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Small strain oscillations of an elastoplastic Kirchhoff plate

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Abstract

The two dimensional equation for transversal vibrations of an elastoplastic plate is derived from a general three dimensional system with a single yield tensorial von Mises plasticity model in the five dimensional deviatoric space. It leads after dimensional reduction to a multiyield three dimensional Prandtl-Ishlinskii hysteresis model whose weight function is explicitly given. The resulting partial differential equation with hysteresis is solved by means of viscous approximations and a monotonicity argument.

1 Introduction

We pursue here the investigation of oscillating lower dimensional elastoplastic structures. We have shown in [13] that in the case of a one dimensional beam, the classical von Mises plasticity criterion with a single yield condition leads after dimensional reduction to a *multiyield model* of Prandtl [20] and Ishlinskii [10] type. This can be explained by the fact that in the 1D model only deformations of longitudinal fibers parameterized by the transversal coordinate are taken into account, and the individual fibers do not switch from the elastic to the plastic regime at the same time. More precisely, the "eccentric" fibers look as if they had higher elasticity modulus and lower yield point than the central ones. Hence, the effect of the existence of plasticized zones is translated into the mathematical language by means of the Prandtl-Ishlinskii combination of elastic, perfectly plastic elements with different yield limits that are not all simultaneously activated.

In this paper, a similar behavior is observed for the Kirchhoff plate model with single yield, von Mises plasticity. Plasticized zones as on Fig. 1 occur as well, although "fibers" have to be replaced with "layers". As a result of dimensional reduction, we obtain again a multiyield Prandtl-Ishlinskii operator in a reduced three dimensional space instead of the original five dimensional space of symmetric tensor deviators. This emerging multiyield character of the elastoplastic plate bending problem does not seem to have been taken into consideration earlier. The multiyield quasistatic model in [1] does not directly refer to plates. In [2, 15, 18], only the quasistatic case is investigated as well, and after dimensional reduction, the yield condition is still described by one sharp surface of plasticity. Methods based on Γ-convergence of energy minimizers ([8, 19]) are indeed more rigorous than a simple scaling analysis, but unfortunately cannot be used for the study of nonequilibrium problems.

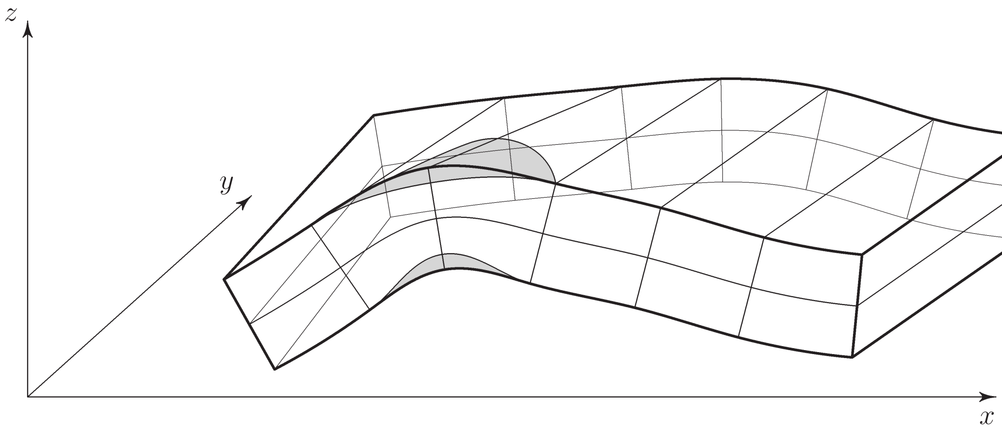

Figure 1. A plate section with grey plasticized zone.

Vectorial Prandtl-Ishlinskii operators are less easy to deal with that the scalar ones. Therefore, at variance with the 1D case in [13], we are not able to solve the resulting PDE without regularizing the constitutive law by an additional kinematic hardening term. After normalizing the physical constants to unity, we thus obtain the transversal deformation $w(x, y, t)$ of a simply supported elastoplastic plate by solving the following variational problem:

$$\int_{\Omega_0} \left(w_{tt}\hat{w} + w_{xtt}\,\hat{w}_x + w_{ytt}\,\hat{w}_y + \tilde{\mathbf{B}}\mathbf{D}_2 w \cdot \mathbf{D}_2\hat{w} \right) dx\,dy$$
$$+ \int_{\Omega_0} \langle \mathcal{F}[\mathbf{D}_2 w], \mathbf{D}_2\hat{w} \rangle \, dx\,dy = \int_{\Omega_0} g\,\hat{w}\,dx\,dy \qquad \forall \hat{w} \in V, \tag{1.1}$$

where $\Omega_0 \subset \mathbb{R}^2$ is the reference shape of the plate, $V = H^2(\Omega_0) \cap H^1_0(\Omega_0)$, $\tilde{\mathbf{B}}$ is a symmetric positive definite (3×3)-matrix that accounts for the kinematic hardening, $\mathcal{F}$ stands for the 3D Prandtl-Ishlinskii operator, $\langle \cdot, \cdot \rangle$ is the generating scalar product of $\mathcal{F}$ in $\mathbb{R}^3$ given by formula (2.39) below, g is a given right hand side, and $\mathbf{D}_2 w = (w_{xx}, w_{yy}, w_{xy})$, $\mathbf{D}_2\hat{w} = (\hat{w}_{xx}, \hat{w}_{yy}, \hat{w}_{xy})$. We may write (1.1) in a more concise form as

$$w_{tt} - \Delta w_{tt} + \mathbf{D}_2^* \left(\tilde{\mathbf{B}}\mathbf{D}_2 w + \mathcal{F}[\mathbf{D}_2 w] \right) = g, \tag{1.2}$$

where $\mathbf{D}_2^* : (L^2(\Omega_0))^3 \to V'$ is the formal adjoint of $\mathbf{D}_2$. The term $-\Delta w_{tt}$ represents the contribution to the momentum balance due to the rotational inertia of the vertical fibers.

We prove the existence and uniqueness under appropriate regularity assumptions on the data. The solution is constructed first for an approximating visco-elasto-plastic problem via contraction in a suitable metric space, the limit as the viscosity coefficient tends to zero is justified by the Minty trick.

The paper is structured as follows. In Section 2, we use the scaling method of [4, 6] to derive the multiyield Prandtl-Ishlinskii plate model. Basic properties of the vectorial Prandtl-Ishlinskii model are summarized in Section 3. Existence and uniqueness of weak solutions to the resulting PDE with a Prandtl-Ishlinskii hysteresis operator is established in Section 4.

2 Derivation of the model

In this section, we derive our model from a general three dimensional system. We focus on the question how the multiyield behavior results from the single yield von Mises model and from the dimensional reduction. This is why we do not look for maximal generality and keep the assumptions as simple as possible. We restrict ourselves to plates of constant thickness, that is, to sets $\Omega \subset \mathbb{R}^3$ of the form $\Omega = \Omega_0 \times (-h, h)$, where $\Omega_0 \subset \mathbb{R}^2$ is the shape of the plate and $2h$ is its thickness. We denote by $(x, y) \in \Omega_0$ the longitudinal coordinates, by $z \in (-h, h)$ the transversal coordinate, and by $t \in [0, T]$ the time, where $T > 0$ is given.

In order to compare the resulting equations, we start with the linear elastic isotropic case (Subsection 2.1), and then pass to the elastoplastic model (Subsection 2.2). We follow the scaling technique of [4, Part A] and [6, Sect. 5.4] in terms of a small parameter $\alpha > 0$ with the intention of keeping only the lowest order terms in α in the resulting equations. In particular, we assume that

$$h = \mathcal{O}(\alpha), \ \Omega_0 = \mathcal{O}(1).$$

Let us consider smooth displacements $\mathbf{u} : \Omega \times (0, T) \to \mathbb{R}^3$, decomposed into

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1^L \\ u_2^L \\ u_3^L \end{pmatrix} + \begin{pmatrix} u_1^H \\ u_2^H \\ u_3^H \end{pmatrix} = \mathbf{u}^L + \mathbf{u}^H ,$$

where the superscripts L and H stand for low order and high order components with respect to α, respectively. We make the following assumptions (cf. Fig. 1).

(A1) The low order displacement of the midsurface $\mathcal{C} = \{(x, y, 0) \in \Omega \ : \ (x, y) \in \Omega_0\}$ is only transversal, that is,

$$\mathbf{u}^L(x, y, 0, t) = \begin{pmatrix} 0 \\ 0 \\ w(x, y, t) \end{pmatrix} \quad \forall\, (x, y) \in \Omega_0 , \quad \forall\, t \in (0, T) , \qquad (2.1)$$

with some function $w : \Omega_0 \times (0, T) \to \mathbb{R}$.

(A2) The low order deformation

$$\mathbf{F}^L(x, y, z, t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \mathbf{u}^L(x, y, z, t)$$

leaves the fibers $\{(x, y)\} \times (-h, h)$ perpendicular to the midsurface, and their deformation is proportional to their distance to it; that is,

$$\mathbf{F}^L(x, y, z, t) = \mathbf{F}^L(x, y, 0, t) + z\, \mathbf{n}(x, y, t) \quad \forall\, (x, y, z, t) \in \Omega \times (0, T) , \qquad (2.2)$$

where $\mathbf{n}(x, y, t)$ is the unit "upward" normal to the deformed midsurface $\mathcal{C}(t) = \mathbf{F}^L(\mathcal{C}, t)$ at time t, see Fig. 1.

(A3) $w_{xx}, w_{xy}, w_{yy} = \mathcal{O}(\alpha)$.

Under the hypothesis (A3), we can linearize the problem by replacing

$$\mathbf{n}(x,y,t) = \frac{1}{\sqrt{1 + w_x^2(x,y,t) + w_y^2(x,y,t)}} \begin{pmatrix} -w_x(x,y,t) \\ -w_y(x,y,t) \\ 1 \end{pmatrix}$$

with

$$\tilde{\mathbf{n}}(x,y,t) := \begin{pmatrix} -w_x(x,y,t) \\ -w_y(x,y,t) \\ 1 \end{pmatrix} . \tag{2.3}$$

This replacement is justified, since an elementary computation yields that

$$|\,\tilde{\mathbf{n}}(x,y,t) - \mathbf{n}(x,y,t)| \,<\, |w_x(x,y,t)|^2 + |w_y(x,y,t)|^2 = \mathcal{O}(\alpha^2)\,.$$

This enables us to write for every $(x,y,z,t) \in \Omega \times (0,T)$ the low order displacement $\mathbf{u}^L(x,y,z,t)$ as

$$\mathbf{u}^L(x,y,z,t) = \begin{pmatrix} -z\, w_x(x,y,t) \\ -z\, w_y(x,y,t) \\ w(x,y,t) \end{pmatrix} . \tag{2.4}$$

The smallness assumptions ensure in particular that the deformation (2.2) is a local homeomorphism. We further compute

$$\nabla \mathbf{u}^L(x,y,z,t) = \begin{pmatrix} -z\, w_{xx}(x,y,t) & -z\, w_{xy}(x,y,t) & -w_x(x,y,t) \\ -z\, w_{xy}(x,y,t) & -z\, w_{yy}(x,y,t) & -w_y(x,y,t) \\ w_x(x,y,t) & w_y(x,y,t) & 0 \end{pmatrix} , \tag{2.5}$$

and the low order strain tensor $\boldsymbol{\varepsilon}^L = (\nabla \mathbf{u}^L + (\nabla \mathbf{u}^L)^T)/2$ becomes

$$\boldsymbol{\varepsilon}^L(x,y,z,t) = \begin{pmatrix} -z\, w_{xx}(x,y,t) & -z\, w_{xy}(x,y,t) & 0 \\ -z\, w_{xy}(x,y,t) & -z\, w_{yy}(x,y,t) & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{2.6}$$

2.1 Small elastic deformations

We denote by " $:$ " the canonical scalar product in the space $\mathbb{T}^{3\times 3}_{\mathrm{sym}}$ of symmetric (3×3)-tensors, i. e.,

$$\boldsymbol{\xi} : \boldsymbol{\eta} = \sum_{i,j=1}^{3} \xi_{ij}\, \eta_{ij}\,, \quad \forall\, \boldsymbol{\xi} = (\xi_{ij})\,,\ \boldsymbol{\eta} = (\eta_{ij})\,, \quad i,j = 1,2,3\,. \tag{2.7}$$

Moreover, we define for any given $\boldsymbol{\xi} \in \mathbb{T}^{3\times 3}_{\mathrm{sym}}$ its (trace free) *deviator* $\mathbf{D}\boldsymbol{\xi}$ by

$$\mathbf{D}\boldsymbol{\xi} = \boldsymbol{\xi} \,-\, \frac{1}{3}\,(\boldsymbol{\xi} : \boldsymbol{\delta})\,\boldsymbol{\delta}\,, \tag{2.8}$$

where $\boldsymbol{\delta} = (\delta_{ij})$ denotes the Kronecker tensor.

