\frac{1}{t_j}Ct_j^{q(\gamma+1/p)-1/p}\,,\\
T_{14,i} &:= h{\sum_{j\le j_*}}' k_M\left(\frac{t_i}{t_j}\right)\frac{1}{t_j}x(t_j)\,.
\end{aligned}
$$

Since $G(\sigma)=H(\sigma)$ over $[0,N-4]$, Lemmata 2.1 and 2.4 imply

$$
\begin{aligned}
T_{13,i} &:= \int_N^\infty k_M\left(\frac{i}{\sigma}\right)\frac{1}{\sigma}C(\sigma h)^{q(\gamma+1/p)-1/p}d\sigma - {\sum_j}''' k_M\left(\frac{i}{j}\right)\frac{1}{j}C(jh)^{q(\gamma+1/p)-1/p}\\
&= \int_{N-4}^\infty [G(\sigma)-H(\sigma)](\partial/\partial\sigma)^4\left\{k_M\left(\frac{i}{\sigma}\right)\frac{1}{\sigma}C(\sigma h)^{q(\gamma+1/p)-1/p}\right\}d\sigma\\
|T_{13,i}| &\le C\int_{N-4}^\infty \left|(\partial/\partial\sigma)^4\left\{\tilde{k}_M\left(\left(\frac{i}{\sigma}\right)^q\right)\left(\frac{i}{\sigma}\right)^{(q-1)/p}\frac{1}{\sigma}(\sigma h)^{q(\gamma+1/p)-1/p}\right\}\right|d\sigma\,.
\end{aligned}
$$

Using (A2) and supposing $q(\gamma-\alpha)<4$, we get

$$
\begin{aligned}
|T_{13,i}| &\le C\int_{N-4}^\infty \left(\frac{i}{\sigma}\right)^{q\alpha+(q-1)/p}(\sigma h)^{q(\gamma+1/p)-1/p}\sigma^{-5}d\sigma\\
&\le Ci^{q\alpha+(q-1)/p}h^{q(\gamma+1/p)-1/p}\int_{N-4}^\infty \sigma^{q(\gamma-\alpha)-5}d\sigma\\
&\le Ci^{q(\alpha+1/p)-1/p}h^{q(\alpha+1/p)-1/p+4}\,.
\end{aligned}
$$

In other words, we get

$$
\|T_{13}\|_{l^p} \le Ch^{4-1/p}\,. \tag{3.10}
$$

For T_{14}, we conclude

$$
\begin{aligned}
T_{14,i} &= h{\sum_{j\le j_*}}' k_M\left(\frac{t_i}{t_j}\right)\frac{1}{t_j}x(t_j)\\
&= h{\sum_{j\le j_*}}' k_M\left(\frac{t_i}{t_j}\right)\frac{1}{t_j}Ct_j^{q(\gamma+1/p)-1/p} + h{\sum_{j\le j_*}}' k_M\left(\frac{t_i}{t_j}\right)\frac{1}{t_j}x_+(t_j)\,,
\end{aligned}
$$

$$
\begin{aligned}
\left|h{\sum_{j\le j_*}}' k_M\left(\frac{t_i}{t_j}\right)\frac{1}{t_j}x_+(t_j)\right| &\le C\left|h{\sum_{j\le j_*}}'\left(\frac{t_i}{t_j}\right)^{-q(\alpha-1/p)-1/p}\frac{1}{t_j}t_j^{q(\gamma_1+1/p)-1/p}\right|\\
&\le Ci^{-q(\alpha-1/p)-1/p}h^{q(\gamma_1+1/p)-1/p}\,.
\end{aligned}
$$

This and assumption (A2) yield

$$\|T_{14} - h^{q(\gamma+1/p)-1/p}\left\{\sum_{j\le j_*}{}' k_M\left(\frac{i}{j}\right)\frac{1}{j}Cj^{q(\gamma+1/p)-1/p}\right\}_{i=1}^{N}\|_{l^p} \le Ch^{q(\gamma_1+1/p)-1/p} \quad (3.11)$$

$$\left|\sum_{j\le j_*}{}' k_M\left(\frac{i}{j}\right)\frac{1}{j}Cj^{q(\gamma+1/p)-1/p}\right| \le Ci^{-q(\alpha-1/p)-1/p} \quad (3.12)$$

Now consider T_{11}. From Lemma 2.4 we get

$$\begin{aligned} T_{11,i} &= h^{q(\gamma+1/p)-1/p}f(i)\,, && (3.13)\\ f(i) &:= \int_0^\infty k_M\left(\frac{i}{\sigma}\right)\frac{1}{\sigma}C\sigma^{q(\gamma+1/p)-1/p}d\sigma - h\sum_j{}'' k_M\left(\frac{i}{j}\right)\frac{1}{j}Cj^{q(\gamma+1/p)-1/p}\,, \\ &= \int_0^\infty G(\sigma)(\partial/\partial\sigma)^4\left\{k_M\left(\frac{i}{\sigma}\right)\frac{1}{\sigma}C\sigma^{q(\gamma+1/p)-1/p}\right\}d\sigma\,. \end{aligned}$$

In view of $G(\sigma) = \frac{1}{24}\sigma^4$ (cf. Lemma 2.4) and assumption (A2), we get

$$\begin{aligned} |f(i)| \le\; & C\int_0^1\left(\frac{i}{\sigma}\right)^{-q\alpha+(q-1)/p}\frac{1}{\sigma}\sigma^{q(\gamma+1/p)-1/p}d\sigma \\ & +C\int_1^i\left(\frac{i}{\sigma}\right)^{-q\alpha+(q-1)/p}\frac{1}{\sigma}\sigma^{q(\gamma+1/p)-1/p}\sigma^{-4}d\sigma \\ & +C\int_i^\infty\left(\frac{i}{\sigma}\right)^{q\alpha+(q-1)/p}\frac{1}{\sigma}\sigma^{q(\gamma+1/p)-1/p}\sigma^{-4}d\sigma \\ \le\; & Ci^{-min\{q(\alpha-1/p)+1/p,4-q(\gamma+1/p)+1/p\}}\begin{cases}\log i & \text{if } 4 = q(\alpha+\gamma)\\ 1 & \text{else}\,,\end{cases} \qquad (3.14) \end{aligned}$$

where we have assumed $q(\gamma-\alpha) < 4$.

Let us turn to T_{12}. Analogously to the estimation of $f(i)$, we obtain

$$\begin{aligned} |T_{12,i}| =\; & \int_0^N H(\sigma)(\partial/\partial\sigma)^4\left\{k_M\left(\frac{i}{\sigma}\right)\frac{1}{\sigma}x_+(\sigma)\right\}d\sigma \\ \le\; & C\int_0^1\left(\frac{i}{\sigma}\right)^{-q\alpha+(q-1)/p}\frac{1}{\sigma}(h\sigma)^{q(\gamma_1+1/p)-1/p}d\sigma \\ & +C\int_1^i\left(\frac{i}{\sigma}\right)^{-q\alpha+(q-1)/p}\frac{1}{\sigma}(h\sigma)^{q(\gamma_1+1/p)-1/p}\sigma^{-4}d\sigma \\ & +C\int_i^N\left(\frac{i}{\sigma}\right)^{q\alpha+(q-1)/p}\frac{1}{\sigma}(h\sigma)^{q(\gamma_1+1/p)-1/p}\sigma^{-4}d\sigma \\ \le\; & Ch^{q(\gamma_1+1/p)-1/p}i^{-q(\alpha-1/p)-1/p} \\ & +Ch^{q(\gamma_1+1/p)-1/p}\begin{cases} i^{-q(\alpha-1/p)-1/p} & \text{if } q(\alpha+\gamma_1) < 4\\ i^{-q(\alpha-1/p)-1/p}\log i & \text{if } q(\alpha+\gamma_1) = 4\\ i^{q(\gamma_1+1/p)-1/p-4} & \text{if } q(\alpha+\gamma_1) > 4\end{cases} \\ & +C\begin{cases} h^{q(\gamma_1+1/p)-1/p}i^{q(\gamma_1+1/p)-1/p-4} & \text{if } q(\gamma_1-\alpha) < 4\\ h^{q(\gamma_1+1/p)-1/p}\log h^{-1}i^{q(\gamma_1+1/p)-1/p-4} & \text{if } q(\gamma_1-\alpha) = 4\\ h^{4+q(\alpha+1/p)-1/p}i^{q(\alpha+1/p)-1/p} & \text{if } q(\gamma_1-\alpha) > 4\,.\end{cases} \end{aligned}$$

