Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Weak approximation for stochastic differential equations with small noises

Grigori N. Milstein[1] , Michael V. Tret'yakov[2]

submitted: 30th November 1994

[1] Weierstraß–Institut
für Angewandte Analysis
und Stochastik
Mohrenstraße 39
D – 10117 Berlin
Germany

[2] Department of Mathematics
Ural State University
Lenin St., 51
Ekaterinburg 620083
Russia

Preprint No. 123
Berlin 1994

1991 Mathematics Subject Classification. 60H10.
Key words and phrases. Stochastic differential equations, small noises, numerical methods, Monte–Carlo technique.

Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2004975
e–mail (X.400): c=de;a=d400;p=iaas-berlin;s=preprint
e–mail (Internet): preprint@iaas-berlin.d400.de

Abstract

New approach to construction of weak numerical methods, which are intended for Monte-Carlo technique, is proposed for a stochastic system with small noises. The theorem on estimate of method error in terms of product $h^i\varepsilon^j$ (h is a time increment, ε is a small parameter) is proved. Various efficient weak schemes are derived for a general system with small noises and for systems with small additive and small colored noises. The Talay-Tubaro expansion of the global error is considered for such systems. The efficient approach to reduction of the Monte-Carlo error is proposed. The derived methods are tested by calculation of Lyapunov exponents and by simulation of a bistable dynamical system for which multiplicative stochastic resonance is observed.

Contents

1. Introduction

Naturally, for specific systems peculiar numerical methods may be more effective than general methods. One of the important particular cases of a stochastic system is stochastic differential equations with small noises because often fluctuations, which affect a dynamical system, are sufficiently small.

In the previous paper [11] mean-square approximation for stochastic differential equations with small noises was systematically considered. As is known, mean-square numerical methods are used for trajectory simulation of a stochastic dynamical system, for the problem of estimating parameters, and they are the basis for construction of weak schemes. Weak methods [7], [8], [9], [10], [14], [18] are simpler for realization than mean-square ones, and they are effectively applied to calculation of the expectation for functionals of the solution of a stochastic system. For example, by weak schemes it is possible to simulate moments, evaluation of Wiener function space integrals, and to solve problems of mathematical physics by Monte-Carlo technique. That is, weak methods are sufficient for most physical problems. Moreover, weak methods are always efficient as to simulation of needed random variables, while in the case of a general system efficient mean-square schemes have been obtained only with time order 1/2 . However, in the case of a general system third order weak methods already require laborious calculations. Besides, there are no sufficiently efficient high-order weak Runge-Kutta schemes.

Herein we systematically consider weak numerical methods for a stochastic system with small noises

$$dX = a(t,X)dt + \varepsilon \sum_{r=1}^{q} \sigma_r(t,X)dW_r,\ X(t_o) = X_o,\ t \in [t_o, T] \tag{1.1}$$

where $X = (X^1, X^2, \dots, X^n)$, $a(t,x) = (a^1(t,x), \dots, a^n(t,x))$ and $\sigma_r(t,x) = (\sigma_r^1(t,x), \dots, \sigma_r^n(t,x))$, $r = 1, \dots, q$, are n-dimensional vectors, W_r, $r = 1, \dots, q$, are independent standard Wiener processes, and ε is a small parameter.

In the paper we propose an approach to construction of weak methods for a system with small noises; we prove a theorem on relation between properties of one-step weak approximation and estimate of error of the corresponding weak method on the whole interval; we derive various peculiar weak schemes for a system with small noises including explicit Taylor-type methods, implicit methods and Runge-Kutta schemes; the concept of the Talay-Tubaro expansion of the global error is applied to a system with small noises; a method for efficient reduction of the Monte-Carlo error is proposed.

As in the case of mean-square approximation [11], here errors of the proposed methods are also estimated in the terms of product $h^i\varepsilon^j$, where h is a step of a discretization. That is, estimates of errors on the whole interval have a form

$$|E[f(X(T)) - f(\bar{X}(T))]| = O(h^p + \sum_{l \in S} h^l \varepsilon^{J(l)})$$

where f is a function from a sufficiently wide class, $\bar{X}(T)$ is an approximation of the exact solution $X(T)$, p is a natural number, S is a subset of positive integers $l < p$, and $J(l)$ is a decreasing function with natural values. Time-step order of such a method is equal to $l_o = \min_{l \in S} l$ (of course, if S is not empty) which may be low. But under small ε the sum $\sum h^l \varepsilon^{J(l)}$ is also small and, therefore, the method error is sufficiently low.

It gives us an opportunity to construct effective weak methods the time-step order of which is not high but which nevertheless have low errors.

The paper is organized as follows. In Section 2 we briefly explain an approach to construction of weak methods for the system (1.1) and in Section 3 the theorem on estimate of a method error is stated (see Appendix (Section 13) for the proof of the theorem).

Section 4 is devoted to Taylor-type methods for a general system with small noises where weak schemes with errors from $O(h^2 + \varepsilon^2 h)$ up to $O(h^4 + \varepsilon^4 h^2)$ are proposed. Section 5 deals with Runge-Kutta methods for a general system with small noises. We write down full (derivative free) Runge-Kutta methods with the errors from $O(h^2+\varepsilon^2 h)$ up to $O(h^4 + \varepsilon^4 h)$. Runge-Kutta schemes without derivatives of the drift coefficients (semi-Runge-Kutta schemes), which may be useful in the case of simple functions $\sigma_r(t,x)$, are also considered. We obtain the semi-Runge-Kutta schemes with the errors from $O(h^3 + \varepsilon^2 h^2)$ up to $O(h^4 + \varepsilon^4 h^2)$. In Section 6 we propose implicit schemes $O(h^2+\ldots)$. In Section 7 a Stratonovich system with small noises is considered. Sections 8 and 9 are devoted to systems with small additive and colored noises. For a system with small additive noises we propose Taylor-type methods from $O(h^2 + \varepsilon^2 h)$ up to $O(h^4 + \varepsilon^4 h^3)$ and full Runge-Kutta methods from $O(h^2 + \varepsilon^2 h)$ up to $O(h^4 + \varepsilon^4 h^2)$. For a system with colored noises we obtain Taylor-type methods from $O(h^2 + \varepsilon^2 h)$ up to $O(h^4 + \varepsilon^4 h^3)$ but we do not write down them because firstly they follow easily from the corresponding schemes for a system with additive noises, and secondary we propose Runge-Kutta schemes from $O(h^2 + \varepsilon^2 h)$ up to $O(h^4 + \varepsilon^2 h^3)$ which are more effective, as to calculation expenses, than the Taylor-type ones. Note that here we obtain full Runge-Kutta method with the error $O(h^3)$, i.e., the scheme which has the third weak order for a general system with colored noises ($\varepsilon = 1$). Such a method has not been proposed in the previous papers (see ref. [12] and refs. therein). Thus, in Sections 4-9 we write down various methods which, as we believe, would be useful for applications. In the paper we do not give derivations of all methods. We restrict ourselves to the detailed derivations only of two methods for a general system with small noises: the Taylor-type scheme with the error $O(h^4 + \varepsilon^4 h^2)$ [see Section 14] and the semi-Runge-Kutta scheme with the error $O(h^4 + \varepsilon^2 h^2)$ [see Section 15]. These derivations, jointly with Section 2, have all typical features and receptions to derive the other methods of the paper.

In Section 10 we apply the Talay-Tubaro concept [20] of expansion of the global error which is similar to the Runge estimation method for a deterministic system. In contrast to the Talay-Tubaro expansion, according to which the global error is expanded in powers of time increment h, we present the global error in terms of $\varepsilon^i h^j$. We prove a theorem on such an expansion for one of our methods in Appendix (Section 16). As it follows from the proof, the similar expansion can be obtained for the other methods.

As is known, dealing with Monte-Carlo technique we have two errors: error of a weak method and the Monte-Carlo error. The second error can be reduced by increasing the number of independent realizations of the solution. However, it leads to heavy calculation expenses. In Section 11 the effective approach to reduction of the Monte-Carlo error by conversion from the original system to another system with small noises is proposed.

In Section 12 numerical tests of the proposed methods are presented.

2. Preliminary consideration

In connection with the system (1.1) let us introduce an equidistant discretization Δ_N of the interval $[t_o, T]$: $\Delta_N = \{t_i : i = 0, 1, \dots, N; t_o < t_1 < \dots < t_N = T\}$; the time increment $h = t_{i+1} - t_i$; the approximation X_k or $\bar{X}(t_k)$ of the exact solution $X(t_k)$; operators

$$
\begin{aligned}
&L = L_1 + \varepsilon^2 L_2, \\
&L_1 = \frac{\partial}{\partial t} + (a, \frac{\partial}{\partial x}) = \frac{\partial}{\partial t} + \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}, \\
&L_2 = \frac{1}{2}\sum_{r=1}^{q} (\sigma_r, \frac{\partial}{\partial x})^2 = \frac{1}{2}\sum_{r=1}^{q}\sum_{i,j=1}^{n} \sigma_r^i \sigma_r^j \frac{\partial^2}{\partial x^i \partial x^j}, \\
&\Lambda_r = (\sigma_r, \frac{\partial}{\partial x}) = \sum_{i=1}^{n} \sigma_r^i \frac{\partial}{\partial x^i};
\end{aligned}
$$

Ito integrals

$$I_{i_1,\dots,i_j}(t,h) = \int_t^{t+h} dW_{i_j}(\vartheta) \int_t^{\vartheta} dW_{i_{j-1}}(\vartheta_1) \int_t^{\vartheta_1} \dots \int_t^{\vartheta_{j-2}} dW_{i_1}(\vartheta_{j-1})$$

where $i_1, \dots, i_j$ are from the set of numbers $\{0, 1, \dots, q\}$ and $dW_o(\vartheta_i)$ designates $d\vartheta_i$.

We assume that restrictions on the coefficients of the system (1.1) are so that they ensure the existence and uniqueness of the solution on the whole time interval $[t_o, T]$. For construction of high-order methods the coefficients must be sufficiently smooth functions.

To estimate a method error on the whole interval we need properties of the corresponding one-step approximation. According to Theorem 3.1, which will be stated and proved below, under one-step approximation error

$$|Ef(X_{t,x}(t+h)) - Ef(\bar{X}_{t,x}(t+h))| \le K(h^{p+1} + \sum_{l \in S} h^{l+1}\varepsilon^{J(l)}) \qquad (2.1)$$

the error of the corresponding method on the whole interval $[t_o, T]$ is estimated as

$$|Ef(X(t_k)) - Ef(X_k)| \le K(h^p + \sum_{l \in S} h^l \varepsilon^{J(l)}) \qquad (2.2)$$

Thus, to prove error of a weak method we need estimate (2.1). By Taylor expansion of function f it is possible to obtain that if the inequalities

$$|E\prod_{j=1}^{m} \Delta^{i_j} - E\prod_{j=1}^{m} \bar{\Delta}^{i_j}| \le K(h^{p+1} + \sum_{l \in S} h^{l+1}\varepsilon^{J(l)}),\ i_j = 1, \dots, n,\ m = 1, \dots, s-1 \quad (2.3)$$

$$\Delta^{i_j} = X^{i_j}(t+h) - x^{i_j},\ \bar{\Delta}^{i_j} = \bar{X}^{i_j}(t+h) - x^{i_j},\ X(t) = \bar{X}(t) = x$$

$$E\prod_{j=1}^{s} |\bar{\Delta}^{i_j}| \le K(h^{p+1} + \sum_{l \in S} h^{l+1}\varepsilon^{J(l)}),\ i_j = 1, \dots, n \qquad (2.4)$$

are fulfilled for the corresponding s which depends on p then the estimate (2.1) holds.

To construct a one-step approximation the expansion of the exact solution in Ito integrals (the stochastic Taylor-type expansion [7], [10], [21]) is usually used. For instance, in refs. [9], [10] on the base of such an expansion one-step approximation with the time-step order 3 was derived for a general system ($\varepsilon = 1$) which in the case of the system (1.1) has the form

$$X(t+h) = x + \varepsilon \sum_{r=1}^{q} \sigma_r I_r + ha + \varepsilon^2 \sum_{i,r=1}^{q} \Lambda_i \sigma_r I_{ir} + \varepsilon \sum_{r=1}^{q} (L_1 + \varepsilon^2 L_2) \sigma_r I_{or} +$$

$$+ \varepsilon \sum_{r=1}^{q} \Lambda_r a I_{ro} + h^2 (L_1 + \varepsilon^2 L_2) a/2 + \rho \tag{2.5}$$

The coefficients σ_r, a, $\Lambda_i \sigma_r$, etc. in (2.5) are calculated at the point (t, x), and the remainder ρ has the properties

$$|E\rho| \leq Kh^3, \; E\rho^2 \leq K\left[h^6 + \varepsilon^6 h^3\right], \; \varepsilon |E\rho I_r| \leq K\varepsilon^2 h^3,$$

$$\varepsilon^2 |E\rho I_{ir}| \leq K\varepsilon^4 h^3, \; \varepsilon^2 |E\rho I_i I_r| \leq K\varepsilon^4 h^3, \; i, r = 1, \ldots, q \tag{2.6}$$

Then, it can be proved that the error of the one-step approximation $\tilde{X}(t+h) = X(t+h) - \rho$ is equal to $O(h^3)$.

As is known, the important advantage of weak approximations is that they give an opportunity to avoid the problem of simulation of complicated random variables. For instance, the approximation $\tilde{X}$ contains multiple Ito integrals which are difficult to simulate. But on the base of the approximation $\tilde{X}$ the following weak method can be derived (see refs. [9], [10])

$$X_{k+1} = X_k + \varepsilon h^{1/2} \sum_{r=1}^{q} (\sigma_r \xi_r)_k + h a_k + \varepsilon^2 h \sum_{i,r=1}^{q} (\Lambda_i \sigma_r \xi_{ir})_k + \varepsilon h^{3/2} \sum_{r=1}^{q} [(L_1 + \varepsilon^2 L_2) \sigma_r (\xi_r - \eta_r)]_k +$$

$$+ \varepsilon h^{3/2} \sum_{r=1}^{q} [\Lambda_r a \eta_r]_k + h^2 (L_1 + \varepsilon^2 L_2) a_k / 2 \tag{2.7}$$

where the random variables ξ_r, η_r and ξ_{ir} are such that the inequalities (2.3), (2.4) hold with the right sides equal to Kh^3. To fulfill these inequalities it is sufficient to simulate $2q$ independent random variables ξ_r and ζ_r according to the simple laws

$$\eta_r = \xi_r / 2, \; \xi_{ir} = (\xi_i \xi_r - \gamma_{ir} \zeta_i \zeta_r)/2, \; \gamma_{ir} = \begin{cases} -1, & i < j \\ 1, & i \geq j \end{cases}$$

$$P(\xi = 0) = 2/3, \; P(\xi = -\sqrt{3}) = P(\xi = \sqrt{3}) = 1/6,$$

$$P(\zeta = -1) = P(\zeta = 1) = 1/2 \tag{2.8}$$

Due to fulfilling of (2.3) and (2.4) with $p = 2$ and empty S, the error of the method (2.7) on the whole interval is equal to $O(h^2)$ [see (2.2)].

As it has been mentioned in Introduction, the system (1.1) is the peculiar one because error of a method, constructed for this system, can be expanded not only in powers of time increment h but also in powers of small parameter ε. Like it was done in ref. [11], we can transfer some sufficiently simply simulated terms from a remainder to a method thereby reducing the error. And vice versa, terms, simulated in a complicated way and multiplied by ε^α, can be transferred from a method to its remainder; such a procedure reduces (that, of course, is important for applications) calculation expenses but according to smallness of ε it does not lead to substantial increasing of the error.

For instance, we can transfer from the remainder ρ to the method (2.7) the term $h^3L_1^2a/6$ (see details in Section 14). As it turns out, by this way we obtain the method

$$X_{k+1} = \tilde{X}_{k+1} + h^3L_1^2a_k/6 \tag{2.9}$$

where $\tilde{X}_{k+1}$ is from (2.7). The error of this method is equal to $O(h^3 + \varepsilon^2h^2)$ on the whole interval. Although the method (2.9) has time-step accuracy order 2, just as the method (2.7), but h^2 in the error of the method (2.9) is multiplied by ε^2. That is why, the new method has lower error than the method (2.7).

By transferring most complicated (from the computational point of view) terms, for instance, $\varepsilon^3h^{3/2}L_2\sigma_r\xi_r/2$ if the diffusion coefficients σ_r are composite functions, from the method (2.9) to its remainder we obtain another method for the system (1.1). It is found that such a method has order $O(h^3 + \varepsilon^4h)$. Moreover, if after transferring of the terms $\varepsilon^3h^{3/2}L_2\sigma_r\xi_r/2$ we omit the terms $\varepsilon^2h\Lambda_i\sigma_r\xi_{ir}$ in the method (2.9) then it can be proved (such a proof essentially uses the expressions $E\xi_{ir} = 0$, $E\xi_{ir}\xi_j = 0$) that the accuracy of the method, in the sense of smallness with respect to both h and ε, does not reduce. As a result we write down the method

$$X_{k+1} = X_k + \varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k + ha_k + \varepsilon h^{3/2}\sum_{r=1}^{q}[L_1\sigma_r\xi_r]_k/2 + \varepsilon h^{3/2}\sum_{r=1}^{q}[\Lambda_r a\xi_r]_k/2+$$
$$+ h^2(L_1 + \varepsilon^2L_2)a_k/2 + h^3L_1^2a_k/6 \tag{2.10}$$

the accuracy order of which is equal to $O(h^3 + \varepsilon^4h)$. For its realization it is sufficient to simulate only q independent random variables ξ_r according to the law $P(\xi = -1) = P(\xi = 1) = 1/2$. Time-step accuracy order of the method (2.10) is equal to 1, i.e., it is lower than time-step order of the methods (2.7) and (2.9). Nevertheless, under small ε the method (2.10) has sufficiently low error. For instance, if we choose time-step h so that $h = C\varepsilon^\alpha$, $0 < \alpha < 4$, the method (2.10) even exceeds the method (2.7) in the sense of smallness order with respect to ε. Further, if we choose time-step h so that $h = C\varepsilon^\alpha$, $0 < \alpha \le 2$, the method (2.10) does not yield to the method (2.9) in the same sense. It is important to emphasize that in addition the method (2.10) has essentially lower calculation expenses on account of both fewer number of simulated random variables and fewer number of calculated operators.

Thus, we briefly explain the concept of construction of weak methods for a system with small noises. Let us mention that to prove new methods strictly one must thoroughly analyze the remainder, prove relations like (2.6) and the inequalities (2.3), (2.4) for the appropriate p, S and $J(l)$, and apply Theorem 3.1.

3. The theorem on estimate of a method error on the whole interval

Let us consider more general Ito system with small noises than the system (1.1)

$$dX = a(t,X)dt + \varepsilon^2b(t,X)dt + \varepsilon\sum_{r=1}^{q}\sigma_r(t,X)dW_r,\ X(t_o) = X_o,\ t \in [t_o,T] \tag{3.1}$$

where $b(t,x)$ is n-dimensional vector, the other notation is the same as in (1.1). A system like (3.1), even with $\sigma_r \equiv 0$, $r = 1, \dots, q$, is significant by itself. Besides, the reason of the generalization is the following: a Stratonovich system with small noises can be easily rewritten in the form of the system (3.1) [see Section 7].

The operator L for the system (3.1) has the form

$$L = L_1 + \varepsilon^2 \tilde{L}_2, \ \tilde{L}_2 = L_2 + (b, \frac{\partial}{\partial x})$$

where L_1 and L_2 are the same as above.

Definition. *A function $f(x)$ belongs to the class* $\mathbf{F}$, $f \in \mathbf{F}$, *if constants $K > 0$ and $\kappa \geq 0$ are such that the inequality*

$$|f(x)| \leq K(1 + |x|^{\kappa}) \tag{3.2}$$

is fulfilled for any $x \in \mathcal{R}^n$. A function $f(s,x)$, which depends both on $x \in \mathcal{R}^n$ and on a parameter $s \in Q$, belongs to the class $\mathbf{F}$ *(with respect to x) if the inequality (3.2) uniformly fulfills with respect to $s \in Q$.*

Note that below the same letter K is used for various constants, and the same notation $K(x)$ is used for various functions.

Theorem 3.1 *Let us assume that the following conditions are fulfilled*

(1) The coefficients of the system (3.1) are continuous and satisfy the Lipschitz condition, they and their partial derivatives up to sufficiently high order belong to the class $\mathbf{F}$;

(2) The error of a one-step approximation $\bar{X}_{t,x}(t+h)$ of the exact solution $X_{t,x}(t+h)$ of the system (3.1) with initial condition $X(t) = \bar{X}(t) = x$ is estimated as

$$|Ef(X_{t,x}(t+h)) - Ef(\bar{X}_{t,x}(t+h))| \leq K(x)[h^{p+1} + \sum_{l \in S} h^{l+1}\varepsilon^{J(l)}], \ K(x) \in \mathbf{F} \tag{3.3}$$

where function $f(x)$ and its partial derivatives up to sufficiently high order belong to the class $\mathbf{F}$, *S is a subset of positive integers l which are less than natural number p, $J(l)$ is a decreasing function with natural values;*

(3) For sufficiently large number m the moments $E|\bar{X}_k|^m$ exist and are uniformly bounded with respect to N, $k = 0, 1, \dots, N$, and $0 \leq \varepsilon \leq \varepsilon_o$ for some number $\varepsilon_o > 0$.

Then for any N and $k = 0, 1, \dots, N$

$$|Ef(X_{t_o,X_o}(t_k)) - Ef(\bar{X}_{t_o,X_o}(t_k))| \leq K[h^p + \sum_{l \in S} h^l \varepsilon^{J(l)}] \tag{3.4}$$

where the constant K depends on the random variable X_o and on T.

Although the proof of Theorem 3.1 is distinguished from the proof of the weak convergence theorem of refs. [9], [10] in a small way, for completeness of the presentation we provide the detail proof of this important theorem in Appendix (see Section 13).

Remark. If a method is so that it satisfies to the inequality (3.3) either with $p \geq 1$ and $\min_{l \in S} l \geq 1$ in the case of not empty subset S or with $p \geq 1$ and empty S then from Theorem 3.1 it follows the convergence of such a method. However, the primary meaning of Theorem 3.1 is to estimate error of a method on the whole interval in terms of h and ε.