To motivate the elastoplastic case treated below, we first study the case of linear isotropic elasticity, in which the strain tensor $\boldsymbol{\varepsilon}$ and the stress tensor $\boldsymbol{\sigma}$ are related to each other through the formula

$$\boldsymbol{\sigma} = 2\mu\,\boldsymbol{\varepsilon} + \lambda\,(\boldsymbol{\varepsilon} : \boldsymbol{\delta})\,\boldsymbol{\delta}\,, \tag{2.9}$$

where μ, λ are the Lamé constants such that $\mu > 0,\ 2\mu + 3\lambda > 0$. The main issue is to choose a proper scaling of $\boldsymbol{\sigma}$. The components $\sigma_{11}, \sigma_{22}, \sigma_{12}$ are of the lowest order, which is $\mathcal{O}(\alpha^2)$ due to (A3), (2.6), and (2.9). Assuming that the motion is "sufficiently slow" and no volume forces act on the body, we may for scaling purposes refer to the elastostatic equilibrium conditions $\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}$ which, by virtue of the natural scaling of the variables $z = \mathcal{O}(\alpha),\ x, y = \mathcal{O}(1)$ and due to the symmetry of $\boldsymbol{\sigma}$, justify the scaling hypothesis

(A4) $\sigma_{13}, \sigma_{23} = \mathcal{O}(\alpha^3),\ \sigma_{33} = \mathcal{O}(\alpha^4)$.

According to (2.9) and Hypothesis (A4), the high order strain tensor $\boldsymbol{\varepsilon}^H$ is scaled as

(A5) $\varepsilon^H_{13}, \varepsilon^H_{23} = \mathcal{O}(\alpha^3),\ \varepsilon^H_{33} = \mathcal{O}(\alpha^2),\ \varepsilon^H_{11}, \varepsilon^H_{22}, \varepsilon^H_{12} = \mathcal{O}(\alpha^4)$.

In terms of the high order displacements $\mathbf{u}^H$, (A5) corresponds to the scaling $u^H_1, u^H_2 = \mathcal{O}(\alpha^4),\ u^H_3 = \mathcal{O}(\alpha^3)$.

Let $\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{\varepsilon}}$ denote the stress and strain components of the order $\mathcal{O}(\alpha^2)$ at most. Then

$$\bar{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\boldsymbol{\varepsilon}} = \begin{pmatrix} \varepsilon^L_{11} & \varepsilon^L_{12} & 0 \\ \varepsilon^L_{12} & \varepsilon^L_{22} & 0 \\ 0 & 0 & \varepsilon^H_{33} \end{pmatrix}. \tag{2.10}$$

We compute ε^H_{33} from the relation

$$0 = \sigma_{33} = 2\mu\varepsilon^H_{33} + \lambda(\varepsilon^L_{11} + \varepsilon^L_{22} + \varepsilon^H_{33})\,,$$

that is,

$$\varepsilon^H_{33} = -\frac{\lambda}{2\mu + \lambda}(\varepsilon^L_{11} + \varepsilon^L_{22})\,.$$

Hence,

$$\bar{\boldsymbol{\varepsilon}} : \boldsymbol{\delta} = \frac{2\mu}{2\mu + \lambda}(\varepsilon^L_{11} + \varepsilon^L_{22})\,.$$

We now rewrite the constitutive relations in terms of the Young modulus E and the Poisson ratio ν, given by the formula

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda} > 0\,, \qquad \nu = \frac{\lambda}{2(\mu + \lambda)} \in \left(-1, \frac{1}{2}\right).$$

In addition to the obvious identity $\bar{\mathbf{u}} = \mathbf{u}^L$, we have

$$\bar{\boldsymbol{\sigma}} = \frac{E}{1-\nu^2} \begin{pmatrix} \varepsilon^L_{11} + \nu\varepsilon^L_{22} & (1-\nu)\varepsilon^L_{12} & 0 \\ (1-\nu)\varepsilon^L_{12} & \nu\varepsilon^L_{11} + \varepsilon^L_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\boldsymbol{\varepsilon}} = \begin{pmatrix} \varepsilon^L_{11} & \varepsilon^L_{12} & 0 \\ \varepsilon^L_{12} & \varepsilon^L_{22} & 0 \\ 0 & 0 & -\frac{\nu}{1-\nu}(\varepsilon^L_{11} + \varepsilon^L_{22}) \end{pmatrix}, \tag{2.11}$$

with $\boldsymbol{\varepsilon}^L$ given by (2.6). On the upper boundary, we prescribe the boundary condition $\boldsymbol{\sigma}(x,y,h,t)\cdot\boldsymbol{\nu}_3 = \mathbf{f}(x,y,t)$, where $\boldsymbol{\nu}_3 = (0,0,1)^T$ is the upward normal vector, and $\mathbf{f} = (f_1, f_2, f_3)^T$ is a given external surface load. In component form, this boundary condition reads $\sigma_{13} = f_1$, $\sigma_{23} = f_2$, $\sigma_{33} = f_3$. In agreement with the scaling hypothesis (A4), we require $f_1, f_2 = \mathcal{O}(\alpha^3), f_3 = \mathcal{O}(\alpha^4)$. On the rest of the boundary, we assume the *vanishing normal stress* boundary conditions $\boldsymbol{\sigma}\cdot\boldsymbol{\nu} = 0$, where $\boldsymbol{\nu}$ is the unit outward normal vector. On $\partial\Omega_0 \times (-h,h)$, we add the boundary condition for w

$$w(x,y,t) = 0 \quad \text{for } (x,y) \in \partial\Omega_0\,, \tag{2.12}$$

in order to eliminate possible transversal rigid body displacements. This corresponds to a *simply supported plate*. In accordance with these boundary conditions, we consider the Sobolev space

$$V = \left\{ w \in H^2(\Omega_0) : \, w\big|_{\partial\Omega_0} = 0 \right\}. \tag{2.13}$$

Finally, suppose that the initial conditions

$$w(x,y,0) = w^0(x,y)\,, \;\; w_t(x,y,0) = w^1(x,y)\,, \tag{2.14}$$

are given. As in [16], we write the *momentum balance equation* in variational form

$$\int_\Omega \rho\, \mathbf{u}_{tt}\cdot\hat{\mathbf{u}}\; dx\,dy\,dz + \int_\Omega \boldsymbol{\sigma} : \hat{\boldsymbol{\varepsilon}}\, dx\,dy\,dz = \int_{\partial\Omega} (\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\cdot\hat{\mathbf{u}}\; ds\,, \tag{2.15}$$

with the unknown vector $\mathbf{u}$ and tensor $\boldsymbol{\sigma}$, for all admissible displacements $\hat{\mathbf{u}}$ and strains $\hat{\boldsymbol{\varepsilon}}$ of the form (2.4), (2.6), and (2.11); i. e., we have

$$\hat{\mathbf{u}}(x,y,z) = \begin{pmatrix} -z\,\hat{w}_x(x,y) \\ -z\,\hat{w}_y(x,y) \\ \hat{w}(x,y) \end{pmatrix}, \quad \hat{\boldsymbol{\varepsilon}}(x,y,z) = \begin{pmatrix} -z\hat{w}_{xx} & -z\hat{w}_{xy} & 0 \\ -z\hat{w}_{xy} & -z\hat{w}_{yy} & 0 \\ 0 & 0 & \frac{\nu}{1-\nu} z\Delta\hat{w} \end{pmatrix}, \tag{2.16}$$

where $\hat{w}$ varies over the space V. It follows from the choice of the boundary conditions that

$$\begin{aligned} \int_{\partial\Omega} (\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\cdot\hat{\mathbf{u}}\; ds &= \int_{\Omega_0} (-h\, f_1\, \hat{w}_x - h\, f_2\, \hat{w}_y + f_3\, \hat{w})\; dx\,dy \\ &= \int_{\Omega_0} (h(f_1)_x + h(f_2)_y + f_3)\, \hat{w}\, dx\,dy\,. \end{aligned}$$

Keeping on the left hand side of (2.15) only terms of the lowest order in α, we may replace $(\mathbf{u}, \boldsymbol{\sigma})$ by $(\bar{\mathbf{u}}, \bar{\boldsymbol{\sigma}})$ from (2.4), (2.6), and (2.11), and obtain

$$\begin{aligned}
&\rho \int_{\Omega_0} \left(w_{tt}\, \hat{w} \,+\, \frac{h^2}{3}\, (w_{xtt}\, \hat{w}_x + w_{ytt}\, \hat{w}_y) \right) dx\, dy \\
&\qquad + \frac{E\, h^2}{3(1+\nu)} \int_{\Omega_0} \left(w_{xx}\, \hat{w}_{xx} + 2w_{xy}\, \hat{w}_{xy} + w_{yy}\, \hat{w}_{yy} + \frac{\nu}{1-\nu} \Delta w\, \Delta \hat{w} \right) dx\, dy \\
&\qquad = \int_{\Omega_0} g\, \hat{w}\, dx\, dy\,,
\end{aligned} \tag{2.17}$$

where we have set

$$g(x,y,t) = \frac{1}{2h} f_3(x,y,t) + \frac{1}{2}((f_1)_x + (f_2)_y)(x,y,t)\,. \tag{2.18}$$

The variational equation (2.17) leads formally to the partial differential equation describing transversal vibrations of a thin elastic plate

$$\rho\, w_{tt} \,-\, \frac{\rho\, h^2}{3}\, \Delta w_{tt} \,+\, \frac{E\, h^2}{3(1-\nu^2)}\, \Delta^2 w \,=\, g\,, \tag{2.19}$$

with boundary conditions

$$\left.\begin{array}{rcl}
w &=& 0 \\
n_1 \left(w_{xx} + \frac{\nu}{1-\nu} \Delta w \right) + n_2 w_{xy} &=& 0 \\
n_1 w_{xy} + n_2 \left(w_{yy} + \frac{\nu}{1-\nu} \Delta w \right) &=& 0
\end{array} \right\} \quad \text{on} \ \ \partial\Omega_0\,, \tag{2.20}$$

where $\mathbf{n} = (n_1, n_2)$ is the outward normal to Ω_0.