Hence, we obtain

$$\|T_{12}\|_{l^p} \leq Ch^{min\{4-1/p, q(\gamma_1+1/p)-1/p\}} \begin{cases} \log\, h^{-1} & \text{if } q(\gamma_1-\alpha)=4 \\ (\log\, h^{-1})^{1/p} & \text{if } q(\gamma_1+1/p)=4 \\ 1 & \text{else .} \end{cases} \quad (3.15)$$

Now (3.5) and (3.6) follow from the Eqs. (3.10)-(3.15) if $q(\gamma-\alpha) < 4$. For $q(\gamma-\alpha) \geq 4$, we can repeat the arguments leading to (3.15) to obtain

$$\begin{aligned} \|T_1\|_{l^p} &\leq Ch^{min\{4-1/p, q(\gamma+1/p)-1/p\}} \begin{cases} \log\, h^{-1} & \text{if } q(\gamma-\alpha)=4 \\ (\log\, h^{-1})^{1/p} & \text{if } q(\gamma+1/p)=4 \\ 1 & \text{else} \end{cases} \\ &\leq Ch^{4-1/p} \begin{cases} \log\, h^{-1} & \text{if } q(\gamma-\alpha)=4 \\ 1 & \text{else .} \end{cases} \end{aligned}$$

This proves Lemma 3.1 i). ∎

Proof of Lemma 3.1 ii): Setting $T_2 = T_{21} + T_{22}$, $T_{2l} = \{T_{2l,i}\}_{i=1}^N$,

$$T_{21,i} := \int_0^1 k_S(t_i,s)x(s)ds - h\sum_j{}' k_S(t_i,t_j)x(t_j)\ , \quad T_{22,i} := h\sum_{j\leq j_*}{}' k_S(t_i,t_j)x(t_j)\ ,$$

we get (cf. Lemma 2.1)

$$\begin{aligned} T_{21,i} &:= h\left\{\int_0^N k_S(t_i,\sigma h)x(\sigma h)d\sigma - \sum_j{}' k_S(t_i,jh)x(jh)\right\} \\ &= h\int_0^N H(\sigma)(\partial/\partial\sigma)^4\{k_S(t_i,\sigma h)x(\sigma h)\}d\sigma\ . \end{aligned}$$

From (A1), (3.2), and (A6) we obtain

$$\begin{aligned} (\partial/\partial\sigma)^4\{k_S(t_i,\sigma h)x(\sigma h)\} &= (\partial/\partial\sigma)^4 \left\{Ct_i^{(q-1)/p}\tilde{k}_S(t_i^q,(\sigma h)^q)(\sigma h)^{(q-1)/p'}\ \cdot \right. \\ &\qquad \left. \cdot\ \tilde{x}((\sigma h)^q)(\sigma h)^{(q-1)/p}\right\} \\ &= (\partial/\partial\sigma)^4 \left\{Ct_i^{(q-1)/p}\tilde{k}_S(t_i^q,(\sigma h)^q)(\sigma h)^{(q-1)}\tilde{x}((\sigma h)^q)\right\}\ , \\ |(\partial/\partial\sigma)^4\{k_S(t_i,\sigma h)x(\sigma h)\}| &\leq Ci^{(q-1)/p}h^{(q-1)(1+1/p)+q\gamma}\sigma^{(q-1)+q\gamma-4}\ . \end{aligned}$$

Inserting this into the last formula for $T_{21,i}$, we get

$$\begin{aligned} |T_{21,i}| &\leq Ci^{(q-1)/p}h^{(q-1)(1+1/p)+q\gamma+1}\left\{\int_0^1 \sigma^{(q-1)+q\gamma}d\sigma + \int_1^N \sigma^{(q-1)+q\gamma-4}d\sigma\right\} \\ &\leq Ci^{(q-1)/p}h^{min\{4+(q-1)/p, (q-1)(1+1/p)+q\gamma+1\}} \begin{cases} \log\, h^{-1} & \text{if } q(\gamma+1)=4 \\ 1 & \text{else ,} \end{cases} \\ \|T_{21}\| &\leq Ch^{min\{4-1/p, q(\gamma+1)-1/p\}} \begin{cases} \log\, h^{-1} & \text{if } q(\gamma+1)=4 \\ 1 & \text{else .} \end{cases} \end{aligned} \quad (3.16)$$

On the other hand,

$$
\begin{aligned}
|T_{22,i}| &\le Ch{\sum_{j\le j_*}}' t_i^{(q-1)/p}(jh)^{(q-1)+q\gamma} \le Ci^{(q-1)/p}h^{q+(q-1)/p+q\gamma} , \\
\|T_{22}\| &\le Ch^{q(\gamma+1)-1/p} . \qquad (3.17)
\end{aligned}
$$

Eqs. (3.16) and (3.17) imply Lemma 3.1 ii). ∎

PROOF OF LEMMA 3.1 iii): We get $\mu_N(t) = \mu_{H,N}(t) - \mu_{R,N}(t) + \mu_{S,N}(t) + \mu_{T,N}(t)$, where $\mu_{H,N}(t) = \mu(t/h)$ is given by (2.9) and

$$
\begin{aligned}
\mu_{R,N}(t) &:= \int_1^\infty k_M\left(\frac{t}{s}\right)\frac{1}{s}\left(\frac{s}{t}\right)^{(q-1)/p} ds - h{\sum_j}''' k_M\left(\frac{t}{t_j}\right)\frac{1}{t_j}\left(\frac{t_j}{t}\right)^{(q-1)/p} , \\
\mu_{S,N}(t) &:= \int_0^1 k_S(t,s)\left(\frac{s}{t}\right)^{(q-1)/p} ds - h{\sum_j}' k_S(t,t_j)\left(\frac{t_j}{t}\right)^{(q-1)/p} , \\
\mu_{T,N}(t) &:= h{\sum_{j\le j_*}}' k_S(t,t_j)\left(\frac{t_j}{t}\right)^{(q-1)/p} .
\end{aligned}
$$

Using (A1), we get

$$
\begin{aligned}
|\mu_{T,N}(t)| &\le Ch{\sum_{j\le j_*}}' t^{(q-1)/p}|\tilde{k}_S(t^q,t_j^q)|t_j^{(q-1)/p'}\left(\frac{t_j}{t}\right)^{(q-1)/p} \\
&\le Ch\sum_{j\le j_*}(jh)^{(q-1)} \le Ch^q . \qquad (3.18)
\end{aligned}
$$

