Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Convergence of Fourier-Wavelet models for Gaussian random processes

O. Kurbanmuradov[1] **and K. Sabelfeld**[2,3]

[1] Center for Phys. Math. Research, Turkmenian State University,
Turkmenbashy av. 31, 744000 Ashgabad, Turkmenistan

[2] Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstrasse 39, D – 10117 Berlin, Germany
sabelfel@wias-berlin.de

[3]Institute of Computational Mathematics and Mathem. Geophysics
Russian Acad. Sci., Lavrentieva str., 6
630090 Novosibirsk, Russia

No. 1239

Berlin 2007

1991 *Mathematics Subject Classification.* 65C05, 65C20, 60G15 .

Key words and phrases. Fourier-Wavelet model, stationary Gaussian random process, Meyer's wavelets, Nikolskiĭ-Besov space, convergence in probability, convergence in mean square..

This work is supported partly by the grants: German DFG Grant GZ: SA 861/5-1, RFBR Grant N 06-01-00498, the program of Leading Scientific Schools under Grant N 4774.2006.1, and NATO Linkage Grant ESP Nr CLG 981426. OK thanks WIAS for the kind hospitality during his stay in May-June 2007.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

Mean square convergence and convergence in probability of Fourier-Wavelet Models (FWM) of stationary Gaussian Random processes in the metric of Banach space of continuously differentiable functions and in Sobolev space are studied. Sufficient conditions for the convergence formulated in the frame of spectral functions are given. It is shown that the given rates of convergence of FWM in the mean square obtained in the Nikolskiĭ-Besov classes cannot be improved.

1 Introduction

Random processes and fields provide a convenient tool for a quantitative description of highly inhomogeneous and multiscale media such as, e.g., atmosphere, ocean and soil [28], [31], [19], [6].

The random fields are defined by all their finite-dimensional distributions (e.g., see [14]). Fortunately, the most often distributions used in practice are Gaussian. The Gaussian random fields are defined uniquely by their expectations and covariances which greatly simplifies the measurement design. Along with the Gaussian random fields, one exploits often random fields which are obtained by some transformations of the Gaussian fields through either an explicit transform or implicitly, through differential or integral equations where the Gaussian random fields are entered as random coefficients, or random source term, or boundary functions. As an important example we mention the hydraulic conductivity which is often modelled as a lognormal random field, e.g., see [6]. The Darcy flow itself is obtained as a random field which solves the Darcy equation with the lognormal hydraulic conductivity.

Another question about the structure of the random field which in practice is of extremely high importance sounds: is the random field homogeneous or not? Unlike the first feature, the gaussinity, the most practically interesting random fields are non-homogeneous. However the homogeneous Gaussian random fields can be treated as perfect approximations in many cases, and they are indeed the most frequently used models in all branches of science and technology since the simulation technique based on the spectral representation is well developed.

Simulation of non-homogeneous random fields is a much more difficult problem. The most developed are the Cholesky (singular) decomposition (e.g., see, [9], [34]) and Karhunen-Loeve expansion techniques(e.g., see, [43]). We mention also the wavelet decomposition method which is applied to many interesting multiscale resolution problems, e.g., see [37], [44].

There is very extensive literature on simulation of homogeneous random fields, and even for a short discussion of recent publications in this field we would need to write a large review article. So we mention here only some publications, and the reader has a possibility to search further leading by the references inside these publications: [8], [9], [11]-[13], [23], [25], [27], [30]-[36], [40].

It should be mentioned that there are also a lot of publications devoted to a comparative analysis of the cost of the simulation algorithms, but as we have concluded in [20], the most competitive simulation method for multiscale random fields are the randomized spectral methods, Fourier-wavelet methods, and wavelet decomposition technique (e.g., see, [2],[12], [13], [24]).

Having an approximate model u_N for our random field $u(x)$ (say, the parameter N is integer), there is only an approximate closeness between the statistical characteristics of u_N and $u(x)$. The convergence $u_N \to u$ as $N \to \infty$ can be defined differently: (i) convergence of the relevant finite-difference distributions, (ii) convergence in a relevant functional space. The convergence in functional spaces is important for practical applications, and they were considered in many publications (e.g., for the spectral models see [1], [21], [22], [42]). In [4], [5], the convergence of wavelet decomposition models is studied. As to the Fourier-wavelet models (FWM), the first convergence results were obtained in [23] where the convergence of the correlation function of the model process FWM $u_N(x)$, $B_N(x,r) = E(u_N(x+r)u_N(x))$, to the correlation function $B(r) = E(u(x+r)u(x))$ of the original Gaussian process $u(x)$ has been established. In the language of spectral function $F(k)$, we have obtained an estimation of $\sup_{x,r\in\mathbb{R}} |B(r) - B_N(x,r)|$ which shows that this quantity can be made arbitrarily small as $N \to \infty$.

In this paper we study the convergence of FWM $u_N \to u$ as $N \to \infty$ in Sobolev spaces $W_2^n[a,b]$ and in the space of n-times continuously differentiable functions $C^n[a,b]$ where n is a nonnegative integer. We study conditions on the spectral function F which are sufficient for $E||\mathcal{E}_N||^2_{W_2^n[a,b]} \to 0$ and $||\mathcal{E}_N||_{C^n[a,b]} \xrightarrow{P} 0$ as $N \to \infty$. Here $[a,b] \subset \mathbf{R}$ is an arbitrary fixed finite interval, and $\mathcal{E}_N(x) = u(x) - u_N(x)$. To make the presentation clear, we first consider the case of a scalar random process $u(x)$ and give sufficient conditions for the convergence $E||\mathcal{E}_N||^2_{L_2[a,b]} \to 0$ and $||\mathcal{E}_N||_{C[a,b]} \xrightarrow{P} 0$ as $N \to \infty$. Then we give a relevant generalization of the results to vector processes and extend the study for some stronger metrics.

The paper is organized as follows. The definition of the FWM for Gaussian random processes is given in Section 2 . Convergence of FWM in $L_2[0,1]$ is presented in section 3, while an analysis of the quality of the derived estimations is given in section 4. Convergence of FWM in the space $C[0,1]$ is given in section 5. In section 6 we study the convergence FWM in some stronger metrics (in spaces $W_p^n[0,1]$ and $C^n[0,1]$) and for vector processes. Conclusions are made in section 7. Finally, we present some known results from the theory of Besov's space we used in our work in the Appendix.

2 Fourier-Wavelet models of Gaussian random processes

2.1 Fourier-Wavelet expansions of Gaussian random processes

Let $\mathbf{u}(x) = (u_1(x), \dots, u_l(x))^T$, $x \in \mathbf{R} = (-\infty, \infty)$ be an l-dimensional real valued Gaussian stationary (homogeneous) random process with zero mean and a given correlation tensor $B(r)$:

$$B_{ij}(r) = E[u_i(x+r)\,u_j(x)], \quad i,j = 1,\dots l, \tag{2.1}$$

or the corresponding spectral tensor F:

$$F_{ij}(k) = \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi k\cdot r} B_{ij}(r)\,dr, \quad B_{ij}(r) = \int_{-\infty}^{\infty} e^{\mathrm{i}2\pi r\cdot k} F_{ij}(k)\,dk, \quad i,j = 1,\dots l\ . \tag{2.2}$$

Here and throughout the paper E stands for the mathematical expectation. We will assume that $\int_{-\infty}^{\infty} \sum_{j=1}^{l} B_{jj}^2(r)\,dr < \infty$ which ensures the existence of the spectral tensor F_{ij} in the space $L_2(\mathbb{R})$ [28]. Moreover, $B_{ii}(0) = Eu_i^2(x) < \infty$ implies $B_{ii}(0) = \int_{-\infty}^{\infty} F_{ii}(k)\,dk < \infty$, that is $F \in L_1(\mathbb{R})$. Hence $F \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$.

For some positive integer n let $Q(k)$ be an $l \times n$-matrix satisfying the condition

$$Q(k)Q^*(k) = F(k), \quad Q(-k) = \bar{Q}(k)\ . \tag{2.3}$$

Here the star stands for the complex conjugate transpose which is equivalent to taking two operations, the transpose ^{T}, and the complex conjugation of each entry.

Let $\phi(x)$ and $\psi(x)$, $x \in \mathbb{R}$ be orthonormal scaling and wavelet functions (for the definitions see, e.g., [7], [26], [15], [3]), respectively, and $\hat{\phi}(k)$, $\hat{\psi}(k)$ are Fourier transformations of these functions:

$$\hat{\phi}(k) = \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi kx}\phi(x)dx, \quad \hat{\psi}(k) = \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi kx}\psi(x)dx.$$

Assume, that $\hat{\phi}(k)$, $\hat{\psi}(k)$ are even functions

$$\hat{\phi}(-k) = \bar{\hat{\phi}}(k), \quad \hat{\psi}(-k) = \bar{\hat{\psi}}(k). \tag{2.4}$$

Then the Fourier-Wavelet expansion of the random process $\mathbf{u}(x)$ reads [11], [23]:

$$\mathbf{u}(x) = \sum_{j=-\infty}^{\infty} \mathcal{F}_{m_0}^{(\phi)}(2^{m_0}x + j)\,\xi_j + \sum_{m=m_0}^{\infty}\sum_{j=-\infty}^{\infty} \mathcal{F}_m^{(\psi)}(2^m x + j)\,\xi_{mj}\ . \tag{2.5}$$

where m_0 is an arbitrary (but fixed) integer, $\xi_j, \xi_{mj} \quad (j \in \mathbb{Z}, m \geq m_0)$ is a family of mutually independent standard real valued Gaussian random vectors of dimension n, and $\mathcal{F}^{(\phi)}(\cdot)$, $\mathcal{F}^{(\psi)}(\cdot)$ are $l \times n$-dimensional matrix functions defined by

$$\begin{aligned} \mathcal{F}_m^{(\phi)}(y) &= \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi ky}\, 2^{m/2}\bar{Q}(2^m k)\hat{\phi}(k)\,dk, \\ \mathcal{F}_m^{(\psi)}(y) &= \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi ky}\, 2^{m/2}\bar{Q}(2^m k)\hat{\psi}(k)\,dk. \end{aligned} \tag{2.6}$$

Since $B(0) = \int F(k)dk < \infty$, and $QQ^* = F$, we conclude that all the entries of the matrix Q belong to $L_2(\mathbb{R})$. Therefore, the functions (2.6) are well defined.

Let us give a comment concerning some difference between the Fourier-Wavelet and Wavelet decompositions. In both methods, the random process is represented as a series of deterministic functions weighted by a random Gaussian coefficients. The crucial difference is that unlike the Wavelet decomposition, in the Fourier-Wavelet method these random coefficients are independent which might be quite convenient in practical simulations, see for instance [20]. The Wavelet decomposition is more general and can be used also to simulate inhomogeneous random fields [44].

Further in Fourier-Wavelet expansion (2.5) we will use Meyer's wavelet functions $\phi(x)$ and $\psi(x)$ which are defined by their Fourier transforms (e.g., see [7]):

$$\phi(x) = \int_{-\infty}^{\infty} e^{i2\pi kx}\hat{\phi}(k)\,dk, \quad \psi(x) = \int_{-\infty}^{\infty} e^{i2\pi kx}\hat{\psi}(k)\,dk, \tag{2.7}$$

where

$$\hat{\phi}(k) = \begin{cases} 1 & |k| \le 1/3\ , \\ \cos[\frac{\pi}{2}\nu(3|k|-1)], & 1/3 \le |k| \le 2/3 \\ 0\ , & otherwise \end{cases} \tag{2.8}$$

$$\hat{\psi}(k) = \begin{cases} e^{-i\pi k}\sin[\frac{\pi}{2}\nu(3|k|-1)], & 1/3 \le |k| \le 2/3 \\ e^{-i\pi k}\cos[\frac{\pi}{2}\nu(\frac{3}{2}\,|k|-1)], & 2/3 \le |k| \le 4/3 \\ 0\ , & otherwise\ . \end{cases} \tag{2.9}$$

Here $\nu(x)$ is a smooth function satisfying the following conditions: $\nu(x) \equiv 0$ for $x \le 0$, $\nu(x) \equiv 1$ for $x \ge 1$, and $\nu(x) + \nu(1-x) = 1$ for $0 < x < 1$. As an example of such a function, we mention a function $\nu(x) = \nu_p(x)$ depending on a positive parameter p (see [11]):

$$\nu_p(x) = \frac{4^{p-1}}{p}\left\{[x-x_0]_+^p + [x-x_p]_+^p + 2\sum_{j=1}^{p-1}(-1)^j\,[x-x_j]_+^p\right\}\ ,$$

where $x_j = (1/2)[\cos(((p-j)/p)\pi)+1]$, and $[a]_+ = \max(a,0)$. The function ν_p is $p-1$ times continuously differentiable, therefore, choosing p sufficiently large, we can make the functions $\hat{\phi}$ and $\hat{\psi}$ smooth enough.

2.2 Fourier-Wavelet models of Gaussian random processes

In the numerical implementation of (2.5) we have to find a reasonable choice of the cut-off parameters m_1 and $b_0, b_m\,(m = m_0,\ldots,m_1)$ in the approximations:

$$\sum_{j=-\infty}^{\infty} \mathcal{F}_{m_0}^{(\phi)}(2^{m_0}x+j)\,\xi_j \simeq \sum_{j=-b-\lfloor 2^{m_0}x\rfloor}^{b-\lfloor 2^{m_0}x\rfloor} \mathcal{F}_{m_0}^{(\phi)}(2^{m_0}x+j)\,\xi_j\ , \tag{2.10}$$

$$\sum_{m=m_0}^{\infty}\sum_{j=-\infty}^{\infty} \mathcal{F}_m^{(\psi)}(2^m x+j)\,\xi_{mj} \simeq \sum_{m=m_0}^{m_1}\sum_{j=-b_m-\lfloor 2^m x\rfloor}^{b_m-\lfloor 2^m x\rfloor} \mathcal{F}_m^{(\psi)}(2^m x+j)\,\xi_{mj} \tag{2.11}$$

where $\lfloor a\rfloor$ stands for the integer part of a. General idea is that b, b_m should be chosen so that supports of the functions $\mathcal{F}_{m_0}^{(\phi)}$ and $\mathcal{F}_m^{(\psi)}$ belong essentially to the intervals $[-b,b]$ and $[-b_m,b_m]$, respectively.