4. Taylor-type weak methods for a general system with small noises

Our aim is to construct weak methods which have low errors under the condition that ε is a small parameter and which are sufficiently effective as to calculation expenses. Herein we present weak methods for the system (3.1) the errors of which are estimated from $O(h^2+\varepsilon^2 h)$ up to $O(h^4+\varepsilon^4 h^2)$. By the developed approach it is possible to derive methods $O(h^5+\ldots)$, $O(h^6+\ldots)$, etc. But we do not write down them because most popular deterministic schemes have orders not higher than four. It is also possible to derive the methods $O(h^3+\varepsilon^6 h^2)$, $O(h^3)$, $O(h^4+\varepsilon^6 h^2)$, $O(h^4+\varepsilon^\alpha h^3)$, $\alpha = 2,\ldots,8$, but they are not proposed for the system (3.1) because of their heavy calculation expenses.

Proofs of the proposed methods are based on estimate of one-step errors and on Theorem 3.1 as it is done in Appendix (see Section 14) for the Taylor-type method $O(h^4+\varepsilon^4 h^2)$.

Let us introduce the notation R for error of a method on the whole interval.

Note that used random variables are mutually independent.

4.1. Euler method

Method $O(h)$ coincides with the well-known weak Euler method.

4.2. Methods $O(h^2+\ldots)$

The first method, which is the simplest one among schemes $O(h^2+\ldots)$, has the form

$$X_{k+1} = X_k + \varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k + h(a+\varepsilon^2 b)_k + h^2 L_1 a_k/2, \tag{4.1}$$

$$R = O(h^2+\varepsilon^2 h)$$

where the random variables ξ_r are simulated as

$$P(\xi=-1) = P(\xi=1) = 1/2 \tag{4.2}$$

The second method is

$$X_{k+1} = X_k + \varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k + h(a+\varepsilon^2 b)_k + \varepsilon h^{3/2}\sum_{r=1}^{q}(L_1\sigma_r\xi_r)_k/2+$$

$$+\varepsilon h^{3/2}\sum_{r=1}^{q}(\Lambda_r a\xi_r)_k/2 + h^2 L_1(a+\varepsilon^2 b)_k/2 + \varepsilon^2 h^2 \tilde{L}_2 a_k/2, \tag{4.3}$$

$$R = O(h^2+\varepsilon^4 h)$$

where the random variables ξ_r are simulated by the law (4.2).

The method $O(h^2)$ has been written down above [see Section 2, the formula (2.7)] for the system (1.1). For the system (3.1) it can be easily rewritten by substituting $a+\varepsilon^2 b$ instead of a (note that the operator L_1 contains a).

4.3. Methods $O(h^3+\ldots)$

The first method, which is the simplest one among schemes $O(h^3+\ldots)$, has the form

$$X_{k+1} = X_k + \varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k + h(a+\varepsilon^2 b)_k + h^2 L_1 a_k/2 + h^3 L_1^2 a_k/6, \tag{4.4}$$

$$R = O(h^3+\varepsilon^2 h)$$

where ξ_r are from (4.2).

The second method is

$$X_{k+1} = X_k + \varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k + h(a+\varepsilon^2 b)_k + \varepsilon h^{3/2}\sum_{r=1}^{q}(L_1\sigma_r\xi_r)_k/2+$$

$$+\,\varepsilon h^{3/2}\sum_{r=1}^{q}(\Lambda_r a\xi_r)_k/2 + h^2 L_1(a+\varepsilon^2 b)_k/2 + \varepsilon^2 h^2\tilde{L}_2 a_k/2 + h^3 L_1^2 a_k/6, \tag{4.5}$$

$$R = O(h^3+\varepsilon^4 h)$$

where ξ_r are from (4.2).

The third method is

$$X_{k+1} = X_k + \varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k + h(a+\varepsilon^2 b)_k + \varepsilon^2 h\sum_{i,r=1}^{q}(\Lambda_i\sigma_r\xi_{ir})_k+$$

$$+\varepsilon h^{3/2}\sum_{r=1}^{q}([L_1+\varepsilon^2\tilde{L}_2]\sigma_r\xi_r)_k/2 + \varepsilon h^{3/2}\sum_{r=1}^{q}(\Lambda_r(a+\varepsilon^2 b)\xi_r)_k/2+$$

$$+\,h^2[L_1+\varepsilon^2\tilde{L}_2](a+\varepsilon^2 b)_k/2 + h^3 L_1^2 a_k/6, \tag{4.6}$$

$$R = O(h^3+\varepsilon^2 h^2)$$

where the random variables ξ_r and ξ_{ir} are simulated according to either [9], [10]

$$P(\xi=0)=2/3,\ P(\xi=-\sqrt{3}) = P(\xi=\sqrt{3}) = 1/6,$$

$$\xi_{ir} = (\xi_i\xi_r - \gamma_{ir}\zeta_i\zeta_r)/2,\ \gamma_{ir} = \begin{cases} -1, & i<r \\ 1, & i\geq r \end{cases},\ P(\zeta=-1) = P(\zeta=1) = 1/2 \tag{4.7}$$

or [18]

$$P(\xi=0)=2/3,\ P(\xi=-\sqrt{3}) = P(\xi=\sqrt{3}) = 1/6,$$

$$\xi_{ir} = (\xi_i\xi_r - \zeta_{ir})/2, \zeta_{ii}=1,\ \zeta_{ir} = -\zeta_{ri}, i\neq r$$

$$P(\zeta_{ir}=-1) = P(\zeta_{ir}=1) = 1/2, i<r \tag{4.8}$$

The fourth method, which is the most accurate one among the proposed schemes $O(h^3+\ldots)$, has the form

$$X_{k+1} = X_k + \varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k + h(a+\varepsilon^2 b)_k + \varepsilon^2 h\sum_{i,r=1}^{q}(\Lambda_i\sigma_r\xi_{ir})_k+$$

$$+\varepsilon h^{3/2}\sum_{r=1}^{q}([L_1+\varepsilon^2\tilde{L}_2]\sigma_r(\xi_r-\mu_r))_k + \varepsilon h^{3/2}\sum_{r=1}^{q}(\Lambda_r(a+\varepsilon^2 b)\mu_r)_k+$$

$$+h^2[L_1+\varepsilon^2\tilde{L}_2](a+\varepsilon^2 b)_k/2+\varepsilon h^{5/2}\sum_{r=1}^{q}((L_1^2\sigma_r+L_1\Lambda_r a+\Lambda_r L_1 a)\xi_r)_k/6+$$

$$+h^3L_1^2(a+\varepsilon^2 b)_k/6+\varepsilon^2h^3(L_1\tilde{L}_2+\tilde{L}_2L_1)a_k/6, \quad (4.9)$$

$$R=O(h^3+\varepsilon^4h^2)$$

where ξ_r, ξ_{ir} and μ_r are simulated according to either

$$P(\xi=0)=2/3,\ P(\xi=-\sqrt{3})=P(\xi=\sqrt{3})=1/6,\ \mu_r=\xi_r/2+\zeta_r/\sqrt{12},$$

$$\xi_{ir}=(\xi_i\xi_r-\gamma_{ir}\zeta_i\zeta_r)/2,\ \gamma_{ir}=\begin{cases}-1, & i<r\\ 1, & i\geq r\end{cases},\ P(\zeta=-1)=P(\zeta=1)=1/2 \quad (4.10)$$

or

$$P(\xi=0)=2/3,\ P(\xi=-\sqrt{3})=P(\xi=\sqrt{3})=1/6,$$

$$\xi_{ir}=(\xi_i\xi_r-\zeta_{ir})/2, \zeta_{ii}=1,\ \zeta_{ir}=-\zeta_{ri}, i\neq r,\ P(\zeta_{ir}=-1)=P(\zeta_{ir}=1)=1/2, i<r,$$

$$\mu_r=\xi_r/2+\eta_r,\ P(\eta=-1/\sqrt{12})=P(\eta=1/\sqrt{12})=1/2 \quad (4.11)$$

4.4. Methods $O(h^4+\ldots)$

The first method, which is the simplest one of the family $O(h^4+\ldots)$, has the form

$$X_{k+1}=X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k+h(a+\varepsilon^2 b)_k+h^2L_1a_k/2+h^3L_1^2a_k/6+h^4L_1^3a_k/24, \quad (4.12)$$

$$R=O(h^4+\varepsilon^2 h)$$

where ξ_r are from (4.2).

The second method is

$$X_{k+1}=X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k+h(a+\varepsilon^2 b)_k+\varepsilon h^{3/2}\sum_{r=1}^{q}(L_1\sigma_r\xi_r)_k/2+\varepsilon h^{3/2}\sum_{r=1}^{q}(\Lambda_r a\xi_r)_k/2+$$

$$+h^2L_1(a+\varepsilon^2 b)_k/2+\varepsilon^2h^2\tilde{L}_2a_k/2+h^3L_1^2a_k/6+h^4L_1^3a_k/24, \quad (4.13)$$

$$R=O(h^4+\varepsilon^2h^2+\varepsilon^4 h)$$

where ξ_r are simulated as in (4.2).

The third method is

$$X_{k+1}=X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k+h(a+\varepsilon^2 b)_k+\varepsilon h^{3/2}\sum_{r=1}^{q}(L_1\sigma_r(\xi_r/2-\eta_r))_k+$$

$$+\varepsilon h^{3/2}\sum_{r=1}^{q}(\Lambda_r a(\xi_r/2+\eta_r))_k+h^2L_1(a+\varepsilon^2 b)_k/2+\varepsilon^2h^2\tilde{L}_2a_k/2+$$

$$+\varepsilon h^{5/2}\sum_{r=1}^{q}((L_1^2\sigma_r+L_1\Lambda_r a+\Lambda_r L_1 a)\xi_r)_k/6+h^3L_1^2(a+\varepsilon^2 b)_k/6+$$

$$+\varepsilon^2h^3(L_1\tilde{L}_2+\tilde{L}_2L_1)a_k/6+h^4L_1^3a_k/24, \quad (4.14)$$

$$R=O(h^4+\varepsilon^4 h)$$

where the random variables ξ_r and η_r are simulated as

$$P(\xi=-1)=P(\xi=1)=1/2,\ P(\eta=-1/\sqrt{12})=P(\eta=1/\sqrt{12})=1/2 \qquad (4.15)$$

The fourth method is

$$X_{k+1}=X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k+h(a+\varepsilon^2 b)_k+\varepsilon^2 h\sum_{i,r=1}^{q}(\Lambda_i\sigma_r\xi_{ir})_k+$$

$$+\varepsilon h^{3/2}\sum_{r=1}^{q}([L_1+\varepsilon^2\tilde{L}_2]\sigma_r\xi_r)_k/2+\varepsilon h^{3/2}\sum_{r=1}^{q}(\Lambda_r(a+\varepsilon^2 b)\xi_r)_k/2+$$

$$+h^2[L_1+\varepsilon^2\tilde{L}_2](a+\varepsilon^2 b)_k/2+h^3L_1^2a_k/6+h^4L_1^3a_k/24, \qquad (4.16)$$

$$R=O(h^4+\varepsilon^2h^2)$$

where the random variables ξ_r and ξ_{ir} are simulated according to either (4.7) or (4.8).

The fifth method, which is the most complicated and the most accurate one among the proposed Taylor-type schemes for the system (3.1), has the form

$$X_{k+1}=X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k+h(a+\varepsilon^2 b)_k+\varepsilon^2 h\sum_{i,r=1}^{q}(\Lambda_i\sigma_r\xi_{ir})_k+$$

$$+\varepsilon h^{3/2}\sum_{r=1}^{q}([L_1+\varepsilon^2\tilde{L}_2]\sigma_r(\xi_r-\mu_r))_k+\varepsilon h^{3/2}\sum_{r=1}^{q}(\Lambda_r(a+\varepsilon^2 b)\mu_r)_k+$$

$$+h^2[L_1+\varepsilon^2\tilde{L}_2](a+\varepsilon^2 b)_k/2+\varepsilon h^{5/2}\sum_{r=1}^{q}((L_1^2\sigma_r+L_1\Lambda_r a+\Lambda_r L_1 a)\xi_r)_k/6+$$

$$+h^3L_1^2(a+\varepsilon^2 b)_k/6+\varepsilon^2h^3(L_1\tilde{L}_2+\tilde{L}_2L_1)a_k/6+h^4L_1^3a_k/24, \qquad (4.17)$$

$$R=O(h^4+\varepsilon^4h^2)$$

where the needed random variables are simulated either as in (4.10) or as in (4.11). In Appendix (see Section 14) the method (4.17) is derived in details in the case of $b\equiv 0$.

4.5. Remark on selection of increment h depending on ε

Let us illustrate selection of time increment h depending on parameter ε by the methods of Subsection 4.4.

Let us choose time increment h so that $h=C\varepsilon^\alpha$. Then error of a method on the whole interval can be estimated in powers of small parameter ε

$$R=O(\varepsilon^\beta)$$

where

$$\beta=\min\left\{\alpha p,\min_{l\in S}(\alpha l+J(l))\right\}$$

If $h=C\varepsilon^\alpha$, the method (4.17) gives $R=O(\varepsilon^{4\alpha}+\varepsilon^{2\alpha+4})$, and the method (4.12) has $R=O(\varepsilon^{4\alpha}+\varepsilon^{\alpha+2})$. In the case of $0<\alpha\le 2/3$ both errors are estimated by $O(\varepsilon^{4\alpha})$, and so, both methods have the same order with respect to ε. However, if $\alpha>2/3$, the method (4.17) has higher order with respect to ε than (4.12) [for instance, if $\alpha=2$, we have $O(\varepsilon^8)$ for (4.17) and $O(\varepsilon^4)$ for (4.12)]. Thus, in the case of selection of comparatively large time increment h with respect to ε (it may be interesting in the case of sufficiently small ε, when an error, estimated by ε^β, is not large), complicated methods like (4.17) and sufficiently simple methods like (4.12) have the same order with respect to ε. And, usually, in such a situation simple methods are preferable because of considerably lower calculation expenses. But, if one wants to reach high order error with respect to ε, complicated methods are preferable.

5. Runge-Kutta weak methods for a general system with small noises

To reduce calculations of derivatives in the methods of Section 4 we propose Runge-Kutta schemes. Herein we consider (i) full (derivative free) Runge-Kutta schemes, (ii) Runge-Kutta schemes without derivatives of the coefficients $a(t,x)$ and $b(t,x)$ but with derivatives of the diffusion coefficients $\sigma_r(t,x)$ (semi-Runge-Kutta schemes) which may be useful in the case of simple functions σ_r.

As is known [7], [10], in the case of a general system ($\varepsilon = 1$) there are no sufficiently constructive high-order Runge-Kutta schemes. Here we obtain full Runge-Kutta methods with the errors $O(h^2+\varepsilon^2 h)$, $O(h^2+\varepsilon^4 h)$, $O(h^3+\varepsilon^2 h)$, $O(h^3+\varepsilon^4 h)$, $O(h^4+\varepsilon^2 h)$, $O(h^4+\varepsilon^2 h^2+\varepsilon^4 h)$ and $O(h^4+\varepsilon^4 h)$. For higher orders we have succeeded in construction of semi-Runge-Kutta schemes with the errors $O(h^3+\varepsilon^2 h^2)$, $O(h^3+\varepsilon^4 h^2)$, $O(h^4+\varepsilon^2 h^2)$ and $O(h^4+\varepsilon^4 h^2)$.

In Appendix (see Section 15) we give the detailed derivation of the semi-Runge-Kutta scheme $O(h^4+\varepsilon^2 h^2)$. The other Runge-Kutta methods are obtained in the same way.

To construct the Runge-Kutta methods for the system (3.1) we use as a subsidiary tool deterministic Runge-Kutta methods. To this end we select most convenient, from our point of view, concrete deterministic schemes. Obviously, it is possible to derive families of stochastic Runge-Kutta methods which are similar to the proposed ones but which use other deterministic Runge-Kutta schemes.

5.1. Methods $O(h^2+\ldots)$

The first method, which is the simplest one, has the form

$$X_{k+1} = X_k + \varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k + \varepsilon^2 h b_k + h[a_k + a(t+h, X_k + ha_k)]/2, \tag{5.1}$$

$$R = O(h^2+\varepsilon^2 h)$$

where ξ_r are from (4.2).

The second method, which has the same error as the Taylor-type method (4.3), is

$$X_{k+1} = X_k + \varepsilon h^{1/2}\sum_{r=1}^{q}[\sigma_r(t_k, X_k) + \sigma_r(t_{k+1}, X_k + ha_k)]\xi_{r_k}/2 + h[a_k +$$

$$+ a(t_{k+1}, X_k + \varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k + h(a+\varepsilon^2 b)_k)]/2 + \varepsilon^2 h[b_k + b(t_{k+1}, X_k + ha_k)]/2, \tag{5.2}$$

$$R = O(h^2+\varepsilon^4 h)$$

where ξ_r are from (4.2).

5.2. Methods $O(h^3+\ldots)$

The first method is

$$X_{k+1}=X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k+\varepsilon^2hb_k+[k_1+4k_2+k_3]/6, \tag{5.3}$$

$$R=O(h^3+\varepsilon^2h)$$

where

$$k_1=ha_k,\ k_2=ha(t_{k+1/2},X_k+k_1/2),\ k_3=ha(t_{k+1},X_k-k_1+2k_2) \tag{5.4}$$

and ξ_r are from (4.2).

The second method is

$$X_{k+1}=X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}[\sigma_r(t_k,X_k)+\sigma_r(t_{k+1},X_k+ha_k)]\xi_{r_k}/2+$$

$$+[k_1+4k_2+k_3]/6+\varepsilon^2h[b_k+b(t_{k+1},X_k+ha_k)]/2, \tag{5.5}$$

$$R=O(h^3+\varepsilon^4h)$$

where

$$k_1=ha_k,\ k_2=ha(t_{k+1/2},X_k+\alpha_1\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k+k_1/2),$$

$$k_3=ha(t_{k+1},X_k+\alpha_2\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k-k_1+2k_2+3\varepsilon^2hb_k),$$

$$\alpha_1=(6\pm\sqrt{6})/10,\ \alpha_2=(3\mp2\sqrt{6})/5 \tag{5.6}$$

and ξ_r are from (4.2).

The third method is

$$X_{k+1}=X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}[\sigma_r(t_k,X_k)+\sigma_r(t_{k+1},X_k)]\xi_{r_k}/2+\varepsilon^2h\sum_{i,r=1}^{q}(\Lambda_i\sigma_r\xi_{ir})_k+$$

$$+\varepsilon h^{3/2}\sum_{r=1}^{q}\sum_{i=1}^{n}\left((a^i+\varepsilon^2b^i)\frac{\partial\sigma_r}{\partial x^i}\xi_r\right)_k/2+\varepsilon^3h^{3/2}\sum_{r=1}^{q}(L_2\sigma_r\xi_r)_k/2+[k_1+4k_2+k_3]/6+$$

$$+\varepsilon^2h[b_k+b(t_{k+1},X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k+h(a+\varepsilon^2b)_k)]/2, \tag{5.7}$$

$$R=O(h^3+\varepsilon^2h^2)$$

where k_i, $i=1,2,3$, are from (5.6) and the needed random variables ξ_r, ξ_{ir} are simulated as in the method (4.6). The method (5.7) contains the operators with first and second derivatives of the diffusion coefficients σ_r with respect to x.

We have obtained the semi-Runge-Kutta method with the error $R=O(h^3+\varepsilon^4h^2)$. But it requires the same number of coefficient recalculations as the semi-Runge-Kutta method $O(h^4+\varepsilon^4h^2)$ which is provided below [see (5.17)]. So, here we do not write down the Runge-Kutta scheme $O(h^3+\varepsilon^4h^2)$.

5.3. Methods $O(h^4 + \ldots)$

The first method, which is the simplest Runge-Kutta scheme among methods $O(h^4 + \ldots)$, has the form

$$X_{k+1} = X_k + \varepsilon h^{1/2} \sum_{r=1}^{q} (\sigma_r \xi_r)_k + \varepsilon^2 h b_k + [k_1 + 2k_2 + 2k_3 + k_4]/6, \tag{5.8}$$

$$R = O(h^4 + \varepsilon^2 h)$$

where

$$k_1 = ha_k,\ k_2 = ha(t_{k+1/2}, X_k + k_1/2),\ k_3 = ha(t_{k+1/2}, X_k + k_2/2)$$

$$k_4 = ha(t_{k+1}, X_k + k_3) \tag{5.9}$$

and ξ_r are from (4.2).

The second full Runge-Kutta method, which has the same error as the Taylor-type method (4.13), is

$$X_{k+1} = X_k + \varepsilon h^{1/2} \sum_{r=1}^{q} [\sigma_r(t_k, X_k) + \sigma_r(t_{k+1}, X_k + ha_k)]\xi_{r_k}/2+$$

$$+ [k_1 + 2k_2 + 2k_3 + k_4]/6 + \varepsilon^2 h[b_k + b(t_{k+1}, X_k + ha_k)]/2, \tag{5.10}$$

$$R = O(h^4 + \varepsilon^2 h^2 + \varepsilon^4 h)$$

where

$$k_1 = ha_k,\ k_2 = ha(t_{k+1/2}, X_k + k_1/2),\ k_3 = ha(t_{k+1/2}, X_k + \varepsilon h^{1/2} \sum_{r=1}^{q} (\sigma_r \xi_r)_k + k_2/2),$$

$$k_4 = ha(t_{k+1}, X_k + \varepsilon h^{1/2} \sum_{r=1}^{q} (\sigma_r \xi_r)_k + k_3 + 3\varepsilon^2 h b_k), \tag{5.11}$$

and ξ_r are from (4.2).