With the help of Lemma 2.1, we arrive at

$$
\begin{aligned}
\mu_{S,N}(t) &= h\left\{\int_0^N k_S(t,\sigma h)\left(\frac{\sigma}{t/h}\right)^{(q-1)/p} d\sigma - {\sum_j}' k_S(t,jh)\left(\frac{j}{t/h}\right)^{(q-1)/p}\right\} \\
&= h\int_0^N H(\sigma)(\partial/\partial\sigma)^4\left\{t^{(q-1)/p}\tilde{k}_S(t^q,(\sigma h)^q)(\sigma h)^{(q-1)/p'}\left(\frac{\sigma}{t/h}\right)^{(q-1)/p}\right\} d\sigma , \\
|\mu_{S,N}(t)| &\le Ch\int_0^N |H(\sigma)| \sum_{\substack{m:\, m\le 4, \\ mq+(q-1)-4\ge 0}} C|\partial_2^{(m)}\tilde{k}_S(t^q,(\sigma h)^q)|(\sigma h)^{mq+(q-1)}\sigma^{-4} d\sigma \\
&\le Ch\int_0^N |H(\sigma)| \sum_{\substack{m:\, m\le 4, \\ mq+(q-1)-4\ge 0}} h^{mq+(q-1)}\sigma^{mq+(q-1)-4} d\sigma \\
&\le C\sum_{\substack{m:\, m\le 4, \\ mq+(q-1)-4\ge 0}} h^{1+mq+(q-1)}\left\{\int_0^1 \sigma^{mq+(q-1)}d\sigma + \int_1^N \sigma^{mq+(q-1)-4}d\sigma\right\} \\
&\le Ch^{min\{q,4\}} . \qquad (3.19)
\end{aligned}
$$

For $\mu_{R,N}(t)$, we obtain (cf. Lemmata 2.1 and 2.4 and use $H(\sigma) = G(\sigma)$ for $\sigma \in [0, N-4]$)

$$\begin{aligned}
\mu_{R,N}(t) &= \int_N^\infty k_M\left(\frac{t/h}{\sigma}\right)\frac{1}{\sigma}\left(\frac{\sigma}{t/h}\right)^{(q-1)/p} d\sigma - \sum_j{}''' k_M\left(\frac{t/h}{j}\right)\frac{1}{j}\left(\frac{j}{t/h}\right)^{(q-1)/p} \\
&= \int_{N-4}^\infty [G(\sigma) - H(\sigma)](\partial/\partial\sigma)^4 \left\{k_M\left(\frac{t/h}{\sigma}\right)\frac{1}{\sigma}\left(\frac{\sigma}{t/h}\right)^{(q-1)/p}\right\} d\sigma \\
&= q\int_{N-4}^\infty [G(\sigma) - H(\sigma)](\partial/\partial\sigma)^4 \left\{\tilde{k}_M\left(\left(\frac{t/h}{\sigma}\right)^q\right)\frac{1}{\sigma}\right\} d\sigma \\
|\mu_{R,N}(t)| &\le \int_{N-4}^\infty \sum_{m\le 4} C\left|\tilde{k}_M^{(m)}\left(\left(\frac{t/h}{\sigma}\right)^q\right)\right|\left(\frac{t/h}{\sigma}\right)^{qm}\frac{1}{\sigma}\sigma^{-4} d\sigma \\
&\le C\int_{N-4}^\infty \left(\frac{t/h}{\sigma}\right)^{q\alpha}\sigma^{-5} d\sigma \le Ch^4 t^{q\alpha}\,. \qquad (3.20)
\end{aligned}$$

From (3.18)-(3.20), (2.10), and (2.9) we conclude

$$|\mu_N(t) - \mu(t/h)| \le Ch^{min\{4,q\}}, \quad |\mu(\tau)| \le \tau^{-min\{4,q\alpha\}}\begin{cases} \log\tau & \text{if } q\alpha = 4 \\ 1 & \text{else .}\end{cases} \qquad (3.21)$$

Together with (3.2) we get

$$\begin{aligned}
\mu_N(t_i)x(t_i) &= \mu(i)\left\{C(ih)^{q(\gamma+1/p)-1/p} + O\left((ih)^{q(\gamma_1+1/p)-1/p}\right)\right\} \\
&\quad + O\left(h^{min\{4,q\}}(ih)^{q(\gamma+1/p)-1/p}\right) \\
&= h^{q(\gamma+1/p)-1/p} f(i) + O\left(|\mu(i)|(ih)^{q(\gamma_1+1/p)-1/p} + h^{min\{4,q\}}(ih)^{q(\gamma+1/p)-1/p}\right),
\end{aligned}$$

where

$$\begin{aligned}
f(i) &:= C\mu(i) i^{q(\gamma+1/p)-1/p}\,, \\
|f(i)| &\le Ci^{-min\{q(\alpha-\gamma-1/p)+1/p,\,4-q(\gamma+1/p)+1/p\}}\begin{cases} \log i & \text{if } q\alpha = 4 \\ 1 & \text{else .}\end{cases}
\end{aligned}$$

Hence, we arrive at

$$\begin{aligned}
&\|\{\mu_N(t_i)x(t_i)\}_{i=1}^N - h^{q(\gamma+1/p)-1/p}\{f(i)\}_{i=1}^N\|_{l^p} \\
&\le Ch^{min\{q,4\}+q(\gamma+1/p)-1/p}\left\{\sum_{i=1}^N i^{q(\gamma+1/p)p-1}\right\}^{1/p} \\
&\quad + Ch^{q(\gamma_1+1/p)-1/p}\left\{\sum_{i=1}^N |\mu(i)|^p i^{q(\gamma_1+1/p)p-1}\right\}^{1/p} \\
&\le Ch^{min\{q,4\}-1/p} + Ch^{q(\gamma_1+1/p)-1/p}\,.
\end{aligned}$$

$$\cdot \left\{ \sum_{i=1}^{N} i^{-min\{4,q\alpha\}p+q(\gamma_1+1/p)p-1} \right\}^{1/p} \cdot \begin{cases} \log h^{-1} & \text{if } q\alpha = 4 \\ 1 & \text{else} \end{cases}$$

$$\leq C h^{min\{4-1/p, q\alpha-1/p, q-1/p, q(\gamma_1+1/p)-1/p\}} \cdot$$

$$\cdot \begin{cases} \log h^{-1} & \text{if } q\alpha = 4 \text{ and } \gamma_1 \neq \alpha - 1/p \\ (\log h^{-1})^{1/p} & \text{if } \gamma_1 = min\{q\alpha, 4\}/q - 1/p \text{ and } q\alpha \neq 4 \\ (\log h^{-1})^{1+1/p} & \text{if } \gamma_1 = \alpha - 1/p \text{ and } q\alpha = 4 \\ 1 & \text{else .} \end{cases}$$

