# Weierstraß-Institut
## für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633


# On the stability of elastic-plastic systems with hardening


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submitted: 8th May 2007



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No. 1223

Berlin 2007

WIAS

2000 *Mathematics Subject Classification.* 34A60, 47H06, 73H99.

*Key words and phrases.* differential inclusions, plasticity, hardening, existence, stability.

This work is part of the project 'New materials, adaptive systems and their nonlinearities; modelling control and numerical simulation' carried out in the framework of the European Community Program Improving the human research potential and socio-economic knowledge base (contract n°HPRN-CT-2002-00284). A.P. was also supported by the Deutsche Forschungsgemeinschaft through the project C18 'Analysis and numerics of multidimensional models for elastic phase transformation in a shape-memory alloys' of the Research Center MATHEON.

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**Abstract**

This paper discusses the stability of quasi-static paths for a continuous elastic-plastic system with hardening in a one-dimensional (bar) domain. Mathematical formulations, as well as existence and uniqueness results for dynamic and quasi-static problems involving elastic-plastic systems with linear kinematic hardening are recalled in the paper. The concept of stability of quasi-static paths used here is essentially a continuity property of the system dynamic solutions relatively to the quasi-static ones, when (as in Lyapunov stability) the size of initial perturbations is decreased and the rate of application of the forces (which plays the role of the small parameter in singular perturbation problems) is also decreased to zero. The stability of the quasi-static paths of these elastic-plastic systems is the main result proved in the paper.


# 1 Introduction

The relation that exists between, on one hand, dynamic and quasi-static problems in mechanics and, on the other hand, the theory of singular perturbations was first discussed by Martins et al. in [11]. Those authors recognized the distinct time scales involved in dynamic and quasi-static problems, and performed a change of variables in the governing system of dynamic equations that consists of replacing the physical time by a loading parameter. This leads to a system of equations where, in some of them, the highest order derivative with respect to the loading parameter appears multiplied by the time rate of that loading parameter. The quasi-static problem and solution are expected to be approached when the time rate of change of that loading parameter is decreased to zero.

The variational formulation of plasticity problems with hardening was developed by Johnson [4, 5]. Existence of a strong solution was proved and, under some additional assumptions, a regularity result for the velocity field was obtained. The variational formulation and some existence results for elastic-perfect-plastic and elastic-visco-plastic systems had already been obtained by Duvaut and Lions [3]. In what concerns the dynamic problems in elasto-plasticity with hardening, we address the reader to the works of Krejčí [7], Showalter and Shi [13, 14], Visintin [15], and the references therein.

After the study of finite dimensional elastic-plastic systems with hardening in [9], we prove here that also in the continuum case the dynamic evolutions remain close to a quasi-static path when the dynamic evolutions start sufficiently close to that quasi-static path and the load is applied sufficiently slowly. In the present paper, the definition of stability given in [11] is adapted to the continuum case.

The structure of the article is the following. In Section 2, the mathematical formulations for dynamic and quasi-static elastic-plastic systems with hardening are presented, and in Section 3, existence and uniqueness results are recalled, which use the theory of *m-accretive* operators (see [1, 3, 13, 14, 16]). The final goal of Section 4 is to prove the main stability result of this paper: Proposition 4.8 in Section 4.4. The definition of stability of a quasi-static path is adapted from [9, 10, 11] in Section 4.1. The relevant distance between a dynamic and a quasi-static path at each value of the (time-like) load parameter involves the $H^1$ (semi-)norm of the displacements and the $L^2$ norms of the stresses in the plastic element and of the time rate of change of the displacements. In order to estimate that distance, an auxiliary special dynamic solution is considered in Section 4.4, which has initial conditions that coincide with the quasi-static solution at the initial time. The distance between the dynamic and the quasi-static solutions at any value of the load parameter is then estimated by the sum of the distance between the dynamic and the special dynamic solutions with the distance between the special dynamic and the quasi-static solutions. In Section 4.3, a priori estimates are obtained that are a little more general than those needed for the distance between the special dynamic and the quasi-static solutions. We observe that: (i) the estimate of the distance between the quasi-static solution and the auxiliary special dynamic solution is used in the proof of the main stability result, instead of a direct estimate of the distance between the quasi-static solution and a dynamic solution with arbitrary initial conditions, because (cf. Proposition 4.6) the latter would involve, on the right hand side, norms of the displacements and the stresses in the plastic element that are stronger than those used for the same quantities on the left hand side; (ii) in order to estimate a term that involves the second derivative of the dynamic displacements with respect to the load parameter, the governing system was differentiated with respect to the load parameter (Lemma 4.4); (iii) this in turn required the use of a classical Yosida regularization of the original elastic-problem, i.e. the elasto-visco-plastic approximation introduced in Section 4.2 together with its finite dimensional (Galerkin) approximation.

Finally note that this is the first mathematical discussion of quasi-static stability in smooth or non-smooth *continuum* problems involving the relation between dynamic and quasi-static solutions and an appropriate functional setting. In fact most related discussions in the mechanical literature are based on definitions of stability involving an energetic (power rate) criterion that has an unclear relationship with dynamics, and excludes from the analysis cases with non-symmetric stiffness operators; moreover at some point of those discussions, finite dimensional approximations are often adopted and some of the arguments used may break down in an infinite dimensional context [12]. On the other hand, some related mathematical results on the convergence of dynamic solutions to quasi-static ones were obtained by Duvaut and Lions [3] by making the mass tend to zero; since the different time scales involved in dynamic and quasi-static problems are not brought into play and since perturbations to the initial quasi-static configuration are not considered, the physical relevance of those results and their relationship with the present study are limited.

## 2 Governing equations

We consider an elastic-plastic bar with linear kinematic hardening that has the length $L$ along the $x$ axis. Geometrical linearity is assumed. The governing dynamic equation can be non-dimensionalized by using the non-dimensional time ($\tau$) and load parameter ($\lambda$, $\lambda = \lambda_1 + \varepsilon\tau$), yielding

$$\varepsilon^2 u'' - \sigma_x(u, r) = f(x, \lambda), \tag{1}$$

where $u$, $r$, $f$ are the non-dimensional axial displacement, stress in the plastic element, and applied force per unit length along the bar respectively; $\sigma$ is the non-dimensional stress in the elastic-plastic element, which depends on $u$ and $r$; and the subscript $x$ denotes a derivative with respect to $x$. The extension $e$ is the derivative in space of the non-dimensional generalized displacement $u$, and it can be decomposed into elastic, $e^e$, and plastic, $e^p$, parts:

$$e = 2u_x = e^e + e^p. \tag{2}$$

The stress $\sigma$ is related to the elastic part $e^e$ of the extension by means of Hooke's law, and is also related by the hardening law to the stress in the plastic element $r$ and the plastic extension $e^p$,

$$\sigma = 2r + e^p = e^e = 2u_x - e^p. \tag{3}$$

Therefore (3) leads to

$$\sigma(u, r) = u_x + r. \tag{4}$$

Carrying (4) into (1), we obtain

$$\varepsilon^2 u'' - u_{xx} - r_x = f. \tag{5}$$

Note that, in order that the non-dimensional relation and equation (4), (5) have simple forms, the factor 2 was introduced in several points of (2) and (3), and unit stiffness moduli were considered in the Hooke and hardening laws in (3). The behavior of the plastic element is characterized by the non-dimensional inequality and flow rule:

$$|r| \leq 1, \;\; (e^p)' \begin{cases} \geq 0 \text{ if } r = +1, \\ = 0 \text{ if } \; -1 < r < +1, \\ \leq 0 \text{ if } r = -1. \end{cases} \tag{6}$$