The third full Runge-Kutta method is

$$X_{k+1} = X_k + \varepsilon h^{1/2} \sum_{r=1}^{q} [\sigma_r(t_k, X_k)(\xi_r + 6\eta_r)_k + 4\sigma_r(t_{k+1/2}, X_k + k_2/2)\xi_{r_k}+$$

$$+\sigma_r(t_{k+1}, X_k + k_1)(\xi_r - 6\eta_r)_k]/6 + h[a(t_k, X_k + \varepsilon h^{1/2} \sum_{r=1}^{q} (\sigma_r \eta_r)_k)-$$

$$- a(t_k, X_k - \varepsilon h^{1/2} \sum_{r=1}^{q} (\sigma_r \eta_r)_k)]/2 + [k_1 + 2k_2 + 2k_3 + k_4]/6 + \varepsilon^2[l_1 + 3l_2]/4, \tag{5.12}$$

$$R = O(h^4 + \varepsilon^4 h)$$

where

$$k_1 = ha_k,\ k_2 = ha(t_{k+1/2}, X_k + k_1/2),$$

$$k_3 = ha(t_{k+1/2}, X_k + \varepsilon h^{1/2} \sum_{r=1}^{q} (\sigma_r \xi_r)_k + k_2/2 + \varepsilon^2 l_1/4 + 3\varepsilon^2 l_2/4),$$

$$k_4 = ha(t_{k+1}, X_k + \varepsilon h^{1/2} \sum_{r=1}^{q} \sigma_r(t_{k+1}, X_k + k_1)\xi_{r_k} + k_3 + \varepsilon^2 l_1),$$

$$l_1 = hb_k,\ l_2 = hb(t_k + 2h/3, X_k + 2k_1/9 + 4k_2/9) \qquad (5.13)$$

and ξ_r, η_r are simulated as in (4.15). This full Runge-Kutta method requires six recalculations of the function $a(t,x)$, three recalculations of the functions $\sigma_r(t,x)$ and two recalculations of the function $b(t,x)$.

The fourth method, which contains first and second derivatives of the functions σ_r with respect to x, is

$$X_{k+1} = X_k + \varepsilon h^{1/2} \sum_{r=1}^{q} [\sigma_r(t_k, X_k) + \sigma_r(t_{k+1}, X_k)]\xi_{r_k}/2 + \varepsilon^2 h \sum_{i,r=1}^{q} (\Lambda_i \sigma_r \xi_{ir})_k +$$

$$+\varepsilon h^{3/2} \sum_{r=1}^{q} \sum_{i=1}^{n} \left((a^i + \varepsilon^2 b^i) \frac{\partial \sigma_r}{\partial x^i} \xi_r \right)_k /2 + \varepsilon^3 h^{3/2} \sum_{r=1}^{q} (L_2 \sigma_r \xi_r)_k/2 + [k_1 + 2k_2 + 2k_3 + k_4]/6 +$$

$$+ \varepsilon^2 h [b_k + b(t_{k+1}, X_k + \varepsilon h^{1/2} \sum_{r=1}^{q} (\sigma_r \xi_r)_k + h(a + \varepsilon^2 b)_k)]/2, \qquad (5.14)$$

$$R = O(h^4 + \varepsilon^2 h^2)$$

where k_i, $i = 1, \ldots, 4$, are from (5.13) and the needed random variables ξ_r, ξ_{ir} are simulated as in the method (4.16). The method (5.14) is derived in Appendix (see Section 15).

Let us note that in the case of *one noise* ($q = 1$) we succeeded in constructing of the full Runge-Kutta method with the error estimated by $O(h^4 + \varepsilon^2 h^2)$. It has the form

$$X_{k+1} = X_k + \varepsilon h^{1/2} \left\{ \sigma(t_k, X_k)(\xi)_k + \sigma(t_{k+1}, X_k - \varepsilon h^{1/2} \sigma(t_k, X_k) + h(a + \varepsilon^2 b)_k)[\xi - \xi^2 + \right.$$

$$\left. +1]_k/2 + \sigma(t_{k+1}, X_k + \varepsilon h^{1/2} \sigma(t_k, X_k) + h(a + \varepsilon^2 b)_k)[\xi + \xi^2 - 1]_k/2 \right\} /2 + [k_1 + 2k_2 +$$

$$+ 2k_3 + k_4]/6 + \varepsilon^2 h [b_k + b(t_{k+1}, X_k + \varepsilon h^{1/2} \sigma(t_k, X_k)(\xi)_k + h(a + \varepsilon^2 b)_k)]/2, \qquad (5.15)$$

$$R = O(h^4 + \varepsilon^2 h^2)$$

where

$$k_1 = ha_k,\ k_2 = ha(t_{k+1/2}, X_k + k_1/2),\ k_3 = ha(t_{k+1/2}, X_k + \varepsilon h^{1/2}(\sigma\xi)_k + k_2/2),$$

$$k_4 = ha(t_{k+1}, X_k + \varepsilon h^{1/2}(\sigma\xi)_k + k_3 + 3\varepsilon^2 h b_k),$$

and the needed random variables ξ are simulated as

$$P(\xi = 0) = 2/3,\ P(\xi = -\sqrt{3}) = P(\xi = \sqrt{3}) = 1/6 \qquad (5.16)$$

The last semi-Runge-Kutta scheme is

$$X_{k+1} = X_k + \varepsilon h^{1/2} \sum_{r=1}^{q} [\sigma_r(t_k, X_k)(\xi_r + 6\eta_r)_k + 4\sigma_r(t_{k+1/2}, X_k + k_2/2)\xi_{r_k} +$$

$$+\sigma_r(t_{k+1},X_k+k_1)(\xi_r-6\eta_r)_k]/6+h[a(t_k,X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\eta_r)_k)-$$

$$-a(t_k,X_k-\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\eta_r)_k)]/2+\varepsilon^2 h\sum_{i,r=1}^{q}(\Lambda_i\sigma_r\xi_{ir})_k+$$

$$+\varepsilon^3 h^{3/2}\sum_{r=1}^{q}[\tilde{L}_2\sigma_r\xi_r/2]_k+[k_1+2k_2+2k_3+k_4]/6+\varepsilon^2[l_1+3l_2]/4, \tag{5.17}$$

$$R=O(h^4+\varepsilon^4h^2)$$

where

$$k_1=ha_k,\ k_2=ha(t_{k+1/2},X_k+k_1/2),$$

$$k_3=ha(t_{k+1/2},X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k+k_2/2+\varepsilon^2l_1/4+3\varepsilon^2l_2/4),$$

$$k_4=ha(t_{k+1},X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}\sigma_r(t_{k+1},X_k+k_1)\xi_{r_k}+k_3+\varepsilon^2l_1),$$

$$l_1=hb(t_k,X_k+\varepsilon h^{1/2}(1+\sqrt{3})\sum_{r=1}^{q}(\sigma_r\xi_r)_k/2),\ l_2=hb(t_k+2h/3,X_k+2\varepsilon^2l_1/3+$$

$$+2k_1/9+4k_2/9+\varepsilon h^{1/2}(3-\sqrt{3})\sum_{r=1}^{q}(\sigma_r\xi_r)_k/6) \tag{5.18}$$

and the needed random variables are simulated as

$$\xi_{ir}=(\xi_i\xi_r-\gamma_{ir}\zeta_i\zeta_r)/2,\ \gamma_{ir}=\begin{cases}-1, & i<r\\ 1, & i\ge r\end{cases},\ \eta_r=\zeta_r/\sqrt{12},$$

$$P(\xi=0)=2/3,\ P(\xi=-\sqrt{3})=P(\xi=\sqrt{3})=1/6,\ P(\zeta=-1)=P(\zeta=1)=1/2 \tag{5.19}$$

6. Implicit weak methods for a general system with small noises

We write down implicit methods only with the orders $O(h^2+\varepsilon^2h)$ and $O(h^2+\varepsilon^4h)$. Some implicit methods for a general system ($\varepsilon=1$) may be found in refs. [7], [10]. We do not write down implicit methods which under $\varepsilon=0$ have time-step orders higher than 2 because, as is known, increasing of time-increment order leads to deterioration of stability properties of a method.

The family of implicit methods with $R=O(h^2{+}\varepsilon^2h)$ is the following

$$X_{k+1}=X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k+h[\alpha a_k+(1-\alpha)a_{k+1}]+\varepsilon^2h[\alpha b_k+(1-\alpha)b_{k+1}]+$$

$$+h^2\beta(2\alpha-1)L_1a_k/2+h^2(1-\beta)(2\alpha-1)L_1a_{k+1}/2 \tag{6.1}$$

where $0\le\alpha\le1$, $0\le\beta\le1$, and ξ_r are from (4.2).

The family of implicit methods with $R = O(h^2 + \varepsilon^4 h)$ has the form

$$X_{k+1} = X_k + \varepsilon h^{1/2} \sum_{r=1}^{q} (\sigma_r \xi_r)_k + h[\alpha a_k + (1-\alpha) a_{k+1}] + \varepsilon^2 h[\alpha b_k + (1-\alpha) b_{k+1}] +$$

$$+\varepsilon h^{3/2} \sum_{r=1}^{q} [(L_1 \sigma_r + (2\alpha - 1)\Lambda_r a)\xi_r]_k/2 + h^2 \beta (2\alpha - 1)(L_1 + \varepsilon^2 \tilde{L}_2) a_k/2 +$$

$$+h^2(1-\beta)(2\alpha - 1)(L_1 + \varepsilon^2 \tilde{L}_2) a_{k+1}/2 + \varepsilon^2 h^2 \beta (2\alpha - 1) L_1 b_k/2 +$$

$$+ \varepsilon^2 h^2 (1-\beta)(2\alpha - 1) L_1 b_{k+1}/2 \tag{6.2}$$

where $0 \le \alpha \le 1$, $0 \le \beta \le 1$, and ξ_r are from (4.2).

These methods follow from the corresponding implicit schemes of ref. [10] by omitting of the terms which do not influence on the one-step errors $O(h^3 + \varepsilon^2 h^2)$ and $O(h^3 + \varepsilon^4 h^2)$ of the methods (6.1) and (6.2) respectively.

Under α equal to $1/2$ we obtain the simplest schemes of the families (6.1) and (6.2). In this case the method (6.1) becomes *the derivative free implicit Runge-Kutta scheme.* Under $\alpha = 1/2$ the method (6.2) contains derivatives of the diffusion coefficients $\sigma_r(t,x)$ but it is possible to obtain *the derivative free implicit Runge-Kutta scheme with* $R = O(h^2 + \varepsilon^4 h)$

$$X_{k+1} = X_k + \varepsilon h^{1/2} \sum_{r=1}^{q} \{\sigma_r(t_k) + \sigma_r(t_{k+1}, X_k + h a_k)\} \xi_{r_k}/2 +$$

$$+ h(a_k + a_{k+1})/2 + \varepsilon^2 h (b_k + b_{k+1})/2 \tag{6.3}$$

where ξ_r are from (4.2).

7. Stratonovich system with small noises

It is known that a stochastic system in the Stratonovich sense (marked by "*")

$$dX = a(t,X)dt + \varepsilon^2 c(t,X)dt + \varepsilon \sum_{r=1}^{q} \sigma_r(t,X) * dW_r, \ X(t_o) = X_o, \ t \in [t_o, T] \tag{7.1}$$

is equivalent to the following system in the Ito sense

$$dX = a(t,X)dt + \varepsilon^2 b(t,X)dt + \varepsilon \sum_{r=1}^{q} \sigma_r(t,X) dW_r \tag{7.2}$$

where

$$b(t,x) = c(t,x) + \frac{1}{2} \sum_{r=1}^{q} \frac{\partial \sigma_r}{\partial x}(t,x)\sigma_r(t,x) \tag{7.3}$$

In Sections 4-6 we have proposed weak methods for the Ito system in the form of (7.2). Thus, the methods of Sections 4-6 are also appropriate for the Stratonovich system (7.1). Let us note that the full Runge-Kutta methods of Section 5 are not full for the system (7.1) because $b(t,x)$ in (7.3) contains derivatives $\dfrac{\partial \sigma_r}{\partial x}$. But if the diffusion coefficients σ_r are simple functions, the methods of Section 5 may be efficient and useful for the Stratonovich system (7.1). Nevertheless, in some cases we obtain the full

Runge-Kutta schemes for (7.1). Here we restrict ourselves to the proposition of the full Runge-Kutta method with $R = O(h^4 + \varepsilon^2 h^2)$ for the Stratonovich system *with one noise*

$$X_{k+1} = X_k + (n_1 + 2n_2 + 2n_3 + n_4)/6 + (k_1 + 2k_2 + 2k_3 + k_4)/6 + (l_1 + l_2)/2 \quad (7.4)$$

where

$$n_1 = \varepsilon h^{1/2}(\sigma\xi)_k,\ n_2 = \varepsilon h^{1/2}\sigma(t_{k+1/2}, X_k + n_1/2 + k_1/2 + l_1/2)(\xi)_k,$$

$$n_3 = \varepsilon h^{1/2}\sigma(t_{k+1}, X_k + n_2/2 + k_1 + l_1)(\xi)_k,\ n_4 = \varepsilon h^{1/2}\sigma(t_k, X_k + n_3)(\xi)_k,$$

$$k_1 = ha_k,\ k_2 = ha(t_{k+1/2}, X_k + k_1/2),\ k_3 = ha(t_{k+1/2}, X_k + k_2/2 + n_2),$$

$$k_4 = ha(t_{k+1}, X_k + k_3 + n_2 + 3l_1),\ l_1 = \varepsilon^2 hc_k, l_2 = \varepsilon^2 hc(t_{k+1}, X_k + l_1 + k_1 + n_2)$$

and ξ_k are simulated as in (5.16).

8. Weak methods for a system with small additive noises

The important particular case of the system with small noises is the system with small additive noises

$$dX = a(t,X)dt + \varepsilon\sum_{r=1}^{q}\sigma_r(t)dW_r,\ X(t_o) = X_o,\ t \in [t_o, T] \quad (8.1)$$

Here for simplicity we restrict ourselves to the system with $b \equiv 0$ (note that in the case of additive noises the Stratonovich system coincides with the Ito one). For the system (8.1) we obtain methods with the errors estimated by $O(h^3 + \varepsilon^6 h^2)$, $O(h^3)$, $O(h^4 + \varepsilon^2 h^3 + \varepsilon^6 h^2)$, $O(h^4 + \varepsilon^6 h^2)$, $O(h^4 + \varepsilon^2 h^3)$, $O(h^4 + \varepsilon^4 h^3)$ and also with the same orders as in Section 4. Let us note that methods $O(h^4 + \varepsilon^6 h^3)$ and $O(h^4 + \varepsilon^8 h^3)$ are too complicated, and therefore we do not write down them.

Methods for the system (8.1) with the same orders as in Section 4 follow from the corresponding methods for a general system with small noises taking into account that for the system (8.1) we have

$$\Lambda_r\sigma_i = 0,\ L_2\sigma_i = 0,\ L_1\sigma_i = \frac{d\sigma_i}{dt},\ b = 0$$

8.1. Taylor-type methods

Methods $O(h^2 + \ldots)$ easily follow from the corresponding methods of Section 4, and here we do not write down them.

Methods $O(h^3 + \ldots)$, except methods $O(h^3 + \varepsilon^6 h^2)$ and $O(h^3)$, can be also written down from the corresponding methods of Section 4.

The method $O(h^3 + \varepsilon^6 h^2)$ is written as

$$X_{k+1} = X_k + \varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k + ha_k + \varepsilon h^{3/2}\sum_{r=1}^{q}\left(\frac{d\sigma_r}{dt}(\xi_r/2 - \eta_r)\right)_k +$$

$$+\varepsilon h^{3/2}\sum_{r=1}^{q}(\Lambda_r a(\xi_r/2+\eta_r))_k + h^2(L_1+\varepsilon^2 L_2)a_k/2+$$

$$+\varepsilon^2 h^2\sum_{r=1}^{q}\sum_{i=1}^{q}(\Lambda_i\Lambda_r a(\xi_i\xi_r-\zeta_i\zeta_r))_k/6+\varepsilon h^{5/2}\sum_{r=1}^{q}((\frac{d^2\sigma_r}{dt^2}+(L_1+\varepsilon^2 L_2)\Lambda_r a+$$

$$+\Lambda_r(L_1+\varepsilon^2 L_2)a)\xi_r)_k/6+h^3(L_1+\varepsilon^2 L_2)^2 a_k/6 \tag{8.2}$$

where the random variables ξ_r, η_r and ζ_r are simulated as

$$P(\xi=0)=2/3, P(\xi=-\sqrt{3})=P(\xi=\sqrt{3})=1/6,$$
$$P(\eta=-1/\sqrt{12})=P(\eta=1/\sqrt{12})=1/2,$$
$$P(\zeta=-1)=P(\zeta=1)=1/2 \tag{8.3}$$

The method $O(h^3)$ has the same form (8.2) but requires simulation of the needed random variables by the laws

$$P(\xi=0)=1/3, P(\xi=-1)=P(\xi=1)=3/10, P(\xi=-\sqrt{6})=P(\xi=\sqrt{6})=1/30,$$
$$P(\eta=-1/\sqrt{12})=P(\eta=1/\sqrt{12})=1/2,\ \ P(\zeta=-1)=P(\zeta=1)=1/2 \tag{8.4}$$

This method coincides with the third order weak method proposed in refs. [9], [10] for a general system with additive noises ($\varepsilon=1$).

Methods $O(h^4+\ldots)$ for the system (8.1), except the methods $O(h^4+\varepsilon^2h^3+\varepsilon^6h^2)$, $O(h^4+\varepsilon^6h^2)$, $O(h^4+\varepsilon^2h^3)$, $O(h^4+\varepsilon^4h^3)$, are obtained from the corresponding methods of Section 4.

For instance, *the method* $O(h^4+\varepsilon^4h^2)$ follows from the scheme (4.17) and has the form

$$X_{k+1}=X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k+ha_k+\varepsilon h^{3/2}\sum_{r=1}^{q}\left(\frac{d\sigma_r}{dt}(\xi_r/2-\eta_r)\right)_k+$$

$$+\varepsilon h^{3/2}\sum_{r=1}^{q}(\Lambda_r a(\xi_r/2+\eta_r))_k+h^2(L_1+\varepsilon^2L_2)a_k/2+\varepsilon h^{5/2}\sum_{r=1}^{q}((\frac{d^2\sigma_r}{dt^2}+L_1\Lambda_r a+$$

$$+\Lambda_r L_1 a)\xi_r)_k/6+h^3L_1^2a_k/6+\varepsilon^2h^3(L_1L_2+L_2L_1)a_k/6+h^4L_1^3a_k/24 \tag{8.5}$$

where ξ_r and η_r are simulated as

$$P(\xi=0)=2/3, P(\xi=-\sqrt{3})=P(\xi=\sqrt{3})=1/6,$$
$$P(\eta=-1/\sqrt{12})=P(\eta=1/\sqrt{12})=1/2, \tag{8.6}$$

The method $O(h^4+\varepsilon^2h^3+\varepsilon^6h^2)$ is written as

$$X_{k+1}=X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k+ha_k+\varepsilon h^{3/2}\sum_{r=1}^{q}\left(\frac{d\sigma_r}{dt}(\xi_r/2-\eta_r)\right)_k+$$

$$+\varepsilon h^{3/2}\sum_{r=1}^{q}(\Lambda_r a(\xi_r/2+\eta_r))_k+h^2(L_1+\varepsilon^2L_2)a_k/2+$$

$$+\varepsilon^2h^2\sum_{r=1}^{q}\sum_{i=1}^{q}(\Lambda_i\Lambda_r a(\xi_i\xi_r-\zeta_i\zeta_r))_k/6+\varepsilon h^{5/2}\sum_{r=1}^{q}((\frac{d^2\sigma_r}{dt^2}+(L_1+\varepsilon^2L_2)\Lambda_r a+$$

$$+\Lambda_r(L_1+\varepsilon^2L_2)a)\xi_r)_k/6+h^3(L_1+\varepsilon^2L_2)^2a_k/6+h^4L_1^3a_k/24 \tag{8.7}$$

where the random variables are simulated by the laws (8.3).

The method $O(h^4+\varepsilon^6h^2)$ has the form

$$X_{k+1}=X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k+ha_k+\varepsilon h^{3/2}\sum_{r=1}^{q}\left(\frac{d\sigma_r}{dt}(\xi_r/2-\eta_r)\right)_k+$$

$$+\varepsilon h^{3/2}\sum_{r=1}^{q}(\Lambda_r a(\xi_r/2+\eta_r))_k+h^2(L_1+\varepsilon^2L_2)a_k/2+$$

$$+\varepsilon^2h^2\sum_{r=1}^{q}\sum_{i=1}^{q}(\Lambda_i\Lambda_r a(\xi_i\xi_r-\zeta_i\zeta_r))_k/6+\varepsilon h^{5/2}\sum_{r=1}^{q}\left((L_1+\varepsilon^2L_2)\Lambda_r a\xi_r\right)_k/6+$$

$$+\varepsilon h^{5/2}\sum_{r=1}^{q}\left(\Lambda_r(L_1+\varepsilon^2L_2)a(\xi_r/6+\eta_r/2)\right)_k+\varepsilon h^{5/2}\sum_{r=1}^{q}\left(\frac{d^2\sigma_r}{dt^2}(\xi_r/6-\eta_r/2)\right)_k+$$

$$+h^3(L_1+\varepsilon^2L_2)^2a_k/6+\varepsilon h^{7/2}\sum_{r=1}^{q}\left[\left(\Lambda_rL_1^2a+L_1\Lambda_rL_1a+L_1^2\Lambda_r a+\frac{d^3\sigma_r}{dt^3}\right)\xi_r\right]_k/24+$$

$$+h^4L_1^3a_k/24+\varepsilon^2h^4(L_2L_1^2a+L_1^2L_2a+L_1L_2L_1a)_k/24 \tag{8.8}$$

where the random variables are simulated according to the laws (8.3).

The method $O(h^4+\varepsilon^2h^3)$ has the form (8.7) but the random variables are simulated as in (8.4). Note that this method distinguishes from the method $O(h^3)$ only by the additional term $h^4L_1^3a_k/24$.

The method $O(h^4+\varepsilon^4h^3)$ has the same form (8.8) as the method $O(h^4+\varepsilon^6h^2)$ but the needed random variables are from (8.4).

8.2. Runge-Kutta methods

Herein we restrict ourselves to full (derivative free) Runge-Kutta schemes.

Methods $O(h^2+\ldots)$ easily follow from the corresponding methods of Section 5.