In previous papers on Fourier-Wavelet models ([2],[11],[20], [23]) the authors use approximations of type (2.10)-(2.11). Unfortunately, functions staying in the right-hand sides of (2.10)-(2.11) are discontinuous in x (with the jump discontinuity). Therefore, to ensure that the samples of the approximation process are continuous we have to modify the model. So let us define the following modification of the Fourier-Wavelet model (FWM) of the random field $\mathbf{u}(x)$. Let $\{\mu_N\}_{N=1}^{\infty}$ be a sequence of positive integers, $\mu_N \geq m_0, N = 1,2,...$ and $\{b_N\}_{N=1}^{\infty}$, $\{b_{mN}\}_{N=1}^{\infty}$, $m = 0, ..., \mu_N$ be sequences of positive real numbers depending on a positive integer N so that

$$\lim_{N\to\infty} \mu_N = \infty, \quad \lim_{N\to\infty} b_N = \infty, \quad \lim_{N\to\infty} b_{mN} = \infty, \quad m = 0,1,...,\mu_N. \tag{2.12}$$

For fixed positive number A denote by $\eta_A(\cdot)$ a cut-off function $\eta_A : [0,\infty) \to [0,1]$ defined by

$$\eta_A(x) = \begin{cases} 0, & 0 \leq x \leq A, \\ 1, & x \geq A+1, \\ x-A, & A < x < A+1. \end{cases} \tag{2.13}$$

Define functions $\chi_N : \mathbb{R} \to [0,1]$ and $\chi_{mN} : \mathbb{R} \to [0,1], \quad m = 0,1,...,\mu_N$ assuming

$$\chi_N(x) = \eta_{b_N}(|x|), \quad \chi_{mN}(x) = \eta_{b_{mN}}(|x|),$$

$$\chi'_N(x) = 1-\chi_N(x), \quad \chi'_{mN}(x) = 1-\chi_{mN}(x), \quad m = 0,1,...,\mu_N\,.$$

Under FWM of the random process $\mathbf{u}(x)$ we understand the following sequence of random functions

$$\begin{aligned} \mathbf{u}_N(x) = \sum_{j\in\mathbb{Z}} \quad & \chi'_N(2^{m_0}x+j)\cdot\mathcal{F}_{m_0}^{(\phi)}(2^{m_0}x+j)\,\xi_j \\ + \sum_{m=m_0}^{\mu_N} \sum_{j\in\mathbb{Z}} \quad & \chi'_{mN}(2^m x+j)\cdot\mathcal{F}_m^{(\psi)}(2^m x+j)\,\xi_{mj}\,, \quad N = 1,2,\dots. \end{aligned} \tag{2.14}$$

Our aim is to study the convergence $\mathbf{u}_N \to \mathbf{u}$ as $N \to \infty$ in functional spaces $W_2^n[a,b]$ and $C^n[a,b]$ where n is a nonnegative integer. Here we suggest sufficient conditions on the spectral tensor F which ensure $E||\mathcal{E}_N||^2_{W_2^n[a,b]} \to 0$ and $||\mathcal{E}_N||_{C^n[a,b]} \xrightarrow{P} 0$ as $N \to \infty$. Here $[a,b] \subset \mathbf{R}$ is an arbitrary fixed finite interval, and $\mathcal{E}_N(x) = u(x) - u_N(x)$. Further, not loosing in generality, we can take $[a,b] = [0,1]$, and $m_0 = 0$. Moreover, for the simplicity of presentation we first consider the case of a scalar random process $u(x)$ (i.e., $l = 1$) and give sufficient conditions for the convergence $E||\mathcal{E}_N||^2_{L_2[a,b]} \to 0$ and $||\mathcal{E}_N||_{C[a,b]} \xrightarrow{P} 0$ as $N \to \infty$. In section 6 we give relevant generalizations of results for vector processes and stronger metrics.

3 Convergence of FWM in L_2 metric

Let $u(x), x \in \mathbb{R}$ be a real-valued scalar stationary zero mean Gaussian random process with a spectral function $F(k)$, and $\phi(x)$, $\psi(x)$ be orthonormal Meyer's scaling and wavelet functions. It is clear that their Fourier transforms $\hat{\phi}(k)$, $\hat{\psi}(k)$ are even functions. In this case FWM of $u(x)$ is (see (2.14), recall that $m_0 = 0$):

$$\begin{aligned} u_N(x) \quad &= \quad \sum_{j\in\mathbb{Z}} \chi'_N(x+j)\cdot\mathcal{F}_0^{(\phi)}(x+j)\,\xi_j \\ &+ \quad \sum_{m=0}^{\mu_N} \quad \sum_{j\in\mathbb{Z}} \chi'_{mN}(2^m x+j)\cdot\mathcal{F}_m^{(\psi)}(2^m x+j)\,\xi_{mj}\,, \end{aligned} \tag{3.1}$$

where $\xi_j, \xi_{mj}, m = 0, 1, ...; j \in \mathbb{Z}$ is a set of mutually independent standard Gaussian random variables.

Let us introduce some notations.
We denote $\Delta = [-4/3, 4/3]$, $\Delta_m = [-4 \cdot 2^m/3, -2^m/3] \cup [2^m/3, 4 \cdot 2^m/3]$, $m = 0, 1, 2, ...$, and $\Delta' = \cup_{m=0}^{\infty} \Delta_m$. For a function $f : \mathbb{R} \to \mathbb{R}$ and $D \subset \mathbb{R}$ we denote by $f|_D$ the restriction of the function f in D. For a measurable $D \subset \mathbb{R}$ we write $|D| = \int_D dx$.

In this paper we use some results from the theory of Nikolskiĭ-Besov spaces (e.g., see [29], [39]). For $r > 0$, a triple (r, j, l) is called admissible if $j \in N$, $l \in N_0$ and $j > r - l > 0$. Here $N_0 = \{0, 1, 2, \ldots\}$ and $N = \{1, 2, \ldots\}$. Let us denote by $\Delta_h^{(j)} g$ the j-th difference of g:

$$\Delta_h g(\cdot) = g(\cdot + h) - g(\cdot), \ldots, \Delta_h^{(j)} g(\cdot) = \Delta_h \Delta_h^{(j-1)} g(\cdot).$$

For $1 \le p, q \le \infty$, $r > 0$, Nikolskiĭ-Besov space $B_{pq}^r(R)$ is defined as a set of all functions $f \in L_p$ such that the norm

$$||f||_{B_{pq}^r} = ||f||_{L_p} + ||f||_{b_{pq}^r},$$

where

$$||f||_{b_{pq}^r} = \left(\int_{-1}^{1} \left(\frac{||\Delta_h^{(j)} f^{(l)}||_{L_p}}{|h|^{r-l}} \right)^q \frac{dh}{|h|} \right)^{\frac{1}{q}}, \quad 1 \le q < \infty, \tag{3.2}$$

$$||f||_{b_{p\infty}^r} = \sup_{0 < |h| \le 1} |h|^{l-r} ||\Delta_h^{(j)} f^{(l)}||_{L_p} \tag{3.3}$$

makes sense and is finite for some admissible triple (r, j, l). Here $f^{(l)}(x) = D^l f(x)$ is the l-th derivative of the function f. The ambiguity in the choice of triple (r, j, l) is not essential: different admissible triples correspond to equivalent norms. For a measurable subset $D \subset \mathbb{R}$ the space $B_{pq}^r(D)$ is defined as above but changing $||\cdot||_{L_p}$ with $||\cdot||_{L_p(D_{jh})}$, where $D_h = \{x \in D : x + \lambda h \in D \text{ for all } \lambda \in [0,1]\}$. For a function $f : \mathbb{R} \to \mathbb{R}$ such that $f|_D \in B_{pq}^r(D)$ we will simply write $||f||_{B_{pq}^r(D)}$ instead of more exact but complex notation $||(f|_D)||_{B_{pq}^r(D)}$.

We will exploit the following imbeddings (e.g., see [29], [39]):

$$B_{pq}^r(R) \hookrightarrow B_{p_1 q}^{\rho}(R), \quad \rho = r \left(1 - \frac{1}{r} \cdot \left[\frac{1}{p} - \frac{1}{p_1} \right] \right), \quad 1 \le p < p_1 < \infty, \tag{3.4}$$

$$B_{p\infty}^{r+\varepsilon}(R) \hookrightarrow B_{pq}^r(R) \hookrightarrow B_{pq_1}^r(R), \quad 1 \le q < q_1 \le \infty, \quad \varepsilon > 0, \tag{3.5}$$

where $X \hookrightarrow Y$ means that $X \subset Y$ for seminormed spaces X and Y, and there exists a constant $c > 0$ such that the inequality $||x||_Y \le c||x||_X$ is fulfilled.

3.1 Convergence of FWM in the mean square

We shall assume
(H0) for some $s > 0$

$$I_s \stackrel{def}{=} \int_{-\infty}^{\infty} F(k)(1 + |k|^2)^s dk < \infty. \tag{3.6}$$

Theorem 1. *Assume, that the spectral function $F(k)$ satisfies the condition (H0). Let $\{\rho_m\}_{m=0}^{\mu_N}$ be a finite sequence of positive numbers such that $\min_m \rho_m > 1/2$. Suppose that $Q|_\Delta \in B_{1\infty}^{\rho_0}(\Delta)$ and $Q|_{\Delta_m} \in B_{1\infty}^{\rho_m}(\Delta_m)$ for $1 \le m \le \mu_N$, and assume that the Meyer wavelet functions ϕ and ψ belong to the class $C^{\nu+1}$ for $\nu = \max\{\lfloor\rho\rfloor, \lfloor s\rfloor\}$, where $\rho = \max_{0\le m\le \mu_N} \rho_m$. Then*

$$\begin{aligned}E(u(x)-u_N(x))^2 \quad &\le \quad \frac{C_1(s,\psi)}{4^{s(\mu_N+1)}} \cdot I_s + \frac{C_2(\rho_0,\phi)}{b_N^{2\rho_0-1}} ||Q||^2_{B^{\rho_0}_{1\infty}(\Delta)} \\ + \sum_{m=0}^{\mu_N} \frac{C_3(\rho_m,\psi)}{b_{mN}^{2\rho_m-1}} \quad &\cdot \quad \left(2^{-m}||Q||^2_{L_1(\Delta_m)} + 2^{2m(\rho_m-1/2)}||Q||^2_{b^{\rho_m}_{1\infty}(\Delta_m)}\right), \end{aligned} \tag{3.7}$$

where constants C_i, $i = 1,2,3$ depend only on the shown arguments.

Proof. From the definition (3.1) we have:

$$u(x) = u_N(x) + v^{(1)}(x) + \sum_{m=0}^{\mu_N} v_m^{(2)}(x) + v^{(3)}(x), \tag{3.8}$$

where

$$\begin{aligned} v^{(1)}(x) = &\quad \sum_{j\in\mathbb{Z}} \chi_N(x+j)\cdot \mathcal{F}_0^{(\phi)}(x+j)\,\xi_j\,, \\ v_m^{(2)}(x) = &\quad \sum_{j\in\mathbb{Z}} \chi_{mN}(2^m x+j)\cdot \mathcal{F}_m^{(\psi)}(2^m x+j)\,\xi_{mj}\,, \\ v^{(3)}(x) = &\quad \sum_{m=\mu_N+1}^{\infty} \quad \sum_{j=-\infty}^{\infty} \mathcal{F}_m^{(\psi)}(2^m x+j)\,\xi_{mj}\,. \end{aligned}$$

The random variables ξ_j, ξ_{mj} are mutually independent, hence the terms in the right-hand side of (3.8) are also independent, therefore

$$E(u(x)-u_N(x))^2 = E(v^{(1)}(x))^2 + \sum_{m=0}^{\mu_N} E(v_m^{(2)}(x))^2 + E(v^{(3)}(x))^2. \tag{3.9}$$

Let us estimate the terms in the right-hand side. First, we have for the last term

$$E(v^{(3)}(x))^2 = \sum_{m=\mu_N+1}^{\infty} \quad \sum_{j=-\infty}^{\infty} |\mathcal{F}_m^{(\psi)}(2^m x+j)|^2. \tag{3.10}$$

Further, from the definition (2.6) of the function $\mathcal{F}$, and since the functions $Q(k)$ and $\psi(k)$ are even we have

$$\mathcal{F}_m^{(\psi)}(y) = \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi k y} 2^{m/2} \bar{Q}(2^m k)\hat{\psi}(k)dk = \int_{-\infty}^{\infty} e^{\mathrm{i}2\pi k y} 2^{m/2} Q(2^m k)\bar{\hat{\psi}}(k)dk.$$

Therefore,

$$\begin{aligned} \mathcal{F}_m^{(\psi)}(2^m x+j) \quad = \quad &\int_{-\infty}^{\infty} e^{\mathrm{i}2\pi k(2^m x+j)} 2^{m/2} Q(2^m k)\bar{\hat{\psi}}(k)dk = \\ &\int_{-\infty}^{\infty} e^{\mathrm{i}2\pi k x} Q(k)\bar{\hat{\psi}}_{mj}(k)\,dk = \int_{-\infty}^{\infty} G(x+y)\psi_{mj}(y)dy, \end{aligned} \tag{3.11}$$

where

$$G(x)=\int_{-\infty}^{\infty} e^{\mathrm{i}2\pi kx}Q(k)\,dk,$$

$$\psi_{mj}(x)=2^{m/2}\psi(2^m x-j),\quad \hat{\psi}_{mj}(k)=2^{-m/2}\,e^{-\mathrm{i}2\pi kj2^{-m}}\,\hat{\psi}(2^{-m}k).$$

Consequently for fixed x, the quantities $\mathcal{F}_m^{(\psi)}(2^m x+j)$ $(m,j\in\mathbb{Z})$ are Fourier coefficients in the expansion of $G(x+\cdot)$ with respect to the orthonormal system ψ_{mj} $(m,j\in\mathbb{Z})$.

Notice that the condition (3.6) can be formulated equivalently that the function G belongs to the Sobolev space $H_2^s(R)$. This is because $G\in H_2^s(R)$ means that the function $Q(k)(1+|k|^2)^{s/2}$ is from $L_2(R)$) and $||G||_{H_2^s}=I_s^{1/2}$(more details in [39]). Further we use the fact that $H_2^s(R)$ coincides with the Besov space $B_{22}^s(R)$, and the norms in these spaces are equivalent (see [39], section 2.3.9). Since $G\in B_{22}^s$, we conclude by virtue of Corollary A3 that

$$\sum_{j=-\infty}^{\infty}|\mathcal{F}_m^{(\psi)}(2^m x+j)|^2\le B_s^2\,4^{-ms}\cdot||G(x+\cdot)||^2_{H_2^s} \tag{3.12}$$

for some $B_s=B_s(\psi)$ depending only on s and ψ. From this we get by $||G(x+\cdot)||^2_{H_2^s}=||G||^2_{H_2^s}=I_s$ that

$$E(v^{(3)}(x))^2=\sum_{m=\mu_N+1}^{\infty}\quad\sum_{j=-\infty}^{\infty}|\mathcal{F}_m^{(\psi)}(2^m x+j)|^2\le\frac{B_s^2}{4^{(\mu_N+1)s}}I_s. \tag{3.13}$$

Now we turn to the estimation of the first two terms in the right- hand side of (3.9). It is obvious, due to independency of random variables $\xi_j,\ j\in\mathbb{Z}$ that

$$E(v^{(1)}(x))^2=\sum_{j\in\mathbb{Z}}|\chi_N(x+j)|^2\cdot|\mathcal{F}_0^{(\phi)}(x+j)|^2\le\sum_{j\in\mathbb{Z}:\,|x+j|\ge b_N}|\mathcal{F}_0^{(\phi)}(x+j)|^2. \tag{3.14}$$

We first estimate each term of this sum. The function $\mathcal{F}_0^{(\phi)}(y)$ has the Fourier transform $\hat{\mathcal{F}}(k)=Q(-k)\hat{\phi}(-k)=Q(k)\hat{\phi}(-k)$ (see (2.6)). Due to the conditions $\phi\in C^{\nu+1}$ and $Q|_\Delta\in B_{1\infty}^{\rho_0}(\Delta)$ it follows from Corollary A2 that $||\hat{\mathcal{F}}||_{B_{1\infty}^{\rho_0}(\mathbb{R})}\le C(\rho_0,\phi)||Q||_{B_{1\infty}^{\rho_0}(\Delta)}$. Then from Lemma A1 we get

$$|\mathcal{F}_0^{(\phi)}(y)|\le\frac{C(\rho_0,\phi)}{|y|^{\rho_0}}\cdot||\hat{\mathcal{F}}||_{B_{1\infty}^{\rho_0}(\mathbb{R})}\le\frac{C(\rho_0,\phi)}{|y|^{\rho_0}}\cdot||Q||_{B_{1\infty}^{\rho_0}(\Delta)}. \tag{3.15}$$

