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Weak-convergence methods for Hamiltonian multiscale problems

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Abstract

We consider Hamiltonian problems depending on a small parameter like in wave equations with rapidly oscillating coefficients or the embedding of an infinite atomic chain into a continuum by letting the atomic distance tend to 0. For general semi-linear Hamiltonian systems we provide abstract convergence results in terms of the existence of a family of joint recovery operators which guarantee that the effective equation is obtained by taking the Γ-limit of the Hamiltonian. The convergence is in the weak sense with respect to the energy norm. Exploiting the well-developed theory of Γ-convergence, we are able to generalize the admissible coefficients for homogenization in the wave equations. Moreover, we treat the passage from a discrete oscillator chain to a wave equation with general L^∞ coefficients.

1 Introduction

Many evolutionary problems are of geometric nature and are described by functionals and geometric structures. Dissipative systems on a state space $\mathcal{Q}$ are typically given by an energy potential $\Phi : \mathcal{Q} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ and a dissipation functional $\mathcal{R} : \mathrm{T}\mathcal{Q}$ giving rise to an equation of the type of a gradient flow:

$$0 = \partial_{\dot u}\mathcal{R}(u(t), \dot u(t)) + \partial_u \Phi(u(t)). \tag{1.1}$$

Here, we will deal with Lagrangian and Hamiltonian systems that are defined on a tangent or cotangent bundle of the configuration space $\mathcal{Q}$. In a mechanical system we have in addition to the energy potential Φ a kinetic energy $\mathcal{K}(u, \dot u) = \frac{1}{2}\langle M(u)\dot u, \dot u\rangle$ on $\mathrm{T}\mathcal{Q}$ and the Lagrangian equations read

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\partial_{\dot u}\mathcal{K}(u, \dot u)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big(M(u)\dot u\big) = -\partial_u \Phi(u).$$

Introducing the conjugate momentum $p = M(u)\dot u$ we obtain the canonical Hamiltonian form

$$\dot u = \partial_p \mathcal{H}(u,p) = M(u)^{-1}p, \quad \dot p = -\partial_u \mathcal{H}(u,p) = -\partial_u \Phi(u),$$

where $\mathcal{H} : \mathrm{T}^*\mathcal{Q} \to \mathbb{R}_\infty : (u,p) \mapsto \frac{1}{2}\langle M(u)^{-1}p, p\rangle + \Phi(u)$. More generally Hamiltonian systems are defined on a general manifold $\mathcal{P}$ and described by a Hamiltonian $\mathcal{H} : \mathcal{P} \to \mathbb{R}$ and a symplectic form Ω (a nondegenerate two-form).

In all these contexts there arises the natural question about the limiting behavior if the functionals and structures depend on a small parameter ε. Assume that we have given

Φ_ε and $\mathcal{R}_\varepsilon$ in the dissipative case, Φ_ε and $\mathcal{K}_\varepsilon$ in the Lagrangian case, or $\mathcal{H}_\varepsilon$ and Ω_ε in the Hamiltonian case, where the range of ε is given as $[0, \varepsilon_1]$, i.e., the desired limit case $\varepsilon = 0$ is included. For each ε we also have solution $u_\varepsilon : [t_1, t_2] \to \mathcal{Q}$. The general aim in this context is to analyze the types of convergence we need to impose such that we can guarantee that the limit $q_0(t) = \lim_{\varepsilon\to 0} q_\varepsilon(t)$ satisfies the limit problem with $\Phi_0 = \lim_{\varepsilon\to 0} \Phi_\varepsilon$ and similarly for $\mathcal{R}_\varepsilon$, etc. Of course, if the dependence in ε is continuous in suitably strong topologies, then the standard theory of continuous dependence provides the desired result.

We are here interested in relatively weak types of convergences for the functionals, namely those that allow us to treat multiscale problems. For instance, for the wave equation

$$\rho(\tfrac{1}{\varepsilon}x)\ddot{u}_\varepsilon = \mathrm{div}(A(\tfrac{1}{\varepsilon}x)\nabla u_\varepsilon) + B(\tfrac{1}{\varepsilon}x)u_\varepsilon \tag{1.2}$$

with highly oscillatory, periodic coefficients the solutions will not converge for $\varepsilon \to 0$ in strong norms. The best we can hope for will be the weak convergence in the energy norm. Under reasonable assumptions, for this case the limiting problem can be constructed and we obtain an effective, macroscopic equation, namely

$$\rho^*\ddot{u}_0 = \mathrm{div}(A_*\nabla u_0) + B^* u_0,$$

where ρ^* and B^* are simple averages while A_* is a more complicated effective stiffness tensor related to the harmonic mean.

Defining the associated potential and kinetic energies

$$\begin{aligned}
&\Phi_\varepsilon(u) = \tfrac{1}{2}\int_\Omega A(\tfrac{1}{\varepsilon}x)\nabla u{:}\nabla u + B(\tfrac{1}{\varepsilon}x)u{\cdot}u\,\mathrm{d}x, \quad \Phi_0(u) = \tfrac{1}{2}\int_\Omega A_*\nabla u{:}\nabla u + B^* u{\cdot}u\,\mathrm{d}x,\\
&\mathcal{K}_\varepsilon(v) = \tfrac{1}{2}\int_\Omega \rho(\tfrac{1}{\varepsilon}x)v\cdot v\,\mathrm{d}x, \quad \mathcal{K}_0(v) = \tfrac{1}{2}\int_\Omega \rho^* v\cdot v\,\mathrm{d}x
\end{aligned}$$

it is the question in what sense we have that Φ_ε and $\mathcal{K}_\varepsilon$ converge to Φ_0 and $\mathcal{K}_0$ respectively. It turns out that the most relevant converge is the so-called Γ-convergence for functionals, see [Dal93, Bra02]. However, since we have two functionals it is not clear that we can do the two limit calculations independently. The determination of effective Hamiltonian in multiscale problems is one of the fundamental issue in many areas such as quantum mechanics, molecular dynamics, fiber optics, or water wave theory [CDMZ91, BS97, SW00b, All03, LT05, GM06, Mie06c, CS07, GHM06b]

In Section 2 we will address these question in an abstract setting. For this we introduce *families of joint recovery operators* $(G_\varepsilon)_{\varepsilon>0}$ that work for both functionals simultaneously. We also provide counterexamples showing that nonexistence of such a family may lead to failure in the limiting procedure, i.e., limits of solutions fail to solve the problem associated with the limiting functionals. In Section 3 we apply the theory to one-dimensional systems of wave equations generalizing (1.2). and in Section 4 we treat the passage from a discrete lattice system to a continuum system.

Before going into details we point to related work that also bases on the idea of identifying the limit problem by passing to the limit in the determining functionals rather than in the equation itself. For gradient flows the dissipation potential $\mathcal{R}_\varepsilon$ relates to a Riemannian metric, i.e., $\mathcal{R}_\varepsilon(u,\dot u)=\frac12\langle g_\varepsilon(u)\dot u,\dot u\rangle$, where $g_\varepsilon(u):\mathrm{T}_u\mathcal{Q}\to\mathrm{T}_u^*\mathcal{Q}$ is symmetric and positive semidefinite. The question of taking the limit for the gradient flows $g_\varepsilon(u)\dot u=-\partial_u\Phi_\varepsilon(u)$ was addressed in [SS04] to derive the limiting behavior for the vortices in a Ginzburg–Landau model, in [Ort05] to analyze convergence of numerical approximations, and in [KMM06] for the limit behavior of domain walls in thin magnetic films. A simple linear counterexample with $\mathcal{Q}=\mathbb{R}^2$ is given in [Mie06b].

Another interesting dissipative situation is the case of rate-independent systems where $\mathcal{R}(q,\cdot)$ is homogeneous of degree 1. Then, $\partial_v\mathcal{R}(q,v)\subset\mathrm{T}_q\mathcal{Q}$ denotes the set-valued subdifferential of the convex function $\mathcal{R}(q,\cdot)$ and (1.1) is a differential inclusion, which may be reformulated as an evolutionary variational inequality, cf. [Mie05]. For *rate-independent systems*, Γ-convergence is studied via the energetic formulation in [MO07, MRS06] using the global distance $\mathcal{D}_\varepsilon:\mathcal{Q}\times\mathcal{Q}\to[0,\infty]$ associated with the infinitesimal metric $\mathcal{R}_\varepsilon$. In addition to the Γ-convergence of Φ_ε and $\mathcal{D}_\varepsilon$ to Φ_0 and $\mathcal{D}_0$, respectively, one has to impose the existence of *joint recovery sequences*:

$$\begin{aligned}&\forall\,u_\varepsilon \text{ with } u_\varepsilon\to u\ \forall\,\widehat u\in\mathcal{Q}\ \exists\,\widehat u_\varepsilon \text{ with } \widehat u_\varepsilon\to\widehat u:\\ &\limsup_{\varepsilon\to\infty}\big(\Phi_\varepsilon(t,\widehat u_\varepsilon)+\mathcal{D}_\varepsilon(u_\varepsilon,\widehat u_\varepsilon)-\Phi_\varepsilon(t,u_\varepsilon)\big)\le\Phi_0(t,\widehat u)+\mathcal{D}_0(u,\widehat u)-\Phi_0(t,u),\end{aligned}$$

Several applications are treated in [MRS06], and [MT06] addresses the two-scale homogenization for linearized elastoplasticity.

We return to our theory concerning Hamiltonian systems. Our theory in Section 2 is based on a Gelfand triple $V\subset X\subset V^*$ of Hilbert spaces and closed subspaces $V_\varepsilon\subset V$. We consider general, coercive, lower semi-continuous quadratic forms of the type

$$\Phi_\varepsilon(u)=\begin{cases}\frac12\langle A_\varepsilon u,u\rangle & \text{for } u\in V_\varepsilon,\\ \infty & \text{otherwise,}\end{cases}$$

and show that $\Phi_\varepsilon\xrightarrow{\Gamma}\Phi_0$ (defined in (2.2) and also written $\Phi_0=\Gamma\text{-}\lim_{\varepsilon\to0}\Phi_\varepsilon$) if and only if there exists a family $(G_\varepsilon)_\varepsilon$ of recovery operators with $G_\varepsilon\in\mathcal{L}(V_0;V_\varepsilon)$ such that

$$\begin{array}{lll}\text{(i)} & \forall\,v_0\in V_0:\quad F_\varepsilon v_0\rightharpoonup v_0 \text{ in } V, & \\ \text{(ii)} & v_\varepsilon\in V_\varepsilon,\ v_\varepsilon\rightharpoonup v_0\in V_0 \quad\Longrightarrow\quad F_\varepsilon^*A_\varepsilon v_\varepsilon\rightharpoonup A_0v_0 \text{ in } V_0^*, & \\ \text{(iii)} & v_\varepsilon\rightharpoonup v_0\notin V_0 \quad\Longrightarrow\quad \Phi_\varepsilon(v_\varepsilon)\to\infty. & \end{array}\tag{1.3}$$

Combining (i) and (ii) it follows immediately that $\Phi_\varepsilon(F_\varepsilon v_0)\to\Phi_0(v_0)$. In case that $V_\varepsilon=V_0$ and that A_ε has a bounded inverse one can choose $F_\varepsilon=A_\varepsilon^{-1}A_0$ and the stronger statement $A_\varepsilon F_\varepsilon v_0\to A_0v_0$ in V_0^*. But applications (cf. Section 4 and [Mie06c]) need the more general context that V_ε is a true subspace that may not be dense.