Methods $O(h^3+\ldots)$ also follow from the corresponding methods of Section 5. Note that the semi-Runge-Kutta method (5.7) in the case of additive noises becomes full Runge-Kutta scheme and has the form

$$X_{k+1}=X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r(t_k)+\sigma_r(t_{k+1}))\xi_{r_k}/2+(k_1+4k_2+k_3)/6 \tag{8.9}$$

$$R=O(h^3+\varepsilon^2h^2)$$

where

$$k_1=ha_k,\ k_2=ha(t_{k+1/2},X_k+\alpha_1\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k+k_1/2),$$

$$k_3=ha(t_{k+1},X_k+\alpha_2\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k-k_1+2k_2),$$

$$\alpha_1=(6\pm\sqrt{6})/10,\ \alpha_2=(3\mp2\sqrt{6})/5 \tag{8.10}$$

and ξ_r are simulated as

$$P(\xi=0)=2/3,P(\xi=-\sqrt{3})=P(\xi=\sqrt{3})=1/6 \tag{8.11}$$

We also obtain the full Runge-Kutta method $O(h^3+\varepsilon^4h^2)$. But it requires five recalculations of the function $a(t,x)$ and three recalculations of the functions $\sigma_r(t)$ while the full Runge-Kutta method $O(h^4+\varepsilon^4h^2)$ (see (8.14) below) requires six recalculations of the function $a(t,x)$ and three recalculations of the functions $\sigma_r(t)$. Therefore, we do not write down it.

Methods $O(h^4+\ldots)$ follow from the corresponding methods of Section 5. Fortunately, the methods (5.14) and (5.17) for the system with additive noises become full Runge-Kutta schemes and for reader's convenience we write down them below.

The full Runge-Kutta method $O(h^4+\varepsilon^2h^2)$ has the form

$$X_{k+1} = X_k + \varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r(t_k)+\sigma_r(t_{k+1}))\xi_{r_k}/2 + (k_1+2k_2+2k_3+k_4)/6 \qquad (8.12)$$

where

$$k_1 = ha_k,\ k_2 = ha(t_{k+1/2}, X_k + k_1/2),\ k_3 = ha(t_{k+1/2}, X_k + k_2/2+$$

$$+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k),\ k_4 = ha(t_{k+1}, X_k + k_3 + \varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k) \qquad (8.13)$$

and ξ_r are simulated according to (8.11).

The full Runge-Kutta method $O(h^4+\varepsilon^4h^2)$ for the system (8.1) is written as

$$X_{k+1} = X_k + \varepsilon h^{1/2}\sum_{r=1}^{q}\left\{\sigma_r(t_k)(\xi_r+6\eta_r)_k + 4\sigma_r(t_{k+1/2})\xi_{r_k} + \sigma_r(t_{k+1})(\xi_r-6\eta_r)_k\right\}/6+$$

$$h\left\{a(t_k, X_k+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\eta_r)_k) - a(t_k, X_k-\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\eta_r)_k)\right\}/2+(k_1+2k_2+2k_3+k_4)/6 \qquad (8.14)$$

where

$$k_1 = ha_k,\ k_2 = ha(t_{k+1/2}, X_k + k_1/2),\ k_3 = ha(t_{k+1/2}, X_k + k_2/2+$$

$$+\varepsilon h^{1/2}\sum_{r=1}^{q}(\sigma_r\xi_r)_k),\ k_4 = ha(t_{k+1}, X_k + k_3 + \varepsilon h^{1/2}\sum_{r=1}^{q}\sigma_r(t_{k+1})\xi_{r_k}) \qquad (8.15)$$

and ξ_r, η_r are simulated as in (8.6).

8.3. Implicit methods

The implicit methods for the system with additive noises (8.1) are easily obtained from the methods of Section 6, where the implicit methods for a general system with small noises have been proposed.

9. Weak methods for a system with small colored noises

For some physical applications it is preferable to model random perturbations by colored noises [13]. In previous papers (for instance, see refs. [11], [12] and refs. therein) various peculiar methods for numerical solution of a system with colored noises were proposed. Herein we consider weak methods for a system with small colored noises

$$dY = f(t,Y)dt + \varepsilon G(t,Y)Zdt,$$

$$dZ = A(t)Zdt + \sum_{r=1}^{q} \gamma_r(t)dW_r \tag{9.1}$$

$$Y(t_o) = Y_o, \ Z(t_o) = Z_o, \ t \in [t_o, T]$$

where Y and $f(t,Y)$ are l-dimensional vectors, Z and $\gamma_r(t)$ are m-dimensional vectors, $A(t)$ is mxm-matrix, $G(t,Y)$ is lxm-matrix, W_r, $r = 1,\ldots,q$, are uncorrelated standard Wiener processes, and ε is a small parameter.

By introducing new variable $U = \varepsilon Z$ we obtain the system (9.1) in the convenient form

$$dY = f(t,Y)dt + G(t,Y)Udt,$$

$$dU = A(t)Udt + \varepsilon \sum_{r=1}^{q} \gamma_r(t)dW_r \tag{9.2}$$

$$Y(t_o) = Y_o, \ U(t_o) = \varepsilon Z_o, \ t \in [t_o, T]$$

The system (9.2) is the simplified version of the system with small additive noises (8.1). For the system (9.2) we have $X = \begin{bmatrix} Y \\ U \end{bmatrix}$ is the $l+m$-dimensional vector, i.e., $n = l+m$, $a = \begin{bmatrix} f + GU \\ AU \end{bmatrix}$ is the n-dimensional vector, $\sigma_r = \begin{bmatrix} 0 \\ \gamma_r \end{bmatrix}$ is also n-dimensional vector, the first l components of which are equal to zero. For convenience we write down the operators

$$L_1 = \frac{\partial}{\partial t} + (f(t,y) + G(t,y)u, \frac{\partial}{\partial y}) + (A(t)u, \frac{\partial}{\partial u}),$$

$$L_2 = \frac{1}{2}\sum_{r=1}^{q}\sum_{i,j=1}^{m} \gamma_r^i \gamma_r^j \frac{\partial^2}{\partial u^i \partial u^j}, \ \Lambda_r = (\gamma_r, \frac{\partial}{\partial u}) \tag{9.3}$$

and the expressions

$$L_1 a = \begin{bmatrix} L_1(f + Gu) \\ \dfrac{dA}{dt}u + A^2 u \end{bmatrix}, \ L_2 a = 0, \ \Lambda_r a = \begin{bmatrix} G\gamma_r \\ A\gamma_r \end{bmatrix},$$

$$L_1 \sigma_r = \begin{bmatrix} 0 \\ \dfrac{d\gamma_r}{dt} \end{bmatrix}, \ L_2 \sigma_r = 0, \ \Lambda_i \sigma_r = 0,$$

$$L_2 \Lambda_r a = 0, \ \Lambda_i \Lambda_r a = 0, \ i, r = 1, \ldots, q, \ etc. \tag{9.4}$$

By the formulas (9.3)-(9.4) the methods of Sections 4-6 and 8 can be rewritten for the system (9.1). That is why, herein we do not write down Taylor-type explicit and implicit methods. Below we provide the most useful full Runge-Kutta methods with the errors $O(h^3+\varepsilon^6h^2)$, $O(h^3)$, $O(h^4+\varepsilon^2h^3+\varepsilon^6h^2)$, $O(h^4+\varepsilon^2h^3)$.

Runge-Kutta methods $O(h^2+\ldots)$ easily follow from the corresponding methods of Section 5, and we do not write down them.

Runge-Kutta methods $O(h^3+\ldots)$ are obtained from the methods of Sections 5 and 8. However, for the system with colored noises (9.1) we obtain sufficiently effective full Runge-Kutta method with the error $O(h^3+\varepsilon^6h^2)$

$$Y_{k+1}=Y_k+(k_1+4k_2+k_3)/6,$$

$$U_{k+1}=U_k+\varepsilon h^{1/2}\sum_{r=1}^{q}\left\{\gamma_r(t_k)(\xi_r+6\eta_r)_k+4\gamma_r(t_{k+1/2})\xi_{rk}+\gamma_r(t_{k+1})(\xi_r-6\eta_r)_k\right\}/6+$$
$$+(l_1+4l_2+l_3)/6 \tag{9.5}$$

where

$$F(t,y,u)\equiv f(t,y)+G(t,y)u,\ k_1=hF_k,\ k_2=hF(t_{k+1/2},Y_k+$$
$$+k_1/2,U_k+l_1/2+\varepsilon h^{1/2}\sum_{r=1}^{q}(\gamma_r\xi_r)_k/2),\ k_3=hF(t_{k+1},Y_k-k_1+2k_2,U_k-$$
$$-l_1+2l_2+\varepsilon h^{1/2}\sum_{r=1}^{q}\gamma_r(t_{k+1})(\xi_r+6\eta_r)_k),\ l_1=h(AU)_k,$$
$$l_2=hA_{k+1/2}(U_k+l_1/2+\varepsilon h^{1/2}\sum_{r=1}^{q}(\gamma_r\xi_r)_k/2),\ l_3=hA_{k+1}(U_k-$$
$$-l_1+2l_2+\varepsilon h^{1/2}\sum_{r=1}^{q}\gamma_r(t_{k+1})(\xi_r+6\eta_r)_k) \tag{9.6}$$

and the random variables are simulated according to (8.6).

The full Runge-Kutta method $O(h^3)$ has the form (9.5) but the needed random variables are simulated as

$$P(\xi=0)=1/3,P(\xi=-1)=P(\xi=1)=3/10,P(\xi=-\sqrt{6})=P(\xi=\sqrt{6})=1/30,$$
$$P(\eta=-1/\sqrt{12})=P(\eta=1/\sqrt{12})=1/2 \tag{9.7}$$

The method (9.5) requires only three recalculations of the functions $f(t,y)$, $G(t,y)$, $A(t)$ and $\gamma_r(t)$. For a general system with colored noises ($\varepsilon=1$) the full Runge-Kutta scheme (9.5) with the random variables simulated by (9.7) also has the accuracy $O(h^3)$. Note that third time-order full Runge-Kutta schemes for a system with colored noises were not obtained in the previous papers (see ref. [12] and refs. therein).

Runge-Kutta methods $O(h^4+\ldots)$ follow from the methods of Sections 5 and 8. But for the system (9.1) we also obtain effective Runge-Kutta methods $O(h^4+\varepsilon^2h^3+\varepsilon^6h^2)$ and $O(h^4+\varepsilon^2h^3)$ which are provided below.

The full Runge-Kutta method $O(h^4+\varepsilon^2h^3+\varepsilon^6h^2)$ is written as

$$Y_{k+1}=Y_k+(k_1+2k_2+2k_3+k_4)/6,$$

$$U_{k+1}=U_k+\varepsilon h^{1/2}\sum_{r=1}^{q}\left\{\gamma_r(t_k)(\xi_r+6\eta_r)_k+4\gamma_r(t_{k+1/2})\xi_{rk}+\gamma_r(t_{k+1})(\xi_r-6\eta_r)_k\right\}/6+$$

$$+ (l_1 + 2l_2 + 2l_3 + l_4)/6 \tag{9.8}$$

where

$$F(t,y,u) \equiv f(t,y) + G(t,y)u, \; k_1 = hF_k, \; k_2 = hF(t_{k+1/2}, Y_k +$$

$$+ k_1/2, U_k + l_1/2), \; k_3 = hF(t_{k+1/2}, Y_k + k_2/2, U_k + l_2/2 + \varepsilon h^{1/2} \sum_{r=1}^{q} (\gamma_r \xi_r)_k),$$

$$k_4 = hF(t_{k+1}, Y_k + k_3, U_k + l_3 + \varepsilon h^{1/2} \sum_{r=1}^{q} \gamma_r(t_{k+1})(\xi_r + 6\eta_r)_k), \; l_1 = h(AU)_k,$$

$$l_2 = hA_{k+1/2}(U_k + l_1/2), \; l_3 = hA_{k+1/2}(U_k + l_2/2 + \varepsilon h^{1/2} \sum_{r=1}^{q} (\gamma_r \xi_r)_k),$$

$$l_4 = hA_{k+1}(U_k + l_3 + \varepsilon h^{1/2} \sum_{r=1}^{q} \gamma_r(t_{k+1})(\xi_r + 6\eta_r)_k) \tag{9.9}$$

and the random variables are simulated according to the laws (8.6).

The Runge-Kutta method $O(h^4 + \varepsilon^2 h^3)$ has the form (9.8) but the random variables are simulated as in (9.7). This method requires three recalculations of the functions $A(t)$ and $\gamma_r(t)$, and four recalculations of the functions $f(t,y)$ and $G(t,y)$.

10. Talay-Tubaro expansion of the global error

In ref. [20] the authors prove that it is possible to expand error of a method for a stochastic system on the whole interval in powers of time increment h. Their approach is analogous to the Runge estimation method for ordinary differential equations and allows to estimate the global error and to improve the method accuracy. Herein we expand the global error not only in powers of time increment h but also in powers of small parameter ε. Therefore, we cannot directly apply Talay-Tubaro theorems.

Theorem 10.1 *The error of the method (4.1) on the whole interval is equal to*

$$R = C_1(\varepsilon)h^2 + \varepsilon^2 C_2(\varepsilon)h + O(h^3 + \varepsilon^2 h^2) \tag{10.1}$$

where the functions $C_i(\varepsilon)$, $i = 1, 2$, *do not depend on* h *and are equal to* $C_i(\varepsilon) = C_i^o + O(\varepsilon^2)$, *and constants* $C_i^o(\varepsilon)$, $i = 1, 2$, *do not depend on both* h *and* ε.

The proof of Theorem 10.1 and the expressions for the coefficients $C_i(\varepsilon)$ see in Appendix (Section 16). It follows obviously from the proof that in the same way as the expansion (10.1) of the global error for the method (4.1) is derived, the expansions of errors for other methods can be obtained. For instance, for the method (4.14) with the error $O(h^4 + \varepsilon^4 h)$ we have

$$R = C_1(\varepsilon)h^4 + \varepsilon^2 C_2(\varepsilon)h^3 + \varepsilon^4 C_3(\varepsilon)h^2 + \varepsilon^4 C_4(\varepsilon)h + O(h^5 + \varepsilon^6 h^2)$$

The expansion like (10.1) can be used in the following way. Let us twice simulate $u^\varepsilon(t_o, X_o) = Ef(X^\varepsilon_{t_o,X_o}(T))$ by the method (4.1) under the given ε and with different time steps h_1, h_2 and obtain $\bar{u}^{\varepsilon,h_1}(t_o, X_o) = Ef(\bar{X}^{\varepsilon,h_1}_{t_o,X_o}(T))$, $\bar{u}^{\varepsilon,h_2}(t_o, X_o) = Ef(\bar{X}^{\varepsilon,h_2}_{t_o,X_o}(T))$. We can write

$$u^\varepsilon = \bar{u}^{\varepsilon,h_1} + C_1(\varepsilon)h_1^2 + \varepsilon^2 C_2(\varepsilon)h_1 + O(h^3 + \varepsilon^2 h^2)$$

$$u^\varepsilon = \bar{u}^{\varepsilon,h_2} + C_1(\varepsilon)h_2^2 + \varepsilon^2 C_2(\varepsilon)h_2 + O(h^3 + \varepsilon^2 h^2)$$

Then the constant $C_2(\varepsilon)$ is calculated as

$$\varepsilon^2 C_2(\varepsilon) = \varepsilon^2 \bar{C}_2(\varepsilon) - \varepsilon^2 C_1^o \cdot (h_1 + h_2) + O(h^2 + \varepsilon^2 h) \tag{10.2}$$

where

$$\varepsilon^2 \bar{C}_2(\varepsilon) = (\bar{u}^{\varepsilon,h_1} - \bar{u}^{\varepsilon,h_2})/(h_2 - h_1)$$

Using the method (4.1) under $\varepsilon = 0$ with different time steps h_1, h_2 we obtain $\bar{u}^{o,h_1}(t_o, X_o) = f(\bar{X}^{o,h_1}_{t_o,X_o}(T))$, $\bar{u}^{o,h_2}(t_o, X_o) = f(\bar{X}^{o,h_2}_{t_o,X_o}(T))$, where $\bar{X}^{o,h_i}_{t_o,X_o}$ is the approximation of the solution $X^o_{t_o,X_o}$ of the deterministic system. Then by the Runge estimation method we find the constant $C_1^o = C_1(0)$

$$C_1^o = \bar{C}_1^o + O(h) \tag{10.3}$$

where

$$\bar{C}_1^o = \left(\bar{u}^{o,h_1} - \bar{u}^{o,h_2}\right)/(h_2^2 - h_1^2)$$

By (10.2) and (10.3) we obtain the improved value $\bar{u}^\varepsilon_{imp}$ with the accuracy order $O(h^3 + \varepsilon^2 h^2)$

$$\bar{u}^\varepsilon_{imp} = \bar{u}^{\varepsilon,h_1} + \varepsilon^2 \bar{C}_2(\varepsilon)h_1 - \varepsilon^2 \bar{C}_1^o h_1 h_2 \tag{10.4}$$

According to the approach to construction of weak methods for a system with small noises, we can transfer some terms, contribution of which to the error is proportional to $h^i\varepsilon^j$, from a method to its remainder and vice versa. By calculation of the constants $C_i(\varepsilon)$ it is possible to estimate real weight of the terms and select the most appropriate scheme for solution of a certain system with small noises in the sense of calculation expenses and accuracy order.

11. Reduction of the Monte-Carlo error

Let us calculate the expectation $Ef(X(T))$ by Monte-Carlo technique using a weak method for solution of the system (3.1). Then, as is known, two errors arise: error of a weak method, which is considered in the previous Sections, and the Monte-Carlo error which is discussed below. We have

$$Ef(\bar{X}(T)) \approx \frac{1}{N}\sum_{m=1}^{N} f(\bar{X}^{(m)}(T)) \pm \frac{c}{\sqrt{N}}\left[Df(\bar{X}(T))\right]^{1/2} \tag{11.1}$$

where N is the number of independent realizations $\bar{X}^{(m)}$ simulated by a weak method, and c is a constant. If the constant c is equal to 1, 2, or 3, the calculated value belongs to the interval defined by (11.1) with confidence probability 0.68, 0.95, or 0.997 correspondingly.

According to the closeness of $Df(\bar{X}(T))$ to $Df(X(T))$, the Monte-Carlo error can be estimated by $[Df(X(T))]^{1/2}$. If $Df(X(T))$ is large, to reach the needed accuracy we must take sufficiently large N which leads to heavy calculation expenses. If instead of $f(X(T))$ we succeed in constructing a variable Z such that $EZ = Ef(X(T))$ but $DZ \ll Df(X(T))$, simulation of the variable Z instead of $f(X(T))$ would make it possible to obtain more accurate results with the same calculation expenses.

One of approaches to constructing Z was proposed in ref. [10]. As is shown below, this approach allows effectively reduce the Monte-Carlo error in the case of a system with small noises.

Together with the system (3.1) let us consider the following system

$$dX = a(t,X)dt + \varepsilon^2 b(t,X)dt - \varepsilon\sum_{r=1}^{q}\mu_r(t,X)\sigma_r(t,X)dt + \varepsilon\sum_{r=1}^{q}\sigma_r(t,X)dW_r,$$

$$dY = \varepsilon\sum_{r=1}^{q}\mu_r(t,X)Ydt \tag{11.2}$$

where μ_r and Y are scalars.

According to the Girsanov theorem for any μ_r we have

$$yEf(X_{s,x}(T))|_{(3.1)} = E\left(Y_{s,x,y}(T)f(X_{s,x}(T))\right)|_{(11.2)} \tag{11.3}$$

The function $u(s,x) = Ef(X_{s,x}(T))|_{(3.1)}$ satisfies the equation

$$Lu \equiv \frac{\partial u}{\partial s} + \sum_{i=1}^{n} a^i\frac{\partial u}{\partial x^i} + \varepsilon^2\sum_{i=1}^{n} b^i\frac{\partial u}{\partial x^i} + \frac{\varepsilon^2}{2}\sum_{r=1}^{q}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_r^i\sigma_r^j\frac{\partial^2}{\partial x^i\partial x^j} = 0 \tag{11.4}$$

with the condition at the instant T

$$u(T,x) = f(x) \tag{11.5}$$

Under sufficiently wide conditions on the coefficients and the function f, the solution $u(s,x) = u^{\varepsilon}(s,x)$ of the problem (11.4)-(11.5) has the form (see ref. [4, Chapt. 2])

$$u^{\varepsilon}(s,x) = u^{o}(s,x) + \varepsilon^2 u^1(s,x;\varepsilon) \tag{11.6}$$

The function u^o satisfies the first-order partial differential equation

$$\frac{\partial u}{\partial s} + \sum_{i=1}^{n} a^i\frac{\partial u}{\partial x^i} = 0 \tag{11.7}$$

with the condition (11.5). Obviously, the solution of (11.7) has the form

$$u^o(s,x) = f(X^o_{s,x}(T)) \tag{11.8}$$

where $X^o_{s,x}$ is the solution of the Cauchy problem for the deterministic system of differential equations

$$\frac{dX}{dt} = a(t,X),\ \ X(s) = x \tag{11.9}$$

Let us assume that the solution $u(s,x)$ of the problem (11.4)-(11.5) exists.

By the Ito formula let us calculate the following expression with respect to the system (11.2) (note that $Lu = 0$)

$$d\left[u(t,X_{s,x}(t))Y_{s,x,y}(t)\right] = LuYdt - \varepsilon\sum_{r=1}^{q}\mu_r(\sigma_r,\frac{\partial u}{\partial x})Ydt + \varepsilon\sum_{r=1}^{q}(\sigma_r,\frac{\partial u}{\partial x})YdW_r(t) +$$

$$+u\sum_{r=1}^{q}\mu_r YdW_r(t) + \varepsilon\sum_{r=1}^{q}(\sigma_r,\frac{\partial u}{\partial x})\mu_r Ydt = \sum_{r=1}^{q}\left(\varepsilon(\sigma_r,\frac{\partial u}{\partial x}) + \mu_r u\right)YdW_r(t)$$

Then

$$u(t, X_{s,x}(t))Y_{s,x,y}(t) = u(s,x)y + \int_s^t \sum_{r=1}^q \left(\varepsilon(\sigma_r, \frac{\partial u}{\partial x}) + \mu_r u\right) Y dW_r(t) \qquad (11.10)$$

If we suppose that $t = T$, $y = 1$, $\mu_r \equiv 0$, we obtain

$$f(X_{s,x}(T)) = u(s,x) + \int_s^T \varepsilon \sum_{r=1}^q (\sigma_r, \frac{\partial u}{\partial x}) dW_r(t)$$

Therefore

$$Df(X_{s,x}(T)) = \varepsilon^2 \int_s^T E\left[\sum_{r=1}^q (\sigma_r, \frac{\partial u}{\partial x})\right]^2 dt \qquad (11.11)$$

because of $u(s,x) = Ef(X_{s,x}(T))|_{(3.1)}$.