Note that from (3.14) and (3.15) we get

$$E(v^{(1)}(x))^2\le\frac{C(\rho_0,\phi)}{b_N^{2\rho_0-1}}\cdot||Q||^2_{B_{1\infty}^{\rho_0}(\Delta)}. \tag{3.16}$$

Let us turn to estimation of the terms $E(v_m^{(2)}(x))^2$, $m=0,...,\mu_N$. Denote by Q_m the function $Q_m:\Delta_0\to\mathbb{R}$ defined as $Q_m(k)=Q(2^m k)$. Then by the same arguments as we used in the derivation of estimation (3.16), we obtain

$$\begin{aligned}E(v_m^{(2)}(x))^2\ &\le\ \sum_{j\in\mathbb{Z}:\,|2^m x+j|\ge b_{mN}}|\mathcal{F}_m^{(\psi)}(2^m x+j)|^2\le\frac{C(\rho_m,\psi)}{b_{mN}^{2\rho_m-1}}\cdot 2^m||Q_m||^2_{B_{1\infty}^{\rho_m}(\Delta_0)}\\ &\le\frac{C(\rho_m,\psi)}{b_{mN}^{2\rho_m-1}}\cdot 2^m\cdot\left(2||Q_m||^2_{L_1(\Delta_0)}+||Q_m||^2_{b_{1\infty}^{\rho_m}(\Delta_0)}\right).\end{aligned} \tag{3.17}$$

Put $l = \lfloor \rho_m \rfloor - 1$. Then

$$||Q_m||_{b^{\rho_m}_{1\infty}(\Delta_0)} = \sup_{|h|\le 1} \frac{1}{|h|^{\rho_m - l}} \int\limits_{\Delta_{0,2h}} |\Delta^{(2)}_h Q^{(l)}_m(k)|\, dk. \tag{3.18}$$

Define a function $g_h : \Delta_m \to \mathbb{R}$ by $g_h(k) = \Delta^{(2)}_h Q^{(l)}(k)$. Then $\Delta^{(2)}_h Q^{(l)}_m(k) = 2^{ml} g_{2^m h}(2^m k)$. Therefore,

$$\begin{aligned} ||Q_m||_{b^{\rho_m}_{1\infty}(\Delta_0)} &= \sup_{|h|\le 1} \frac{1}{|h|^{\rho_m - l}} \int\limits_{\Delta_{0,2h}} 2^{ml} |g_{2^m h}(2^m k)|\, dk \\ &= \sup_{|h|\le 1} \frac{2^{-m}}{|h|^{\rho_m - l}} \int\limits_{\Delta_{m,2^{m+1}h}} 2^{ml} |g_{2^m h}(k)|\, dk. \end{aligned} \tag{3.19}$$

By the definition we have

$$\int\limits_{\Delta_{m,2h}} |g_h(k)|\, dk = \int\limits_{\Delta_{m,2h}} |\Delta^{(2)}_h Q^{(l)}(k)| \le ||Q||_{b^{\rho_m}_{1\infty}(\Delta_m)} |h|^{\rho_m - l}, \quad h \in \mathbb{R}. \tag{3.20}$$

Therefore it follows from (3.19) and (3.20) that

$$||Q_m||_{b^{\rho_m}_{1\infty}(\Delta_0)} \le 2^{m(\rho_m - 1)} ||Q||_{b^{\rho_m}_{1\infty}(\Delta_m)}. \tag{3.21}$$

Now taking into account that $||Q_m||_{L_1(\Delta_0)} \le 2^{-m} \cdot ||Q||_{L_1(\Delta_m)}$ and using the estimates (3.17) and (3.21) we obtain

$$E(v^{(2)}_m(x))^2 \le \frac{2C(\rho_m, \psi)}{b^{2\rho_m - 1}_{mN}} \cdot \left(2^{-m} ||Q||^2_{L_1(\Delta_m)} + 2^{m(2\rho_m - 1)} ||Q||^2_{b^{\rho_m}_{1\infty}(\Delta_m)} \right) \tag{3.22}$$

for $m = 0, 1, ..., \mu_N$. This completes the proof of Theorem 1. $\square$

We shall assume

(H1) there exist positive constants c_0, ε, $\rho_0 > 1/2$, $\rho_1 > 1/2$ and positive integer m_0 such that

$$\begin{aligned} &\text{(i)} && Q|_\Delta \in B^{\rho_0}_{1\infty}(\Delta), Q|_{\Delta_0} \in B^{\rho_0}_{1\infty}(\Delta_0), \ldots, Q|_{\Delta_{m_0-1}} \in B^{\rho_0}_{1\infty}(\Delta_{m_0-1}); \\ &\text{(ii)} && Q|_{\Delta_m} \in B^{\rho_1}_{1\infty}(\Delta_m), \quad m \ge m_0; \\ &\text{(iii)} && ||Q||_{b^{\rho_1}_{1\infty}(\Delta_m)} \le c_0 \cdot 2^{-m(\rho_1 - 1/2 + \varepsilon)}, \quad m \ge m_0. \end{aligned} \tag{3.23}$$

Proposition 1. *Assume*

$$\lim_{N\to\infty} \mu_N = \infty, \quad \lim_{N\to\infty} b_N = \infty, \quad \lim_{N\to\infty} \min_{0\le m\le \mu_N} \{b_{mN}\} = \infty. \tag{3.24}$$

Then under the assumptions (H0)-(H1),

$$\sup_{x\in[0,1]} E\mathcal{E}^2_N(x) \to 0 \quad as \quad N \to \infty. \tag{3.25}$$

Proof. Indeed, as a simple consequence of Theorem 1 we have

$$\begin{aligned} \sup_{x\in[0,1]} E\mathcal{E}^2_N(x) \le{}& \frac{C_1(s,\psi)}{4^{s(\mu_N+1)}} \cdot I_s + \frac{C_2(\rho_0, \phi)}{b^{2\rho_0 - 1}_N} ||Q||^2_{B^{\rho_0}_{1\infty}(\Delta)} \\ &+ \sum_{m=0}^{m_0-1} \frac{C_3(\rho_0, \psi)}{b^{2\rho_0 - 1}_{mN}} \cdot \left(2^{-m} ||Q||^2_{L_1(\Delta_m)} + 2^{2m(\rho_0 - 1/2)} ||Q||^2_{b^{\rho_0}_{1\infty}(\Delta_m)} \right) \\ &+ \sum_{m=m_0}^{\mu_N} \frac{C_3(\rho_1, \psi)}{b^{2\rho_1 - 1}_{mN}} \cdot \left(2^{-m} ||Q||^2_{L_1(\Delta_m)} + 2^{2m(\rho_1 - 1/2)} ||Q||^2_{b^{\rho_1}_{1\infty}(\Delta_m)} \right). \end{aligned} \tag{3.26}$$

To show (3.25), it is sufficient to check that the last sum in the r.h.s of this inequality converges to zero as $N \to \infty$. For this sake, for given $\delta > 0$, let us choose $N_0 = N_0(\delta)$ so that

$$\min_{0 \le m \le \mu_N} \frac{1}{b_{mN}^{2\rho_1 - 1}} \le \delta \quad N \ge N_0.$$

Denote by S_N the lust sum in r.h.s. of (3.26). Then due to the condition (3.23) we get for $N \ge N_0$

$$S_N \le C_3(\rho_1, \psi) \cdot \delta \cdot \left(\sum_{m=0}^{\infty} 2^{-m} ||Q||^2_{L_1(\Delta_m)} + c_0 \cdot \sum_{m=0}^{\infty} 2^{-2\varepsilon m} \right).$$

From

$$||Q||^2_{L_1(\Delta_m)} \le |\Delta_m| \cdot \int_{\Delta_m} F(k)\, dk,$$

and due to the assumption (H0) we get

$$||Q||^2_{L_1(\Delta_m)} \le |\Delta_m| (3 \cdot 2^{-m})^{2s} \cdot I_s = 2 \cdot 2^m (3 \cdot 2^{-m})^{2s} \cdot I_s.$$

This implies that $S_N \le C \cdot \delta$ for $N \ge N_0(\delta)$ with a constant C which is not dependent on N. Since the parameter δ can be chosen arbitrarily small, $S_N \to 0$ as $N \to \infty$. The proof of the proposition is complete. □

A straightforward consequence of Theorem 1 is the following assertion on the rate of the mean square convergence of FWM (3.1).

Corollary 1. *Assume that the hypothesis (H0)-(H1) are valid. Choose in the model (3.1)*

$$\mu_N = \left\lfloor \frac{(2\rho_0 - 1)}{2s} \cdot \log_2 N \right\rfloor + 1, \quad b_N = N, \quad b_{mN} = N, 0 \le m \le m_0 - 1;$$
$$b_{mN} = N^{\frac{2\rho_0 - 1}{2\rho_1 - 1}}, \; m_0 \le m \le \mu_N. \tag{3.27}$$

Assume, that $\phi, \psi \in C^{\nu+1}$ where $\nu = \max\{\lfloor s \rfloor, \lfloor \rho_0 \rfloor, \lfloor \rho_1 \rfloor\}$. Then for each positive integer N the following estimation is valid:

$$\sup_{x \in [0,1]} E\mathcal{E}_N^2(x) \le C \cdot N^{-(2\rho_0 - 1)} \tag{3.28}$$

where C is a constant not depending on N.

From the results given above it follows that the condition (3.23) is crucial in the analysis of the mean square convergence of FWM.

3.2 Sufficient conditions for validity of (3.23)

Proposition 2. *For a nonnegative k_0 and a nonnegative integer l, assume that $Q \in W_1^l(k_0, \infty)$, and*

$$|\Delta_h^{(2)} Q^{(l)}(k)| \le \frac{C|h|^\gamma}{k^{l+1/2+\gamma+\varepsilon}}, \quad \forall h \in [-1,1], \quad \textit{and} \quad \forall k \ge k_0 \tag{3.29}$$

for some positive numbers C, ε and $\gamma \in (0,2)$. Then the following estimate is true

$$||Q||_{b^{\rho_1}_{1\infty}(\Delta_m)} \le c_0 \cdot 2^{-m(\rho_1 - 1/2 + \varepsilon)}, \quad \textit{for all} \quad m \ge \log_2(3k_0), \tag{3.30}$$

where $\rho_1 = l+\gamma$ *and* $c_0 = c_0(\rho_1,\varepsilon)$.

Proof. Since the triple $(l+\gamma,2,l)$ is admissible,

$$||Q||_{b^{\rho_1}_{1\infty}(\Delta_m)} \le C(\rho_1)\sup_{|h|\le 1}\frac{1}{|h|^\gamma}\int_{\Delta_m}|\Delta_h^{(2)}Q^{(l)}(k)|\,dk.$$

Therefore, for $2^m > 3k_0$ we have

$$||Q||_{b^{\rho_1}_{1\infty}(\Delta_m)} \le C(\rho_1)\sup_{|h|\le 1}\frac{1}{|h|^\gamma}\int_{\Delta_m}\frac{C|h|^\gamma}{k^{l+1/2+\gamma+\varepsilon}}\,dk \le c_0(\rho_1,\varepsilon)\cdot 2^{-m(\rho_1-1/2+\varepsilon)}.$$

Corollary 2. *Assume that for some nonnegative* k_0 *and positive integer* n, $Q\in W_1^n(k_0,\infty)$, *and*

$$|Q^{(n)}(k)| \le \frac{C}{k^{n+1/2+\varepsilon}}, \quad \forall k\ge k_0 \tag{3.31}$$

for some positive numbers C,ε. *Then the following estimate is valid*

$$||Q||_{b^{n}_{1\infty}(\Delta_m)} \le c_0\cdot 2^{-m(n-1/2+\varepsilon)}, \quad \textit{for all} \quad m\ge \log_2(3(k_0+2)), \tag{3.32}$$

for some positive $c_0 = c_0(n,\varepsilon)$.

Indeed,

$$|\Delta_h^{(2)}Q^{(n-1)}(k)| \le |\Delta_h Q^{(n-1)}(k+h)| + |\Delta_h Q^{(n-1)}(k+h)| \le h\cdot|Q^{(n)}(k+\beta h)| + h\cdot|Q^{(n)}(k+\alpha h)|$$

for some $\alpha\in[0,1]$ and $\beta\in[1,2]$. Hence, by virtue of (3.31)

$$|\Delta_h^{(2)}Q^{(n-1)}(k)| \le \frac{C_1 h}{k^{n+1/2+\varepsilon}}, \quad \text{for all} \quad k\ge k_0+2\,.$$

Therefore for $l=n-1$ and $\gamma=1$ all the conditions of Proposition 2 are fulfilled. $\square$

Remark. Conditions (3.29) and (3.31) can be replaced with the following weaker conditions:

$$\int_{k_0}^{\infty} k^{l+\gamma-1/2+\varepsilon}\cdot\frac{|\Delta_h^{(2)}Q^{(l)}(k)|}{|h|^\gamma}\cdot dk < \infty, \tag{3.33}$$

and

$$\int_{k_0}^{\infty} k^{n-1/2+\varepsilon}|Q^{(n)}(k)|\,dk < \infty, \tag{3.34}$$

respectively.

3.3 Cost estimations for FWM

Now let us discuss the efficiency of FWM. Let us denote by T_N the number of arithmetic operations for calculation of the value of FWM (3.1) in one point $x\in\mathbb{R}$. It is obvious that $T_N \sim b_N + \sum_{m=0}^{\mu_N} b_{mN}$. Here $T_N\sim a_N$ means that there exist constants $0<C_1\le C_2$ not depending on N, such that $C_1 a_N \le T_N \le C_2\cdot a_N$.

Therefore, if we choose the parameters of the FWM in accordance with (3.27), then under the conditions of Corollary 1 it follows that

$$T_N \sim N \qquad \text{if} \quad \rho_1 > \rho_0; \tag{3.35}$$

$$T_N \sim N \ln N \qquad \text{if} \quad \rho_1 = \rho_0; \tag{3.36}$$

$$T_N \sim N^{(2\rho_0-1)/(2\rho_1-1)} \ln N \qquad \text{if} \quad \rho_1 < \rho_0. \tag{3.37}$$

Let us denote

$$\varepsilon_N = \left(\int_0^1 E(u(x) - u_N(x))^2 dx\right)^{1/2} = \left(E||\mathcal{E}_N||^2_{L_2[0,1]}\right)^{1/2},$$

the root mean square (r.m.s.) discrepancy of FWM (3.1) in the metric of $L_2[0,1]$. Then under the conditions of Corollary 1 the following estimation holds:

$$\varepsilon_N \leq C \cdot N^{-(\rho_0-1/2)}, \qquad N = 1,2,\ldots \tag{3.38}$$

where C is some positive constant not depending on N. Hence under the conditions of Corollary 1 the number of operations, T_ε, to achieve a given value of the r.m.s. error $\varepsilon_N = \varepsilon$, satisfies the estimation

$$T_\varepsilon \lesssim \varepsilon^{-1/(\rho_0-1/2)} \qquad \text{if} \quad \rho_1 > \rho_0; \tag{3.39}$$

$$T_\varepsilon \lesssim \varepsilon^{-1/(\rho_0-1/2)} \ln \varepsilon \qquad \text{if} \quad \rho_1 = \rho_0; \tag{3.40}$$

$$T_\varepsilon \lesssim \varepsilon^{-1/(\rho_1-1/2)} \ln \varepsilon \qquad \text{if} \quad \rho_1 < \rho_0. \tag{3.41}$$

Here $T_\varepsilon \lesssim a(\varepsilon)$ means that there exists a constant C not depending on ε, such that $T_\varepsilon \leq C \cdot a(\varepsilon)$.

