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On the construction of bifurcation curves related to limit cycles of multiplicity three for planar vector fields

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Abstract

For plane vector fields depending on three parameters we describe an algorithm to construct a curve in the parameter space such that to each point of this curve there belongs a vector field possessing a limit cycle of multiplicity three. One point of this curve is related to the bifurcation of a limit cycle of multiplicity three from an equilibrium point. The underlying procedure is a continuation method.

1 Introduction

We consider systems of two scalar autonomous differential equations

$$\frac{dx}{dt} = P(x, y, \lambda), \quad \frac{dy}{dt} = Q(x, y, \lambda) \tag{1.1}$$

depending on a parameter vector $\lambda \in R^m$ in some region Ω of the phase plane. Under certain conditions, the phase portrait of system (1.1) in Ω is determined by the so-called singular trajectories, namely the equilibria, separatrices and limit cycles of (1.1) in Ω (see, e.g., [1]). The most difficult problem in studying these singular trajectories is to localize multiple limit cycles and to estimate the number of limit cycles. This problem is still unsolved even in the case of polynomial systems

$$\frac{dx}{dt} = \sum_{i+j=0}^{n} a_{ij} x^i y^j, \quad \frac{dy}{dt} = \sum_{i+j=0}^{n} b_{ij} x^i y^j,$$

where a_{ij} and b_{ij} are real coefficients, and it represents the second part of the famous 16-th problem of D. Hilbert [4]. As already D. Hilbert indicated, the investigation of the dependence of limit cycles on parameters should play a fundamental role in solving the posed problem.

By our understanding, a complete solution of Hilbert's 16-th problems should not only give an upper estimate of the number of limit cycles, it should also provide a constructive method to localize the limit cycles. This is similar to the problem of estimating and localizing the real roots of a given polynomial in some interval.

Our main goal in this note is to contribute to the constructive methods for estimating the number of limit cycles of some classes of planar polynomial systems depending on parameters.

2 Prelimineries

Let Ω be some connected region in R^2 and Λ a simply connected region in R^m. We consider system (1.1) in Ω under the following smoothness assumption.

(A_1). P and Q are n-times ($n \geq 1$) continuously differentiable with respect to x and y, and continuously differentiable with respect to λ for $(x, y, \lambda) \in \Omega \times \Lambda$.

System (1.1) defines the planar vector field $f := (P, Q)$ on Ω. We denote f as smooth if assumption (A_1) is satisfied.
In what follows we recall the definitions of some basic tools of the qualitative theory of autonomous differential systems.

An isolated periodic solution $(x, y) = (x_p(t, \lambda), y_p(t, \lambda))$ of system (1.1) with finite minimal period $T(\lambda) > 0$ is called a limit cycle. We set

$$\Gamma(\lambda) := \{(x, y) \in R^2 : x = x_p(t, \lambda), y = y_p(t, \lambda), 0 \leq t \leq T(\lambda)\}.$$

Let p_0 be any point on $\Gamma(\lambda)$, let Σ be a small segment (open connected set) of the normal to $\Gamma(\lambda)$ through p_0 containing p_0 with the following properties:

(i) All trajectories of (1.1) which meet Σ intersect Σ transversally.

(ii) There is an open connected subset Σ_0 of Σ containing p_0 such that to each point $q \in \Sigma_0$ there is a unique minimal positive number $\tau(q) \approx T(\lambda)$ such that the trajectory of (1.1) starting for $t = 0$ at q intersects Σ for $t = \tau(q)$.

By this way we define a map $\Pi(\lambda, .) : \Sigma_0 \to \Sigma$ which is called the first return map or Poincaré map associated with $\Gamma(\lambda)$. Obviously, p_0 is a fixed point of $\Pi(\lambda, .)$.

The following properties of the Poincaré map are well known (see [3]).

(i) $\Pi(\lambda, .)$ is a diffeomorphism with the same smoothness as the vector field f.

(ii)

$$\Pi(\lambda, p_0) = p_0, \quad \Pi'_\xi(\lambda, p_0) = exp\Big\{\int_0^{T(\lambda)} div f(x_p(t, \lambda), y_p(t, \lambda), \lambda)\, dt\Big\}. \qquad (2.1)$$

Together with $\Pi(\lambda, .)$ we introduce the displacement function $\delta(\lambda, .) : \Sigma_0 \to R$ by

$$\delta(\lambda, \xi) := \Pi(\lambda, \xi) - \xi.$$

The multiplicity of a limit cycle is usually introduced by means of the Poincaré or of the displacement function:

Definition 2.1 *Suppose hypothesis* (A_1) *holds with* $n \geq k \geq 1$. $\Gamma(\lambda)$ *is called a limit cycle of multiplicity* 1 (*or simple limit cycle or hyperbolic limit cycle*), *if we have*

$$\delta(\lambda, p_0) = 0,\ \delta'_\xi(\lambda, p_0) \neq 0.$$

$\Gamma(\lambda)$ *is called a limit cycle of multiplicity* $k, k \geq 2$, *if it holds*

$$\delta(\lambda, p_0) = \delta'_\xi(\lambda, p_0) = 0, ..., \delta_\xi^{(k-1)}(\lambda, p_0) = 0, \delta_\xi^{(k)}(\lambda, p_0) \neq 0.$$

The importance of the concept of the multiplicity of a limit cycle is based on the fact that under appropriate perturbations of (1.1), from a limit cycle of multiplicity k may bifurcate up to k simple limit cycles [3, 5].