Thus, if we calculate $Ef(X(T))$ by Monte-Carlo technique using a weak method for solution of the system (3.1), the Monte-Carlo error, evaluated by $c[Df(\bar{X}(T))/N]^{1/2}$ and close to $c[Df(X(T))/N]^{1/2}$, has the small factor equal to ε.

As it follows from (11.3), the mean value $EZ = E\left(Y_{s,x,y}(T)f(X_{s,x}(T))\right)|_{(11.2)}$ does not depend on μ_r but $D\left(Y_{s,x,y}(T)f(X_{s,x}(T))\right)|_{(11.2)}$ depends on μ_r. So, now our aim is to select functions μ_r, $r = 1, \ldots, q$, in such a way that the variance DZ would be less than the variance (11.11).

Let us assume that $f > 0$. Then $u^o > 0$. Note that if function f is not greater than zero but there exist constants K and C such that $Kf + C > 0$ then for the function $g = Kf + C$ this assumption holds. In this case we can simulate Eg and then easily obtain Ef.

If we suppose that in the formula (11.10) $t = T$, $y = 1$ and

$$\mu_r = -\frac{\varepsilon}{u^o}\left(\sigma_r, \frac{\partial u^o}{\partial x}\right), \quad r = 1, \ldots, q \qquad (11.12)$$

we obtain

$$f(X_{s,x}(T))Y = u(s,x) + \int_s^T \varepsilon^3 \sum_{r=1}^q \left[(\sigma_r, \frac{\partial u^1}{\partial x}) - (\sigma_r, \frac{\partial u^o}{\partial x})\frac{u^1}{u^o}\right] dW_r(t)$$

Therefore

$$D\left[f(X_{s,x}(T))Y\right] = \varepsilon^6 \int_s^T E\left(\sum_{r=1}^q \left[(\sigma_r, \frac{\partial u^1}{\partial x}) - (\sigma_r, \frac{\partial u^o}{\partial x})\frac{u^1}{u^o}\right]\right)^2 dt$$

That is, the Monte-Carlo error for the system (11.2) with μ_r from (11.12) has the small factor equal to ε^3.

The system (11.2) with μ_r from (11.12) is again the system with small noises, and all methods, proposed in the paper, are suitable for it. And even for not large number of simulations N the Monte-Carlo error for this system is small. Of course, to apply the approach we must know the function $u^o(s,x)$.

12. Numerical tests

12.1. Simulation of Lyapunov exponent of a linear system with small noises

Lyapunov exponents are useful for investigation of stability of a dynamic stochastic system [1], [6]. The negativeness of upper Lyapunov exponent is an indication of system stability. Usually, it is impossible to obtain analytical expressions for Lyapunov exponents. That is why, a numerical approach to calculation of Lyapunov exponents is needed. Previously D.Talay [19] proposed such an approach which was based on ergodic property and allowed to calculate Lyapunov exponent by simulation of a one trajectory with the help of weak methods. This method is attractive because of its visuality and low calculation expenses. But it is difficult to analyze the errors arising in this approach.

Herein we calculate Lyapunov exponent as a convenient example to illustrate correctness and effectiveness of the proposed methods. In addition we pay a certain attention to analysis of the errors.

For our numerical tests we take the following two-dimensional linear Ito stochastic system

$$dX = AXdt + \varepsilon \sum_{r=1}^{q} B_r X dW_r \tag{12.1}$$

where X is two-dimensional vector, A and B_r are constant 2x2-matrices, W_r are independent standard Wiener processes, $\varepsilon > 0$ is a small parameter.

In ergodic case the unique Lyapunov exponent λ of the system (12.1) exists [6] and

$$\lambda = \lim_{t\to\infty} \frac{1}{t} E\rho(t) = \lim_{t\to\infty} \frac{1}{t}\rho(t) \ a.s.$$

where $\rho(t) = \ln |X(t)|$, and $X(t)$, $t \geq 0$, is a non-trivial solution of the system (12.1).

If $D(\rho(t)) \to \infty$ in the limit of $t \to \infty$ then [6]

$$E\left(\frac{\rho(t)}{t} - \lambda\right)^2 = D\left(\frac{\rho(t)}{t}\right)\left(1 + \varphi^2(t)\right) \tag{12.2}$$

where $\varphi(t) \to 0$ in the limit of $t \to \infty$. And it is not difficult to estimate that $D(\rho(t)/t) \to 0$ in the limit of $t \to \infty$. From (12.2) and the equality

$$D\left(\frac{\rho(t)}{t}\right) = E\left(\frac{\rho(t)}{t} - \lambda\right)^2 - \left[E\left(\frac{\rho(t)}{t}\right) - \lambda\right]^2$$

we have

$$\left|E\left(\frac{\rho(t)}{t}\right) - \lambda\right| = \varphi(t)\left[D\left(\frac{\rho(t)}{t}\right)\right]^{1/2} \tag{12.3}$$

In refs. [2], [15] the expansion of Lyapunov exponent of the system (12.1) in powers of small parameter ε was obtained. Herein we consider the system (12.1) with the matrices A and B_r which are such that

$$A = \begin{pmatrix} a & c \\ -c & a \end{pmatrix}, \quad B_r = \begin{pmatrix} b_r & d_r \\ -d_r & b_r \end{pmatrix}, \quad r = 1, 2 \tag{12.4}$$

In this case the Lyapunov exponent is exactly equal to [2]

$$\lambda = a + \frac{\varepsilon^2}{2}\sum_{r=1}^{2}[(d_r)^2 - (b_r)^2] \tag{12.5}$$

By Monte-Carlo technique we numerically calculate the function

$$\lambda(T) = \frac{1}{T}E\rho(T) \approx \bar{\lambda}(T) = \frac{1}{T}E\bar{\rho}(T),\ \bar{\rho}(T) = \ln|\bar{X}(T)| \tag{12.6}$$

Let us remind that $\bar{X}(T)$ denotes an approximation of the exact solution $X(T)$. The function $\lambda(t)$ in the limit of large time ($t \to \infty$) tends to the Lyapunov exponent λ. In this case three errors arise: (1) method error, i.e., $|E\rho(T)/T - E\bar{\rho}(T)/T|$, which is estimated by Theorem 3.1 (the function $f(x)$ from the theorem statement is equal to $\ln|x|/T$), (2) the Monte -Carlo error which is estimated by $[D(\bar{\rho}(T)/T)]^{1/2}/\sqrt{N}$ [see (11.1)], and (3) the error with respect to the choice of integration time T [see (12.3)].

As it follows from the computational results, in our tests the third error, i.e., $|\lambda(T) - \lambda| = |E(\rho(T)/T) - \lambda|$, is negligibly small, at any rate for $T \geq 2$, in comparison with the method error or/and the Monte-Carlo error.

In our case the function $[\bar{D}(\rho(T)/T)]^{1/2}$ tends to zero with the rate as $1/\sqrt{T}$. So, the Monte-Carlo error is proportional to $1/\sqrt{TN}$. Therefore, to reduce the Monte-Carlo error we can increase N or T. As to calculation expenses it does not matter to increase N or T. But in practice for large time T some computational problems arise according to the fact that $|X(T)|$ for the system (12.1) decreases to zero or increases to infinity exponentially fast. That is why, we prefer to increase N. It is clear that in our case the Talay's approach requires the same calculation expenses as simulation of Lyapunov exponent by Monte-Carlo technique. And by Monte-Carlo simulations we find not only $E\bar{\rho}(T)/T$ but also $D(\bar{\rho}(T)/T)$ which is useful for estimation of the errors.

We estimate the system (12.1) by four weak schemes. They are (1) the method (4.1) with the error $O(h^2 + \varepsilon^2 h)$ which is the simplest method among the schemes proposed in the paper, (2) the method (4.3) with the error $O(h^2 + \varepsilon^4 h)$, (3) the ordinary method (2.7) with the error $O(h^2)$, (4) the semi-Runge-Kutta scheme (5.17) with the error $O(h^4 + \varepsilon^4 h^2)$ which is the most accurate (in the sense of product $\varepsilon^i h^j$) scheme among the methods proposed in the paper for a general system with small noises.

From Table I and Figures 12.1, 12.2 it follows that the proposed methods for a system with small noises require lower calculation expenses than ordinary methods.

Table I. Lyapunov exponent. Computational results for Lyapunov exponent $\bar{\lambda}(T)$ for $a = -2$, $c = 1$, $b_1 = b_2 = 2$, $d_1 = 1$, $d_2 = -1$, $\varepsilon = 0.2$, $X^1(0) = 0$, $X^2(0) = 1$, $T = 10$, and for various steps h with averages over N realizations, where $N = 4 \cdot 10^4$ for the methods $O(h^2 + ...)$ and $N = 1 \cdot 10^6$ for the method $O(h^4 + \varepsilon^4 h^2)$. The exact solution is $\lambda = -2.12$.

	$\frac{1}{N}\sum_{m=1}^{q}\bar{\rho}^{(m)}(T)/T \pm \frac{2}{\sqrt{N}}\left(\frac{1}{N}\sum_{m=1}^{q}[\bar{\rho}^{(m)}(T)/T]^2 - \left[\frac{1}{N}\sum_{m=1}^{q}\bar{\rho}^{(m)}(T)/T\right]^2\right)^{1/2}$			
h	$O(h^2+\varepsilon^2 h)$	$O(h^2+\varepsilon^4 h)$	$O(h^2)$	$O(h^4+\varepsilon^4 h^2)$
0.3	-2.461 ± 0.004	-2.067 ± 0.002	-2.067 ± 0.002	-2.1228 ± 0.0004
0.2	-2.290 ± 0.003	-2.106 ± 0.002	-2.097 ± 0.002	-2.1195 ± 0.0004
0.1	-2.186 ± 0.002	-2.1198 ± 0.0018	-2.1140 ± 0.0017	-2.1192 ± 0.0004
0.05	-2.150 ± 0.002	-2.1219 ± 0.0018	-2.1186 ± 0.0018	-2.1197 ± 0.0004

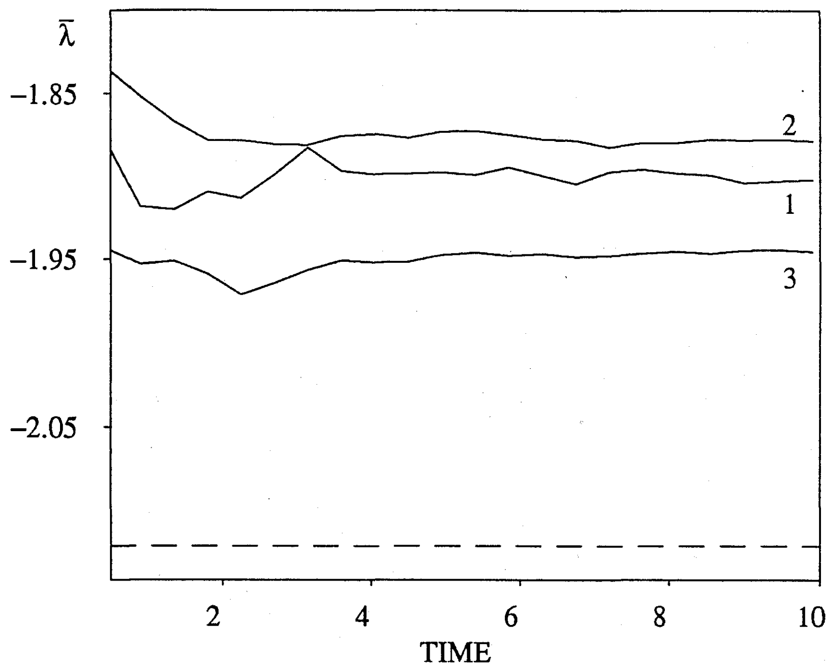

Figure 12.1: *Lyapunov exponent.* Time dependence of the function $\bar{\lambda}(T) = E\bar{\rho}(T)/T$ for time step $h = 0.45$. Other parameters are the same as in Table I. The solution of the system (12.1), (12.4) is approximated by (1) the method (4.1), (2) the method (4.3), (3) the method (2.7). Dashed line is the exact value of the Lyapunov exponent λ ($\lambda = -2.12$). The number of realizations $N = 400$ which ensures that the Monte-Carlo errors at $T \geq 7$ are not greater than 0.04 for the curve 1 and not greater than 0.02 for the curves 2,3 and are less than the method errors.

From Fig. 12.1 one can conclude that the methods $O(h^2 + \ldots)$ with the time step h equal to $\varepsilon^{1/2}$, i.e., their errors are estimated by $O(\varepsilon)$, give similar results but our method (4.1) has the lowest calculation expenses and, therefore, is preferable. As is shown in Fig. 12.2, the methods (4.3) and (2.7) under $h = \varepsilon$ and the method (5.17) under $h = \varepsilon^{1/2}$ (the errors of these methods are estimated by $O(\varepsilon^2)$) give similar results. Obviously, in this case the semi-Runge-Kutta method (5.17) is preferable because it permits to safe CPU time.

By the data of Table I it is possible to improve the methods $O(h^2+\varepsilon^2 h)$, $O(h^2+\varepsilon^4 h)$ and $O(h^2)$ by the Talay-Tubaro expansion (see Section 10). For instance, one can calculate the constants C_1 and C_2 from the expansion of the global error of the method (4.1) [see Section 10, Theorem 10.1] and obtain that $C_1 \approx 2.1$ and $C_2 \approx 10.2$. Let us note that if constants of the error expansion have opposite signs, the error becomes non-monotone function of time step h and can increase with decreasing of h. Such a behaviour is demonstrated in Table I [see the methods $O(h^2 + \varepsilon^4 h)$ and $O(h^4 + \varepsilon^4 h^2)$].

In Figure 12.3 we show the time dependence of the function $\bar{\rho}(T)/T$ in the case of the Talay's approach to calculation of Lyapunov exponents, i.e., along a one weak trajectory. One can see that in this case our methods also give accurate results and allow to reduce computation expanses. However, we must mentioned that the Talay's approach would give the same accuracy, as we reach by the Monte-Carlo simulations

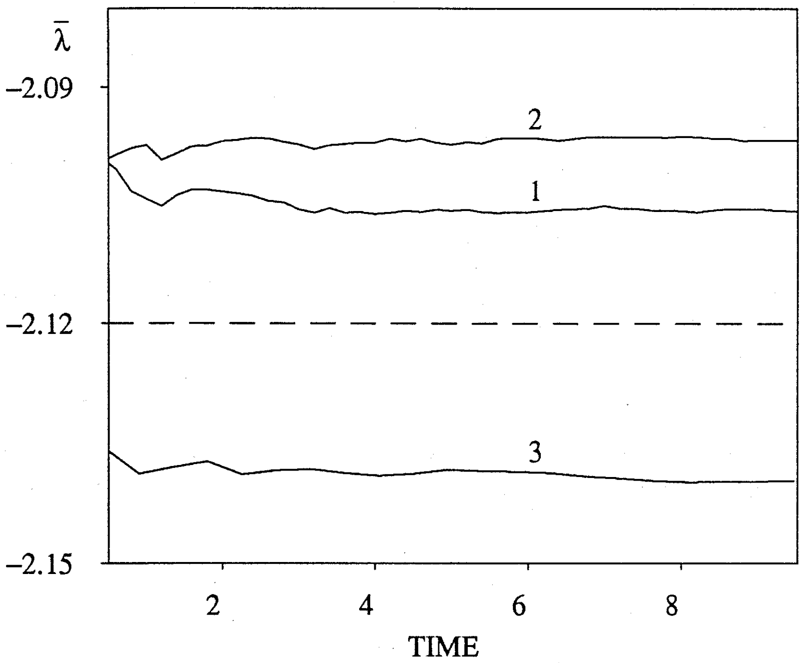

Figure 12.2: *Lyapunov exponent.* Time dependence of the function $\bar{\lambda}(T) = E\bar{\rho}(T)/T$ simulated by (1) the method (4.3) with $h = 0.2$, (2) the method (2.7) with $h = 0.2$, (3) the method (5.17) with $h = 0.45$. Other parameters are the same as in Table I. Dashed line is the exact value of the Lyapunov exponent λ ($\lambda = -2.12$). The Monte-Carlo errors at $T \geq 7$ are not greater than 0.002 ($N = 40000$) and are less than the method errors.

using the method (5.17) under $h = 0.2$ (see Table I), if the system (12.1) is solved during a period $T \sim 10^6$, which is difficult task from the computational point of view.

In our tests to generate uniform random numbers we use the procedure RAN1 from ref. [16].

Remark. Note that the function $\ln|x|$ does not belong to the class $\mathbf{F}$. If $\lambda > 0$ then to provide strictness it is possible to consider the function $\ln(1+|x|)$ instead of $\ln|x|$. The function $\ln(1+|x|)$ already belongs to the class $\mathbf{F}$, and $\lim_{t\to\infty}\ln(1+|X(t)|)/t = \lim_{t\to\infty}\ln(|X(t)|)/t$. As it follows from numerical tests, simulations of the function $\ln(1+|x|)$ give the same results as simulations of $\ln|x|$. Under $\lambda < 0$ it is possible to consider either the function $\ln(1+1/|x|)$ or the system

$$dX = (\gamma I + AX)dt + \varepsilon \sum_{r=1}^{q} B_r X dW_r \qquad (12.7)$$

instead of the system (12.1). The Lyapunov exponent of the system (12.7) is equal to $\gamma + \lambda$, and if we choose γ such that $\gamma + \lambda > 0$, we can again consider the function $\ln(1+|x|)$.

12.2. Stochastic resonance

Stochastic resonance (SR) is a term which describes a cooperative effect of noise and periodic forcing in a bistable system. It finds applications in a wide variety of physical,

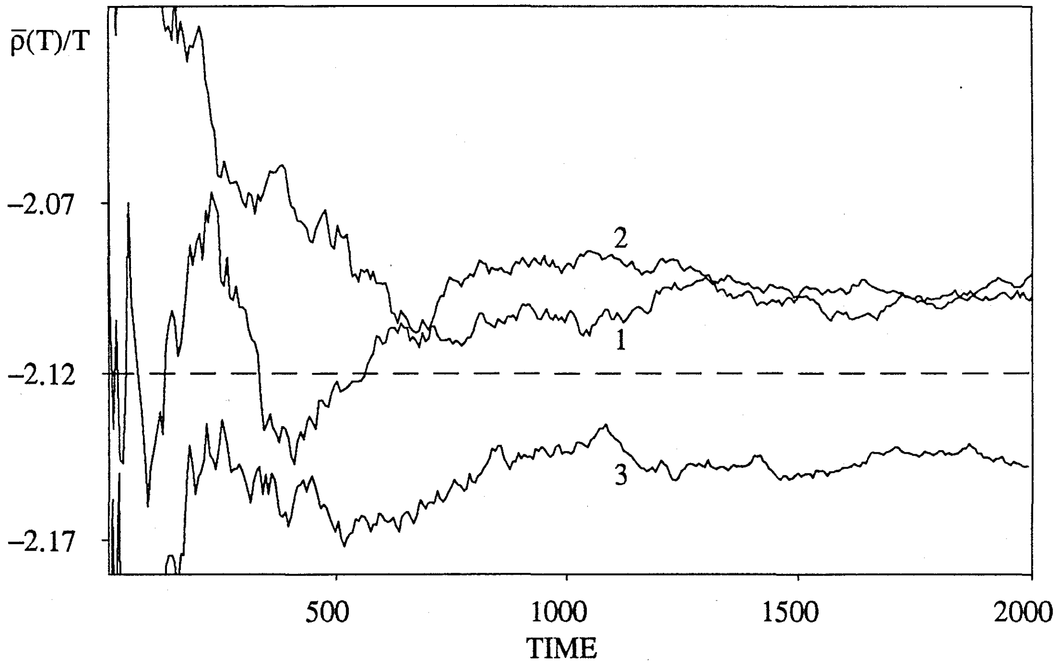

Figure 12.3: *Lyapunov exponent.* Time dependence of the function $\bar{\rho}(T)/T$ computated along a one trajectory using (1) the method (4.3) with $h = 0.1$, (2) the method (2.7) with $h = 0.1$, (3) the method (5.17) with $h = 0.3$. Other parameters are the same as in Table I. Dashed line is the exact value of the Lyapunov exponent λ ($\lambda = -2.12$).

physicochemical and biological systems (see, for instance, [17] and refs. therein). As is known, SR includes a variety of resonant effects which usually can be observed under small noises. Herein we illustrate effectiveness of our methods by simulation of the Stratonovich system with small noises which is similar to the system proposed in ref. [3] for description of multiplicative stochastic resonance in optical bistable system. For the solution of this system, which has the expansion $X(t) = \sum_{n=0}^{\infty} \alpha_n \cos(n\varepsilon^2\omega t - \phi_n)$ on the interval $[0, T]$, $T = 2\pi/\varepsilon^2\omega$, we calculate the mean values of amplitude α_1 and phase ϕ_1 of the first harmonic. Under some parameters these mean values have non-monotone behaviour with increasing of noise intensity. In ref. [3] the amplitude was investigated by numerical solution of the Fokker-Planck equation, and non-monotone behaviour of the amplitude was found. Phase shifts in SR is a subject of many year's studying (for instance, see [17, M.I.Dykman et. al. pp.463-478]). Previously, it was shown for bistable systems with additive noises that phase shifts accompany SR, and phase lag in SR varies non-monotonically with increasing of noise level. In our example we also observe these features of SR.

We consider the following one-dimensional Stratonovich equation

$$dX = (y_o - X - \frac{2cX}{1+X^2} + A\cos(\varepsilon^2\omega t))dt + \varepsilon(\sigma_1 \frac{X}{1+X^2} * dW_1 + \sigma_2 dW_2) \qquad (12.8)$$

where W_i, $i = 1, 2$, are independent standard Wiener processes and ε is a small parameter.