The governing dynamic equations (5), together with the conditions (6) can be put in the form of a singular perturbation system of first order differential equation and inclusion. For that purpose, let $\mathcal{C}$ denote the following closed convex set in $L^2(0, L)$

$$\mathcal{C} = \{r \in L^2(0, L) : |r| \leq 1\}, \tag{7}$$

and let $\mathrm{sign}^{-1}(r)$ be the normal cone to $\mathcal{C}$ at $r \in L^2(0,L)$. Then we observe that (6) can be written in the differential inclusion form:

$$(e^p)' \in \mathrm{sign}^{-1}(r). \tag{8}$$

Relations (3) lead to

$$(e^p)' = u_x' - r'. \tag{9}$$

Substituting (9) in (8), we get

$$u_x' - r' \in \mathrm{sign}^{-1}(r). \tag{10}$$

We now introduce the following spaces

$$H = L^2(0,L),\ V = H^1(0,L),\ V_0 = H_0^1(0,L),$$

and the set

$$W = \{(u,r) \in V_0 \times \mathcal{C} : \sigma = u_x + r \in V\}.$$

We will denote the norm in $H$ (resp. $V$) by $|\cdot|$ (resp. $\|\cdot\|$) and the scalar product in $H$ by $(\,\cdot\,,\,\cdot\,)$. From (5) and (10) we finally obtain the governing dynamic system

$$\begin{cases} \varepsilon u' - v = 0, \\ \varepsilon v' - u_{xx} - r_x = f, \\ u_x' - r' \in \mathrm{sign}^{-1}(r), \end{cases} \tag{11}$$

together with the Dirichlet boundary conditions

$$u = v = 0 \text{ on } \{0, L\} \times (\lambda_1, \lambda_2), \tag{12}$$

and the initial conditions

$$\big(v(\lambda_1), u(\lambda_1), r(\lambda_1)\big) = (v_1, u_1, r_1) \in V_0 \times W. \tag{13}$$

The corresponding quasi-static system is then (let $\varepsilon = 0$ in (11))

$$\begin{cases} -\bar{u}_{xx} - \bar{r}_x = f, \\ \bar{u}_x' - \bar{r}' \in \mathrm{sign}^{-1}(\bar{r}), \end{cases} \tag{14}$$

with the Dirichlet boundary conditions

$$\bar{u} = 0 \text{ on } \{0, L\} \times (\lambda_1, \lambda_2), \tag{15}$$

and the initial conditions

$$\big(\bar{u}(\lambda_1), \bar{r}(\lambda_1)\big) = (\bar{u}_1, \bar{r}_1) \in W. \tag{16}$$

Note that, consistently with the above, the quasi-static displacement rate with respect to the physical time vanishes ($\bar{v} \equiv 0$). Besides, if $X$ is a space of scalar functions, the bold-face notation $\boldsymbol{X}_d$ will denote the space $X^d$.

# 3 Existence and uniqueness of solution for the dynamic and the quasi-static systems

We observe that the dynamic and the quasi-static systems introduced in Section 2 can be rewritten in a form that may be studied with the theory of *m-accretive* operators. The definition and some properties of *m-accretive* operators are recalled in Section 3.1. Existence and uniqueness results for the systems of Section 2 are presented in Section 3.2.

## 3.1 Reminder about m-accretive operators

We recall now the definition of *m-accretive* operators which is contained in many textbooks, see, e.g., [1] or [16]. Let $(\cdot\,,\,\cdot)_Y$ denote the scalar product in $Y$. An operator $\mathbb{A}$ in $Y$ is a collection of related pairs $(x, y) \in Y \times Y$ denoted by $y \in \mathbb{A}(x)$; the *domain* $\mathcal{D}(\mathbb{A})$ is the set of such $x$.

**Definition 3.1** *An operator $\mathbb{A}$ in $Y$ with domain $\mathcal{D}(\mathbb{A})$ is called* ***m-accretive****, if it is monotone,*

$$\forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathcal{D}(\mathbb{A}),\, \forall \boldsymbol{w}_1 \in \mathbb{A}\boldsymbol{v}_1, \forall \boldsymbol{w}_2 \in \mathbb{A}\boldsymbol{v}_2,\, (\boldsymbol{w}_1 - \boldsymbol{w}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2)_Y \geq 0,$$

*and maximal in the set of monotone operators, i.e. for all $[\boldsymbol{v}, \boldsymbol{w}] \in Y \times Y$ such that*

$$(\boldsymbol{w} - \boldsymbol{h}, \boldsymbol{v} - \boldsymbol{\zeta})_Y \geq 0,\ \forall \boldsymbol{\zeta} \in \mathcal{D}(\mathbb{A}),\, \boldsymbol{h} \in \mathbb{A}\boldsymbol{\zeta} \text{ then } \boldsymbol{w} \in \mathbb{A}\boldsymbol{v}.$$

If $\boldsymbol{\varphi}$ is a convex proper and lower semi-continuous function from $Y$ to $(-\infty, +\infty]$, we can define its sub-differential $\boldsymbol{\partial\varphi}$ as the operator in $Y$ such that, for any pair $(\boldsymbol{v}, \boldsymbol{w}) \in Y \times Y$,

$$\boldsymbol{w} \in \boldsymbol{\partial\varphi}(\boldsymbol{v}) \Leftrightarrow \forall \boldsymbol{h} \in Y,\ \boldsymbol{\varphi}(\boldsymbol{v} + \boldsymbol{h}) - \boldsymbol{\varphi}(\boldsymbol{v}) \geq (\boldsymbol{w}, \boldsymbol{h})_Y.$$

Notice that $\boldsymbol{\partial\varphi}$ is an *m-accretive* operator. In particular, we remark that the *indicator function* of interval $[-1, 1]$, $\varphi(r) = \chi_{[-1,1]}(r)$ for $r \in \mathbb{R}$, given by $\chi_{[-1,1]}(r) = 0$ if $r \in [-1, 1]$ and $\chi_{[-1,1]}(r) = +\infty$ otherwise, is a convex proper and lower semi-continuous function and its sub-differential is $\partial\varphi(v) = \text{sign}^{-1}(v)$. For more details, the reader can see the example 2.3.4, p.25 of [1].

## 3.2 Existence and uniqueness of solution

The dynamic and the quasi-static systems introduced in Section 2 can be rewritten in a form that may be studied with the theory of *m-accretive* operators. Consider the differential inclusion problem that involves a multivalued operator $\mathbb{A}$ in the Hilbert space $Y$, with domain $\mathcal{D}(\mathbb{A}) = \{\boldsymbol{x} \in Y : \mathbb{A}\boldsymbol{x} \neq \emptyset\}$:

$$\boldsymbol{x} \in \mathcal{D}(\mathbb{A}),\ \forall \lambda \in [\lambda_1, \lambda_2], \tag{17a}$$

$$\boldsymbol{x}' + \mathbb{A}\boldsymbol{x} \ni \boldsymbol{g} \text{ a.e. on } (\lambda_1, \lambda_2), \tag{17b}$$

$$\boldsymbol{x}(\lambda_1) = \boldsymbol{x}_1. \tag{17c}$$

Recall that existence and uniqueness of solution to this problem can be obtained from the following Proposition:

**Proposition 3.2** *Assume that $\mathbb{A}$ is an m-accretive operator in the Hilbert space $Y$, $\boldsymbol{g}$ belongs to $W^{1,\infty}(\lambda_1, \lambda_2; Y)$ and $\boldsymbol{x}_1 \in \mathcal{D}(\mathbb{A})$. Then there exists a unique solution $\boldsymbol{x}$ of* (17) *belonging to $W^{1,\infty}(\lambda_1, \lambda_2; Y)$.*

By applying Proposition 3.2, we prove existence and uniqueness of solution for the dynamic system (11)–(13) and for the corresponding quasi-static system (14)–(16). Differentiating with respect to $x$ the first equation in the system (11), performing a change of unknown function by using $e = 2u_x$ and denoting $\boldsymbol{x} = (e/2, v, r)$, we get the inclusion (17b) with

$$\mathbb{A} = \frac{1}{\varepsilon}\begin{pmatrix} 0 & -\partial/\partial x & 0 \\ -\partial/\partial x & 0 & -\partial/\partial x \\ 0 & -\partial/\partial x & \varepsilon \operatorname{sign}^{-1}(\,\cdot\,) \end{pmatrix} \text{ and } \boldsymbol{g} = \frac{1}{\varepsilon}\begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}. \tag{18}$$

First it can be checked that $\mathbb{A}$ is a monotone operator. Second, if $(g_1, f, g_2) \in \boldsymbol{H}_3$ and $(v, e, r) \in \boldsymbol{H}_3$, there exists $h(r) \in \operatorname{sign}^{-1}(r) \in H$ for which the resolvent equation $(\mathbf{1} + \mathbb{A})(e, v, r)^T \ni (g_1, f/\varepsilon, g_2)^T$ is equivalent to solving the system

$$\begin{cases} \varepsilon e - 2v_x = \varepsilon g_1, \\ \varepsilon v - \dfrac{1}{2}e_x - r_x = f, \\ \varepsilon r - v_x + \varepsilon h(r) = \varepsilon g_2. \end{cases} \tag{19}$$

