Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Universality of the REM for dynamics of mean-field spin glasses

Gérard Ben Arous[1], Anton Bovier[2] , Jiří Černý[3]

submitted: 21. Juni 2007

[1] Courant Institute of the Mathematical Sciences
New York University
251 Mercer Street
New York, NY 10012, USA
E-Mail:benarous@cims.nyu.edu

[2] Weierstraß-Institut
für Angewandte Analysis und Stochastik
Mohrenstr. 39
10117 Berlin, Germany
and
Technische Universität Berlin
Institut für Mathematik
Straße des 17. Juni 136
10623 Berlin, Germany
E-Mail:bovier@wias-berlin.de

[3] École Polytechnique
Fédérale de Lausanne
1015 Lausanne, Switzerland
E-Mail:jiri.cerny@epfl.ch

No. 1241
Berlin 2007

2000 *Mathematics Subject Classification.* 82C44, 60K35, 60G70.

Key words and phrases. aging, universality, spin glasses, SK model, random walk.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider a version of a Glauber dynamics for a p-spin Sherrington–Kirkpatrick model of a spin glass that can be seen as a time change of simple random walk on the N-dimensional hypercube. We show that, for any $p \geq 3$ and any inverse temperature $\beta > 0$, there exist constants $\gamma_0 > 0$, such that for all exponential time scales, $\exp(\gamma N)$, with $\gamma \leq \gamma_0$, the properly rescaled *clock process* (time-change process), converges to an α-stable subordinator where $\alpha = \gamma/\beta^2 < 1$. Moreover, the dynamics exhibits aging at these time scales with time-time correlation function converging to the arcsine law of this α-stable subordinator. In other words, up to rescaling, on these time scales (that are shorter than the equilibration time of the system), the dynamics of p-spin models ages in the same way as the REM, and by extension Bouchaud's REM-like trap model, confirming the latter as a universal aging mechanism for a wide range of systems. The SK model (the case $p = 2$) seems to belong to a different universality class.

1 Introduction and results

Aging has become one of the main paradigms to describe the long-time behavior of complex and/or disordered systems. Systems that have strongly motivated this research are *spin glasses*, where aging was first observed experimentally in the anomalous relaxation patterns of the magnetization [LSNB83, Cha84]. The theoretical modeling of aging phenomena took a major leap with the introduction of so-called *trap models* by Bouchaud and Dean in the early 1990'ies [Bou92, BD95] (see [BCKM98] for a review). These models reproduce the characteristic power law behavior seen experimentally while being sufficiently simple to allow for detailed analytical treatment. While trap models are heuristically motivated to capture the behavior of the dynamics of spin glass models, there is no clear theoretical, let alone mathematical derivation of these from an underlying spin-glass dynamics. The first attempt to establish such a connection was made in [BBG02, BBG03a, BBG03b] where it was shown that starting from a particular Glauber dynamics of the Random Energy Model (REM), at low temperatures and at the time scale slightly shorter than the equilibration time of the dynamics, the aging of the time-time correlation function of the dynamics converged to that given by Bouchaud's REM-like trap model.

On the other hand, in a series of papers [BČ05, BČM06, BČ07a, BČ07b] a systematic investigation of a variety of trap models was initiated. In this process, it emerged that there appears to be an almost universal aging mechanism based on α-stable subordinators that governs aging in most of the trap models. It was also shown that the same feature holds for the dynamics of the REM at shorter time scales than those considered in [BBG03a, BBG03b], and that this also happens at high temperature

provided appropriate time scales are considered [BČ07a]. For a general review on trap models see [BČ06].
In all models considered so far, however, the random variables describing the quenched disorder were considered to be independent, be it in the REM or in the trap models. Aging in correlated spin glass models was investigated rigorously only in some cases of spherical SK models and at very short time scales [BDG01]. In the present paper we show for the first time that the same type of aging mechanism is relevant also in correlated spin glasses, at least on time scales that are short compared to equilibration time (but exponentially large in the volume of the system).
Let us first describe the class of models we are considering. Our state spaces will be the N-dimensional hypercube, $\mathcal{S}_N \equiv \{-1,1\}^N$. $R_N : \mathcal{S}_N \times \mathcal{S}_N \to [-1,1]$ denotes as usual the normalized overlap, $R_N(\sigma,\tau) \equiv N^{-1}\sum_{i=1}^N \sigma_i\tau_i$. The Hamiltonian of the p-spin SK-model is defined as $\sqrt{N}H_N$, where $H_N : \mathcal{S}_N \to \mathbb{R}$ is the centered normal process indexed by $\mathcal{S}_N$ with covariance

$$\mathbb{E}[H_N(\sigma)H_N(\tau)] = R_N(\sigma,\tau)^p, \tag{1.1}$$

and $p \in \mathbb{N}$, $p > 2$. We will denote by $\mathcal{H}$ the σ-algebra generated by the random variables $H_N(\sigma), \sigma \in \mathcal{S}_N, N \in \mathbb{N}$. The corresponding Gibbs measure is then given by

$$\mu_{\beta,N}(\sigma) \equiv Z_{\beta,N}^{-1} e^{\beta\sqrt{N}H_N(\sigma)}, \tag{1.2}$$

where $Z_{\beta,N}$ denotes the normalizing partition function.
We define the classical trap-model dynamics as a nearest neighbor continuous time Markov chain $\sigma_N(\cdot)$ on $\mathcal{S}_N$ with transition rates

$$w_N(\sigma,\tau) = \begin{cases} N^{-1}e^{-\beta\sqrt{N}H_N(\sigma)}, & \text{if } \operatorname{dist}(\sigma,\tau) = 1, \\ 0, & \text{otherwise;} \end{cases} \tag{1.3}$$

here $\operatorname{dist}(\cdot,\cdot)$ is the graph distance on the hypercube,

$$\operatorname{dist}(\sigma,\tau) = \frac{1}{2}\sum_{i=1}^N |\sigma_i - \tau_i|. \tag{1.4}$$

A simple way to construct this dynamics is as a time change of a simple random walk on $\mathcal{S}_N$: We denote by $Y_N(k) \in \mathcal{S}_N$, $k \in \mathbb{N}$, the simple unbiased random walk (SRW) on $\mathcal{S}_N$ started at some fixed point of $\mathcal{S}_N$, say at $\{1,\dots,1\}$. For $\beta > 0$ we define the *clock-process* by

$$S_N(k) = \sum_{i=0}^{k-1} e_i \exp\left\{\beta\sqrt{N}H_N\big(Y_N(i)\big)\right\}, \tag{1.5}$$

where $\{e_i, i \in \mathbb{N}\}$ is a sequence of mean-one i.i.d. exponential random variables. We denote by $\mathcal{Y}$ the σ-algebra generated by the SRW random variables $Y_N(k)$, $k \in \mathbb{N}$, $N \in \mathbb{N}$. The σ-algebra generated by the random variables $e_i, i \in \mathbb{N}$ will be denoted by $\mathcal{E}$. Then the process $\sigma_N(\cdot)$ can be written as

$$\sigma_N(t) \equiv Y_N(S_N^{-1}(t)). \tag{1.6}$$

Obviously, σ_N is reversible with respect to the measure $\mu_{\beta,N}$. We will consider all random processes to be defined on an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that the three σ-algebras $\mathcal{H}$, $\mathcal{Y}$, and $\mathcal{E}$ are all independent under $\mathbb{P}$.
We will systematically use the definition of the dynamics given by (1.3) or (1.6). This is the same as was used in the analysis of the REM and in most work on trap models. It differs substantially from more popular dynamics such as the Metropolis or the heat-bath algorithm. The main difference is that in these dynamics the trajectories are not independent of the environment and are biased against going up in energy. This may have a substantial effect on the dynamics, and we do not know whether our results will apply also (with some modifications) in these cases. The fact is that we currently do not have the tools to analyze these dynamics even in the case of the REM!

Let $V_\alpha(t)$ be the α-stable subordinator with the Laplace transform given by

$$\mathbb{E}[e^{-\lambda V_\alpha(t)}] = \exp(-t\lambda^\alpha). \tag{1.7}$$

The main technical result on the dynamics will be the following theorem that provides the asymptotic behavior of the clock process.

Theorem 1.1. *There exists a function $\zeta(p)$ such that for all $p \geq 3$ and γ satisfying*

$$0 < \gamma < \min\left(\beta^2, \zeta(p)\beta\right), \tag{1.8}$$

under the conditional distribution $\mathbb{P}[\cdot|\mathcal{Y}]$ the law of the stochastic process

$$\bar{S}_N(t) = e^{-\gamma N} S_N\left(\lfloor tN^{1/2} e^{N\gamma^2/2\beta^2} \rfloor\right), \qquad t \geq 0, \tag{1.9}$$

defined on the the space of càdlàg functions equipped with the Skorokhod M_1-topology, converges, $\mathcal{Y}$-a.s., to the law of γ/β^2-stable subordinator $V_{\gamma/\beta^2}(Kt), t \geq 0$, where K is a positive constant depending on γ, β and p.
Moreover, the function $\zeta(p)$ is increasing and it satisfies

$$\zeta(3) \simeq 1.0291 \qquad \text{and} \qquad \lim_{p\to\infty} \zeta(p) = \sqrt{2\log 2}. \tag{1.10}$$

We will explain in Section 5 what the M_1-topology is. Roughly, it is a weak topology that does not convey much information at the jumps of the limiting process: it can be the case that the approximating processes jumps several times at rather short distances to produce one bigger jump of the limit process. This will actually be the case in our models for $p < \infty$, while it is not the case in the REM. Therefore we cannot replace the M_1 topology with the stronger J_1-topology in Theorem 1.1.
To control the behavior of spin-spin correlation functions that are commonly used to characterize aging, we need to know more on how these jumps occur at finite N. What we will show, is that if we the slightly coarse-grain the process $\bar{S}_N$ over blocks of size $o(N)$, the rescaled process does converge in the J_1-topology. What this says, is that the jumps of the limiting process are compounded by smaller jumps that are made over $\leq o(N)$ steps of the SRW. In other words, the jumps of the limiting process come from waiting times accumulated in one slightly extended trap, and during this entire time only a negligible fraction of the spins are flipped. That will imply the following aging result.

Theorem 1.2. *Let $A_N^\varepsilon(t,s)$ be the event defined by*

$$A_N^\varepsilon(t,s) = \left\{R_N\big(\sigma_N\big(te^{\gamma N}\big), \sigma_N\big((t+s)e^{\gamma N}\big)\big) \geq 1-\varepsilon\right\}. \tag{1.11}$$

Then, under the hypothesis of Theorem 1.1, for all $\varepsilon \in (0,1)$, $t>0$ and $s>0$,

$$\lim_{N\to\infty} \mathbb{P}[A_N^\varepsilon(t,s)] = \frac{\sin \alpha\pi}{\pi} \int_0^{t/(t+s)} u^{\alpha-1}(1-u)^{-\alpha}\,\mathrm{d}u. \tag{1.12}$$

Remark. We will in fact prove the stronger statement that aging in the above sense occurs along almost every random walk trajectory, that is

$$\lim_{N\to\infty} \mathbb{P}[A_N^\varepsilon(t,s)|\mathcal{Y}] = \frac{\sin \alpha\pi}{\pi} \int_0^{t/(t+s)} u^{\alpha-1}(1-u)^{-\alpha}\,\mathrm{d}u, \qquad \mathcal{Y}\text{-a.s.} \tag{1.13}$$

Let us discuss the meaning of these results. $e^{\gamma N}$ is the time-scale at which we want to observe the process. According to Theorem 1.1, at this time the random walk will make of the order of $N^{1/2}e^{N\gamma^2/2\beta^2} \ll e^{\gamma N}$ steps. Since this number is also much smaller than 2^N (as follows from (1.10)), the random walk will essentially visit that number of sites.
If the random process H_N was i.i.d., then the maximum of H_N along the trajectory would be $\big(2\ln(N^{1/2}e^{N\gamma^2/2\beta^2})\big)^{1/2} \sim N^{1/2}\gamma/\beta$, and the time spent in that site would be of order $e^{\gamma N}$. Since Theorem 1.1 holds also in the i.i.d. case, that is in the REM (see [BČ07a]), the time spent in the maximum is comparable to the total time and the convergence to the α-stable subordinator implies that the total accumulated time is composed of pieces of order $e^{\gamma N}$ that are collected along the trajectory. In fact, each jump of the subordinator corresponds to one visit to a site that has waiting times of that order. In a common metaphor, the sites are referred to as traps and the mean waiting times as their depths.
The theorem in the general case states that in the p-spin model, the same is essentially true. The difference will be that the traps here will not consist of a single site, but consist of a deep valley (along the trajectory) whose bottom that has approximately the same energy as in the i.i.d. case and whose shape and width we will be able to describe quite precisely. Remarkably, the number of sites contributing significantly to the residence time in the valley is essentially finite, and different valleys are statistically independent.
The fact that traps are finite may appear quite surprising to those familiar with the statics of p-spin models. From the results there (see [Tal03, Bov06]), it is known that the Gibbs measure concentrates on "lumps" whose diameter is of order $N\epsilon_p$, with $\epsilon_p > 0$. The mystery is however solved easily: the process $H_N(\sigma)$ does indeed decreases essentially linearly with speed $N^{-1/2}$ from a local maximum. Thus, the residence times in such sites decrease geometrically, so that the contributions of a neighborhood of size K of a local maximum amounts to a fraction of $(1-c^{-K})$ of the total time spend in that valley ; for the support of the Gibbs measure, one needs however to take into account the entropy, that is that the volumes of the balls of radius r increases like N^r. For the dynamics, at least at our time-scales, this is, however, irrelevant, since the SRW leaves a local minimum essentially ballistically.

The proof of Theorem 1.1 relies on the combination of detailed information on the properties of simple random walk on the hypercube, which is provided in Section 4 (but see also [Mat89, BG06, ČG06]), and comparison of the process H_N on the trajectory of the SRW to a simpler Gaussian process using interpolation techniques à la Slepian, familiar from extreme value theory of Gaussian processes.
Let us explain this in more detail. On the time scales we are considering, the SRW makes $tN^{1/2}\exp(N\gamma^2/2\beta^2) \ll tN^{1/2}\exp(N\zeta(p)^2/2) \ll 2^N$ steps. In this regime the SRW is extremely "transient", in the sense that (i) starting from a given point x, for a times $t \leq \nu \sim N^\omega, \omega < 1$, the distance from x grows essentially linearly with speed one, that is there are no backtrackings with high probability; (ii) the SRW will *never* return to a neighborhood of size ν of the starting point x, with high probability. The upshot is that we can think of the trajectory of the SRW essentially as of a straight line.
Next we consider the Gaussian process restricted to the SRW trajectory. We expect that the main contributions to the sums $S_N(k)$ come from places where Y_N is maximal (on the trajectory). We expect that the distribution of these extremes do not feel the correlation between points farther than ν apart. On the other hand, for points closer than ν, the correlation function $R_N(Y_N(i), Y_N(j))^p$ can be well approximated by a linear function $1-2p|i-j|/N$ (using that $R_N(Y_N(i), Y_N(j)) \sim 1-2|i-j|/N$). This is convenient since this process has an explicit representation in terms of i.i.d. random variables that allow for explicit computations (in fact, this is one of the famous Slepian processes for which the extremal distribution can be computed explicitly [Sle61, She71]). Thus the idea is to cut the SRW trajectory into blocks of length ν and to replace the original process $H_N(Y_N(i))$ by a new one U_i, where U_i and U_j are independent, if i, j are not in the same block, and $\mathbb{E}[U_i U_j] = 1 - 2p|i-j|/N$ if they are. For the new process, Theorem 1.1 is relatively straightforward. The main step is the computation of Laplace transforms in Section 2. Comparing the real process with the auxiliary one is the bulk of the work and is done in Section 3. The properties of SRW needed are established in Section 4. In Section 5 we present the proofs of the main theorems.
Our results here show some universality of the REM for dynamics of p-spin models with $p \geq 3$. This dynamic universality is close to the static universality of the REM, which shows that various features of the landscape of energies (that is of the Hamiltonian H_N) are insensitive to correlations. This static universality in a microcanonical context has been introduced by [BM04] (see [BK06a, BK06b] for rigorous results on spin-glasses). The static results closest to our dynamics question are given in [BGK06, BK07] where it is shown that the statistics of extreme values for the restriction of H_N to a random sets $X_N \subset \mathcal{S}_N$ are universal, for $p \geq 3$ and $|X_N| = e^{cN}$, for c small enough.
This work was initiated during a concentration period on metastability and aging at the Max-Planck Institute for Mathematics in the Sciences in Leipzig. GBA and AB thank the MIP-MIS and Stefan Müller for kind hospitality during this event. AB's research is supported in part by DFG in the Dutch-German Bilateral Research Group "Mathematics of Random Spatial Models from Physics and Biology".