In Sections 2.3 and 2.4 we consider linear and semilinear mechanical systems of the form

$$M_\varepsilon\ddot u_\varepsilon+\mathrm{D}_u\Phi_\varepsilon(u_\varepsilon)=0,\quad u_\varepsilon\in V_\varepsilon,\tag{1.4}$$

where the kinetic energies $\mathcal{K}_\varepsilon(v)=\frac12\langle M_\varepsilon v,v\rangle$ and the potentials Φ_ε are uniformly coercive on X and V respectively. Moreover, we assume $\Phi_\varepsilon\in\mathrm{C}^1(V_\varepsilon;\mathbb{R})$ for some closed subspace

$V_\varepsilon \subset V$. Using the coercivity any family of solutions $u_\varepsilon : \mathbb{R} \to V_\varepsilon$ with bounded energy has a subsequence and a limit function $u_0 \in \mathrm{C}^\infty_\mathrm{w}(\mathbb{R}, V) \cap \mathrm{C}^{\mathrm{Lip}}(\mathbb{R}, X)$ satisfying

$$\forall\, t \in \mathbb{R}: \; u_\varepsilon(t) \rightharpoonup u_0(t) \text{ in } V \quad \text{ and } \; \dot u_\varepsilon \overset{*}{\rightharpoonup} \dot u_0 \text{ in } \mathrm{L}^\infty(\mathbb{R}, X).$$

In Theorem 2.10 we show that u_0 satisfies the limit problem if there exists a family $(G_\varepsilon)_{\varepsilon>0}$ of joint recovery operators $G_\varepsilon \in \mathcal{L}(V_0; V_\varepsilon)$ such that the following holds: If $u_\varepsilon \in V_\varepsilon$ with $u_\varepsilon \rightharpoonup u_0$ and $\sup \Phi_\varepsilon(u_\varepsilon) < \infty$, then we have

$$\text{(a)} \;\; u_0 \in V_0, \quad \text{(b)} \;\; G_\varepsilon^* M_\varepsilon u_\varepsilon \rightharpoonup M_0 u_0 \text{ in } V_0^*, \quad \text{(c)} \;\; G_\varepsilon^* \mathrm{D}\Phi_\varepsilon(u_\varepsilon) \rightharpoonup \mathrm{D}\Phi_0(u_0) \text{ in } V_0^*. \tag{1.5}$$

We also discuss the question whether convergence of the initial conditions $(u_\varepsilon(t_1), \dot u_\varepsilon(t))$ implies convergence at other times. Example 2.8 shows that this is wrong in general, and Theorem 2.7(c) provides sufficient conditions. In general, the convergence

$$(G_\varepsilon^* M u_\varepsilon(t_1), G_\varepsilon^* M \dot u_\varepsilon(t_1)) \to (M_0 u_0(t_1), M_0 \dot u_0(t_1)) \text{ in } V_0^* \to V_0^*$$

for some t_1 implies the same convergence for all $t \in \mathbb{R}$.

Section 2.5 provides the corresponding results for Hamiltonian systems of the form

$$\Omega_\varepsilon \dot z_\varepsilon = \mathrm{D}H_\varepsilon(z_\varepsilon), \quad z_\varepsilon \in Z_\varepsilon.$$

The joint recovery condition reads exactly as (1.5) if u, V, Φ, and M are replaced by z, Z, H, and Ω, respectively, see (2.29). Similar statements for the initial values hold. Finally, Section 2.6 provides some results concerning strong convergence for the case the energy $H_0(z_0(t))$ of the limit functions is the limit $\lim_{\varepsilon\to 0} H_\varepsilon(z_\varepsilon(t_\varepsilon))$ of the energies. In this case it is possible to show that $G_\varepsilon z_0 - z_\varepsilon$ converges strongly to 0 in V a.e. in $\mathbb{R}$.

For semigroups generated from equations of the type $\dot u = A_\varepsilon u$ or $\ddot u + A_\varepsilon u = 0$ similar convergence results are known. There convergence can be expressed in terms of the convergence of the resolvent operators $R_\varepsilon(\lambda, A) = (A_\varepsilon - \lambda I)^{-1}$ for $\varepsilon \to 0$. Our setting $M_\varepsilon \ddot u + A_\varepsilon u = 0$ involves two physically relevant structures depending on ε and convergence is asked in the physically relevant quantities. The transformation $u_\varepsilon = M_\varepsilon^{1/2} \widetilde u_\varepsilon$ would not transfer convergence properties of u_ε to those of v_ε.

Section 3 is devoted to the homogenization of systems of hyperbolic equations as in (1.2). For simplicity we restrict ourselves to the one-dimensional setting but allow for vector-valued $u(t, x) \in \mathbb{R}^m$. The main advantage of the present theory is that it uses very weak convergence notions. Thus, we are able to homogenize equations with general L^∞ coefficients. For general periodicity structures in $\mathbb{R}^d$ we let $Y = [0, 1)^d \sim (\mathbb{R}/_{\mathbb{Z}})^d$ and $[\,\cdot\,] : \mathbb{R}^d \to \mathbb{Z}^d$ for the componentwise Gauß bracket. For a general function $a \in \mathrm{L}^\infty(\mathbb{R}^d \times Y; \mathbb{R}^{m\times m})$ the usual ansatz for the oscillation coefficients would be $\widetilde a_\varepsilon(x) = a(x, \frac{1}{\varepsilon}x)$, but this is not well-defined, as $\{\, (x, \frac{1}{\varepsilon}x) \in \mathbb{R}^d \times Y \mid x \in \mathbb{R}^d \,\}$ is a null set in $\mathbb{R}^d \times Y$. Thus, the usual work assumes additional smoothness for a, cf. [CDMZ91, All03] and the references therein. We avoid this problem by using

$$a_\varepsilon(x) := \int_Y a\big(\varepsilon([\tfrac{1}{\varepsilon}x] + y), \tfrac{1}{\varepsilon}x\big)\, \mathrm{d}y,$$

which is well defined due to the averaging in the first variable.

Using this concept, we show that the solutions u_ε of the oscillatory wave equation

$$\rho_\varepsilon(x)\ddot u_\varepsilon(t,x) = \big(a_\varepsilon(x)u'(t,x)\big)' - \partial_u F_\varepsilon(x,u), \quad u(t,\cdot) \in \mathrm{H}^1_0((0,l);\mathbb{R}^m).$$

have weak limits that solve the effective wave equation

$$\rho_*(x)\ddot u_\varepsilon(t,x) = \big(a^*(x)u'(t,x)\big)' - \partial_u F_*(x,u), \quad u(t,\cdot) \in \mathrm{H}^1_0((0,l);\mathbb{R}^m),$$

where the subscript $_*$ denotes the harmonic mean $a_*(x) = \big(\int_Y a(x,y)^{-1}\,\mathrm{d}y\big)^{-1}$ while the superscript * is the arithmetic mean, e.g., $F^*(x,u) = \int_Y F(x,y,u)\,\mathrm{d}y$. Note that the effective tensors and nonlinearity may have arbitrary jumps for many $x \in (0,l)$. This leads to reflection effects in the wave equation homogenized wave equation and it is not at all clear that these effects are present in the original oscillatory system. The present theory shows that the homogenization works in this case if we use weak convergence in the energy topology.

Note also that the associated potential energy Φ_ε converges to the correct limit energy Φ_0 in the weak H^1 topology. The same holds true for the kinetic energies $\mathcal{K}_\varepsilon(v) = \frac{1}{2}\int_0^l \rho_\varepsilon v\cdot v\,\mathrm{d}y$, when we again use the weak H^1 norm or the strong L^2 norm. However, using weak L^2 convergence, which would be suggested from energetic considerations, would give a Γ-limit defined via the harmonic mean ρ_*, see Proposition 3.1(b).

This shows that the idea of using the Γ-limits cannot be applied naïvely. On the one hand, the joint-recovery condition (1.5) justifies that in the Lagrangian setting the topologies for the energy recovery and for the momentum recovery have to be the same. On the other hand the Hamiltonian approach (see Section 3.3) defines the kinetic energy in terms of the momentum p giving $\widehat{\mathcal{K}}_\varepsilon(p) = \frac{1}{2}\int_0^l \rho_\varepsilon^{-1} p\cdot p\,\mathrm{d}x$. In this setting the symplectic structure enforces that the weak L^2 convergence has to be used for the calculation of the Γ-limit. The correct effective density matrix is obtained as the inverse of the harmonic mean of the inverse, which is of course the arithmetic mean.

In Section 4 we provide another example arising from the discrete system. For such atomic systems there is some literature concerning the passage to the continuum limit, but only in the exactly periodic case: For the general linear setting the derivation of the elastodynamical wave equation was done in [Mie06c], using methods from Fourier transforms and series, which may be generalized to the slowly varying case, but not to cases with jumps in the coefficients, which occur for instance at the phase boundaries in crystals. For some work in the nonlinear setting we refer to [SW00a, DHM06, GM06, GHM06a, GHM06b] and the references therein.

The methods developed here will be useful in much more general contexts. For simplicity we have restricted ourselves to the following model of an atomic chain for $(u_\gamma(t))_{\gamma\in\mathbb{Z}} \in \ell^2(\mathbb{Z};\mathbb{R}^m)$:

$$m_\varepsilon(\varepsilon\gamma)\ddot u_\gamma = a_\varepsilon(\varepsilon\gamma)(u_{\gamma+1}-u_\gamma) + a_\varepsilon(\varepsilon\gamma+\varepsilon)(u_{\gamma-1}-u_\gamma) - \varepsilon^2\mathrm{D}_u\psi_\varepsilon(x,u_\gamma), \ \gamma \in \mathbb{Z}, \tag{1.6}$$

Our main result states that, if we embed the discrete solutions into $\mathrm{H}^1(\mathbb{R};\mathbb{R}^m)$ via $\widehat{u}_\varepsilon = E_\varepsilon((u_\gamma)_\gamma)$ such that $\widehat{u}_\varepsilon$ is piecewise linear with $\widehat{u}_\varepsilon(\varepsilon\gamma) = u_\gamma$, then any accumulation point u_0 of families of solutions solves the macroscopic effective wave equation

$$m^*(x)\tfrac{\partial^2}{\partial\tau^2}u_0(\tau) = \tfrac{\partial}{\partial\tau}\big(a_*(x)\tfrac{\partial}{\partial\tau}u\big) - \mathrm{D}_u\psi^*(x,u), \quad u(\tau,\cdot)\in \mathrm{H}^1(\mathbb{R};\mathbb{R}^m).$$

2 Abstract convergence results

Here we provide a general, abstract framework that allow us to pass to multiscale limits in several applications. The idea is to use the fact that Hamiltonian systems are driven by a function, namely Hamiltonian H_ε, and a symplectic structure Ω_ε. We study the question in what sense H_ε and Ω_ε have to converge to their limits H_0 and Ω_0. Here, we are interested in rather weak convergence notions like Γ-convergence.