Analogously to the concept of a multiple limit cycle, we introduce the concept of a multiple equilibrium of focus type.
An equilibrium point (x_e, y_e) of system (1.1) is said to be an equilibrium of focus type, if the eigenvalues of the Jacobian matrix $J(\lambda)$ of the right hand side of (1.1) at (x_e, y_e) have non-vanishing imaginary parts. In a neighborhood of an equilibrium point of focus type we may also introduce the Poincaré map $\Pi(\lambda, .)$ and the displacement function $\delta(\lambda, .)$.
If we assume that the origin is an equilibrium point of focus type and that the displacement function δ can be represented in the form

$$\delta(\lambda, \xi) = \sum_{i=1}^{n} \alpha_i(\lambda)\xi^i + o(\xi^n), \tag{2.2}$$

and if we assume that for $\lambda = \lambda_0$ the Jacobian matrix $J(\lambda_0)$ at the origin has non-vanishing pure imaginary eigenvalues, then the origin is called a focus of multiplicity 1 if we have $\alpha_1(\lambda_0) \neq 0$. It is said to be a focus of multiplicity k with $2k + 1 \leq n$, if it holds

$$\alpha_1(\lambda_0) = \alpha_2(\lambda_0) = ... = \alpha_{2k-2}(\lambda_0) = 0, \alpha_{2k-1}(\lambda_0) \neq 0.$$

Analogously to the perturbation of a multiple limit cycle we have the fact that under appropriate perturbations of (1.1), from a focus of multiplicity k may bifurcate up to k simple limit cycles [3, 5].
Finally, we introduce the so-called Poincaré functional, which is a simple tool to establish the absence of a periodic solution in some region $D \subset R^n$ for an n-dimensional system of autonomous differential equations

$$\frac{dx}{dt} = g(x), \tag{2.3}$$

where g maps D into R^n.

Definition 2.2 *Let $G : D \to R$ be a continuously differentiable functional and such that the scalar product of gradG and g does not change sign in D and does not vanish on any nontrivial closed orbit in D, then G is called a Poincaré functional to system* (2.3).

The following result is well-known

Proposition 2.1 *Suppose the function g maps D continuously into R^n and that there is a Poincaré functional G to system* (2.3) *in D. Then system* (2.3) *has no nontrivial periodic solution in D.*

3 Determining relations for a limit cycle of multiplicity 3

In the sequel we reformulate the multiplicity conditions in Definition 2.1 for a limit cycle. For this purpose we introduce the following notations.

$$
\begin{aligned}
&H(x,y,\lambda) := P^2(x,y,\lambda) + Q^2(x,y,\lambda),\\
&H_1(x,y,\lambda) := divf(x,y,\lambda) \equiv \frac{\partial P}{\partial x}(x,y,\lambda) + \frac{\partial Q}{\partial y}(x,y,\lambda),\\
&H_i(x,y,\lambda) = \frac{\partial}{\partial y}\left(\frac{PH_{i-1}}{H}\right)(x,y,\lambda) - \frac{\partial}{\partial x}\left(\frac{QH_{i-1}}{H}\right)(x,y,\lambda) \quad \text{for} \quad i = 2,3.\\
&\hat{H}_i(t,\lambda) := H_i(x_p(t,\lambda), y_p(t,\lambda), \lambda) \quad \text{for} \quad i = 1,2,3,\\
&h_1(\lambda) = \int_0^{T(\lambda)} \hat{H}_1(t,\lambda)dt, \quad h_2(\lambda) = \int_0^{T(\lambda)} \hat{H}_2(t,\lambda) exp\left(\int_0^t \hat{H}_1(\tau,\lambda)d\tau\right)dt,\\
&h_3(\lambda) = \int_0^{T(\lambda)} \hat{H}_3(t,\lambda) \exp\left(2\int_0^t \hat{H}_1(\tau,\lambda)d\tau\right)dt.
\end{aligned}
\tag{3.1}
$$

Taking into account relation (2.1) we have the following lemma.