2.2 Elastoplastic oscillations

We still consider here $\bar{\mathbf{u}} = \mathbf{u}^L$, $\bar{\boldsymbol{\sigma}}$, and $\bar{\boldsymbol{\varepsilon}}$ as in (2.4) and (2.10), with $\boldsymbol{\varepsilon}^L$ given by (2.6). In addition, following [7, 9], we make further specific hypotheses.

(B1) The strain tensor $\bar{\boldsymbol{\varepsilon}}$ is decomposed in elastic and plastic components $\bar{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$.

(B2) The elastic constitutive law is as in (2.9), that is,

$$\bar{\boldsymbol{\sigma}} = 2\mu \boldsymbol{\varepsilon}^e + \lambda (\boldsymbol{\varepsilon}^e : \boldsymbol{\delta}) \boldsymbol{\delta}\,. \tag{2.21}$$

(B3) The plastic deformations are *volume preserving* in the sense that

$$\boldsymbol{\varepsilon}^p : \boldsymbol{\delta} \,=\, 0\,. \tag{2.22}$$

The von Mises plastic yield condition is stated in terms of the stress deviator $\mathbf{D}\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}} - \frac{1}{3}(\bar{\boldsymbol{\sigma}} : \boldsymbol{\delta})\, \boldsymbol{\delta}$ as

(B4) $\mathbf{D}\bar{\boldsymbol{\sigma}} : \mathbf{D}\bar{\boldsymbol{\sigma}} \le \frac{2}{3}R^2$,

where $R > 0$ is a given *yield limit*. Using the formula $\mathbf{D}\bar{\boldsymbol{\sigma}} : \mathbf{D}\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}} : \mathbf{D}\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{\sigma}} - \frac{1}{3}(\bar{\boldsymbol{\sigma}} : \boldsymbol{\delta})^2$, we may rewrite (B4) as

$$\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} + 3\sigma_{12}^2 \le R^2 . \tag{2.23}$$

For the plastic strain, we prescribe the *normality flow rule*

(B5) $\boldsymbol{\varepsilon}_t^p : (\bar{\boldsymbol{\sigma}} - \boldsymbol{\theta}) \ \ge \ 0 \quad \forall\, \boldsymbol{\theta} \in \mathbb{T}^{3\times 3}_{\mathrm{sym}} : \quad \mathbf{D}\boldsymbol{\theta} : \mathbf{D}\boldsymbol{\theta} \le \frac{2}{3}R^2$,

where the subscript $_t$ denotes the time derivative. Introducing the set

$$K = \left\{ \boldsymbol{\theta} \in \mathbb{T}^{3\times 3}_{\mathrm{sym}} \, : \, \mathbf{D}\boldsymbol{\theta} : \mathbf{D}\boldsymbol{\theta} \le \frac{2}{3}R^2 \right\}$$

of admissible stresses and using the convex analysis formalism of e. g. [21], we can rewrite (B4)+(B5) in subdifferential form as

$$\boldsymbol{\varepsilon}_t^p \ \in \ \partial I_K(\bar{\boldsymbol{\sigma}}) , \tag{2.24}$$

where I_K is the indicator function of K and ∂I_K its subdifferential. For the sake of completeness, we recall other equivalent formulations of the von Mises criterion, cf. also [17].

Proposition 2.1. *Each of the following two conditions is equivalent to (B4)+(B5).*

(i) (multiplier formulation) *Condition (B4) holds, and there exists a multiplier* $\ell_t \ge 0$ *such that* $\ell_t = 0$ *if* $\mathbf{D}\bar{\boldsymbol{\sigma}} : \mathbf{D}\bar{\boldsymbol{\sigma}} < \frac{2}{3}R^2$, *and*

$$\boldsymbol{\varepsilon}_t^p \ = \ \ell_t \, \mathbf{D}\bar{\boldsymbol{\sigma}} ; \tag{2.25}$$

(ii) (dissipation formulation) *Let*

$$\Psi(\boldsymbol{\xi}) = \begin{cases} \sqrt{\frac{2}{3}}R\,\sqrt{\boldsymbol{\xi} : \boldsymbol{\xi}} & \text{if } \boldsymbol{\xi} : \boldsymbol{\delta} = 0 , \\ +\infty & \text{if } \boldsymbol{\xi} : \boldsymbol{\delta} \ne 0 , \end{cases}$$

be the pseudopotential of dissipation. *Then*

$$\bar{\boldsymbol{\sigma}} \ \in \ \partial\Psi(\boldsymbol{\varepsilon}_t^p) , \tag{2.26}$$

that is,

$$\bar{\boldsymbol{\sigma}} : (\boldsymbol{\varepsilon}_t^p - \boldsymbol{\xi}) \ge \Psi(\boldsymbol{\varepsilon}_t^p) - \Psi(\boldsymbol{\xi}) \quad \forall \xi \in \mathbb{T}^{3\times 3}_{\mathrm{sym}} . \tag{2.27}$$

Sketch of the proof. Choosing in (2.27) consecutively $\boldsymbol{\xi} = 2\boldsymbol{\varepsilon}_t^p$ and $\boldsymbol{\xi} = 0$, we see that (2.26) is equivalent to the system

$$\bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}_t^p - \Psi(\boldsymbol{\varepsilon}_t^p) \ = \ 0 , \tag{2.28}$$

$$\bar{\boldsymbol{\sigma}} : \boldsymbol{\xi} - \Psi(\boldsymbol{\xi}) \ \le \ 0 \quad \forall \xi \in \mathbb{T}^{3\times 3}_{\mathrm{sym}} . \tag{2.29}$$

The implication (i)$\Rightarrow$(B4)+(B5) is straightforward. Assume now that (B4)+(B5) holds. We obtain (2.29) from (B4) and the Cauchy-Schwarz inequality. Putting in (B5) $\boldsymbol{\theta} = \bar{\boldsymbol{\sigma}} \pm \boldsymbol{\delta}$, we see that $\boldsymbol{\varepsilon}^p_t : \boldsymbol{\delta} = 0$. Identity (2.28) holds automatically if $\boldsymbol{\varepsilon}^p_t = 0$, otherwise we set in (B5) $\boldsymbol{\theta} = \frac{2}{3}R^2\boldsymbol{\varepsilon}^p_t/\Psi(\boldsymbol{\varepsilon}^p_t)$, and (2.28) follows again from the Cauchy-Schwarz inequality.

It remains to check the implication (ii)$\Rightarrow$(i). To this end, we choose $\boldsymbol{\xi} = \mathbf{D}\bar{\boldsymbol{\sigma}}$ in (2.29), and obtain (B4). This and (2.28) imply in turn that

$$\mathbf{D}\bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}^p_t \ = \ \sqrt{\frac{2}{3}}R\sqrt{\boldsymbol{\varepsilon}^p_t : \boldsymbol{\varepsilon}^p_t} \ \geq \ \sqrt{\mathbf{D}\bar{\boldsymbol{\sigma}} : \mathbf{D}\bar{\boldsymbol{\sigma}}}\sqrt{\boldsymbol{\varepsilon}^p_t : \boldsymbol{\varepsilon}^p_t}\,,$$

and (2.25) follows from the reverse Cauchy-Schwarz inequality. □

Note that both (2.24) and (2.26) can be interpreted as a *maximal dissipation principle*. In (2.24), for a given stress $\bar{\boldsymbol{\sigma}}$, the strain rate $\boldsymbol{\varepsilon}^p_t$ is chosen so as to maximize the dissipation rate $\bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}^p_t$ among all stress values $\boldsymbol{\theta} \in K$; in (2.27), for a given strain rate $\boldsymbol{\varepsilon}^p_t$, the stress $\bar{\boldsymbol{\sigma}}$ is required to maximize the reduced dissipation rate $\bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}^p_t - \Psi(\boldsymbol{\varepsilon}^p_t)$ over the set of all values $\boldsymbol{\xi}$ of the strain rate.

Similarly as in (2.11), we have

$$\bar{\boldsymbol{\sigma}} = \frac{E}{1-\nu^2}\begin{pmatrix} \varepsilon^e_{11} + \nu\varepsilon^e_{22} & (1-\nu)\varepsilon^e_{12} & 0 \\ (1-\nu)\varepsilon^e_{12} & \nu\varepsilon^e_{11} + \varepsilon^e_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\varepsilon}^e = \begin{pmatrix} \varepsilon^e_{11} & \varepsilon^e_{12} & 0 \\ \varepsilon^e_{12} & \varepsilon^e_{22} & 0 \\ 0 & 0 & -\frac{\nu}{1-\nu}(\varepsilon^e_{11} + \varepsilon^e_{22}) \end{pmatrix}. \tag{2.30}$$

Assume that $\varepsilon^p_{13} = \varepsilon^p_{23} = 0$ at initial time $t = 0$. Then we have by (B3) and (2.25) that

$$\boldsymbol{\varepsilon}^p = \begin{pmatrix} \varepsilon^p_{11} & \varepsilon^p_{12} & 0 \\ \varepsilon^p_{12} & \varepsilon^p_{22} & 0 \\ 0 & 0 & -(\varepsilon^p_{11} + \varepsilon^p_{22}) \end{pmatrix}. \tag{2.31}$$

It is convenient to consider $\bar{\boldsymbol{\sigma}}$, $\boldsymbol{\varepsilon}^e$, and $\boldsymbol{\varepsilon}^p$ as vectors with three components. To this end, we introduce the notation

$$\bar{\boldsymbol{\sigma}}_* = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}, \quad \boldsymbol{\varepsilon}^e_* = \begin{pmatrix} \varepsilon^e_{11} \\ \varepsilon^e_{22} \\ \varepsilon^e_{12} \end{pmatrix}, \quad \boldsymbol{\varepsilon}^p_* = \begin{pmatrix} \varepsilon^p_{11} \\ \varepsilon^p_{22} \\ \varepsilon^p_{12} \end{pmatrix}, \quad \bar{\boldsymbol{\varepsilon}}_* = \begin{pmatrix} -zw_{xx} \\ -zw_{yy} \\ -zw_{xy} \end{pmatrix}. \tag{2.32}$$

According to (B1) and (2.30), we have

$$\bar{\boldsymbol{\varepsilon}}_* = \boldsymbol{\varepsilon}^p_* + \boldsymbol{\varepsilon}^e_*\,, \quad \bar{\boldsymbol{\sigma}}_* = \mathbf{C}\boldsymbol{\varepsilon}^e_*\,, \tag{2.33}$$

where $\mathbf{C}$ is the positive definite matrix

$$\mathbf{C} = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix}. \tag{2.34}$$

Let $\mathbf{D}_*, \mathbf{J}$ be the matrices

$$\mathbf{D}_* = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2.35}$$

In view of (2.23), condition (B4) can be restated as

$$\bar{\boldsymbol{\sigma}}_* \cdot \mathbf{D}_* \bar{\boldsymbol{\sigma}}_* \le R^2 ,$$

and (B5) reads

$$\mathbf{J}(\boldsymbol{\varepsilon}_*^p)_t \cdot (\bar{\boldsymbol{\sigma}}_* - \boldsymbol{\theta}_*) \ \ge \ 0 \quad \forall \boldsymbol{\theta}_* \in K_* , \tag{2.36}$$

where

$$K_* = \{\boldsymbol{\theta}_* \in \mathbb{R}^3 \, : \, \boldsymbol{\theta}_* \cdot \mathbf{D}_* \boldsymbol{\theta}_* \le R^2\}. \tag{2.37}$$