2.1 Quadratic forms

The basic objects for the linear theory are quadratic forms $Q : X \to \mathbb{R}_\infty$. We always assume that these forms are homogeneous of degree 2 and uniformly convex. This implies the coercivity

$$\exists\, c>0\ \forall\, u\in X: \quad Q(u)\geq c\|u\|^2.$$

We allow for the value $+\infty$ such that the domain $\mathrm{dom}Q = \{\, u\in X \mid Q(u)<\infty\,\}$ may be a proper subspace of X. Moreover, we do not impose density, i.e.,

$$X_Q = \overline{\mathrm{dom}Q}^X$$

may be a nontrivial closed subspace of X.

Finally, we define a self-adjoint operator $L_Q : \ D(L_Q) \subset X_Q \subset X_Q$ in the usual way. Using the bilinear form $B : \mathrm{dom}Q\times\mathrm{dom}Q \to \mathbb{R}, (u,v)\mapsto \frac14 Q(u{+}v) - \frac14 Q(u{-}v)$ we let

$$\mathcal{D}(L_Q) = \{\, u\in\mathrm{dom}Q \mid \exists\, C>0\ \forall\, v\in\mathrm{dom}Q: \ |B(u,v)|\leq C\|v\|\,\}$$

and define the linear operator L_Q via

$$L_Q u = w \quad\text{if}\quad B(u,v) = \langle w,v\rangle \quad\text{for all } v\in\mathrm{dom}Q.$$

The classical theory of quadratic forms and of selfadjoint operators says that L_Q is self-adjoint if and only if the subspace $\mathrm{dom}Q$ equipped with the energy norm $\|\cdot\|_Q : \ v\mapsto Q(v)^{1/2}$ is complete. Obviously, the latter condition is equivalent to the property that $Q: \ X\to\mathbb{R}_\infty$ is weakly lower semicontinuous. Under these conditions Q takes the form

$$Q: \ X\to\mathbb{R}; \ u\mapsto \begin{cases} \langle L_Q u,u\rangle & \text{for } u\in\mathrm{dom}Q,\\ \infty & \text{otherwise.}\end{cases} \tag{2.1}$$

Now, $\mathrm{dom}Q = \mathcal{D}(L_Q^{1/2})$ and $L_Q \in \mathcal{L}(\mathcal{D}(L_Q^{1/2}); \mathcal{D}(L_Q^{-1/2}))$ and we denote by

$$\mathcal{S}(X) = \{ L : \mathcal{D}(L) \subset X_L \to X_L \mid X_L \subset X \text{ closed}, L \text{ selfadjoint} \}$$

the set of all such operators. The associated quadratic form for $L \in \mathcal{S}(x)$ is then denoted by Q_L and defined as in (2.1).

2.2 Γ-convergence and recovery operators

We consider a Banach space X and denote by $\to$ and $\rightharpoonup$ the strong and weak convergence respectively. The notion of Γ-convergence is adjusted to the convergence of functionals $\Phi_\varepsilon : X \to \mathbb{R}_\infty$ related to the direct method of the calculus of variations, see [Dal93, Bra02]. We say that Φ_ε Γ-converges to Φ_0 for $\varepsilon \to 0$ with respect to the weak topology on X, and shortly write $\Phi_0 = \Gamma\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon$ or $\Phi_\varepsilon \xrightarrow{\Gamma} \Phi_0$, if the following two conditions hold:

$$\begin{array}{ll} \text{(G1)} & \text{Liminf estimate:} \\ & u_\varepsilon \rightharpoonup u \text{ in } X \quad \Longrightarrow \quad \Phi_0(u) \leq \liminf_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon). \\ \text{(G2)} & \text{Recovery sequence:} \\ & \forall\, \widehat{u} \in X\ \exists\, (\widehat{u}_\varepsilon)_{\varepsilon>0} :\ \widehat{u}_\varepsilon \rightharpoonup \widehat{u} \text{ in } X \quad \text{and} \quad \Phi_\varepsilon(\widehat{u}_\varepsilon) \to \Phi_0(\widehat{u}). \end{array} \tag{2.2}$$

We first deal with families of quadratic forms $\Phi_\varepsilon = Q_{A_\varepsilon}$ as defined in (2.1), namely

$$\Phi_\varepsilon(u) = \begin{cases} \frac{1}{2}\langle A_\varepsilon u, u\rangle & \text{for } u \in V_\varepsilon, \\ \infty & \text{for } u \in X \backslash V_\varepsilon. \end{cases} \tag{2.3}$$

Here, V is a Hilbert space with dual V^*, V_ε are closed subspaces, and $A_\varepsilon \in \mathcal{L}(V_\varepsilon, V_\varepsilon^*)$ with $A_\varepsilon^* = A_\varepsilon$ satisfy the uniform coercivity assumption

$$\exists\, c_0 > 0\ \forall \varepsilon \in [0,1]\ \forall\, v \in V :\ \Phi_\varepsilon(v) \geq \frac{c_0}{2}\|v\|_V^2. \tag{2.4}$$

We also introduce the V-orthogonal projectors $P_\varepsilon \in \mathcal{L}(V,V)$ with $P_\varepsilon V = V_\varepsilon$ and their adjoints $P_\varepsilon^* \in \mathcal{L}(V^*, V^*)$ with $P_\varepsilon^* V^* = V_\varepsilon^*$.

For quadratic forms we reformulate Γ-convergence using *families of recovery operators.*

Definition 2.1 *Assume that V is a Hilbert space with dual V^*. Moreover, let $(V_\varepsilon)_{\varepsilon\in[0,1]}$ be a family of closed subspaces of V and assume $K_\varepsilon \in \mathcal{L}(V_\varepsilon, V_\varepsilon^*)$. Then, $(G_\varepsilon)_{\varepsilon\in(0,1]}$ with $G_\varepsilon \in \mathcal{L}(V_0, V)$ is called a* family of recovery operators *for $(K_\varepsilon)_{\varepsilon\in[0,1]}$ if*

(R1) $G_\varepsilon V_0 \subset V_\varepsilon$,

(R2) $\forall\, v_0 \in V_0 :\quad G_\varepsilon v_0 \rightharpoonup v_0 \quad$ *in V,*

(R3) $v_\varepsilon \in V_\varepsilon$ *for* $\varepsilon \in [0,1]$ *and* $v_\varepsilon \rightharpoonup v_0$ *in* $V \quad \Longrightarrow \quad G_\varepsilon^* K_\varepsilon v_\varepsilon \rightharpoonup K_0 v_0$ *in* V_0^*.

The following conditions are either equivalent or sufficient for the recovery property. They will be used in the sequel since they wherever they are easier to handle. However, we refer to Example 2.3 to see that (R3)* is strictly stronger and not appropriate in situations where V_ε is not strongly dense.

Lemma 2.2 *: Let $V, V_\varepsilon, K_\varepsilon$ and G_ε be as in Definition 2.1 except for* (R2) *and* (R3)*. Then we have* (R2) $\Longleftrightarrow$ (R2)* *and* (R3)* $\Longrightarrow$ (R3)*, where* (R2)* *and* (R3)* *are given by*

$$\begin{aligned} &(\mathrm{R2})^* \quad \forall\, \zeta \in V^* : \quad G_\varepsilon^* \zeta \overset{*}{\rightharpoonup} P_0^* \zeta \quad \textit{in } V_0^*, \\ &(\mathrm{R3})^* \quad \forall\, v_0 \in V_0 : \quad K_\varepsilon^* G_\varepsilon v_0 \;\to\; K_0^* v_0 \textit{ in } V^*. \end{aligned}$$

If additionally $V_\varepsilon = V$ for all $\varepsilon \in [0,1]$, then (R3)* $\Longleftrightarrow$ (R3)*,*

Proof: The equivalence between (R2) and (R2)* follows easily since (R2) means that $\langle G_\varepsilon v_0, \zeta\rangle$ converges to $\langle v_0, \zeta\rangle$ for all $v_0 \in V_0$ and all $\zeta \in V^*$. Using $\langle G_\varepsilon v_0, \zeta\rangle = \langle v_0, G_\varepsilon^* \zeta\rangle$ the desired equivalence follows with $\langle v_0, \zeta\rangle = \langle v_0, P_0^* \zeta\rangle$.

Next we show that (R3)* implies (R3). For this take any family $(v_\varepsilon)_{\varepsilon\in[0,1]}$ with $v_\varepsilon \in V_\varepsilon$ and $v_\varepsilon \rightharpoonup v_0$ in V. Then, for arbitrary $w_0 \in V_0$ condition (R3)* gives

$$\langle w_0, G_\varepsilon^* K_\varepsilon v_\varepsilon\rangle = \langle K_\varepsilon^* G_\varepsilon w_0, v_\varepsilon\rangle \to \langle K_0^* w_0, v_0\rangle = \langle w_0, K_0 v_0\rangle,$$

since the first term in the duality pairing converges strongly whereas the second term converges weakly. Thus, (R3) is established.

For the opposite implication (R3)$\Rightarrow$(R3)* we assume $V_\varepsilon = V$ and use a standard result: A family $(\eta_\varepsilon)_{\varepsilon\in[0,1]}$ satisfies $\eta_\varepsilon \to \eta_0$ in V^* if and only if for all $(v_\varepsilon)_{\varepsilon\in[0,1]}$ in V with $v_\varepsilon \rightharpoonup v_0$ we have $\langle \eta_\varepsilon, v_\varepsilon\rangle \to \langle \eta_0, v_0\rangle$, see Lemma A.1 for a proof. ∎

Example 2.3 *We consider $V = V^* = V_0 = \mathrm{L}^2((0,1))$ and for all $\varepsilon \in (0,1]$ and fixed $\alpha \in (0,1)$ we define $X(\varepsilon) = (0,1) \cap \cup_{k=0}^\infty (\varepsilon k, \varepsilon(k{+}\alpha))$ and $V_\varepsilon = \{\, u \in V \,|\, \mathrm{sppt}\, v \subset X(\varepsilon)\,\}$. Finally, we let $\Phi_\varepsilon(u) = \int_0^1 u(x)^2\,\mathrm{d}x$ for $u \in V_\varepsilon$ and ∞ otherwise.*

The Γ-limit reads $\Phi_0(u) = \frac{1}{\alpha}\int_0^1 u(x)^2\,\mathrm{d}x$ and as recovery operators we may choose $G_\varepsilon u = \frac{1}{\alpha}\chi_\varepsilon u$ with $\chi_\varepsilon = \chi_{X(\varepsilon)}$, since $\frac{1}{\alpha}\chi_\varepsilon$ converges weak* to 1. Note that (R3)* cannot hold for any family of recovery operators, since $A_\varepsilon G_\varepsilon v_0 \in V_\varepsilon^*$ and no element in $V_0^*\backslash\{0\}$ is a strong limit of points $\sigma_\varepsilon \in V_\varepsilon^*$.