Lemma 3.1 *Suppose assumption (A_1) holds with $n = 1$. Then the limit cycle $\Gamma(\lambda)$ is simple (or hyperbolic) if and only if it holds*

$$h_1(\lambda) \neq 0.$$

Using this lemma and the introduced notations we get

Lemma 3.2 *Suppose the assumption (A_1) to be valid for $n = 3$. Let $\Gamma(\lambda)$ be a limit cycle of system* (1.1). *Then it holds:*

(i) $\Gamma(\lambda)$ *has multiplicity* 2 *if and only if*

$$h_1(\lambda) = 0, \quad h_2(\lambda) \neq 0.$$

(ii) $\Gamma(\lambda)$ *has multiplicity* 3 *if and only if*

$$h_1(\lambda) = 0, \quad h_2(\lambda) = 0, \quad h_3(\lambda) \neq 0.$$

The following propositions essentially represent a reformulation of the lemmata 3.1 and 3.2.

Proposition 3.1 *Suppose assumption* (A_1) *is valid for* $n = 2$. *If the system*

$$\begin{aligned} &\frac{dx}{dt} = P(x, y, \lambda), \ \frac{dy}{dt} = Q(x, y, \lambda), \\ &\frac{dz}{dt} = H_1(x, y, \lambda), \ \frac{dw_1}{dt} = e^z H_2(x, y, \lambda) \end{aligned} \tag{3.2}$$

has no nontrivial periodic solution, then the multiplicity of any limit cycle of system (1.1) *is bounded by* 2.

Proof. If we suppose that system (1.1) has no limit cycle, then Proposition 3.1 is obviously true. If we assume that system (1.1) has a limit cycle $(x_p(t, \lambda), y_p(t, \lambda))$ with period $T(\lambda)$ such that

$$z(T(\lambda)) - z(0) = \int_0^{T(\lambda)} H_1(t, \lambda) dt = h_1(\lambda) \neq 0,$$

then this limit cycle has multiplicity 1 and Proposition 3.1 is valid.
The last possibility is that system (1.1) has a limit cycle such that $h_1(\lambda) = 0$ and

$$w_1(T(\lambda)) - w_1(0) = \int_0^{T(\lambda)} e^{z(t)} H_2(t, \lambda)\, dt = h_2(\lambda) \neq 0,$$

then this limit cycle has multiplicity 2. This completes the proof of Proposition 3.1. □

Proposition 3.2 *Suppose assumption* (A_1) *is valid for* $n = 3$. *If the system*

$$\begin{aligned} &\frac{dx}{dt} = P(x, y, \lambda), \ \frac{dy}{dt} = Q(x, y, \lambda), \\ &\frac{dz}{dt} = H_1(x, y, \lambda), \ \frac{dw_2}{dt} = e^{2z} H_3(x, y, \lambda) \end{aligned} \tag{3.3}$$

has no nontrivial periodic solution, then the multiplicity of any limit cycle of system (1.1) *is not larger than* 3.

Proof. If system (1.1) has no limit cycle or only a simple limit cycle, then Proposition 3.2 is obviously true. If we assume that system (1.1) has a limit cycle such that $h_1(\lambda) = 0$ and

$$w_2(T(\lambda)) - w_2(0) = \int_0^{T(\lambda)} e^{2z(t)} H_3(t, \lambda) dt = h_3(\lambda) \neq 0,$$

then this limit cycle has either multiplicity 2 or multiplicity 3. This completes the proof of Proposition 3.2. □

Remark 3.1 *If we increase the differentiability of P and Q, then Proposition* 3.2 *excludes also the existence of limit cycles of multiplicity larger than* 3.

4 An algorithm to determine families of planar vector fields possessing a limit cycle with multiplicity 3

In this section we consider systems (1.1) depending on three parameters ($m = 3$). Our goal is to describe an algorithm yielding a finite set Λ_3 of points in the three-dimensional parameter space Λ such that to each point of this set there corresponds a system (1.1) with a limit cycle of multiplicity 3. We denote by $\mathcal{K}_3$ the curve defined by the set Λ_3.

The first step of our algorithm in constructing such a set consists in finding a point $\lambda_0 \in \Lambda$ which corresponds to a system (1.1) having a focus of multiplicity three. That means we are looking for a point $\lambda_0 \in \Lambda$ which corresponds to a system (1.1) having a degenerate limit cycle of multiplicity 3 (vanishing amplitude). Taking into account that the relation $\alpha_{2i+1} = 0$ implies $\alpha_{2i+2} = 0$ for $i = 0, 1, ...$ (see [3, 5]) we consider the equations

$$P(x, y, \lambda) = 0, Q(x, y, \lambda) = 0, \alpha_1(x, y, \lambda) = 0,\ \alpha_3(x, y.\lambda) = 0,\ \alpha_5(x, y, \lambda) = 0 \qquad (4.1)$$

to determine an equilibrium point $(x_e, y_e) \in \Omega$ and a parameter value $\lambda_0 \in \Lambda \subset R^3$ such that (x_e, y_e) is a focus of multiplicity at least 3 of the corresponding system (1.1). We assume that system (4.1) has such a solution. Without loss of generality, we may suppose that the corresponding equilibrium coincides with the origin. Then we check, whether $\alpha_7(\lambda_0)$ is different from zero in order to be able to say that the focus has exactly multiplicity 3.