2 Behavior the one-block sums

In this section we analyze the distribution of the block-sums $\sum_{i=1}^{\nu} e_i e^{\beta\sqrt{N}U_i}$, where e_i are mean-one i.i.d. exponential random variables, and $\{U_i, i = 1, \dots, \nu\}$ is a centered Gaussian process with the covariance $\mathbb{E}U_iU_j = 1 - 2p|i-j|/N$; $\nu = \nu_N$ is a function of N of the form

$$\nu = \lfloor N^{\omega} \rfloor, \qquad \text{with } \omega \in (1/2, 1). \tag{2.1}$$

As explained in the introduction, this process will serve as a local approximation of the corresponding block sums along a SRW trajectory. We characterize the distribution of the block-sums in terms of its Laplace transform

$$\mathcal{F}_N(u) = \mathbb{E}\Big[\exp\Big\{-ue^{-\gamma N}\sum_{i=1}^{\nu} e_i e^{\beta\sqrt{N}U_i}\Big\}\Big]. \tag{2.2}$$

Proposition 2.1. *For all γ such that $\gamma/\beta^2 \in (0,1)$ there exists a constant, $K = K(\gamma, \beta, \omega, p)$, such that, uniformly for u in compact subsets of $[0, \infty)$,*

$$\lim_{N\to\infty} N^{1/2}\nu^{-1}e^{N\gamma^2/2\beta^2}\left[1 - \mathcal{F}_N(u)\right] = Ku^{\gamma/\beta^2}. \tag{2.3}$$

Proof. We first compute the conditional expectation in (2.2) given the σ-algebra, $\mathcal{U}$, generated by the Gaussian process U,

$$\begin{aligned}\mathbb{E}\Big[\exp\Big\{-ue^{-\gamma N}\sum_{i=1}^{\nu} e_i e^{\beta\sqrt{N}U_i}\Big\}\Big|\mathcal{U}\Big] &= \prod_{i=1}^{\nu}\frac{1}{1+ue^{-\gamma N}e^{\beta\sqrt{N}U_i}}\\ &= \exp\Big\{-\sum_{i=1}^{\nu} g\Big(ue^{-\gamma N}e^{\beta\sqrt{N}U_i}\Big)\Big\},\end{aligned} \tag{2.4}$$

where

$$g(x) \equiv \ln(1+x). \tag{2.5}$$

Note that importantly, $g(x)$ is monotone increasing and non-negative for $x \in \mathbb{R}_+$. We use the well-known fact (see e.g. [Sle61]) that the random variables U_i can be expressed using a sequence of i.i.d. standard normal variables, Z_i, as follows. Set $Z_1 = (U_1 + U_\nu)/(4 - 4p(\nu-1)/N)^{1/2}$ and $Z_k = (U_k - U_{k-1})/(4p/N)^{1/2}$, $k = 2, \dots, \nu$. Then Z_i are i.i.d. standard normal and

$$U_i = \Gamma_1 Z_1 + \dots + \Gamma_i Z_i - \Gamma_{i+1}Z_{i+1} - \Gamma_\nu Z_\nu, \tag{2.6}$$

where

$$\Gamma_1 = \sqrt{1 - \frac{p}{N}(\nu - 1)} \qquad \text{and} \qquad \Gamma_2 = \dots = \Gamma_\nu = \sqrt{\frac{p}{N}}. \tag{2.7}$$

Observe that $\sum_{i=1}^{\nu}\Gamma_i^2 = 1$. Let us define $G_i(z) = G_i(z_1, \dots, z_\nu)$ as

$$G_i(z) = \Gamma_1 z_1 + \dots + \Gamma_i z_i - \Gamma_{i+1}z_{i+1} - \dots - \Gamma_\nu z_\nu. \tag{2.8}$$

Using this notation we get

$$1 - \mathcal{F}_N(u) = \int_{\mathbb{R}^\nu}\frac{\mathrm{d}z}{(2\pi)^{\nu/2}}e^{-\frac{1}{2}\sum_{i=1}^{\nu} z_i^2}\Big\{1 - \exp\Big[-\sum_{i=1}^{\nu} g\Big(ue^{-\gamma N}e^{\beta\sqrt{N}G_i(z)}\Big)\Big]\Big\}. \tag{2.9}$$

We divide the domain of integration into several parts according to which of the $G_i(z)$ is maximal. Define $D_k = \{z : G_k(z) \geq G_i(z) \forall i \neq k\}$. On D_k we use the substitution

$$\begin{aligned} z_i &= b_i + \Gamma_i(\gamma N - \log u)/(\beta\sqrt{N}), \qquad &&\text{if } i \leq k, \\ z_i &= b_i - \Gamma_i(\gamma N - \log u)/(\beta\sqrt{N}), \qquad &&\text{if } i > k. \end{aligned} \tag{2.10}$$

It will be useful to define $\sum_{j=i+1}^{k} a_j$ as $\sum_{j=1}^{k} a_j - \sum_{j=1}^{i} a_j$, which is meaningful also for $k < i+1$. Using this definition

$$G_k(b) - G_i(b) = 2\sum_{j=i+1}^{k} \Gamma_\nu b_j. \tag{2.11}$$

Set $\theta = -\log(u)/(\gamma N)$ and define

$$D'_k = \Big\{ b : \sum_{j=i+1}^{k} b_j + \frac{\gamma\sqrt{p}}{\beta}|k-i|(1+\theta) \geq 0 \,\forall i \neq k \Big\}. \tag{2.12}$$

After a straightforward computation we find that (2.9) equals

$$\begin{aligned} &e^{-N\gamma^2/2\beta^2} u^{\gamma/\beta^2} \sum_{k=1}^{\nu} \int_{D'_k} \frac{\mathrm{d}b}{(2\pi)^{\nu/2}} e^{-\frac{1}{2}\sum_{i=1}^{\nu} b_i^2} e^{-\frac{\gamma}{\beta}\sqrt{N} G_k(b)(1+\theta)} \\ &\quad \times \Big\{ 1 - \exp\Big(-\sum_{i=1}^{\nu} g\Big(e^{\beta\sqrt{N} G_k(b) - 2\beta\sqrt{p}\sum_{j=i+1}^{k} b_j - 2p\gamma|k-i|(1+\theta)} \Big) \Big) \Big\}. \end{aligned} \tag{2.13}$$

To finish the proof we have to show that u^{γ/β^2} is asymptotically the only dependence of (2.13) on u (or on θ) and that the sum is of order $\nu N^{-1/2}$. We change variables once more to $a_j = b_j/(1+\theta)$ in order to remove the dependence of the integration domains on u. Then the sum (without the prefactor) in (2.13) can be expressed as

$$\begin{aligned} &\sum_{k=1}^{\nu} \int_{D''_k} \frac{(1+\theta)^\nu \mathrm{d}a}{(2\pi)^{\nu/2}} e^{-\frac{1}{2}(1+\theta)^2 \sum_{i=1}^{\nu} a_i^2} \Big[e^{-\frac{\gamma}{\beta}\sqrt{N} G_k(a)(1+\theta)^2} \\ &\quad \times \Big\{ 1 - \exp\Big(-\sum_{i=1}^{\nu} g\Big(e^{(\beta\sqrt{N} G_k(a) - 2\beta\sqrt{p}\sum_{j=i+1}^{k} a_j - 2p\gamma|k-i|)(1+\theta)} \Big) \Big) \Big\} \Big], \end{aligned} \tag{2.14}$$

where $D''_k = \big\{ a : \sum_{j=i+1}^{k} a_j + \frac{\gamma\sqrt{p}}{\beta}|k-i| \geq 0 \,\forall i \neq k \big\}$.
Let $\delta > 0$ be such that $(1+\delta)\gamma/\beta^2 < 1$, and let $N > \log(u)/(\gamma\delta)$, so that $|\theta| \leq \delta$. We first examine the bracket in the above expression for a fixed k. On D''_k

$$\exp\Big\{ -\sum_{i=1}^{\nu} g\big(e^{(\beta\sqrt{N} G_k(a) - 2\beta\sqrt{p}\sum_{j=i+1}^{k} a_j - 2p\gamma|k-i|)(1+\theta)} \big) \Big\} \geq \exp\big\{ -\nu g\big(e^{\beta\sqrt{N} G_k(a)(1+\theta)} \big) \big\}. \tag{2.15}$$

Write $G_k(a)$ as (recall (2.1))

$$G_k(a) = \frac{\xi - \omega\log N}{(1+\theta)\beta\sqrt{N}}. \tag{2.16}$$

The bracket of (2.14) is then smaller than

$$
\begin{aligned}
&e^{-\frac{\gamma}{\beta^2}(\xi-\omega\log N)(1+\theta)}\big\{1-\exp\big(-\nu g\big(e^{\xi-\omega\log N}\big)\big)\big\}\\
&\quad= N^{\frac{\gamma\omega(1+\theta)}{\beta^2}} e^{-\frac{\gamma\xi}{\beta^2}(1+\theta)}\big\{1-\exp\big(-\nu g\big(e^{\xi}/\nu\big)\big)\big\}.
\end{aligned}
\tag{2.17}
$$

The function $e^{-\frac{\gamma\xi}{\beta^2}(1+\theta)}\big\{1-\exp\big(-\nu g\big(e^{\xi}/\nu\big)\big)\big\}$ is bounded for $\xi\in\mathbb{R}$, uniformly in ν, if $(1+\theta)\gamma/\beta^2<1$. Namely, if $\xi\geq 0$,

$$
e^{-\frac{\gamma\xi}{\beta^2}(1+\theta)}\big\{1-\exp\big(-\nu g\left(e^{\xi}/\nu\right)\big)\big\}\leq e^{-\frac{\gamma\xi}{\beta^2}(1+\theta)}\leq 1. \tag{2.18}
$$

If $\xi<0$, then, since $g(x)\leq x$,

$$
\big\{1-\exp\big(-\nu g\big(e^{\xi}/\nu\big)\big)\big\}\leq\big\{1-\exp\big(-e^{\xi}\big)\big\}, \tag{2.19}
$$

which behaves like e^{ξ}, as $\xi\to-\infty$. This compensates the exponentially growing prefactor, if $(1+\theta)\gamma/\beta^2<1$. Thus, under this condition, the bracket of (2.14) increases at most polynomially with N.

In view of this at most polynomial increase, there exist $\delta>0$ small, such that the domain of integration in (2.14) may be restricted to a_i's satisfying

$$
\nu^{-1}\sum_{i=1}^{\nu}a_i^2\in(1-\delta,1+\delta),\quad |a_1|\leq N^{1/4},\quad \sum_{i=1}^{\nu}|a_i|\leq\nu^{1+\delta}. \tag{2.20}
$$

The integral over the remaining a_i's decays at least as $e^{-N^{\delta'}}$ for some $\delta'>0$ (by a simple large deviation argument). For all a satisfying (2.20), $|G_k(a)|\leq N^{1/4}+N^{-1/2}\nu^{1+\delta'}\ll N^{1/2}$ and thus, for any fixed u, uniformly in a,

$$
\frac{e^{-\frac{\gamma}{\beta}\sqrt{N}G_k(a)(1+\theta)}}{e^{-\frac{\gamma}{\beta}\sqrt{N}G_k(a)}}\xrightarrow{N\to\infty}1,\quad\text{and}\quad\frac{e^{-\frac{1}{2}(1+\theta)^2\sum_{i=1}^{\nu}a_i^2}}{e^{-\frac{1}{2}\sum_{i=1}^{\nu}a_i^2}}\xrightarrow{N\to\infty}1. \tag{2.21}
$$

Also, $(1+\theta)^{\nu}\xrightarrow{N\to\infty}1$. Hence, up to a small error, we can remove all but the last occurrence of θ in (2.14).

Finally, taking $x_i=a_i$ for $i\geq 2$, $x_1=N^{1/2}G_k(a)$, and thus

$$
a_1=\frac{x_1-4p(x_2+\cdots+x_k-x_{k+1}-\cdots-x_{\nu})}{\Gamma_1\sqrt{N}}, \tag{2.22}
$$

(2.14) equals, up to a small error,

$$
\begin{aligned}
&\sum_{k=1}^{\nu}\int_{D_k''}\frac{\mathrm{d}x\,e^{-\frac{1}{2}\sum_{i=2}^{\nu}x_i^2}}{\Gamma_1N^{1/2}(2\pi)^{\nu/2}}\exp\Big(-\frac{\gamma}{\beta}x_1-\frac{x_1^2}{2\Gamma_1^2N}\Big)\exp\Big(-\frac{a_1^2}{2}+\frac{x_1^2}{2\Gamma_1^2N}\Big)\\
&\times\Big\{1-\exp\Big(-\sum_{i=1}^{\nu}g\Big(e^{(1+\theta)\beta x_1}e^{-\left(2\beta\sqrt{p}\sum_{j=i+1}^{k}x_j-2p\gamma|k-i|\right)(1+\theta)}\Big)\Big)\Big\}.
\end{aligned}
\tag{2.23}
$$

The last exponential term on the first line can be omitted. Indeed,

$$
-\frac{a_1^2}{2}+\frac{x_1^2}{2\Gamma_1^2N}=\frac{4}{\Gamma_1^2N}\big[px_1(x_2+\cdots-x_{\nu})-2p^2(x_2+\cdots-x_{\nu})^2\big]\xrightarrow{N\to\infty}0 \tag{2.24}
$$

uniformly for all $|x_1| \le N^{(1+\delta)/2}$ and $|x_2 + \cdots - x_\nu| \le \nu^{(1+\delta)/2}$, if $\delta > 0$ sufficiently small. The integral over the remaining x is again at most $e^{-N^{\delta'}}$.
Now we estimate the integral over $x_2, \ldots, x_\nu$,

$$\int_{\bar{D}_k''} \frac{\mathrm{d}x e^{-\frac{1}{2}\sum_{i=2}^{\nu} x_i^2}}{(2\pi)^{(\nu-1)/2}} \exp\Big(-\sum_{i=1}^{\nu} g\Big(e^{(1+\theta)\beta x_1} e^{-\left(2\beta\sqrt{p}\sum_{j=i+1}^{k} x_j + 2p\gamma|k-i|\right)(1+\theta)}\Big)\Big), \tag{2.25}$$

where $\bar{D}_k''$ is the restriction of D_k'' to the last $\nu - 1$ coordinates (which does not depend on the value of the first one). Let $V = (V_2, \ldots, V_\nu)$ be a sequence of i.i.d. standard normal random variables. Then, (2.25) equals

$$\mathbb{P}[V \in \bar{D}_k'']\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{\nu} g\left(e^{(1+\theta)\beta x_1} e^{-\left(2\beta\sqrt{p}\sum_{j=i+1}^{k} V_j + 2p\gamma|k-i|\right)(1+\theta)}\right)\Big)\Big| V \in \bar{D}_k''\Big]. \tag{2.26}$$