This is equivalent to solve for $v \in V$ the following equation:

$$v - \frac{\partial}{\partial x}\Big(\frac{1}{\varepsilon^2}v_x + \frac{1}{2\varepsilon}g_1\Big) - \frac{\partial}{\partial x}\Big(\big(1 + h(\,\cdot\,)\big)^{-1}\Big(\frac{1}{\varepsilon^2}v_x + \frac{1}{\varepsilon}g_2\Big)\Big) = \frac{1}{\varepsilon}f \text{ in } V'.$$

The form is coercive, and existence of a solution follows. The components of $(e, r) \in \boldsymbol{H}_2$ are obtained directly from the first and third terms in (19) respectively. Hence, we conclude that $\mathbb{A}$ is *m-accretive*. For more details, see [13] and [14]. Then Proposition 3.2, with $Y = \boldsymbol{H}_3$ and $\mathcal{D}(\mathbb{A}) = \{(e, v, r) \in Y : v \in V_0,\ e/2 + r \in V,\ r \in \mathcal{C}\}$ yields the following Corollary:

**Corollary 3.3** *Assume that $f$ belongs to $W^{1,\infty}(\lambda_1, \lambda_2; H)$ and that* (13) *holds. Then there exists a unique solution $\boldsymbol{x} = (v, e, r)$ belonging to $W^{1,\infty}(\lambda_1, \lambda_2; \boldsymbol{H}_3)$ that solves* (17) *with $\mathbb{A}$ and $\boldsymbol{g}$ given by* (18)*, and with $r(\lambda) \in \mathcal{C}$ for all $\lambda \in [\lambda_1, \lambda_2]$, $v$ and $\sigma(e, r)$ belong respectively to $L^\infty(\lambda_1, \lambda_2; V_0)$ and $L^\infty(\lambda_1, \lambda_2; V)$.*

**Remark 3.4** *According to Corollary 3.3 and since* $e = 2u_x$, $u = 0$ *on* $\{0, L\}$, $u$ *belongs to* $W^{1,\infty}(\lambda_1, \lambda_2, V_0)$.

In what concerns the *quasi-static problem*, we differentiate the first identity in (14) with respect to the load parameter $\lambda$ and we get

$$-\bar{u}'_{xx} = \bar{r}'_x + f', \tag{20}$$

together with the Dirichlet boundary conditions (15). Since this is an elliptic problem for $\bar{u}'$ we conclude that there exists a unique solution. For such solution $\bar{u}'_x + \bar{r}'$ depends linearly and continuously on $f'$, i.e.

$$\bar{u}'_x + \bar{r}' = Bf', \tag{21}$$

where $B$ is a continuous linear operator between the appropriate spaces. Inserting this in the inclusion in (14) we finally get the differential inclusion

$$\bar{r}' + \mathrm{sign}^{-1}(\bar{r}) \ni Bf'. \tag{22}$$

The sub-differential $\partial\varphi(\bar{r}) = \mathrm{sign}^{-1}(\bar{r})$ is an *m-accretive* operator since $\varphi(\bar{r})$ is a proper convex and lower semi-continuous function. For $\boldsymbol{x} = \bar{r}$, $\mathbb{A} = \mathrm{sign}^{-1}$, $\boldsymbol{g} = Bf'$ and $Y = H$, we apply Proposition 3.2 and we obtain the following Corollary:

**Corollary 3.5** *Assume that* $f$ *belongs to* $W^{1,\infty}(\lambda_1, \lambda_2; H)$ *and* (16) *holds. Then there exists a unique solution* $(\bar{u}, \bar{r})$ *of* (14)–(16) *such that* $(\bar{u}, \bar{r})$ *and* $(\bar{u}', \bar{r}')$ *belong both to* $L^\infty(\lambda_1, \lambda_2; V_0 \times H)$ *and* $\bar{\sigma}(\bar{u}_x, \bar{r})$ *belongs to* $L^\infty(\lambda_1, \lambda_2; V)$.

# 4 Stability of quasi-static paths of elastic-plastic systems

In Section 4.1, we adapt the definition of stability of a quasi-static path [9, 10, 11] to the present elastic-plastic problem with hardening, which appears as a limit case of an elasto-visco-plastic problem. In Section 4.2 we introduce an elasto-visco-plastic problem and we recall existence and uniqueness results for that problem. The Galerkin approximation to that problem is also introduced. In Section 4.3, *a priori* estimates on the elasto-visco-plastic system are obtained which, in Section 4.4, lead to the proof that those two solutions remain close to each other if the dynamic solution of (11) is initially close to the quasi-static solution of (14) and the loading rate $\varepsilon$ is sufficiently small.

## 4.1 Definition of stability of a quasi-static path

The mathematical definition of stability of a quasi-static path at an equilibrium point is presented in the context of the governing dynamic system (11)–(13) and the quasi-static system (14)–(16).

**Definition 4.1** *The quasi-static path $(\bar{u}(\lambda), \bar{r}(\lambda))$ is said to be stable at $\lambda_1$ if there exists $0 < \Delta\lambda \leq \lambda_2 - \lambda_1$, such that, for all $\delta > 0$ there exists $\bar{\rho}(\delta) > 0$ and $\bar{\varepsilon}(\delta) > 0$ such that for all initial conditions $u_1$, $v_1$, $r_1$ and $\bar{u}_1$, $\bar{r}_1$ and all $\varepsilon > 0$ such that*

$$|v_1|^2 + |u_{1x} - \bar{u}_{1x}|^2 + |r_1 - \bar{r}_1|^2 \leq \bar{\rho}(\delta) \text{ and } \varepsilon \leq \bar{\varepsilon}(\delta),$$

*the solution $(u(\lambda), v(\lambda), r(\lambda))$ of the dynamic system (11)–(13) satisfies*

$$|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 \leq \delta,$$

*for all $\lambda \in [\lambda_1, \lambda_1 + \Delta\lambda]$.*

For more details, the reader is referred to [11].

## 4.2 Existence and uniqueness of solution for the elasto-visco-plastic systems

We introduce here the elasto-visco-plastic systems:

$$\begin{cases} \varepsilon^2 u''_\mu - u_{\mu xx} - r_{\mu x} = f, \\ u'_{\mu x} - r'_\mu = \mathcal{J}_\mu(r_\mu), \end{cases} \quad \text{where} \quad \mathcal{J}_\mu(r_\mu) = \frac{1}{\mu}\left(r_\mu - \mathrm{proj}_{\mathcal{C}} r_\mu\right), \tag{23}$$

with the Dirichlet boundary conditions

$$u_\mu = 0 \text{ on } \{0, L\} \times (\lambda_1, \lambda_2), \tag{24}$$

and the initial conditions

$$\big(v_\mu(\lambda_1), u_\mu(\lambda_1), r_\mu(\lambda_1)\big) = (v_1, u_1, r_1) \in V_0 \times W. \tag{25}$$

Here $\mu > 0$ is the viscosity parameter and $\mathrm{proj}_{\mathcal{C}}$ denotes the projection on the convex $\mathcal{C}$.

The variational formulation of the problem (23)–(25) is the following:

$$\begin{cases} \text{Find } (u_\mu, r_\mu) \in V_0 \times H \text{ such that } \forall (u^*, r^*) \in V_0 \times H, \\ (\varepsilon^2 u''_\mu, u^*) + (u_{\mu x}, u^*_x) + (r_\mu, u^*_x) = (f, u^*), \\ (r'_\mu, r^*) - (u'_{\mu x}, r^*) + (\mathcal{J}_\mu(r_\mu), r^*) = 0, \end{cases} \tag{26}$$

with the initial conditions (25). Note that this elasto-visco-plastic problem is an Yosida regularization of the original elastic-plastic problem. For a similar approximation in the corresponding finite-dimensional system see [9]. Whenever convenient we shall use in the following the notation $v_\mu = \varepsilon u'_\mu$.