In the next step we consider the extended system (3.2) for λ near λ_0. We assume that to a given sequence of small positive numbers $x_1 < x_2 < ... < x_N << 1$ there is a sequence of parameter values $\lambda_1, \lambda_2, ..., \lambda_N$ such that system (3.2) has a limit cycle $\Gamma(\lambda_i)$ intersecting the positive x-axis in the point x_i. That means that at least for λ_i near λ_0 the set Λ_3 can

be parameterized by the intersection point of the corresponding limit cycle $\Gamma(\lambda_i)$ with the positive x-axis and that this set is related to the bifurcation of a limit cycle of multiplicity 3 from a focus of the same multiplicity. In what follows we describe an algorithm which computes to given x_i the corresponding value λ_i.

First we modify system (3.2) slightly. By means of the transformation $t = \frac{T}{2\pi}\,\tau = \mu\tau$ with $\mu = \frac{T}{2\pi}$ we introduce a new time τ such that the primitive period of the limit cycle $\Gamma(\lambda_i)$ is 2π for any λ_i. But then we have to include the parameter μ into the set of parameters to be determined. Using the new time τ, we get from (3.2) the system

$$\begin{aligned} &\frac{dx}{d\tau} = \mu P(x,y,\lambda),\ \frac{dy}{d\tau} = \mu Q(x,y,\lambda),\\ &\frac{dz}{d\tau} = \mu H_1(x,y,\lambda),\ \frac{dw_1}{d\tau} = \mu e^z H_2(x,y,\lambda). \end{aligned} \tag{4.2}$$

If we denote by

$$(\tilde{x}(\tau,x_0,0,0,0,\mu,\lambda),\tilde{y}(\tau,x_0,0,0,0,\mu,\lambda),\tilde{z}(\tau,x_0,0,0,0,\mu,\lambda),\tilde{w_1}(\tau,x_0,0,0,0,\mu,\lambda))$$

the solution of system (4.2) satisfying

$$\tilde{x}(0,x_0,0,0,0,\mu,\lambda) = x_0, \quad \tilde{y}(0,x_0,0,0,0,\mu,\lambda) = 0,$$

$$\tilde{z}(0,x_0,0,0,0,\mu,\lambda) = 0, \quad \tilde{w_1}(0,x_0,0,0,0,\mu,\lambda) = 0,$$

then the system of equations which determines the parameters λ and μ such that system (4.2) has a nontrivial periodic solution has the form

$$\begin{aligned} \varphi_1(x_0,\lambda,\mu) &\equiv \tilde{x}(2\pi,x_0,0,0,0,\mu,\lambda) - x_0 = 0,\\ \varphi_2(x_0,\lambda,\mu) &\equiv \tilde{y}(2\pi,x_0,0,0,0,\mu,\lambda) = 0,\\ \varphi_3(x_0,\lambda,\mu) &\equiv \tilde{z}(2\pi,x_0,0,0,0,\mu,\lambda) = 0,\\ \varphi_4(x_0,\lambda,\mu) &\equiv \tilde{w_1}(2\pi,x_0,0,0,0,\mu,\lambda) = 0. \end{aligned} \tag{4.3}$$

Suppose we have determined to the sequence $x_1,...,x_{i-1}$ the values $\mu_1^*,...,\mu_{i-1}^*$, $\lambda_1^*,...,\lambda_{i-1}^*$ approximating the corresponding values $\mu(x_1),...,\mu(x_{i-1})$, and $\lambda(x_1),...,\lambda(x_{i-1})$.

In order to determine to x_i the corresponding approximating values (μ_i^*,λ_i^*) we apply Newton's method to system (4.3) yielding the sequence (μ_i^k,λ_i^k) defined by

$$\begin{pmatrix}\lambda_i^{k+1}\\ \mu_i^{k+1}\end{pmatrix} = \begin{pmatrix}\lambda_i^{k}\\ \mu_i^{k}\end{pmatrix} - J^{-1}(x_i)\varphi(x_i,\mu_i^k,\lambda_i^k), \quad k = 0,1,...$$