The probability $\mathbb{P}[V \in \bar{D}_k'']$ is bounded from below by the probability that the two-sided random walk, $R_i = \sum_{j=0}^{i} V_j$, $i \in \mathbb{Z}$, with standard normal increments is larger than $-\gamma\sqrt{p}|i|/\beta$ for all i. This probability is positive and does not depend on N, which implies that, for all k,

$$1 > \mathbb{P}[V \in \bar{D}_k''] \ge c > 0. \tag{2.27}$$

The expectation in (2.26) is bounded by one, since the functions g is positive on the domain of integration. Moreover, as $x_1 \to -\infty$, the argument of g in (2.26) tends to zero (since the first exponential does, and the second is bounded by one on D_k''). Hence

$$g\left(e^{(1+\theta)\beta x_1} e^{-(2\beta\sqrt{p}\sum_{j=i+1}^{k} V_j + 2p\gamma|k-i|)(1+\theta)}\right) \sim e^{(1+\theta)\beta x_1} e^{-(2\beta\sqrt{2}\sum_{j=i+1}^{k} V_j + 2p\gamma|k-i|)(1+\theta)}. \tag{2.28}$$

Therefore, as $x_i \to -\infty$,

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{\nu} g\left(e^{(1+\theta)\beta x_1} e^{-\left(2\beta\sqrt{p}\sum_{j=i+1}^{k} V_j + 2p\gamma|k-i|\right)(1+\theta)}\right)\Big)\Big| V \in \bar{D}_k''\Big] \\
&\sim 1 - e^{(1+\theta)\beta x_1}\mathbb{E}\Big[\sum_{i=1}^{\nu} e^{-\left(2\beta\sqrt{p}\sum_{j=i+1}^{k} V_j + 2p\gamma|k-i|\right)(1+\theta)}\Big| V \in D_k''\Big] \\
&= 1 - e^{(1+\theta)\beta x_1}\sum_{i=1}^{\nu}\mathbb{E}\Big[e^{-\left(2\beta\sqrt{p}R_{k-i} + 2p\gamma|k-i|\right)(1+\theta)}\Big| R_{k-i} \ge -\frac{\gamma\sqrt{p}}{\beta}|k-i|\Big].
\end{aligned} \tag{2.29}$$

Since R_i is a centered normal random variable with variance $|i|$, a straightforward Gaussian calculation implies that

$$\mathbb{E}\Big[e^{-\left(2\beta\sqrt{p}R_{k-i} + 2p\gamma|k-i|\right)(1+\theta)}\Big| R_{k-i} \ge -\frac{\gamma\sqrt{p}}{\beta}|k-i|\Big] \sim \frac{C_{\beta,\gamma,p}}{\sqrt{|k-i|}} e^{-\gamma^2 p|k-i|/(2\beta^2)}. \tag{2.30}$$

Hence, (2.29) is essentially a summation of a geometrical sequence and therefore there exists constants c_1, c_2 independent of k, such that

$$1 - c_1 e^{(1+\theta)\beta x_1} \le (2.29) \le 1 - c_2 e^{(1+\theta)\beta x_1}, \qquad \forall x_1 < 0. \tag{2.31}$$

Bounds (2.27) and (2.31) imply that (2.25) is bounded from above and from below (with different constants) by

$$CN^{-1/2}\exp\Big(-\frac{\gamma}{\beta}x_1-\frac{x_1^2}{2\Gamma_1^2N}\Big)(1\wedge ce^{(1+\theta)\beta x_1}). \tag{2.32}$$

and hence (2.23) is bounded from above and below by

$$C\nu N^{-1/2}\int_{\mathbb{R}}\mathrm{d}x_1\exp\Big(-\frac{\gamma}{\beta}x_1-\frac{x_1^2}{2\Gamma_1^2N}\Big)(1\wedge ce^{(1+\theta)\beta x_1})=C\nu N^{-1/2}. \tag{2.33}$$

Moreover, (2.25) is decreasing as function of $\min(k,\nu-k)$. As this minimum tends to infinity, (2.25) behaves as $f(x_1)N^{-1/2}$ which is of course satisfy the bound (2.32). Due to this convergence, the constants in the lower and the upper bound of (2.33) can be made arbitrarily close. This completes the proof of Proposition 2.1. $\square$

We close this section with a short description of the shape of the valleys mentioned in the introduction. First, it follows from (2.10) and the following computations that the most important contribution to the Laplace transform comes from realizations for which $\max\{U_i : 1\le i\le\nu\}\sim\gamma\sqrt{N}/\beta$ with an error of order $N^{-1/2}$. It is the "geometrical" sequence in (2.29) which shows that only finitely many neighbors of the maximum actually contribute to the Laplace transform. The same can be seen, at least heuristically, from a simple calculation

$$\mathbb{E}\Big[U_{k+i}\Big|U_k=\frac{\gamma}{\beta}\sqrt{N}\Big]=\frac{\gamma\sqrt{N}}{\beta}-C_{\beta,\gamma,p}\frac{|i|}{\sqrt{N}}. \tag{2.34}$$

Which means that, disregarding the fluctuations, the energy decreases linearly with the distance from the local maximum and thus the mean waiting times decrease exponentially.

3 Comparison of the real and the block process

We now come to the main task, the comparison of the clock-process sums with those in which the real Gaussian process is replaced by a simplified process. For a given realization, Y_N, of the SRW, we set $X_N^0(i)=H_N\big(Y_N(i)\big)$ (the dependence on Y_N will be suppressed in the notation). Then $X_N^0(i)$ is a centered Gaussian process indexed by $\mathbb{N}$ with covariance matrix

$$\Lambda_{ij}^0=\mathbb{E}[X_N^0(i)X_N^0(j)]=R_N\big(Y_N(i),Y_N(j)\big)^p. \tag{3.1}$$

Now we define the comparison process, $X_N^1(i)$, as the centered Gaussian process with the covariance matrix

$$\Lambda_{ij}^1=\mathbb{E}[X_N^1(i)X_N^1(j)]=\begin{cases}1-2p|i-j|/N, & \text{if } \lfloor i/\nu\rfloor=\lfloor j/\nu\rfloor,\\ 0, & \text{otherwise.}\end{cases} \tag{3.2}$$

For $h\in[0,1]$ we define the interpolating process $X_N^h(i)\equiv\sqrt{1-h}X_N^0(i)+\sqrt{h}X_N^1(i)$.

Let $\ell \in \mathbb{N}$, $0 = t_0 < \cdots < t_\ell = T$ and $u_1, \ldots, u_\ell \in \mathbb{R}_+$ be fixed. For any Gaussian process X we define a function $F_N(X) = F_N\big(X; \{t_i\}, \{u_i\}\big)$ as

$$
\begin{aligned}
F_N\big(X; \{t_i\}, \{u_i\}\big) &\equiv \mathbb{E}\Big[\exp\Big(-\sum_{k=1}^{\ell} \frac{u_k}{e^{\gamma N}} \sum_{i=t_{k-1}r(N)+1}^{t_k r(N)} e_i e^{\beta\sqrt{N}X(i)}\Big)\Big|\mathcal{X}\Big](X) \\
&= \exp\Big(-\sum_{k=1}^{\ell} \sum_{i=t_{k-1}r(N)}^{t_k r(N)-1} g\Big(\frac{u_k}{e^{\gamma N}} e^{\beta\sqrt{N}X(i)}\Big)\Big),
\end{aligned} \tag{3.3}
$$

where $r(N) = N^{1/2} e^{N\gamma^2/2\beta^2}$. Observe that $\mathbb{E}[F(X^0; t, u)|\mathcal{Y}]$ is a joint Laplace transform of the distribution of the properly rescaled clock process at times t_i. The following approximation is the crucial step of the proof.

Proposition 3.1. *If the assumptions of Theorem 1.1 are satisfied, then for all sequences $\{t_i\}$ and $\{u_i\}$,*

$$
\lim_{N\to\infty} \mathbb{E}\big[F_N\big(X_N^0; \{t_i\}, \{u_i\}\big)\big|\mathcal{Y}\big] - \mathbb{E}\big[F_N\big(X_N^1; \{t_i\}, \{u_i\}\big)\big] = 0, \qquad \mathcal{Y}\text{-}a.s. \tag{3.4}
$$

Proof. We use the well-known interpolation formula for functionals of two Gaussian processes due (probably) to Slepian and Kahane (see e.g. [LT91]

$$
\mathbb{E}[F_N(X_N^1) - F_N(X_N^0)|\mathcal{Y}] = \frac{1}{2}\int_0^1 \mathrm{d}h \sum_{\substack{i,j=1\\ i\neq j}}^{tr(N)} (\Lambda_{ij}^0 - \Lambda_{ij}^1)\mathbb{E}\Big[\frac{\partial^2 F_N(X_N^h)}{\partial X(i)\partial X(j)}\Big|\mathcal{Y}\Big]. \tag{3.5}
$$

We will show that the integral in (3.5) converges to 0.
Let $k(i)$ be defined by $t_{k(i)-1}r(N) < i \le t_{k(i)}r(N)$. The second derivative in (3.5) is equal to

$$
\begin{aligned}
&\frac{u_{k(i)}u_{k(j)}\beta^2 N}{e^{2\gamma N}} e^{\beta\sqrt{N}(X_N^h(i)+X_N^h(j))} g'\Big(\frac{u_{k(i)}}{e^{\gamma N}} e^{\beta\sqrt{N}X_N^h(i)}\Big) g'\Big(\frac{u_{k(j)}}{e^{\gamma N}} e^{\beta\sqrt{N}X_N^h(j)}\Big) F_N(X_N^h) \\
&\qquad \le \frac{u_{k(i)}u_{k(j)}\beta^2 N}{e^{2\gamma N}} e^{\beta\sqrt{N}(X_N^h(i)+X_N^h(j))} \\
&\qquad\qquad \times \exp\Big[-2g\Big(\frac{u_{k(i)}}{e^{\gamma N}} e^{\beta\sqrt{N}X_N^h(i)}\Big) - 2g\Big(\frac{u_{k(j)}}{e^{\gamma N}} e^{\beta\sqrt{N}X_N^h(j)}\Big)\Big],
\end{aligned} \tag{3.6}
$$

where we used that $g'(x) = (1+x)^{-1} = \exp(-g(x))$ (recall (2.5)), and we omitted in the summation of $F_N(X_N^h)$ all terms different from i and j. To estimate the expected value of this expression we need the following technical lemma.

Lemma 3.2. *Let $c \in [-1, 1]$ and let U_1, U_2 be two standard normal variables with the covariance $\mathbb{E}[U_1 U_1] = c$ and λ a small constant, $0 < \lambda < 1 - \gamma/\beta^2$ (which will stay fixed). Define $\Xi_N(c) = \Xi_N(c, \beta, \gamma, u, v)$ and $\bar{\Xi}_N(c) = \bar{\Xi}_N(c, \beta, \gamma, u, v, \lambda)$ by*

$$
\Xi_N(c) = \frac{uv\beta^2 N}{e^{2\gamma N}} \mathbb{E}\Big[\exp\Big\{\beta\sqrt{N}(U_1+U_2) - 2g\big(ue^{\beta\sqrt{N}U_1 - \gamma N}\big) - 2g\big(ve^{\beta\sqrt{N}U_2-\gamma N}\big)\Big\}\Big] \tag{3.7}
$$

and

$$\bar{\Xi}_N(c) = \begin{cases} \frac{C(\gamma,\beta,u,v,\lambda)}{(1-c)^{1/2}} \exp\left\{ -\frac{\gamma^2 N}{\beta^2(1+c)}\right\}, & \text{if } c > (\gamma/\beta^2)+\lambda-1, \\ C'(\gamma,\beta,u,v)N\exp\left\{N(\beta^2(1+c)-2\gamma)\right\}, & \text{if } c \leq (\gamma/\beta^2)+\lambda-1, \end{cases} \tag{3.8}$$

where $C(\gamma,\beta,u,v,\lambda)$ and $C'(\gamma,\beta,u,v)$ are suitably chosen constants, independent of N and c. Then

$$\Xi_N(c) \leq \bar{\Xi}_N(c). \tag{3.9}$$

Proof. Define $\kappa_\pm = \sqrt{2(1\pm c)}$. Let $\bar{U}_1$, $\bar{U}_2$ be two independent standard normal variables. Then U_1 and U_2 can be written as

$$U_1 = \frac{1}{2}(\kappa_+\bar{U}_1 + \kappa_-\bar{U}_2), \qquad U_2 = \frac{1}{2}(\kappa_+\bar{U}_1 - \kappa_-\bar{U}_2). \tag{3.10}$$

Hence, $U_1+U_2 = \kappa_+\bar{U}_1$. Using $g(x)+g(y) = g(x+y+xy) \geq g(x+y)$ and $ue^x + ve^{-x} \geq \min(u,v)e^{|x|}$, we get

$$\begin{aligned} &g\big(ue^{\beta\sqrt{N}U_1-\gamma N}\big) + g\big(ve^{\beta\sqrt{N}U_2-\gamma N}\big) \\ &\quad \geq g\Big(\min(u,v)\exp\Big(\frac{\kappa_+\beta\sqrt{N}\bar{U}_1}{2} + \Big|\frac{\kappa_-\beta\sqrt{N}\bar{U}_2}{2}\Big| - \gamma N\Big)\Big). \end{aligned} \tag{3.11}$$

Denoting $\min(u,v)$ by $\bar{u}$, we find that $\Xi_N(c)$ is bounded from above by

$$\frac{uv\beta^2 N}{e^{2\gamma N}} \int_{\mathbb{R}^2} \frac{\mathrm{d}y}{2\pi} \exp\Big\{ -\frac{y_1^2+y_2^2}{2} + \beta\sqrt{N}\kappa_+ y_1 - 2g\big(\bar{u}e^{\kappa_+\beta\sqrt{N}y_1/2+\kappa_-\beta\sqrt{N}|y_2|/2-\gamma N}\big)\Big\}. \tag{3.12}$$

Substituting $z_1 = y_1 - \beta\sqrt{N}\kappa_+$, $z_2 = y_2$ we get

$$\begin{aligned} &\frac{uv\beta^2 N}{e^{2\gamma N}} e^{\beta^2\kappa_+^2 N/2} \int_{\mathbb{R}^2} \frac{\mathrm{d}z}{2\pi} \exp\Big(-\frac{z_1^2+z_2^2}{2}\Big) \\ &\times \exp\Big(-2g\Big(\bar{u}\exp\Big\{\sqrt{N}\Big[\Big(\frac{\beta^2\kappa_+^2}{2}-\gamma\Big)\sqrt{N} + \frac{\beta\kappa_+}{2}z_1 + \frac{\beta\kappa_-}{2}|z_2|\Big]\Big\}\Big)\Big). \end{aligned} \tag{3.13}$$