We consider now a *finite dimensional approximation* of the above elastic-visco-plastic problem, which is obtained in the following classical manner. Let $\{w_j\}_{j=1}^\infty$ be a complete orthonormal sequence in $H$ whose elements belong to $H^2(0, L)$. Let $u_{\mu_n} =$

$\sum_{i=1}^n g_{in}(\lambda)w_i(x)$ and $r_{\mu_n} = \sum_{i=1}^n h_{in}(\lambda)w_i(x)$ satisfying the following variational formulation

$$\begin{cases} \text{For all } u^* = \sum_{i=1}^n g_{in}^*(\lambda)w_i(x) \text{ and } r^* = \sum_{i=1}^n h_{in}^*(\lambda)w_i(x), \\ (\varepsilon^2 u''_{\mu_n}, u^*) + (u_{\mu_n x}, u_x^*) + (r_{\mu_n}, u_x^*) = (f, u^*), \\ (r'_{\mu_n}, r^*) - (u'_{\mu_n x}, r^*) + (\mathcal{J}_\mu(r_{\mu_n}), r^*) = 0, \end{cases} \tag{27}$$

with

$$\varepsilon \sum_1^\infty g'_{in}(\lambda_1)w_i(x) = v_1,\ \sum_1^\infty g_{in}(\lambda_1)w_i(x) = u_1,\ \sum_1^\infty h_{in}(\lambda_1)w_i(x) = r_1. \tag{28}$$

The following results can be proved for the above approximations, when the dimension parameter $n$ tends to $\infty$, and the viscosity parameter $\mu$ tends to 0.

**Proposition 4.2** *Assume that $f$ belongs to $W^{1,\infty}(\lambda_1, \lambda_2; \mathcal{H})$ and that* (25) *holds. Then there exists a unique solution $(v_\mu, u_\mu, r_\mu)$ of* (23)–(25) *such that $(v_\mu, u_\mu, r_\mu)$ and $(v'_\mu, u'_\mu, r'_\mu)$ belong respectively to $L^\infty(\lambda_1, \lambda_2; V_0 \times V_0 \times H)$ and $L^\infty(\lambda_1, \lambda_2; \boldsymbol{H}_3)$ and $\sigma_\mu(u_{\mu x}, r_\mu)$ belongs to $L^\infty(\lambda_1, \lambda_2; V)$. Moreover, as $\mu$ tends to zero, $u_\mu$ and $\sigma_\mu(u_{\mu x}, r_\mu)$ converge strongly to their limits.*

The Galerkin approximation described above together with *a priori* estimates based on the variational formulations (26), (27) can be used to prove these results. The reader can find detailed proofs in the Appendix or in [3]. This Proposition can also be proved using the theory of *m-accretive* operators.

### 4.3 A priori estimates

**Lemma 4.3** *Assume that* (25) *holds and $f$ belongs to $W^{1,\infty}(\lambda_1, \lambda_2; H)$. Then independently of $\mu > 0$, for all $\lambda$ belonging to $(\lambda_1, \lambda_2)$, $v_\mu(\lambda)$, $u_{\mu x}(\lambda)$ and $r_\mu(\lambda)$ are bounded in $H$.*

*Proof.* This estimate results from the application of Gronwall's lemma to energy estimates. Choosing $u^* = u'_\mu$ and $r^* = r_\mu$ in (26), and adding both identities, we obtain

$$(\varepsilon^2 u''_\mu, u'_\mu) + (u_{\mu x}, u'_{\mu x}) + (r'_\mu, r_\mu) + (\mathcal{J}_\mu(r_\mu), r_\mu) = (f, u'_\mu). \tag{29}$$

Let us remark that $(\mathcal{J}_\mu(r_\mu), r_\mu)$ is non negative then we deduce from (29) that

$$\frac{d}{d\xi}\left(|\varepsilon u'_\mu|^2 + |u_{\mu x}|^2 + |r_\mu|^2\right) \leq 2(f, u'_\mu). \tag{30}$$

We integrate (30) over $(\lambda_1, \lambda)$, $\lambda \in [\lambda_1, \lambda_2]$, and since $v_\mu = \varepsilon u'_\mu$, we get

$$\left[|v_\mu|^2 + |u_{\mu x}|^2 + |r_\mu|^2\right]_{\lambda_1}^{\lambda} \leq 2\int_{\lambda_1}^{\lambda} (f, u'_\mu)\, d\xi. \tag{31}$$

Integrating by parts in time the right hand side of (31), we obtain

$$\big[|v_\mu|^2 + |u_{\mu x}|^2 + |r_\mu|^2\big]_{\lambda_1}^{\lambda} \leq 2\big[(f, u_\mu)\big]_{\lambda_1}^{\lambda} - 2\int_{\lambda_1}^{\lambda} (f', u_\mu)\, d\xi.$$

We estimate the product $(z, y)$ by $|z|^2/2\gamma_i + \gamma_i|y|^2/2$, and, choosing different values for $\gamma_i$, $i = 1, 2, 3$, in different terms, we have

$$|v_\mu(\lambda)|^2 + |u_{\mu x}(\lambda)|^2 + |r_\mu(\lambda)|^2 \leq c_1 + \frac{1}{\gamma_1}\,|u_\mu(\lambda)|^2 + \frac{1}{\gamma_3}\int_{\lambda_1}^{\lambda} |u_\mu|^2\, d\xi, \tag{32}$$

where

$$\begin{aligned} c_1 =& |v_1|^2 + |u_{1x}|^2 + |r_1|^2 + \gamma_2|u_1|^2 + \frac{1}{\gamma_2}\,|f(\lambda_1)|^2 \\ &+ \gamma_1\|f\|^2_{L^\infty(\lambda_1,\lambda_2;H)} + \gamma_3\|f'\|^2_{L^2(\lambda_1,\lambda_2;H)}. \end{aligned}$$

On the other hand, the Poincaré inequality (see [2, 6]) shows that there exists a strictly positive constant $c$ such that

$$|u_\mu(\xi)|^2 \leq c|u_{\mu x}(\xi)|^2, \;\; \forall \xi \in (\lambda_1, \lambda_2). \tag{33}$$

Using (33) in (32) and choosing $\gamma_1 = \gamma_3 = 2c$ and $\gamma_2 = 1$ in (32), we may infer that

$$|v_\mu(\lambda)|^2 + \frac{1}{2}\,|u_{\mu x}(\lambda)|^2 + |r_\mu(\lambda)|^2 \leq c_1 + \frac{1}{2}\int_{\lambda_1}^{\lambda} |u_{\mu x}|^2 d\xi. \tag{34}$$

By classical Gronwall's lemma, we get

$$|u_{\mu x}(\lambda)|^2 \leq 2c_1 \exp(\lambda_2 - \lambda_1). \tag{35}$$

As the last term on the right hand side of (34) is now easily estimated, we finally obtain

$$|v_\mu(\lambda)|^2 + |u_{\mu x}(\lambda)|^2 + |r_\mu(\lambda)|^2 \leq c_1\big(1 + (1 + (\lambda_2 - \lambda_1))\exp(\lambda_2 - \lambda_1)\big),$$

from which the desired result follows. □

**Lemma 4.4** *Assume that* (25) *holds and $f$ belongs to $W^{2,\infty}(\lambda_1, \lambda_2; H)$. Then there exists a subsequence, still denoted by $v'_{\mu_n}$, such that*

$$v'_{\mu_n} \rightharpoonup v'_\mu \;\; \textit{weakly} \; * \;\; \textit{in } L^\infty(\lambda_1, \lambda_2; H). \tag{36}$$

*Moreover there exists a positive constant $c(\lambda_1, \lambda_2)$ that depends on the interval of $\lambda$ and such that*

$$\begin{aligned} &|\varepsilon v'_{\mu_n}(\lambda)|^2 \leq c(\lambda_1, \lambda_2)\big(\|v_1\|^2 + |(u_{1xx} + r_{1x}) - (\bar{u}_{1xx} + \bar{r}_{1x})|^2 \\ &\quad + \varepsilon^2|f'(\lambda_1)|^2 + \varepsilon^2\|f'\|^2_{L^\infty(\lambda_1,\lambda_2;H)} + \varepsilon^2\|f''\|^2_{L^2(\lambda_1,\lambda_2;H)}\big). \end{aligned} \tag{37}$$