4 Analysis of the estimations (3.16)-(3.17)

In this section we show that the exponents $2\rho_0 - 1$ and $2\rho_m - 1$ appearing in the estimations (3.16)-(3.17) cannot be improved in Nikolskiĭ-Besov spaces $B^{\rho_0}_{1\infty}$ and $B^{\rho_m}_{1\infty}$, respectively. This will be done by construction of a function Q which makes it possible to write $E(v^{(1)}(x))^2 \sim b_N^{-(2\rho_0-1)}$ and $E(v^{(2)}(x))^2 \sim b_{mN}^{-(2\rho_m-1)}$.

Let us consider the following example.

Example 1. Let $k_0 \geq 0$ and $\rho > 0$. Define

$$Q_0(k) = \begin{cases} 0, & 0 \leq k \leq k_0, \\ (k-k_0)^{\rho-1}, & k > k_0. \end{cases} \tag{4.1}$$

$$Q_0(k) = Q_0(-k) \quad \text{if} \quad k < 0.$$

Let $q : \mathbb{R} \to [0,\infty)$ be an arbitrary $(l+1)$ -times differentiable even function with compact support satisfying the condition $q(k_0) \neq 0$. Within this section we denote $l = \lfloor \rho \rfloor$. Define $Q(k) = q(k)Q_0(k)$. By the definition it follows that $Q \in B^{\rho}_{1\infty}$.

Now let us consider asymptotic behavior of functions $\mathcal{F}_0^{(\phi)}(y)$, $\mathcal{F}_m^{(\psi)}(y)$ as $y \to \infty$. We will separately consider two cases: (i) ρ is noninteger and (ii) ρ is integer.

4.1 Asymptotics of $\mathcal{F}_0^{(\phi)}$ and $\mathcal{F}_m^{(\psi)}$ for a noninteger ρ

Let us first assume that $k_0 < 2/3$, $\hat{\phi} \in C^l$, and consider the function $\mathcal{F}_0^{(\phi)}(y)$. If we denote $g = q\hat{\phi}$ then

$$\mathcal{F}_0^{(\phi)}(y) = \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi ky} \cdot Q(k)\hat{\phi}(k)\,dk = \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi ky} \cdot Q_0(k)g(k)\,dk$$

$$= \frac{1}{(-\mathrm{i}2\pi y)^l} \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi ky} \cdot (Q_0 g)^{(l)}(k)\,dk = \frac{1}{(-\mathrm{i}2\pi y)^l} \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi ky} \cdot [Q_0^{(l)}(k)g(k) + \mathcal{R}(k)]\,dk,$$

where $\mathcal{R}(k) = \sum_{j=0}^{l-1} C_l^j Q_0^{(j)}(k) g^{(l-j)}(k)$. It is obvious that $\mathcal{R}$ is a function from $B^1_{1\infty}$. Therefore, due to Lemma A1 the integral

$$R(y) = \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi ky} \cdot \mathcal{R}(k)\,dk$$

satisfies the following estimation

$$|R(y)| \leq \frac{A_1}{1+|y|}, \qquad y \in \mathbb{R}.$$

Now let us consider the integral

$$F_0(y) = \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi ky} \cdot Q_0^{(l)}(k)g(k)\,dk$$

$$= \begin{cases} 2 \cdot \int_0^{\infty} \cos(2\pi ky) \cdot Q_0^{(l)}(k) \cdot g(k)\,dk, & \text{if } \; l \text{ is even,} \\ -2\mathrm{i} \cdot \int_0^{\infty} \sin(2\pi ky) \cdot Q_0^{(l)}(k) \cdot g(k)\,dk, & \text{if } \; l \text{ is odd.} \end{cases} \tag{4.2}$$

Let us assume that l is even. Then,

$$F_0(y) = 2C_1 \cos(2\pi k_0 y) F_c(2\pi y) - 2C_1 \sin(2\pi k_0 y) F_s(2\pi y),$$

where $C_1 = C_1(\rho) = (l-\alpha)\cdot(l-1-\alpha)\cdot\ldots\cdot(1-\alpha)$, $\alpha = 1-\rho+l$, and

$$F_c(x) = \int_0^{\infty} \cos(kx) \cdot f(k)\,dk, \quad F_s(x) = \int_0^{\infty} \sin(kx) \cdot f(k)\,dk \tag{4.3}$$

are the respective Fourier cosine and sine transformations of the function $f(k) = g(k+k_0)k^{-\alpha}$. Now we will use the following assertion:

Theorem ([38], Theorem 126). *Let $f(k) = k^{-\alpha}\varphi(k)$, where $0 < \alpha < 1$, and $\varphi(k)$ is of bounded variation in $(0,\infty)$. Let F_c and F_s are Fourier sine and cosine transformations of f. Then*

$$F_c(x) = \varphi(+0) \cdot \left(\frac{2}{\pi}\right)^{1/2} \Gamma(1-\alpha)\sin(\pi\alpha/2) \cdot x^{\alpha-1} \cdot (1+o(1)) \quad \text{as} \quad x \to \infty$$

and

$$F_c(x) = \varphi(\infty) \cdot \left(\frac{2}{\pi}\right)^{1/2} \Gamma(1-\alpha)\sin(\pi\alpha/2) \cdot x^{\alpha-1} \cdot (1+o(1)) \quad \text{as} \quad x \to 0.$$

$F_s(x)$ *satisfies similar conditions with* $\sin(\pi\alpha/2)$ *replaced by* $\cos(\pi\alpha/2)$.

For even l it follows from this theorem that

$$F_0(y) = C_2(\rho)g(k_0)\cdot\frac{\sin(\pi\alpha/2 - 2\pi k_0 y)}{y^{1-\alpha}}\cdot(1+o(1)), \quad \text{as} \quad y\to\infty. \tag{4.4}$$

For odd l it can be obtained a similar asymptotic:

$$F_0(y) = -\mathrm{i}\cdot C_2(\rho)g(k_0)\cdot\frac{\cos(\pi\alpha/2 + 2\pi k_0 y)}{y^{1-\alpha}}\cdot(1+o(1)), \quad \text{as} \quad y\to\infty. \tag{4.5}$$

Thus we have obtained the following asymptotical result for $\mathcal{F}_0^{(\phi)}$:

$$\mathcal{F}_0^{(\phi)}(y) = C_3(\rho)\cdot\Phi(k_0 y)\cdot\frac{q(k_0)\cdot\hat{\phi}(k_0)}{y^{\rho}}\cdot(1+o(1)), \quad \text{as} \quad y\to\infty \tag{4.6}$$

where $\Phi(x) = \sin(\pi\alpha/2 - 2\pi x)$ if $l = \lfloor\rho\rfloor$ is even, and $\Phi(x) = \cos(\pi\alpha/2 + 2\pi x)$ if l is odd.

Recall that we have assumed that $k_0 \in [0, 2/3]$. If $k_0 \geq 2/3$ then $\mathcal{F}_0^{(\phi)} \equiv 0$. By the same arguments it can be shown that for m such that $2^m < 3k_0 < 2^{m+2}$ the following asymptotic is valid

$$\mathcal{F}_m^{(\psi)}(y) = C_3(\rho)\cdot 2^{m(\rho-1/2)}\Phi(2^{-m}k_0 y)\cdot\frac{q(2^{-m}k_0)\cdot\hat{\psi}(2^{-m}k_0)}{y^{\rho}}\cdot(1+o(1)) \tag{4.7}$$

as $\quad y\to\infty$, provided $q(2^{-m}k_0) \neq 0$.

4.2 Asymptotics of $\mathcal{F}_0^{(\phi)}$ and $\mathcal{F}_m^{(\psi)}$ for an integer ρ

Let $l = \rho = 1$. Then

$$\begin{aligned} \mathcal{F}_0^{(\phi)}(y) &= 2\int_{k_0}^{\infty}\cos(2\pi ky)g(k)\,dk = \frac{1}{\pi y}\int_{k_0}^{\infty} g(y)\cdot d(\sin(2\pi ky)) \\ &= -\frac{g(k_0)}{\pi y}\cdot(1+O(1/|y|)), \quad \text{as} \quad y\to\infty. \end{aligned} \tag{4.8}$$

If $l \geq 2$ then

$$\mathcal{F}_0^{(\phi)}(y) = \frac{1}{(-\mathrm{i}2\pi y)^{l-1}}\int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi ky}\cdot[Q_0^{(l-1)}(k)g(k) + \mathcal{R}(k)]\,dk,$$

where $\mathcal{R} \in B^2_{1\infty}$. Therefore

$$\mathcal{F}_0^{(\phi)}(y) = \frac{1}{(-\mathrm{i}2\pi y)^{l-1}}\cdot(F_0(y) + O(1/|y|^2), \quad \text{as} \quad y\to\infty$$

where

$$\begin{aligned} F_0(y) &= \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi ky}\cdot Q_0^{(l-1)}(k)g(k)\,dk \\ &= \begin{cases} 2\cdot\int\limits_{k_0}^{\infty}\cos(2\pi ky)\cdot g(k)\,dk, & \text{if} \quad l \text{ is odd,} \\ -2\mathrm{i}\cdot\int\limits_{k_0}^{\infty}\sin(2\pi ky)\cdot g(k)\,dk, & \text{if} \quad l \text{ is even.} \end{cases} \end{aligned} \tag{4.9}$$

Using the integration by part and using Lemma A1 we can easily show that

$$\mathcal{F}_0^{(\phi)}(y) = C(\rho)\cdot\Phi(k_0 y)\cdot\frac{q(k_0)\cdot\hat{\phi}(k_0)}{y^l}\cdot(1+o(1)), \quad \text{as} \quad y\to\infty \tag{4.10}$$

where $\Phi(x) = \cos(2\pi x)$ if $l = \rho$ is even, and $\Phi(x) = \sin(2\pi x)$ if l is odd.

By the same arguments can be established that (4.7) is valid in the case of integer ρ with $\Phi(x) = \cos(2\pi x)$ if $l = \rho$ is even, and $\Phi(x) = \sin(2\pi x)$ if l is odd.

4.3 Lower bounds for the estimations (3.16)-(3.17)

Now let us turn to the estimations (3.16)-(3.17). Assume that ρ is not integer, and $l = \lfloor\rho\rfloor$ is even. Let $k_0 = 1/2$. Then, by virtue of (4.6) we get for some positive integer N_0 and for each $N \geq N_0$

$$\begin{aligned} E\,(v^{(1)}(x))^2 \quad &\geq \sum_{j\in\mathbb{Z}:\,|x+j|\geq b_N+1} |\mathcal{F}_0^{(\phi)}(x+j)|^2 \\ &\geq C(\rho)\cdot\sum_{j\in\mathbb{Z}:\,|x+j|\geq b_N+1} \sin^2(\pi\alpha/2-\pi\cdot(x+j))\cdot\frac{1}{|x+j|^{2\rho}} \\ &\geq C_1(\rho)\sin^2(\pi\alpha/2-\pi x)\frac{1}{b_N^{2\rho-1}}, \end{aligned} \tag{4.11}$$

where $C_1(\rho) > 0$. By the same way similar estimation can be established for odd l and noninteger ρ. For integer ρ one can use (4.10) and establish analogous result.

If, for nonnegative m_0, one put $k_0 = 2^{m_0-1}$ then by (4.7)

$$\begin{aligned} E\,(v_{m_0}^{(2)}(x))^2 \quad \geq \quad & C(\rho)\sin^2(\pi\alpha/2-\pi 2^{m_0}x)2^{2m_0(\rho-1/2)} \sum_{j\in\mathbb{Z}:\,|2^{m_0}x+j|\geq b_{m_0N}+1} \frac{1}{|2^{m_0}x+j|^{2\rho}} \\ &\geq C_1(\rho)\sin^2(\pi\alpha/2-\pi 2^{m_0}x)2^{2m_0(\rho-1/2)}\frac{1}{b_{m_0N}^{2\rho-1}}, \quad N\geq N_0 \end{aligned} \tag{4.12}$$

for noninteger ρ with even $l = \lfloor\rho\rfloor$. For other values of ρ ($\lfloor\rho\rfloor$ is odd or ρ is integer) similar estimations can be obtained by analogous way.

Therefore in Nikolskiĭ-Besov classes $B_{1\infty}^{\rho_m}$, $m = 0,1,\ldots,\mu_N$, exponents $2\rho_m - 1$, $m = 0,1,\ldots,\mu_N$ in r.h.s. of estimation (3.7) are best possible values. In this connection should be pointed out that for Sobolev spaces $W_1^{\rho_m}$, this is not the case. More precisely, if, for positive integers $l_0,\ldots,l_{\mu_N}$ we replace the conditions $Q|_\Delta \in B_{1\infty}^{\rho_0}(\Delta)$ and $Q|_{\Delta_m} \in B_{1\infty}^{\rho_m}(\Delta_m)$ in Theorem 1 with the conditions $Q|_\Delta \in W_1^{l_0}(\Delta)$ and $Q|_{\Delta_m} \in W_1^{l_m}(\Delta_m)$ then the following estimation holds true

$$\begin{aligned} E\,(u(x)-u_N(x))^2 \quad \leq \quad & \frac{C_1(s,\psi)}{4^{s(\mu_N+1)}}\cdot I_s + \frac{C_2(l_0,\phi)}{b_N^{2l_0-1}}||Q||^2_{W_1^{l_0}(\Delta)} \\ + \sum_{m=0}^{\mu_N}\frac{C_3(l_m,\psi)}{b_{mN}^{2l_m-1}} \quad \cdot \quad & \left(2^{-m}||Q||^2_{L_1(\Delta_m)} + 2^{2m(l_m-1/2)}||Q||^2_{W_1^{l_m}(\Delta_m)}\right). \end{aligned} \tag{4.13}$$

But exponents $2l_m - 1$, $m = 0,1,\ldots,\mu_N$ in this estimation might be not best possible. To illustrate this idea let us consider the following

Example 2. For $\rho \in (1,2)$ let us consider the function $Q(k)$ constructed as above in Example 1 with $k_0 = 1/2$ and $q \in C^2(\mathbb{R})$ such, that $supp(q) \subset (-\frac{2}{3}, \frac{2}{3})$. Then a maximum possible value of an integer l_0 in the condition $Q|_\Delta \in W_1^{l_0}(\Delta)$ is $l_0 = 1$. Therefore taking into account that $Q|_{\Delta_m} = 0$, $m = 1, 2, ...$ we get by (4.13)

$$E(u(x) - u_N(x))^2 \leq C \cdot \left(\frac{1}{b_N} + \frac{1}{b_{0N}} \right)$$

while the estimation (3.7) ensures more exact inequality is true

$$E(u(x) - u_N(x))^2 \leq C \cdot \left(\frac{1}{b_N^{2\rho-1}} + \frac{1}{b_{0N}^{2\rho-1}} \right).$$

5 Convergence of FWM in C metric

Now let us turn to the problem of convergence of FWM (3.1) in probability, in the metric of $C[0,1]$. Recall $\mathcal{E}_N(x) = u(x) - u_N(x)$. Denote by $D = [0,1]$ a unit interval, and $C(D)$ is the space of continuous scalar functions on D with the uniform norm $||f||_{C(D)} = \max_{x \in D} |f(x)|$. Let us mention that the functional convergence $u_N \xrightarrow{P} u$ in $C(D)$ as $N \to \infty$ means $||\mathcal{E}_N||_{C(D)} \xrightarrow{P} 0$ as $N \to \infty$, that is, for each nonnegative ε and δ there exists $N_0 = N_0(\varepsilon, \delta)$ such that $\mathcal{P}\{||\mathcal{E}_N||_{C(D)} > \varepsilon\} < \delta$ for each $N \geq N_0$. Thus to study the convergence in probability $u_N \xrightarrow{P} u$ in $C(D)$ as $N \to \infty$ we need to estimate the probability $\mathcal{P}\{||\mathcal{E}_N||_{C(D)} > \varepsilon\}$.