The function $\exp(-2g(\bar{u}e^{\sqrt{N}x}))$ converges to the indicator function $\mathbf{1}_{x<0}$, as $N\to\infty$. The rôle of x will be played by the bracket in the expression (3.13).
If this bracket remains negative for z close to zero, that is if $\gamma \geq -\lambda' + \beta^2\kappa_+^2/2$ (or equivalently $c \leq (\gamma/\beta^2)+\lambda-1$), then the integral in (3.13) is bounded from above by 1. This yields the claim of the lemma for such c:

$$\Xi_N(c) \leq \frac{uv\beta^2 N}{e^{2\gamma N}} e^{\beta^2\kappa_+^2 N/2} = C'(\gamma,\beta,u,v)N\exp\left\{N(\beta^2(1+c)-2\gamma)\right\} = \bar{\Xi}_N(c). \tag{3.14}$$

If this is not the case, that is $\gamma < -\lambda' + \beta^2\kappa_+^2/2$, then we need another substitution,

$$\begin{aligned} z_1 &= \frac{1}{\sqrt{N}}\Big[v_1 - \frac{\kappa_-}{\kappa_+}|v_2| - N\Big(\beta\kappa_+ - \frac{2\gamma}{\beta\kappa_+}\Big)\Big], \\ z_2 &= \frac{v_2}{\sqrt{N}}. \end{aligned} \tag{3.15}$$

This substitution transforms the domain where the bracket of (3.13) is negative into the half-plain $v_1 < 0$: The expression inside of the braces in (3.13) equals $\beta\kappa_+ v_1/2$. Substituting (3.15) into $(z_1^2+z_2^2)/2$ produces an additional exponential prefactor $\exp\big(-\frac{(\beta^2\kappa_+^2-2\gamma)^2N}{2\beta^2\kappa_+^2}\big)$. Another prefactor N^{-1} comes from the Jacobian. The remaining terms can be bounded from above by

$$\int_{\mathbb{R}^2}\frac{\mathrm{d}v}{2\pi}\exp\Big\{\Big(\beta\kappa_+-\frac{2\gamma}{\beta\kappa_+}\Big)\Big(v_1-\frac{\kappa_-}{\kappa_+}|v_2|\Big)-2g(\bar{u}e^{\beta\kappa_+/2})\Big\}, \tag{3.16}$$

which can be separated into a product of two integrals. The integration over v_2 gives a factor

$$\Big(\Big(\beta\kappa_+-\frac{2\gamma}{\beta\kappa_+}\Big)\frac{\kappa_-}{\kappa_+}\Big)^{-1}\le C(\lambda)\kappa_-^{-1}\le C(\lambda)(1-c)^{-1/2}. \tag{3.17}$$

Using properties of g, the integrand of (3.16) behaves as $\exp\{-2v_1\gamma/\beta\kappa_+\}$ as $v_1\to\infty$, and as $\exp\{(\beta\kappa_+-(2\gamma/\beta\kappa_+))v_1\}$ as $v_1\to-\infty$. Therefore, the integral over v_1 is bounded uniformly by some λ-dependent constant for all values of $c\ge -1+(\gamma/\beta^2)+\lambda$. Putting everything together

$$\begin{aligned}\Xi_N(c)&\le C(1-c)^{-1/2}\frac{uv\beta^2N}{e^{2\gamma N}}e^{\beta^2\kappa_+^2N/2}\frac{1}{N}\exp\Big(-\frac{(\beta^2\kappa_+^2-2\gamma)^2N}{2\beta^2\kappa_+^2}\Big)\\&=C(\gamma,\beta,u,v,\lambda)(1-c)^{-1/2}\exp\Big\{-\frac{\gamma^2N}{\beta^2(1+c)}\Big\}=\bar{\Xi}_N(c).\end{aligned} \tag{3.18}$$

This finishes the proof of Lemma 3.2. □

Let $\|d\|=\min(d,N-d)$ and $D_{ij}=\mathrm{dist}(Y_N(i),Y_N(j))$. Define, with a slight abuse of notation, $\Lambda_d^0=(1-2dN^{-1})^p$. That is Λ_d^0 is the covariance of $X_N^0(i)$ and $X_N^0(j)$ if $D_{ij}=d$. The next proposition, which will be proved in Section 4, will be used to control the correlations of the process X_N^0.

Proposition 3.3. *Let γ and β satisfy the hypothesis of Theorem 1.1, and let ν be as in* (2.1)*. Then, for any $\eta>0$, there exists a constant, $C=C(\beta,\gamma,\nu,\eta)$, such that, $\mathcal{Y}$-a.s. for N large enough, for all $d\in\{0,\dots,N\}$*

$$\sum_{\substack{i,j=1\\ \lfloor i/\nu\rfloor\ne\lfloor j/\nu\rfloor}}^{tr(N)}\mathbf{1}\{D_{ij}=d\}\le C\Big[t^2r(N)^22^{-N}\binom{N}{d}+tr(N)\nu^{-1}e^{\eta\|d\|}\Big], \tag{3.19}$$

$$\sum_{\substack{i,j=1,i\ne j\\ \lfloor i/\nu\rfloor=\lfloor j/\nu\rfloor}}^{tr(N)}\mathbf{1}\{D_{ij}=d\}(\Lambda_d^0-\Lambda_{ij}^1)\le\frac{Cd^2tr(N)}{N^2}\mathbf{1}\{d\le\nu\}. \tag{3.20}$$

We now conclude the proof of Proposition 3.1, that is we prove that the right-hand side of (3.5) tends to 0. Observe first that D_{ij} is smaller than $|i-j|$. Hence, for $\lfloor i/\nu\rfloor=\lfloor j/\nu\rfloor$

$$\Lambda_{ij}^0=\big[1-2N^{-1}D_{ij}\big]^p\ge[1-2N^{-1}|i-j|]^p\ge\Lambda_{ij}^1. \tag{3.21}$$

Since $\Lambda^1_{ij} = 0$ for (i,j) with $\lfloor i/\nu \rfloor \neq \lfloor j/\nu \rfloor$, $\Lambda^0_{ij} - \Lambda^1_{ij} < 0$ if and only if $\Lambda^0_{ij} < 0$. The summands on the right-hand side of (3.5) can be written as differences of two non-negative terms:

$$(\Lambda^0_{ij} - \Lambda^1_{ij})_+ \mathbb{E}\Big[\frac{\partial^2 F_N(X^h_N)}{\partial X(i)\partial X(j)}\Big|\mathcal{Y}\Big] - (\Lambda^0_{ij})_- \mathbb{E}\Big[\frac{\partial^2 F_N(X^h_N)}{\partial X(i)\partial X(j)}\Big|\mathcal{Y}\Big]. \tag{3.22}$$

We bound this expression using Lemma 3.2. For given $\{u_i\}$ let

$$\tilde{\bar{\Xi}}_N(c) = \max\{\bar{\Xi}_N(c,\beta,\gamma,u_i,u_j) : 1 \leq i,j \leq \ell\}. \tag{3.23}$$

Then $\tilde{\bar{\Xi}}_N(c)$ satisfies (3.8) for some constants C and C' and it is therefore increasing in c. The absolute value of the right-hand side of (3.5) is then bounded from above by

$$\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{tr(N)} (\Lambda^0_{ij} - \Lambda^1_{ij})_+ \mathbb{E}\Big[\frac{\partial^2 F_N(X^0_N)}{\partial X(i)\partial X(j)}\Big|Y_N\Big] + \sum_{\substack{i,j=1\\ i\neq j}}^{tr(N)} (\Lambda^0_{ij})_- \mathbb{E}\Big[\frac{\partial^2 F_N(X^1_N)}{\partial X(i)\partial X(j)}\Big] \\
&\qquad \leq \sum_{d=0}^{N} \Bigg\{ \sum_{\substack{i,j=1\\ \lfloor i/\nu\rfloor \neq \lfloor j/\nu \rfloor}}^{tr(N)} \mathbf{1}\{D_{ij} = d\}(\Lambda^0_d)_+ \int_0^1 \tilde{\bar{\Xi}}(h\Lambda^0_d)\mathrm{d}h \\
&\qquad + \sum_{\substack{i,j=1, i\neq j\\ \lfloor i/\nu\rfloor = \lfloor j/\nu \rfloor}}^{tr(N)} \mathbf{1}\{D_{ij} = d\}(\Lambda^0_d - \Lambda^1_{ij})\tilde{\bar{\Xi}}\big(\Lambda^0_d\big) \\
&\qquad + \sum_{i,j:|i-j|\geq N/2}^{tr(N)} \mathbf{1}\{D_{ij} = d\}(\Lambda^0_d)_- \tilde{\bar{\Xi}}\big(0\big) \Bigg\}.
\end{aligned} \tag{3.24}$$

From the definition of $\tilde{\bar{\Xi}}$ it follows that,

$$\int_0^1 \tilde{\bar{\Xi}}(hc)\mathrm{d}h \leq C \exp\Big\{-\frac{\gamma^2 N}{\beta^2(1+c)}\Big\} \int_0^1 (1-hc)^{-1/2}\mathrm{d}h. \tag{3.25}$$

The last integral can be easily evaluated and is smaller than 2 for all $c \in [-1,1]$. Using Proposition 3.3, the first line of (3.24) is smaller than the sum of the following two terms:

$$C \sum_{d=0}^{N} t^2 r(N)^2 2^{-N} \binom{N}{d} \Lambda^0_d \exp\Big\{-\frac{\gamma^2 N}{\beta^2(1+\Lambda^0_d)}\Big\} \tag{3.26}$$

and

$$C \sum_{d=0}^{N} \frac{tr(N)e^{\eta\|d\|}}{\nu} \Lambda^0_d \exp\Big\{-\frac{\gamma^2 N}{\beta^2(1+\Lambda^0_d)}\Big\}. \tag{3.27}$$

The second line of (3.24) is bounded by

$$C \sum_{d=0}^{\nu} \frac{tr(N)d^2}{N^2} \tilde{\bar{\Xi}}(\Lambda^0_d). \tag{3.28}$$

The third line is non-zero only if p is odd, and in that case it is bounded by

$$\sum_{d=N/2}^{N} C\Big[t^2 r(N)^2 2^{-N}\binom{N}{d} + tr(N)\nu^{-1}e^{\eta\|d\|}\Big]\Big(\frac{2d}{N}-1\Big)^p\tilde{\Xi}(0), \tag{3.29}$$

We estimate (3.26) first. Let $I(u)$ be defined by

$$I(u) = u\log u + (1-u)\log(1-u) + \log 2, \tag{3.30}$$

and let

$$J_N(u) = 2^{-N}\binom{N}{\lfloor Nu\rfloor}\sqrt{\frac{\pi N}{2}}e^{NI(u)}. \tag{3.31}$$

Stirling's formula yields $J_N(u) \xrightarrow{N\to\infty} (4u(1-u))^{-1}$ uniformly in u on compact subsets of $(0,1)$. Further, $J_N(u) \le CN^{1/2}$ for all $u \in [0,1]$. From the definitions of $r(N)$ and $\tilde{\Xi}$, we find that

$$(3.26) = C\sum_{d=0}^{N} t^2 N^{1/2}\Big(1-\frac{2d}{N}\Big)^p \exp\Big\{N\Upsilon_{p,\beta,\gamma}\Big(\frac{d}{N}\Big)\Big\}J_N\Big(\frac{d}{N}\Big), \tag{3.32}$$

where

$$\Upsilon_{p,\beta,\gamma}(u) = \begin{cases} \frac{\gamma^2}{\beta^2} - I(u) - \frac{\gamma^2}{\beta^2(1+(1-2u)^p)}, & \text{if } (1-2u)^p \ge \frac{\gamma}{\beta^2}+\lambda-1, \\ \frac{\gamma^2}{\beta^2} - I(u) + \beta^2(1+(1-2u)^p) - 2\gamma, & \text{if } (1-2u)^p \le \frac{\gamma}{\beta^2}+\lambda-1. \end{cases} \tag{3.33}$$

Lemma 3.4. *There exists a function $\zeta(p)$ such that for all $p \ge 2$, and γ, β satisfying $\gamma \le \zeta(p)\beta$ and $\gamma < \beta^2$, there exist positive constants δ, δ' and c such that*

$$\Upsilon_{p,\beta,\gamma}(u) \le -\delta \qquad \text{for all } u \in [0,1]\setminus(1/2-\delta', 1/2+\delta'), \tag{3.34}$$

and

$$\Upsilon_{p,\beta,\gamma}(u) \le -c(u-1/2)^2 \qquad \text{for all } u \in (1/2-\delta', 1/2+\delta'). \tag{3.35}$$

Moreover $\zeta(p)$ is increasing and satisfies (1.10)*, that is*

$$\zeta(2) = 2^{-1/2}, \quad \zeta(3) = 1.0291, \qquad \text{and} \qquad \lim_{p\to\infty}\zeta(p) = \sqrt{2\log 2}. \tag{3.36}$$

Proof. Since $\gamma/\beta^2 < 1$, the second line of the definition of $\Upsilon_{p,\beta,\gamma}$ is used only for p odd and $u \ge u_c(p,\beta,\gamma,\lambda) = (1+(1-\lambda-\gamma/\beta^2)^{1/p})/2 > 1/2$. Furthermore, $\Upsilon_{p,\beta,\gamma}(1/2) = \Upsilon'_{p,\beta,\gamma}(1/2) = 0$ and

$$\Upsilon''_{p,\beta,\gamma}(1/2) = \begin{cases} 4\big(\frac{2\gamma^2}{\beta^2}-1\big), & \text{if } p=2, \\ -4 & \text{otherwise.} \end{cases} \tag{3.37}$$

The second derivative is always negative for β, γ, p satisfying the assumptions of Theorem 1.1. Therefore (3.35) holds.
The second line of the definition of $\Upsilon_{p,\beta,\gamma}(u)$ is decreasing in u. Hence for $u \ge u_c$

$$\Upsilon_{p,\beta,\gamma}(u) \le \Upsilon_{p,\beta,\gamma}(u_c) = -\gamma(1-\gamma/\beta^2) - I(u_c) \tag{3.38}$$

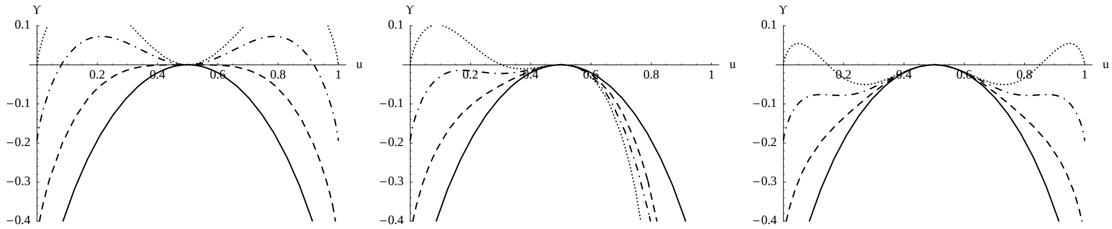

Figure 1: Function $\Upsilon_{p,\gamma,\beta}$ for $p = 2, 3, 4$ and various values of γ/β.

which is obviously strictly negative and (3.34) is proved for $u \geq u_c$.
For any $\delta' > 0$ and $u < 1/2-\delta'$ the function $I(u)$ is strictly positive, and the function $\Phi(u) \equiv 1 - 1/(1+(1-2u)^p)$ is bounded. Therefore, if γ/β is sufficiently small, then $\Upsilon_{p,\beta,\gamma}(u) < -\delta$. If p is even, the function $\Upsilon_{p,\beta,\gamma}$ is symmetric around $u = 1/2$. If $1/2 < u < u_c(p, \beta, \gamma)$ and p is odd, then

$$\Upsilon_{p,\beta,\gamma}(u) < \Upsilon_{p,1,0}(u) = -I(u) < 0 \tag{3.39}$$

and the proof of (3.35) is finished.
To prove the first part of (3.36) we should check that (3.35) holds for all $\gamma \leq 2^{-1/2}\beta$. However, $\Upsilon_{2,\beta,\gamma}(u)$ is increasing in γ^2/β^2 and $I(u) \geq (1-2u)^2/2$. Thus, for $\gamma \leq 2^{-1/2}\beta$,

$$\Upsilon_{2,\beta,\gamma}(u) \leq \frac{1}{2}\Big(1 - \frac{1}{1+(1-2u)^2}\Big) - \frac{1}{2}(1-2u)^2. \tag{3.40}$$