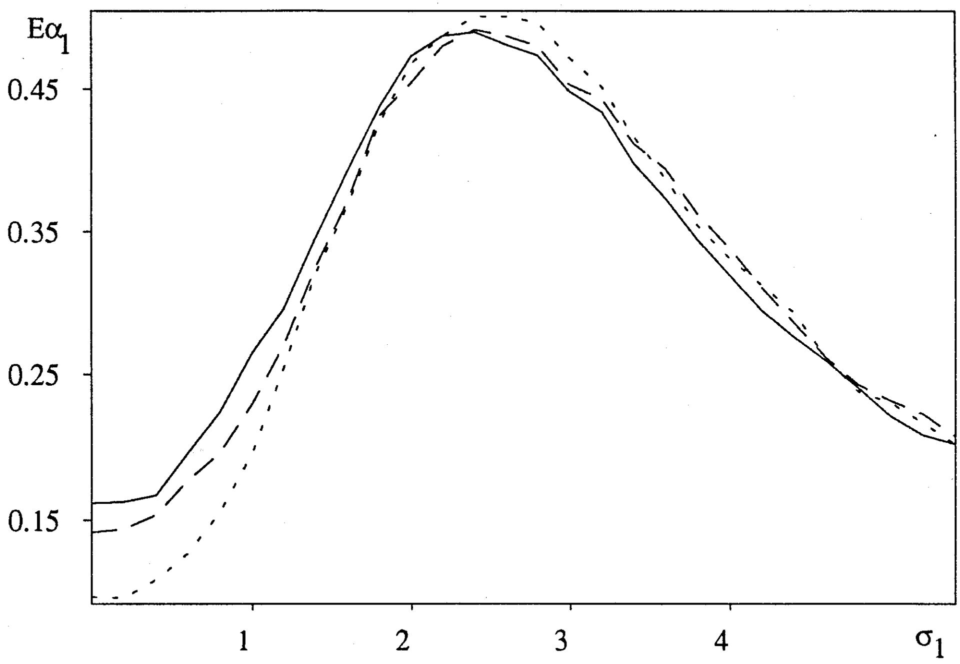

Figure 12.4: *Stochastic resonance.* The dependence of the mean amplituda $E\alpha_1$ on the noise intensity σ_1 for $y_o = 7$, $c = 6$, $\varepsilon = 0.1$, $\sigma_2 = 1.2$, $\omega = 2$, $T = \dfrac{4\pi}{\varepsilon^2\omega}$, $X(0) = 1$, $u(0) = 1$, $v(0) = 0$. Dotted line - the Euler method with $h = 0.5$, dashed line - the Euler method with $h = 0.1$, and solid line - the semi-Runge-Kutta method (5.17) with $h = 0.5$. The Monte-Carlo error is not greater than 0.01 ($N = 4000$).

To find the mean values of α_1 and ϕ_1 we additionally consider the equations

$$dZ_1 = X \cdot \cos(\varepsilon^2 \omega t)dt,$$
$$dZ_2 = X \cdot \sin(\varepsilon^2 \omega t)dt \tag{12.9}$$

If we choose the integration time T such that $T = \dfrac{2\pi}{\varepsilon^2\omega}$ then

$$E\alpha_1 = 2E(Z_1^2(T) + Z_2^2(T))^{1/2}/T,$$
$$E\varphi_1 = E \arctan(Z_2(T)/Z_1(T)) \tag{12.10}$$

To reduce calculation expanses we simulate $\cos(\varepsilon^2\omega t)$ and $\sin(\varepsilon^2\omega t)$ by the system

$$du = -\varepsilon^2 \omega v dt,$$
$$dv = \varepsilon^2 \omega u dt \tag{12.11}$$

To find parametric dependence of $E\alpha_1$ and $E\phi_1$ on the noise intensity σ_1 we use the Euler method with the error $O(h)$ and the proposed in the paper semi-Runge-Kutta method (5.17) with the error $O(h^4 + \varepsilon^4 h^2)$. The computational results are presented in Figures 12.4 and 12.5. We can add that under $h = 0.1$ the semi-Runge-Kutta method (5.17) gives visually almost the same curves as under $h = 0.5$. Thus, the curves of Figures 12.4 and 12.5, which correspond to the method (5.17) with $h = 0.5$, may be

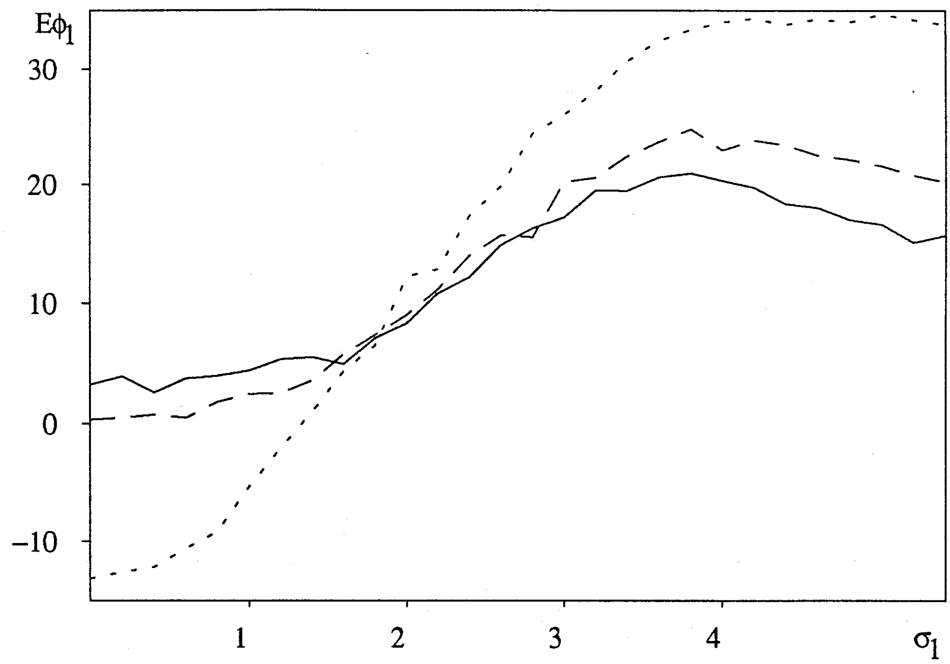

Figure 12.5: *Stochastic resonance.* The dependence of the mean phase $E\phi_1$, *grad*, on the noise intensity σ_1. The parameters and the notation are the same as in Figure 12.4. The Monte-Carlo error is not greater than 1.3 ($N = 4000$).

considered as sufficiently accurate. For the Euler method such an accuracy is attained for the amplitude under $h \approx 0.05$ and for the phase under $h \approx 0.01$. The computational results are in a good quality agreement with the results of other authors. One can see that the proposed method (5.17) gives quite accurate results under large time step and, therefore, allows to safe CPU time. It is important for this problem because to investigate effects of SR one must integrate a system during a long period of time and simulate sufficiently large number of independent realizations to reduce Monte-Carlo error.

13. Appendix. Proof of Theorem 3.1

Let us involve the function

$$u(s,x) = Ef(X_{s,x}(t_N)) \tag{13.1}$$

According to the conditions (1) and (2) of Theorem 3.1 the function u has partial derivatives with respect to x up to sufficiently high order, and the function u and its derivatives belong to the class $\mathbf{F}$ [5]. The function $u(s,x)$ uniformly (with respect to $s \in [t_o, t_N]$ and $0 \leq \varepsilon \leq \varepsilon_o$ for some number $\varepsilon_o > 0$) satisfies such an inequality as (3.3).

Because of $\bar{X}_o = X_o$, $X_{t_o,\bar{X}_o}(t_1) = X(t_1)$, $X_{t_1,X_{t_o,\bar{X}_o}(t_1)}(t_N) = X(t_N)$, we have

$$Ef(X(t_N)) = Ef(X_{t_1,X_{t_o,\bar{X}_o}(t_1)}(t_N)) - Ef(X_{t_1,\bar{X}_1}(t_N)) + Ef(X_{t_1,\bar{X}_1}(t_N)) \tag{13.2}$$

From $X_{t_1,\bar{X}_1}(t_N) = X_{t_2,X_{t_1,\bar{X}_1}(t_2)}(t_N)$ it follows

$$Ef(X_{t_1,\bar{X}_1}(t_N)) = Ef(X_{t_2,X_{t_1,\bar{X}_1}(t_2)}(t_N)) - Ef(X_{t_2,\bar{X}_2}(t_N)) + Ef(X_{t_2,\bar{X}_2}(t_N)) \quad (13.3)$$

Substituting (13.3) in (13.2) we obtain

$$Ef(X(t_N)) = Ef(X_{t_1,X_{t_o,\bar{X}_o}(t_1)}(t_N)) - Ef(X_{t_1,\bar{X}_1}(t_N)) +$$

$$+Ef(X_{t_2,X_{t_1,\bar{X}_1}(t_2)}(t_N)) - Ef(X_{t_2,\bar{X}_2}(t_N)) + Ef(X_{t_2,\bar{X}_2}(t_N))$$

Continuing further in the manner as above, we write

$$Ef(X(t_N)) = \sum_{i=0}^{N-2} \left[Ef(X_{t_{i+1},X_{t_i,\bar{X}_i}(t_{i+1})}(t_N)) - Ef(X_{t_{i+1},\bar{X}_{i+1}}(t_N)) \right] + Ef(X_{t_{N-1},\bar{X}_{N-1}}(t_N)) \quad (13.4)$$

Then

$$Ef(X(t_N)) - Ef(\bar{X}_N) = \sum_{i=0}^{N-2} \left\{ EE\left[f(X_{t_{i+1},X_{t_i,\bar{X}_i}(t_{i+1})}(t_N) | X_{t_i,\bar{X}_i}(t_{i+1}) \right] - \right.$$

$$\left. -EE\left[f(X_{t_{i+1},\bar{X}_{i+1}}(t_N) | X_{t_i,\bar{X}_i}(t_{i+1}) \right] \right\} + Ef(X_{t_{N-1},\bar{X}_{N-1}}(t_N)) - Ef(\bar{X}_{t_{N-1},\bar{X}_{N-1}}(t_N)) \quad (13.5)$$

According to the definition of the function u (see (13.1)), from the expression (13.5) we have

$$|Ef(X(t_N)) - Ef(\bar{X}_N)| =$$

$$= \left| \sum_{i=0}^{N-2} \left[Eu(t_{i+1}, X_{t_i,\bar{X}_i}(t_i + h)) - Eu(t_{i+1}, \bar{X}_{t_i,\bar{X}_i}(t_i + h)) \right] + \right.$$

$$\left. +Ef(X_{t_{N-1},\bar{X}_{N-1}}(t_N)) - Ef(\bar{X}_{t_{N-1},\bar{X}_{N-1}}(t_N)) \right| \le$$

$$\le \sum_{i=0}^{N-2} E\left| E\left[u(t_{i+1}, X_{t_i,\bar{X}_i}(t_i + h)) - u(t_{i+1}, \bar{X}_{t_i,\bar{X}_i}(t_i + h)) | \bar{X}_i \right] \right| +$$

$$+ E\left| E\left[f(X_{t_{N-1},\bar{X}_{N-1}}(t_N)) - f(\bar{X}_{t_{N-1},\bar{X}_{N-1}}(t_N)) | \bar{X}_{N-1} \right] \right| \quad (13.6)$$

Let us assume that both for $u(s,x)$ and for $f(x)$ the function $K(x)$ of the inequality (3.3) has $\kappa = m$ (see κ in the definition of the class **F**, Section 3). Then from (3.3) and (13.6) we obtain

$$|Ef(X(t_N)) - Ef(\bar{X}_N)| \le \sum_{i=0}^{N-2} K(1 + E|\bar{X}_i|^m) \left[h^{p+1} + \sum_{l \in S} h^{l+1} \varepsilon^{J(l)} \right] +$$

$$+K(1 + E|\bar{X}_{N-1}|^m) \left[h^{p+1} + \sum_{l \in S} h^{l+1} \varepsilon^{J(l)} \right]$$

Hence, from the third condition of the theorem we have

$$|Ef(X(t_N)) - Ef(\bar{X}_N)| \le K \left[h^p + \sum_{l \in S} h^l \varepsilon^{J(l)} \right]$$

It is obvious that for any $k < N$ the procedure, proposed above, can be carried out and the inequality (3.4) holds. Theorem 3.1 is proved.

14. Appendix. Construction of the weak method with the order $O(h^4+\varepsilon^4 h^2)$

14.1. Construction of the one-step approximation

As the point of departure we take the stochastic Taylor-type expansion [7], [10], [21] for the solution of the system (1.1) in the form

$$X_{t,x}(t+h) = x+\varepsilon\sum_{r=1}^{q}\sigma_r I_r + ah + \varepsilon^2\sum_{r=1}^{q}\sum_{i=1}^{q}\Lambda_r\sigma_i I_{ri} + \varepsilon\sum_{r=1}^{q}(L_1+\varepsilon^2 L_2)\sigma_r I_{or} +$$

$$+\varepsilon\sum_{r=1}^{q}\Lambda_r a I_{ro} + \varepsilon^3\sum_{r=1}^{q}\sum_{i=1}^{q}\sum_{s=1}^{q}\Lambda_s\Lambda_i\sigma_r I_{sir} + \frac{1}{2}(L_1+\varepsilon^2 L_2)h^2 +$$

$$+\rho_1+\rho_2+\int_t^{t+h}\Big(\int_t^{\vartheta}\Big(\int_t^{\vartheta_1} L^2 a(\vartheta_2, X(\vartheta_2))d\vartheta_2\Big)d\vartheta_1\Big)d\vartheta \tag{14.1}$$

where

$$\rho_1 = \varepsilon^4\sum_{r=1}^{q}\sum_{i=1}^{q}\sum_{s=1}^{q}\sum_{j=1}^{q}\int_t^{t+h}\Big(\int_t^{\vartheta}\Big(\int_t^{\vartheta_1}\Big(\int_t^{\vartheta_2}\Lambda_j\Lambda_s\Lambda_i\sigma_r(\vartheta_3, X(\vartheta_3))dW_j(\vartheta_3)\Big)\times$$

$$\times dW_s(\vartheta_2)\Big)dW_i(\vartheta_1)\Big)dW_r(\vartheta)+$$

$$+\varepsilon^2\sum_{r=1}^{q}\sum_{i=1}^{q}\int_t^{t+h}\Big(\int_t^{\vartheta}\Big(\int_t^{\vartheta_1} L\Lambda_i\sigma_r(\vartheta_2, X(\vartheta_2))d\vartheta_2\Big)dW_i(\vartheta_1)\Big)dW_r(\vartheta)+$$

$$+\varepsilon^2\sum_{r=1}^{q}\sum_{i=1}^{q}\int_t^{t+h}\Big(\int_t^{\vartheta}\Big(\int_t^{\vartheta_1} \Lambda_i L\sigma_r(\vartheta_2, X(\vartheta_2))dW_i(\vartheta_2)\Big)d\vartheta_1\Big)dW_r(\vartheta)+$$

$$+\varepsilon^2\sum_{r=1}^{q}\sum_{i=1}^{q}\int_t^{t+h}\Big(\int_t^{\vartheta}\Big(\int_t^{\vartheta_1} \Lambda_i\Lambda_r a(\vartheta_2, X(\vartheta_2))dW_i(\vartheta_2)\Big)dW_r(\vartheta_1)\Big)d\vartheta+$$

$$+\varepsilon^3\sum_{r=1}^{q}\sum_{i=1}^{q}\sum_{s=1}^{q}\int_t^{t+h}\Big(\int_t^{\vartheta}\Big(\int_t^{\vartheta_1}\Big(\int_t^{\vartheta_2} L\Lambda_s\Lambda_i\sigma_r(\vartheta_3, X(\vartheta_3))d\vartheta_3\Big)dW_s(\vartheta_2)\Big)dW_i(\vartheta_1)\Big)dW_r(\vartheta), \tag{14.2}$$

$$\rho_2 = \varepsilon\sum_{r=1}^{q}\int_t^{t+h}\Big(\int_t^{\vartheta}\Big(\int_t^{\vartheta_1} L^2\sigma_r(\vartheta_2, X(\vartheta_2))d\vartheta_2\Big)d\vartheta_1\Big)dW_r(\vartheta)+$$

$$+\varepsilon\sum_{r=1}^{q}\int_t^{t+h}\Big(\int_t^{\vartheta}\Big(\int_t^{\vartheta_1} L\Lambda_r a(\vartheta_2, X(\vartheta_2))d\vartheta_2\Big)dW_r(\vartheta_1)\Big)d\vartheta+$$

$$+\varepsilon\sum_{r=1}^{q}\int_t^{t+h}\Big(\int_t^{\vartheta}\Big(\int_t^{\vartheta_1} \Lambda_r La(\vartheta_2, X(\vartheta_2))dW_r(\vartheta_2)\Big)d\vartheta_1\Big)d\vartheta \tag{14.3}$$

Here $x = X(t)$, $\sigma_r = \sigma_r(t,x)$, $a = a(t,x)$, $I_{i_1,\dots,i_j} = I_{i_1,\dots,i_j}(t,h)$, $\Lambda_r a = \Lambda_r a(t,x)$, etc.

Let us rewrite the expression ρ_2 by the Ito formula in the form

$$\rho_2 = \varepsilon\sum_{r=1}^{q} L_1^2\sigma_r I_{oor} + \varepsilon\sum_{r=1}^{q} L_1\Lambda_r a I_{oro} + \varepsilon\sum_{r=1}^{q}\Lambda_r L_1 a I_{roo} + \tilde{\rho}_2 \tag{14.4}$$

where

$$\tilde{\rho}_2 = \varepsilon^3\sum_{r=1}^{q}(L_1L_2+L_2L_1)\sigma_r I_{oor} + \varepsilon^5\sum_{r=1}^{q} L_2^2\sigma_r I_{oor}+$$

$$+\varepsilon^2\sum_{r=1}^{q}\sum_{i=1}^{q}\int_t^{t+h}(\int_t^{\vartheta}(\int_t^{\vartheta_1}(\int_t^{\vartheta_2}\Lambda_i L^2\sigma_r(\vartheta_3,X(\vartheta_3))dW_i(\vartheta_3))d\vartheta_2)d\vartheta_1)dW_r(\vartheta)+$$

$$+\varepsilon\sum_{r=1}^{q}\int_t^{t+h}(\int_t^{\vartheta}(\int_t^{\vartheta_1}(\int_t^{\vartheta_2}L^3\sigma_r(\vartheta_3,X(\vartheta_3))d\vartheta_3)d\vartheta_2)d\vartheta_1)dW_r(\vartheta)+\varepsilon^3\sum_{r=1}^{q}L_2\Lambda_r aI_{oro}+$$

$$+\varepsilon^2\sum_{r=1}^{q}\sum_{i=1}^{q}\int_t^{t+h}(\int_t^{\vartheta}(\int_t^{\vartheta_1}(\int_t^{\vartheta_2}\Lambda_i L\Lambda_r a(\vartheta_3,X(\vartheta_3))dW_i(\vartheta_3))d\vartheta_2)dW_r(\vartheta_1))d\vartheta+$$

$$+\varepsilon\sum_{r=1}^{q}\int_t^{t+h}(\int_t^{\vartheta}(\int_t^{\vartheta_1}(\int_t^{\vartheta_2}L^2\Lambda_r a(\vartheta_3,X(\vartheta_3))d\vartheta_3)d\vartheta_2)dW_r(\vartheta_1))d\vartheta+\varepsilon^3\sum_{r=1}^{q}\Lambda_r L_2 aI_{roo}+$$

$$+\varepsilon^2\sum_{r=1}^{q}\sum_{i=1}^{q}\int_t^{t+h}(\int_t^{\vartheta}(\int_t^{\vartheta_1}(\int_t^{\vartheta_2}\Lambda_i\Lambda_r La(\vartheta_3,X(\vartheta_3))dW_i(\vartheta_3))dW_r(\vartheta_2))d\vartheta_1)d\vartheta+$$

$$+\varepsilon\sum_{r=1}^{q}\int_t^{t+h}(\int_t^{\vartheta}(\int_t^{\vartheta_1}(\int_t^{\vartheta_2}L\Lambda_r La(\vartheta_3,X(\vartheta_3))d\vartheta_3)dW_r(\vartheta_2))d\vartheta_1)d\vartheta \tag{14.5}$$

The last integral in the formula (14.1) we rewrite as

$$\int_t^{t+h}(\int_t^{\vartheta}(\int_t^{\vartheta_1}L^2a(\vartheta_2,X(\vartheta_2))d\vartheta_2)d\vartheta_1)d\vartheta=$$

$$=\frac{1}{6}\left[L_1^2+\varepsilon^2(L_1L_2+L_2L_1)\right]ah^3+\frac{1}{24}L_1^3ah^4+\tilde{\rho}_3 \tag{14.6}$$

where

$$\tilde{\rho}_3=\frac{1}{6}\varepsilon^4L_2^2ah^3+\frac{1}{24}\left[\varepsilon^2(L_1^2L_2+L_2L_1^2+L_1L_2L_1)+\varepsilon^4(L_1L_2^2+L_2L_1L_2+L_2^2L_1)+\right.$$

$$\left.+\varepsilon^6L_2^3\right]ah^4+\varepsilon\sum_{r=1}^{q}\int_t^{t+h}(\int_t^{\vartheta}(\int_t^{\vartheta_1}(\int_t^{\vartheta_2}\Lambda_r L^2a(\vartheta_3,X(\vartheta_3))dW_r(\vartheta_3))d\vartheta_2)d\vartheta_1)d\vartheta+$$

$$+\varepsilon\sum_{r=1}^{q}\int_t^{t+h}(\int_t^{\vartheta}(\int_t^{\vartheta_1}(\int_t^{\vartheta_2}(\int_t^{\vartheta_3}\Lambda_r L^3a(\vartheta_4,X(\vartheta_4))dW_r(\vartheta_4))d\vartheta_3)d\vartheta_2)d\vartheta_1)d\vartheta+$$

$$+\int_t^{t+h}(\int_t^{\vartheta}(\int_t^{\vartheta_1}(\int_t^{\vartheta_2}(\int_t^{\vartheta_3}L^4a(\vartheta_4,X(\vartheta_4))d\vartheta_4)d\vartheta_3)d\vartheta_2)d\vartheta_1)d\vartheta \tag{14.7}$$

By (14.1)-(14.7) we obtain

$$X_{t,x}(t+h)=x+\varepsilon\sum_{r=1}^{q}\sigma_rI_r+ah+\varepsilon^2\sum_{r=1}^{q}\sum_{i=1}^{q}\Lambda_i\sigma_rI_{ir}+\varepsilon\sum_{r=1}^{q}(L_1+\varepsilon^2L_2)\sigma_rI_{or}+$$

$$+\varepsilon\sum_{r=1}^{q}\Lambda_raI_{ro}+\frac{1}{2}(L_1+\varepsilon^2L_2)ah^2+\varepsilon\sum_{r=1}^{q}L_1^2\sigma_rI_{oor}+\varepsilon\sum_{r=1}^{q}L_1\Lambda_raI_{oro}+\varepsilon\sum_{r=1}^{q}\Lambda_rL_1aI_{roo}+$$

$$+\frac{1}{6}\left[L_1^2+\varepsilon^2(L_1L_2+L_2L_1)\right]ah^3+\frac{1}{24}L_1^3ah^4+\rho, \tag{14.8}$$

where

$$\rho=\varepsilon^3\sum_{r=1}^{q}\sum_{i=1}^{q}\sum_{s=1}^{q}\Lambda_s\Lambda_i\sigma_rI_{sir}+\bar{\rho}, \tag{14.9}$$

$$\bar{\rho}=\rho_1+\tilde{\rho}_2+\tilde{\rho}_3 \tag{14.10}$$

Thus, in comparison with the ordinary approximation (14.1), which does not take into account smallness of noises, the main part of (14.8) contains the following new terms $\varepsilon L_1^2\sigma_r I_{oor}$, $\varepsilon L_1\Lambda_r a I_{oro}$, $\varepsilon\Lambda_r L_1 a I_{roo}$, $[L_1^2+\varepsilon^2(L_1L_2+L_2L_1)]\, ah^3/6$ and $L_1^3 ah^4/24$. As it will turn out below these additional terms do not lead to substantial increasing of calculation expenses but in the case of small noises essentially increase accuracy of the method in the sense of product $\varepsilon^i h^j$. So, the peculiar features of the system with small noises give an opportunity to construct effective and accurate special numerical methods.