For a family $(A_\varepsilon)_{\varepsilon\in[0,1]}$ of symmetric operators as above having a family of recovery operators $(_\varepsilon)_{\varepsilon\in(0,1]}$ we may define the symmetric operators $A_0^\varepsilon :\ V_0 \to V_0^*;\ v_0 \mapsto G_\varepsilon^* A_\varepsilon G_\varepsilon v_0$ and the associated quadratic forms $\Phi_\varepsilon^0 : V \to \mathbb{R}_\infty$. Then, for $v_0 \in V_0$ we have

$$\Phi_\varepsilon(G_\varepsilon v_0) = \frac{1}{2}\langle A_\varepsilon G_\varepsilon v_0, G_\varepsilon v_0\rangle = \Phi_\varepsilon^0(v_0) = \frac{1}{2}\langle A_0^\varepsilon v_0, v_0\rangle \to \frac{1}{2}\langle A_0 v_0, v_0\rangle. \tag{2.5}$$

This leads to the first result concerning the sufficiency of recovery operators for the proof of Γ-convergence.

Proposition 2.4 *For $\varepsilon \in [0,1]$ let $V_\varepsilon, A_\varepsilon$ and Φ_ε be given as above and satisfying (2.4). Moreover let $(G_\varepsilon)_{\varepsilon>0}$ be a family of recovery operators as in Definition 2.1. If additionally*

$$v_\varepsilon \rightharpoonup v \text{ and } v \notin V_0 \quad \Longrightarrow \quad \Phi_\varepsilon(v_\varepsilon) \to \infty, \tag{2.6}$$

then we have $\Phi_0 = \Gamma\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon$.

Proof: Because of $\Phi_0(v) = \infty$ for $v \notin V_0$, condition (2.6) shows that (G1) and (G2) in (2.2) hold for all $v \notin V_0$.

It remains to consider $v_0 \in V_0$. Using $v_\varepsilon = G_\varepsilon v_0$ we have a recovery sequence, as $\Phi_\varepsilon(G_\varepsilon v_0) \to \Phi_0(v_0)$, see (2.5). Thus, (G2) is established. For (G1) consider an arbitrary family with $v_\varepsilon \to v_0$ and use the identity

$$\Phi_\varepsilon(v_\varepsilon) = \Phi_\varepsilon(G_\varepsilon v_0 - v_\varepsilon) + \langle G_\varepsilon^* A_\varepsilon v_\varepsilon, v_0\rangle - \Phi_\varepsilon(G_\varepsilon v_0).$$

We have just seen that the last term converges to $\Phi_0(v_0)$. The second last term converges because of (R3), i.e., $G_\varepsilon^* A_\varepsilon v_\varepsilon \to A_0 v_0$, and the limit is $\langle A_0 v_0, v_0\rangle = 2\Phi_0(v_0)$. Since the first term after the equality sign is nonnegative we can take the liminf and obtain (G1). ■

We also want to show that under the assumption that $\Phi_\varepsilon \xrightarrow{\Gamma} \Phi_0$ we always have at least one such recovery operator. Our construction provides a canonical version but we hasten to emphasize that this is not useful for practical purposes, since usually the proof of Γ-convergence has to be done first and therefore recovery sequences are needed to start with. Nevertheless the following result clears the structures and provides further insight.

The construction of the recovery operators $F_\varepsilon : V_0 \to V_\varepsilon$ involves the functionals

$$J_{\varepsilon,v_0} : \ V \to \mathbb{R}_\infty; \ v \mapsto \Phi_\varepsilon(v) - \langle A_0 v_0, v\rangle.$$

Clearly, J_{ε,v_0} is coercive, lower semi-continuous and uniformly convex. Hence, J_{ε,v_0} has a unique minimizer $\widetilde{v}_\varepsilon(v_0)$ in V_ε, and we set

$$F_\varepsilon : \begin{cases} V_0 & \to \quad V_\varepsilon, \\ v_0 & \mapsto \quad \widetilde{v}_\varepsilon(v_0) = \operatorname{argmin} J_{\varepsilon,v_0}. \end{cases} \tag{2.7}$$

Using $0 = \mathrm{D}J_{\varepsilon,v_0}(\widetilde{v}_\varepsilon) = A_\varepsilon v_\varepsilon - P_\varepsilon^* A_0 v_0$ we easily find $F_\varepsilon = A_\varepsilon^{-1} P_\varepsilon^* A_0 \in \mathcal{L}(V_0, V_\varepsilon)$ and

$$\|F_\varepsilon\|_{V_\varepsilon \leftarrow V_0} \le \|A_\varepsilon^{-1}\|_{V_\varepsilon \leftarrow V_\varepsilon^*} \|P_\varepsilon^*\|_{V_\varepsilon^* \leftarrow V^*} \|A_0\|_{V^* \leftarrow V_0} \le \frac{1}{c_0} \|A_0\|_{V^* \leftarrow V_0}. \tag{2.8}$$

Proposition 2.5 *Let $\Phi_\varepsilon, V_\varepsilon, P_\varepsilon$ and A_ε be defined as above such that (2.4) holds. If $\Phi_0 = \Gamma\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon$, then $(F_\varepsilon)_{\varepsilon>0}$ defines a family of recovery operators.*

Proof: To show $v_\varepsilon := F_\varepsilon v_0 \rightharpoonup v_0$ we use that v_ε minimizes J_{ε,v_0}. By (2.8) we know that $\|v_\varepsilon\|_V$ is bounded, hence for a subsequence we have $v_{\varepsilon_k} \rightharpoonup \widetilde{v}$. By $\widehat{v}_\varepsilon$ we denote a recovery sequence for v_0 as postulated by (G2), i.e., $\widehat{v}_\varepsilon \rightharpoonup v_0$ and $\Phi_\varepsilon(\widehat{v}_\varepsilon) \to \Phi_0(v_0) < \infty$. Thus,

$$\begin{aligned} \Phi_0(\widetilde{v}) &\le \liminf_{k\to\infty} \Phi_{\varepsilon_k}(v_{\varepsilon_k}) = \lim_{k\to\infty} \langle A_0 v_0, v_{\varepsilon_k}\rangle + \liminf_{k\to\infty} J_{\varepsilon,v_0}(v_{\varepsilon_k}) \\ &\le \langle A_0 v_0, \widetilde{v}\rangle + \liminf_{k\to\infty} J_{\varepsilon,v_0}(\widehat{v}_{\varepsilon_k}) = \langle A_0 v_0, \widetilde{v}\rangle + \Phi_0(v_0) - \langle A_0 v_0, v_0\rangle. \end{aligned}$$

Rearranging this inequality gives

$$0 \geq \Phi_0(\widetilde{v}) + \Phi_0(v_0) - \langle A_0 v_0, \widetilde{v}\rangle = \tfrac{1}{2}\langle A_0(v_0-\widetilde{v}), v_0-\widetilde{v}\rangle \geq c_0\|v_0-\widetilde{v}\|_V^2.$$

Hence, $\widetilde{v} = v_0$ and thus the only accumulation point of the family $F_\varepsilon v_0$ is v_0 and (R2) is established.

The convergence (R3) follows easily since a small computation shows $F_\varepsilon^* A_\varepsilon = A_0 P_0$. Because of $A_0 P_0$ lies in $\mathcal{L}(V; V_0^*)$ and is independent of ε, the desired weak convergence follows from $v_\varepsilon \rightharpoonup v_0$ due to the weak continuity of bounded linear operators. ■

For $V_\varepsilon = V$ we have the simplification $F_\varepsilon = A_\varepsilon^{-1} A_0$ and we see that Γ-convergence reduces to the weak convergence of the resolvent with respect to the energy norm. The generalization presented here allows us to avoid assumptions that involve a joint upper bound like $\langle A_\varepsilon v, v\rangle \leq C_{\mathrm{upp}}\|v\|_V^2$ and, thus, are more flexible in applications.

Remark 2.6 *Our construction of recovery operators is not restricted to the linear setting. For strictly convex functionals Φ_ε for $\varepsilon > 0$ and for differentiable $\Phi_0 : V_0 \to \mathbb{R}$ the functional J_{ε,v_0} takes the form $J_{\varepsilon,v_0}(v) = \Phi_\varepsilon(v) - \langle \mathrm{D}\Phi_0(v_0), v\rangle$. It is interesting to note that such recovery sequences do not recover the energy level but rather the derivative, namely the minimizer v_ε of J_{ε,v_0} satisfies $\mathrm{D}\Phi_\varepsilon(v_\varepsilon) = P_\varepsilon^* \mathrm{D}\Phi_0(v_0)$. This is quite close to what we need for our nonlinear theory, cf.* (2.19).

2.3 Linear mechanical systems

Since the kinetic and the potential energies in mechanical systems associate with different topologies we use a Gelfand triple $V \subset X \cong X^* \subset V^*$ of Hilbert spaces. We denote by $\langle\cdot,\cdot\rangle$ the scalar product in X as well as the duality product on $V^* \times V$ and distinguish the norms by a subscript. For each $\varepsilon \in [0,1]$ we consider functions K_ε and Φ_ε denoting the kinetic and the potential energies, respectively. In this section we assume that both functionals are quadratic:

$$K_\varepsilon(u) = \tfrac{1}{2}\langle \overline{M}_\varepsilon u, u\rangle \quad \text{and} \quad \Phi_\varepsilon(u) = \begin{cases} \tfrac{1}{2}\langle A_\varepsilon u, u\rangle & \text{for} \quad u \in V_\varepsilon, \\ \infty & \text{otherwise,} \end{cases}$$

where $V_\varepsilon \subset V$ is a closed subspace, $\overline{M}_\varepsilon \in \mathcal{L}(X, X^*)$ with $\overline{M}_\varepsilon^* = \overline{M}_\varepsilon$, and $A_\varepsilon \in \mathcal{L}(V_\varepsilon, V_\varepsilon^*)$ with $A_\varepsilon^* = A_\varepsilon$. We will use the following coercivity assumption:

$$\begin{aligned} &\exists\, c_0 > 0\ \forall\, u \in V: \ \Phi_\varepsilon(u) \geq \tfrac{c_0}{2}\|u\|_V^2, \\ &\exists\, c_1 > 0\ \forall\, v \in X: \ \tfrac{1}{c_1}\|v\|_V^2 \geq \langle \overline{M}_\varepsilon v, v\rangle \geq c_1\|v\|_X^2. \end{aligned} \tag{2.9}$$