Alternatively, we can write this variational inequality in the form

$$\begin{aligned} &\boldsymbol{\varepsilon}_*^e \in \mathbf{C}^{-1}(K_*) , \\ &\mathbf{J}\mathbf{C}(\bar{\boldsymbol{\varepsilon}}_* - \boldsymbol{\varepsilon}_*^e)_t \cdot (\boldsymbol{\varepsilon}_*^e - \boldsymbol{\eta}_*) \ \ge \ 0 \quad \forall \boldsymbol{\eta}_* \in \mathbf{C}^{-1}(K_*) . \end{aligned} \tag{2.38}$$

Let us choose in $\mathbb{R}^3$ the scalar product

$$\langle \boldsymbol{\xi}_*, \boldsymbol{\eta}_* \rangle = \mathbf{J}\mathbf{C}\boldsymbol{\xi}_* \cdot \boldsymbol{\eta}_* . \tag{2.39}$$

This is meaningful, since $\mathbf{JC} = \mathbf{CJ}$ is a symmetric positive definite matrix. We then prescribe the canonical initial condition

$$\boldsymbol{\varepsilon}_*^e(0) = Q_{\mathbf{C}^{-1}(K_*)}(\bar{\boldsymbol{\varepsilon}}_*(0)) , \tag{2.40}$$

where $Q_{\mathbf{C}^{-1}(K_*)}$ is the orthogonal projection onto $\mathbf{C}^{-1}(K_*)$ with respect to the scalar product (2.39). For every $\bar{\boldsymbol{\varepsilon}}_* \in W^{1,1}(0,T;\mathbb{R}^3)$, Problem (2.38)–(2.40) has a unique solution $\boldsymbol{\varepsilon}_*^e$ in the metric space $W^{1,1}(0,T;\mathbf{C}^{-1}(K_*))$, and the solution mapping

$$\mathcal{S}_{\mathbf{C}^{-1}(K_*)} : W^{1,1}(0,T;\mathbb{R}^3) \to W^{1,1}(0,T;\mathbf{C}^{-1}(K_*)) : \ \bar{\boldsymbol{\varepsilon}}_* \mapsto \boldsymbol{\varepsilon}_*^e$$

introduced in [11] is called the *stop with characteristic* $\mathbf{C}^{-1}(K_*)$. The set $\mathbf{C}^{-1}(K_*)$ is an ellipsoid, hence $\mathcal{S}_{\mathbf{C}^{-1}(K_*)}$ is locally Lipschitz continuous and admits a $1/2$-Hölder continuous extension to $C([0,T];\mathbb{R}^3) \to C([0,T];\mathbf{C}^{-1}(K_*))$, see [5, Chapter 2]. More about the vectorial stop will be said in Section 3. This concept enables us to rewrite (2.38) as

$$\boldsymbol{\varepsilon}_*^e \ = \ \mathcal{S}_{\mathbf{C}^{-1}(K_*)}[\bar{\boldsymbol{\varepsilon}}_*] ,$$

or, equivalently,

$$\bar{\boldsymbol{\sigma}}_* \ = \ \mathbf{C}\mathcal{S}_{\mathbf{C}^{-1}(K_*)}[\bar{\boldsymbol{\varepsilon}}_*] . \tag{2.41}$$

The stop $\mathcal{S}_Z$ with any symmetric convex closed characteristic Z has the following elementary scaling property:

$$\mathcal{S}_Z[\boldsymbol{\varepsilon}_*] = -\mathcal{S}_Z[-\boldsymbol{\varepsilon}_*] = \frac{1}{c}\mathcal{S}_{cZ}[c\boldsymbol{\varepsilon}_*] \tag{2.42}$$

for every $c > 0$ and every $\boldsymbol{\varepsilon}_* \in W^{1,1}(0,T;\mathbb{R}^3)$, where $cZ = \{\boldsymbol{\theta}_* \in \mathbb{R}^3 \,:\, \frac{1}{c}\boldsymbol{\theta}_* \in Z\}$.

Notice first that we obtain from (2.41), (2.32), and (2.42) that

$$\bar{\boldsymbol{\sigma}}_* \;=\; -z\mathbf{C}\mathcal{S}_{\frac{1}{|z|}\mathbf{C}^{-1}(K_*)}\begin{bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{bmatrix}. \tag{2.43}$$

We see that in view of (2.34) and (2.37), at distance $|z|$ from the midsurface, the virtual elasticity modulus is $|z|E$ and the virtual yield limit is $R/|z|$. This is precisely the behavior mentioned in the introduction. The eccentric layers seem harder in elasticity and softer in plasticity than the central ones. This produces the multiyield effect when integrating over the thickness of the plate.

To derive a counterpart of the partial differential equation (2.19), we consider test functions $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\varepsilon}}$ as in (2.16), and set in agreement with (2.32)

$$\hat{\boldsymbol{\varepsilon}}_* = \begin{pmatrix} -z\hat{w}_{xx} \\ -z\hat{w}_{yy} \\ -z\hat{w}_{xy} \end{pmatrix}. \tag{2.44}$$

The first and the third integral in (2.15) are evaluated in the same way as in (2.17). The remaining one has to be treated more carefully. Using (2.42), we obtain

$$\begin{aligned}
\int_\Omega \bar{\boldsymbol{\sigma}} : \hat{\boldsymbol{\varepsilon}}\, dx\, dy\, dz \;&=\; \int_\Omega \mathbf{J}\bar{\boldsymbol{\sigma}}_* \cdot \hat{\boldsymbol{\varepsilon}}_*\, dx\, dy\, dz \\
&=\; \int_{-h}^{h}\int_{\Omega_0} \mathbf{J}\mathbf{C}\mathcal{S}_{\mathbf{C}^{-1}(K_*)}\begin{bmatrix} -zw_{xx} \\ -zw_{yy} \\ -zw_{xy} \end{bmatrix} \cdot \begin{pmatrix} -z\hat{w}_{xx} \\ -z\hat{w}_{yy} \\ -z\hat{w}_{xy} \end{pmatrix} dx\, dy\, dz \\
&=\; 2\int_0^h \int_{\Omega_0} z^2 \mathbf{J}\mathbf{C}\mathcal{S}_{\frac{1}{z}\mathbf{C}^{-1}(K_*)}\begin{bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{bmatrix} \cdot \begin{pmatrix} \hat{w}_{xx} \\ \hat{w}_{yy} \\ \hat{w}_{xy} \end{pmatrix} dx\, dy\, dz \\
&=\; \int_{\Omega_0} \mathbf{J}\mathbf{C}\left(\int_{1/h}^{\infty} 2q^{-4}\mathcal{S}_{q\mathbf{C}^{-1}(K_*)}\begin{bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{bmatrix} dq\right) \cdot \begin{pmatrix} \hat{w}_{xx} \\ \hat{w}_{yy} \\ \hat{w}_{xy} \end{pmatrix} dx\, dy\,.
\end{aligned}$$

The mapping

$$\mathcal{F} : \begin{pmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{pmatrix} \mapsto \int_{1/h}^{\infty} 2q^{-4}\mathcal{S}_{q\mathbf{C}^{-1}(K_*)}\begin{bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{bmatrix} dq$$

is called the *vectorial Prandtl-Ishlinskii operator*. The equation for oscillations of an elastoplastic plate can thus be written in the form

$$\rho \int_{\Omega_0} \left(w_{tt}\hat{w} + \frac{h^2}{3}\left(w_{xtt}\,\hat{w}_x + w_{ytt}\,\hat{w}_y\right)\right) dx\,dy$$
$$+ \int_{\Omega_0} \mathbf{JC}\mathcal{F}\left[\begin{array}{c} w_{xx} \\ w_{yy} \\ w_{xy} \end{array}\right] \cdot \left(\begin{array}{c} \hat{w}_{xx} \\ \hat{w}_{yy} \\ \hat{w}_{xy} \end{array}\right) dx\,dy \;=\; \int_{\Omega_0} g\,\hat{w}\,dx\,dy \qquad \forall \hat{w} \in V\,, \tag{2.45}$$

with g as in (2.17)–(2.18).

2.3 Kinematic hardening

In order to model kinematic hardening, we assume that the stress $\bar{\boldsymbol{\sigma}}$ of the form (2.10) is decomposed into the sum $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^{ep} + \boldsymbol{\sigma}^b$ of a purely elastoplastic stress tensor $\boldsymbol{\sigma}^{ep}$ satisfying hypotheses (B1)–(B6), and the so-called *backstress* $\boldsymbol{\sigma}^b$, which, in the three dimensional representation (2.32), is assumed to obey an elastic constitutive law

$$\boldsymbol{\sigma}^b_* \;=\; \mathbf{JB}\bar{\boldsymbol{\varepsilon}}_*\,, \tag{2.46}$$

where $\mathbf{B}$ is a constant symmetric (3×3)-matrix such that $\mathbf{JB} = \mathbf{BJ}$, and the inequality

$$\mathbf{JB}\boldsymbol{\xi}_* \cdot \boldsymbol{\xi}_* \;\ge\; \beta(\xi_{11}^2 + \xi_{22}^2) \qquad \forall \boldsymbol{\xi}_* = \left(\begin{array}{c} \xi_{11} \\ \xi_{22} \\ \xi_{12} \end{array}\right) \tag{2.47}$$

holds with some $\gamma > 0$. Repeating the computation from the previous subsection, we obtain, as a counterpart of (2.45), the equation for w in the form

$$\rho \int_{\Omega_0} \left(w_{tt}\hat{w} + \frac{h^2}{3}\left(w_{xtt}\,\hat{w}_x + w_{ytt}\,\hat{w}_y + \mathbf{JBD}_2 w \cdot \mathbf{D}_2\hat{w}\right)\right) dx\,dy$$
$$+ \int_{\Omega_0} \mathbf{JC}\mathcal{F}\left[\mathbf{D}_2 w\right] \cdot \mathbf{D}_2\hat{w}\,dx\,dy \;=\; \int_{\Omega_0} g\,\hat{w}\,dx\,dy \qquad \forall \hat{w} \in V\,, \tag{2.48}$$

where the vector-valued differential operator $\mathbf{D}_2$ is defined as

$$\mathbf{D}_2 w \;=\; \left(\begin{array}{c} w_{xx} \\ w_{yy} \\ w_{xy} \end{array}\right).$$

3 Vectorial Prandtl-Ishlinskii operator

The original Prandtl-Ishlinskii construction ([20, 10]) is one dimensional. A vector Prandtl-Ishlinskii model in connection with phase transitions was considered in [14]. It is based on the concept of *stop operator*, which we recall here in an abstract framework. Consider a real, separable Hilbert space X endowed with a scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$ (in application to our model, we will consider $X = \mathbb{R}^3$ with scalar product (2.39)), and assume that a convex closed set $Z \subset X$ containing the origin is given. For each function $v \in W^{1,1}(0,T;X)$, we define $\chi \in W^{1,1}(0,T;X)$ as the unique solution of the variational inequality

$$\left.\begin{aligned} &\chi(t) \in Z \quad \forall t \in [0,T], \\ &\chi(0) = Q_Z(v(0)), \\ &\langle \dot{v}(t) - \dot{\chi}(t), \chi(t) - y \rangle \geq 0 \ \text{ a.e.} \quad \forall y \in Z, \end{aligned}\right\} \tag{3.1}$$

where $Q_Z : X \to Z$ is the orthogonal projection onto Z, and the dot denotes differentiation with respect to t. The solution mapping

$$\mathcal{S}_Z : W^{1,1}(0,T;X) \to W^{1,1}(0,T;X) : v \mapsto \chi \tag{3.2}$$

is called the *stop with characteristic* Z. It was introduced in [11] and its analytical properties were studied in detail in [5]. We list here some properties of the stop that are needed in the sequel.