where $\lambda_i^0 = \lambda_{i-1}^*, \mu_i^0 = \mu_{i-1}^*$,

$$J(x_i) = \begin{pmatrix} \frac{\partial \tilde{x}}{\partial \lambda}(2\pi, x_i, 0, 0, 0, \mu_{i-1}^*, \lambda_{i-1}^*) & \frac{\partial \tilde{x}}{\partial \mu}(2\pi, x_i, 0, 0, 0, \mu_{i-1}^*, \lambda_{i-1}^*) \\ \frac{\partial \tilde{y}}{\partial \lambda}(2\pi, x_i, 0, 0, 0, \mu_{i-1}^*, \lambda_{i-1}^*) & \frac{\partial \tilde{y}}{\partial \mu}(2\pi, x_i, 0, 0, 0, \mu_{i-1}^*, \lambda_{i-1}^*) \\ \frac{\partial \tilde{z}}{\partial \lambda}(2\pi, x_i, 0, 0, 0, \mu_{i-1}^*, \lambda_{i-1}^*) & \frac{\partial \tilde{z}}{\partial \mu}(2\pi, x_i, 0, 0, 0, \mu_{i-1}^*, \lambda_{i-1}^*) \\ \frac{\partial \tilde{w_1}}{\partial \lambda}(2\pi, x_i, 0, 0, 0, \mu_{i-1}^*, \lambda_{i-1}^*) & \frac{\partial \tilde{w_1}}{\partial \mu}(2\pi, x_i, 0, 0, 0, \mu_{i-1}^*, \lambda_{i-1}^*) \end{pmatrix}.$$

Remark 4.1 *Under the assumption that the Jacobian matrix $J(x_i)$ is invertible and that the difference $|x_i - x_{i-1}|$ is sufficiently small for any i, the sequences $\{\mu_i^k\}, \{\lambda_i^k\}$, converge to μ_i^*, λ_i^*, respectively, as k tends to infinity.*

The entries of the matrix J can be calculated by solving the initial value problem

$$\frac{dx}{d\tau} = \mu P(x, y, \lambda), \qquad \frac{dy}{d\tau} = \mu Q(x, y, \lambda),$$

$$\frac{dz}{d\tau} = \mu H_1(x, y, \lambda), \qquad \frac{dw_1}{d\tau} = \mu e^z H_2(x, y, \lambda),$$

$$\frac{d(\frac{\partial x}{\partial \lambda})}{d\tau} = \mu \left(\frac{\partial P}{\partial \lambda} + \frac{\partial P}{\partial x}\frac{\partial x}{\partial \lambda} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial \lambda} \right),$$

$$\frac{d(\frac{\partial y}{\partial \lambda})}{d\tau} = \mu \left(\frac{\partial Q}{\partial \lambda} + \frac{\partial Q}{\partial x}\frac{\partial x}{\partial \lambda} + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial \lambda} \right),$$

$$\frac{d(\frac{\partial z}{\partial \lambda})}{d\tau} = \mu \left(\frac{\partial H_1}{\partial \lambda} + \frac{\partial H_1}{\partial x}\frac{\partial x}{\partial \lambda} + \frac{\partial H_1}{\partial y}\frac{\partial y}{\partial \lambda} \right),$$

$$\frac{d(\frac{\partial w_1}{\partial \lambda})}{d\tau} = \mu \left(\frac{\partial (e^z H_2)}{\partial \lambda} + \frac{\partial (e^z H_2)}{\partial x}\frac{\partial x}{\partial \lambda} + \frac{\partial (e^z H_2)}{\partial y}\frac{\partial y}{\partial \lambda} + \frac{\partial (e^z H_2)}{\partial z}\frac{\partial z}{\partial \lambda} \right),$$

$$\frac{d(\frac{\partial x}{\partial \mu})}{d\tau} = P + \mu \left(\frac{\partial P}{\partial x}\frac{\partial x}{\partial \mu} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial \mu} \right),$$

$$\frac{d(\frac{\partial y}{\partial \mu})}{d\tau} = Q + \mu \left(\frac{\partial Q}{\partial x}\frac{\partial x}{\partial \mu} + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial \mu} \right),$$

$$\frac{d(\frac{\partial z}{\partial \mu})}{d\tau} = H_1 + \mu \left(\frac{\partial H_1}{\partial x}\frac{\partial x}{\partial \mu} + \frac{\partial H_1}{\partial y}\frac{\partial y}{\partial \mu} \right),$$

$$\frac{d(\frac{\partial w_1}{\partial \mu})}{d\tau} = e^z H_2 + \mu \left(\frac{\partial (e^z H_2)}{\partial x}\frac{\partial x}{\partial \mu} + \frac{\partial (e^z H_2)}{\partial y}\frac{\partial y}{\partial \mu} + \frac{\partial (e^z H_2)}{\partial z}\frac{\partial z}{\partial \mu} \right),$$

$$x(0, x_i, 0, 0, 0, \mu_{i-1}^*, \lambda_{i-1}^*) = x_i, \quad y(0, x_i, 0, 0, 0, \mu_{i-1}^*, \lambda_{i-1}^*) = 0,$$