We set $X_\varepsilon = \overline{V_\varepsilon}^X$ and define Q_ε as the X-orthogonal projector from X into X_ε. Letting $M_\varepsilon = Q_\varepsilon^* \overline{M}_\varepsilon Q_\varepsilon : X_\varepsilon \to X_\varepsilon^* \cong X_\varepsilon$ we now consider solutions of the associated Hamiltonian system

$$M_\varepsilon \ddot{u} + A_\varepsilon u = 0, \quad u(t) \in V_\varepsilon, \tag{2.10}$$

where we always assume that the energy

$$E_\varepsilon(u,u) = \tfrac{1}{2}\langle M_\varepsilon \dot u, \dot u\rangle + \Phi_\varepsilon(u) \tag{2.11}$$

is finite and constant along solutions. According to (2.9) we consider weak solutions $u_\varepsilon : \mathbb{R} \to V$ of (2.10) with $u_\varepsilon \in \mathrm{C}^0(\mathbb{R}, V_\varepsilon) \cap \mathrm{C}^1(\mathbb{R}, X_\varepsilon) \cap \mathrm{C}^2(\mathbb{R}, V_\varepsilon^*)$ satisfying

$$\left.\begin{aligned} &\textstyle\int_S^T \langle M_\varepsilon u_\varepsilon(t), \ddot\varphi_\varepsilon(t)\rangle + \langle A_\varepsilon u_\varepsilon(t), \varphi_\varepsilon(t)\rangle \,\mathrm{d}t \\ &\quad + \big[\langle M_\varepsilon \dot u_\varepsilon(t), \varphi_\varepsilon(t)\rangle - \langle M_\varepsilon u_\varepsilon(t), \dot\varphi_\varepsilon(t)\rangle\big]_S^T = 0. \end{aligned}\right\} \quad \begin{aligned} &\text{for all } \varphi_\varepsilon \in \mathrm{C}^2(\mathbb{R}, V_\varepsilon) \\ &\text{and } S<T. \end{aligned} \tag{2.12}$$

This notion looks very weak, but using the selfadjointness of M_ε and A_ε it is easy to see that each solution of (2.12) satisfies $u_\varepsilon \in \mathrm{BC}^0(\mathbb{R}, V) \cap \mathrm{BC}^1(\mathbb{R}, X) \cap \mathrm{BC}^2(\mathbb{R}, V^*)$ and that it satisfies energy conservation $E_\varepsilon(u_\varepsilon(t), \dot u_\varepsilon(t)) = \text{const}$.

We now consider a family $(u_\varepsilon)_{\varepsilon>0}$ of solutions such that the energy $e_\varepsilon = E_\varepsilon(u_\varepsilon(t), \dot u_\varepsilon(t))$ is bounded. We are interested in passing to the limit $\varepsilon \to 0$ under weak conditions. The coercivity assumptions (2.9) show that u_ε is bounded in $\mathrm{BC}^0(\mathbb{R}, V) \cap \mathrm{BC}^1(\mathbb{R}, X)$. Since V is continuously embedded into X, we have boundedness of u_ε in $\mathrm{BC}^1(\mathbb{R}, X)$ and we may apply the Arzela–Ascoli theorem in $\mathrm{C}^0([-T,T], X_{\text{weak}})$ to obtain a subsequence $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ with $\varepsilon_k \searrow 0$ and a limit function $u_0 \in \mathrm{BC}^0(\mathbb{R}, X)$, such that

$$\forall\, t \in \mathbb{R}: \quad u_{\varepsilon_k}(t) \rightharpoonup u_0(t) \quad \text{in } V, \qquad \text{and } \dot u_{\varepsilon_k} \overset{*}{\rightharpoonup} \dot u_0 \quad \text{in } \mathrm{L}^\infty(\mathbb{R}, X). \tag{2.13}$$

Note that the boundedness of u_ε in $\mathrm{BC}^0(\mathbb{R}, V)$ implies that the pointwise weak convergence in X can be improved to weak convergence in V. The weak* convergence of $\dot u_{\varepsilon_k}$ follows by the Banach–Alaoglu theorem as $\mathrm{L}^\infty(\mathbb{R}, X)$ is the dual of the separable space $\mathrm{L}^1(\mathbb{R}, X)$.

The following result provides a first sufficient condition such that u_0 obtained in (2.13) solves (2.10) for $\varepsilon = 0$.

Theorem 2.7 *For $\varepsilon \in [0,1]$ let $V_\varepsilon, M_\varepsilon, A_\varepsilon$ be given as above. Assume $\Phi_0 = \Gamma\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon$ and that $(F_\varepsilon)_{\varepsilon>0}$ as defined in (2.7) is a family of recovery operators satisfying*

$$v_\varepsilon \in V_\varepsilon \text{ for } \varepsilon \in [0,1] \text{ and } v_\varepsilon \rightharpoonup v_0 \quad \Longrightarrow \quad F_\varepsilon^* M_\varepsilon v_\varepsilon \rightharpoonup M_0 v_0 \quad \text{in } V_0^*. \tag{2.14}$$

Now let $(u_\varepsilon)_{\varepsilon>0}$ be a family of solutions of (2.12) with bounded energy and u_0 any limit as postulated in (2.13).
(a) Then, u_0 lies in $\mathrm{BC}^0(\mathbb{R}, V_0) \cap \mathrm{BC}^1(\mathbb{R}, X_0) \cap \mathrm{BC}^2(\mathbb{R}, V_0^*)$ and satisfies (2.12) for $\varepsilon = 0$. Moreover, $F_{\varepsilon_k}^* M_{\varepsilon_k} \dot u_{\varepsilon_k}(t) \rightharpoonup M_0 \dot u_0(t)$ for all $t \in \mathbb{R}$.
(b) If in addition to (a) we have that $(F_\varepsilon^* M_\varepsilon u_\varepsilon(t), F_\varepsilon^* M_\varepsilon \dot u_\varepsilon(t)) \rightharpoonup (M_0 u_0(t), M_0 \dot u_0(t))$ in $V_0^* \times V_0^*$ holds for one $t \in \mathbb{R}$, then it holds for all other $t \in \mathbb{R}$ as well.
(c) Under the additional upper bound

$$\exists\, C_{\text{upp}} > 0\ \forall\, \varepsilon \in [0,1]: \ \|M_\varepsilon^{-1}\|_{V_\varepsilon^* \to V_\varepsilon^*} + \|A_\varepsilon\|_{V_\varepsilon \to V_\varepsilon^*} \le C_{\text{upp}} \tag{2.15}$$

the additional convergence $(u_\varepsilon(t), \dot u_\varepsilon(t)) \rightharpoonup (u_0(t), \dot u_0(t))$ in $V \times X$ for some $t \in \mathbb{R}$ implies the same convergence for all other $t \in \mathbb{R}$ as well.

Example 2.8 *Here we show that the assertion in Part (b) cannot be improved without further condition as in Part (c). Let $X = V_\varepsilon = \mathbb{R}^2$ with $M_\varepsilon = I$ and $A_\varepsilon = \operatorname{diag}(1, 1/\varepsilon)$, for $\varepsilon > 0$. Then, for $\varepsilon = 0$ we obtain $V_0 = \operatorname{span}\left\{\binom{1}{0}\right\}$ and $\Phi_0\big(\binom{q_1}{q_2}\big) = \frac{1}{2}q_1^2$ if $q_2 = 0$ and $+\infty$ else. For $\varepsilon > 0$ we have the solutions $u_\varepsilon(t) = \binom{a\sin(t+\alpha_\varepsilon)}{\varepsilon b\sin(t/\varepsilon)}$, which have the bounded energy $e_\varepsilon = E_\varepsilon(u_\varepsilon, \dot u_\varepsilon) = \frac{1}{2}(a^2 + b^2)$. We have $u_\varepsilon(t) \to u_0(t) = \binom{a\sin(t+\alpha_0)}{0}$ uniformly in $t \in \mathbb{R}$. Moreover, $\dot u_\varepsilon(t) = \binom{a\cos(t+\alpha_\varepsilon)}{b\cos(t/\varepsilon)}$ satisfies $\dot u_\varepsilon \overset{*}{\rightharpoonup} \dot u_0$. Note that we have $\dot u_\varepsilon(0) \to \binom{a\cos\alpha_0}{b}$ but for $t \neq 0$ the second component of $\dot u_\varepsilon(t)$ does not converge. As $F_\varepsilon : V_0 \to V_\varepsilon$ takes the form $F_\varepsilon\binom{\alpha}{0} = \binom{\alpha}{0}$ we find $F_\varepsilon^* M_\varepsilon\binom{\alpha}{\beta} = \binom{\alpha}{0} \in V_0^*$. Thus, we are able to confirm statement (b), as the convergence of the first component of $u_\varepsilon(t)$ and $\dot u_\varepsilon(t)$ for some t implies the convergence for all over t as well.*

Proof: First, note that the limit function u_0 from (2.13) must lie in V_0, as $\Phi_0(u_0(t)) \leq \liminf_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon(t))$ by (G1). However, $\Phi_\varepsilon(u_\varepsilon(t)) \leq E_\varepsilon(u_\varepsilon, \dot u_\varepsilon) \leq E_*$.

Part (a) follows by inserting $\varphi_\varepsilon(t) = F_\varepsilon\varphi_0(t)$ into (2.12) for $\varepsilon > 0$. Here, $\varphi_0 \in \mathrm{C}^2([0,T], V_0)$ is arbitrary. Pushing F_ε to the other side in the duality pairing we can use (R3) to obtain $\langle F_\varepsilon^* A_\varepsilon u_\varepsilon(t), \varphi_0(t)\rangle \to \langle A_0 u_0(t), \varphi_0(t)\rangle$ for all $t \in \mathbb{R}$. Similarly we have $\langle M_\varepsilon u_\varepsilon(t), F_\varepsilon\ddot\varphi_0(t)\rangle = \langle F_\varepsilon^* M_\varepsilon u_\varepsilon(t), \ddot\varphi_0(t)\rangle \to \langle M_0 u_0(t), \ddot\varphi_0(t)\rangle$ for all $t \in \mathbb{R}$. Thus, we obtain (2.12) for $S < T$, and $\varphi_0 \in \mathrm{C}^2_\mathrm{c}((S,T), V_0)$. From this and from $M_0, M_0^{-1} \in \mathcal{L}(X_0, X_0)$ and $A_0 \in \mathcal{L}(V_0, V_0^*)$ it follows that u_0 satisfies $u_0 \in \mathrm{BC}^0(\mathbb{R}, V_0) \cap \mathrm{BC}^1(\mathbb{R}, X) \cap \mathrm{BC}^2(\mathbb{R}, V_0^*)$, i.e., (2.10) holds pointwise for u_0 as an equation in V_0^*. Then, it follows again that (2.12) holds including boundary terms.