Lemma 14.1 *Let us assume that the Lipschitz condition*

$$|a(t,x)-a(t,y)|+\sum_{r=1}^{q}|\sigma_r(t,x)-\sigma_r(t,y)|\le K(x-y) \tag{14.11}$$

holds, and the coefficients a, σ_r, $r=1,...,q$, and their partial derivatives up to sufficiently high order belong to the class $\mathbf{F}$. *Then*

$$|E\rho|\le K(x)\left[h^5+\varepsilon^4h^3\right],\ K(x)\in\mathbf{F},$$

$$|E\rho^2|\le K(x)\left[h^{10}+\varepsilon^4h^4+\varepsilon^6h^3\right],\ K(x)\in\mathbf{F},$$

$$\varepsilon|E\rho I_j|\le K(x)\left[\varepsilon^2h^4+\varepsilon^4h^3\right],\ K(x)\in\mathbf{F},$$

$$\varepsilon^2|E\rho I_{jl}|\le K(x)\varepsilon^4h^3,\ K(x)\in\mathbf{F},$$

$$\varepsilon|E\rho I_{oj}|\le K(x)\left[\varepsilon^2h^5+\varepsilon^4h^3\right],\ K(x)\in\mathbf{F},$$

$$\varepsilon|E\rho I_{jo}|\le K(x)\left[\varepsilon^2h^5+\varepsilon^4h^3\right],\ K(x)\in\mathbf{F},$$

$$\varepsilon^2|E\rho I_jI_l|\le K(x)\left[\varepsilon^2h^6+\varepsilon^4h^3\right],\ K(x)\in\mathbf{F},\ j,l=1,\ldots,r \tag{14.12}$$

Proof of Lemma 14.1 is similar to the proof of the corresponding lemma in ref.[10] and is based on the following properties of Ito integrals

$$EI_{i_1,...,i_j}=0,\ \text{if at least one of the indices } i_k\ne 0,$$

$$EI_{i_1,...,i_j}=O(h^j),\ \text{if all indices } i_k=0,$$
$$[E(I_{i_1,...,i_j})^2]^{1/2}=O(h^r),\ r=l_1+l_2/2 \tag{14.13}$$

where l_1 is the number of zero indices i_k and l_2 is the number of non-zero indices i_k; and

$$E(I_{i_1,...,i_m}\cdot I_{j_1,...,j_l})=0,\ E(I_{i_1,...,i_m}\cdot I_{j_1,...,j_l}\cdot I_{r_1,...,r_p})=0 \tag{14.14}$$

if the number of non-zero integers among all indices is the odd number;

$$E(I_{sir}\cdot I_j)=0 \tag{14.15}$$

where s, i, r and j are from the set $\{1,...,q\}$.

By Lemma 14.1 we obtain the following lemma.

Lemma 14.2. *If the conditions of Lemma 14.1 are fulfilled, then we have the inequalities*

$$|E(\prod_{j=1}^{s} \Delta^{i_j} - \prod_{j=1}^{s} \tilde{\Delta}^{i_j})| \le K(x)(h^5 + \varepsilon^4 h^3),\ i_j = 1, ..., n,\ s = 1, ..., 5,\ K(x) \in \mathbf{F} \quad (14.16)$$

where Δ^{i_j} *is the* i_j-*component of the vector* $\Delta = X_{t,x}(t+h) - x$, *and* $\tilde{\Delta}^{i_j}$ *is the* i_j-*component of the vector* $\tilde{\Delta} = \tilde{X}_{t,x}(t+h) - x$, $\tilde{X}_{t,x}(t+h) = X_{t,x}(t+h) - \rho$.

Lemma 14.2 is proved by the following reasons: (1) the inequalities (14.16) for $s = 1$ and $s = 2$ directly follow from (14.12); (2) it is clear that the odd (third and fifth) moments have not less smallness orders than the first moments, and the fourth moments have not less smallness order than the second.

The approximation $\tilde{X} = X - \rho$ [see (14.8)] contains multiple Ito integrals I_{ri} which cannot be easily simulated. That is why, we shall try to obtain another approximation $\bar{X}$ such that the inequalities

$$|E(\prod_{j=1}^{s} \tilde{\Delta}^{i_j} - \prod_{j=1}^{s} \bar{\Delta}^{i_j})| \le K(x)(h^5 + \varepsilon^4 h^3),\ i_j = 1, ..., n,\ s = 1, ..., 5,\ K(x) \in \mathbf{F}, \quad (14.17)$$

$$\bar{\Delta} = \bar{X}_{t,x}(t+h) - x,$$

$$E\prod_{j=1}^{6} |\bar{\Delta}^{i_j}| \le K(x)(h^5 + \varepsilon^4 h^3),\ i_j = 1, ..., n,\ K(x) \in \mathbf{F} \quad (14.18)$$

hold, but $\bar{X}$ contains only sufficiently simple random variables. It is clear that the inequalities

$$|E(\prod_{j=1}^{s} \Delta^{i_j} - \prod_{j=1}^{s} \bar{\Delta}^{i_j})| \le K(x)(h^5 + \varepsilon^4 h^3),\ i_j = 1, ..., n,\ s = 1, ..., 5,\ K(x) \in \mathbf{F}, \quad (14.19)$$

follow from Lemma 14.2 and (14.17). As it will turn out below in Theorem 14.1, the inequalities (14.18), (14.19) are sufficient to estimate error of a one-step weak approximation.

Now let us construct the approximation $\bar{X}$ on the base of $\tilde{X}$ in the form

$$\bar{X} = x + \varepsilon h^{1/2} \sum_{r=1}^{q} \sigma_r \xi_r + ah + \varepsilon^2 h \sum_{r=1}^{q} \sum_{i=1}^{q} \Lambda_i \sigma_r \xi_{ir} + \varepsilon h^{3/2} \sum_{r=1}^{q} (L_1 + \varepsilon^2 L_2) \sigma_r [\xi_r - \mu_r] +$$

$$+ \varepsilon h^{3/2} \sum_{r=1}^{q} \Lambda_r a \mu_r + \frac{1}{2}(L_1 + \varepsilon^2 L_2) a h^2 + \varepsilon h^{5/2} \sum_{r=1}^{q} L_1^2 \sigma_r \left[\frac{1}{2}\xi_r - \vartheta_r\right] + \varepsilon h^{5/2} \sum_{r=1}^{q} L_1 \Lambda_r a \left[2\vartheta_r - \mu_r\right] +$$

$$+ \varepsilon h^{5/2} \sum_{r=1}^{q} \Lambda_r L_1 a \left[\mu_r - \vartheta_r\right] + \frac{1}{6}\left[L_1^2 + \varepsilon^2 (L_1 L_2 + L_2 L_1)\right] a h^3 + \frac{1}{24} L_1^3 a h^4 \quad (14.20)$$

where the random variables ξ_r, ξ_{ir}, μ_r, ϑ_r are such that the following lemma takes place.

Let us introduce the notation

$$G_r = \int_t^{t+h} (W_r(\vartheta) - W_r(t))(\vartheta - t) d\vartheta$$

and remind that δ_{ij} is the Kronecker delta.

Lemma 14.3. *Under the conditions of Lemma 14.1 the inequalities (14.18) and (14.19) hold if the random variables ξ_r, ξ_{ir}, μ_r, and ϑ_r satisfy the equalities*

$$E\xi_r h^{1/2} = EI_r = 0,\ E\xi_{ir}h = EI_{ir} = 0,\ E\mu_r h^{3/2} = EI_{ro} = 0,$$

$$E\vartheta_r h^{5/2} = EG_r = 0; \tag{14.21}$$

$$E\xi_i\xi_r h = EI_iI_r = \delta_{ir}h,\ E\xi_i\xi_{rj}h^{3/2} = EI_iI_{rj} = 0,$$

$$E\xi_r\mu_j h^2 = EI_rI_{jo} = \frac{1}{2}\delta_{rj}h^2,$$

$$E\xi_{ir}\xi_{js}h^2 = EI_{ir}I_{js} = \begin{cases} \frac{1}{2}h^2, \text{ if } i=j,\ r=s, \\ 0, \text{ otherwise}, \end{cases}$$

$$E\xi_{ir}\mu_j h^{5/2} = EI_{ir}I_{jo} = 0,\ E\mu_r\mu_j h^3 = EI_{ro}I_{jo} = \frac{1}{3}\delta_{rj}h^3,$$

$$E\xi_r\vartheta_j h^3 = EI_rG_j = \frac{1}{3}\delta_{rj}h^3; \tag{14.22}$$

$$E\xi_i\xi_r\xi_j h^{\frac{3}{2}} = EI_iI_rI_j = 0,\ E\xi_i\xi_{jr}\xi_{sl}h^{\frac{5}{2}} = EI_iI_{jr}I_{sl} = 0,$$

$$E\xi_i\xi_r\xi_{js}h^2 = EI_iI_rI_{js} = \begin{cases} \frac{1}{2}h^2, \text{ if } j \neq s \text{ and either } i=j,\ r=s \text{ or } i=s,\ r=j, \\ h^2, \text{ if } i=r=j=s, \\ 0, \text{ otherwise}, \end{cases}$$

$$E\xi_i\xi_r\mu_j h^{\frac{5}{2}} = EI_iI_rI_{jo} = 0; \tag{14.23}$$

$$E\xi_i\xi_r\xi_j\xi_s h^2 = EI_iI_rI_jI_s = \begin{cases} h^2, \text{ if } \{i,r,j,s\} \text{ consists} \\ \text{of two pairs of equal numbers}, \\ 3h^2, \text{ if } i=r=j=s, \\ 0, \text{ otherwise}, \end{cases}$$

$$E\xi_i\xi_r\xi_j\xi_{sl}h^{5/2} = EI_iI_rI_jI_{sl} = 0; \tag{14.24}$$

$$E\xi_i\xi_r\xi_j\xi_s\xi_l h^{5/2} = EI_iI_rI_jI_sI_l = 0 \tag{14.25}$$

Proof. The inequality (14.18) follows obviously from the expression (14.20) because each term of $\bar{\Delta} = \bar{X} - x$ has a smallness order with respect to h which is equal or greater than 1/2, and the term with order 1/2 is multiplied by ε.

The equalities (14.21)-(14.25) have two parts. The right parts, which we need for construction of the random variables ξ_r, ξ_{ir}, μ_r and ϑ_r, are proved in ref.[10]. The left parts of (14.21) are sufficient to fulfill the inequality (14.17) with $s = 1$, the left parts of (14.21)-(14.22) give (14.17) with $s = 2$, and the left parts of (14.21)-(14.23) are sufficient for $s = 3$; (14.21)-(14.24) for $s = 4$; and (14.21)-(14.25) for $s = 5$. As it has been mentioned above, the inequalities (14.19) follow from (14.17) and Lemma 14.2. Lemma 14.3 is proved.

14.2. The theorem on estimate of the error of the approximation (14.20)

Theorem 14.1. *If the inequalities (14.18), (14.19) and the conditions of Lemma 14.1 are fulfilled, a function f and its partial derivatives up to the sixth order belong to the class* $\mathbf{F}$, *and the random variables ξ_r, ξ_{ir}, μ_r and ϑ_r have sufficiently high finite moments then for $\bar{X}$ of (14.20), we have*

$$|Ef(X) - Ef(\bar{X})| \leq K(x)(h^5 + \varepsilon^4 h^3), \; K(x) \in \mathbf{F} \tag{14.26}$$

Proof. The proof is similar to the proof of the corresponding theorem on one-step approximation for a general system, i.e., $\varepsilon = 1$, with one-step weak accuracy order 3 [10].

In the same way as (14.18) has been proved in Lemma 14.3, we obtain

$$E\prod_{j=1}^{6} |\Delta^{i_j}| \leq K(x)(h^5 + \varepsilon^4 h^3), \; i_j = 1, ..., n, \; K(x) \in \mathbf{F} \tag{14.27}$$

Let us expand $f(x)$ in powers of $\Delta^i = X^i - x^i$ about the point x by the Taylor formula with the Larangian remainder containing sixth-order terms. Analogously, let us expand $f(\bar{X})$ in powers of $\bar{\Delta}^i = \bar{X}^i - x^i$. Then by the inequalities (14.18), (14.19) and (14.27) we obtain (14.26). Theorem 14.1 is proved.

14.3. Simulation of the needed random variables

Now we shall form the random variables ξ_i, μ_i, ξ_{ij}, and ϑ_i, $i, j = 1, ..., q$, so that the expressions (14.21)-(14.25) hold. We propose the following two ways.

The first way [9], [10]. Let us present the random variables ξ_i, μ_i, ξ_{ij} and ϑ_i in the form

$$\mu_i = \frac{1}{2}\xi_i + \frac{1}{\sqrt{12}}\zeta_i, \; \xi_{ij} = \frac{1}{2}(\xi_i\xi_j - \gamma_{ij}\zeta_i\zeta_j), \; \gamma_{ij} = \begin{cases} -1, \; i < j, \\ 1, \; i \geq j \end{cases}, \; \vartheta_i = \frac{1}{3}\xi_i \tag{14.28}$$

where the random variables ξ_i and ζ_i have all needed moments.

Lemma 14.4. *If the independent random variables ξ_i and ζ_i of (14.28) have the properties*

$$E\xi_i = E\xi_i^3 = E\xi_i^5 = 0, \; E\xi_i^2 = 1, \; E\xi_i^4 = 3 \tag{14.29}$$

and

$$E\zeta_i = E\zeta_i^3 = 0, \; E\zeta_i^2 = E\zeta_i^4 = 1 \tag{14.30}$$

then the expressions (14.21)-(14.25) hold.

The random variables, which satisfy Lemma 14.4, can be simulated as

$$P(\xi = 0) = \frac{2}{3}, \; P(\xi = -\sqrt{3}) = P(\xi = \sqrt{3}) = \frac{1}{6}, \; P(\zeta = 1) = P(\zeta = -1) = \frac{1}{2} \tag{14.31}$$

The second way [18]. Let us present the random variables ξ_i, μ_i, ξ_{ij} and ϑ_i in the form

$$\mu_i = \frac{1}{2}\xi_i + \eta_i, \; \xi_{ij} = \frac{1}{2}(\xi_i\xi_j - \zeta_{ij}), \; \vartheta_i = \frac{1}{3}\xi_i \tag{14.32}$$

where the random variables ξ_i, η_i, and ζ_{ij} have all needed moments.

Lemma 14.5. *If the random variables ξ_i, η_i, and ζ_{ij} of (14.32) satisfy expressions (14.29) and*

$$E\eta_i = 0,\ E\eta_i^2 = \frac{1}{12},\ E\zeta_{ij} = 0,\ E\zeta_{ij}^2 = 1,\ i \neq j, \tag{14.33}$$

$$\zeta_{ij} = -\zeta_{ji},\ i \neq j,\ \zeta_{ii} = 1 \tag{14.34}$$

where the random variables ξ_i, η_i, and ζ_{ij}, $i < j$, are independent then the expressions (14.21)-(14.25) hold.

One of the simplest ways to satisfy (14.29), (14.33) is to simulate the random variables so as

$$P(\xi = 0) = \frac{2}{3},\ P(\xi = -\sqrt{3}) = P(\xi = \sqrt{3}) = \frac{1}{6},$$
$$P(\eta = -\frac{1}{\sqrt{12}}) = P(\eta = \frac{1}{\sqrt{12}}) = \frac{1}{2},$$
$$P(\zeta_{ij} = 1) = P(\zeta_{ij} = -1) = \frac{1}{2},\ i < j \tag{14.35}$$

Proofs of both lemmas consist in direct checking of the expressions (14.21)-(14.25) and are distinguished from the proof of the corresponding lemma of ref.[10] in a small way.

The first way requires to simulate $2r$ independent random variables at one step, and the second way requires $r(r+3)/2$ independent random variables at one step. That is why, under large r the first way is preferable.

As a result we obtain the following theorem.

Theorem 14.2. *Let us assume that the conditions of Lemma 14.1 hold and function f has the appropriate properties (see Theorem 14.1). Then the one-step approximation*

$$\bar{X}(t+h) = x + \varepsilon h^{1/2}\sum_{r=1}^{q}\sigma_r\xi_r + ah + \varepsilon^2 h\sum_{r=1}^{q}\sum_{i=1}^{q}\Lambda_i\sigma_r\xi_{ir} + \varepsilon h^{3/2}\sum_{r=1}^{q}(L_1+\varepsilon^2 L_2)\sigma_r[\xi_r - \mu_r] +$$
$$+\varepsilon h^{3/2}\sum_{r=1}^{q}\Lambda_r a\mu_r + \frac{1}{2}(L_1+\varepsilon^2 L_2)ah^2 + \frac{1}{6}\varepsilon h^{5/2}\sum_{r=1}^{q}[L_1^2\sigma_r + L_1\Lambda_r a + \Lambda_r L_1 a]\xi_r +$$
$$+\frac{1}{6}\left[L_1^2 + \varepsilon^2(L_1L_2 + L_2L_1)\right]ah^3 + \frac{1}{24}L_1^3 ah^4 \tag{14.36}$$

where the random variables ξ_r, ξ_{ir} and μ_r are simulated according to either (14.28), (14.31) or (14.32), (14.34), (14.35), has the error

$$|Ef(X) - Ef(\bar{X})| \leq K(x)(h^5 + \varepsilon^4 h^3),\ K(x) \in \mathbf{F}$$

From Theorem 14.2 and Theorem 3.1 it follows that the method, which is based on the one-step approximation (14.36), has the error estimated by $O(h^4 + \varepsilon^4 h^2)$ on the whole interval.

15. Appendix. Derivation of the error estimate for the Runge-Kutta method (5.14)

In connection with the semi-Runge-Kutta method (5.14) let us consider the one-step approximation $\bar{X} = \bar{X}(t+h)$ of the exact solution $X(t+h)$ with the initial value $X(t) = \bar{X}(t) = x$ and introduce the notation

$$\Delta = X(t+h) = x,\ \bar{\Delta} = \bar{X} - x,\ \tilde{\Delta} = \tilde{X} - x$$

where $\tilde{X}$ is the one-step approximation corresponding to the Taylor-type method (4.16).

We shall prove the inequalities

$$|E(\prod_{j=1}^{s}\bar{\Delta}^{i_j}-\prod_{j=1}^{s}\tilde{\Delta}^{i_j})|\leq K(h^5+\varepsilon^2h^3),\ i_j=1,...,n,\ s=1,...,5 \tag{15.1}$$

$$E\prod_{j=1}^{6}|\bar{\Delta}^{i_j}|\leq K(h^5+\varepsilon^2h^3),\ i_j=1,...,n \tag{15.2}$$

because by these inequalities it is easy to obtain the error of the method (5.14).

Indeed, the one-step approximation $\tilde{X}$ can be estimated in the same way as the one-step error of the approximation (14.36) has been obtained in Section 14. So, for the approximation $\tilde{X}$ we can derive

$$|E(\prod_{j=1}^{s}\Delta^{i_j}-\prod_{j=1}^{s}\tilde{\Delta}^{i_j})|\leq K(h^5+\varepsilon^2h^3),\ i_j=1,...,n,\ s=1,...,5 \tag{15.3}$$

Then, as it has been done in Section 2, from (15.1)-(15.3) we obtain the estimate of one-step error of the approximation $\bar{X}$ in the form

$$|Ef(X(t+h))-Ef(\bar{X}(t+h))|\leq K(h^5+\varepsilon^2h^3) \tag{15.4}$$

From (15.4) and Theorem 3.1 it follows that the semi-Runge-Kutta method (5.14) has the error on the whole interval estimated as

$$|Ef(X(t_k))-Ef(X_k)|\leq K(h^4+\varepsilon^2h^2) \tag{15.5}$$

for any N and $k=0,1,...,N$.

Now let us prove the inequalities (15.1) and (15.2). The inequalities (15.2) evidently follow from the form of the one-step approximation $\bar{X}$ by reasoning similar to proving of the inequalities (14.18) in Lemma 14.3.