The right-hand side of the last inequality is equal 0 for $u = 1/2$ and its derivative

$$2(1-2u)\Big(1 - \frac{1}{(1+(1-2u)^2)^2}\Big) > 0 \quad \text{for all } u < 1/2. \tag{3.41}$$

The symmetry of $\Upsilon_{2,\beta,\gamma}$ around 1/2 then implies the first part of (3.36).
Obviously, $\Phi(0) = 1/2$, $\Phi'(0) = -2p$, $I(0) = \log 2$ and $I'(0) = -\infty$. Hence, for $\gamma/\beta = \sqrt{\log 2}$ there exists u small such that $\Upsilon_{p,\beta,\gamma}(u)$ is positive. This implies $\zeta(p) < \sqrt{2\log 2}$. If $u \in (0, 1/2)$ then $\lim_{p\to\infty} \Phi(u) = 0$. This yield the second half of (3.36).
For illustration you find the graphs of function $\Upsilon_{p,\beta,\gamma}$ for $p = 2, 3, 4$, $\beta = 1$, and $\gamma = 0$ (solid lines), $\gamma = \sqrt{1/2}$ (dashed lines), $\gamma = 1$ (dash-dotted lines) and $\gamma = \sqrt{2\log 2}$ (dotted lines) on Figure 1. The value of $\zeta(3)$ was calculated numerically using the figure for $p = 3$. □

We can now finish the bound on (3.26). Lemma 3.4 and bounds on the function J_N yield that for $d/N \notin (1/2-\delta', 1/2+\delta')$ the summands decrease exponentially in N. Therefore they can be neglected. The remaining part can be bounded by

$$\begin{aligned} C &\sum_{d=(1/2-\delta')N}^{(1/2+\delta')N} t^2 N^{1/2}\Big(1-\frac{2d}{N}\Big)^p \exp(-cN(d/N-1/2)^2) \\ &\leq Ct^2N^{3/2}\int_{-\delta'}^{\delta'} x^p e^{-c'Nx^2}\mathrm{d}x \\ &\leq Ct^2N^{3/2}N^{-(p+1)/2}\int_{-\infty}^{\infty} u^p e^{-c'u^2}\mathrm{d}u \xrightarrow{N\to\infty} 0, \end{aligned} \tag{3.42}$$

if $p \geq 3$.
Similarly, for (3.27) we have

$$(3.27) \leq C \sum_{d=0}^{N/2} t N^{1/2} \nu^{-1} \Big(1 - \frac{2d}{N}\Big)^p \exp(N \tilde{\Upsilon}(d/N)), \tag{3.43}$$

where, setting $\|u\| = \min(u, 1-u)$,

$$\tilde{\Upsilon}_{p,\beta,\gamma}(u) = \begin{cases} \frac{\gamma^2}{2\beta^2} - \frac{\gamma^2}{\beta^2(1+(1-2u)^p)} + \eta\|u\|, & \text{if } (1-2u)^p \geq \frac{\gamma}{\beta^2} + \lambda - 1, \\ \frac{\gamma^2}{2\beta^2} + \beta^2(1+(1-2u)^p) - 2\gamma + \eta\|u\|, & \text{if } (1-2u)^p \leq \frac{\gamma}{\beta^2} + \lambda - 1. \end{cases} \tag{3.44}$$

Observe first that the second part of the definition of $\tilde{\Upsilon}_{p,\beta,\gamma}$ is always strictly negative. It is also easy to be checked that it is possible to choose δ, δ' and η small such that the first part of the definition of $\tilde{\Upsilon}(u) < \delta$ for all $\|u\| \geq \delta'$. Therefore such d can be neglected. Around $d = 0$ the function $\tilde{\Upsilon}(x)$ can be approximated by a linear function $-cx$, $c > 0$, and the summation by an integration. As an upper bound we get

$$CtN^{3/2}\nu^{-1} \int_0^{\delta'} e^{-cNx} \mathrm{d}x \leq CtN^{1/2}\nu^{-1} \xrightarrow{N\to\infty} 0. \tag{3.45}$$

An analogous bound works for d close to N and p even.
For (3.28) we have

$$(3.28) \leq C \sum_{d=0}^{\nu} t N^{-3/2} d^2 [1 - (1 - 2dN^{-1})^p]^{-1/2} \exp(N\tilde{\Upsilon}(d/N)). \tag{3.46}$$

The linear approximation of $\tilde{\Upsilon}$ and of the bracket in the last expression yields an upper bound

$$CtN^{3/2} \int_0^{\varepsilon} x^{3/2} e^{-c'Nx} \mathrm{d}x \leq CtN^{-1} \xrightarrow{N\to\infty} 0. \tag{3.47}$$

Finally, since $\tilde{\Xi}(0) = Ce^{-N\gamma^2/\beta^2}$, it is easy to see that the second half of (3.29) tends to 0. The first half equals (up to constant)

$$\begin{aligned} &\sum_{d=N/2}^{N} \Big(\frac{2d}{N} - 1\Big)^p t^2 N 2^{-N} \binom{N}{d} \\ &\leq Ct^2 \Big\{ \sum_{d \geq N/2 + N^{3/5}} N 2^{-N} \binom{N}{d} + \sum_{i=1}^{2N^{3/5}} \Big(\frac{N+i}{N} - 1\Big)^p N^{1/2} e^{-i^2/2N} \Big\}, \end{aligned} \tag{3.48}$$

where we used the known approximation of $\binom{N}{d} \leq CN^{-1/2} 2^N e^{-i^2/2N}$ for $d = (N+i)/2$ and $i \ll N^{2/3}$. The first term in (3.48) tends to 0 by a standard moderate deviation argument. The second one can be approximated by

$$Ct^2 N^{1-(p/2)} \int_0^{\infty} x^p e^{-x^2/2} \mathrm{d}x \xrightarrow{N\to\infty} 0 \tag{3.49}$$

for $p \geq 3$. This completes the proof of Proposition 3.1. □

4 Random walk properties

In this section we prove Proposition 3.3. For $A \subset \mathcal{S}_N$ let $T_A = \min\{k \geq 1 : Y_N(k) \in A\}$ be the hitting time of A. We write $\mathbb{P}_x$ for the law of the simple random walk Y_N conditioned on $Y_N(0) = x$. Let $Q = Q_i$, $i \in \mathbb{N}$, be a birth-death process on $\{0, \dots, N\}$ with transition probabilities $p_{i,i-1} = 1 - p_{i,i+1} = i/N$. We use P_k and E_k to denote the law of (the expectation with respect to) Q conditioned on $Q_0 = k$. Under P_0, Q_i has the same law as $\mathrm{dist}(Y_N(0), Y_N(i))$. Define $T_k = \min\{i \geq 1 : Q_i = k\}$ the hitting time of k by Q. It is well-known fact that for $k < l < m$

$$P_l[T_m < T_k] = \frac{\sum_{i=k}^{l-1} \binom{N-1}{i}^{-1}}{\sum_{i=k}^{m-1} \binom{N-1}{i}^{-1}}. \tag{4.1}$$

Finally, let $p_k(d) = P_0(Q_k = d)$. We need the following lemma for estimating $p_k(d)$ for large k.

Lemma 4.1. *There exists K large enough such that for all $k \geq KN^2 \log N =: \mathcal{K}(N)$ and $x, y \in \mathcal{S}_N$*

$$\left| \frac{\mathbb{P}_y[Y_N(k) = x \cup Y_N(k+1) = x]}{2} - 2^{-N} \right| \leq 2^{-8N} \tag{4.2}$$

and thus

$$\left| \frac{p_k(d) + p_{k+1}(d)}{2} - 2^{-N} \binom{N}{d} \right| \leq 2^{-4N}. \tag{4.3}$$

Proof. The beginning of the argument is the same as in [Mat87]. We construct coupling between Y_N (which by definition starts at site $\mathbf{1} = (1, \dots, 1) \in \mathcal{S}_N$) and another process $Y_N^\star$. This process is a simple random walk on $\mathcal{S}_N$ with the initial distribution $\mu_N^\star$ being uniform on those $x \in \mathcal{S}_N$ with $\mathrm{dist}(x, \mathbf{1})$ even. The coupling is the same as in [Mat87]. This coupling gives certain random time $\mathcal{T}_N$ which can be used to bound the variational distance between $\mu^\star$ and the distribution μ_N^k of $Y_N(k)$: for k even

$$d_\infty(\mu_N^\star, \mu_N^k) \equiv \max_{A \subset \mathcal{S}_N} |\mu_N^\star(A) - \mu_N^k(A)| \leq \mathbb{P}[\mathcal{T}_N > k]. \tag{4.4}$$

The law of $\mathcal{T}_N$ is as follows. Let $U = \mathrm{dist}(Y_N^\star(0), \mathbf{1})$. That is U is a binomial random variable with parameters N and $1/2$ conditioned on being even. Consider another simple random walk $\tilde{Y}_U$ on $\mathcal{S}_U$ started from $\mathbf{1}$. The distribution of $\mathcal{T}_N$ is then the same as the distribution of the hitting time of $\{x \in \mathcal{S}_U : \mathrm{dist}(\mathbf{1}, x) = U/2\}$. It is proved in [Mat87] that $P(\mathcal{T}_N > N \log N) \to c < 1$. It is then easy to see that,

$$\mathbb{P}[\mathcal{T}_N \geq \mathcal{K}(N)] \leq c^{KN/2} \leq 2^{-8N}, \tag{4.5}$$

if K is large enough. Thus, for even $k \geq \mathcal{K}(N)$, $d_\infty(\mu_N^\star, \mu_N^k) \leq 2^{-8N}$ and thus $|\mu_N^\star(x) - \mu_N^k(x)| \leq 2^{-8N}$ for all $x \in \mathcal{S}_N$. A similar claim for k odd is then not difficult to prove. The second part of the lemma is a direct consequence of the first part. $\square$

Lemma 4.2. *Let γ, β, ν satisfy the hypothesis of Proposition 3.3. Then, there exists a constant, $C = C(\beta, \gamma, \nu)$, such that for all N large enough, $\mathcal{Y}$-a.s.*

$$\sum_{\substack{i,j=1, i\neq j \\ \lfloor i/\nu \rfloor = \lfloor j/\nu \rfloor}}^{tr(N)} \mathbf{1}\{D_{ij} = d\} \leq Ctr(N)\mathbf{1}\{d \leq \nu\}, \tag{4.6}$$

and for all $d \in \{0, \dots, N\}$.

Proof. The lemma is trivially true for $d > \nu$. For $d \leq \nu$, let

$$\rho(d) = E_0 \sum_{i=1}^{\nu} \mathbf{1}\{Q_i = d\}. \tag{4.7}$$

We have $\rho(0) \geq N^{-1}$ and $\rho(d) \geq P_0[T_d \leq \nu]$. This probability is decreasing in d and

$$P_0[T_\nu \leq \nu] = \frac{N}{N} \cdot \frac{N-1}{N} \cdots \frac{N-\nu+1}{N} \geq e^{-\nu^2/N}. \tag{4.8}$$

Thus $\rho(d) \geq e^{-\nu^2/N}$ for all $d \leq \nu$. To get an upper bound on $\rho(d)$ we write

$$\rho(d) \leq E_0 \Big[\sum_{i=1}^{T_\nu} \mathbf{1}\{Q_i = d\} \Big] = 1 + E_d \Big[\sum_{i=1}^{T_\nu} \mathbf{1}\{Q_i = d\} \Big] = 1 + \frac{1}{P_d[T_\nu < T_d]}. \tag{4.9}$$

However, using (4.1),

$$P_d[T_\nu < T_d] = \frac{N-d}{N} P_{d+1}[T_\nu < T_d] = \frac{N-d}{N} \frac{\binom{N-1}{d}^{-1}}{\sum_{i=d}^{\nu-1} \binom{N-1}{i}^{-1}} = 1 - O(\nu N^{-1}). \tag{4.10}$$

Since $\nu \ll N$, $\rho(d) \leq 2$.
Consider now one-block contribution to (4.6),

$$\sum_{i,j=1}^{\nu} \mathbf{1}\{D_{ij} = d\} =: \nu^2 \tilde{Z}. \tag{4.11}$$

Of course, $\tilde{Z} \in [0, 1]$ and, using the results of the previous paragraph,

$$e^{-\nu^2/N} (2\nu)^{-2} \leq \mathbb{E}[\tilde{Z}] \leq 2\nu^{-1}. \tag{4.12}$$

The left-hand side of (4.6) is stochastically smaller than $\nu^2 \sum_{k=1}^{m} \tilde{Z}_k$, where $\tilde{Z}_k$ are i.i.d. copies of $\tilde{Z}$ and $m = \lceil tr(N)/\nu \rceil$. By Hoeffding's inequality [Hoe63],

$$\mathbb{P}\Big[\sum_{i=1}^{m} \tilde{Z}_k \geq 2m\mathbb{E}[\tilde{Z}_k] \Big] \leq \exp\{-2m^2 \mathbb{E}[\tilde{Z}_k]^2\} \leq \exp\{-m^2 e^{-2\nu^2/N} (2\nu)^{-4}\}, \tag{4.13}$$

where we used the lower bound from (4.12). Since $\nu/N^2 \ll N$, by the Borel-Cantelli lemma, the left-hand side of (4.6) is a.s. bounded by

$$\nu^2 2m\mathbb{E}[\tilde{Z}] \leq Ctr(N) \tag{4.14}$$

for all N large enough and $d \leq \nu$. This completes the proof of Lemma 4.2. $\square$

Proof of Proposition 3.3. We prove (3.20) first. Observe that for i, j in the same block

$$\Lambda_d^0 - \Lambda_{ij}^1 = \Big(1 - \frac{2d}{N}\Big)^p - \Big(1 - \frac{2p|i-j|}{N}\Big) = \frac{2p(|i-j|-d)}{N} + O\Big(\frac{d^2}{N^2}\Big). \tag{4.15}$$

The contribution of the error term is smaller than the right-hand side of (3.20), as follows from Lemma 4.2.
To compute the contribution of the main term, let

$$\tilde\rho(d) = E_0\Big[\sum_{i=1}^{\nu}(i-d)\mathbf{1}\{Q_i = d\}\Big]. \tag{4.16}$$

Let $T_d^1 = T_d$ and $T_d^k = \min\{i > T_d^{k-1} : Q_i = d\}$. Then

$$\begin{aligned}\tilde\rho(d) &= E_0\Big[\sum_{j=1}^{\infty}(T_d^j - d)\mathbf{1}\{T_d^j < \nu\}\Big] = E_0\Big[\sum_{j=1}^{\infty}(T_d^j - T_d^1 + T_d^1 - d)\mathbf{1}\{T_d^j < \nu\}\Big] \\ &\le E_0[(T_d - d)\mathbf{1}\{T_d < \nu\}]\Big(1 + \sum_{i=1}^{\infty} E_d[T_d^i\mathbf{1}\{T_d^i < \nu - d\}]\Big).\end{aligned} \tag{4.17}$$