Let $\xi(x)$, $x \in D = [0,1]$ be a Gaussian random process with zero mean. Let

$$\varphi_\xi(h) = \sup_{\substack{x,y \in D \\ |x-y| \leq h}} (E|\xi(x) - \xi(y)|^2)^{1/2}; \quad h \in [0,1].$$

The following assertion is a 1-dimensional variant of the well known Fernique's inequality.
Theorem ([10]). *If* $\int\limits_1^\infty \varphi_\xi(e^{-x^2})dx < \infty$ *then almost all samples of the random process* $\xi(x)$ *are continuous. Moreover, for each* $t \geq \sqrt{5}$ *the following estimation is valid*

$$\mathcal{P}\{\sup_{x \in D} |\xi(x)| \geq qt\} \leq 10 \cdot \int_t^\infty e^{-x^2/2}\, dx, \tag{5.1}$$

where

$$q = \sup_{x \in D} (E|\xi(x)|^2)^{1/2} + (2 + \sqrt{2}) \cdot \int\limits_1^\infty \varphi_\xi(2^{-x^2})dx.$$

Taking into account the following inequality

$$\int_t^\infty e^{-x^2/2}\, dx \leq \frac{e^{-t^2/2}}{t} \leq \frac{e^{-t^2/2}}{\sqrt{5}}, \qquad t \geq \sqrt{5}$$

and the estimation (5.1), one can derive the following estimation:

$$\mathcal{P}\{\sup_{x \in D} |\xi(x)| \geq qt\} \leq 2\sqrt{5} \cdot e^{-t^2/2}, \qquad t \geq \sqrt{5}. \tag{5.2}$$

Thus, in order to use this estimation for $\xi(x) = \mathcal{E}_N(x)$ we have to analyze the function

$$\varphi_{\mathcal{E}_N}(h) = \sup_{\substack{x,y\in D\\ |x-y|\le h}} (E|\mathcal{E}_N(x) - \mathcal{E}_N(y)|^2)^{1/2}; \quad h \in [0,1].$$

Let us turn to estimation of this function.

For a fixed positive integer N we denote

$$\varepsilon_N^{(1)} = \frac{||Q||^2_{B^{\rho_0}_{1\infty}(\Delta)}}{b_N^{2\rho_0-1}} \cdot, \quad \varepsilon_N^{(3)} = \frac{I_s}{4^{(\mu_N+1)s}},$$

$$\varepsilon_{mN}^{(2)} = \frac{1}{b_{mN}^{2\rho_m-1}} \cdot \left(2^{-m}||Q||^2_{L_1(\Delta_m)} + 2^{2m(\rho_m-1/2)}||Q||^2_{b^{\rho_m}_{1\infty}(\Delta_m)}\right), \quad m = 0,1,...,\mu_N.$$

Theorem 2. *Assume that all the conditions of Theorem 1 are satisfied. Then for each* $\alpha \in (0,1)$ *and* $h \in [0,1]$ *the following inequality is valid*

$$E|\mathcal{E}_N(x+h) - \mathcal{E}_N(x)|^2 \le C(s) \cdot \left(I_s \cdot |h|^{s\wedge 1}\right)^{\alpha} \cdot \varepsilon_N^{1-\alpha}, \quad N = 1,2,..., \tag{5.3}$$

where $s \wedge 1 = \min\{s,1\}$, *and*

$$\varepsilon_N = C_1(\rho_0,\phi)\cdot \varepsilon_N^{(1)} + \sum_{m=0}^{\mu_N} C_2(\rho_m,\psi)\cdot \varepsilon_{mN}^{(1)} + C_3(s,\psi)\cdot \varepsilon_N^{(3)},$$

and C, C_i, $i = 1,2,3$ *are some constants depending on the shown arguments.*
Proof. From independency of random variables ξ_j, ξ_{mj}, $j \in \mathbb{Z}$, $m \ge 0$ it follows that (see (3.8))

$$\begin{aligned} E[\mathcal{E}_N(x+h) - \mathcal{E}_N(x)]^2 = E[v^{(1)}(x+h) - v^{(1)}(x)]^2 + \\ \sum_{m=0}^{\mu_N} E[v_m^{(2)}(x+h) - v_m^{(2)}(x)]^2 + E[v^{(3)}(x+h) - v^{(3)}(x)]^2. \end{aligned} \tag{5.4}$$

Let us first estimate the first term in r.h.s. of the last equality.
Lemma 1. *For each* $\alpha \in (0,1)$ *the following estimation holds*

$$E[v^{(1)}(x+h) - v^{(1)}(x)]^2 \le \left(C(s)I_s \cdot |h|^{(s\wedge 1)}\right)^{\alpha} \cdot (C_1(\rho_0,\phi)\cdot \varepsilon_N^{(1)})^{1-\alpha}. \tag{5.5}$$

Proof. Indeed, for each $\alpha \in (0,1)$ one has

$$E[v^{(1)}(x+h) - v^{(1)}(x)]^2 = \sum_{j\in\mathbb{Z}} [\chi_N(x+h+j)\cdot \mathcal{F}^{(\phi)}(x+h+j) - \chi_N(x+j)\cdot \mathcal{F}^{(\phi)}(x+j)]^2$$

$$= \sum_{j\in\mathbb{Z}} |a_j - b_j|^2 \le \left(\sum_{j\in\mathbb{Z}} |a_j - b_j|^2\right)^{\alpha} \cdot \left(\sum_{j\in\mathbb{Z}} (|a_j| + |b_j|)^2\right)^{1-\alpha}. \tag{5.6}$$

Here we denote

$$a_j = \chi_N(x+h+j)\cdot \mathcal{F}_0^{(\phi)}(x+h+j), \quad b_j = \chi_N(x+j)\cdot \mathcal{F}_0^{(\phi)}(x+j),$$

and in the second line of (5.6) we use Hölder's inequality:

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|a_j-b_j|^2 \quad & \le \sum_{j\in\mathbb{Z}}\left\{|a_j-b_j|^{2\alpha}\cdot(|a_j|+|b_j|)^{2\cdot(1-\alpha)}\right\}\\ & \le \left(\sum_{j\in\mathbb{Z}}|a_j-b_j|^{2\alpha p}\right)^{1/p}\cdot\left(\sum_{j\in\mathbb{Z}}(|a_j|+|b_j|)^{2\cdot(1-\alpha)q}\right)^{1/q} \qquad (5.7)\end{aligned}$$

with parameters $p=1/\alpha$, $q=1/(1-\alpha)$. Let us estimate each factor in the right-hand side of (5.6). First note that

$$\begin{aligned}|a_j-b_j| &= |\chi_N(x+h+j)\cdot\mathcal{F}_0^{(\phi)}(x+h+j)-\chi_N(x+j)\cdot\mathcal{F}_0^{(\phi)}(x+j)|\\ &\le |\chi_N(x+h+j)\cdot\mathcal{F}_0^{(\phi)}(x+h+j)-\chi_N(x+h+j)\cdot\mathcal{F}_0^{(\phi)}(x+j)|\\ &\quad +|\chi_N(x+h+j)\cdot\mathcal{F}_0^{(\phi)}(x+j)-\chi_N(x+j)\cdot\mathcal{F}_0^{(\phi)}(x+j)|\end{aligned}$$

$$\le |\mathcal{F}_0^{(\phi)}(x+h+j)-\mathcal{F}_0^{(\phi)}(x+j)|+|\mathcal{F}_0^{(\phi)}(x+j)|\cdot|\chi_N(x+h+j)-\chi_N(x+j)|. \qquad (5.8)$$

Therefore,

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|a_j-b_j|^2 \quad & \le 2\cdot\sum_{j\in\mathbb{Z}}[\mathcal{F}_0^{(\phi)}(x+h+j)-\mathcal{F}_0^{(\phi)}(x+j)]^2\\ & +2\cdot\sum_{j\in\mathbb{Z}}|\mathcal{F}_0^{(\phi)}(x+j)|^2\cdot|\chi_N(x+h+j)-\chi_N(x+j)|^2. \qquad (5.9)\end{aligned}$$

Note that the sequences $\left\{\mathcal{F}_0^{(\phi)}(x+h+j)-\mathcal{F}_0^{(\phi)}(x+j)\right\}_{j\in\mathbb{Z}}$ and $\left\{\mathcal{F}_0^{(\phi)}(x+j)\right\}_{j\in\mathbb{Z}}$ are the Fourier coefficients of $\Delta_h G(x+\cdot)$ and $G(x+\cdot)$, respectively, in the orthonormal system $\{\phi(\cdot+j)\}_{j\in\mathbb{Z}}$. Therefore

$$\sum_{j\in\mathbb{Z}}|\mathcal{F}_0^{(\phi)}(x+j)|^2\le ||G(x+\cdot)||_{L_2}^2=||G(\cdot)||_{L_2}^2\le I_s\,, \qquad (5.10)$$

and

$$\sum_{j\in\mathbb{Z}}[\mathcal{F}_0^{(\phi)}(x+h+j)-\mathcal{F}_0^{(\phi)}(x+j)]^2\le ||\Delta_h G(x+\cdot)||_{L_2}^2=||\Delta_h G(\cdot)||_{L_2}^2. \qquad (5.11)$$

Note that $G\in B^s_{22}\hookrightarrow B^s_{2\infty}\hookrightarrow B^\gamma_{2\infty}$ for each $0<\gamma\le s$. Therefore, taking $\gamma=\min\{1/2,s/2\}$ we have

$$\begin{aligned}||\Delta_h G(\cdot)||_{L_2}\le C_1(s)\cdot|h|^\gamma\cdot||\Delta_h G(\cdot)||_{B^\gamma_{2\infty}} \quad & \le \quad C_2(s)\cdot|h|^\gamma\cdot||\Delta_h G(\cdot)||_{B^s_{22}}\\ & \le C_3(s)\cdot|h|^\gamma\cdot I_s^{1/2} \qquad (5.12)\end{aligned}$$

for each $h\in[-1,1]$. From (5.9)-(5.12), taking into account that $|\chi_N(x+h+j)-\chi_N(x+j)|\le|h|$, we obtain

$$\sum_{j\in\mathbb{Z}}|a_j-b_j|^2\le C(s)\,I_s\cdot|h|^{s\wedge 1}, \qquad (5.13)$$

where $s\wedge 1=\min\{1,s\}$.

Now, let us estimate the second factor in r.h.s. of (5.6). From

$$\sum_{j\in\mathbb{Z}}(|a_j|+|b_j|)^2\le 2\cdot\sum_{j\in\mathbb{Z}}|a_j|^2+2\cdot\sum_{j\in\mathbb{Z}}|b_j|^2=2\cdot E(v^{(1)}(x+h))^2+2\cdot E(v^{(1)}(x))^2$$

and (3.16) we obtain

$$\sum_{j\in\mathbb{Z}}(|a_j|+|b_j|)^2 \leq \frac{4\cdot C(\rho_0,\phi)}{b_N^{2\rho_0-1}}\cdot||Q||^2_{B^{\rho_0}_{1\infty}(\Delta)}. \tag{5.14}$$

Thus by virtue of (5.6) and (5.13)-(5.14) the estimation (5.5) follows. The proof of Lemma 1 is complete. □

Lemma 2. *For each* $\alpha\in(0,1)$ *the following estimation is valid*

$$\sum_{m=0}^{\mu_N} E[v_m^{(2)}(x+h) - v_m^{(2)}(x)]^2 \leq \left(C(s)\cdot I_s\cdot|h|^{s\wedge 1}\right)^{\alpha} \cdot\left(\sum_{m=0}^{\mu_N} C_2(\rho_m,\psi)\cdot\varepsilon^{(2)}_{mN}\right)^{1-\alpha}, \quad N=1,2,\dots. \tag{5.15}$$

Proof. For each $\alpha\in(0,1)$, by the same arguments as we used in proof of Lemma 1, we arrive at

$$E[v_m^{(2)}(x+h)-v_m^{(2)}(x)]^2 \leq \left(\sum_{j\in\mathbb{Z}}|a_{mj}-b_{mj}|^2\right)^{\alpha}\cdot\left(\sum_{j\in\mathbb{Z}}(|a_{mj}|+|b_{mj}|)^2\right)^{1-\alpha}, \tag{5.16}$$

where

$$a_{mj}=\chi_{mN}(2^m\cdot(x+h)+j)\cdot\mathcal{F}_m^{(\psi)}(2^m\cdot(x+h)+j),\quad b_{mj}=\chi_{mN}(2^m\cdot x+j)\cdot\mathcal{F}_m^{(\psi)}(2^m\cdot x+j).$$

From (5.16) it follows that

$$\sum_{m=0}^{\mu_N}E[v_m^{(2)}(x+h)-v_m^{(2)}(x)]^2 \leq \left(\sum_{m=0}^{\mu_N}\sum_{j\in\mathbb{Z}}|a_{mj}-b_{mj}|^2\right)^{\alpha}\left(\sum_{m=0}^{\mu_N}\sum_{j\in\mathbb{Z}}(|a_{mj}|+|b_{mj}|)^2\right)^{1-\alpha}.$$

Now obviously

$$\begin{aligned}|a_{mj} - b_{mj}| &\leq |\mathcal{F}_m^{(\psi)}(2^m\cdot(x+h)+j)-\mathcal{F}_m^{(\psi)}(2^m\cdot x+j)| \\ &+|\mathcal{F}_m^{(\psi)}(2^m\cdot x+j)|\cdot|\chi_{mN}(2^m\cdot(x+h)+j)-\chi_{mN}(2^m\cdot x+j)|.\end{aligned} \tag{5.17}$$

Since $\left\{\mathcal{F}_m^{(\psi)}(2^m\cdot(x+h)+j)-\mathcal{F}_m^{(\psi)}(2^m\cdot x+j)\right\}_{j\in\mathbb{Z}}$, $m=0,\dots,\mu_N$ are Fourier coefficients in the expansion of $\Delta_h G(x+\cdot)$ with respect to the orthonormal system $\{\psi_{mj}\}_{j\in\mathbb{Z}}$,
$m=0,\dots,\mu_N$, then (see (5.12)-(5.13))

$$\sum_{m=0}^{\mu_N}\sum_{j\in\mathbb{Z}}|\mathcal{F}_m^{(\psi)}(2^m(x+h)+j)-\mathcal{F}_m^{(\psi)}(2^m x+j)|^2\leq||\Delta_h G||_{L_2}\leq C(s)I_s|h|^{s\wedge 1}. \tag{5.18}$$

When we estimate the second term in r.h.s. of (5.17) we consider two cases: (i) $2^m|h|\leq 1$, and (ii) $2^m|h|>1$. In the first case, taking into account that $|\chi_{mN}(2^m\cdot(x+h)+j)-\chi_{mN}(2^m\cdot x+j)|^2\leq(2^m|h|)^2$ (see (5.12) and (3.12)) we have

$$\begin{aligned}&\sum_{m=0}^{\mu_N}\sum_{j\in\mathbb{Z}}|\mathcal{F}_m^{(\psi)}(2^m\cdot x+j)|^2\cdot|\chi_{mN}(2^m\cdot(x+h)+j)-\chi_{mN}(2^m\cdot x+j)|^2\\ &\leq(2^m|h|)^2\cdot\sum_{m=0}^{\mu_N}\sum_{j\in\mathbb{Z}}|\mathcal{F}_m^{(\psi)}(2^m\cdot x+j)|^2\leq(2^m|h|)^{s\wedge1}\cdot\sum_{m=0}^{\mu_N}\sum_{j\in\mathbb{Z}}|\mathcal{F}_m^{(\psi)}(2^m\cdot x+j)|^2\\ &\leq B_s^2\cdot I_s\cdot|h|^{s\wedge 1},\qquad \text{if}\quad 2^m|h|\leq 1.\end{aligned}$$