$$z(0,x_i,0,0,0,\mu^*_{i-1},\lambda^*_{i-1})=0,\quad w_1(0,x_i,0,0,0,\mu^*_{i-1},\lambda^*_{i-1})=0,$$
$$\frac{\partial x}{\partial\mu}(0,x_i,0,0,0,\mu^*_{i-1},\lambda^*_{i-1})=\frac{\partial y}{\partial\mu}(0,x_i,0,0,0,\mu^*_{i-1},\lambda^*_{i-1})=0,$$
$$\frac{\partial z}{\partial\mu}(0,x_i,0,0,0,\mu^*_{i-1},\lambda^*_{i-1})=\frac{\partial w_1}{\partial\mu}(0,x_i,0,0,0,\mu^*_{i-1},\lambda^*_{i-1})=0,$$
$$\frac{\partial x}{\partial\lambda}(0,x_i,0,0,0,\mu^*_{i-1},\lambda^*_{i-1})=\frac{\partial y}{\partial\lambda}(0,x_i,0,0,0,\mu^*_{i-1},\lambda^*_{i-1})=0,$$
$$\frac{\partial z}{\partial\lambda}(0,x_i,0,0,0,\mu^*_{i-1},\lambda^*_{i-1})=\frac{\partial w_1}{\partial\lambda}(0,x_i,0,0,0,\mu^*_{i-1},\lambda^*_{i-1})=0.$$

Remark 4.2 *As we mentioned above, the presented variant of the procedure is based on the assumption that the set Λ_3 of parameter points can be parameterized by the x-coordinate of the intersection point of the corresponding limit cycle $\Gamma(\lambda)$ with the positive x-axis. If this assumption is not longer fulfilled, we can reparametrize the remaining subset by one of the components of the parameter vector λ.*

In the last step we show that the limit cycle $\Gamma(\lambda_i)$ of system (1.1) has multiplicity not greater than 3. For this purpose we consider system (3.3) for $\lambda=\lambda_i$ and construct a Poincaré functional $G(x,y,z,w_2,\lambda_i)$ for (x,y) in an annulus $\Omega(\lambda_i)$ containing the limit cycle $\Gamma(\lambda_i)$ such that system (3.3) has no periodic solution in $\Omega(\lambda_i)$. Thus, according to Proposition 3.2, the multiplicity of $\Gamma(\lambda_i)$ is exactly 3.

To construct the Poincaré functional G we make the ansatz

$$G(x,y,z,w_2,\lambda_i):=\psi(x,y,\lambda_i)e^{2z}+C_{n+1}(\lambda_i)w_2, \tag{4.4}$$

where ψ is the linear combination of some base functions ψ_j in $\Omega(\lambda_i)$

$$\psi(x,y,\lambda_i)=\sum_{j=1}^{n}C_j(\lambda_i)\psi_j(x,y,\lambda_i), \tag{4.5}$$

and $C_{n+1}(\lambda_i)$ is some additional parameter. In case that (1.1) is a polynomial system, we can take monomials in x and y as base functions ψ_j, $j=1,2,...,n$. If we differentiate the functional G along system (3.3) we get

$$\frac{dG}{dt}_{|(3.3)}=e^{2z}\big(2\psi\,divf+\psi_xP+\psi_yQ+C_{n+1}H_3\big), \tag{4.6}$$

where H_3 is defined in (3.1). If we are able to given λ_i to find a function $\psi(x,y,\lambda_i)$ and a parameter function $C_{n+1}(\lambda_i)$ such that it holds in $\Omega(\lambda_i)$

$$\Phi(x,y,\lambda_i):=2\psi\,divf+\psi_xP+\psi_yQ+C_{n+1}H_3\neq 0, \tag{4.7}$$

then according to Proposition 3.2 the limit cycle $\Gamma(\lambda_i)$ has multiplicity 3.

We summarize this result in the following theorem.

Theorem 4.1 *Let the assumption* (A_1) *be satisfied for* $n = 3$. *Furthermore, we suppose that to given* $x_i > 0$, $\lambda_i \in \Lambda_3$, *system* (3.2) *has a limit cycle* $\Gamma(\lambda_i)$ *located in the annulus* $\Omega(\lambda_i)$ *and intersecting the positive* x*-axis in the point* $(x_i, 0)$ *and that there are base functions* $\psi_j(x, y, \lambda_i), j = 1, ..., n$, *and coefficient functions* $C_k(\lambda_i), k = 1, ..., n+1$, *such that for* $(x, y) \in \Omega(\lambda_i)$ *we have*

$$\begin{aligned}\Phi(x, y, \lambda_i) \equiv & 2\ div f(x, y, \lambda_i) \sum_{j=1}^{n} C_j(\lambda_i)\psi_j(x, y, \lambda_i) + P(x, y, \lambda_i) \sum_{j=1}^{n} C_j(\lambda_i)\frac{\partial \psi_j}{\partial x}(x, y, \lambda_i) \\ & + Q(x, y, \lambda_i) \sum_{j=1}^{n} C_j(\lambda_i)\frac{\partial \psi_j}{\partial y}(x, y, \lambda_i) + C_{n+1}(\lambda_i)H_3(x, y, \lambda_i) \neq 0.\end{aligned} \tag{4.8}$$

Then the multiplicity of the limit cycle $\Gamma(\lambda_i)$ *of system* (1.1) *is exactly* 3.