*Proof.* This estimate results from the energy estimate, Gronwall's lemma and the proof can be completed by a classical Galerkin method. We drop now the subscript $n$. We start by differentiating the governing system (27) with respect to $\lambda$, taking $u^* = \varepsilon^2 u''_\mu$ and $r^* = \varepsilon^2 r'_\mu$ and finally adding both identities, we get

$$(\varepsilon^2 u'''_\mu, \varepsilon^2 u''_\mu) + (u'_{\mu x}, \varepsilon^2 u''_{\mu x}) + (r''_\mu, \varepsilon^2 r'_\mu) + \big((\mathcal{J}_\mu(r_\mu))', \varepsilon^2 r'_\mu\big) = (f', \varepsilon^2 u''_\mu). \tag{38}$$

The monotonicity of $r_\mu \mapsto \mathcal{J}_\mu(r_\mu)$ leads to

$$\begin{aligned}&\big((\mathcal{J}_\mu(r_\mu(\xi)))', r'_\mu(\xi)\big)\\ &= \lim_{\Delta\xi\to 0} \frac{1}{(\Delta\xi)^2}\big(\mathcal{J}_\mu(r_\mu(\xi+\Delta\xi)) - \mathcal{J}_\mu(r_\mu(\xi)), r_\mu(\xi+\Delta\xi) - r_\mu(\xi)\big) \geq 0.\end{aligned}$$

Then we deduce from (38) that

$$\frac{d}{d\xi}\big(|\varepsilon^2 u''_\mu|^2 + |\varepsilon u'_{\mu x}|^2 + |\varepsilon r'_\mu|^2\big) \leq 2(f', \varepsilon^2 u''_\mu). \tag{39}$$

We integrate (39) over $(\lambda_1, \lambda)$, $\lambda \in [\lambda_1, \lambda_2]$, and since $v_\mu = \varepsilon u'_\mu$, we get

$$\big[|\varepsilon v'_\mu|^2 + |v_{\mu x}|^2 + |\varepsilon r'_\mu|^2\big]_{\lambda_1}^{\lambda} \leq 2\int_{\lambda_1}^{\lambda} (\varepsilon f', v'_\mu)\, d\xi. \tag{40}$$

On one hand, we subtract the first equation in (23) at $\lambda_1$ to the first one in (14) at $\lambda_1$. From (25), we deduce that

$$|\varepsilon v'(\lambda_1)|^2 = |(u_{1xx} + r_{1x}) - (\bar{u}_{1xx} + \bar{r}_{1x})|^2. \tag{41}$$

Moreover the initial condition $r_\mu(\lambda_1) = r_1 \in \mathcal{C}$ implies that $\mathcal{J}_\mu(r_1) = 0$ and then the second identity in (23) leads to the following identity

$$|\varepsilon r'_\mu(\lambda_1)|^2 = |v_{1x}|^2. \tag{42}$$

On the other hand, we integrate by parts the right hand side of (40), and we estimate the product $(z, y)$ by $|z|^2/2\gamma_i + \gamma_i|y|^2/2$, and, choosing different values for $\gamma_i$, $i = 1, 2, 3$, we get

$$\begin{aligned}2\int_{\lambda_1}^{\lambda} (\varepsilon f', v'_\mu)\, d\xi &\leq \varepsilon^2\gamma_1 \|f'\|^2_{L^\infty(\lambda_1,\lambda_2;H)} + \frac{1}{\gamma_1}|v_\mu(\lambda)|^2\\ &+ \frac{1}{\gamma_2}|v_1|^2 + \varepsilon^2\gamma_2|f'(\lambda_1)|^2 + \varepsilon^2\gamma_3\|f''\|^2_{L^2(\lambda_1,\lambda_2;H)} + \frac{1}{\gamma_3}\int_{\lambda_1}^{\lambda}|v_\mu|^2\, d\xi.\end{aligned} \tag{43}$$

Since $v = \varepsilon u'$ then the Dirichlet boundary conditions and the Poincaré inequality show that there exists a strictly positive constant $c$ such that

$$|v_\mu(\xi)|^2 \leq c|v_{\mu x}(\xi)|^2, \ \forall \xi \in (\lambda_1, \lambda_2). \tag{44}$$

Carrying (44) into (43) and choosing $\gamma_1 = \gamma_3 = 2c$ and $\gamma_2 = 1$, we have

$$\begin{aligned} 2\int_{\lambda_1}^{\lambda} (\varepsilon f', v'_\mu)\, d\xi &\le 2c\varepsilon^2 \|f'\|^2_{L^\infty(\lambda_1,\lambda_2;H)} + \frac{1}{2}\,|v_{\mu x}(\lambda)|^2 \\ &+ |v_1|^2 + \varepsilon^2 |f'(\lambda_1)|^2 + 2c\varepsilon^2 \|f''\|^2_{L^2(\lambda_1,\lambda_2;H)} + \frac{1}{2}\int_{\lambda_1}^{\lambda} |v_{\mu x}|^2\, d\xi. \end{aligned} \tag{45}$$

Introducing (41), (42) and (45) in (40), we obtain

$$|\varepsilon v'_\mu(\lambda)|^2 + \frac{1}{2}\,|v_{\mu x}(\lambda)|^2 + |\varepsilon r'_\mu(\lambda)|^2 \le g(\varepsilon) + \frac{1}{2}\int_{\lambda_1}^{\lambda} |v_{\mu x}|^2\, d\xi, \tag{46}$$

where

$$\begin{aligned} g(\lambda_1,\varepsilon) &= |v_1|^2 + 2|v_{1x}| + |(u_{1xx} + r_{1x}) - (\bar{u}_{1xx} + \bar{r}_{1x})|^2 \\ &+ \varepsilon^2\big(|f'(\lambda_1)|^2 + 2c\|f'\|^2_{L^\infty(\lambda_1,\lambda_2;H)} + 2c\|f''\|^2_{L^2(\lambda_1,\lambda_2;H)}\big). \end{aligned}$$

By classical Gronwall's lemma, it is clear that

$$|v_{\mu x}(\lambda)|^2 \le 2g(\lambda_1,\varepsilon)\exp(\lambda_2 - \lambda_1). \tag{47}$$

Therefore the last term on the right hand side of (46) is now easily estimated. We finally obtain

$$|\varepsilon v'_\mu(\lambda)|^2 + \frac{1}{2}\,|v_{\mu x}(\lambda)|^2 + |\varepsilon r'_\mu(\lambda)|^2 \le g(\lambda_1,\varepsilon)\big(1 + (\lambda_2 - \lambda_1)\exp(\lambda_2 - \lambda_1)\big),$$

which proves the Lemma. □

Let us remark that the differential inclusion system (11) can be written in a slightly different but equivalent form:

$$\begin{cases} \text{Find } (u,r) \in V_0 \times \mathcal{C} \text{ such that } \forall (u^*, r^*) \in V_0 \times \mathcal{C}, \\ (\varepsilon^2 u'', u^*) + (u_x, u^*_x) + (r, u^*_x) = (f, u^*), \\ (r', r - r^*) - (u'_x, r - r^*) \le 0, \end{cases} \tag{48}$$

with the initial conditions (13).

**Lemma 4.5** *Assume that* (25) *holds and $f$ belongs to $W^{1,\infty}(\lambda_1,\lambda_2;H)$. Then for all $\lambda$ belonging to $(\lambda_1,\lambda_2)$,*

$$\begin{gathered} v_\mu(\lambda) \to v(\lambda) \text{ strongly in } H, \\ u_{\mu x}(\lambda) \to u_x(\lambda) \text{ strongly in } H, \\ r_\mu(\lambda) \to r(\lambda) \text{ strongly in } H, \end{gathered}$$

*as $\mu$ tends to* 0.