In the case (ii) (i.e. $2^m|h|>1$) it is obvious, that $(2^m|h|)^s>1$ and $|\chi_{mN}(2^m\cdot(x+h)+j)-\chi_{mN}(2^m\cdot x+j)|^2\le 1$. Therefore, in this case

$$\begin{aligned}
&\sum_{m=0}^{\mu_N}\sum_{j\in\mathbb{Z}}|\mathcal{F}_m^{(\psi)}(2^m\cdot x+j)|^2\cdot|\chi_{mN}(2^m\cdot(x+h)+j)-\chi_{mN}(2^m\cdot x+j)|^2\\
&\le\sum_{m=0}^{\mu_N}\sum_{j\in\mathbb{Z}}|\mathcal{F}_m^{(\psi)}(2^m\cdot x+j)|^2\le(2^m|h|)^{2s}\cdot\sum_{m=0}^{\mu_N}\sum_{j\in\mathbb{Z}}|\mathcal{F}_m^{(\psi)}(2^m\cdot x+j)|^2\\
&\le B_s 2\cdot I_s\cdot|h|^{2s},\qquad\text{if}\quad 2^m|h|>1.
\end{aligned}\tag{5.19}$$

Thus in general case we have

$$\begin{aligned}
\sum_{m=0}^{\mu_N}\sum_{j\in\mathbb{Z}}|\mathcal{F}_m^{(\psi)}(2^m\cdot x+j)|^2\quad\cdot\quad&|\chi_{mN}(2^m\cdot(x+h)+j)-\chi_{mN}(2^m\cdot x+j)|^2\\
&\le B_s^2\cdot I_s\cdot|h|^{s\wedge 1},\qquad\text{if}\quad|h|\le 1.
\end{aligned}\tag{5.20}$$

Using (5.18), (5.20) and taking into account (5.17) we obtain

$$\sum_{m=0}^{\mu_N}\sum_{j\in\mathbb{Z}}|a_j-b_j|^2\le C(s)\cdot I_s\cdot|h|^{s\wedge 1},\qquad\text{if}\quad|h|\le 1.\tag{5.21}$$

To estimate the second factor on r.h.s. of inequality (5.16) we first notice that

$$\sum_{j\in\mathbb{Z}}(|a_j|+|b_j|)^2\le 2\cdot E(v_m^{(2)}(x+h))^2\;+\;2\cdot E(v_m^{(2)}(x))^2.$$

Then using the inequality (3.22) we get

$$\sum_{j\in\mathbb{Z}}(|a_j|+|b_j|)^2\le\frac{C(\rho_m,\psi)}{b_{mN}^{2\rho_m-1}}\cdot 2^{-m}||Q||^2_{L_1(\Delta_m)}+\frac{C(\rho_m,\psi)}{b_{mN}^{2\rho_m-1}}\cdot 2^{m(2\rho_m-1)}||Q||^2_{b_{1\infty}^{\rho_m}(\Delta_m)}\tag{5.22}$$

for $m=0,1,...,\mu_N$.

From (5.16), (5.21) and (5.22) we get (5.15). The proof of Lemma 2 is complete. $\square$

It is easy to see that

$$E(v^{(3)}(x+h)-v^{(3)}(x))^2=\sum_{m=\mu_N+1}^{\infty}\sum_{j\in\mathbb{Z}}|a_{mj}-b_{mj}|^2,$$

where

$$a_{mj}=\mathcal{F}_m^{(\psi)}(2^m\cdot(x+h)+j),\quad b_{mj}=\mathcal{F}_m^{(\psi)}(2^m\cdot x+j).$$

Therefore using

$$\sum_{j\in\mathbb{Z}}|a_{mj}-b_{mj}|^2\le||\Delta_h G||^2_{L_2},$$

for each integer m, and

$$\sum_{m=\mu_N+1}^{\infty}\sum_{j\in\mathbb{Z}}(|a_{mj}|+|b_{mj}|)^2\le 2\cdot[E(v^{(3)}(x+h))^2+E(v^{(3)}(x))^2]$$

and taking into account (3.13) and (5.12) we get

$$E(v^{(3)}(x+h)-v^{(3)}(x))^2\le\left(C(s)\cdot I_s\cdot|h|^{s\wedge 1}\right)^{\alpha}\cdot\left(C_3(s,\psi)\cdot\varepsilon_N^{(3)}\right)^{1-\alpha}.\tag{5.23}$$

Finally, using (5.4), (5.5), (5.15) and the last inequality we obtain

$$
\begin{aligned}
& E|\mathcal{E}_N(x+h)-\mathcal{E}_N(x)|^2 \\
& \leq \left(C(s)\cdot I_s\cdot|h|^{s\wedge 1}\right)^{\alpha}\cdot\left[(C_1\cdot\varepsilon_N^{(1)})^{1-\alpha}+(\sum_{m=0}^{\mu_N}C_{2m}\cdot\varepsilon_{mN}^{(2)})^{1-\alpha}+(C_3\cdot\varepsilon_N^{(3)})^{1-\alpha}\right] \\
& \leq 3^{\alpha}\left(C(s)\cdot I_s\cdot|h|^{s\wedge 1}\right)^{\alpha}\cdot\left[C_1\cdot\varepsilon_N^{(1)}+\sum_{m=0}^{\mu_N}C_{2m}\cdot\varepsilon_{mN}^{(2)}+C_3\cdot\varepsilon_N^{(3)}\right]^{1-\alpha}.
\end{aligned}
\tag{5.24}
$$

Theorem 2 is proven. $\square$

Now we are in a position to formulate an assertion on convergence of FWM in probability:

Theorem 3. *Assume that all the conditions of Corollary 1 are fulfilled. Then for each* $\alpha\in(0,1)$ *and* $t>\sqrt{5}$

$$
\mathcal{P}\{\sup_{x\in[0,1]}|\mathcal{E}_N(x)|>C\cdot\varepsilon_N(\alpha)\cdot t\}\leq 2\sqrt{5}\cdot e^{-t^2/2}, \tag{5.25}
$$

where C *is some positive constant not depending on* α *and* N*, and*

$$
\varepsilon_N(\alpha)=\frac{1}{\sqrt{\alpha}}\cdot N^{-\frac{(2\rho_0-1)\cdot(1-\alpha)}{2}}.
$$

Proof. From the estimation (5.3), taking into account (3.23) and (3.27) we get

$$
\sup_{x,y\in[0,1],\,|x-y|\leq h}E\left(\mathcal{E}_N(x)-\mathcal{E}_N(y)\right)^2\leq C_1|h|^{\alpha(s\wedge 1)}N^{-(2\rho_0-1)\cdot(1-\alpha)}
$$

for each $\alpha\in(0,1)$ and $|h|\leq 1$, where C_1 is some positive constant not depending on α, h and N. Now let $\xi(x)=\mathcal{E}_N(x)$. Then

$$
\varphi_\xi(h)=\sup_{\substack{x,y\in D\\|x-y|\leq h}}E^{1/2}(|\xi(x)-\xi(y)|^2)\leq C_2\cdot|h|^{\alpha(s\wedge 1)/2}N^{-(2\rho_0-1)\cdot(1-\alpha)/2}.
$$

Therefore, by (3.28)

$$
\begin{aligned}
q=\sup_{x\in D}E^{1/2}|\xi(x)|^2+(2+\sqrt{2})\cdot\int_1^\infty\varphi_\xi(2^{-x^2})dx\leq C_3\cdot N^{-\frac{(2\rho_0-1)}{2}} \\
+C_4\cdot N^{-\frac{(2\rho_0-1)\cdot(1-\alpha)}{2}}\cdot\int_1^\infty 2^{-x^2\alpha(s\wedge 1)/2)}dx\leq C\varepsilon_N(\alpha).
\end{aligned}
$$

Now (5.25) is a direct consequence of estimation (5.2). Theorem is proved. $\square$

Corollary 3. *Under the conditions of Theorem 3,* $u_N\xrightarrow{P}u$ *in the metric of* $C[0,1]$ *as* $N\to\infty$.

Indeed, by (5.25) for each $\varepsilon>0$ and $\delta>0$ it can be found a positive integer N_0 such that

$$
\mathcal{P}\{\sup_{x\in[0,1]}|\mathcal{E}_N(x)|>\varepsilon\}\leq\delta,\qquad N\geq N_0.
$$

6 Generalization of results to vector processes and some stronger metrics

6.1 Convergence in L_p metric

Recall the well known relation between different moments of zero mean Gaussian random variables. Let ξ be a zero mean real-valued Gaussian random variable. Then (e.g., see, [18])

$$E|\xi|^p = c_p(E\xi^2)^{p/2}, \quad p \geq 1, \quad c_p = \frac{2^{p/2}}{\sqrt{\pi}}. \tag{6.1}$$

Therefore, under the conditions of Theorem 1, using the estimation (3.7) for $E\mathcal{E}_N^2(x) = E(u(x) - u_N(x))^2$ it is easy to obtain an estimation for the convergence rate $E||\mathcal{E}_N||^p_{L_p[0,1]} = \int_0^1 E(\mathcal{E}_N(x))^p\,dx$ in the metric of $L_p[0,1]$, $p \geq 1$.

6.2 Convergence in the metric of Sobolev space W_p^n

For a positive integer n and $p \geq 1$ let $W_p^n[0,1]$ be a Sobolev space with the norm

$$||f||_{W_p^n[0,1]} = ||f||_{L_p[0,1]} + ||f^{(n)}||_{L_p[0,1]}.$$

In the study of convergence $E||\mathcal{E}_N||^p_{W_p^n[0,1]} \to 0$, due to the relation (6.1), we can restrict ourselves to the case $p = 2$. Therefore let us consider the convergence of FWM (3.1) in Sobolev space W_2^n. In order to analyze the convergence of FWM, $u_N(x)$ to $u(x)$ as $N \to \infty$ in $W_2^n[0,1]$ it is sufficient to study: (i) convergence $u_N \to u$ in $L_2[0,1]$, and (ii) convergence $u_N^{(n)} \to u^{(n)}$ in $L_2[0,1]$. The first question was already studied in section 3. Therefore we will concentrate on the second question: $u_N^{(n)} \to u^{(n)}$ in $L_2[0,1]$. Recall that for a function $f : \mathbb{R} \to \mathbb{R}$ we denote by $f^{(n)}$ the n-th derivative $D^n f$ (if it exists).

First of all note that a necessary and sufficient condition that samples of a random process $u(x)$ belong to $W_2^n[0,1]$ (with probability 1) is (e.g., see, [17])

$$\int_{-\infty}^{\infty} |k|^{2n} F(k)\,dk < \infty\,. \tag{6.2}$$

If this condition is satisfied, then $D^n u(x) = u^{(n)}$ have the spectral function $k^{2n}F(k)$. In order to ensure that u_N is n-times differentiable we have to choose the cut-off function η_A n-times differentiable. Therefore we will assume, that $\eta_A \in C^n$, and

$$\eta_A(x) = \begin{cases} 0, & 0 \leq x \leq A, \\ 1, & x \geq A+1. \end{cases} \tag{6.3}$$

Thus, using results of section 3 one can formulate sufficient conditions for $u_N^{(n)} \to u^{(n)}$ as $N \to \infty$ in $L_2[0,1]$. Denote by Q_n the function $Q_n(k) = |k|^n Q(k) = |k|^n F^{1/2}(k)$. Then the following assertion is valid:

Theorem 1a. *Assume that the spectral function $F(k)$ satisfies the condition*

$$I_{n+s} \stackrel{def}{=} \int_{-\infty}^{\infty} F(k)(1+|k|^2)^{n+s} dk < \infty \tag{6.4}$$

for some $s > 0$. Let $\{\rho_m\}_{m=0}^{\mu_N}$ be a finite sequence of positive numbers such that $\min_m \rho_m > 1/2$. Assume that $Q_n|_\Delta \in B_{1\infty}^{\rho_0}(\Delta)$ and $Q|_{\Delta_m} \in B_{1\infty}^{\rho_m}(\Delta_m)$ for $1 \le m \le \mu_N$. Assume that the Meyer wavelet functions ϕ and ψ belong to the class $C^{\nu+1}$ for $\nu = \max\{\lfloor\rho\rfloor, \lfloor s\rfloor\}$, where $\rho = \max_{0 \le m \le \mu_N} \rho_m$. Then

$$\begin{aligned} E(u^{(n)}(x) - u_N^{(n)}(x))^2 &\le \frac{C_1(s,\psi)}{4^{s(\mu_N+1)}} \cdot I_{n+s} + \frac{C_2(\rho_0,\phi)}{b_N^{2\rho_0-1}} ||Q_n||^2_{B_{1\infty}^{\rho_0}(\Delta)} \\ + \sum_{m=0}^{\mu_N} \frac{C_3(\rho_m,\psi)}{b_{mN}^{2\rho_m-1}} &\cdot \left(2^{-2ms} \cdot I_{n+s} + 2^{2m(\rho_m-1/2)} ||Q_n||^2_{b_{1\infty}^{\rho_m}(\Delta_m)}\right), \end{aligned} \tag{6.5}$$

where C_i, $i = 1,2,3$ are some positive constants depending only on the shown arguments.

Proof. Indeed, this assertion is a direct consequence of Theorem 1 applied to the random process $u^{(n)}(x)$ with the spectral function $|k|^{2n}F(k)$. Then the estimation (6.5) is a consequence of (3.7) and the following simple inequality

$$||Q_n||_{L_1(\Delta_m)} = \int_{\Delta_m} |k|^n Q(k) dk \le |\Delta_m|^{1/2} \left(\int_{\Delta_m} |k|^{2n} F(k) dk\right)^{1/2} \le 2^{(m+1)/2 - ms} I_{n+s}^{1/2}.$$

Proposition 3. *Assume that for some nonnegative k_0 and positive integer l $\quad Q \in W_1^l(k_0,\infty)$, and for each nonnegative integer j such that $0 \le j \le n \wedge l$ the following condition is valid:*

$$|Q^{(l-j)}(k)| \le \frac{A_\varepsilon}{k^{n+l-j+1/2+\varepsilon}}, \quad \forall k \ge k_0 \tag{6.6}$$

for some positive numbers A_ε and ε. Then

$$||Q_n||_{b_{1\infty}^l(\Delta_m)} \le A_\varepsilon \cdot c_0 \cdot 2^{-m(l-1/2+\varepsilon)}, \quad \textit{for all} \quad m \ge \log_2(3(k_0+2)), \tag{6.7}$$

for some positive $c_0 = c_0(n,l,\varepsilon)$.