This proves Lemma 3.1 iii). ∎

3.3 Proof of the asymptotic error estimate

From (3.4), the stability of A_N, and Lemma 3.1 we get

$$\begin{aligned} e_N &= h^{q(\gamma+1/p)-1/p} A_N^{-1} \{g(i)\}_{i=1}^N + O\Big(h^{\varrho_{30}} (\log h^{-1})^{\varrho_{22}}\Big) , \qquad (3.22) \\ \varrho_{30} &:= min\{4-1/p,\ q-1/p,\ q\alpha-1/p,\ q(\gamma_1+1/p)-1/p,\ q(1+\gamma)-1/p\} , \\ \varrho_{22} &:= \begin{cases} \varrho_{14} & \text{if } \varrho_{13} < min\{\varrho_{16},\ \varrho_{20}\} \\ \varrho_{17} & \text{if } \varrho_{16} < min\{\varrho_{13},\ \varrho_{20}\} \\ \varrho_{21} & \text{if } \varrho_{20} < min\{\varrho_{13},\ \varrho_{16}\} \\ max\{\varrho_{14},\ \varrho_{17}\} & \text{if } \varrho_{13} = \varrho_{16} < \varrho_{20} \\ max\{\varrho_{14},\ \varrho_{21}\} & \text{if } \varrho_{13} = \varrho_{20} < \varrho_{16} \\ max\{\varrho_{17},\ \varrho_{21}\} & \text{if } \varrho_{16} = \varrho_{20} < \varrho_{13} \\ max\{\varrho_{14},\ \varrho_{17},\ \varrho_{21}\} & \text{if } \varrho_{13} = \varrho_{16} = \varrho_{20} , \end{cases} \\ \varrho_{20} &:= min\{4-1/p,\ q(\gamma+1)-1/p\} \quad , \quad \varrho_{21} := \begin{cases} 1 & \text{if } q(\gamma+1) = 4 \\ 0 & \text{else ,} \end{cases} \end{aligned}$$

where

$$\begin{aligned} |g(i)| &\leq C i^{-\varrho_{23}} (\log h^{-1})^{\varrho_{25}} , \\ \varrho_{23} &:= min\{q(\alpha-\gamma-1/p)+1/p,\ 4-q(\gamma+1/p)+1/p\} , \\ \varrho_{25} &:= \begin{cases} \varrho_6 & \text{if } q(\gamma-\alpha) \geq 4 \\ \varrho_6 & \text{if } q(\gamma-\alpha) < 4 \text{ and } \varrho_{15} < \varrho_{12} \\ max\{\varrho_{24},\ \varrho_6\} & \text{if } q(\gamma-\alpha) < 4 \text{ and } \varrho_{12} = \varrho_{15} , \end{cases} \\ \varrho_6 &:= \begin{cases} 1 & \text{if } q\alpha = 4 \\ 0 & \text{else} \end{cases} \quad , \quad \varrho_{24} := \begin{cases} 1 & \text{if } q(\gamma+\alpha) = 4 \\ 0 & \text{else .} \end{cases} \end{aligned}$$

We observe that (A7) implies $min\{q(\alpha-\gamma-1/p),\ 4-q(\gamma+1/p)\} > 0$ and $\{g(i)\}_{i=1}^{\infty} \in l^p$. Hence, Theorem 2.7 applies and we get

$$\begin{aligned} e_N &= h^{q(\gamma+1/p)-1/p} \{f(t_i/h)\}_{i=1}^N + O\left(h^{\varrho_{26}} (\log h^{-1})^{\varrho_{27}}\right) , \qquad (3.23) \\ \varrho_{26} &:= min\{4-1/p,\ q-1/p,\ q\alpha-1/p,\ q(\gamma_1+1/p)-1/p,\ q(1+\gamma)-1/p, \\ &\qquad q2\gamma-1/p\} , \end{aligned}$$

$$\varrho_{27} \;:=\; \begin{cases} \varrho_{22} & \text{if } \varrho_{30} < \varrho_{31} \\ \varrho_{32} & \text{if } \varrho_{31} < \varrho_{30} \\ max\{\varrho_{22},\, \varrho_{32}\} & \text{if } \varrho_{30} = \varrho_{31} \end{cases}$$

as well as

$$\begin{aligned} |f(i)| \;&\leq\; C i^{-\varrho_{28}} (\log\, h^{-1})^{\varrho_{29}} \,, \qquad (3.24) \\ \varrho_{28} \;&:=\; min\{4 - q(\gamma + 1/p) + 1/p,\, q(\gamma - 1/p) + 1/p,\, q(\alpha - \gamma - 1/p) + 1/p\} \,. \end{aligned}$$

Here we have set

$$\varrho_{31} \;:=\; q(\gamma + 1/p) - 1/p + \varrho_3 \,, \quad \varrho_{32} := \varrho_4 \,, \qquad \varrho_{29} := \varrho_2,$$

where ϱ_2, ϱ_3, and ϱ_4 are the numbers defined as in Sect. 2.3 under the special choice $\beta := \varrho_{23}$, $\omega := \varrho_{25}$. Eqs. (3.23) and (3.24) imply Theorem 1.3.

4 APPLICATION TO THE DOUBLE LAYER EQUATION OVER POLYGONAL DOMAINS AND NUMERICAL TESTS

4.1 The quadrature method for the double layer equation

In this section we shall apply the results of Sect. 1 to the numerical solution of the double layer integral equation over polygonal boundaries. For definiteness, we shall restrict ourselves to the case $p = \infty$. Before we consider the equation over the polygonal boundary let us have a look at a model problem. This model problem is the equation (1.1) with the kernel function (1.3) corresponding to the angle $\theta \neq \pi$, $0 < \theta < 2\pi$. We note that the double layer equation over a polygonal boundary can be written as a system of equations, where the main part of the matrix operator is a diagonal matrix the entries of which take the form of our model operator (cf. e.g. [4, 3, 26]).

It is not hard to derive from (1.3) that the kernel $\tilde{k}_M$ satisfies (A2) with the parameters $\alpha = 1$, $\alpha_1 = 2$. From the asymptotics of solutions to Mellin convolution equations (cf. [4, 6, 20]) we conclude the validity of (A6) with

$$\gamma = \frac{\pi}{max\{2\pi - \theta,\, \theta\}} \,, \qquad \gamma_1 = \begin{cases} \min\{\frac{2\pi}{2\pi-\theta},\, \frac{\pi}{\theta}\} & \text{if } \theta \leq \pi \\ \min\{\frac{\pi}{2\pi-\theta},\, \frac{2\pi}{\theta}\} & \text{if } \theta > \pi \,. \end{cases}$$

The formula (cf. e.g. [4])

$$(\mathcal{M}\tilde{k}_M)(\xi) = \mp \frac{\sin([\pi - \theta]\xi)}{\sin(\pi\xi)}$$

for the Mellin symbol implies (A3) i) and ii). Moreover, because (1.1) is a "Wiener-Hopf" equation with Mellin convolution and because either the null space or the cokernel of such an operator is trivial, we conclude that (A3) iii) is satisfied too. As mentioned in Sect. 1.2, the resolvent kernel $\tilde{l}_M$ is the solution of $(I_H + \tilde{K}_H)\tilde{l}_M = -\tilde{k}_M$. Thus the asymptotics in (A6) implies the relation (A4) for $\sigma \longrightarrow 0$. To obtain the relation for

$\sigma \longrightarrow \infty$, we perform the transformation of variables $\tilde{t} \mapsto \tilde{t}^{-1}$, $\tilde{s} \mapsto \tilde{s}^{-1}$ in the equation $(I_H + \tilde{K}_H)\tilde{l}_M = -\tilde{k}_M$. Observing $\tilde{k}_M(1/\tau) = \tilde{k}_M(\tau)$ and $(\tilde{K}_H \tilde{f})(\tilde{t}^{-1}) = (\tilde{K}_H \tilde{g})(\tilde{t})$ for $\tilde{g}(\tilde{s}) = \tilde{f}(\tilde{s}^{-1})$, we conclude $(I_H + \tilde{K}_H)\tilde{h} = -\tilde{k}_M$ for $\tilde{h}(\tilde{s}) = \tilde{l}_M(\tilde{s}^{-1})$. Since the operator $(I_H + \tilde{K}_H)$ is injective, we get $\tilde{l}_M(\tilde{s}) = \tilde{l}_M(\tilde{s}^{-1})$ and the asymptotics for $\sigma \longrightarrow \infty$ in (A4) follows from that for $\sigma \longrightarrow 0$. Assumption (A7) is obvious for $q = 1, 2, 3, 4$. In other words, all the assumptions (A1)-(A7) are fulfilled for the choice $p = \infty$ and $q = 1, 2, 3, 4$.