*Proof.* These convergence properties are obtained by energy estimating the difference between the elastic-visco-plastic system and the elastic-plastic system with hardening. Choosing $u_\mu^* = u_\mu' - u'$ and $u^* = u' - u_\mu'$ respectively the first identities in (52) and (48), and adding both identities, we get

$$(\varepsilon^2 u_\mu'' - \varepsilon^2 u'', u_\mu' - u') + (u_{\mu x} - u_x, u_{\mu x}' - u_x') + (r_\mu - r, u_{\mu x}' - u_x') = 0. \tag{49}$$

Observing that the second identity in the system (23) implies that

$$(r_\mu - r, u_{\mu x}' - u_x') = (r_\mu' - r', r_\mu - r) + (\mathcal{J}_\mu(r_\mu), r_\mu - r) + (r' - u_x', r_\mu - r). \tag{50}$$

Carrying (50) into (49), integrating over $(\lambda_1, \lambda)$, $\lambda \in [\lambda_1, \lambda_2]$, and using the initial conditions (25) and (12) leads to the following identity

$$\begin{aligned} &|\varepsilon(u_\mu'(\lambda) - u'(\lambda))|^2 + |u_{\mu x}(\lambda) - u_x(\lambda)|^2 + |r_\mu(\lambda) - r(\lambda)|^2 \\ &+ 2\int_{\lambda_1}^{\lambda} (\mathcal{J}_\mu(r_\mu), r_\mu - r)\, d\xi + 2\int_{\lambda_1}^{\lambda} (r' - u_x', r_\mu - r)\, d\xi = 0. \end{aligned} \tag{51}$$

Since $(\mathcal{J}_\mu(r_\mu), r_\mu - r)$ is non negative, $v_\mu = \varepsilon u_\mu'$ and $v = \varepsilon u'$, then we may deduce from (51) that

$$|v_\mu(\lambda) - v(\lambda)|^2 + |u_{\mu x}(\lambda) - u_x(\lambda)|^2 + |r_\mu(\lambda) - r(\lambda)|^2 \le 2\int_{\lambda_1}^{\lambda} (r' - u_x', r - r_\mu)\, d\xi.$$

The conclusion follows from Lemma 4.3. $\square$

On the other hand, the differential inclusion system (14) can be written in a slightly different but equivalent form:

$$\begin{cases} \text{Find } (\bar{u}, \bar{r}) \in V_0 \times \mathcal{C} \text{ such that } \forall (\bar{u}^*, \bar{r}^*) \in V_0 \times \mathcal{C}, \\ (\bar{u}_x, \bar{u}_x^*) + (\bar{r}, \bar{u}_x^*) = (f, \bar{u}^*), \\ (\bar{r}', \bar{r} - \bar{r}^*) - (\bar{u}_x', \bar{r} - \bar{r}^*) \le 0, \end{cases} \tag{52}$$

with the initial conditions (16).

**Proposition 4.6** *Assume that $f$ belongs to $W^{2,\infty}(\lambda_1, \lambda_2; H)$ and that* (13) *and* (16) *hold. Then there exist $\gamma_i > 0$, $i = 1, 2$, such that*

$$\begin{aligned} &|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 \le \gamma_1\big(\|v_1\|^2 \\ &+ |u_{1x} - \bar{u}_{1x}|^2 + |r_1 - \bar{r}_1|^2 + |(u_{1xx} + r_{1x}) - (\bar{u}_{1xx} + \bar{r}_{1x})|^2\big) + \varepsilon\gamma_2. \end{aligned} \tag{53}$$

*Proof.* This result follows from an energy estimate of the difference between the dynamic elastic-visco-plastic system and the quasi-static elastic-plastic system. Choosing $u^* = u_\mu' - \bar{u}'$ and $r^* = r_\mu - \bar{r}$ in (26), $\bar{u}^* = \bar{u}' - u_\mu'$ and $\bar{r}^* = r$ in (52), and adding the resulting expressions, we obtain the following inequality:

$$\begin{aligned} &(\varepsilon^2 u_\mu'', u_\mu') + (u_{\mu x} - \bar{u}_x, u_{\mu x}' - \bar{u}_x') + (r_\mu' - \bar{r}', r_\mu - \bar{r}) \\ &+ (\bar{r}' - \bar{u}_x', r_\mu - r) + (\mathcal{J}_\mu(r_\mu), r_\mu - \bar{r}) \le (\varepsilon^2 u_\mu'', \bar{u}'). \end{aligned} \tag{54}$$

Since $\bar{r} \in \mathcal{C}$ then $\mathcal{J}_\mu(\bar{r}) = 0$, and due to the monotonicity of $\mathcal{J}_\mu$, we get

$$(\mathcal{J}_\mu(r_\mu), r_\mu - \bar{r}) = (\mathcal{J}_\mu(r_\mu) - \mathcal{J}_\mu(\bar{r}), r_\mu - \bar{r}) \geq 0. \tag{55}$$

Using (55) in (54) and since $v_\mu = \varepsilon u'_\mu$, we infer that

$$\frac{d}{d\xi}\big(|v_\mu|^2 + |u_{\mu x} - \bar{u}_x|^2 + |r_\mu - \bar{r}|^2\big) + 2(\bar{r}' - \bar{u}'_x, r_\mu - r) \leq 2(\varepsilon v'_\mu, \bar{u}'). \tag{56}$$

We integrate (56) over $(\lambda_1, \lambda)$, $\lambda \in [\lambda_1, \lambda_2]$ and we obtain

$$\begin{aligned} &|v_\mu(\lambda)|^2 + |u_{\mu x}(\lambda) - \bar{u}_x(\lambda)|^2 + |r_\mu(\lambda) - \bar{r}(\lambda)|^2 \\ &+ 2\int_{\lambda_1}^{\lambda} (\bar{r}' - \bar{u}'_x, r_\mu - r)\, d\xi \leq c(\lambda_1) + 2\int_{\lambda_1}^{\lambda} (\varepsilon v'_\mu, \bar{u}')\, d\xi, \end{aligned} \tag{57}$$

where

$$c(\lambda_1) = |v_1|^2 + |u_{1x} - \bar{u}_{1x}|^2 + |r_1 - \bar{r}_1|^2.$$

Let us observe that

$$\begin{aligned} &\frac{1}{2}\big(|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2\big) - g_\mu(\lambda) \\ &\leq |v_\mu(\lambda)|^2 + |u_{\mu x}(\lambda) - \bar{u}_x(\lambda)|^2 + |r_\mu(\lambda) - \bar{r}(\lambda)|^2, \end{aligned} \tag{58}$$

where

$$g_\mu(\lambda) = |v_\mu(\lambda) - v(\lambda)|^2 + |u_{\mu x}(\lambda) - u_x(\lambda)|^2 + |r_\mu(\lambda) - r(\lambda)|^2.$$

Carrying (58) into (57) and using Cauchy-Schwarz's inequality we have

$$\begin{aligned} &\frac{1}{2}\big(|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2\big) + h_{\mu,n}(\lambda_1, \lambda) \\ &\leq c(\lambda_1) + 2\Big(\int_{\lambda_1}^{\lambda} |\varepsilon v'_{\mu_n}|^2\, d\xi\Big)^{1/2}\Big(\int_{\lambda_1}^{\lambda} |\bar{u}'|^2\, d\xi\Big)^{1/2}, \end{aligned} \tag{59}$$

where

$$h_{\mu,n}(\lambda_1, \lambda) = 2\int_{\lambda_1}^{\lambda} (\bar{r}' - \bar{u}'_x, r_\mu - r)\, d\xi + 2\int_{\lambda_1}^{\lambda} \big(\varepsilon(v'_{\mu_n} - v'_\mu), \bar{u}'\big)\, d\xi - g_\mu(\lambda).$$

Introducing (37), the estimate obtained in Lemma 4.4, in (59), we deduce that there exist $\gamma_i > 0$, $i = 1, 2$, such that

$$\begin{aligned} &\frac{1}{2}\big(|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2\big) + h_{\mu,n}(\lambda_1, \lambda) \leq \gamma_1\big(\|v_1\|^2 \\ &+ |u_{1x} - \bar{u}_{1x}|^2 + |r_1 - \bar{r}_1|^2 + |(u_{1xx} + r_{1x}) - (\bar{u}_{1xx} + \bar{r}_{1x})|^2\big) + \varepsilon\gamma_2. \end{aligned}$$

The conclusion follows then from Lemma 4.5. $\square$

As observed in the Introduction, a direct estimation of the distance between the quasi-static solution and an arbitrary dynamic solution of the elastic-plastic system, leads (because of the required differentiation with respect to the load parameter and the additional initial conditions) to an estimate (53) where, on the right hand side, the initial conditions on $u$ and $r$ are affected by norms that are not the same as those on the left hand side. This situation is overcome in the next section by decomposing that distance into two parts: the distance between an arbitrary dynamic solution and a special dynamic solution, and the distance between the special dynamic solution and the quasi-static solution.