Proof. Indeed,

$$|\Delta_h^{(2)} Q_n^{(l-1)}(k)| \le |\Delta_h Q_n^{(l-1)}(k+h)| + |\Delta_h Q_n^{(l-1)}(k+h)| \le h \cdot |Q_n^{(l)}(k+\beta h)| + h \cdot |Q_n^{(l)}(k+\alpha h)|$$

for some $\alpha \in [0,1]$ and $\beta \in [1,2]$. Since

$$Q_n^{(l)}(k) = \sum_{j=0}^{n\wedge l} C_l^j \cdot n \cdot (n-1) \cdot \ldots \cdot (n-j+1) \cdot k^{n-j} Q^{(l-j)}(k),$$

we find for each $k \in \Delta_m$ and $m \ge \log_2(3(k_0+2))$

$$|Q_n^{(l)}(k)| \le C(n,l) 2^{-m(l+1/2+\varepsilon)}$$

by virtue of condition (6.6). Therefore,

$$\int_{\Delta_m} |\Delta_h^{(2)} Q^{(l-1)}(k)| \le C(n,l) 2^{-m(l+1/2+\varepsilon)} \cdot h \cdot |\Delta_m| \le C_1(n,l) \cdot h \cdot 2^{-m(l-1/2+\varepsilon)}.$$

This completes the proof of (6.7). □

Corollary 4. *Assume that the spectral function $F(k)$ satisfies the condition (6.4) for some $s>0$ and a positive integer n, the function $Q_n(k)=|k|^n\cdot F^{1/2}(k)=|k|^n\cdot Q(k)$ satisfies the conditions*

$$Q_n|_\Delta\in B^{\rho_0}_{1\infty}(\Delta),\quad Q|_{\Delta'}\in W^l_1(\Delta') \tag{6.8}$$

for some positive $\rho_0>1/2$ and positive integer l. Assume that for each nonnegative integer $j\in[0,n\wedge l]$ the function Q satisfies the condition (6.6) and suppose that $\phi,\psi\in C^{\nu+1}$ where $\nu=\max\{\lfloor s\rfloor,\lfloor\rho_0\rfloor,l\}$. Let us choose in FWM (3.1)

$$\mu_N=\left\lfloor\frac{(2\rho_0-1)}{s\ln 4}\cdot\ln N\right\rfloor,\quad b_N=N,\quad b_{mN}=N^{\frac{2\rho_0-1}{2l-1}},\ m=0,1,...,\mu_N. \tag{6.9}$$

Then for each positive integer N and $x\in\mathbb{R}$ the following estimation is valid:

$$\begin{aligned}E(u^{(n)}(x)-u^{(n)}_N(x))^2&\le N^{-(2\rho_0-1)}\\&\times\left[C_1(s,\psi)I_{n+s}+C_2(\rho_0,\phi)\cdot||Q_n||^2_{B^{\rho_0}_{1\infty}(\Delta)}+C_3(n,\rho_1,\varepsilon,\psi)\cdot C^2_\varepsilon\right],\end{aligned} \tag{6.10}$$

where C_i, $i=1,2,3$ are some positive constants depending only on the shown arguments.

Thus if the spectral function $F(\cdot)$ satisfies all the conditions of the last corollary and $Q|_\Delta\in B^{\rho_0}_{1\infty}(\Delta)$ then the estimates (3.38) and (6.10) imply the following result on convergence of FWM in W^n_2:

Corollary 5. *Let all the conditions of Corollary 4 be fulfilled and $Q|_\Delta\in B^{\rho_0}_{1\infty}(\Delta)$. Then*

$$E||\mathcal{E}_N||^2_{W^n_2[0,1]}\le\frac{C}{N^{2\rho_0-1}},\quad N=1,2,\dots$$

for some positive C not depending on N.

6.3 Convergence in $C^n[0,1]$

For positive integer n, let $C^n[0,1]$ be the space of all n-times continuously differentiable real valued scalar functions $f:[0,1]\to\mathbb{R}$ with the norm

$$||f||_{C^n[0,1]}=\sup_{x\in[0,1]}|f(x)|+\sup_{x\in[0,1]}|f^{(n)}(x)|.$$

By the definition of this norm it follows that $\{u_N(x)\}_{N=1}^\infty$, $x\in[0,1]$ converges in the metric of $C^n[0,1]$ to a random field $u(x)$, $x\in[0,1]$ as $N\to\infty$ iff: (i) $u_N\to u$ in $C[0,1]$ as $N\to\infty$, and (ii) $u^{(n)}_N\to u^{(n)}$ in $C[0,1]$ as $N\to\infty$. Since the first question was studied in section 5, we concentrate on the second one. To ensure the samples of FWM (3.1) are of $C^n[0,1]$, we assume below that $\eta_A\in C^n(\mathbb{R})$ and the condition (6.3) is satisfied.

The following assertion is an analog of Theorem 3 and can be proved by the similar way.

Theorem 3a. *Let all the conditions of Corollary 4 are fulfilled. Then for each $\alpha\in(0,1)$ and $t>\sqrt{5}$,*

$$\mathcal{P}\{\sup_{x\in[0,1]}|\mathcal{E}^{(n)}_N(x)|>C\varepsilon_N(\alpha)\cdot t\}\le 2\sqrt{5}\cdot e^{-t^2/2}, \tag{6.11}$$

where C is a positive constant not depending on α and N, and

$$\varepsilon_N(\alpha) = \frac{1}{\sqrt{\alpha}} \cdot N^{-\frac{(2\rho_0-1)\cdot(1-\alpha)}{2}}.$$

Now let all the conditions of Corollary 5 be valid. Then using the simple inequality $\mathcal{P}\{\xi+\eta > x+y\} \leq \mathcal{P}\{\xi > x\} + \mathcal{P}\{\eta > y\}$ and the results of Theorems 3 and 3a we arrive at the following:
Corollary 6. *Let all the conditions of Corollary 5 be satisfied. Then for each $\alpha \in (0,1)$ and $t > \sqrt{5}$*

$$\mathcal{P}\left\{ ||\mathcal{E}_N||_{C^n[0,1]} > \frac{C\cdot t}{\sqrt{\alpha}} \cdot N^{-\frac{(2\rho_0-1)\cdot(1-\alpha)}{2}} \right\} \leq 4\sqrt{5}\cdot e^{-t^2/2},$$

where C is some positive constant not depending on α.

Under the conditions of this Corollary it follows immediately that $u_N \xrightarrow{P} u$ in the metric of $C^n[0,1]$ as $N \to \infty$.

6.4 Generalizations for vector processes

Let $\mathbf{u}(x) = (u_1(x),\ldots,u_l(x))^T$, $x \in \mathbf{R}$ be an l-dimensional real valued stationary Gaussian random process with mean zero and spectral tensor $F(k) = (F_{ij}(k))$, $i = 1,...,l$; $j = 1,...,l$, and $\mathbf{u}_N(x)$ is its FWM given by (2.14). Denote by $|\mathbf{a}| = (\sum_{i=1}^l a_i^2)^{1/2}$ the Euclidean norm of an l-dimensional real valued vector $\mathbf{a}$, and by $|A| = (\sum_{i=1}^l \sum_{j=1}^n a_{ij}^2)^{1/2}$ the Euclidean norm of a matrix $A = (a_{ij})$, $i = 1,...l$; $j = 1,...,n$ which can be realized as the Euclidean norm of the $l\times n$-vector A. For an $l\times n$ dimensional function $Q(k) = (q_{ij}(k))$, $i = 1,...l$; $j = 1,...,n$, and $D \subset \mathbb{R}$, denote (if it makes a sense)

$$|||Q|||_{L_1(D)} = \left(\sum_{i=1}^{l} \sum_{j=1}^{n} ||q_{ij}||^2_{L_1(D)} \right)^{1/2}$$

and

$$|||Q|||_{b^{\rho}_{1\infty}(D)} = \left(\sum_{i=1}^{l} \sum_{j=1}^{n} ||q_{ij}||^2_{b^{\rho}_{1\infty}(D)} \right)^{1/2}.$$

Finally, denote by $SpF(k)$ the trace $\sum_{i=1}^l F_{ii}(k)$ of the spectral tensor F. Then by the same arguments as in the scalar case one can establish the following result:
Theorem 1b. *Assume that the spectral tensor $F(k)$ satisfies the condition*

$$I_s \stackrel{def}{=} \int_{-\infty}^{\infty} SpF(k)(1+|k|^2)^s dk < \infty \tag{6.12}$$

for some $s > 0$. Let $\{\rho_m\}_{m=0}^{\mu_N}$ be a finite sequence of positive numbers such that $\min_m \rho_m > 1/2$. Assume that

$$|||Q|||_{L_1(\Delta)} < \infty, \quad |||Q|||_{b^{\rho_0}_{1\infty}(\Delta)} < \infty,$$

and

$$|||Q|||_{L_1(\Delta_m)} < \infty, \quad |||Q|||_{b^{\rho_m}_{1\infty}(\Delta_m)} < \infty, \quad m = 0,1,\ldots,\mu_N.$$

Assume that the Meyer wavelet functions ϕ *and* ψ *belong to the class* $C^{\nu+1}$ *for* $\nu = \max\{\lfloor\rho\rfloor, \lfloor s\rfloor\}$*, where* $\rho = \max_{0\le m\le \mu_N} \rho_m$*. Then*

$$E|u(x)-u_N(x)|^2 \leq \frac{C_1(s,\psi)}{4^{s(\mu_N+1)}}\cdot I_s + \frac{C_2(\rho_0,\phi)}{b_N^{2\rho_0-1}}\cdot\left\{|||Q|||^2_{L_1(\Delta)} + |||Q|||^2_{b^{\rho_0}_{1\infty}(\Delta)}\right\}$$
$$+\sum_{m=0}^{\mu_N}\frac{C_3(\rho_m,\psi)}{b_{mN}^{2\rho_m-1}} \cdot \left(2^{-m}|||Q|||^2_{L_1(\Delta_m)} + 2^{2m(\rho_m-1/2)}|||Q|||^2_{b^{\rho_m}_{1\infty}(\Delta_m)}\right), \qquad (6.13)$$

where C_i, $i = 1,2,3$ *are some constants depending only on the shown arguments.*

Analogous convergence results can be obtained in stronger metrics.

7 Conclusion and discussion

Functional convergence of Fourier-Wavelet Models (FWM) for stationary Gaussian random processes is studied in Sobolev spaces $W_p^n[0,1]$ and in the space of n-times continuously differentiable functions $C^n[0,1]$. Conditions sufficient for the convergence of FWMs are formulated in terms of spectral functions F (spectral tensors, in the case of vector random processes). Two kind of assumptions on the behaviour of the spectral functions are made: (i) finiteness of certain spectral moments (ii) a generalized smoothness of the function $Q = F^{1/2}$ in different wave bands. This is formulated in terms of smoothness in Besov's space $B^{\rho}_{1\infty}$.

The condition (i) is related to the behaviour of the high-frequency part of the spectrum which is standard in the convergence studies. Condition (ii) is related to the rate of convergence of the tales of the wavelet function $\mathcal{F}_m^{\psi}$ for a given spectral band. For these tales, we obtained upper estimations which are improvable in the sense that for some functions these estimations are exact to within a constant factor.

This analysis is new and provides a constructive algorithm for choosing the cut-off parameters for all spectral bands to ensure a uniform behaviour of the error on the whole spectral interval.

We give also estimations of the cost needed to guarantee the desired root mean square error ε in $L_2[0,1]$ depending on the smoothness parameter of the relevant Besov space. The typical behaviour of the cost has the form $T_\varepsilon \lesssim \frac{\ln\varepsilon}{\varepsilon^{1/(\rho_{min}-0.5)}}$ where ρ_{min} is the minimal smoothness parameter.

The Fourier-Wavelet models are well suited for simulation of random processes with smooth spectral functions as is clearly seen from our presentation. However in case the spectral function is not smooth in an isolated point, it is reasonable to use a hybrid method as follows. The spectral function is decomposed into two parts, the first being smooth, and the second is nonsmooth, and has a compact support in the neighbourhood of the isolated point. Then, the random process is represented respectively as a sum of two independent processes, the first having the smooth spectral function, and the second with the nonsmooth spectral function. The smooth part is simulated by FWM, and the second part, by a standard deterministic spectral method which takes into account the singularity.

8 Appendix

We present here some technical results which we use in the main part of this paper.

Lemma A1. *Let $\hat{f} \in B^r_{1\infty}(R)$, $r > 1/2$. Then f is uniformly continuous, $f \in L_2(R)$ and there exists a positive constant C_r depending only on r such that for all $x \in R$*

$$|x|^r\,|f(x)| \leq C_r||\hat{f}||_{b^r_{1\infty}}. \tag{8.1}$$

Proof. The uniform continuity of f follows from the fact that $\hat{f} \in L_1(R)$, since $B^r_{1\infty}(R) \subset L_1(R)$. Then, for a positive $\varepsilon \in (0, r-1/2)$ by (3.4) and (3.5),

$$B^r_{1\infty}(R) \hookrightarrow B^{r-1/2}_{2\infty}(R) \hookrightarrow B^{r-1/2-\varepsilon}_{22}(R). \tag{8.2}$$

From this we get that $\hat{f} \in L_2(R)$, hence $f \in L_2(R)$. Let $l = \lfloor r \rfloor$ be the integer part of r. From

$$\Delta_h^{(l+2)}\hat{f}(k) = \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi kx}(e^{-\mathrm{i}2\pi hx}-1)^{l+2} f(x)\,dx \tag{8.3}$$

and $f \in L_2(R)$ it follows by the inverse Fourier transform

$$f(x)(e^{-\mathrm{i}2\pi hx}-1)^{l+2} = \int_{-\infty}^{\infty} e^{\mathrm{i}2\pi kx}\Delta_h^{(l+2)}\hat{f}(k)\,dk. \tag{8.4}$$

Taking the absolute values and dividing this equation by $|h|^r$ we then take the supremum over $h \in R$. This yields

$$|x|^r\,|f(x)|C'_r \leq \sup_{h\in\mathbb{R}} |h|^{-r}\,||\Delta_h^{(l+2)}\hat{f}||_{L_1}, \tag{8.5}$$

where

$$C'_r = \sup_{t\in\mathbb{R}} \left\{ \frac{|e^{-\mathrm{i}2\pi t}-1|^{l+2}}{|t|^r} \right\}. \tag{8.6}$$

Since the triple $(r, l+2, 0)$ is admissible, we have $\sup_{h\in\mathbb{R}} |h|^{-r}\,||\Delta_h^{(l+2)}\hat{f}||_{L_1} \leq C''_r||\hat{f}||_{b^r_{1\infty}}$ for some C''_r depending only on r, which completes the proof of Lemma A1. $\square$

Lemma A2. *Assume $\phi \in C^2(\mathbb{R})$ is chosen so that $\phi, \phi' = D^1\phi, \phi'' = D^2\phi \in L_\infty(\mathbb{R})$. Then for $\varepsilon \in [0,1)$ and $Q \in B^{1+\varepsilon}_{1\infty}(\mathbb{R})$ the product $\phi \cdot Q \in B^{1+\varepsilon}_{1\infty}(\mathbb{R})$ and $||\phi Q||_{b^{1+\varepsilon}_{1\infty}} \leq C||Q||_{B^{1+\varepsilon}_{1\infty}}$ for some constant $C = C(\varepsilon, \phi)$ depending only on ε and ϕ.*

Proof. From the obvious equalities

$$\phi(x+h) = \phi(x) + h\phi'(x) + \frac{h^2}{2}\phi''(x+\alpha\cdot h), \quad \alpha \in [0,1];$$

$$\phi(x+2h) = \phi(x) + 2h\phi'(x) + \frac{4h^2}{2}\phi''(x+\beta\cdot 2h), \quad \beta \in [0,1];$$

it follows that

$$\Delta_h^{(2)}(\phi Q)(x) = \phi(x)\cdot\Delta_h^{(2)}Q(x) + 2h\cdot\phi'(x)\Delta_h Q(x+h)$$
$$+2h^2\phi''(x+\beta\cdot 2h)Q(x+2h) - h^2\phi''(x+\alpha\cdot h)Q(x+h).$$

Therefore

$$||\Delta_h^{(2)}(\phi Q)||_{L_1} \le ||\phi||_{L_\infty} \cdot ||\Delta_h^{(2)} Q||_{L_1} + 2|h| \cdot ||\phi'||_{L_\infty} \cdot ||\Delta_h Q||_{L_1} + 3h^2||\phi''||_{L_\infty} \cdot ||Q||_{L_1}.$$

Since

$$||\Delta_h^{(2)} Q||_{L_1} \le ||Q||_{B_{1\infty}^{1+\varepsilon}}|h|^{1+\varepsilon}, \quad ||\Delta_h Q||_{L_1} \le ||Q||_{B_{1\infty}^{\varepsilon}} \cdot |h|^{\varepsilon} \le C(\varepsilon)||Q||_{B_{1\infty}^{1+\varepsilon}} \cdot |h|^{\varepsilon}$$

then for $|h| \le 1$ we have

$$||\Delta_h^{(2)}(\phi Q)||_{L_1} \le (||\phi||_{L_\infty} + 2C(\varepsilon)||\phi'||_{L_\infty} + 3||\phi''||_{L_\infty}) \cdot ||Q||_{B_{1\infty}^{1+\varepsilon}}|h|^{1+\varepsilon}.$$

This completes the proof. $\square$

Corollary A1. *For $r > 0$, let $\psi \in C^{\lfloor r \rfloor + 1}$ be chosen so that*

$$\max_{n=0,1,\ldots,\lfloor r \rfloor + 1} ||D^n \psi||_{L_\infty} < \infty.$$

Then there exists a constant $C = C(r, \psi)$ such that $||\psi F||_{b_{1\infty}^r} \le C||F||_{B_{1\infty}^r}$ for each $F \in B_{1\infty}^r(\mathbb{R})$.