To prove (15.1) let us expand the terms of $\bar{X}$ in powers of h at the point (t,x). For the first term we have

$$\varepsilon h^{\frac{1}{2}}(\sigma_r(t,x)+\sigma_r(t+h,x))\frac{\xi_r}{2}=\varepsilon h^{\frac{1}{2}}\sigma_r+\varepsilon h^{\frac{3}{2}}\frac{\partial\sigma_r}{\partial t}\frac{\xi_r}{2}+\rho_{1_r} \tag{15.6}$$

where

$$\rho_{1_r}=\varepsilon h^{\frac{5}{2}}\frac{\partial^2\sigma_r}{\partial t^2}\frac{\xi_r}{4}+O(\varepsilon h^{\frac{7}{2}})\xi_r$$

It is clear that ρ_{1_r} has the properties

$$E\rho_{1_r}=0,\ r=1,...,q$$

$$|E\rho_{1_r}\rho_{1_s}|\leq K\varepsilon^2h^5,\ r,s=1,...,q$$

$$\varepsilon h^{\frac{1}{2}}|E\rho_{1_r}\xi_s|\leq K\varepsilon^2h^3,\ r,s=1,...,q \tag{15.7}$$

The term

$$B\equiv\frac{\varepsilon^2h}{2}\left\{b(t,x)+b(t+h,x+\varepsilon h^{\frac{1}{2}}\sum_{j=1}^{q}\sigma_j\xi_j+h(a+\varepsilon^2b))\right\}$$

is expanded as

$$B = \frac{\varepsilon^2 h}{2}\left\{2b + h\frac{\partial b}{\partial t} + \varepsilon h^{\frac{1}{2}}\sum_{j=1}^{q}\sum_{i=1}^{n}\sigma_j^i\xi_j\frac{\partial b}{\partial x^i} + h\sum_{i=1}^{n}(a+\varepsilon^2 b)^i\frac{\partial b}{\partial x^i} + \right.$$

$$\left. + \frac{\varepsilon^2 h}{2}\sum_{j=1}^{q}\sum_{s=1}^{q}\sum_{i=1}^{n}\sum_{l=1}^{n}\sigma_j^i\sigma_s^l\xi_j\xi_s\frac{\partial^2 b}{\partial x^i\partial x^l}\right\} + A + O(\varepsilon^2 h^3) \qquad (15.8)$$

Here A contains terms like $A_{jsl}\varepsilon^5 h^{\frac{5}{2}}\xi_i\xi_j\xi_l$, $A_j\varepsilon^3 h^{\frac{5}{2}}\xi_j$, where A_{jsl} and A_j are constants containing a, b, σ_r and their derivatives at the point (t,x).

Let us remain the mean value

$$\frac{1}{2}E\sum_{j=1}^{q}\sum_{s=1}^{q}\sum_{i=1}^{n}\sum_{l=1}^{n}\sigma_j^i\sigma_s^l\xi_j\xi_s\frac{\partial^2 b}{\partial x^i\partial x^l} = L_2 b$$

in the braces of (15.8) instead of the term $\frac{1}{2}\sum_{j=1}^{q}\sum_{s=1}^{q}\sum_{i=1}^{n}\sum_{l=1}^{n}\sigma_j^i\sigma_s^l\xi_j\xi_s\frac{\partial^2 b}{\partial x^i\partial x^l}$. Then the expression (15.8) is rewritten as

$$B = \frac{\varepsilon^2 h}{2}\left\{2b + h\frac{\partial b}{\partial t} + \varepsilon h^{\frac{1}{2}}\sum_{j=1}^{q}\sum_{i=1}^{n}\sigma_j^i\xi_j\frac{\partial b}{\partial x^i} + h\sum_{i=1}^{n}(a+\varepsilon^2 b)^i\frac{\partial b}{\partial x^i} + \varepsilon^2 h L_2 b\right\} + \rho_2 \quad (15.9)$$

where

$$\rho_2 = \frac{\varepsilon^2 h}{2}\left\{-\varepsilon^2 h L_2 b + \frac{\varepsilon^2 h}{2}\sum_{j=1}^{q}\sum_{s=1}^{q}\sum_{i=1}^{n}\sum_{l=1}^{n}\sigma_j^i\sigma_s^l\xi_j\xi_s\frac{\partial^2 b}{\partial x^i\partial x^l}\right\} + A + O(\varepsilon^2 h^3)$$

For the remainder ρ_2 we have

$$|E\rho_2| \le K\varepsilon^2 h^3,$$

$$|E\rho_2^2| \le K(\varepsilon^4 h^6 + \varepsilon^8 h^4),$$

$$\varepsilon h^{\frac{1}{2}}|E\rho_2\xi_m| \le K\varepsilon^4 h^3,\ m = 1,\dots,q,$$

$$\varepsilon^2 h|E\rho_2\xi_{mr}| \le K(\varepsilon^4 h^4 + \varepsilon^6 h^3),\ m,r = 1,\dots,q,$$

$$\varepsilon^2 h|E\rho_2\xi_m\xi_r| \le K(\varepsilon^4 h^4 + \varepsilon^6 h^3),\ m,r = 1,\dots,q,$$

$$|E\rho_{1_r}\rho_2| \le K\varepsilon^4 h^5,\ r = 1,\dots,q \qquad (15.10)$$

For k_i, $i = 1,\dots,4$, of (5.13) we have

$$\frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = ha + \frac{1}{2}h^2 L_1 a + \frac{1}{6}h^3 L_1^2 a + \frac{1}{24}h^4 L_1^3 a +$$

$$+ \frac{1}{2}\varepsilon h^{\frac{3}{2}}\sum_{r=1}^{q}\sum_{i=1}^{n}\sigma_r^i\xi_r\frac{\partial a}{\partial x^i} + \frac{1}{2}\varepsilon^2 h^2\sum_{i=1}^{n}b^i\frac{\partial a}{\partial x^i} + \frac{1}{2}\varepsilon^2 h^2 L_2 a + \rho_3 \qquad (15.11)$$

where

$$\rho_3 = -\frac{1}{2}\varepsilon^2 h^2 L_2 a + \varepsilon^2 h^2\sum_{r=1}^{q}\sum_{s=1}^{q}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_r^i\sigma_s^j\xi_r\xi_s\frac{\partial^2 a}{\partial x^i\partial x^j} + A + O(h^5 + \varepsilon^2 h^3) \qquad (15.12)$$

The expression A of (15.12) contains $\varepsilon^3 h^{\frac{5}{2}} A_{rsj}\xi_r\xi_s\xi_j$, $\varepsilon h^{\frac{5}{2}} A_r\xi_r$, where A_{rsj}, and A_r are constants containing a, b, σ_r and their derivatives at the point (t,x).

The remainder ρ_3 has the properties

$$|E\rho_3| \le K(h^5+\varepsilon^2 h^3),$$

$$|E\rho_3^2| \le K(h^{10}+\varepsilon^2 h^5+\varepsilon^4 h^4),$$

$$\varepsilon h^{\frac{1}{2}}|E\rho_3\xi_m| \le K\varepsilon^2 h^3, \ m=1,...,q,$$

$$\varepsilon^2 h|E\rho_3\xi_{mr}| \le K\varepsilon^4 h^3, \ m,r=1,...,q,$$

$$\varepsilon^2 h|E\rho_3\xi_m\xi_r| \le K\varepsilon^4 h^3, \ m,r=1,...,q,$$

$$|E\rho_3\rho_{1_r}| \le K\varepsilon^2 h^5, \ r=1,...,q,$$

$$|E\rho_3\rho_2| \le K(\varepsilon^4 h^5+\varepsilon^6 h^4) \tag{15.13}$$

From (15.6), (15.9), (15.11) and from the forms of the one-step approximations $\bar{X}$ and $\tilde{X}$ we have

$$\bar{\Delta}-\tilde{\Delta}=\sum_{r=1}^{q}\rho_{1_r}+\rho_2+\rho_3 \tag{15.14}$$

Then from (15.7), (15.10), (15.11), (15.13) and (15.14) we obtain (15.1).

16. Appendix. Proof of Theorem 10.1

Let us introduce the notation

$$u^\varepsilon(s,x)=Ef(X^\varepsilon_{s,x}(T))$$

and note that

$$u^\varepsilon(s,x)=u^o(s,x)+\varepsilon^2 u^1(s,x;\varepsilon) \tag{16.1}$$

Similarly to the formula (13.6) we obtain

$$R\equiv Ef(X^\varepsilon(T))-Ef(\bar{X}^\varepsilon(T))=Ef(X^\varepsilon(T_N))-Ef(X^\varepsilon_N)=$$

$$=\sum_{i=0}^{N-1}\left\{Eu(t_{i+1},X^\varepsilon_{t_i,\bar{X}_i}(t_{i+1}))-Eu(t_{i+1},\bar{X}^\varepsilon_{t_i,\bar{X}_i}(t_{i+1}))\right\}=$$

$$=E\sum_{i=0}^{N-1}E(u(t_{i+1},X^\varepsilon_{t_i,\bar{X}_i}(t_{i+1}))-u(t_{i+1},\bar{X}^\varepsilon_{t_i,\bar{X}_i}(t_{i+1}))|\bar{X}^\varepsilon_i) \tag{16.2}$$

From ref.[10] we have

$$Eu(t+h,X_{t,x}(t+h))=u+hLu+\frac{1}{2}h^2L^2u+\frac{1}{6}h^3L^3u+O(h^4)$$

and taking into account that $L=L_1+\varepsilon^2\tilde{L}_2$, we write

$$Eu(t+h,X_{t,x}(t+h))=u+h(L_1+\varepsilon^2\tilde{L}_2)u+\frac{1}{2}h^2(L_1+\varepsilon^2\tilde{L}_2)^2u+\frac{1}{6}h^3L_1^3u+O(h^4+\varepsilon^2h^3) \tag{16.3}$$

At one step the method (4.1) has the form

$$\bar{X} = x + \varepsilon h^{\frac{1}{2}} \sum_{r=1}^{q} \sigma_r \xi_r + h(a + \varepsilon^2 b) + \frac{1}{2} h^2 L_1 a, \tag{16.4}$$

$$P(\xi = -1) = P(\xi = 1) = \frac{1}{2}$$

Expanding $Eu(t+h, \bar{X}_{t,x}(t+h))$ by Taylor formula in powers of h, we obtain

$$Eu(t+h, \bar{X}_{t,x}(t+h)) = u + h(L_1 + \varepsilon^2 \tilde{L}_2)u + \frac{1}{2} h^2 L_1^2 u + \varepsilon^2 h^2 A_2 + \varepsilon^4 h^2 A_3 + h^3 A_1 + O(h^4 + \varepsilon^2 h^3) \tag{16.5}$$

where

$$A_1 = \frac{1}{2} \sum_{i=1}^{n} (L_1 a)^i \frac{\partial^2 u}{\partial t \partial x^i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a^i (L_1 a)^j \frac{\partial^2 u}{\partial x^i \partial x^j} + \frac{1}{6} \frac{\partial^3 u}{\partial t^3} + \frac{1}{6} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} a^i a^j a^l \frac{\partial^3 u}{\partial x^i \partial x^j \partial x^l},$$

$$A_2 = \sum_{i=1}^{n} b^i \frac{\partial^2 u}{\partial t \partial x^i} + \sum_{i=1}^{n} \sum_{j=1}^{n} b^i a^j \frac{\partial^2 u}{\partial x^i \partial x^j} + \frac{1}{2} \sum_{r=1}^{q} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \sigma_r^i \sigma_r^j a^l \frac{\partial^3 u}{\partial x^i \partial x^j \partial x^l} +$$

$$+ \frac{1}{2} \sum_{r=1}^{q} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_r^i \sigma_r^j \frac{\partial^3 u}{\partial t \partial x^i \partial x^j},$$

$$A_3 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b^i b^j \frac{\partial^2 u}{\partial x^i \partial x^j} + \frac{1}{2} \sum_{r=1}^{q} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \sigma_r^i \sigma_r^j b^l \frac{\partial^3 u}{\partial x^i \partial x^j \partial x^l} +$$

$$+ \frac{1}{24} \sum_{r=1}^{q} \sum_{s=1}^{q} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} \sigma_r^i \sigma_r^j \sigma_s^l \sigma_s^m \frac{\partial^4 u}{\partial x^i \partial x^j \partial x^l \partial x^m} \tag{16.6}$$

All coefficients in (16.5) and (16.6) are calculated at (t, x) and depend on ε. By (16.3) and 16.5) we have

$$Eu(t+h, X_{t,x}(t+h)) - Eu(t+h, \bar{X}_{t,x}(t+h)) = \varepsilon^2 h^2 B_2 + \varepsilon^4 h^2 B_3 + h^3 B_1 + O(h^4 + \varepsilon^2 h^3) \tag{16.7}$$

where

$$B_1 = \frac{1}{6} L_1^3 u - A_1, \; B_2 = \frac{1}{2} L_1 \tilde{L}_2 u + \frac{1}{2} \tilde{L}_2 L_1 u - A_2, \; B_3 = \frac{1}{2} \tilde{L}_2^2 u - A_3 \tag{16.8}$$

Substituting the conditional variant of (16.7) in (16.2) we obtain

$$R = E \sum_{i=0}^{N-1} \left\{ B_2(\varepsilon; t_i, \bar{X}_i^\varepsilon) \varepsilon^2 h^2 + B_3(\varepsilon; t_i, \bar{X}_i^\varepsilon) \varepsilon^4 h^2 + B_1(\varepsilon; t_i, \bar{X}_i^\varepsilon) h^3 \right\} + O(h^3 + \varepsilon^2 h^2) \tag{16.9}$$

Let us for each $j = 1, 2, 3$ consider the $(n+1)-$dimensional system

$$dX = (a + \varepsilon^2 b) dt + \sum_{r=1}^{q} \sigma_r dW_r, \; X(t_o) = X_o$$

$$dY = B_j dt, \; Y(t_o) = 0 \tag{16.10}$$

where the first n equations are the original ones (see (3.1)) and the last equation describes Y.

Solving the system (16.10) by the method (4.1) we obtain

$$E\int_{t_o}^{T} B_j(\varepsilon;\vartheta,X^\varepsilon(\vartheta))d\vartheta = EY(T) = E\left\{\sum_{i=0}^{N-1}\left[B_j(\varepsilon;t_i,\bar{X}_i^\varepsilon)h + \frac{1}{2}L_1 B_j(\varepsilon;t_i,\bar{X}_i^\varepsilon)h^2\right]\right\} +$$

$$+O(h^2+\varepsilon^2 h) = E\sum_{i=0}^{N-1} B_j(\varepsilon;t_i,\bar{X}_i^\varepsilon)h + O(h)$$

Therefore

$$E\sum_{i=0}^{N-1} B_j(\varepsilon;t_i,\bar{X}_i^\varepsilon)h = E\int_{t_o}^{T} B_j(\varepsilon;\vartheta,X^\varepsilon(\vartheta))d\vartheta + O(h) \tag{16.11}$$

Substituting (16.11) in (16.9) we obtain

$$R = C_1(\varepsilon)h^2 + C_2(\varepsilon)\varepsilon^2 h + O(h^3+\varepsilon^2 h^2) \tag{16.12}$$

where

$$C_1(\varepsilon) = E\int_{t_o}^{T} B_1(\varepsilon;\vartheta,X^\varepsilon(\vartheta))d\vartheta,$$

$$C_2(\varepsilon) = E\int_{t_o}^{T} B_2(\varepsilon;\vartheta,X^\varepsilon(\vartheta))d\vartheta + \varepsilon^2 E\int_{t_o}^{T} B_3(\varepsilon;\vartheta,X^\varepsilon(\vartheta))d\vartheta \tag{16.13}$$

The coefficients $C_i(\varepsilon)$ do not depend on time-increment h but depend on ε (see (16.1)) as

$$C_i(\varepsilon) = C_i^o + O(\varepsilon^2) \tag{16.14}$$

Theorem 10.1 is proved.

17. Acknowledgements

The second author is grateful to Deutscher Akademischer Austauschdienst for support of this work by Forschungsstipendium aus Sondermitteln der NATO.

References

[1] L.Arnold, V.Wihstutz (Eds.), *Lyapunov Exponents.* Lecture Notes in Mathematics 1186 (Springer, 1986)

[2] E.I.Auslender, G.N.Milstein, *Asymptotic expansions of the Lyapunov index for linear stochastic systems with small noises.* Prikl.Mat.Mekh. 46:358 (1982) (in Russian); J.Appl.Math.Mech. pp.277-283 (1983) (in English)

[3] R.Bartussek, P.Hänggi, *Stochastic resonance in optical bistable systems.* Phys.Rev.E 49:3930 (1994)

[4] M.I.Freidlin, A.D.Wentzell, *Random Perturbations of Dynamical Systems.* (Springer, 1984)

[5] I.I.Gihman, A.V.Scorohod, *Stochastic Differential Equations.* (Springer, 1972)

[6] R.Z.Hasminski, *Stochastic Stability of Differential Equations.* (Sijthoff&Noordhoff, 1980)

[7] P.E.Kloeden, E.Platen, *Numerical Solution of Stochastic Differential Equations.* (Springer-Verlag, Berlin, 1992)

[8] G.N.Milstein, *A method of second-order accuracy integration of stochastic differential equations.* Theor.Prob.Appl. 23:396 (1978)

[9] G.N.Milstein, *Weak approximation of solutions of systems of stochastic differential equations.* Theor.Prob.Appl. 30:750 (1985)

[10] G.N.Milstein, *Numerical Integration of Stochastic Differential Equations.*(Ural Univ. Press, Sverdlovsk, 1988; Kluwer Academic Publishers 1995)

[11] G.N.Milstein, M.V.Tret'yakov, *Mean-square approximation for stochastic differential equations with small noises.* IAAS Preprint No. 102 (Berlin, 1994) (submitted for publication)

[12] G.N.Milstein, M.V.Tret'yakov, *Numerical solution of differential equations with colored noise.* Journal of Stat. Physics 77:691 (1994)

[13] *Noise in Nonlinear Dynamical Systems.* F.Moss and P.V.E.McClintock eds. vol. I,II,III (Cambridge Univ. Press, Cambridge, 1989)

[14] E.Pardoux, D.Talay, *Discretization and simulation of stochastic differential equations.* Acta Appl. Math. 3:23 (1985)

[15] E.Pardoux, V.Wihstutz, *Lyapunov exponent and rotation number of two-dimensional linear stochastic systems with small diffusion.* SIAM J.Appl.Math. 48:442 (1988)

[16] W.H.Press, S.A.Teukolsky, W.T.Vetterling, B.P.Flannery, *Numerical Recipes in Fortran: The Art of Scientific Computing.* (Cambridge Univ. Press, Cambridge, 1992)

[17] *Proceedings of the NATO Advanced Research Workshop: Stochastic Resonance in Physics and Biology.* Journal of Stat.Phys. v.70, Nos 1/2 (1993)

[18] D.Talay, *Efficient numerical schemes for the approximation of expectations of functionals of the solution of an SDE and applications.* Springer Lecture Notes in Control and Inform. Sc. 61:294 (1984)

[19] D.Talay, *Approximation of upper Lyapunov exponents of bilinear stochastic differential equations.* SIAM J.Numer.Anal. 28:1141 (1991)

[20] D.Talay, L.Tubaro, *Expansion of the global error for numerical schemes solving stochastic differential equations.* Stoch. Anal.Appl. 8:483 (1990)

[21] W.Wagner, E.Platen, *Approximation of Ito integral equations.* Preprint ZIMM (Akad. der Wiss. der DDR, Berlin, 1978)

Recent publications of the Weierstraß–Institut für Angewandte Analysis und Stochastik

Preprints 1994

94. Hermann Haaf: Existence of periodic travelling waves to reaction–diffusion equations with excitable–oscillatory kinetics.

95. Anton Bovier, Véronique Gayrard, Pierre Picco: Large deviation principles for the Hopfield model and the Kac–Hopfield model.

96. Wolfgang Wagner: Approximation of the Boltzmann equation by discrete velocity models.

97. Anton Bovier, Véronique Gayrard, Pierre Picco: Gibbs states of the Hopfield model with extensively many patterns.

98. Lev D. Pustyl'nikov, Jörg Schmeling: On some estimations of Weyl sums.

99. Michael H. Neumann: Spectral density estimation via nonlinear wavelet methods for stationary non–Gaussian time series.

100. Karmeshu, Henri Schurz: Effects of distributed delays on the stability of structures under seismic excitation and multiplicative noise.

101. Jörg Schmeling: Estimates of Weyl sums over subsequences of natural numbers.

102. Grigori N. Milstein, Michael V. Tret'yakov: Mean–square approximation for stochastic differential equations with small noises.

103. Valentin Konakov: On convergence rates of suprema in the presence of non–negligible trends.

104. Pierluigi Colli, Jürgen Sprekels: On a Penrose–Fife model with zero interfacial energy leading to a phase–field system of relaxed Stefan type.

105. Anton Bovier: Self–averaging in a class of generalized Hopfield models.

106. Andreas Rathsfeld: A wavelet algorithm for the solution of the double layer potential equation over polygonal boundaries.

107. Michael H. Neumann: Bootstrap confidence bands in nonparametric regression.

108. Henri Schurz: Asymptotical mean square stability of an equilibrium point of some linear numerical solutions with multiplicative noise.

109. Gottfried Bruckner: On the stabilization of trigonometric collocation methods for a class of ill–posed first kind equations.

110. Wolfdietrich Müller: Asymptotische Input–Output–Linearisierung und Störgrößenkompensation in nichtlinearen Reaktionssystemen.

111. Vladimir Maz'ya, Gunther Schmidt: On approximate approximations using Gaussian kernels.

112. Henri Schurz: A note on pathwise approximation of stationary Ornstein–Uhlenbeck processes with diagonalizable drift.

113. Peter Mathé: On the existence of unbiased Monte Carlo estimators.

114. Kathrin Bühring: A quadrature method for the hypersingular integral equation on an interval.

115. Gerhard Häckl, Klaus R. Schneider: Controllability near Takens–Bogdanov points.

116. Tatjana A. Averina, Sergey S. Artemiev, Henri Schurz: Simulation of stochastic auto–oscillating systems through variable stepsize algorithms with small noise.

117. Joachim Förste: Zum Einfluß der Wärmeleitung und der Ladungsträgerdiffusion auf das Verhalten eines Halbleiterlasers.

118. Herbert Gajewski, Konrad Gröger: Reaction–diffusion processes of electrically charged species.

119. Johannes Elschner, Siegfried Prössdorf, Ian H. Sloan: The qualocation method for Symm's integral equation on a polygon.

120. Sergej Rjasanow, Wolfgang Wagner: A stochastic weighted particle method for the Boltzmann equation.

121. Ion G. Grama: On moderate deviations for martingales.

122. Klaus Fleischmann, Andreas Greven: Time–space analysis of the cluster–formation in interacting diffusions.