### 4.4 Stability of a quasi-static path

We start by estimating the distance between an arbitrary dynamic solution of the elastic-plastic problem and a special dynamic solution $(\widetilde{v}, \widetilde{u}, \widetilde{r})$ that solves (23) with the the Dirichlet boundary conditions (12) and with initial conditions that coincide with the quasi-static solution at the initial time:

$$\big(\widetilde{v}(\lambda_1), \widetilde{u}(\lambda_1), \widetilde{r}(\lambda_1)\big) = (\varepsilon \bar{u}_1', \bar{u}_1, \bar{r}_1) \in V_0 \times W. \tag{60}$$

Let us remark that the variational formulation of that problem is the following:

$$\begin{cases} \text{Find } (\widetilde{u}, \widetilde{r}) \in V_0 \times \mathcal{C} \text{ such that } \forall (\widetilde{u}^*, \widetilde{r}^*) \in V_0 \times \mathcal{C}, \\ (\varepsilon^2 \widetilde{u}'', \widetilde{u}^*) + (\widetilde{u}_x, \widetilde{u}_x^*) + (\widetilde{r}, \widetilde{u}_x^*) = (f, \widetilde{u}^*), \\ (\widetilde{r}', \widetilde{r} - \widetilde{r}^*) - (\widetilde{u}_x', \widetilde{r} - \widetilde{r}^*) \leq 0, \end{cases} \tag{61}$$

with the initial conditions (60).

**Lemma 4.7** *Assume that* (13) *and* (60) *hold and that* $f$ *belongs to* $W^{1,\infty}(\lambda_1, \lambda_2; H)$. *Then*

$$\begin{aligned} &|v(\lambda) - \widetilde{v}(\lambda)|^2 + |u_x(\lambda) - \widetilde{u}_x(\lambda)|^2 + |r(\lambda) - \widetilde{r}(\lambda)|^2 \\ &\leq |v_1 - \widetilde{v}(\lambda_1)|^2 + |u_{1x} - \bar{u}_{1x}|^2 + |r_1 - \bar{r}_1|^2. \end{aligned} \tag{62}$$

*Proof.* Once again we use energy techniques to compare two dynamic elastic-plastic problems with hardening that have the same boundary conditions but different initial conditions. Choosing $u^* = u' - \widetilde{u}'$ and $\widetilde{u}^* = \widetilde{u}' - u'$ in (48) and (61), respectively, we have

$$\big(\varepsilon^2(u'' - \widetilde{u}''), u' - \widetilde{u}'\big) + (u_x - \widetilde{u}_x, u_x' - \widetilde{u}_x') + (r - \widetilde{r}, u_x' - \widetilde{u}_x') = 0. \tag{63}$$

On the other hand, taking $r^* = \widetilde{r}$ and $\widetilde{r}^* = r$ in (48) and (61), respectively, we get

$$(r' - \widetilde{r}', r - \widetilde{r}) \leq (r - \widetilde{r}, u_x' - \widetilde{u}_x'). \tag{64}$$

Carrying (64) into (63) and since $v = \varepsilon u'$ and $\widetilde{v} = \varepsilon \widetilde{u}'$, we obtain

$$\frac{d}{d\xi}\left(|v - \widetilde{v}|^2 + |u_x - \widetilde{u}_x|^2 + |r - \widetilde{r}|^2\right) \leq 0. \tag{65}$$

Integrating (65) over $(\lambda_1, \lambda)$, $\lambda \in [\lambda_1, \lambda_2]$, and using the initial conditions (13) and (60), leads to the result in the Lemma. $\square$

**Proposition 4.8** *(Stability). Assume that* (13) *and* (16) *hold and that $f$ belongs to $W^{2,\infty}(\lambda_1, \lambda_2; H)$. Then there exist $\gamma > 0$ such that for $0 < \varepsilon < 1$,*

$$\begin{aligned}&|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 \\ &\leq \gamma\big(|v_1|^2 + |u_{1x} - \bar{u}_{1x}|^2 + |r_1 - \bar{r}_1|^2 + \varepsilon\big).\end{aligned}$$

*Proof.* The stability result follows from the estimates obtained in Proposition 4.6 and Lemma 4.7. Let us remark that (62) leads to the following inequality

$$\begin{aligned}&\frac{1}{2}\big(|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2\big) \\ &\leq c(\lambda_1) + |\widetilde{v}(\lambda)|^2 + |\widetilde{u}_x(\lambda) - \bar{u}_x(\lambda)|^2 + |\widetilde{r}(\lambda) - \bar{r}(\lambda)|^2,\end{aligned} \tag{66}$$

where

$$c(\lambda_1) = |v_1 - \widetilde{v}(\lambda_1)|^2 + |u_{1x} - \bar{u}_{1x}|^2 + |r_1 - \bar{r}_1|^2.$$

On the other hand, choosing $u = \widetilde{u}$, $v = \widetilde{v}$ and $r = \widetilde{r}$ in (53) and since $\widetilde{u}(\lambda_1) = \bar{u}_1$ and $\widetilde{r}(\lambda_1) = \bar{r}_1$, we obtain

$$|\widetilde{v}(\lambda)|^2 + |\widetilde{u}_x(\lambda) - \bar{u}_x(\lambda)|^2 + |\widetilde{r}(\lambda) - \bar{r}(\lambda)|^2 \leq \gamma_1\|\widetilde{v}(\lambda_1)\|^2 + \varepsilon\gamma_2. \tag{67}$$

Introducing (67) in (66), we get

$$|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 \leq \gamma_1\|\widetilde{v}(\lambda_1)\|^2 + 2c(\lambda_1) + \varepsilon\gamma_2.$$

Since $\bar{u}'(\lambda_1)$ and $\bar{u}'_x(\lambda_1)$ are bounded in $H$ and $\widetilde{v}(\lambda_1) = \varepsilon\bar{u}'_1$ then the Proposition follows. □

# Appendix

*Proof. (Proposition 4.2).* We drop the subscript $\mu$ in the proof and without loss of generality, we may assume here that $\varepsilon = 1$ and using the fact that $e = 2u_x$, then (26) with the initial conditions (25) become

$$\begin{cases}\frac{1}{2}(e', e^*) + (v, e^*_x) = 0, \\ (v', v^*) + \frac{1}{2}(e, v^*_x) + (r, v^*_x) = (f, v^*), \\ (r', r^*) - (v_x, r^*) + (\mathcal{J}_\mu(r), r^*) = 0,\end{cases} \tag{68}$$

with the initial conditions

$$\big(v_\eta(\lambda_1), e_\eta(\lambda_1), r_\eta(\lambda_1)\big) = (v_1, u_{1x}, r_1) \in V_0 \times W. \tag{69}$$

Let us first prove the uniqueness. Let $(v, e, r)$ and $(v_*, e_*, r_*)$ be two possible solutions of (68).

$$\widehat{v} = v - v_*, \;\; \widehat{e} = (e - e_*)/2, \;\; \widehat{r} = r - r_*.$$

Therefore we deduce from (68) and from the analogous equations for $v_*$, $e_*$, $r_*$, that

$$(\widehat{e}', e^*) + (\widehat{v}, e^*_x) + (\widehat{v}', v^*) + (\widehat{e}, v^*_x) + (\widehat{r}', r^*) + (\mathcal{J}_\mu(r) - \mathcal{J}_\mu(r_*), r^*) = 0. \tag{70}$$

Choosing $v^* = \widehat{v}$, $e^* = \widehat{e}$, $r^* = \widehat{r}$ in (70), we obtain

$$\frac{1}{2}\frac{d}{d\xi}\big(|\widehat{v}|^2 + |\widehat{e}|^2 + |\widehat{r}|^2\big) + (\widehat{v}, \widehat{e}_x) + (\widehat{e}, \widehat{v}_x) + (\mathcal{J}_\mu(r) - \mathcal{J}_\mu(r_*), r - r_*) = 0. \tag{71}$$

Since $(\widehat{v}, \widehat{e}_x) + (\widehat{e}, \widehat{v}_x) = 0$ and $r \mapsto \mathcal{J}_\mu(r)$ is monotone then we infer from (71) that

$$\frac{d}{d\xi}\big(|\widehat{v}|^2 + |\widehat{e}|^2 + |\widehat{r}|^2\big) \le 0,$$

which, together with the fact that $\widehat{v}(\lambda_1) = \widehat{e}(\lambda_1) = \widehat{r}(\lambda_1) = 0$, leads to $\widehat{v} = \widehat{e} = \widehat{r} = 0$ and the uniqueness follows.