Let us suppose $p = \infty$ and $q = 1, 2, 3, 4$ and consider the solution x of (1.5) with the kernel given by (1.7) and (1.3). Let x_N stand for the approximate solution obtained from (1.9) or (1.10) and let the extrapolated solution x_N^e be given by (1.23). Furthermore, let us consider a smooth function $\tilde{g} : [0,1] \longrightarrow \mathbb{R}$. From Theorem 1.1, Corollary 1.5 and Theorem 1.6 we get

$$\|x - x_N\|_{L^\infty[0,1]} \leq C h^{q\gamma}, \tag{4.1}$$

$$\sup_{\epsilon \leq t \leq 1} |x(t) - x_N(t)| \leq C h^q \begin{cases} \log\, h^{-1} & \text{if } q = 4 \\ 1 & \text{else}\, , \end{cases} \tag{4.2}$$

$$\left| \int_0^1 \tilde{x}\tilde{g} - h {\sum_j}' x_N(t_j) g(t_j) \right| \leq C h^q \begin{cases} \log\, h^{-1} & \text{if } q = 4 \\ 1 & \text{else}\, , \end{cases} \tag{4.3}$$

$$\sup_{i=1,\dots,N} |x(t_i) - x_N^e(t_i)| \leq C h^q \begin{cases} \log\, h^{-1} & \text{if } q = 4 \\ 1 & \text{else}\, . \end{cases} \tag{4.4}$$

The estimates (4.2)-(4.4), however, can be improved. Namely, using the special form (1.3), it is not hard to conclude that

$$\left| (\partial/\partial\sigma)^4 k_M \left(\frac{\tau}{\sigma}\right) \frac{1}{\sigma} \right| \leq C \frac{\tau^q \sigma^{9q-5} + \tau^{10q-5}}{\tau^{10q} + \sigma^{10q}}\, ,$$
$$\int_0^\infty \left| (\partial/\partial\sigma)^4 k_M \left(\frac{\tau}{\sigma}\right) \frac{1}{\sigma} \right| d\sigma \leq C \tau^{-4}\, .$$

Hence, instead of (2.10) we even have $|\mu(\tau)| \leq C\tau^{-4}$. On the other hand, the choice $j_* = 0$ and $p = \infty$ leads to $\mu_{T,N} = 0$ and $\mu_{S,N} = O(h^4)$ (cf. (3.18) and (3.19)). This means that Lemma 3.1 iii) holds without the factor $\log i$ in the estimate for $|f(i)|$ and with ϱ_{15}, ϱ_{16}, and ϱ_{17} replaced by $4 - q\gamma$, $min\{4,\, q\gamma_1\}$, and 0, respectively. Applying this result in the proof of Sect. 3.3, we arrive at (3.22), (3.23), and (3.24) with ϱ_{23}, ϱ_{25}, ϱ_{30}, ϱ_{22}, ϱ_{28}, ϱ_{29}, ϱ_{26}, and ϱ_{27} replaced by $min\{4 - q\gamma,\, q\}$, 0, $min\{4,\, q\gamma_1\}$, 0, $min\{4 - q\gamma,\, q\gamma\}$,

$$\varrho_{29} = \begin{cases} 1 & \text{if } q = 3 \text{ and } \theta \in \{\frac{\pi}{2},\, \frac{3\pi}{2}\} \\ 0 & \text{else}\, , \end{cases}$$

$min\{4,\, q\gamma_1\}$, and ϱ_{29}, respectively. Finally, from this improved error expansion, we get

$$\sup_{\epsilon \leq t \leq 1} |x(t) - x_N(t)| \leq C h^{min\{4, q\gamma_1\}} \begin{cases} \log\, h^{-1} & \text{if } q = 3 \text{ and } \theta \in \{\frac{\pi}{2},\, \frac{3\pi}{2}\} \\ 1 & \text{else}\, , \end{cases} \tag{4.5}$$

$$\left| \int_0^1 \tilde{x}\tilde{g} - h {\sum_j}' x_N(t_j) g(t_j) \right| \leq C h^{min\{4, q\gamma_1\}} \begin{cases} \log\, h^{-1} & \text{if } q = 3 \text{ and } \theta \in \{\frac{\pi}{2},\, \frac{3\pi}{2}\} \\ 1 & \text{else}\, , \end{cases} \tag{4.6}$$

$$\sup_{i=1,\dots,N} |x(t_i) - x_N^e(t_i)| \quad \le \quad C h^{min\{4, q\gamma_1\}} \begin{cases} \log\, h^{-1} & \text{if } q=3 \text{ and } \theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\} \\ 1 & \text{else .} \end{cases} \tag{4.7}$$

Now we turn to the double layer equation over the polygonal boundary and introduce the same quadrature method as in [26]. Let Ω be a bounded simply connected polygon, and let Γ denote its boundary. The Dirichlet problem for Laplace's equation

$$\begin{aligned} \triangle U(t) &= 0, \quad t \in \Omega, \\ U|_\Gamma &= g \end{aligned} \tag{4.8}$$

with a continuous function g, can be reduced to the second kind integral equation (cf. e. g. [20, 3])

$$(I - 2W)x \quad = \quad -2g, \tag{4.9}$$

$$(Wx)(t) \quad := \quad \frac{1}{2\pi} \int_\Gamma \frac{\nu(s) \cdot (t-s)}{|t-s|^2} x(s) d_s\Gamma - \frac{1}{2}\chi(t)x(t), \quad t \in \Gamma, \tag{4.10}$$

where $\nu(s)$ is the exterior normal of Ω at $s \in \Gamma = \partial\Omega$ and $\chi(s) \in (-1,1)$ is chosen such that $[1+\chi(s)]\pi$ is the exterior angle between the tangents to Γ at t as $t \to s\pm$. Especially, $\chi(s) = 0$ if s is not a corner point of Γ. We shall consider (4.9) in the space of continuous functions. Taking into account that the constant functions are eigenfunctions of W corresponding to the eigenvalue -1/2, we can write (4.9) as

$$2x(t) - \frac{1}{\pi} \int_\Gamma \frac{\nu(s) \cdot (t-s)}{|t-s|^2} [x(s) - x(t)] d_s\Gamma = -2g(t), \quad t \in \Gamma. \tag{4.11}$$

Let N_C stand for the number of corners of Γ and suppose Γ is parameterized by the function $[0, N_C] \ni \sigma \mapsto \Gamma(\sigma)$ such that $\{\Gamma(l),\, l = 0, 1, \dots, N_C\}$ are the corner points of Γ. Moreover, we suppose $\Gamma(0) = \Gamma(N_C)$ and that $[l, l+1] \ni \sigma \mapsto \Gamma(\sigma)$ is linear for $l = 1, 2, \dots, N_c$. We fix an integer $N \ge 1$ and a real number $q \ge 1$ and introduce the graded mesh

$$\begin{aligned} t_j^{(N)} \quad := \quad & \left(\frac{j}{N}\right)^q, \quad s_{2l-1,j}^{(N)} := \Gamma(l - 1 + t_j^{(N)}/2), \quad s_{2l,j}^{(N)} := \Gamma(l - t_j^{(N)}/2), \\ & l = 1, 2, \dots, N_C, \quad j = 0, 1, \dots, N\ . \end{aligned}$$