Since

$$||\psi F||_{b_{1\infty}^r} = \sup_{h:\,|h| \le 1} \frac{1}{|h|^{1+\varepsilon}} \cdot ||\Delta_h^{(2)}(\psi F)^{(l)}||_{L_1},$$

where $l = \lfloor r \rfloor - 1$, and $\varepsilon = r - \lfloor r \rfloor$, it is sufficient to show that

$$||\Delta_h^{(2)}(\psi F)^{(l)}||_{L_1} \le C(r, \psi)||F||_{B_{1\infty}^r} \cdot |h|^{1+\varepsilon}.$$

Indeed,

$$\Delta_h^{(2)}(\psi F)^{(l)} = \sum_{n=0}^{l} C_l^n \Delta_h^{(2)}(\psi^{(n)} F^{(l-n)}).$$

For all n ($0 \le n \le l$), the functions $\phi = \psi^{(n)}$ and $Q = F^{(l-n)}$ satisfy all the conditions of Lemma A2. Therefore, taking into account that $||\Delta_h^{(2)} Q||_{B_{1\infty}^{1+\varepsilon}} \le c(r)h^{1+\varepsilon}||Q||_{B_{1\infty}^r}$ and using the result of Lemma A2 one completes the proof of the corollary. $\square$

Corollary A2. *Let $r > 0$ and $\psi \in C^{\lfloor r \rfloor + 1}$ be a function with a compact support $\Delta = supp\{\psi\}$. Then there exists a constant $C = C(r, \psi)$ such that $||\psi f||_{B_{1\infty}^r(\mathbb{R})} \le C||f||_{B_{1\infty}^r(\Delta)}$ for each $f \in B_{1\infty}^r(\Delta)$.*

Indeed, let us denote by A the extension operator $A : B_{1\infty}^r(\Delta) \to B_{1\infty}^r(\mathbb{R})$, i.e. $(Af)(x) = f(x)$ for $x \in \Delta$ and $f \in B_{1\infty}^r(\Delta)$. Existence of a bounded linear extension operator is well known (e.g., see [29]). The corollary then follows from

$$||\psi f||_{B_{1\infty}^r(\mathbb{R})} = ||\psi \cdot Af||_{B_{1\infty}^r(\mathbb{R})} \le C(r, \psi)||Af||_{B_{1\infty}^r(\mathbb{R})} \le C(r, \psi) \cdot ||A|| \cdot ||f||_{B_{1\infty}^r(\Delta)}.$$

Below we give a result about characterization of the Besov space norms through wavelet coefficients.

For a nonnegative integer L we denote by $\mathcal{R}^L$ the class of $(L+1)$-times continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

$$\int_{\mathbb{R}} x^n f(x)dx = 0, \quad \text{for} \quad n = 0, 1, 2, \ldots, L; \tag{8.7}$$

$$\exists \gamma > 0, C > 0 \quad \text{such that} \quad |f(x) \leq \frac{C}{(1+|x|)^{2+L+\gamma}} \quad \forall x \in \mathbb{R}; \tag{8.8}$$

$$\exists \varepsilon > 0, C_1 > 0 \quad \text{such that} \quad \max_{1 \leq n \leq L+1} |D^n f(x)| \leq \frac{C}{(1+|x|)^{1+\varepsilon}} \quad \forall x \in \mathbb{R}. \tag{8.9}$$

It is known(e.g., see, [26], [16], [41]), that under the assumption $\psi \in \mathcal{R}^{\lfloor s \rfloor}$ on a wavelet function ψ, the norm $||f||_{B^s_{pq}}$ in Besov's space $B^s_{pq}(R)$, $1 \leq p, q \leq \infty$, $s > 0$ can be equivalently defined through the wavelet coefficients $\beta_{mj}(f) = \int_{\mathbb{R}} f(x)\psi_{mj}(x)dx$, m, $j \in \mathbb{Z}$, where $\psi_{mj}(x) = 2^{m/2}\psi(2^m x - j)$. Namely, let us introduce the norm $||\cdot||^{(\psi)}_{B^s_{pq}}$ by

$$||f||^{(\psi)}_{B^s_{pq}} = ||f||_{L_p} + \left\{ \sum_{m=-\infty}^{\infty} \left[\sum_{j=-\infty}^{\infty} \left(|\beta_{mj}(f)| \cdot 2^{m(s+\frac{1}{2}-\frac{1}{p})} \right)^p \right]^{q/p} \right\}^{1/q} \qquad (1 \leq q < \infty),$$

$$||f||^{(\psi)}_{B^s_{p\infty}} = ||f||_{L_\infty} + \sup_{m \in \mathbb{Z}} \left[\sum_{j=-\infty}^{\infty} \left(|\beta_{mj}(f)| \cdot 2^{m(s+\frac{1}{2}-\frac{1}{p})} \right)^p \right]^{1/p} \quad .$$

Then there exist constants $C_1 = C_1(p,q,s,\psi) > 0, C_2 = C_2(p,q,s,\psi) > 0$ such that

$$C_1 \cdot ||f||^{(\psi)}_{B^s_{pq}} \leq ||f||_{B^s_{pq}} \leq C_2 \cdot ||f||^{(\psi)}_{B^s_{pq}} \qquad \forall f \in B^s_{pq}(\mathbb{R}). \tag{8.10}$$

As an immediate consequence of the last inequality we arrive at

Corollary A3. For $s > 0$, let ψ be an orthonormal wavelet function satisfying the condition $\psi \in \mathcal{R}^{\lfloor s \rfloor}$. Then there exists a constant $C = C(p,q,s,\psi)$ such that for each $f \in B^s_{pq}(R)$

$$\left(\sum_{j=-\infty}^{\infty} |\beta_{mj}(f)|^p \right)^{1/p} \leq C \cdot 2^{-m(s+\frac{1}{2}-\frac{1}{p})} \cdot ||f||_{B^s_{pq}}. \tag{8.11}$$

References

[1] Buglanova N.A. and Kurbanmuradov O. Convergence of the Randomized Spectral Models of Homogeneous Gaussian Random Fields. *Monte Carlo Methods and Appl.*, vol. 1, No-3, 173-201, 1995.

[2] Chris Cameron. Relative efficiency of Gaussian stochastic process sampling procedures. *J. Comput. Phys.*, 192(2):546–569, 2003.

[3] Charles K.Chui. *An Introduction to Wavelets*. Academic Press, Inc., 1992.

[4] A. Cohen, J. Froment, and J. Istas. Analyse multiresolution de signaux aleatories. C.R. Acad. Sci. Paris Ser. I. Math., 312, pp 567-570, 1991.

[5] A. Cohen and J.-P. D'Ales. Nonlinear approximation of random functions. SIAM J. Appl. Math., Vol. 57, No. 2, pp 518-540, 1997.

[6] G. Dagan. Flow and Transport in Porous Formations. Springer-Verlag, Berlin-Heidelberg, Germany, 1989.

[7] I.Daubechies. *Ten Lectures On Wavelets*. CBS-NSF Regional Conferences in Applied Mathematics, **61**, SIAM, 1992

[8] G. Deodatis and M. Shinozuka. Simulation of stochastic processes by spectral representation. Applied Mechanics Reviews, **44**, No.4, 1991, 191-204.

[9] C.R. Dietrich and G.N. Newsam. Fast and Exact Simulation of Stationary Gaussian Processes through Circulant Embedding of the Covariance Matrix, Siam J. Sci. Comput., **18**, No. 4, 1997, 1088-1107.

[10] Fernique X., Regularite des trajectories des functions aleatoires gaussiennes, *Lectures Notes in Mathematics*, -vol. 480, Berlin, 1975.

[11] Frank W. Elliott, Jr, David J. Horntrop, and Andrew J. Majda. A Fourier-wavelet Monte Carlo method for fractal random fields. *J. Comp. Phys*, 132(2):384–408, 1997.

[12] Frank W. Elliott, Jr and Andrew J. Majda. A wavelet Monte Carlo method for turbulent diffusion with many spatial scales. *J. Comp. Phys*, 113(1):82–111, July 1994.

[13] Frank W. Elliott, Jr and Andrew J. Majda. A new algorithm with plane waves and wavelets for random velocity fields with many spatial scales. *J. Comp. Phys*, 117:146–162, 1995.

[14] Gikhman I.I. and Skorohod A.V. *Introduction to the Theory of Random Processes.* Philadelphia, W.B., Sounders Company, 1969.

[15] Eugenio Hernandez and Guido Weiss. *A First Course on Wavelets*. CRC Press, Inc., 1996.

[16] W. Hardle, G. Kerkyacharian, D. Picard and A. Tsybakov. *Wavelets, Approximation, and Statistical Applications.* Lecture Notes in Statistics, v. 129, Springer-Verlag, 1998.

[17] Ibragimov I.A., Properties of sample functions for stochastic processes and embedding theorems. Theory Probab. Appl., Vol. 18, 442, 1974 (Russian version: Teoriya Veroyatn. Primeneniya, tom 18, 468-480, 1973)

[18] Korolyuk V.S., Portenko N.I., A.V. Skorohod and Turbin A.F., *Handbook of probability theory and mathematical statistics*, Nauka, Moscow, 1985 (In Russian).

[19] Peter R. Kramer. A Review of Some Monte Carlo Simulation Methods for Turbulent Systems. Monte Carlo Methods and Appl., **7**, No. 3-4, 229-244, 2001.

[20] P. Kramer, O. Kurbanmuradov and K. Sabelfeld. Extensions of multiscale Gaussian random field simulation algorithms. Preprint No. 1040, WIAS, Berlin, 2005 (To appear in J. Comp. Physics, 2007).

[21] Kurbanmuradov O. Weak Convergens of Approximate Models of Random Fields. *Russian Journal of Numerical Analysis and Mathematical Modelling*, **10** (1995), N6, 500-517.

[22] Kurbanmuradov O. Weak Convergence of Randomized Spectral Models of Gaussian Random Vector Fields. *Bull.Novosibirsk Computing Center, Numerical Analysis*, issue 4, 19-25, 1993.

[23] Kurbanmuradov O., Sabelfeld K. Stochastic spectral and Fourier-Wavelet methods for vector Gaussian random fields. Monte Carlo Methods and Applications. Vol. 12, No.5-6, 395-446, 2006.

[24] O. Kurbanmuradov, K. Sabelfeld, and D. Koluhin. Stochastic Lagrangian models for two-particle motion in turbulent flows. Numerical results. *Monte Carlo Methods Appl.*, 3(3):199–223, 1997.

[25] A. Majda. Random shearing direction models for isotropic turbulent diffusion. J. Stat. Phys., **75** (1994), 1153.

[26] Y. Meyer. *Ondelettes et opérateurs. I: Ondelettes*, Hermann, Paris, 1990. (English translation: *Wavelets and operators*, Cambridge University Press, 1992.)

[27] Mikhailov G.A. Approximate models of random processes and fields. Russian J. Comp. Mathem. and mathem. Physics, vol.23 (1983), N3, 558-566. (in Russian).

[28] A.S. Monin and A.M. Yaglom. *Statistical Fluid Mechanics: Mechanics of Turbulence*, Volume 2. The M.I.T. Press, 1981.

[29] Nikolskiĭ S.M. *Approximation of function of several variables and imbedding theorems*: 2nd edition, Nauka, Moscow, 1977 (In Russian).

[30] K.K. Phoon, H.W. Huang and S.T. Quek, Comparison betveen Karhunen-Loeve and wavelet expansions for simulation of Gaussian processes. Computers and Structures, **82**, 2004, 985-991.

[31] L. Piterbarg and A.G. Ostrovskii, *Advection and Diffusion in Random Media. Implification for Sea Surface Temperature Anomalies.* Kluwer Acad. Press, 1997.

[32] F. Poirion and C. Soize. Numerical methods and mathematical aspects for simulation of homogenous and non homogenous Gaussian vector fields. In Paul Kree and Walter Wedig, editors, *Probabilistic methods in applied physics*, volume 451 of *Lecture Notes in Physics*, pages 17–53. Springer-Verlag, Berlin, 1995.

[33] Karl K. Sabelfeld. *Monte Carlo methods in boundary value problems,* Springer-Verlag, New York, Berlin, 1991.

[34] M.K. Schneider and A.S. Willsky, A Krylov Subspace Method for Covariance Approximation and Simulation of Random Processes and Fields.,Multidimensional Systems and Signal Processing, Volume 14 , Issue 4 , 2003, 295-318.

[35] M. Shinozuka. Simulation of multivariate and multidimensional random processes. J. of Acoust. Soc. Am. **49** (1971), 357-368.

[36] P. Spanos and R. Ghanem, Stochastic finite element expansion for random media. *Journal of Engineering Mechanics, ASCE,*, **115**, 1989, 1035-1053.

[37] P.D. Spanos and V. Ravi S. Rao. Random Field Representation in a Biorthogonal Wavelet Basis. *Journal of Engineering Mechanics*, **127**, No.2 (2001), 194-205.

[38] Titchmarsh, E.C., Introduction to the theory of Fourier integrals, Second Edition, Oxford, Clarendon press, 1975.

[39] H. Triebel. *Theory of function spaces*. Birkhäuser, Basel-Boston-Stuttgart, Academische Verl. Ges., Leipzig, 1983.

[40] E. Van Marcke, *Random Fields: Analysis and Synthesis*, The MIT Press, Cambrige (Mass.), 1988.

[41] Wojtaszczyk P. A Mathematical Introduction to Wavelets. London Math. Soc., Student Texts 37, Cambridge University Press, 1997.

[42] A.W. Woitishek and S.M. Prigarin, Moment conditions on functional convergence of randomized numerical spectral models of homogeneous Gaussian fields. Theory and Applications of Statistical modeling. Computing Center of Siberian Branch of Acad. Sci. USSR, Novosibirsk, pp. 41-46, 1988.

[43] A. M. Yaglom. *Correlation theory of stationary and related random functions. Volume I: Basic results*. Springer-Verlag, Berlin, 1987.

[44] B.A. Zeldin and P.D. Spanos. Random Field Representation and Synthesis Using Wavelet Basis. *ASME Journal of Applied Mechanics*, **63**, (1996), 946-952.