To show the pointwise weak convergence of $F_\varepsilon^* M_\varepsilon \dot u_\varepsilon(t)$ towards $M_0\dot u_0(t)$ in V_0^* we choose a function $\rho \in \mathrm{C}^2(\mathbb{R})$ with $\rho(0) = 1$ and $\rho(-1) = 0 = \dot\rho(0) = \dot\rho(-1)$. For any $q_0 \in V_0$ we let $\varphi_\varepsilon(t) = \rho(t{-}T)F_\varepsilon q_0$ and $S = T - 1$ in (2.12) to obtain

$$\begin{aligned}\langle F_\varepsilon^* M_\varepsilon \dot u_\varepsilon(T), q_0\rangle &= \langle M_\varepsilon \dot u_\varepsilon(T), \varphi_\varepsilon(T)\rangle \\ &= \textstyle\int_{T-1}^T \langle F_\varepsilon^* M_\varepsilon u_\varepsilon(t), q_0\rangle \ddot\rho(t{-}T) + \langle A_0 P_0 u_\varepsilon(t), q_0\rangle \rho(t{-}T)\,\mathrm{d}t.\end{aligned}$$

The uniform weak convergence of u_ε allows us to pass to the limit in the right-hand side. Thus, the limit $\mu(t) = \lim_{\varepsilon\to 0}\langle F_\varepsilon^* M_\varepsilon \dot u_\varepsilon(t), q_0\rangle$ exists for all $t \in \mathbb{R}$ and we have

$$\mu(t) = \textstyle\int_{T-1}^T \langle M_0 u_0(t), q_0\rangle \ddot\rho(t{-}T) + \langle A_0 u_0(t), q_0\rangle \rho(t{-}T)\,\mathrm{d}t.$$

However, as u_0 solves (2.12) for $\varepsilon = 0$ we may test with $\varphi_0(t) = \rho(t{-}T)q_0$ to find that $\mu(t) = \langle M_0 \dot u_0(t), q_0\rangle$. Thus, $F_\varepsilon^* M_\varepsilon \dot u_\varepsilon(t) \rightharpoonup M_0\dot u_0(t)$ in V_0^* is established.

To prove Part (b) we simply use the fact that u_0 is uniquely specified if $(u_0(t_*), \dot u_0(t_*)) \in V_0 \times X_0$ is prescribed. Thus, if $u_\varepsilon(t_*) \rightharpoonup \widetilde u_0$ in V and $F_\varepsilon^* M_\varepsilon \dot u_\varepsilon(t_*) \rightharpoonup M_0 \widetilde v_0$ holds, then any limit u_0 of a subsequence in the sense of (2.13) satisfies, by Part (a), the initial condition $u_0(t_*) = \widetilde u_0$ and $\dot u_0(t_*) = \widetilde v_0$. Thus, the whole sequence converges in the sense of (2.13) and Part (a) yields $F_\varepsilon M_\varepsilon \dot u_\varepsilon(t) \rightharpoonup M_0 \dot u_0(t)$ for all $t \in \mathbb{R}$.

In Part (c) we have a uniform upper bound on all operators A_ε and M_ε^{-1}. Hence, from $\ddot u_\varepsilon = -M_\varepsilon^{-1} A_\varepsilon u_\varepsilon$ we obtain a uniform bound for u_ε in $\mathrm{BC}^2(\mathbb{R}, V^*)$. Thus, the Arzela–Ascoli theorem is also applicable to $\dot u_\varepsilon \in \mathrm{C}^{\mathrm{Lip}}(\mathbb{R}, V^*)$. Together with the pointwise bound of $(\dot u_\varepsilon(t))_{\varepsilon\in[0,1]}$ in X we obtain pointwise weak convergence in X. Arguing as in Part (b) by using uniqueness of the limit solution, we obtain the desired result. ■

Example 2.9 *We consider the finite dimensional example with $X = V = V_\varepsilon = \mathbb{R}^2$ with*

$$M_\varepsilon \ddot u + A_\varepsilon u = 0 \quad \text{with } M_\varepsilon = \left(\begin{smallmatrix}1 & 0\\ 0 & \varepsilon^{-\alpha}\end{smallmatrix}\right) \text{ and } \left(\begin{smallmatrix}2 & -1/\varepsilon\\ -1/\varepsilon & 1/\varepsilon^2\end{smallmatrix}\right), \tag{2.16}$$

where $\alpha > 0$ is a fixed parameter. We have $V_0 = \mathrm{span}\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$, $\Phi_\varepsilon \xrightarrow{\Gamma} \Phi_0$, and $\mathcal{K}_\varepsilon \xrightarrow{\Gamma} \mathcal{K}_0$ with

$$\Phi_0 = \mathcal{K}_0 : \mathbb{R}^2 \to \mathbb{R}_\infty;\ \left(\begin{smallmatrix}u^{(1)}\\ u^{(2)}\end{smallmatrix}\right) \mapsto \begin{cases} \frac12 (u^{(1)})^2 & \text{for } u^{(2)}=0,\\ \infty & \text{otherwise.}\end{cases}$$

Thus, the limit problem reads $M_0\ddot u + A_0 u = 0$ with $M_0 = A_0 = I$ on V_0. The solutions of the limit problem are $u(t) = a\cos(t+\alpha)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ for $a, \alpha \in \mathbb{R}$.

The exact solutions of (2.16) for $\varepsilon > 0$ can be written in the form

$$u_\varepsilon(t) = a_1 \cos(\omega_1(\varepsilon)t + \beta_1)\varphi_1(\varepsilon) + a_2\cos(\omega_2(\varepsilon)t+\beta_2)\varphi_2(\varepsilon),$$

where the eigenfunctions $\varphi_j(\varepsilon) \in \mathbb{R}^2$ and the eigenfrequencies $\omega_j(\varepsilon) > 0$ satisfy

$$(A_\varepsilon - \omega_j^2(\varepsilon)M_\varepsilon)\varphi_j(\varepsilon) = 0, \quad \langle M_\varepsilon\varphi_j(\varepsilon), \varphi_k(\varepsilon)\rangle = \delta_{jk}.$$

For $\alpha \in (0,2)$ we find $\omega_1(\varepsilon) = 1 + O(\varepsilon^{2-\alpha}), \varphi_1(\varepsilon) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + O(\varepsilon^{2-\alpha}), \omega_2(\varepsilon) = 1/\varepsilon^{2-\alpha} + O(1)$, and $|\varphi_2(\varepsilon)| \leq 1$. Hence, any convergent subsequence of solutions with bounded energies $E_\varepsilon(u_\varepsilon, \dot u_\varepsilon) = \frac{|a|^2}{2}\omega_1(\varepsilon)^2 + \frac{|a|^2}{2}\omega_2(\varepsilon)^2$ converges to a solution of the limit problem.

For $\alpha = 2$ we find $\varphi_j(\varepsilon) \to \left(\begin{smallmatrix}\rho_j\\0\end{smallmatrix}\right)$ and $\omega_j^2(\varepsilon) = (3\pm\sqrt5)/2$. For $\alpha > 2$ we find $\varphi_1(\varepsilon) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + O(\varepsilon^{\alpha-2}), \omega_1(\varepsilon) = \sqrt2 + O(\varepsilon^{\alpha-2})$, and $\omega_2(\varepsilon) = \varepsilon^{\alpha/2-1}/\sqrt2 + \text{h.o.t.}$ Hence, for $\alpha \geq 2$ the limits of subsequences of energy-bounded solutions u_ε have the form

$$u_0(t) = (a_1\cos(\omega_1^* t + \beta_1) + a_2\cos(\omega_2^* t + \beta_2))\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),$$

*where $\omega_{1,2}^* = ((3\pm\sqrt5)/2)^{1/2}$ for $\alpha = 2$ and $(\omega_1^*, \omega_2^*) = (\sqrt2, 0)$ for $\alpha > 2$. These functions certainly do **not** satisfy the limit problem.*

We now check in what regime for α our sufficient conditions hold. Note that the recovery operator $F_\varepsilon :\ V_0 \to V_\varepsilon = \mathbb{R}^2$ constructed in (2.7) for $(A_\varepsilon)_{\varepsilon\in[0,1]}$ depends only on A_ε and is, thus, independent of α. We have $F_\varepsilon = A_\varepsilon^{-1}P_\varepsilon^0 A_0 :\ \left(\begin{smallmatrix}\delta\\0\end{smallmatrix}\right) \mapsto \delta\left(\begin{smallmatrix}1\\ \varepsilon\end{smallmatrix}\right)$ and $F_\varepsilon^* = \left(\begin{smallmatrix}1 & \varepsilon\\ 0 & 0\end{smallmatrix}\right)$. The condition (2.14) reads $M_\varepsilon F_\varepsilon\left(\begin{smallmatrix}\delta\\0\end{smallmatrix}\right) = \delta\left(\begin{smallmatrix}1\\ \varepsilon^{1-\alpha}\end{smallmatrix}\right) \to \left(\begin{smallmatrix}\delta\\0\end{smallmatrix}\right)$ and holds only for $\alpha \in (0,1)$. In the next section we will weaken the condition (cf. (2.19)) to

$$u_\varepsilon \rightharpoonup u \ \text{ and } \ \langle A_\varepsilon u_\varepsilon, u_\varepsilon\rangle \leq C \quad \Longrightarrow \quad F_\varepsilon^* M_\varepsilon u_\varepsilon \to M_0 u_0.$$

This condition holds for all $\alpha \in (0,2)$, since $F_\varepsilon^* M_\varepsilon = \left(\begin{smallmatrix}1 & \varepsilon^{1-\alpha}\\ 0 & 0\end{smallmatrix}\right)$ and $\langle A_\varepsilon u_\varepsilon, u_\varepsilon\rangle \leq C$ implies $|\langle u_\varepsilon, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\rangle| \leq \widetilde C\varepsilon$.

2.4 Nonlinear mechanical systems

We now generalize the above theory to the nonlinear setting. The new conditions are even more general for the linear case. We treat abstract systems of the form

$$M_\varepsilon \ddot{u}_\varepsilon + \mathrm{D}\Phi_\varepsilon(u_\varepsilon) = 0, \quad u_\varepsilon \in V_\varepsilon, \tag{2.17}$$

where now $\Phi_\varepsilon : V \to \mathbb{R}_\infty$ is such that $\Phi_\varepsilon(u) = +\infty$ for $u \notin V_\varepsilon$ and $\Phi_\varepsilon|_{V_\varepsilon} \in \mathrm{C}^1(V_\varepsilon;\mathbb{R})$. Moreover, we assume the coercivity

$$\begin{aligned} &\Phi_\varepsilon(u) \to +\infty \quad \text{for } \|u\|_V \to \infty \quad \text{and} \\ &\exists\, c_0 > 0\ \forall\, \varepsilon \in [0,1]\ \forall\, u \in X :\ \langle M_\varepsilon u, u\rangle \geq c_0 \|u\|_X^2. \end{aligned} \tag{2.18}$$

The main observation about the theory in Section 2.1 is that the specific choice of F_ε for the recovery operator is not necessary. All what we use for proving Theorem 2.7 can be put into the following condition:

$$\begin{aligned} &\forall\, \varepsilon \in (0,1]\ \exists\, G_\varepsilon \in \mathcal{L}(V_0;V_\varepsilon) : \\ &\text{if } u_\varepsilon \rightharpoonup u_0 \text{ in } V \text{ and } \sup_{\varepsilon\in[0,1]} \Phi_\varepsilon(u_\varepsilon) < \infty, \text{ then} \\ &\qquad \text{(i)} \quad G_\varepsilon^* \mathrm{D}\Phi_\varepsilon(u_\varepsilon) \rightharpoonup \mathrm{D}\Phi_0(u_0) \text{ in } V_0^*, \\ &\qquad \text{(ii)} \quad G_\varepsilon^* M_\varepsilon u_\varepsilon \rightharpoonup M_0 u_0 \text{ in } V_0^*. \end{aligned} \tag{2.19}$$

Even for linear systems this condition is weaker than the classical recovery condition, since we only need to consider sequences that have bounded energies (cf. also Example 2.9). Note that we do not impose that Φ_0 is the Γ-limit of the family $(\Phi_\varepsilon)_{\varepsilon>0}$ for $\varepsilon \to 0$. Condition (2.19)(i) asks that the derivatives are "recovered" correctly, cf. also Remark 2.6. However, having a weakly convergent sequence u_ε inside the nonlinear term $\mathrm{D}\Phi_\varepsilon(\cdot)$ roughly means that we are restricted to semilinear cases.