Proposition 3.1. *The mapping $\mathcal{S}_Z$ defined by (3.1)–(3.2) has the following properties.*

(i) *$\mathcal{S}_Z$ is continuous in the strong topology of $W^{1,1}(0,T;X)$;*

(ii) *If the boundary of Z is of class $W^{2,\infty}$ (that is, the outward normal mapping is Lipschitz continuous), then $\mathcal{S}_Z$ is locally Lipschitz continuous in $W^{1,1}(0,T;X)$;*

(iii) *If Z has a nonempty interior, then $\mathcal{S}_Z$ can be extended to a continuous mapping $C([0,T];X) \to C([0,T];X)$;*

(iv) *If Z is uniformly strictly convex, then $\mathcal{S}_Z : C([0,T];X) \to C([0,T];X)$ is $1/2$-Hölder continuous;*

(v) *The mapping $\mathcal{S}_Z$ is monotone in the sense that*

$$\langle \mathcal{S}_Z[v_1](t) - \mathcal{S}_Z[v_2](t), \dot{v}_1(t) - \dot{v}_2(t) \rangle \geq \frac{1}{2}\frac{d}{dt} \left|\mathcal{S}_Z[v_1](t) - \mathcal{S}_Z[v_2](t)\right|^2 \quad \textit{a. e.} \tag{3.3}$$

for every $v_1, v_2 \in W^{1,1}(0,T;X)$;

(vi) *The mapping $\mathcal{S}_Z$ is locally monotone, i. e.*

$$\left\langle \frac{d}{dt}\mathcal{S}_Z[v](t), \dot{v}(t) \right\rangle = \left| \frac{d}{dt}\mathcal{S}_Z[v](t) \right|^2 \quad \textit{a. e.} \tag{3.4}$$

for every $v \in W^{1,1}(0,T;X)$;

(vii) *The "second order energy inequality"*

$$\left\langle \frac{d}{dt}\mathcal{S}_Z[v](t), \ddot{v}(t) \right\rangle \geq \frac{1}{2}\frac{d}{dt} \left\langle \frac{d}{dt}\mathcal{S}_Z[v](t), \dot{v}(t) \right\rangle \tag{3.5}$$

holds in the sense of distributions for every $v \in W^{2,1}(0,T;X)$.

Detailed proofs of the above statements are given in [5, Chapter 2] except for the inequality (3.5) which is derived in [12, pp. 37–38]. Notice a certain similarity of (3.5) with the "real" physical energy inequality

$$\langle \mathcal{S}_Z[v](t), \dot{v}(t)\rangle \geq \frac{1}{2}\frac{d}{dt}\left|\mathcal{S}_Z[v](t)\right|^2 , \quad \text{a. e.} \tag{3.6}$$

which follows immediately from (3.3) by choosing $v_2 = 0$. In (3.6), the right hand side is the time derivative of the potential energy associated with the stop, and the (nonnegative) difference between the left hand and the right hand sides is the *dissipation rate*. If $\dim X = 1$, then it can be identified with the area of the corresponding hysteresis loops. Instead, the "dissipation" in (3.5) is related to the *curvature* of the hysteresis branches. A detailed discussion on this subject can be found in [12, Section II. 4].

As another consequence of Proposition 3.1 (v) we have

$$\frac{d}{dt}\left|\mathcal{S}_Z[v_1](t) - \mathcal{S}_Z[v_2](t)\right| \leq \left|\dot{v}_1(t) - \dot{v}_2(t)\right| \quad \text{a. e.}, \tag{3.7}$$

hence

$$\left|\mathcal{S}_Z[v_1](t) - \mathcal{S}_Z[v_2](t)\right| \leq \left|\mathcal{S}_Z[v_1](0) - \mathcal{S}_Z[v_2](0)\right| + \int_0^t \left|\dot{v}_1(\tau) - \dot{v}_2(\tau)\right| d\tau \tag{3.8}$$

for all $t \in [0, T]$.

We now assume additionally that Z is a bounded, convex, closed set containing 0 in its interior, that is, there exist $M > m > 0$ such that

$$B_m(0) \subset Z \subset B_M(0)\,, \tag{3.9}$$

where $B_\rho(x)$ for $\rho > 0$ and $x \in X$ denotes the open ball centered at x with radius ρ. Given a nonnegative function $\varphi \in L^1(0, \infty)$, we define the *Prandtl-Ishlinskii operator* $\mathcal{F}$ with characteristic Z and density φ by the formula

$$\mathcal{F}[v](t) = \int_0^\infty \mathcal{S}_{qZ}[v](t)\,\varphi(q)\,dq \tag{3.10}$$

for $v \in W^{1,1}(0, T; X)$. The definition is meaningful due to the fact that, setting $v_\infty = \max\{|v(t)| : t \in [0, T]\}$, we have $\mathcal{S}_{qZ}[v](t) = v(t)$ for all $q > v_\infty/m$ and all $t \in [0, T]$, so that, using the elementary estimate $|\mathcal{S}_{qZ}[v](t)| \leq qM$, we have

$$|\mathcal{F}[v](t)| \leq v_\infty \frac{M}{m}$$

for all $t \in [0, T]$. As a direct consequence of Proposition 3.1, the mapping $\mathcal{F}$ has the following properties.

Proposition 3.2. *Let $\varphi \in L^1(0, \infty)$ be given, $\varphi(q) \geq 0$ a. e., not identically zero, and let $\mathcal{F}$ be defined by (3.10). Then we have:*

(i) *Both* $\mathcal{F} : W^{1,1}(0,T;X) \to W^{1,1}(0,T;X)$ *and* $\mathcal{F} : C([0,T];X) \to C([0,T];X)$ *are continuous with respect to the strong topologies;*

(ii) *The mapping* $\mathcal{F}$ *is monotone in the sense that the inequality*

$$\langle \mathcal{F}[v_1](t) - \mathcal{F}[v_2](t), \dot{v}_1(t) - \dot{v}_2(t)\rangle \geq \frac{1}{2}\frac{d}{dt}\int_0^\infty |\mathcal{S}_Z[v_1](t) - \mathcal{S}_Z[v_2](t)|^2 \varphi(q)\, dq \tag{3.11}$$

holds for every $v_1, v_2 \in W^{1,1}(0,T;X)$ *and a. e.* $t \in (0,T)$*;*

(iii) *The mapping* $\mathcal{F}$ *is locally monotone, i. e.*

$$|\dot{v}(t)|^2 \int_0^\infty \varphi(q)\, dq \geq \left\langle \frac{d}{dt}\mathcal{F}[v](t), \dot{v}(t)\right\rangle \geq \left|\frac{d}{dt}\mathcal{F}[v](t)\right|^2 \left(\int_0^\infty \varphi(q)\, dq\right)^{-1} \quad a.\ e. \tag{3.12}$$

for every $v \in W^{1,1}(0,T;X)$*;*

(iv) *The "second order energy inequality"*

$$\left\langle \frac{d}{dt}\mathcal{F}[v](t), \ddot{v}(t)\right\rangle \geq \frac{1}{2}\frac{d}{dt}\left\langle \frac{d}{dt}\mathcal{F}[v](t), \dot{v}(t)\right\rangle \tag{3.13}$$

holds in the sense of distributions for every $v \in W^{2,1}(0,T;X)$*.*

The canonical choice of initial conditions in (3.1) makes it possible to evaluate explicitly $\mathcal{F}[v](0)$ at time $t = 0$. We have

$$\mathcal{F}[v](0) = \int_0^\infty Q_{qZ}(v(0))\, \varphi(q)\, dq\,. \tag{3.14}$$

We thus can define the *initial value mapping*

$$\mathbf{A}_{\mathcal{F}}(v) : \mathbb{R}^3 \to \mathbb{R}^3 : v \mapsto \int_0^\infty Q_{qZ}(v)\, \varphi(q)\, dq\,. \tag{3.15}$$

Since Q_{qZ} is nonexpansive, we see that $\mathbf{A}_{\mathcal{F}}$ is Lipschitz continuous in $\mathbb{R}^3$.

4 Existence and uniqueness of solutions

Let us first fix the hypotheses and notation. We assume that $\Omega_0 \subset \mathbb{R}^2$ is a Lipschitzian domain, and denote in agreement with Section 2.1

$$\begin{aligned} H &= L^2(\Omega_0)\,, \\ W &= H_0^1(\Omega_0) := \{w \in H^1(\Omega_0) \,:\, w\big|_{\partial\Omega_0} = 0\}\,, \\ V &= H^2(\Omega_0) \cap H_0^1(\Omega_0)\,. \end{aligned}$$

By $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ we denote the outward normal vector to Ω_0. For functions $\mathbf{v} : \Omega_0 \to \mathbb{R}^3$ with components (v^1, v^2, v^3), we define the differential operator

$$\mathbf{D}_1 \mathbf{v} \;=\; \begin{pmatrix} v^1_x + v^3_y \\ v^3_x + v^2_y \end{pmatrix} . \tag{4.1}$$

For $\hat{w} \in V$ and $\mathbf{v} \in L^2(\Omega_0; \mathbb{R}^3)$ such that $\mathbf{D}_1 \mathbf{v} \in L^2(\Omega_0; \mathbb{R}^3)$, we have the following Green/Gauss-type formula

$$\int_{\Omega_0} (\mathbf{J}\mathbf{v} \cdot \mathbf{D}_2 \hat{w} + \mathbf{D}_1 \mathbf{v} \cdot \nabla \hat{w}) \, dx \, dy = \int_{\partial\Omega_0} (\mathbf{v} \bullet \mathbf{n}) \cdot \nabla \hat{w} \, ds \,, \tag{4.2}$$

where we denote

$$\mathbf{v} \bullet \mathbf{n} \;=\; n_1 \begin{pmatrix} v^1 \\ v^3 \end{pmatrix} + n_2 \begin{pmatrix} v^3 \\ v^2 \end{pmatrix} . \tag{4.3}$$

Note also the formal identity

$$\mathbf{D}_1 \mathbf{D}_2 \hat{w} \;=\; \nabla \Delta \hat{w} \quad \forall \hat{w} \in H^3(\Omega_0) \,. \tag{4.4}$$