5 Application to a polynomial Liénard system with a unique equilibrium point

We consider the class of Liénard systems

$$\frac{dx}{dt} = y - (x^7 - cx^5 + bx^3 - ax) \equiv P(x, y, \lambda),\ \frac{dy}{dt} = -x \equiv Q(x, y, \lambda) \tag{5.1}$$

depending on the real parameter vector $\lambda = (a, b, c)$. Our goal is to determine a set of parameter tuples $\{\lambda_i\}$ by applying the procedure described above such that the corresponding system (5.1) has a limit cycle of multiplicity 3. The following lemma can be easily verified.

Lemma 5.1 *For any tuple* $(a, b, c) \in R^3$, *the origin* $(0, 0)$ *is the unique equilibrium point of system* (5.1) *in any bounded part of the phase plane. For* $|a| < 2$ *it represents a focus which is exponentially attracting (repelling) for* $a > 0$ $(a < 0)$.
For $a = 0$, *the origin is a weak focus whose Lyapunov numbers* α_i *defined in* (2.2) *satisfy*

$$\alpha_1 = \alpha_2 = 0,\ sign\, \alpha_3 = -sign\, b,$$

i.e., it is asymptotically stable for $b > 0$.
In case $\alpha_1 = \alpha_2 = \alpha_3 = 0$ *we have*

$$\alpha_4 = 0,\ sign\, \alpha_5 = -sign\, c.$$

Supposing $\alpha_1 = ... = \alpha_5 = 0$, *we have*

$$\alpha_6 = 0, \alpha_7 \neq 0.$$

Using the notion of multiplicity of an equilibrium point we get from Lemma 5.1 that in case $a = b = c = 0$ the origin is a focus of multiplicity 3. Thus, we can use the parameter tuple $(0, 0, 0)$ as starting point for our procedure. As a result we get the following set of parameter points $\{\lambda_i\}$ (see Table 1) for which system (4.2) has a limit cycle $\Gamma(\lambda_i)$ whose multiplicity is at least 3. We note that the limit cycle $\Gamma(\lambda_i)$ shrinks to the origin as $|\lambda_i|$ tends to zero, that means, there appears Hopf bifurcation of a limit cycle of higher multiplicity from the origin.

In the final step we construct to each λ_i an annulus $\Omega(\lambda_i)$ containing the limit cycle $\Gamma(\lambda_i)$ and a Poincaré functional $G(x, y, z, w_2, \lambda_i)$ according to the ansatz (4.4) such that the condition (4.8) is fulfilled.
First we construct an annulus $\Omega_\varepsilon(\lambda_i)$ containing the limit cycle $\Gamma(\lambda_i)$ and depending on some parameter ε.

i	x_0	a	b	c	μ
1	0.2	0.000035	0.003500	0.105000	1.000000
2	0.3	0.000399	0.017719	0.236250	1.000000
3	0.4	0.002240	0.056000	0.420000	1.000000
4	0.5	0.008545	0.136719	0.656250	1.000001
5	0.6	0.025515	0.283499	0.944999	1.000007
6	0.7	0.064336	0.525207	1.286239	1.000047
7	0.8	0.143327	0.895901	1.679930	1.000231
8	0.9	0.290354	1.434568	2.125885	1.000950
9	1	0.545023	2.183981	2.623400	1.003359
10	1.1	0.958499	3.186438	3.170108	1.010496
11	1.2	1.583387	4.469510	3.758703	1.029399
12	1.3	2.433288	6.005218	4.368477	1.073128
13	1.4	3.413452	7.654568	4.957041	1.154102
14	1.5	4.414707	9.299753	5.500792	1.270515

Table 1.