A function $u_\varepsilon \in \mathrm{L}^\infty((t_1,t_2);V_\varepsilon) \cap \mathrm{W}^{1,\infty}((t_1,t_2);X)$ is called a *weak solution* of (2.17) if for all $\varphi \in \mathrm{C}^2_\mathrm{c}((t_1,t_t);V)$ we have

$$\int_{t_1}^{t_2} \langle M_\varepsilon u_\varepsilon(t), \ddot{\varphi}(t)\rangle + \langle \mathrm{D}\Phi_\varepsilon(u_\varepsilon(t)), \varphi(t)\rangle \,\mathrm{d}t = 0. \tag{2.20}$$

We additionally impose in this abstract setting that for all $\varepsilon \in [0,1]$

$$\begin{aligned} &\text{all weak solutions } u_\varepsilon \text{ of (2.17) satisfy} \\ &u_\varepsilon \in \mathrm{C}^0((t_1,t_2);V_\varepsilon) \cap \mathrm{C}^1((t_1,t_2);X), &\text{(2.21a)} \\ &E_\varepsilon(u_\varepsilon(t),\dot{u}_\varepsilon(t)) = \tfrac{1}{2}\langle M_\varepsilon \dot{u}_\varepsilon(t), \dot{u}_\varepsilon(t)\rangle + \Phi_\varepsilon(u_\varepsilon(t)) = \text{const.} &\text{(2.21b)} \end{aligned}$$

For a family $(u_\varepsilon)_{\varepsilon>0}$ of weak solutions of (2.17) on a common interval (t_1,t_2) that have bounded energies $\sup_{\varepsilon>0} e_\varepsilon(t) < \infty$ the coercivity assumption (2.18) provides a priori bounds for u_ε in $\mathrm{C}^0((t_1,t_2);V_\varepsilon) \cap \mathrm{C}^1((t_1,t_2);X)$. Thus, as in the previous section, we are able to extract a subsequence $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ and a limit function $u \in \mathrm{L}^\infty((t_1,t_2);V_\varepsilon) \cap \mathrm{W}^{1,\infty}((t_1,t_2);X)$ such that

$$\text{(i)}\ \ \forall\, t \in (t_1,t_2) :\ u_{\varepsilon_k}(t) \rightharpoonup u(t) \text{ in } V, \qquad \text{(ii)}\ \ \dot{u}_{\varepsilon_k} \overset{*}{\rightharpoonup} \dot{u} \quad \text{in } \mathrm{L}^\infty((t_1,t_2);X). \tag{2.22}$$

The following result provides sufficient conditions that guarantee that any such limit provides a weak solution of (2.17) for $\varepsilon = 0$.

Theorem 2.10 *Let $X, V, V_\varepsilon, M_\varepsilon$ and Φ_ε be such that (2.18), (2.19), and (2.21) hold. Then, any limit u as obtained in (2.22) satisfies (2.17) for $\varepsilon = 0$. Moreover, for all $t \in (t_1, t_2)$ we additionally have $G^*_{\varepsilon_k} M_{\varepsilon_k} \dot{u}_{\varepsilon_k}(t) \rightharpoonup M_0 \dot{u}_0(t)$ in V_0^* for $k \to \infty$.*

If furthermore the limit problem has the property that for each $(w_0, v_0) \in V_0 \times X_0$ and each $t_* \in (t_1, t_2)$ there exists at most one weak solution u_0 with $(u_0(t_*), \dot{u}_0(t_*)) = (w_0, v_0)$, then the convergence $(G^*_\varepsilon M_\varepsilon u_\varepsilon(t), G^*_\varepsilon M_\varepsilon \dot{u}_\varepsilon(t)) \rightharpoonup (u_0(t), \dot{u}_0(t))$ in $V_0^* \times V_0^*$ for one t implies the same convergence for all other $t \in (t_1, t_2)$.

Proof: The proof is essentially the same as for the linear case. Start from the weak solutions $(u_\varepsilon)_{\varepsilon\in(0,1]}$ we test with $\varphi = G_\varepsilon \varphi_0(t)$. Our a priori bounds allow us to apply the recovery conditions (2.19). Thus, we can pass to the limit and obtain that u_0 is a weak solution. Applying the regularity assumption we have $u_0 \in \mathrm{C}^0((t_1, t_2); V_\varepsilon) \cap \mathrm{C}^1((t_1, t_2); X)$. Thus, for all $\varepsilon \in [0, 1]$ we may integrate by parts in (2.20) and obtain

$$\forall\, \varphi_0 \in \mathrm{C}^2_\mathrm{c}((t_1, t_2); V_0): \quad \int_{t_1}^{t_2} \langle G^*_\varepsilon \mathrm{D}\Phi_\varepsilon(u_\varepsilon(t)), \varphi_0(t)\rangle - \langle G^*_\varepsilon M_\varepsilon \dot{u}_\varepsilon(t), \dot{\varphi}_0(t)\rangle \,\mathrm{d}t = 0.$$

Now consider S and T with $t_1 < S < T < t_2$ and let $\chi = \chi_{[S,T]}$ be the characteristic function. Choose a sequence $(\chi_k)_{k\in N}$ with $\chi_k \in \mathrm{C}^2_\mathrm{c}((t_1, t_2))$ and $\chi_k' \overset{*}{\rightharpoonup} \delta_S - \delta_T$ in the sense of Radon measures (the dual of $\mathrm{C}^0([t_1, t_2])$). Replacing φ_0 in the above identity by $\chi_k \varphi_0$ we may pass to the limit and obtain, for all $\varphi_0 \in \mathrm{C}^2_\mathrm{c}((t_1, t_2); V_0)$,

$$\int_S^T \langle G^*_\varepsilon \mathrm{D}\Phi_\varepsilon(u_\varepsilon(t)), \varphi_0(t)\rangle - \langle G^*_\varepsilon M_\varepsilon \dot{u}_\varepsilon(t), \dot{\varphi}_0(t)\rangle \,\mathrm{d}t + \langle G^*_\varepsilon M_\varepsilon \dot{u}_\varepsilon(t), \varphi_0(t)\rangle\Big|_S^T = 0.$$

Now we may undo the integration by parts again and see that weak solutions even satisfy the weak form on subintervals including the boundary terms as given in (2.11).

Based on (2.11), the arguments about the convergence of $G^*_{\varepsilon_k} M_{\varepsilon_k} \dot{u}_{\varepsilon_k}(t)$ and the convergence of $(G_\varepsilon M_\varepsilon u_\varepsilon(t), G_\varepsilon M_\varepsilon \dot{u}_\varepsilon(t))$ works as in the proof of Theorem 2.7. ■

2.5 Hamiltonian systems

Here, we consider general Hamiltonian system. We will mainly restrict to the linear case and address the nonlinear case only shortly at the end of this subsection. We consider a Hilbert space Z, closed subspaces Z_ε and Hamiltonians $H_\varepsilon : Z \to \mathbb{R}_\infty$ with $H_\varepsilon|_{V_\varepsilon} \in \mathrm{C}^1(V_\varepsilon; \mathbb{R})$ and $H_\varepsilon = \infty$ on $V \backslash V_\varepsilon$. The linear case is given by symmetric linear operators $L_\varepsilon \in \mathcal{L}(Z_\varepsilon, Z^*_\varepsilon)$ defining the Hamiltonians

$$H_\varepsilon(z) = \begin{cases} \frac{1}{2}\langle L_\varepsilon z, z\rangle & \text{for } z \in Z_\varepsilon, \\ \infty & \text{otherwise.} \end{cases} \tag{2.23}$$

As above, we assume uniform coercivity:

$$\exists\, c>0\ \forall\, \varepsilon\in[0,1]\ \forall\, z\in Z:\ \ H_\varepsilon(z)\geq c\|z\|_Z^2. \tag{2.24}$$

To define the Hamiltonian flow via a differential equation we have to specify symplectic structures $\Omega_\varepsilon\in\mathcal{L}(Z_\varepsilon,Z_\varepsilon^*)$, i.e., Ω_ε is skew symmetric ($\Omega_\varepsilon^*=-\Omega_\varepsilon$) and nondegenerate:

$$\text{If } \langle\Omega_\varepsilon^* z_\varepsilon, v_\varepsilon\rangle=0 \text{ for all } z_\varepsilon\in Z_\varepsilon,\ \text{ then } v_\varepsilon=0. \tag{2.25}$$

The Hamiltonian system now takes the strong form

$$\Omega_\varepsilon\dot z_\varepsilon=\mathrm{D}H_\varepsilon(z_\varepsilon),\quad z_\varepsilon\in Z_\varepsilon. \tag{2.26}$$

Again we define the notion of weak solutions $z_\varepsilon\in\mathrm{L}^\infty((t_1,t_2);Z_\varepsilon)$ by test functions:

$$\forall\,\varphi_\varepsilon\in\mathrm{C}^1_\mathrm{c}((t_1,t_2);Z_\varepsilon):\quad \textstyle\int_{t_1}^{t_2}\langle\Omega_\varepsilon z_\varepsilon(t),\dot\varphi(t)\rangle+\langle\mathrm{D}H_\varepsilon(z_\varepsilon(t)),\varphi_\varepsilon(t)\rangle\,\mathrm{d}t=0. \tag{2.27}$$

As in the case of mechanical systems we assume that every weak solution is slightly smoother and conserves energy:

$$\begin{array}{l}\text{All weak solutions } z_\varepsilon \text{ of (2.26) satisfy}\\ z_\varepsilon\in\mathrm{C}^0((t_1,t_2);Z_\varepsilon)\quad\text{and}\quad H_\varepsilon(z_\varepsilon(t))=\text{const.}\end{array} \tag{2.28}$$

The above linear mechanical systems can be put into this Hamiltonian form by introducing $p=N_\varepsilon^{-1}\dot u_\varepsilon$ and setting $Z=V\times X$, $H_\varepsilon(u,p)=\frac12\langle A_\varepsilon u,u\rangle_V+\frac12\langle N_\varepsilon^*M_\varepsilon N_\varepsilon p,p\rangle$ and $\Omega_\varepsilon=\left(\begin{smallmatrix}0 & -M_\varepsilon N_\varepsilon\\ N_\varepsilon^*M_\varepsilon & 0\end{smallmatrix}\right)$. In the case $N_\varepsilon=M_\varepsilon$ we obtain the canonical setting while $N_\varepsilon=I$ gives the Lagrangian setting. In general, the weak-convergence properties of these two systems might be different.