We now restate Equation (2.48) in a slightly more general form. Removing the positive constants that have no influence on the existence and uniqueness result, and keeping the matrices $\mathbf{J}, \mathbf{C}, \mathbf{B}$ from Section 2, we consider the variational problem

$$\begin{aligned} &\int_{\Omega_0} \big(w_{tt} (\hat{w} - \Delta \hat{w}) + \mathbf{J} \left(\mathbf{C} \mathcal{F} \left[\mathbf{D}_2 w \right] + \mathbf{B} \mathbf{D}_2 w \right) \cdot \mathbf{D}_2 \hat{w} \big) \, dx \, dy \\ &\qquad = \int_{\Omega_0} \big(g \, \hat{w} + \nabla \mathbf{G} \cdot \nabla \hat{w} \big) \, dx \, dy \qquad \forall \hat{w} \in V \,, \end{aligned} \tag{4.5}$$

where g and $\mathbf{G}$ are given functions, and $\mathcal{F}$ is a Prandtl-Ishlinskii operator as in (3.10) associated with a convex constraint $Z \subset \mathbb{R}^3$ satisfying (3.9). We prescribe initial conditions

$$w(x, y, 0) = w^0(x, y) \,, \quad w_t(x, y, 0) = w^1(x, y) \quad \text{ for } (x, y) \in \Omega_0, \tag{4.6}$$

and boundary condition

$$w(x, y, t) = 0 \quad \text{ for } (x, y) \in \partial\Omega_0 \,. \tag{4.7}$$

Indeed, smooth solutions of the variational equation (4.5) satisfy a second ("no stress") boundary condition (cf. (4.2), (2.20))

$$(\mathbf{C} \mathcal{F} \left[\mathbf{D}_2 w \right] + \mathbf{B} \mathbf{D}_2 w) \bullet \mathbf{n} = 0 \quad \text{on } \partial\Omega_0 \,. \tag{4.8}$$

Hypothesis 4.1. *The data of Problem (4.5)–(4.7) fulfill the following conditions.*

(i) *The function φ in (3.10) is nonnegative and belongs to $L^1(0, \infty)$;*

(ii) $w^0, w^1 \in H^3(\Omega_0) \cap V$, *and the compatibility conditions*

$$\big(\mathbf{C}\mathbf{A}_{\mathcal{F}}(\mathbf{D}_2 w^0) + \mathbf{B}\mathbf{D}_2 w^0\big) \bullet \mathbf{n} = 0\,, \qquad \mathbf{D}_2 w^1 \bullet \mathbf{n} = 0\,,$$

hold a. e. on $\partial\Omega_0$, *where* $\mathbf{A}_{\mathcal{F}}$ *is the initial value mapping (3.15) of the operator* $\mathcal{F}$*;*

(iii) $g \in L^2(0,T;H)$ *and* $\mathbf{G} \in L^2(0,T;H^1(\Omega_0))$ *are such that* $g_t \in L^2(0,T;H)$ *and* $\mathbf{G}_t \in L^2(0,T;H^1(\Omega_0))$

The main result of this section reads as follows.

Theorem 4.2. *Let Hypothesis 4.1 hold. Then Problem (4.5)–(4.7) admits a unique solution* $w \in L^2(0,T;V)$ *such that*

$$w_t \in L^2(0,T;V) \cap C([0,T];W)\,, \quad w_{tt} \in L^2(0,T;W)\,, \tag{4.9}$$

and Eq. (4.5) holds for a. e. $t \in (0,T)$.

The uniqueness proof is easy. Let w^1, w^2 be two solutions, and let $\bar{w} = w^1 - w^2$. Then for every $\hat{w} \in V$ we have

$$\int_{\Omega_0} \big(\bar{w}_{tt}(\hat{w} - \Delta\hat{w}) + \mathbf{J}\left(\mathbf{C}(\mathcal{F}\left[\mathbf{D}_2 w^1\right] - \mathcal{F}\left[\mathbf{D}_2 w^2\right]) + \mathbf{B}\mathbf{D}_2\bar{w}\right) \cdot \mathbf{D}_2\hat{w}\big)\, dx\, dy = 0\,. \tag{4.10}$$

We may set $\hat{w} = \bar{w}_t$ in (4.10), and use (3.11) together with (2.39) to obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega_0} \Big(|\bar{w}_t|^2 + |\nabla\bar{w}_t|^2 + \mathbf{J}\mathbf{B}\mathbf{D}_2\bar{w} \cdot \mathbf{D}_2\bar{w} \\ &\quad + \int_0^\infty \varphi(q)\mathbf{J}\mathbf{C}\left(\mathcal{S}_{qZ}\left[\mathbf{D}_2 w^1\right] - \mathcal{S}_{qZ}\left[\mathbf{D}_2 w^2\right]\right) \cdot \left(\mathcal{S}_{qZ}\left[\mathbf{D}_2 w^1\right] - \mathcal{S}_{qZ}\left[\mathbf{D}_2 w^2\right]\right) dq\Big)\, dx\, dy \\ &\leq \; 0\,. \end{aligned}$$

Both w^1 and w^2 satisfy the same initial conditions, hence $w^1 = w^2$.

The existence proof is more complicated and we carry it out in several steps in Subsections 4.1–4.3 below, following in principle the standard methods of solving PDEs with hysteresis as in [3, 12, 22].

We first introduce an artificial viscosity parameter $\gamma > 0$, then solve an auxiliary linear problem, and use the Banach fixed point argument to construct a "viscous" approximation of the solution. In the last step, we let γ tend to 0 and check that the limit is the desired solution to Problem (4.5)–(4.7).

4.1 A linear parabolic equation

Let $\mathbf{v} \in (L^2(0,T;H))^3$, $g^* \in L^2(0,T;H)$, $G^* \in L^2(0,T;H^1(\Omega_0))$, and $u_0 \in W$ be given functions. We fix a parameter $\gamma > 0$, and consider the problem

$$\int_{\Omega_0} \Big(u_t(\hat{w} - \Delta\hat{w}) + \mathbf{J}\,(\mathbf{v} + \gamma\mathbf{D}_2 u)\cdot\mathbf{D}_2\hat{w}\Big)\,dx\,dy$$
$$= \int_{\Omega_0} \Big(g^*\,\hat{w} + \nabla\mathbf{G}^*\cdot\nabla\hat{w}\Big)\,dx\,dy \qquad \forall \hat{w}\in V\,, \tag{4.11}$$

$$u(x,y,0) = u^0(x,y) \quad \text{ for } (x,y)\in\Omega_0. \tag{4.12}$$

Proposition 4.3. *There exists a unique solution $u \in L^2(0,T;V)\cap C([0,T];W)$ to Problem (4.11)–(4.12) such that $u_t \in L^2(0,T;H)$.*

Proof. Uniqueness is easy again. Let u^1, u^2 be two solutions, and let $\bar{u} = u^1 - u^2$. Then

$$\int_{\Omega_0} \Big(\bar{u}_t(\bar{u} - \Delta\bar{u}) + \gamma\mathbf{J}\mathbf{D}_2\bar{u}\cdot\mathbf{D}_2\bar{u}\Big)\,dx\,dy \;=\; 0\,.$$

Using the formula (note that both $\bar{u}_t$ and $\Delta\bar{u}$ belong to $L^2(0,T;H)$)

$$\int_0^t\int_{\Omega_0} -\,\bar{u}_t\Delta\bar{u}(x,y,\tau)\,dx\,dy\,d\tau = \frac{1}{2}\int_{\Omega_0}|\nabla\bar{u}|^2(x,y,t)\,dx\,dy\,,$$

we obtain $\bar{u} = 0$. Existence follows immediately by considering for example Galerkin approximations

$$u^{(m)}(x,y,t) = \sum_{k=1}^{m} u_k^{(m)}(t)\,e_k(x,y)\,,$$

where $\{e_k : k\in\mathbb{N}\}$ is the complete orthonormal system in H of eigenfunctions of the problem

$$-\Delta e_k = \lambda_k\,e_k \;\text{ in }\; \Omega_0, \quad e_k\big|_{\partial\Omega_0} = 0\,.$$

□

4.2 Viscous approximation

We now consider data $g, \mathbf{G}, w^0, w^1$ as in Hypothesis 4.1, and fix a function $\mathbf{v} \in (L^2(0,T;H))^3$ such that $\mathbf{v}_t \in (L^2(0,T;H))^3$, $v(0) = \mathbf{C}\mathbf{A}_{\mathcal{F}}(\mathbf{D}_2 w^0) + \mathbf{B}\mathbf{D}_2 w^0$. We define w as the solution to the problem

$$\int_{\Omega_0} \Big(w_{tt}(\hat{w} - \Delta\hat{w}) + \mathbf{J}\,(\mathbf{v} + \gamma\mathbf{D}_2 w_t)\cdot\mathbf{D}_2\hat{w}\Big)\,dx\,dy$$
$$= \int_{\Omega_0} \Big(g\,\hat{w} + \nabla\mathbf{G}\cdot\nabla\hat{w}\Big)\,dx\,dy \qquad \forall \hat{w}\in V\,, \tag{4.13}$$

with initial and boundary conditions (4.6)–(4.7), still keeping $\gamma > 0$ constant. We have $w_{tt}(0) \in W$ by Hypothesis 4.1 (ii). Hence, by Proposition 4.3, there exists a unique $w \in L^2(0,T;V)$ such that

$$w_{tt} \in L^2(0,T;V) \cap C([0,T];W)\,,\ w_{ttt} \in L^2(0,T;H)\,, \tag{4.14}$$

and (4.13), (4.6)–(4.7) hold for all $t \in [0,T]$.

Let us introduce the space

$$L \ = \ \{\ell \in L^2(0,T;V) \ : \ \ell(x,y,0) = w^0(x,y)\,,\ \ell_t \in L^2(0,T;V)\}\,. \tag{4.15}$$

We define the mapping $R : L \to L$, which with each $\ell \in L$ associates the solution w to the problem

$$\begin{aligned} &\int_{\Omega_0} \big(w_{tt}(\hat{w} - \Delta\hat{w}) + \mathbf{J}\left(\mathcal{F}_B\left[\mathbf{D}_2\ell\right] + \gamma\mathbf{D}_2 w_t\right)\cdot \mathbf{D}_2\hat{w}\big)\,dx\,dy \\ &\qquad\qquad = \ \int_{\Omega_0} \big(g\,\hat{w} + \nabla\mathbf{G}\cdot\nabla\hat{w}\big)\,dx\,dy \qquad \forall \hat{w} \in V\,, \end{aligned} \tag{4.16}$$

where we set

$$\mathcal{F}_B\left[\mathbf{D}_2\ell\right] = \mathbf{C}\mathcal{F}\left[\mathbf{D}_2\ell\right] + \mathbf{B}\mathbf{D}_2\ell \tag{4.17}$$

with initial and boundary conditions (4.6)–(4.7). By (3.12), we see that $\mathcal{F}_B\left[\mathbf{D}_2\ell\right]_t$ belongs to $(L^2(0,T;H))^3$; hence, w fulfills (4.14). Our goal now is to show that R is a contraction on L endowed with a suitable metric.