We regularize (68) in the space variable. Let $\eta$ be a strictly positive parameter, which will tend toward zero. We consider now the regularized problem

$$\begin{cases} \frac{1}{2}(e'_\eta, e^*) + (v_\eta, e^*_x) + \eta(e_{\eta x}, e^*_x) = 0, \\ (v'_\eta, v^*) + \frac{1}{2}(e_\eta, v^*_x) + (r_\eta, v^*_x) + \eta(v_{\eta x}, v^*_x) = (f, v^*), \\ (r'_\eta, r^*) - (v_{\eta x}, r^*) + (\mathcal{J}_\mu(r_\eta), r^*) + \eta(r_{\eta x}, r^*_x) = 0. \end{cases} \tag{72}$$

with the initial conditions

$$\big(v_\eta(\lambda_1), e_\eta(\lambda_1), r_\eta(\lambda_1)\big) = (v_1, u_{1x}, r_1) \in V_0 \times W. \tag{73}$$

Recall that for every $\eta > 0$, (72)–(73) has a unique solution. This follows from the general theory of monotone parabolic problems (cf. [8]). We establish now *a priori* estimates and then we pass to the limit when $\eta$ tends to zero.

*First a priori estimate.* Taking $v^* = v_\eta$, $e^* = e_\eta/2$, $r^* = r_\eta$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\xi}\Big(\frac{1}{2}|v_\eta|^2 + |e_\eta|^2 + |r_\eta|^2\Big) + \frac{1}{2}\big((v_\eta, e_{\eta x}) + (e_\eta, v_{\eta x})\big) \\ &+ \eta\Big(\frac{1}{2}|v_{\eta x}|^2 + |e_{\eta x}|^2 + |r_{\eta x}|^2\Big) + (\mathcal{J}_\mu(r_\eta), r_\eta) = (f, v_\eta). \end{aligned} \tag{74}$$

Since

$$(v_\eta, e_{\eta x}) + (e_\eta, v_{\eta x}) = 0 \quad \text{and} \quad (\mathcal{J}_\mu(r_\eta), r_\eta) \ge 0,$$

then we deduce from (74) that

$$\frac{1}{2}\frac{d}{d\xi}\Big(\frac{1}{2}|v_\eta|^2 + |e_\eta|^2 + |r_\eta|^2\Big) + \eta\Big(\frac{1}{2}|v_{\eta x}|^2 + |e_{\eta x}|^2 + |r_{\eta x}|^2\Big) \le (f, v_\eta). \tag{75}$$

From (75), we may conclude, when $\eta$ tends to zero, that

$$(v_\eta, e_\eta, r_\eta) \text{ is bounded in } \boldsymbol{L}^\infty_3(\lambda_1, \lambda_2; H), \tag{76a}$$

$$(\eta^{1/2} v_\eta, \eta^{1/2} e_\eta, \eta^{1/2} r_\eta) \text{ is bounded in } \boldsymbol{L}^2_3(\lambda_1, \lambda_2; V). \tag{76b}$$

Moreover, setting $\lambda = \lambda_1$ in (72), we conclude thanks to the initial conditions (73) that

$$\begin{cases} \dfrac{1}{2}(e'_\eta(\lambda_1), e^*) = (v_{1x}, e^*) - \eta(u_{1xx}, e^*_x), \\ (v'_\eta(\lambda_1), v^*) = (f(\lambda_1) + u_{1xx} + r_{1x}, v^*) - \eta(v_{1x}, v^*_x), \\ (r'_\eta(\lambda_1), r^*) = (v_{1x}, r^*) - \eta(r_{1x}, r^*_x), \end{cases}$$

from which it follows, when $\eta$ tends to zero, that

$$\big(v'_\eta(\lambda_1), e'_\eta(\lambda_1), r'_\eta(\lambda_1)\big) \text{ is bounded in } \boldsymbol{H}_3. \tag{77}$$

*Second a priori estimate.* Differentiating (72) with respect to $\lambda$ (this is legitimate if (72) is approximated by Galerkin method), and replacing $e^*$, $v^*$ and $r^*$ respectively by $e_\eta$, $v_\eta$ and $r_\eta$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\xi}\Big(\frac{1}{2}|v'_\eta|^2 + |e'_\eta|^2 + |r'_\eta|^2\Big) + \frac{1}{2}\big((v'_\eta, e'_{\eta x}) + (e'_\eta, v'_{\eta x})\big) \\ &+ \eta\Big(\frac{1}{2}|v'_{\eta x}|^2 + |e'_{\eta x}|^2 + |r'_{\eta x}|^2\Big) + (\mathcal{J}'_\mu(r_\eta), r'_\eta) = (f', v_\eta). \end{aligned} \tag{78}$$

Using the same argument already given in Lemma 4.4, it is clear that the last term on the right hand side of (78) is non negative, on the other hand, $(v'_\eta, e'_{\eta x}) + (e'_\eta, v'_{\eta x}) = 0$, then we deduce from (78) that

$$\frac{1}{2}\frac{d}{d\xi}\Big(\frac{1}{2}|v'_\eta|^2 + |e'_\eta|^2 + |r'_\eta|^2\Big) + \eta\Big(\frac{1}{2}|v'_{\eta x}|^2 + |e'_{\eta x}|^2 + |r'_{\eta x}|^2\Big) \le (f', v_\eta),$$

from which it follows, taking into account of (77), when $\eta$ tends to zero, that

$$(v'_\eta, e'_\eta, r'_\eta) \text{ is bounded in } \boldsymbol{L}^\infty_3(\lambda_1, \lambda_2; H), \tag{79a}$$

$$(\eta^{1/2} v'_\eta, \eta^{1/2} e'_\eta, \eta^{1/2} r'_\eta) \text{ is bounded in } \boldsymbol{L}^2_3(\lambda_1, \lambda_2; V). \tag{79b}$$

On the other hand, from the preceding estimates, we may deduce that

$$\mathcal{J}_\mu(r_\eta) \text{ is bounded in } L^\infty(\lambda_1, \lambda_2; H). \tag{80}$$

Then (76), (79) and (80) imply that we can extract from the sequence $(v_\eta, e_\eta, r_\eta)$, a subsequence still denoted by $(v_\eta, e_\eta, r_\eta)$, when $\eta$ tends to zero, such that

$$(v_\eta, e_\eta, r_\eta) \rightharpoonup (v, e, r) \text{ weakly} * \text{ in } \boldsymbol{L}^\infty_3(\lambda_1, \lambda_2; H), \tag{81a}$$

$$(v'_\eta, e'_\eta, r'_\eta) \rightharpoonup (v', e', r') \text{ weakly} * \text{ in } \boldsymbol{L}^\infty_3(\lambda_1, \lambda_2; H), \tag{81b}$$

$$\mathcal{J}_\mu(r_\eta) \rightharpoonup \Psi_\mu \text{ weakly} * \text{ in } L^\infty(\lambda_1, \lambda_2; H). \tag{81c}$$

By a monotonicity argument, we show that $\Psi_\mu = \mathcal{J}_\mu(r)$ (cf. Chapter 2 of [8]). Then we may pass to the limit in (72) using (76b), (79b), (81) and finally we obtain (68). We have proved existence of $(v, e, r)$ that satisfy (68)–(69) such that $(v, e, r)$

and $(v', e', r')$ belong respectively to $\boldsymbol{L}_3^\infty(\lambda_1, \lambda_2; H)$ and $\boldsymbol{L}_3^\infty(\lambda_1, \lambda_2; H)$. On the other hand, we infer from (55) that in the sense of distribution, we get

$$\begin{cases} v_x = \dfrac{1}{2} e', \\ \dfrac{1}{2} e_x + r_x = v' - f, \\ v_x = r' + \mathcal{J}_\mu(r). \end{cases}$$

Since $e = 2u_x$ then the Proposition follows. □

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