Using (4.8), $P_0[T_d = d] \le Ce^{-d^2/N}$ and further

$$P_0[T_d \ge d + 2k] \le \binom{d+2k}{k}\Big(\frac{d}{N}\Big)^k \le C\frac{d^{2k}}{N^k}. \tag{4.18}$$

Hence, $cd^2N^{-1} \le E_0[(T_d - d)\mathbf{1}\{T_d < \nu\}] \le Cd^2N^{-1}$.
For the second term in (4.17) we write

$$\begin{aligned}1 + &\sum_{i=1}^{\infty} E_d[T_d^i\mathbf{1}\{T_d^i < \nu - d\}] \\ &\le 1 + E_d[T_d\mathbf{1}\{T_d < \nu - d\}]\Big(1 + \sum_{i=1}^{\infty} E_d[T_d^i\mathbf{1}\{T_d^i < \nu - d\}]\Big) \\ &= \sum_{k=0}^{\infty}\big\{E_d[T_d\mathbf{1}\{T_d < \nu - d\}]\big\}^k.\end{aligned} \tag{4.19}$$

Using the well-known estimate $\binom{2k}{k} \le Ck^{-1/2}2^k$ and $k < 2^k$,

$$E_d[T_d\mathbf{1}\{T_d < \nu - d\}] \le \sum_{k=1}^{\nu/2} 2k\binom{2k}{k}\Big(\frac{\nu}{N}\Big)^k \le C\sum_{k=1}^{\infty}\Big(\frac{4\nu}{N}\Big)^k \le C\frac{\nu}{N} \tag{4.20}$$

and (4.19) is finite. Thus $\tilde\rho(d) \le Cd^2N^{-1}$ for all $d \in \{0, \ldots, \nu\}$.
The one-block contribution of the first term of (4.15) to (3.20) is then given by

$$\frac{2p}{N}\sum_{i,j=1}^{\nu}(|i-j| - d)\mathbf{1}\{D_{ij} = d\} =: \frac{2p}{N}\nu^3\tilde Z, \tag{4.21}$$

with $\tilde{Z} \in [0,1]$ and

$$cd^2 N^{-1}\nu^{-3} \le \mathbb{E}[\tilde{Z}] \le Cd^2 N^{-1}\nu^{-2}. \tag{4.22}$$

Therefore, as in the proof of Lemma 4.2, Hoeffding's inequality and (4.22) imply that the contribution of the first term of (4.15) to (3.20) is smaller than $Ctr(N)d^2N^{-2}$, which was to be shown.

Finally, we prove (3.19). Since we are interested in an upper bound only we can, without loss of generality, restrict the summation on $i < j$. We first consider the contribution of pairs (i,j) such that $j - i \ge \mathcal{K}(N)$. Then necessarily, $\lfloor i/n \rfloor \ne \lfloor j/n \rfloor$. Let $R = tr(n)$. Lemma 4.1 yields

$$\mathbb{E}\Big[\sum_{j-i\ge\mathcal{K}(N)}^{R} \mathbf{1}\{D_{ij} = d\}\Big] = \sum_{j-i\ge\mathcal{K}(N)}^{R} p_{j-i}(d) \le CR^2 2^{-N}\binom{N}{d}. \tag{4.23}$$

Further,

$$\begin{aligned} &\mathrm{Var}\Big[\sum_{j-i\ge\mathcal{K}(N)}^{R} \mathbf{1}\{D_{ij} = d\}\Big] \\ &= \sum_{j_1-i_1\ge\mathcal{K}(N)}^{R}\ \sum_{j_2-i_2\ge\mathcal{K}(N)}^{R} \mathbb{P}\big[D_{i_1,j_1} = D_{i_2,j_2} = d\big] - \mathbb{P}\big[D_{i_1,j_1} = d\big]\mathbb{P}\big[D_{i_2,j_2} = d\big]. \end{aligned} \tag{4.24}$$

We can again suppose that $i_1 \le i_2$. The right-hand side of (4.24) is non-null only if $i_1 \le i_2 \le j_1 < j_2$ or $i_1 \le i_2 < j_2 \le j_1$. We will consider only the first case. The second one can be treated analogously. In is not difficult to see using Lemma 4.1 that if $i_2 - i_j \ge \mathcal{K}(N)$ or $j_2 - j_1 \ge \mathcal{K}(N)$ then the difference of probabilities in the above summation is at most 2^{-4N}. Therefore, the contribution of such (i_1, i_2, j_1, j_2) to the variance is at most $R^4 2^{-4N}$.

If $i_2 - i_1 < \mathcal{K}(N)$ and $j_2 - j_1 < \mathcal{K}(N)$ then, using Lemma 4.1 again,

$$\mathbb{P}\big[D_{i_1,j_1} = D_{i_2,j_2} = d\big] \le C2^{-N}\binom{N}{d}. \tag{4.25}$$

We choose $\varepsilon > 0$. For $\|d\| \le (1-\varepsilon)N/2$ we have

$$\begin{aligned} &\sum_{\substack{j_1-i_1\ge\mathcal{K}(N)\\ i_2-i_1<\mathcal{K}(N)}}\ \sum_{\substack{j_2-i_2\ge\mathcal{K}(N)\\ j_2-j_1<\mathcal{K}(N)}} \mathbb{P}\big[D_{i_1,j_1} = D_{i_2,j_2} = d\big] \\ &\qquad \le C\mathcal{K}(N)^2 R^2 2^{-N}\binom{N}{d} \le C\mathcal{K}(N)^2 R^2 e^{-NI((1-\varepsilon/2)/2)} \ll N^{-3}R^2\nu^{-2}, \end{aligned} \tag{4.26}$$

say. For $\|d\| \ge (1-\varepsilon)N/2$, that is $|d - N/2| \le \varepsilon N/2$, we have for ε small enough (how small depend on γ and β) that $2^{-N}\binom{N}{d} \gg N^7 R^{-2}$. Then,

$$\begin{aligned} &\sum_{\substack{j_1-i_1\ge\mathcal{K}(N)\\ i_2-i_1<\mathcal{K}(N)}}\ \sum_{\substack{j_2-i_2\ge\mathcal{K}(N)\\ j_2-j_1<\mathcal{K}(N)}} \mathbb{P}\big[D_{i_1,j_1} = D_{i_2,j_2} = d\big] \\ &\qquad \le CN^4 R^2 2^{-N}\binom{N}{d} \ll N^{-3}R^4 2^{-2N}\binom{N}{d}^2. \end{aligned} \tag{4.27}$$

We have thus found that the expectation of the summation over $j - i > \mathcal{K}(N)$ is smaller than the right-hand side of (3.19) and the variance of the same summation is much smaller than N^{-3} times the right-hand side of (3.19) squared. A straightforward application of the Chebyshev inequality and the Borel-Cantelli Lemma then gives the desired a.s. bound for pairs $j - i \geq \mathcal{K}(N)$ and all $d \in \{0, \ldots, N\}$.
Choose again $\varepsilon > 0$. For $j - i < \mathcal{K}(N)$, observe first that if $\|d\| \geq (\log N)^{1+\varepsilon} \gg \log N$ then the summation over such pairs (i, j) in (3.19) is always smaller than $\mathcal{K}(N)R \ll R\nu^{-1}e^{\eta\|d\|}$ for all $\eta > 0$. For the remaining d's, that is $\|d\| < (\log N)^{1+\varepsilon'}$, let $K_N \geq K$ be the smallest constant such that $K_N N^2 \log N$ is a multiple of ν. Since $\nu \ll N^2$, $K_N - K \ll 1$. As the difference between K and K_N is negligible, we will use the same notation $\mathcal{K}(N)$ for $K_N N^2 \log N$ and we will simply suppose that $\mathcal{K}(N)$ is a multiple of ν. The summation in (3.19) for $j - i \leq \mathcal{K}(N)$ can be bounded from above by

$$\sum_{\substack{0<j-i<\mathcal{K}(N)\\ \lfloor i/\nu\rfloor \neq \lfloor j/\nu \rfloor}}^{tr(N)} \mathbf{1}\{D_{ij} = d\} \leq \sum_{k=0}^{\mathcal{K}(N)-1} \sum_{\ell=0}^{\lceil R/\mathcal{K}(N)\rceil} \sum_{m=j_k}^{\mathcal{K}(N)} \mathbf{1}\{D_{\mathcal{K}(N)\ell+k,\mathcal{K}(N)\ell+k+m} = d\}, \qquad (4.28)$$

where j_k is the smallest integer such that $\lfloor(\mathcal{K}(N)\ell + k)/\nu\rfloor \neq \lfloor(\mathcal{K}(N)\ell + k + j_k)/\nu\rfloor$, which does not depend on ℓ. We define random variables $Z_\ell(j, d)$ by

$$Z_\ell(j, d) = \frac{1}{\mathcal{K}(N)} \sum_{m=j}^{\mathcal{K}(N)} \mathbf{1}\{D_{\mathcal{K}(N)\ell+k,\mathcal{K}(N)\ell+k+m} = d\}. \qquad (4.29)$$

The sequence $\{Z_\ell(j, d) : \ell \geq 0\}$ for fixed j and d is a sequence of i.i.d. variables with values in $[0, 1]$.
Let $E_N = \{d : \|d\| < (\log N)^{1+\varepsilon'}, d \geq N/2\}$. For $d \in E_N$

$$\mathbb{P}[Z_\ell(k, d) > 0] \leq \binom{N}{d} P_d(T_{\mathbf{1}} < \mathcal{K}(N)) \leq \binom{N}{d} e^{\lambda K} E_d\big[e^{-\lambda T_{\mathbf{1}}/N^2 \log N}\big]. \qquad (4.30)$$

According to Lemma 3.4 of [ČG06],

$$E_d\big[\exp(-\lambda T_{\mathbf{1}} m(N)^{-1})\big] \leq (2^{-N} m(N)\lambda^{-1} + \xi_N(d))(1 + o(1)), \qquad (4.31)$$

for $N \log N \ll m(N) \ll 2^N$, with $\xi_n(k) = 2^{-n}\frac{n}{2}\binom{n}{k}^{-1}\sum_{j=1}^{n-k}\binom{n}{k+j}\frac{1}{j}$. Taking $m(N) = N^2$ and $d \in E_N$ it is not difficult to check that for ε small enough

$$\mathbb{E}_{z_d}\big[e^{-\lambda T_{\mathbf{1}}/N^2}\big] \leq 2^{-N(1-\varepsilon)}. \qquad (4.32)$$

Hence,

$$\mathbb{P}\Big[\bigcup_{d\in E_N}\Big\{\sum_{k=0}^{\mathcal{K}(N)-1}\sum_{\ell=0}^{\lceil R/\mathcal{K}(N)\rceil} Z_\ell(j_k, d) > 0\Big\}\Big]$$
$$\leq C\binom{N}{\lceil(\log N)^{1+\varepsilon}\rceil} R(\log N)^{1+\varepsilon} 2^{-N(1-\varepsilon)} \leq C2^{-\varepsilon' N}, \qquad (4.33)$$

for some ε' small. Hence, $d \in E_N$ do not pose any problem, by the Borel-Cantelli lemma again.
To treat $d \leq (\log N)^{1+\varepsilon'}$ we will distinguish two cases: $j_k \leq 2d$ and $j_k > 2d$. For the first case, observe that for any $d < \nu$ there are at most $d\mathcal{K}(N)/\nu$ values of $k \in \{0, \ldots, \mathcal{K}(N) - 1\}$ such that $j_k \leq d$. Further, as before, $Z_\ell(j_k, d) \leq Z_\ell(0, d)$, $\mathbb{E}[Z_\ell(0, d)] \geq 1/(N\mathcal{K}(N))$, and $\mathbb{E}[Z_\ell(0, d)] \leq C/\mathcal{K}(N)$. Hence, by Hoeffding's inequality, the probability

$$\mathbb{P}\Big[\mathcal{K}(N) \sum_{\ell=0}^{\lceil R/\mathcal{K}(N)\rceil} Z_\ell(0, d) \geq \frac{R}{\mathcal{K}(N)}\Big] \tag{4.34}$$

decreases at least exponentially with N and thus for $j_k < 2d$, a.s,

$$\mathcal{K}(N) \sum_{\ell=0}^{\lceil R/\mathcal{K}(N)\rceil} Z_\ell(0, d) \geq \frac{R}{\mathcal{K}(N)}. \tag{4.35}$$

For $j \geq 2d$ and N large enough, $Z_\ell(j, d) \leq Z_\ell(d + 6, d)$. We have,

$$cN^{-6} \leq \mathcal{K}(N)\mathbb{E}[Z_\ell(d + 6, d)] \leq CN^{-3}. \tag{4.36}$$

Indeed, the lower bound is trivial and for the upper bound we use the fact that the probability that Y_N reaches d before returning to $d + 6$ is smaller than CN^{-5} and before the time $\mathcal{K}(N)$ there are at most $\mathcal{K}(N)$ tries. Hence, for $j \geq 2d$ the probability

$$\mathbb{P}\Big[\mathcal{K}(N) \sum_{\ell=0}^{\lceil R/\mathcal{K}(N)\rceil} \tilde{Z}_\ell(k, d) \geq \frac{R}{N^3\mathcal{K}(N)}\Big] \tag{4.37}$$

decreases at least exponentially in N and thus the interior inequality is not valid a.s. for all N large. Summing over k we get

$$\sum_{k=0}^{\mathcal{K}(N)-1} \sum_{\ell=0}^{\lceil R/\mathcal{K}(N)\rceil} \mathcal{K}(N)Z_\ell(j_k, d) \leq d\mathcal{K}(N)\nu^{-1}\frac{R}{\mathcal{K}(N)} + \mathcal{K}(N)\frac{R}{N^3\mathcal{K}(N)} \leq CR\nu^{-1}e^{\eta d}, \tag{4.38}$$

since $\gamma/\beta^2 < 1$. □

5 Convergence of clock process

We will prove the convergence of the rescaled clock process to the stable subordinator on space $D([0, T], \mathbb{R})$ equipped with the Skorokhod M_1-topology. This topology is not commonly used in the literature, therefore we shortly recall some of its properties and compare it with the more standard Skorokhod J_1-topology, which we will need later, too. For more details the reader is referred to [Whi02] for both topologies and to [Bil68] for detailed account on J_1-topology.