Let $\ell_1, \ell_2 \in L$ be given, and let w_1, w_2 be the corresponding solutions. Put $\bar{\ell} = \ell_1 - \ell_2$, $\bar{w} = w_1 - w_2$, and $\hat{w} = \bar{w}_t$ (this choice is admissible by virtue of (4.14)). We obtain, using also (3.8), for all $t \in (0,T)$ that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\Omega_0} \left(|\bar{w}_t|^2 + |\nabla\bar{w}_t|^2\right)(x,y,t)\,dx\,dy + \gamma\int_{\Omega_0} \left(\mathbf{J}\mathbf{D}_2\bar{w}_t \cdot \mathbf{D}_2\bar{w}_t\right)(x,y,t)\,dx\,dy \\ &\qquad = -\int_{\Omega_0} \left(\mathbf{J}\left(\mathcal{F}_B\left[\mathbf{D}_2\ell_1\right] - \mathcal{F}_B\left[\mathbf{D}_2\ell_2\right]\right)\cdot\mathbf{D}_2\bar{w}_t\right)(x,y,t)\,dx\,dy \\ &\qquad \le \ C_{\varphi,\mathbf{B}}\int_{\Omega_0} |\mathbf{D}_2\bar{w}_t|(x,y,t)\int_0^t |\mathbf{D}_2\bar{\ell}_t|(x,y,\tau)d\tau\ dx\,dy\,, \end{aligned}$$

with a constant $C_{\varphi,\mathbf{B}}$ depending only on φ, $\mathbf{C}$, and $\mathbf{B}$. Hence, there exists a constant $C^* > 0$ depending only on γ, φ, $\mathbf{C}$, and $\mathbf{B}$, such that

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega_0} \left(|\bar{w}_t|^2 + |\nabla\bar{w}_t|^2\right)(x,y,t)\,dx\,dy + \gamma\int_{\Omega_0} \left(\mathbf{J}\mathbf{D}_2\bar{w}_t \cdot \mathbf{D}_2\bar{w}_t\right)(x,y,t)\,dx\,dy \\ &\qquad \le \ C^* t\int_0^t\int_{\Omega_0} \big(\mathbf{J}\mathbf{D}_2\bar{\ell}_t\cdot\mathbf{D}_2\bar{\ell}_t\big)(x,y,\tau)\,dx\,dy\,d\tau\,. \end{aligned} \tag{4.18}$$

We now multiply (4.18) by $\mathrm{e}^{-\kappa t^2}$ with $\kappa = C^*/\gamma$, and integrate $\int_0^T dt$. This yields

$$
\begin{aligned}
&\mathrm{e}^{-\kappa T^2} \int_{\Omega_0} \left(|\bar{w}_t|^2 + |\nabla \bar{w}_t|^2\right)(x,y,T)\,dx\,dy \\
&\qquad + \int_0^T 2\kappa t \mathrm{e}^{-\kappa t^2} \int_{\Omega_0} \left(|\bar{w}_t|^2 + |\nabla \bar{w}_t|^2\right)(x,y,t)\,dx\,dy\,dt \\
&\qquad + \gamma \int_0^T \mathrm{e}^{-\kappa t^2} \int_{\Omega_0} \left(\mathbf{J}\mathbf{D}_2 \bar{w}_t \cdot \mathbf{D}_2 \bar{w}_t\right)(x,y,t)\,dx\,dy\,dt \\
&\le\ C^* \int_0^T t\mathrm{e}^{-\kappa t^2} \int_0^t \int_{\Omega_0} \left(\mathbf{J}\mathbf{D}_2 \bar{\ell}_t \cdot \mathbf{D}_2 \bar{\ell}_t\right)(x,y,\tau)\,dx\,dy\,d\tau\,dt \\
&=\ C^* \int_0^T \left(\int_\tau^T t\mathrm{e}^{-\kappa t^2}\,dt\right) \int_{\Omega_0} \left(\mathbf{J}\mathbf{D}_2 \bar{\ell}_t \cdot \mathbf{D}_2 \bar{\ell}_t\right)(x,y,\tau)\,dx\,dy\,d\tau \\
&\le\ \frac{C^*}{2\kappa} \int_0^T \mathrm{e}^{-\kappa \tau^2} \int_{\Omega_0} \left(\mathbf{J}\mathbf{D}_2 \bar{\ell}_t \cdot \mathbf{D}_2 \bar{\ell}_t\right)(x,y,\tau)\,dx\,dy\,d\tau\,.
\end{aligned}
\tag{4.19}
$$

We now introduce in L (see (4.15)) the metric

$$
d_L(\ell_1,\ell_2)\ =\ \left(\int_0^T \mathrm{e}^{-\kappa t^2} \int_{\Omega_0} \left(\mathbf{J}\mathbf{D}_2 \bar{\ell}_t \cdot \mathbf{D}_2 \bar{\ell}_t\right)(x,y,t)\,dx\,dy\,dt\right)^{1/2}, \quad \bar{\ell} = \ell_1 - \ell_2\,.
$$

This is indeed a metric, since it is induced by a weighted norm in $L^2(\Omega_0 \times (0,T))$. To check that $d_L(\ell_1,\ell_2) = 0$ implies $\ell_1 = \ell_2$, note that if $\mathbf{D}_2\ell_1 = \mathbf{D}_2\ell_2$ a. e., then $\Delta\bar{\ell} = 0$ a. e., which together with the homogeneous Dirichlet boundary condition on $\partial\Omega_0$ yields $\ell_1 = \ell_2$. Moreover, (L, d_L) is a complete metric space. From (4.19) we obtain

$$
d_L(w_1,w_2)\ \le\ \frac{1}{\sqrt{2}} d_L(\ell_1,\ell_2)\,,
$$

hence $R : L \to L$ is a contraction. The Banach Contraction Principle then yields the following result.

Proposition 4.4. *Let Hypothesis 4.1 hold, and let $\gamma > 0$ be given. Let $\mathcal{F}_B$ be given by (4.17). Then there exists a unique solution $w^{(\gamma)}$ to the problem*

$$
\begin{aligned}
&\int_{\Omega_0} \left(w_{tt}^{(\gamma)}(\hat{w} - \Delta\hat{w}) + \mathbf{J}\left(\mathcal{F}_B\left[\mathbf{D}_2 w^{(\gamma)}\right] + \gamma \mathbf{D}_2 w_t^{(\gamma)}\right) \cdot \mathbf{D}_2 \hat{w}\right) dx\,dy \\
&\qquad =\ \int_{\Omega_0} \left(g\,\hat{w} + \nabla\mathbf{G} \cdot \nabla\hat{w}\right) dx\,dy \qquad \forall \hat{w} \in V\,,
\end{aligned}
\tag{4.20}
$$

with the regularity (4.14), and satisfying the initial and boundary conditions (4.6)–(4.7).

4.3 Passage to the limit

We first derive estimates for $w^{(\gamma)}$ that enable us to pass to the limit as $\gamma \to 0+$. In what follows, we denote successively by $C_1, C_2, \ldots$ constants independent of γ and depending possibly on the other data of the problem.

By virtue of (4.14), we are allowed to differentiate (4.20) with respect to t and set $\hat{w} = w_{tt}^{(\gamma)}$. Invoking (3.13), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_0} \left(|w_{tt}^{(\gamma)}|^2 + |\nabla w_{tt}^{(\gamma)}|^2 + \mathbf{J}\mathcal{F}_B\left[\mathbf{D}_2 w^{(\gamma)}\right]_t \cdot \mathbf{D}_2 w_t^{(\gamma)}\right)(x,y,t)\,dx\,dy \\
&\qquad + \gamma \int_{\Omega_0} \left(\mathbf{J}\mathbf{D}_2 w_{tt}^{(\gamma)} \cdot \mathbf{D}_2 w_{tt}^{(\gamma)}\right)(x,y,t)\,dx\,dy \\
&\quad \leq \int_{\Omega_0} \left(g_t w_{tt}^{(\gamma)} + \nabla \mathbf{G}_t \cdot \nabla w_{tt}^{(\gamma)}\right)(x,y,t)\,dx\,dy
\end{aligned} \tag{4.21}$$

in the sense of distributions. From (3.12) and (2.47), it follows for all $t \in [0,T]$ that

$$\begin{aligned}
&\int_{\Omega_0} \left(|w_{tt}^{(\gamma)}|^2 + |\nabla w_{tt}^{(\gamma)}|^2 + \beta\,|\Delta w_t^{(\gamma)}|^2\right)(x,y,t)\,dx\,dy \\
&\quad \leq C_1\Bigg(\int_{\Omega_0} \left(|w_{tt}^{(\gamma)}|^2 + |\nabla w_{tt}^{(\gamma)}|^2 + \mathbf{J}\mathbf{D}_2 w_t^{(\gamma)} \cdot \mathbf{D}_2 w_t^{(\gamma)}\right)(x,y,0)\,dx\,dy \\
&\qquad + \int_0^T \int_{\Omega_0} \left(|g_t|^2 + |\nabla \mathbf{G}_t|^2\right)(x,y,t)\,dx\,dy\,dt\Bigg).
\end{aligned} \tag{4.22}$$

We now estimate the right hand side of (4.22). We have indeed

$$\int_{\Omega_0} \left(\mathbf{J}\mathbf{D}_2 w_t^{(\gamma)} \cdot \mathbf{D}_2 w_t^{(\gamma)}\right)(x,y,0)\,dx\,dy \;\leq\; C_2\,|w^1|_V^2 \;=:\; C_3\,. \tag{4.23}$$

The estimate for $w_{tt}^{(\gamma)}(x,y,0)$ and $\nabla w_{tt}^{(\gamma)}(x,y,0)$ is more delicate. The functions $\nabla w_{tt}^{(\gamma)}$, $\mathbf{D}_2 w_t^{(\gamma)}$, and $\mathbf{D}_2 w^{(\gamma)}$ belong to $C([0,T];H)$ by virtue of (4.14). We thus may set $t=0$ in (4.20) and obtain (omitting the arguments x,y for simplicity)

$$\begin{aligned}
&\int_{\Omega_0} \left(w_{tt}^{(\gamma)}(0)\,\hat{w} + \nabla w_{tt}^{(\gamma)}(0) \cdot \nabla \hat{w} + \mathbf{J}\left(\mathbf{C}\mathbf{A}_{\mathcal{F}}(\mathbf{D}_2 w^0) + \mathbf{B}\mathbf{D}_2 w^0 + \gamma \mathbf{D}_2 w^1\right) \cdot \mathbf{D}_2 \hat{w}\right)dx\,dy \\
&\qquad = \int_{\Omega_0} \left(g(0)\,\hat{w} + \nabla \mathbf{G}(0) \cdot \nabla \hat{w}\right)dx\,dy \qquad \forall \hat{w} \in V\,,
\end{aligned} \tag{4.24}$$

where $\mathbf{A}_{\mathcal{F}}$ is the initial value mapping (3.15) of the operator $\mathcal{F}$. We now use the compatibility conditions in Hypothesis 4.1 (ii) and identities (4.2), (4.4), and integrate by parts in (4.24). This yields

$$\begin{aligned}
&\int_{\Omega_0} \left(w_{tt}^{(\gamma)}(0)\,\hat{w} + \nabla w_{tt}^{(\gamma)}(0) \cdot \nabla \hat{w} - \mathbf{D}_1\left(\mathbf{C}\mathbf{A}_{\mathcal{F}}(\mathbf{D}_2 w^0) + \mathbf{B}\mathbf{D}_2 w^0 + \gamma \mathbf{D}_2 w^1\right) \cdot \nabla \hat{w}\right)dx\,dy \\
&\qquad = \int_{\Omega_0} \left(g(0)\,\hat{w} + \nabla \mathbf{G}(0) \cdot \nabla \hat{w}\right)dx\,dy \qquad \forall \hat{w} \in V\,.
\end{aligned} \tag{4.25}$$

Since V is dense in W, identity (4.25) holds for all $\hat{w} \in W$, and in particular for $\hat{w} = w_{tt}^{(\gamma)}(0)$. The mapping $\mathbf{A}_{\mathcal{F}}$ is Lipschitz continuous, hence the L^2-norm of

$$\mathbf{D}_1 \left(\mathbf{C}\mathbf{A}_{\mathcal{F}}(\mathbf{D}_2 w^0) + \mathbf{B}\mathbf{D}_2 w^0 + \gamma \mathbf{D}_2 w^1 \right)$$

can be estimated from above by the H^3-norms of w^0 and w^1, and we obtain

$$\int_{\Omega_0} \left(|w_{tt}^{(\gamma)}(0)|^2 + |\nabla w_{tt}^{(\gamma)}(0)|^2 \right) dx\, dy \ \le\ C_4 . \tag{4.26}$$