For the quadrature, we introduce the rule (cf. (1.8))

$$\begin{aligned} & \int_\Gamma f(s) d_s\Gamma \\ = \quad & \sum_{l=1}^{N_C} \frac{|\Gamma'(l+1/2)|}{2} \left\{ \int_0^1 f\Big(\Gamma(l-1+\sigma^q/2)\Big) q\sigma^{q-1} d\sigma + \int_0^1 f\Big(\Gamma(l-\sigma^q/2)\Big) q\sigma^{q-1} d\sigma \right\} \\ \sim \quad & \sum_{l=1}^{N_C} \frac{|\Gamma'(l+1/2)|}{2} \left\{ h \sum_j{}' f(s_{2l-1,j}^{(N)}) q \left(\frac{j}{N}\right)^{q-1} + h \sum_j{}' f(s_{2l,j}^{(N)}) q \left(\frac{j}{N}\right)^{q-1} \right\} \\ =: \quad & \sum_{l,j}{}^{*} f(s_{l,j}^{(N)}) \omega_{l,j}\ . \end{aligned}$$

If we substitute in (4.11) the point t by $s_{m,i}^{(N)}$ and replace the integration by the quadrature, then we arrive at the following generalization of the quadrature method (1.9).

$$2x_N(s_{m,i}^{(N)}) - \frac{1}{\pi}\sum_{l,j}{}^{*} \frac{\nu(s_{l,j}^{(N)}) \cdot (s_{m,i}^{(N)} - s_{l,j}^{(N)})}{|s_{m,i}^{(N)} - s_{l,j}^{(N)}|^2}[x_N(s_{l,j}^{(N)}) - x_N(s_{m,i}^{(N)})]\omega_{l,j} = -2g(s_{m,i}^{(N)}), \quad (4.12)$$
$$m = 1, 2, \ldots, 2N_C, \quad i = 1, \ldots, N \ .$$

This quadrature method together with a modification (cf. method (1.9) and its modification (1.10)) is well-known to be stable and convergent (cf. e.g. [26]). Now we introduce the interior angle $\theta_l := [1 - \chi(l)]\pi$ at the l-th corner and set

$$\begin{aligned} \gamma^l &:= \frac{\pi}{max\{2\pi - \theta_l,\, \theta_l\}}\ , \\ \gamma_1^l &:= \begin{cases} \min\{\frac{2\pi}{2\pi-\theta_l},\, \frac{\pi}{\theta_l}\} & \text{if } \theta_l \le \pi \\ \min\{\frac{\pi}{2\pi-\theta_l},\, \frac{2\pi}{\theta_l}\} & \text{if } \theta_l > \pi\ , \end{cases} \\ \gamma_\Gamma &:= min\{\gamma^l,\ l = 1, 2, \ldots, N_C\}\ , \\ \gamma_{1,\Gamma} &:= min\{\gamma_1^l,\ l = 1, 2, \ldots, N_C\}\ . \end{aligned}$$

Using the approximate solution x_N from (4.12), we define the extrapolated solution by (cf. (1.23))

$$x_N^e(s_{m,i}^{(N)}) := x_N(s_{m,i}^{(N)}) + \sum_{l=1}^{L_i} 2^{-(l-1)q\gamma^m}\left\{x_{2N}(s_{m,2^{l-1}i}^{(N)}) - x_N(s_{m,2^{l-1}i}^{(N)})\right\}, \qquad (4.13)$$

where L_i is again the largest non-negative integer such that $i \le 2^{-L_i}N$. Furthermore, let $\tilde{g}$ denote a smooth function over Γ. Then, analogously to the estimates (4.1),(4.5)-(4.7), we obtain

$$\|x - x_N\|_{L^\infty(\Gamma)} \le Ch^{q\gamma_\Gamma}, \qquad (4.14)$$

$$\sup_{\epsilon \le |t - \Gamma(l)|,\, l=1,\ldots,N_C} |x(t) - x_N(t)| \le Ch^{min\{4,q\gamma_{1,\Gamma}\}}\begin{cases} \log h^{-1} & \text{if } q = 3 \text{ and } \gamma_{1,\Gamma} = 4/3 \\ 1 & \text{else}\ , \end{cases} \qquad (4.15)$$

$$\left|\int_\Gamma x\,\tilde{g} - \sum_{l,j}{}^{*} x_N(s_{l,j}^{(N)})\tilde{g}(s_{l,j}^{(N)})\omega_{l,j}\right| \le Ch^{min\{4,q\gamma_{1,\Gamma}\}}\begin{cases} \log h^{-1} & \text{if } q = 3 \text{ and } \gamma_{1,\Gamma} = 4/3 \\ 1 & \text{else}\ , \end{cases} \qquad (4.16)$$

$$\sup_{\substack{l=1,\ldots,2N_C,\\ i=1,\ldots,N}} |x(s_{l,i}^{(N)}) - x_N^e(s_{l,i}^{(N)})| \le Ch^{min\{4,q\gamma_{1,\Gamma}\}}\begin{cases} \log h^{-1} & \text{if } q = 3 \text{ and } \gamma_{1,\Gamma} = 4/3 \\ 1 & \text{else}\ . \end{cases} \qquad (4.17)$$

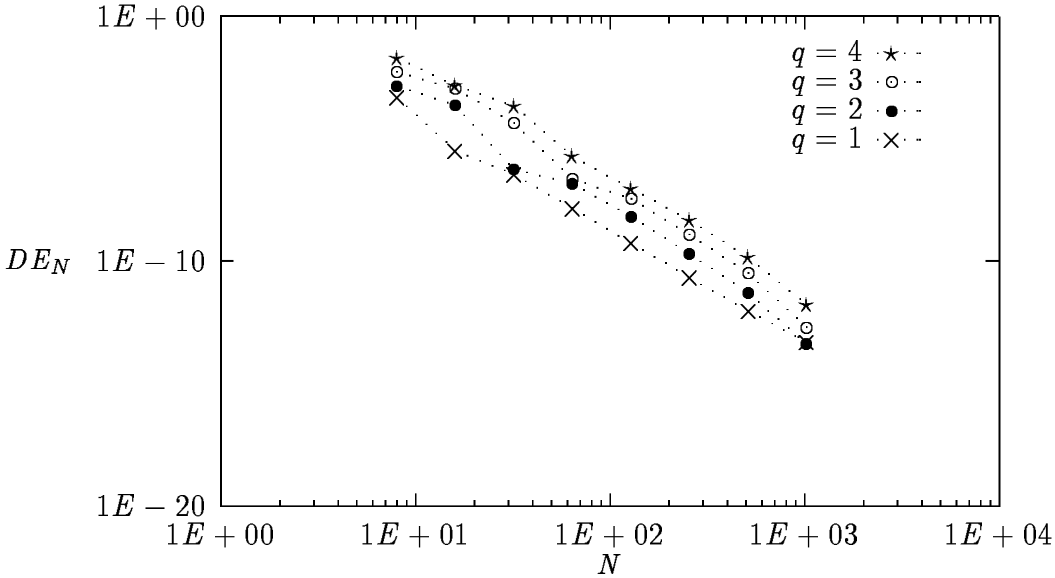

Figure 1: Dirichlet error.