For this purpose we imbed system (5.1) into the system

$$\frac{dx}{dt} = y - (x^7 - c_i x^5 + b_i x^3 - a_i x) + \kappa x, \ \frac{dy}{dt} = -x + \kappa\big(y - (x^7 - c_i x^5 + b_i x^3 - a_i x)\big), \quad (5.2)$$

where the parameter κ strictly rotates the field at $(x, y) \neq (0, 0)$ and which represents for $\kappa = 0$ system (5.1) possessing the limit cycle $\Gamma(\lambda_i)$. Now we set $\kappa = \pm\varepsilon$, where ε is a small

positive number. Then system (5.2) has for $\kappa = \varepsilon$ and $\kappa = -\varepsilon$ the limit cycle $\Gamma_\varepsilon(\lambda_i)$ and $\Gamma_{-\varepsilon}(\lambda_i)$, respectively. Since κ rotates the vector field, $\Gamma_\varepsilon(\lambda_i)$ and $\Gamma_{-\varepsilon}(\lambda_i)$ form an annulus $\Omega_\varepsilon(\lambda_i)$ containing the limit cycle $\Gamma(\lambda_i)$. We use this annulus as an approximation of the wanted annulus $\Omega(\lambda_i)$.
In the next step we construct a functional G_ε on the annulus $\Omega_\varepsilon(\lambda_i)$ according to the ansatz (4.4). The main term in the expression for G_ε is the function $\psi(x, y, \lambda_i)$ which we represent as a linear combination (4.5) of the base functions $x^k y^l$:

$$\psi(x, y, \lambda_i) = \sum_{0 \le k+l \le N} C_{kl}(\lambda_i) x^k y^l. \tag{5.3}$$

Substituting this relation into (4.8) we get

$$\begin{aligned} \Phi(x, y, \lambda_i) \equiv & 2\ div f(x, y, \lambda_i) \sum_{0 \le k+l \le N} C_{kl}(\lambda_i) x^k y^l \\ & + P(x, y, \lambda_i) \sum_{\substack{1 \le k+l \le N \\ k \ge 1\ l \le N-1}} k\ C_{kl}(\lambda_i) x^{k-1} y^l \\ & + Q(x, y, \lambda_i) \sum_{\substack{1 \le k+l \le N \\ l \ge 1\ k \le N-1}} l\ C_{kl}(\lambda_i) x^k y^{l-1} + C_{N+1}(\lambda_i) H_3(x, y, \lambda_i). \end{aligned} \tag{5.4}$$

Our goal is to choose the coefficients $C_{kl}(\lambda_i)$ and the coefficient $C_{N+1}(\lambda_i)$ in (5.4) in such a way that the expression $\Phi(x, y, \lambda_i)$ does not vanish in $\Omega_\varepsilon(\lambda_i)$. This problem can be reduced to a linear programming problem as described in [2]. If this problem has a solution, then we take the annulus $\Omega_\varepsilon(\lambda_i)$ as the wanted annulus $\Omega(\lambda_i)$ and the functional G_ε as Poincaré functional G defined on $\Omega(\lambda_i)$. In case that we cannot find coefficients $C_{kl}(\lambda_i)$ such that (4.8) holds, we decrease the parameter ε or increase the degree N and repeat the programming procedure.

In what follows we construct the annulus $\Omega(\lambda_7)$ and the Poincaré functional G to the parameter tuple λ_7 in Table 1 belonging to $x_0 = 0.8$. As annulus $\Omega_\varepsilon(\lambda_7)$ we choose the region bounded by the limit cycles $\Gamma_\varepsilon(\lambda_7)$ and $\Gamma_{-\varepsilon}(\lambda_7)$ of system (5.2) with $\varepsilon = 0.03$. As base functions we use monomials with maximal degree 4, i.e. $N = 4$ in (5.3). If we apply the linear programming algorithm as described in [2], we get the following result

$C_{00}(\lambda_7) = 1.242085, C_{10}(\lambda_7) = 0.909316, C_{01}(\lambda_7) = 0.909316, C_{20}(\lambda_7) = 1.818632,$
$C_{11}(\lambda_7) = 1.4134, C_{02}(\lambda_7) = 0.683876, C_{30}(\lambda_7) = 0.909316, C_{21}(\lambda_7) = 0.909316,$
$C_{12}(\lambda_7) = 0.909316, C_{03}(\lambda_7) = 0.909316, C_{40}(\lambda_7) = 0, C_{31}(\lambda_7) = 1.8186326,$
$C_{22}(\lambda_7) = 1.077677, C_{13}(\lambda_7) = 0, C_{04}(\lambda_7) = 1.818632, C_5(\lambda_7) = 0.885257,$
where the coefficient $C_5(\lambda_7)$ multiplies the variable w_2. With these coefficients we have

$$\Phi(x, y, \lambda_7) \geq 0.0012 > 0 \quad \text{for} \quad (x, y) \in \Omega(\lambda_7).$$

Thus, the functional

$$G(x, y, z, w_2, \lambda_7) = e^{2z} \sum_{0 \leq k+l \leq 4} C_{kl}(\lambda_7) x^k y^l + C_5(\lambda_7) w_2$$

is a Poincaré functional in the annulus $\Omega(\lambda_7)$ and we can conclude that $\Gamma(\lambda_7)$ is a limit cycle of multiplicity 3 of system (5.1) with $\lambda = \lambda_7$.

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