From (4.22), (4.23), and (4.26) we conclude that

$$w_{tt}^{(\gamma)}, \nabla w_{tt}^{(\gamma)}, \mathbf{D}_2 w_t^{(\gamma)} \quad \text{are bounded in } \ L^\infty(0,T;H) \ \text{ independently of } \gamma . \tag{4.27}$$

Hence, there exist a sequence $\gamma_n \to 0+$ as $n \to \infty$, and functions $\boldsymbol{\sigma} \in (L^\infty(0,T;H))^3$, $w \in L^\infty(0,T;V)$ such that $w_{tt} \in L^\infty(0,T;H)$, $\nabla w_{tt} \in (L^\infty(0,T;H))^2$, and $\mathbf{D}_2 w_t \in (L^\infty(0,T;H))^3$, with the properties

$$\left. \begin{array}{rcl} w_{tt}^{(\gamma_n)} & \to & w_{tt} \\ \nabla w_{tt}^{(\gamma_n)} & \to & \nabla w_{tt} \\ \mathbf{D}_2 w_t^{(\gamma_n)} & \to & \mathbf{D}_2 w_t \\ \mathbf{C}\mathcal{F}\left[\mathbf{D}_2 w^{(\gamma_n)}\right] & \to & \boldsymbol{\sigma} \end{array} \right\} \qquad \text{weakly-}*\,, \tag{4.28}$$

$$\left. \begin{array}{rcl} w_t^{(\gamma_n)} & \to & w_t \\ w^{(\gamma_n)} & \to & w \end{array} \right\} \qquad \text{uniformly.} \tag{4.29}$$

Passing to the limit in (4.20) as $n \to \infty$ yields

$$\begin{aligned} &\int_{\Omega_0} \Big(w_{tt}(\hat{w} - \Delta \hat{w}) + \mathbf{J}\left(\boldsymbol{\sigma} + \mathbf{B}\mathbf{D}_2 w \right) \cdot \mathbf{D}_2 \hat{w} \Big)(x,y,t)\, dx\, dy \\ &\qquad = \int_{\Omega_0} \Big(g\, \hat{w} + \nabla \mathbf{G} \cdot \nabla \hat{w} \Big)(x,y,t)\, dx\, dy \qquad \forall \hat{w} \in V \end{aligned} \tag{4.30}$$

for a.e. $t \in (0,T)$. The initial and boundary conditions (4.6)–(4.7) are indeed satisfied. Hence, the existence proof will be complete if we check that

$$\boldsymbol{\sigma} \ = \ \mathbf{C}\mathcal{F}\left[\mathbf{D}_2 w\right] . \tag{4.31}$$

To prove (4.31), we use a variant of Minty's trick based on the monotonicity property (3.11) of $\mathcal{F}$. We first put $\hat{w} = w_t^{(\gamma)}$ in (4.20), so that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\Omega_0} \left(|w_t^{(\gamma)}|^2 + |\nabla w_t^{(\gamma)}|^2 + \mathbf{J}\mathbf{B}\mathbf{D}_2 w^{(\gamma)} \cdot \mathbf{D}_2 w^{(\gamma)} \right)(x,y,t)\, dx\, dy \\ &\qquad + \int_{\Omega_0} \mathbf{J}\left(\mathbf{C}\mathcal{F}\left[\mathbf{D}_2 w^{(\gamma)}\right] + \gamma \mathbf{D}_2 w_t^{(\gamma)} \right) \cdot \mathbf{D}_2 w_t^{(\gamma)}(x,y,t)\, dx\, dy \\ &\quad = \int_{\Omega_0} \left(g w_t^{(\gamma)} + \nabla \mathbf{G} \cdot \nabla w_t^{(\gamma)} \right)(x,y,t)\, dx\, dy . \end{aligned} \tag{4.32}$$

Hence, for all $t \in [0,T]$ we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_0} \left(|w_t^{(\gamma)}|^2 + |\nabla w_t^{(\gamma)}|^2 + \mathbf{JBD}_2 w^{(\gamma)} \cdot \mathbf{D}_2 w^{(\gamma)} \right)(x,y,t)\,dx\,dy \\
&\qquad + \int_0^t \int_{\Omega_0} \mathbf{J}\left(\mathbf{C}\mathcal{F}\left[\mathbf{D}_2 w^{(\gamma)}\right] + \gamma \mathbf{D}_2 w_t^{(\gamma)} \right) \cdot \mathbf{D}_2 w_t^{(\gamma)}(x,y,\tau)\,dx\,dy\,d\tau \\
&\quad = \int_0^t \int_{\Omega_0} \left(g w_t^{(\gamma)} + \nabla \mathbf{G} \cdot \nabla w_t^{(\gamma)} \right)(x,y,\tau)\,dx\,dy\,d\tau \\
&\qquad + \frac{1}{2}\int_{\Omega_0} \left(|w^1|^2 + |\nabla w^1|^2 + \mathbf{JBD}_2 w^0 \cdot \mathbf{D}_2 w^0 \right)(x,y)\,dx\,dy\,. \qquad (4.33)
\end{aligned}
$$

Setting $\gamma = \gamma_n$ in the above identity and passing to the limit as $n \to \infty$, we see, using (4.28)–(4.29), that a. e. in $(0,T)$ we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_0} \left(|w_t|^2 + |\nabla w_t|^2 + \mathbf{JBD}_2 w \cdot \mathbf{D}_2 w \right)(x,y,t)\,dx\,dy \\
&\qquad + \limsup_{n\to\infty} \int_0^t \int_{\Omega_0} \left(\mathbf{JC}\mathcal{F}\left[\mathbf{D}_2 w^{(\gamma_n)}\right] \cdot \mathbf{D}_2 w_t^{(\gamma_n)} \right)(x,y,\tau)\,dx\,dy\,d\tau \\
&\quad \le \int_0^t \int_{\Omega_0} \left(g w_t + \nabla \mathbf{G} \cdot \nabla w_t \right)(x,y,\tau)\,dx\,dy\,d\tau \\
&\qquad + \frac{1}{2}\int_{\Omega_0} \left(|w^1|^2 + |\nabla w^1|^2 + \mathbf{JBD}_2 w^0 \cdot \mathbf{D}_2 w^0 \right)(x,y)\,dx\,dy\,. \qquad (4.34)
\end{aligned}
$$

We now set $\hat{w} = w_t$ in (4.30) and integrate over t. This yields for each $t \in [0,T]$ that

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_0} \left(|w_t|^2 + |\nabla w_t|^2 + \mathbf{JBD}_2 w \cdot \mathbf{D}_2 w \right)(x,y,t)\,dx\,dy \\
&\qquad + \int_0^t \int_{\Omega_0} \left(\mathbf{J}\boldsymbol{\sigma} \cdot \mathbf{D}_2 w_t \right)(x,y,\tau)\,dx\,dy\,d\tau \\
&\quad \le \int_0^t \int_{\Omega_0} \left(g w_t + \nabla \mathbf{G} \cdot \nabla w_t \right)(x,y,\tau)\,dx\,dy\,d\tau \\
&\qquad + \frac{1}{2}\int_{\Omega_0} \left(|w^1|^2 + |\nabla w^1|^2 + \mathbf{JBD}_2 w^0 \cdot \mathbf{D}_2 w^0 \right)(x,y)\,dx\,dy\,. \qquad (4.35)
\end{aligned}
$$

It follows from the comparison of (4.34) with (4.35) that

$$
\begin{aligned}
&\limsup_{n\to\infty} \int_0^t \int_{\Omega_0} \left(\mathbf{JC}\mathcal{F}\left[\mathbf{D}_2 w^{(\gamma_n)}\right] \cdot \mathbf{D}_2 w_t^{(\gamma_n)} \right)(x,y,\tau)\,dx\,dy\,d\tau \\
&\qquad \le \int_0^t \int_{\Omega_0} \left(\mathbf{J}\boldsymbol{\sigma} \cdot \mathbf{D}_2 w_t \right)(x,y,\tau)\,dx\,dy\,d\tau \qquad (4.36)
\end{aligned}
$$

for all $t \in [0,T]$. On the other hand, for an arbitrary function $\mathbf{v} \in (L^2(0,T;H))^3$ such that $\mathbf{v}_t \in (L^2(0,T;H))^3$ and $\mathbf{v}(x,y,0) = \mathbf{D}_2 w^0(x,y)$, we have by (3.11) that

$$\int_0^t \int_{\Omega_0} \mathbf{JC}\left(\mathcal{F}\left[\mathbf{D}_2 w^{(\gamma_n)}\right] - \mathcal{F}[\mathbf{v}]\right) \cdot \left(\mathbf{D}_2 w_t^{(\gamma_n)} - \mathbf{v}_t\right)(x,y,\tau)\, dx\, dy\, d\tau \;\geq\; 0\,.$$

Passing to the limit and using (4.36), we obtain

$$\int_0^T \int_{\Omega_0} \mathbf{J}\left(\boldsymbol{\sigma} - \mathbf{C}\mathcal{F}[\mathbf{v}]\right) \cdot \left(\mathbf{D}_2 w_t - \mathbf{v}_t\right)(x,y,t)\, dx\, dy\, dt \;\geq\; 0\,. \tag{4.37}$$

We now choose $\mathbf{v}$ in the form

$$\mathbf{v} \;=\; \mathbf{D}_2 w - \delta\, \zeta(t)\, \mathbf{s}\,,$$

where $\delta > 0$ is a parameter, $\zeta \in W^{1,1}(0,T)$ is an arbitrary, nondecreasing function such that $\zeta(0) = 0$, and $\mathbf{s} \in H \times H \times H$ is an arbitrary element. Then

$$\int_0^T \dot{\zeta}(t) \int_{\Omega_0} \mathbf{J}\left(\boldsymbol{\sigma} - \mathbf{C}\mathcal{F}\left[\mathbf{D}_2 w - \delta\zeta(\cdot)\,\mathbf{s}\right]\right)(x,y,t) \cdot \mathbf{s}(x,y)\; dx\, dy\, dt \;\geq\; 0 \quad \forall \mathbf{s} \in H \times H \times H\,. \tag{4.38}$$

We have by (3.8) for a. e. $(x,y) \in \Omega_0$ and every $t \geq 0$ that

$$\left|\mathcal{F}\left[\mathbf{D}_2 w - \delta\zeta(\cdot)\,\mathbf{s}\right](x,y,t) - \mathcal{F}\left[\mathbf{D}_2 w\right](x,y,t)\right| \;\leq\; \delta\, \zeta(t)\, |\mathbf{s}(x,y)| \int_0^\infty \varphi(q)\, dq\,.$$

Letting δ tend to 0 in (4.38), we conclude that $\boldsymbol{\sigma} = \mathbf{C}\mathcal{F}[\mathbf{D}_2 w]$, hence (4.31) holds and the proof of Theorem 4.2 is complete.

Conclusion. The two dimensional partial differential equation for transversal vibrations of an elastoplastic plate is derived from a general three dimensional system with a single yield tensorial von Mises plasticity model in the five dimensional deviatoric space. The multiyield behavior is due to the fact that not all layers parallel to the midsurface become plastic at the same time. The resulting partial differential equation with a Prandtl-Ishlinskii hysteresis operator is solved via viscous approximations and a monotonicity argument.

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