4.2 Numerical tests

For a numerical example, we take the equilateral triangle $\Omega = \triangle ABC$ with corner points $A := (-1/2, 0)$, $B := (1/2, 0)$, and $C := (0, \sqrt{3}/2)$. We consider the harmonic function $U(t) := U({}_1t, {}_2t) := \log\sqrt{({}_1t - 0.1)^2 + ({}_2t - e - 0.15)^2}$ and get

$$U(t) = \frac{1}{2\pi}\int_\Gamma \frac{\nu(s)\cdot(t-s)}{|t-s|^2} x(s) d_s\Gamma, \quad t \in \Omega, \tag{4.18}$$

where x is the solution of $(I - 2W)x = y := 2U|_\Gamma$. In accordance with Sect.4.1 we determine an approximate solution x_N of x by the quadrature method (4.12). Note that the number of linear equations in (4.12) is equal to $3 \cdot (2N - 1)$. For the interior point $t^\# = (0.1, 0.15)$, we compute the approximation

$$U_N(t^\#) = \frac{1}{2\pi}\sum_{l,j}{}^{*} \frac{\nu(s_{l,j}^{(N)})\cdot(t^\# - s_{l,j}^{(N)})}{|t^\# - s_{l,j}^{(N)}|^2} x_N(s_{l,j}^{(N)})\omega_{l,j} \tag{4.19}$$

of $U(t^\#) = 1$. By DE_N we denote the error $|U_N(t^\#) - U(t^\#)|$ of the Dirichlet solution U at $t^\#$ and by SE_N the supremum norm error $\|x_N - x_{N/2}\|_{L^\infty} \sim \|x - x_N\|_{L^\infty}$ of the solution x to the integral equation. The last supremum is computed over the coarser grid $\{s_{l,j}^{(N/2)}\}$. Furthermore, for the orders β^S and β^D of the errors $SE_N \sim h^{\beta^S}$ and $DE_N \sim h^{\beta^D}$, we determine the approximate values $\beta_N^S := -[\log SE_N - \log SE_{N/2}]/\log 2$ and $\beta_N^D := -[\log DE_N - \log DE_{N/2}]/\log 2$. Finally, we compute the extrapolated solution x_N^e following (4.13) and consider the supremum norm error $EE_N := \|x_{N/2}^e - x_{N/4}^e\|_{L^\infty} \sim \|x - x_N^e\|_{L^\infty}$ and the convergence order $\beta_N^E := -[\log EE_N - \log EE_{N/2}]/\log 2$. In Table 1 (cf. Figure 1-3) we present the corresponding numerical results. Numerical test over other triangles and with different kind of Dirichlet data yield similar errors.

The results of Table 1 (cf. Figure 1) show that the approximate values $U_N(t^\#)$ of the linear functional $U(t^\#)$ of x converge with an order which is much higher than the

q	N	DE_N	β_N^D	SE_N	β_N^S	EE_N	β_N^E
1	8	0.00047					
	16	0.0000032	7.19	0.1148		0.08355	
	32	0.00000035	3.20	0.0735	0.64	0.01569	2.41
	64	0.000000014	4.62	0.0477	0.62	0.00674	1.22
	128	0.00000000054	4.72	0.0311	0.61	0.00292	1.21
	256	0.000000000021	4.65	0.0204	0.61	0.00127	1.20
	512	0.00000000000093	4.52	0.0134	0.60	0.00055	1.20
	1024	0.000000000000047	4.31	0.0089	0.60	0.00024	1.20
2	8	0.0014					
	16	0.00023	2.63	0.02744		0.0255233380	
	32	0.00000056	8.66	0.01195	1.20	0.0000950151	8.07
	64	0.00000015	1.96	0.00520	1.20	0.0000079896	3.57
	128	0.0000000063	4.51	0.00226	1.20	0.0000014369	2.47
	256	0.00000000020	4.99	0.00099	1.20	0.0000002683	2.42
	512	0.0000000000048	5.37	0.00043	1.20	0.0000000506	2.41
	1024	0.000000000000042	6.86	0.00019	1.20	0.0000000096	2.40
3	8	0.00544					
	16	0.00109	2.32	0.059912		0.0386	
	32	0.000042	4.70	0.016680	1.84	0.0023330037	4.05
	64	0.00000023	7.47	0.004753	1.81	0.0001926554	3.60
	128	0.000000034	2.78	0.001362	1.80	0.0000158881	3.60
	256	0.0000000012	4.73	0.000391	1.80	0.0000013096	3.60
	512	0.000000000032	5.31	0.000112	1.80	0.0000001080	3.60
	1024	0.00000000000019	7.39	0.000032	1.80	0.0000000089	3.60
4	8	0.0197					
	16	0.00148	3.73	0.1024866		0.051365	
	32	0.000212	2.80	0.0190687	2.42	0.001789	4.84
	64	0.00000195	6.77	0.0035983	2.41	0.00006329517	4.82
	128	0.000000093	4.39	0.0006812	2.40	0.00000223962	4.82
	256	0.0000000048	4.28	0.0001290	2.40	0.00000008362	4.74
	512	0.00000000014	5.11	0.0000245	2.40	0.00000000453	4.21
	1024	0.0000000000017	6.38	0.0000046	2.40	0.00000000025	4.17

Table 1: Errors and orders of convergence.

predicted one (cf. (4.16)). Moreover, it turns out that quite good approximations can be obtained already with the choice $q = 1$. This high order is due to the cancellation of low order terms arising in the error expansion from different sides of the triangle. We suggest that such a cancellation happens for all polygons and for all computations of function values at fixed points in the interior of the domain. However, numerical tests for the Mellin convolution equation over the interval with kernel (1.3) show that the order in (4.6) cannot be improved.

If the equation $(I-2W)x = y$ corresponds to a direct boundary integral formulation for the Neumann problem, then the solution x itself is of interest. For this case or for the computation of $U(t)$ with t close to the boundary Γ, small errors SE_n or EE_N are required. Table 1 (cf. Figure 2) shows that the convergence order β_N^S of SE_N tends to

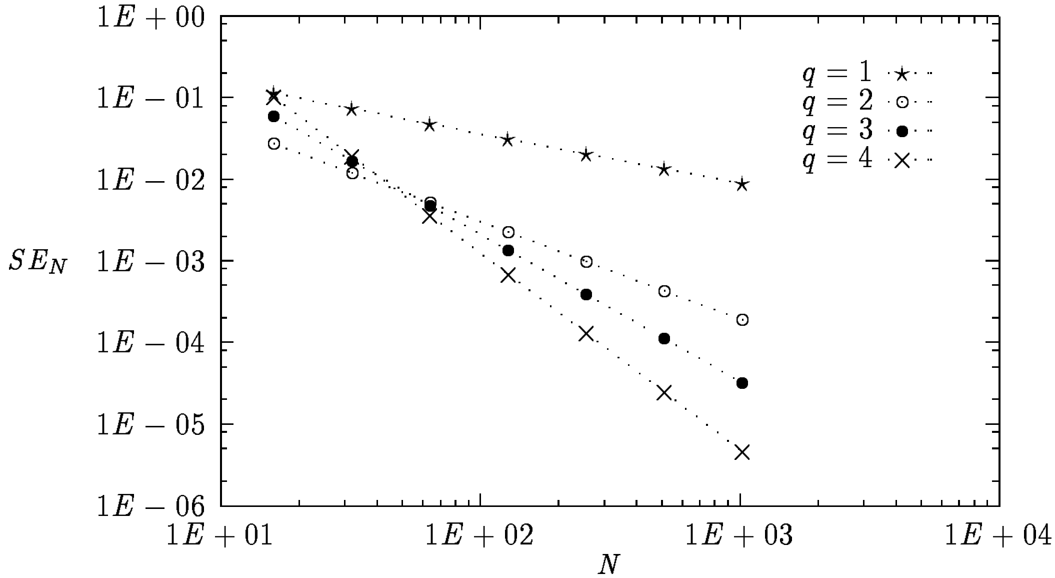

Figure 2: Supremum norm error.