5.1 Topologies on the Skorokhod space

Consider space $D = D([0,T],\mathbb{R})$ of càdlàg functions. The J_1-topology is the topology given by the J_1-metric: for $f, g \in D$

$$d_{J_1}(f,g) = \inf_{\lambda\in\Lambda}\{\|f\circ\lambda - g\|_\infty \vee \|\lambda - e\|_\infty\}, \tag{5.1}$$

where Λ is the set of strictly increasing functions mapping $[0,T]$ onto itself such that both λ and its inverse are continuous, and e is the identity map on $[0,T]$.
Also the M_1-topology is given by a metric. For $f \in D$ let Γ_f be its completed graph,

$$\Gamma_f = \{(z,t)\in\mathbb{R}\times[0,T] : z = \alpha f(t-) + (1-\alpha)f(t), \alpha\in[0,1]\}. \tag{5.2}$$

A parametric representation of the completed graph Γ_f (or of f) is a continuous bijective mapping $\phi(s) = (\phi_1(s),\phi_2(s))$, $[0,1]\mapsto\Gamma_f$ whose first coordinate ϕ_1 is increasing. If $\Pi(f)$ is set of all parametric representation of f, then the M_1-metric is defined by

$$d_{M_1}(f,g) = \inf\{\|\phi_1-\psi_1\|_\infty \vee \|\phi_2-\psi_2\|_\infty : \phi\in\Pi(f), \psi\in\Pi(g)\}. \tag{5.3}$$

The space D equipped with both M_1- and J_1-topologies is Polish. The M_1-topology is weaker than the J_1-topology: As an example, consider the sequence

$$f_n = \mathbf{1}\{[1-1/n,1)\} + 2\cdot\mathbf{1}\{[1,T]\}, \tag{5.4}$$

which converges to $f = 2\cdot\mathbf{1}\{[1,T]\}$ in the M_1-topology but not in the J_1-topology. One often says that the M_1-topology allows "intermediate jumps".
We will need a criterion for tightness of probability measures on D. To this end we define several moduli of continuity,

$$\begin{aligned} w_f(\delta) &= \sup\big\{\min\big(|f(t)-f(t_1)|, |f(t_2)-f(t)|\big) : t_1\le t\le t_2\le T, t_2-t_1\le\delta\big\},\\ w'_f(\delta) &= \sup\big\{\inf_{\alpha\in[0,1]}|f(t)-(\alpha f(t_1)+(1-\alpha)f(t_2))| : t_1\le t\le t_2\le T, t_2-t_1\le\delta\big\},\\ v_f(t,\delta) &= \sup\big\{|f(t_1)-f(t_2)| : t_1,t_2\in[0,T]\cup(t-\delta,t+\delta)\big\}. \end{aligned} \tag{5.5}$$

The following result is a restatement of Theorem 12.12.3 of [Whi02] and Theorem 15.3 of [Bil68].

Theorem 5.1. *The sequence of probability measures $\{P_n\}$ on $D([0,T],\mathbb{R})$ is tight in the J_1-topology if*

(i) For each positive ε there exist c such that

$$P_n[f : \|f\|_\infty > c] \le \varepsilon, \qquad n\ge 1. \tag{5.6}$$

(ii) For each $\varepsilon > 0$ and $\eta > 0$, there exist a δ, $0<\delta<T$, and an integer n_0 such that

$$P_n[f : w_f(\delta)\ge\eta] \le \varepsilon, \qquad n\ge n_0, \tag{5.7}$$

and

$$P_n[f : v_f(0,\delta)\ge\eta]\le\varepsilon \text{ and } P_n[f : v_f(T,\delta)\ge\eta]\le\varepsilon, \qquad n\ge n_0. \tag{5.8}$$

The same claim hold for the M_1-topology with $w_f(\delta)$ in (5.7) replaced by $w'_f(\delta)$.

5.2 Proof of Theorem 1.1

To prove the convergence of the rescaled clock process $\bar{S}_N(\cdot) = e^{-\gamma N} S_N(\cdot r(N))$ to the stable subordinator V_{γ/β^2}, we check first the convergence of finite-dimensional marginals. As can be guessed, Proposition 3.1 will serve to this purpose. Let ℓ, $\{u_i\}$ and $\{t_i\}$ be as above. Then,

$$\begin{aligned}\mathbb{E}\Big[\exp\Big\{-\sum_{i=1}^{\ell} u_i\big(\bar{S}_N(t_k)-\bar{S}_N(t_{k-1})\big)\Big\}\Big|Y_N\Big]\\ = \mathbb{E}\big[F_N(X_N^0;\{t_i\},\{u_i\})\big|Y_N\big] = \mathbb{E}\big[F_N(X_N^1;\{t_i\},\{u_i\})\big] + o(1),\end{aligned} \tag{5.9}$$

as follows from Proposition 3.1.
The value of $\mathbb{E}\big[F_N(X_N^1;\{t_i\},\{u_i\})\big]$ is not difficult to calculate. Define $j_N(i) = \lfloor t_i r(N)/\nu \rfloor$. Then

$$\begin{aligned}\mathbb{E}\big[F_N(X_N^1;\{t_i\},\{u_i\})\big] &= \mathbb{E}\Big[\exp\Big(-\sum_{k=1}^{\ell}\frac{u_k}{e^{\gamma N}}\sum_{i=t_{k-1}r(N)}^{t_k r(N)-1} e_i e^{\beta\sqrt{N}X_N^1(i)}\Big)\Big]\\ &\geq \mathbb{E}\Big[\prod_{k=1}^{\ell}\prod_{j=j(k-1)+1}^{j(k)}\exp\Big(-\frac{u_k}{e^{\gamma N}}\sum_{i=0}^{\nu-1} e_{j\nu+i}e^{\beta\sqrt{N}X_N^1(j\nu+i)}\Big)\Big]\end{aligned} \tag{5.10}$$

Since the process X_N^1 is a piece-wise independent process, the product in (5.10) is a product of independent random variables. Then expectations of all of them can be then bounded using Proposition 2.1. We get, for $\delta > 0$ fixed and N large enough,

$$\begin{aligned}\mathbb{E}\big[F_N(X_N^1;\{t_i\},\{u_i\})\big] &\geq \prod_{k=1}^{\ell}\prod_{j=j_N(k-1)+1}^{j_N(k)}\mathcal{F}_N(u_k)\\ &\geq \prod_{k=1}^{\ell}\big(1-(1+\delta)\nu N^{-1/2}e^{-N\gamma^2/2\beta^2}Ku_k^{\gamma/\beta^2}\big)^{j_N(k)-j_N(k-1)-1}\\ &\geq \prod_{k=1}^{\ell}\exp\big\{-(1+2\delta)(t_k-t_{k-1})Ku^{\gamma/\beta^2}\big\},\end{aligned} \tag{5.11}$$

which is (up to $1+2\delta$ term) the Laplace transform of $V_{\gamma/\beta^2}(K\cdot)$. A corresponding upper bound can be constructed analogously.
To check the tightness for $\bar{S}_N$ in $D([0,T],\mathbb{R})$ equipped with the Skorokhod M_1-topology we use Theorem 5.1. Since the processes $\bar{S}_N$ are increasing, it is easy to see that condition (i) is equivalent to the tightness of the distribution of $\bar{S}_N(T)$, which can be checked easily from the convergence of the Laplace transform of the marginal at time T (the limiting Laplace transform tends to 1 as $u \to 0$).
In order to check condition (ii), remark that for increasing functions the oscillation function $w'_{\bar{S}_N}(\delta)$ is always equal to zero. So checking (ii) boils down to controlling the boundary oscillations $v_{\bar{S}_N}(0,\delta)$ and $v_{\bar{S}_N}(T,\delta)$. For the first quantity (using again the monotonicity of $\bar{S}_N$) this amounts to check that $\mathbb{P}[\bar{S}_N(\delta) \geq \eta] < \varepsilon$ if δ is small enough and N large enough. Using the convergence of of marginal at time δ, it is

sufficient to take δ such that $\mathbb{P}[V_{\gamma/\beta^2}(K\delta) \geq \eta] \leq \varepsilon/2$, and take n_0 such that for all $n \geq n_0$

$$\left|\mathbb{P}[\bar{S}_N(\delta) \geq \eta] - \mathbb{P}[V_{\gamma/\beta^2}(K\delta) \geq \eta]\right| \leq \varepsilon/2. \tag{5.12}$$

The reasoning for $v_{\bar{S}_N}(T,\delta)$ is analogous. □

5.3 Coarse-grained clock process

To prove our aging result, that is Theorem 1.2, we need to modify the result of Theorem 1.1 slightly. Let $\tilde{S}_N$ be the "coarse-grained" clock processes,

$$\tilde{S}_N(t) = \frac{1}{e^{\gamma N}} S_N(\nu \lfloor tr(N)\nu^{-1} \rfloor). \tag{5.13}$$

For these processes we can strengthen the topology used in Theorem 1.1, that is we can replace the M_1- by the J_1-topology.

Theorem 5.2. *If the hypothesis of Theorem 1.1 is satisfied, then*

$$\tilde{S}_N(t) \xrightarrow{N\to\infty} V_{\gamma/\beta^2}(Kt) \qquad \mathcal{Y} - a.s., \tag{5.14}$$

weakly in the J_1-topology on the space of càdlàg functions $D([0,T],\mathbb{R})$.

Unfortunately, we cannot prove the theorem with estimates we have already at disposition. We should return back and improve some of them. First we show that traps with energies "much smaller" than $\gamma\sqrt{N}/\beta$ almost do not contribute to the clock process. Let $B_m = \gamma\sqrt{N}/\beta - m/(\beta\sqrt{N})$ and let

$$\bar{S}^m_N(t) = e^{-\gamma N} \sum_{i=0}^{\lfloor tr(N) \rfloor} e_i \exp\left\{\beta\sqrt{N} X^0_N(i)\right\} \mathbf{1}\{X^0_N(i) \leq B_m\}. \tag{5.15}$$

Lemma 5.3. *For every T and η, $\varepsilon > 0$ there exists m large enough such that*

$$\mathbb{P}[\bar{S}^m_N(T) \geq \eta | \mathcal{Y}] \leq \varepsilon, \qquad \mathcal{Y}\text{-}a.s. \tag{5.16}$$

Proof. To prove this lemma we should improve/modify slightly the calculations of Sections 2 and 3. With the notation of Section 2 define

$$\mathcal{F}^m_N = \mathbb{E}\Big[\exp\Big\{ - e^{-\gamma N} \sum_{i=1}^{\nu} e_i e^{\beta\sqrt{N} U_i} \mathbf{1}\{U_i \leq B_m\}\Big\}\Big]. \tag{5.17}$$

(comparing with (2.2) observe that we set $u = 1$). We will show that

$$\lim_{N\to\infty} f(N) e^{N\gamma^2/2\beta^2} \left[1 - \mathcal{F}^m_N\right] = K_m, \tag{5.18}$$

with $K_m \to 0$ as $m \to \infty$. The proof of this claim is completely analogous to the proof of Proposition 2.1. One should only modify the domains of integrations. More precisely, the definition of D_k which appears after (2.9) should be replaced by $D^m_k = D_k \cap \{z : G_k(z) \leq B_m\}$. Hence, D'_k becomes $D'^m_k = D'_k \cap \{b : G_k(b) \leq -m/(\beta/\sqrt{N})\}$,

which then restricts the domain of integration in (2.33) to $(-\infty, -m/\beta]$. Hence, the constant K_m can be made arbitrarily small by choosing m large.
Further, as in Section 3, define

$$F_N^m(X) = \exp\Big(-\sum_{i=0}^{Tr(N)-1} g\Big(e^{-\gamma N} e^{\beta\sqrt{N}X(i)} \mathbf{1}\{X(i) \le B_m\}\Big)\Big). \tag{5.19}$$

Then, as in Proposition 3.1, we will show

$$\lim_{N\to\infty} \mathbb{E}\big[F_N^m(X_N^0)\big|\mathcal{Y}\big] - \mathbb{E}\big[F_N^m(X_N^1)\big] = 0, \qquad \mathcal{Y}\text{-a.s.} \tag{5.20}$$

We use again (3.5) to show this claim. Although the indicator function is not differentiable, we will proceed as if it was, setting $(\mathbf{1}\{x \le B\})' = -\delta(x-M)$, where δ denotes the Dirac delta function. As usual, this can be justified e.g. by using smooth approximations of the indicator function. The second derivative of $F_N^m(X)$ equals

$$\begin{aligned}
&\frac{u^2\beta^2 N}{e^{2\gamma N}} e^{\beta\sqrt{N}(X(i)+X(j))} g'\big(ue^{\beta\sqrt{N}X(i)-\gamma N}\big) g'\big(ue^{\beta\sqrt{N}X(j)-\gamma N}\big) F_N^m(X)\\
&\qquad \times \Big(\mathbf{1}\{X(i) \le B_m\} - \frac{\delta_{B_m}(X(i))}{\beta\sqrt{N}}\Big)\Big(\mathbf{1}\{X(j) \le B_m\} - \frac{\delta_{B_m}(X(j))}{\beta\sqrt{N}}\Big)\\
&\le u^2\beta^2 N e^{\beta\sqrt{N}(X_N^h(i)+X_N^h(j))-2\gamma N} \exp\big(-2g\big(ue^{\beta\sqrt{N}X_N^h(i)-\gamma N}\big) - 2g\big(ue^{\beta\sqrt{N}X_N^h(j)-\gamma N}\big)\big)\\
&\qquad \times \Big(\mathbf{1}\{X(i) \le B_m\} - \frac{\delta_{B_m}(X(i))}{\beta\sqrt{N}}\Big)\Big(\mathbf{1}\{X(j) \le B_m\} - \frac{\delta_{B_m}(X(j))}{\beta\sqrt{N}}\Big).
\end{aligned} \tag{5.21}$$

We should now bound the contributions of four terms. The one with the product of two indicator functions is easy, because we can use directly the result of Lemma 3.2. For remaining three terms, those with the product of one indicator and one delta function, and this with two delta function, the calculation should be repeated. However, in the end we find that (5.21) is bounded by $\bar{\Xi}(\mathrm{Cov}(X(i), X(j)))$ as before. The presence of the delta functions makes actually the calculations slightly less complicated. The proof then proceed as in Section 3.
We can now finish the proof of Lemma 5.3. By (5.17) and (5.20),

$$\begin{aligned}
\mathbb{E}\big[\exp(-\bar{S}_N^m(T))\big|\mathcal{Y}\big] &= \mathbb{E}\big[F_N^m(X_N^0)\big|\mathcal{Y}\big] = \mathbb{E}\big[F_N^m(X_N^1)\big|\mathcal{Y}\big] + o(1)\\
&= (1 - K_m f(N)^{-1} e^{-N\gamma^2/2\beta^2})^{Tr(N)/\nu} + o(1) = e^{-K_m T} + o(1).
\end{aligned} \tag{5.22}$$

Since $K_m \to 0$ as $m \to \infty$,

$$\mathbb{P}[\bar{S}_N^m(T) \ge \eta|\mathcal{Y}] \le \frac{1 - \mathbb{E}\big[\exp(-\bar{S}_N^m(T))\big|\mathcal{Y}\big]}{1 - e^{-\eta}} \tag{5.23}$$

can be made arbitrarily small by taking m large enough. $\square$

We study now how the blocks where the process visits sites with energies larger than B_m are distributed along the trajectory. To this end we set for any Gaussian process X

$$s_N^m(i; X) = \mathbf{1}\{\exists j : i\nu < j \le (i+1)\nu, X(j) > B_m\}. \tag{5.24}$$

and we define point process $H_N^m(X)$ on $[0,T]$ by

$$H_N^m(X;\mathrm{d}x) = \sum_{i=0}^{Tr(N)/\nu} s_N^m(i;X)\delta_{i\nu/r(N)}(\mathrm{d}x). \tag{5.25}$$

Lemma 5.4. *For every $m \in \mathbb{R}$ the point processes $H_N^m(X_N^0)$ converge to a homogeneous Poisson point process on $[0,T]$ with intensity $\rho_m \in (0,\infty)$, $\mathcal{Y}$-a.s.*

Proof. To show this lemma we use Proposition 16.17 of Kallenberg [Kal02]. According to it, to prove the convergence of $H_N^m(X_N^0)$ to a Poisson point process with intensity ρ_m it is sufficient to check that for any interval $I \subset [0,T]$

$$\lim_{N\to\infty} \mathbb{P}[H_N^m(X_N^0;I) = 0|\mathcal{Y}] = e^{-\rho_m|I|} \tag{5.26}$$

and

$$\limsup_{N\to\infty} \mathbb{E}[H_N^m(X_N^0;I)|\mathcal{Y}] \le \rho_m|I|, \tag{5.27}$$

where $|I|$ denotes the Lebesgue measure of I.
The proof of the first claim is completely similar to the previous ones. We start with a one-block estimate for (5.26):

$$\lim_{N\to\infty} N^{1/2}\nu^{-1}e^{N\gamma^2/2\beta^2}\,\mathbb{E}[s_N^m(0,U)] = \rho_m, \tag{5.28}$$