$0.6 \cdot q$. Since $\gamma_\Gamma = 0.6$, the estimate (4.14) is confirmed. The extrapolated solution x_N^e converges (cf. Figure 3) also with the predicted order $1.2 \cdot q$ (cf. (4.17)).

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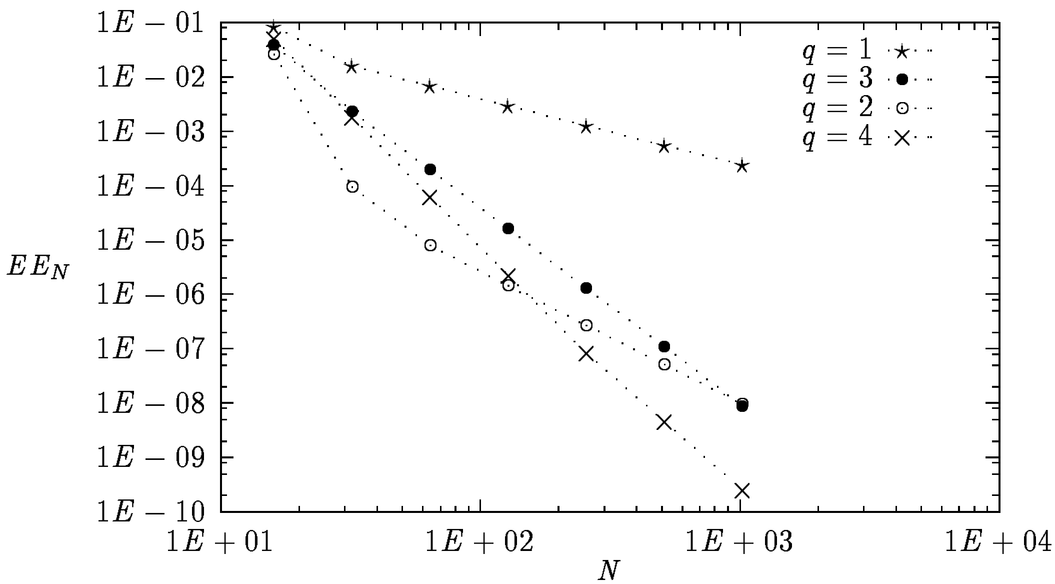

Figure 3: Extrapolation error.

REFERENCES

1. S. Amini and I.H. Sloan, Collocation methods for second kind integral equations with non-compact operators, *J. Int. Equ. and Appl.* **2**, No.1, 1–30 (1989).

2. C.T.H. Baker, *The numerical treatment of integral equations*, Claredon Press, Oxford (1977).

3. G. Chandler and I.G. Graham, High order methods for linear functionals of solutions of second kind integral equations, *SIAM J. Numer. Anal.* **25**, 1118–1137 (1988).

4. M. Costabel and E. Stephan, Curvature terms in the asymptotic expansion for solutions of boundary integral equations on curved polygons, *J. Int. Equ.* **5**, 353–371 (1983).

5. D. Elliott and S. Prößdorf, An algorithm for the approximate solution of integral equations of Mellin type, to appear.

6. J. Elschner, Asymptotics of solutions to pseudodifferential equations of Mellin type, *Math. Nachr.* **130**, 267–305(1987).

7. J. Elschner, On spline approximation for a class of non–compact integral equations, *Math. Nachr.* **146**, 271–321 (1990).

8. J. Elschner, The h-p-version of spline approximation methods for Mellin convolution equations, *J. Int. Equ. and Appl.* **5**, No.1, 47–73 (1993).

9. I. Gohberg and N. Krupnik, *One-dimensional linear singular integral equations, Vols.I and II* , Operator Theory: Advances and Applications, Vols.53 and 54, Birkhäuser Verlag, Basel, (1992), originally in russian, Shtiintsa, Kishinev (1973).

10. Q. Lin, I.G. Graham, and R. Xie, Extrapolation of Nyström solutions of boundary integral equations on non-smooth domains, *J. Comput. Math.* **10**, 231–244 (1992).

11. W. Hackbusch, *Integralgleichungen, Theorie und Numerik*, Teubner Studienbücher Mathematik, B.G.Teubner, Stuttgart (1989).

12. U. Heise, Combination of the boundary integral equation method and the extrapolation method, *Computer Methods in Applied Mechanics and Engineering* **38**, 291–346 (1983).

13. Y. Jeon, A Nyström method for boundary integral equations on domains with piecewise smooth boundary, *J. Int. Equ. Appl.* **5**, 221–242 (1993).

14. Б.Н. Хоромский и Е.П. Жидков, Численные алгоритмы на последовательности сеток и их приложения в задачах магнитостатики и теоретичецкой Физики, Физика элементарных частиц и атомного ядра **19**, 3, 622–668 (1988).

15. А.В. Козак, Локальный принцип в теории проекционных методов, ДАН СССР **212**, 6, 1287–1289 (1973).

16. R. Kreß, A Nyström method for boundary integral equations in domains with corners, *Numer. Math.* **58**, 145–161 (1990).

17. Q. Lin, I.H. Sloan, and R. Xie, Extrapolation of the iterated collocation method for integral equations of the second kind, *SIAM J. Numer. Anal.* **6**, 1535–1541 (1990).

18. Q. Lin and R. Xie, Extrapolation for the approximations to the solution of a boundary integral equation on polygonal domains, *J. Comput. Math.* **7**, 2, 174–181 (1989).

19. G.I. Marchuk and V.V. Shaidurov, *Difference methods and their extrapolation*, Springer-Verlag, New York, Berlin, Heidelberg (1983).

20. V.G. Mazya, Boundary integral equations, in: *Encyclopaedia of Math. Sciences, vol.* **27**, *Analysis* I (eds.: V.G.Mazya and S.M.Nikol'skii), Springer-Verlag, Berlin, Heidelberg (1991).

21. G. Mastroianni and G. Monegato, Nyström interpolants based on the zeros of Legendre polynomials for a non-compact integral operator equation, *IMA J.Numer.Anal.* **14**, 81-95 (1993).

22. W. McLean, Asymptotic error expansions for numerical solutions of integral equations, *IMA J. Numer. Anal.* **9**, 373–384 (1989).

23. W. McLean, Boundary integral methods for the Laplace equation, Ph.D.Thesis, Canberra (1985).

24. S. Prößdorf and B. Silbermann, *Numerical analysis for integral and related operator equations*, Birkhäuser Verlag, Basel (1991).

25. A. Rathsfeld, Eine Quadraturformelmethode für Mellin-Operatoren nullter Ordnung, *Math. Nachr.* **137**, 321–354 (1988).

26. A. Rathsfeld, Iterative solution of linear systems arising from the double-layer potential equation over curves with corners, *Math. Meth. Appl. Sci.* **16**, 443–455 (1993).

27. S. Roch, Lokale Theorie des Reduktionverfahrens für singuläre Integraloperatoren mit Carlemanschen Verschiebungen, Dissertation A, TU Karl-Marx-Stadt, Karl-Marx-Stadt (1988).

28. J. Saranen, Extrapolation methods for spline collocation solutions of pseudodifferential equations on curves, *Numer. Math.* **56**, 385–407 (1989).

29. J. Shi, The extrapolation method for a class of non-smooth Fredholm integral equations, *Systems Science and Mathematical Sciences* **5**, 1, 33–41 (1992).

30. B. Silbermann, Lokale Theorie des Reduktionsverfahrens, *Math. Nachr.* **104**, 137–146 (1981).