Using the notation of Section 2, we get

$$\mathbb{E}[s_N^m(0,U)] = \int_{A_m} \frac{\mathrm{d}z}{(2\pi)^{\nu/2}} e^{-\frac{1}{2}\sum_{i=1}^{\nu} z_i^2}, \tag{5.29}$$

where $A_m = \{z : \exists k \in \{1,\dots,\nu\} G_k(z) > B_m\}$. Dividing the domain of integration according to the maximal $G_k(z)$, this is equal

$$\sum_{k=1}^{\nu} \int_{D_k} \frac{\mathrm{d}z}{(2\pi)^{\nu/2}} e^{-\frac{1}{2}\sum_{i=1}^{\nu} z_i^2}, \tag{5.30}$$

where $D_k = \{z : G_k(z) > B_m, G_i(z) \le G_k(z) \forall i \ne k\}$. Using the substitution $z_i = b_i \pm \Gamma_i B_m$ on D_k (where $+$ sign is used for $i \le k$ and $-$ sign for $i > k$) we get

$$e^{-N\gamma^2/2\beta^2} e^{m\gamma/\beta^2} \sum_{k=1}^{\nu} \int_{D'_k} \frac{\mathrm{d}b}{(2\pi)^{\nu/2}} e^{-\frac{1}{2}\sum_{i=1}^{\nu} b_i^2} e^{-B_m G_k(b)}, \tag{5.31}$$

where $D'_k = \{b : G_k(b) > 0, \sum_{j=i+1}^{k} b_j + |k-i|\Gamma_\nu B_m \ge 0 \forall i \ne k\}$. The same reasoning as before then allows to show that the last expression behaves like $\rho_m \nu N^{-1/2} e^{-\gamma^2 N/2\beta^2}$ as $N \to \infty$.
To compare the real process with the block-independent process, let

$$F_N(I;X) = \mathbf{1}\{\max\{X(i) : i\nu/r(N) \in I\} \le B_m\}. \tag{5.32}$$

The difference between $\mathbb{E}[F_N(I;X_N^0)|\mathcal{Y}]$ and $\mathbb{E}[F_N(I;X_N^1)]$ is again given by the Gaussian comparison formula (3.5). This time the second derivative equals

$$\delta(X(i)-B_m)\delta(X(j)-B_m)\prod_{k\neq i,j}\mathbf{1}\{X(k)\leq B_m\}\leq\delta(X(i)-B_m)\delta(X(j)-B_m). \quad (5.33)$$

If covariance of $X(i)$ and $X(j)$ equals c, the expectation of the last expression is given by the value of the joint density of $X(i)$, $X(j)$ at point (B_m,B_m) which is

$$(2\pi(1-c^2))^{-1}e^{-B_m^2/(1+c)}\leq C(1-c^2)^{-1}\exp\Big\{-\frac{\gamma^2 N}{\beta^2(1+c)}\Big\}. \quad (5.34)$$

The exponential term is the same as in $\bar{\Xi}(c)$. The polynomial prefactor is however different, it diverges faster as $c\to 1$. We should thus return to (3.24) with $\tilde{\Xi}$ replaced by the right-hand side of (5.34). First

$$\int_0^1(1-c^2)^{-1}=c^{-1}\operatorname{arg}\tanh(c)\approx-\frac{1}{2}\log(1-c) \quad (5.35)$$

as $c\to 1$, which is not bounded for all c as before. The estimates (3.26) and (3.27) are influenced by this change. For (3.26) we can actually neglect this change, because the main contribution to this term came from the neighborhood of $d=N/2$ (or $c=0$) and was exponentially small in the neighborhood of $d=1$ (or $c\sim 1/N$). In the treatment of (3.27), the change has more effect, after some computations (3.45) becomes

$$CtN^{3/2}\nu^{-1}\int_0^{\delta'}\log(c/x)e^{-cNx}\mathrm{d}x\leq CtN^{1/2}\nu^{-1}\log N\xrightarrow{N\to\infty}0. \quad (5.36)$$

Finally, the change of polynomial prefactor of Ξ implies change in the control of (3.28). The equation (3.46) becomes

$$(3.28)\leq C\sum_{d=0}^{\nu}tN^{-3/2}d^2[1-(1-2dN^{-1})^2p]^{-1}\exp(N\tilde{\Upsilon}(d/N)). \quad (5.37)$$

and the linearization of $\tilde{\Upsilon}$ gives new form of (3.47)

$$CtN^{3/2}\int_0^{\varepsilon}xe^{-c'Nx}\mathrm{d}x\leq CtN^{-1/2}\xrightarrow{N\to\infty}0. \quad (5.38)$$

Therefore, using (5.28)

$$\begin{aligned}\mathbb{P}[H_N^m(X_N^0;I)=0|\mathcal{Y}]=\mathbb{E}[F_N(I;X_N^0)|\mathcal{Y}]&=\mathbb{E}[F_N(I;X_N^1)]+o(1)\\&=(1-\mathbb{E}[s_N^m(0,U)])^{|I|r(N)/\nu}\to e^{-\rho_m|I|}.\end{aligned} \quad (5.39)$$

This completes the proof of (5.26).
It is easy to check (5.27). By definition,

$$\mathbb{E}[H_N^m(X_N^0;I)|\mathcal{Y}]=\sum_{i:i\nu/R\in I}\mathbb{E}[s_N^m(i,X_N^0)|\mathcal{Y}]. \quad (5.40)$$

Since $\Lambda_{ij}^0\geq\Lambda_{ij}^1$ for i, j in the same block, $\mathbb{E}[s_N^m(i,X_N^0)|\mathcal{Y}]\leq\mathbb{E}[s_N^m(i,X_N^1)]$. Therefore,

$$(5.40)\leq|I|r(N)/\nu\mathbb{E}[s_N^m(0,U)]=\rho_m|I|. \quad (5.41)$$

This completes the proof of Lemma 5.4. ☐

Proof of Theorem 5.2. Checking the convergence of finite-dimensional marginals as well of condition (i) and the second part of (ii) of Theorem 5.1 is analogous as for the original clock process $\bar{S}_N$. We should thus only prove the first part of condition (ii). Namely that, for any η and ε there exist δ such that

$$\mathbb{P}[w_{\bar{S}_N}(\delta) \geq \eta] \leq \varepsilon, \tag{5.42}$$

for all N large enough.
Let

$$w_f([\tau, \tau+\delta]) = \sup\{\min(|f(t_2) - f(t)|, |f(t) - f(t_1)|) : \tau \leq t_1 \leq t \leq t_2 \leq \tau + \delta\}. \tag{5.43}$$

Fix m such that $\mathbb{P}[\bar{S}_N^m(T) \geq \eta/2] \leq \varepsilon/2$, which is possible according to Lemma 5.3. If $H_N^m(X_n^0; [\tau, \tau+\delta]) \leq 1$ then

$$w_{\bar{S}_N}([\tau, \tau+\delta]) \leq \bar{S}_N^m(\tau+\delta) - \bar{S}_N^m(\tau) \leq \bar{S}_N^m(T). \tag{5.44}$$

Hence,

$$\mathbb{P}[w_{\bar{S}_N}([\tau, \tau+\delta]) \geq \eta | i\bar{S}_N^m(T) \leq \eta/2] \leq \mathbb{P}[H_N^m(X_N^0; [\tau, \tau+\delta]) \geq 2] \leq C\rho_m\delta^2. \tag{5.45}$$

We can now show (5.42). Estimate

$$w_{\tilde{S}_N}(\delta) \leq \max\{w_{\tilde{S}_N}([\tau, \tau+2\delta]) : 0 \leq \tau \leq T, \tau = k\delta, k \in \mathbb{N}\} \tag{5.46}$$

yields

$$\begin{aligned}
\mathbb{P}[w_{\tilde{S}_N}(\delta) \geq \eta | \mathcal{Y}] &\leq \sum_{k=0}^{T\delta^{-1}} \mathbb{P}[w_{\tilde{S}_N}([k\delta, (k+2)\delta]) \geq \varepsilon | \mathcal{Y}] \\
&\leq \mathbb{P}[\bar{S}_N^m(T) \geq \eta/2] + \sum_{k=0}^{T\delta^{-1}} \mathbb{P}[H_N^m(X_N^0; [k\delta, (k+2)\delta]) \geq 2] \\
&\leq \varepsilon/2 + CT\delta^{-1}\rho_m\delta^2 \leq \varepsilon
\end{aligned} \tag{5.47}$$

if δ is chosen small enough. This completes the proof. ☐

Proof of Theorem 1.2. Let $\mathcal{R}_N$ be the range of the coarse grained process $\tilde{S}_N$. Obviously, for any $1 > \varepsilon > 0$,

$$A_N^\varepsilon(t, s) \supset \{\mathcal{R}_N \cap (t, s) = \emptyset\}, \tag{5.48}$$

because if the above intersection is empty, then σ_N makes less than ν steps in time interval $[te^{\gamma N}, se^{\gamma N}]$, and thus the overlap of $\sigma_N(te^{\gamma N})$ and $\sigma_N(se^{\gamma N})$ is $O(\nu/N)$.
If $\mathcal{R}_N \cap (t, s) \neq \emptyset$, than there exist u such that $\tilde{S}_N(u) \in (t, s)$. Moreover, it follows from Theorem 5.2 that for any δ there exist η such than

$$\mathbb{P}[\tilde{S}_N(u+\eta) \in (s, t)] \geq 1 - \delta. \tag{5.49}$$

This however means that the process σ_N make at least $\eta r(N)$ steps between times t and s and thus the overlap between $\sigma_N(te^{\gamma N})$ and $\sigma_N(se^{\gamma N})$ is with high probability close to 0.

Hence $\mathbb{P}[A_N^\varepsilon(t,s)|\mathcal{Y}]$ is very well approximated by $\mathbb{P}[\mathcal{R}_N \cap (t,s) = \emptyset|\mathcal{Y}]$. Since stable subordinator does not hit points, that is $\mathbb{P}[\exists u : V_{\gamma/\beta^2}(u) = t] = 0$, and $\tilde{S}_N$ converge in J_1-topology,

$$\mathbb{P}[\mathcal{R}_N \cap (t,s) = \emptyset|\mathcal{Y}] \xrightarrow{N\to\infty} \mathbb{P}[\{V_{\gamma/\beta^2}(u) : u \geq 0\} \cap (s,t) = \emptyset], \tag{5.50}$$

which, as follows from the arc-sine law for stable subordinators, is given by the formula (1.13). □

References

[BBG02] G. Ben Arous, A. Bovier, and V. Gayrard. Aging in the random energy model. *Physical Review Letters*, 88(8):087201, February 2002.

[BBG03a] G. Ben Arous, A. Bovier, and V. Gayrard. Glauber dynamics of the random energy model. I. Metastable motion on the extreme states. *Comm. Math. Phys.*, 235(3):379–425, 2003.

[BBG03b] G. Ben Arous, A. Bovier, and V. Gayrard. Glauber dynamics of the random energy model. II. Aging below the critical temperature. *Comm. Math. Phys.*, 236(1):1–54, 2003.

[BČ05] G. Ben Arous and J. Černý. Bouchaud's model exhibits two aging regimes in dimension one. *Ann. Appl. Probab.*, 15(2):1161–1192, 2005.

[BČ06] G. Ben Arous and J. Černý. Dynamics of trap models. In *École d'Été de Physique des Houches, Session LXXXIII "Mathematical Statistical Physics"*, pages 331–394. Elsevier, 2006.

[BČ07a] G. Ben Arous and J. Černý. The arcsine law as a universal aging scheme for trap models. to appear in Communications on Pure and Applied Mathematics, 2007.

[BČ07b] G. Ben Arous and J. Černý. Scaling limit for trap models on $\mathbb{Z}^d$. to appear in Annals of Probability, 2007.

[BCKM98] J.-P. Bouchaud, L. Cugliandolo, J. Kurchan, and M. Mézard. Out of equilibrium dynamics in spin-glasses and other glassy systems. In A. P. Young, editor, *Spin glasses and random fields*. World Scientific, Singapore, 1998.

[BČM06] G. Ben Arous, J. Černý, and T. Mountford. Aging in two-dimensional Bouchaud's model. *Probab. Theory Related Fields*, 134(1):1–43, 2006.

[BD95] J.-P. Bouchaud and D. S. Dean. Aging on Parisi's tree. *J. Phys I(France)*, 5:265, 1995.

[BDG01] G. Ben Arous, A. Dembo, and A. Guionnet. Aging of spherical spin glasses. *Probab. Theory Related Fields*, 120(1):1–67, 2001.

[BG06] G. Ben Arous and V. Gayrard. Elementary potential theory on the hypercube. preprint, 2006.

[BGK06] G. Ben Arous, V. Gayrard, and A. Kuptsov. A new REM conjecture. preprint, arXiv math.PR 0612373, 2006.

[Bil68] P. Billingsley. *Convergence of probability measures.* John Wiley & Sons Inc., New York, 1968.

[BK06a] A. Bovier and I. Kurkova. Local energy statistics in disordered systems: a proof of the local REM conjecture. *Comm. Math. Phys.*, 263(2):513–533, 2006.

[BK06b] A. Bovier and I. Kurkova. A tomography of the GREM: beyond the REM conjecture. *Comm. Math. Phys.*, 263(2):535–552, 2006.

[BK07] G. Ben Arous and A. Kuptsov. The limits of REM universality. In preparation, 2007.

[BM04] H. Bauke and S. Mertens. Universality in the level statistics of disordered systems. *Physical Review E (Statistical, Nonlinear, and Soft Matter Physics)*, 70(2):025102, 2004.

[Bou92] J.-P. Bouchaud. Weak ergodicity breaking and aging in disordered systems. *J. Phys. I (France)*, 2:1705–1713, september 1992.

[Bov06] A. Bovier. *Statistical mechanics of disordered systems.* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2006.

[ČG06] J. Černý and V. Gayrard. Hitting time of large subsets of the hypercube. submitted, 2006.

[Cha84] R. V. Chamberlin. Time decay of the thermoremanent magnetization in spin-glasses as a function of the time spent in the field-cooled state. *Phys. Rev. B*, 30(9):5393–5395, Nov 1984.

[Hoe63] W. Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, 58:13–30, 1963.

[Kal02] O. Kallenberg. *Foundations of modern probability.* Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[LSNB83] L. Lundgren, P. Svedlindh, P. Nordblad, and O. Beckman. Dynamics of the relaxation-time spectrum in a CuMn spin-glass. *Phys. Rev. Lett.*, 51(10):911–914, Sep 1983.

[LT91] M. Ledoux and M. Talagrand. *Probability in Banach spaces*, volume 23 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1991.

[Mat87] P. Matthews. Mixing rates for a random walk on the cube. *SIAM J. Algebraic Discrete Methods*, 8(4):746–752, 1987.

[Mat89] P. Matthews. Some sample path properties of a random walk on the cube. *J. Theoret. Probab.*, 2(1):129–146, 1989.

[She71] L. A. Shepp. First passage time for a particular Gaussian process. *Ann. Math. Statist.*, 42:946–951, 1971.

[Sle61] D. Slepian. First passage time for a particular Gaussian process. *Ann. Math. Statist.*, 32:610–612, 1961.

[Tal03] M. Talagrand. *Spin glasses: a challenge for mathematicians*, volume 46 of *Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2003.

[Whi02] W. Whitt. *Stochastic-process limits*. Springer Series in Operations Research. Springer-Verlag, New York, 2002.