The crucial assumption to obtain the desired convergence result is again the existence of a family of joint recovery operators, i.e.,

$$\begin{array}{l}\forall\,\varepsilon\in(0,1]\ \exists\, G_\varepsilon\in\mathcal{L}(Z_0;Z_\varepsilon):\\ \text{if } z_\varepsilon\rightharpoonup z_0 \text{ in } Z\ \text{ and } \sup_{\varepsilon\in[0,1]}H_\varepsilon(z_\varepsilon)<\infty,\ \text{then}\\ \qquad\text{(i)}\ \ G_\varepsilon^*\Omega_\varepsilon z_\varepsilon\rightharpoonup\Omega_0 z_0 \text{ in } Z_0^*,\\ \qquad\text{(ii)}\ \ G_\varepsilon^*\mathrm{D}H_\varepsilon(z_\varepsilon)\rightharpoonup\mathrm{D}H_0(z_0) \text{ in } Z_0^*.\end{array} \tag{2.29}$$

Thus, if we have a sequence $(z_\varepsilon)_{\varepsilon\in(0,1]}$ of solutions of (2.26) with bounded energy this sequence is bounded in $\mathrm{L}^\infty((t_1,t_2);Z)$. Thus, we may extract a subsequence that converges weak* to a limit function, namely

$$z_{\varepsilon_k}\overset{*}{\rightharpoonup} z\quad\text{in}\quad \mathrm{L}^\infty((t_1,t_2);Z). \tag{2.30}$$

Note that this convergence is equivalent to the weak convergence

$$\textstyle\int_{\tau_1}^{\tau_2} z_{\varepsilon_k}(s)\,\mathrm{d}s\rightharpoonup\int_{\tau_1}^{\tau_2} z(s)\,\mathrm{d}s\quad\text{in } Z \text{ for all } \tau_1,\tau_2 \text{ with } t_1\leq\tau_1<\tau_2\leq t_2. \tag{2.31}$$

However, weak* convergence is not compatible with nonlinearities occurring in $\mathrm{D}H_\varepsilon$. To exploit (2.29)(ii) we would need weak convergence pointwise in t. How this can be obtained we discuss at the end of this section. At present we restrict to the linear case, where weak* convergence is sufficient.

Theorem 2.11 *Let $Z, Z_\varepsilon, L_\varepsilon$, and Ω_ε be as above and assume that H_ε is given through* (2.23) *such that* (2.28) *holds. Moreover, let the joint recovery condition* (2.29) *be satisfied. Then, every limit z_0 obtained as in* (2.30) *from a sequence of the weak solutions z_ε of* (2.26) *is a solution of* (2.26) *for $\varepsilon \to 0$.*

Moreover, if $G_\varepsilon^* \Omega_\varepsilon z_\varepsilon(t) \rightharpoonup \Omega_0 z_0(t)$ for some $t \in \mathbb{R}$, then this convergence holds for all $t \in \mathbb{R}$ without extracting a subsequence.

Proof: First, by using the linearity $\mathrm{D}H_\varepsilon(z_\varepsilon) = L_\varepsilon z_\varepsilon$ and the characterization (2.31) for weak* convergence, the recovery conditions (2.29) yield

$$G_\varepsilon^* \Omega_\varepsilon z_\varepsilon \overset{*}{\rightharpoonup} \Omega_0 z \quad \text{and} \quad G_\varepsilon^* L_\varepsilon z_\varepsilon \overset{*}{\rightharpoonup} L_0 z \text{ in } \mathrm{L}^\infty(\mathbb{R}; Z_0^*).$$

Second, we use the weak form of (2.27) for the solutions z_ε and test it with $\varphi_\varepsilon(t) = G_\varepsilon \varphi_0(t)$ for $\varphi \in \mathrm{C}^1(\mathbb{R}, Z_0)$. Pushing G_ε to the other side we can pass to the limit and find that z_0 is again a weak solution.

As in the proof of Theorem 2.10 we may now restrict the weak form to intervals $[S, T] \subset (t_1, t_2)$ giving

$$0 = -\langle G_\varepsilon^* \Omega_\varepsilon z, \varphi_0 \rangle \Big|_S^T + \int_S^T \langle G_\varepsilon^* \Omega_\varepsilon z_\varepsilon(t), \dot\varphi_0(t) \rangle + \langle G_\varepsilon^* L_\varepsilon z_\varepsilon(t), \varphi_0(t) \rangle \,\mathrm{d}t. \tag{2.32}$$

From this the results concerning the convergence of $G_\varepsilon^* \Omega_\varepsilon z_\varepsilon(t)$ follows as above. We use here that the linear limit problem $\Omega_0 \dot z_0 = L_0 z_0$ has at most one solution for a given value $w = \Omega_0 z_0(t_*)$, see the following lemma. ■

In the following result we include the case that Ω_0 has a nontrivial kernel. Hence, $z_0(0)$ will not be uniquely determined through $\eta_0 = \Omega_0 z_0$.

Lemma 2.12 *Let $\Omega_0, L_0 \in \mathcal{L}(Z_0, Z_0^*)$ with $\Omega_0 = -\Omega_0^*, L_0 = L_0^*$, and $\langle L_0 z, z \rangle \geq c\|z\|_X^2$. Then, $\Omega_0 \dot z_0 = L_0 z_0$ has at most one solution for a given value $\eta_0 = \Omega_0 z_0(0)$.*

Proof: By linearity it suffices to show that $\eta = 0$ implies $z \equiv 0$. We use (2.32) for $\varepsilon = 0$ with $\varphi_0(t) = \psi$ for $t \in [0, t_*]$ and obtain $\langle \Omega_0 z(t_*) - \Omega_0 z(0) - L_0 \int_0^{t_*} z(s)\,\mathrm{d}s, \psi \rangle = 0$ for all ψ. Using $\Omega_0 z(0) = 0$ and letting $w(t) = \int_0^t z(s)\,\mathrm{d}s$ we find $w \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\mathbb{R}, Z_0)$ and $\Omega_0 \dot w = L_0 w$. From $\frac{\mathrm{d}}{\mathrm{d}t} H_0(w) = \langle L_0 w, \dot w \rangle = \langle \Omega_0 \dot w, \dot w \rangle = 0$ we conclude $H_0(w(t)) = H_0(w(0)) = H_0(0) = 0$ for all t. This implies $w \equiv 0$ and, hence, $z = \dot w \equiv 0$, which is the desired result. ■

Example 2.13 *Consider the case $Z = Z = \mathbb{R}^4$ with $\Omega_\varepsilon = \left(\begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix}\right)$, where $I_2 \in \mathbb{R}^{2\times 2}$. The Hamiltonians are given via $L_\varepsilon = \mathrm{diag}(1, 1, 1/\varepsilon^2, 1)$. We find $Z_0 = \mathrm{span}\{e_1, e_2, e_4\} \subset \mathbb{R}^4$ and $L_0 = \mathrm{id}_{Z_0}$. As recovery operators we may take the constant family $G_\varepsilon : Z_0 \to \mathbb{R}^4$, $z_0 \mapsto z_0$ which is the simple embedding. The above results are applicable and using the coordinates $z_0 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_4$ we find the limit problem*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \dot\alpha = \alpha,$$

that has the solution $\alpha(t) = (0, b\cos(t+\beta), b\sin(t+\beta))^\mathsf{T}$.

Note that the original problem has the solutions

$$z_\varepsilon(t) = (c_\varepsilon \cos(\gamma_\varepsilon + t/\varepsilon), b_\varepsilon \cos(t+\beta_\varepsilon), \varepsilon c_\varepsilon \sin(\gamma_\varepsilon + t/\varepsilon), b_\varepsilon \sin(t+\beta_\varepsilon))^\mathsf{T},$$

with energy $H_\varepsilon(z_\varepsilon(t)) \equiv \frac{1}{2}(c_\varepsilon^2 + b_\varepsilon^2)$. *Boundedness of energy implies boundedness of* b_ε *and* c_ε. *Hence, we may assume convergence of* $(b_\varepsilon, c_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon)$ *to* (b, c, β, γ), *by passing to a suitable subsequence. Then, we obtain uniform convergence of the second, third, and fourth component of* z_ε. *However, the first component converges to* 0 *only weak* in* $\mathrm{L}^\infty(\mathbb{R})$. *Note that* $G_\varepsilon^* \Omega_\varepsilon z_\varepsilon(t)$ *also converges in* Z_0^*, *since* $\Omega_\varepsilon = \left(\begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix}\right)$ *moves the first component into the third one, and* $G_\varepsilon^* = \mathrm{diag}(1,1,0,1)$ *projects out the third component.*

We finally address the question how nonlinear problems can be treated in the Hamiltonian setting. To improve the weak* convergence into a weak pointwise convergence we need some control over the temporal behavior. One natural way of doing this is to impose a bound on the inverses of Ω_ε. For this we assume that Z is continuously embedded into a bigger space Y such that we have

$$\exists\, C_\Omega > 0 \;\forall\, \varepsilon \in [0,1]: \quad \|\Omega_\varepsilon^1\|_{Y \leftarrow Z_\varepsilon} \leq C_\Omega.$$

For the energy we impose the existence of a continuous and nondecreasing function $R_\mathrm{upp} : \mathbb{R} \to [0, \infty)$, such that

$$\forall\, \varepsilon \in [0,1] \;\forall\, z \in Z_\varepsilon : \; \|\mathrm{D}H_\varepsilon(z)\|_{Z^*} \leq R_\mathrm{upp}(H_\varepsilon(z)).$$

Now an energetic bound $H_\varepsilon(z_\varepsilon(\cdot)) \leq E_*$ provides the bound $\|\mathrm{D}H_\varepsilon(z_\varepsilon(\cdot))\|_{\mathrm{L}^\infty(\mathbb{R};Z_\varepsilon^*)} \leq R_* = R_\mathrm{upp}(E_*)$ and moreover $\|\dot{z}_\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R};Y)} \leq C_\Omega R_*$. Thus, Arzela-Ascoli can be applied in $\mathrm{C}^*([t_1, t_2], Y_\mathrm{weak})$ and the boundedness on Z then provides pointwise weak convergence in Z as well.

2.6 Strong convergence

In general, we should not expect strong convergence of u_ε to u_0, since this is usually incompatible with Γ-convergence (except in the case of Mosco convergence, where condition (G2) in (2.2) is strengthened by asking $\widehat{u}_\varepsilon \to \widehat{u}$). However, weak convergence as well as convergence of the energy implies a stronger convergence involving the recovery operators.

Lemma 2.14 *Let* $(K_\varepsilon)_{\varepsilon \in [0,1]}$ *be a family of operators in* $\mathcal{S}(V)$ *with* $Q_{K_\varepsilon}(v) \geq c\|v\|^2$ *for* $c > 0$ *and all* $v \in V$, *and let* $(G_\varepsilon)_{\varepsilon \in (0,1]}$ *be recovery operators, then we have the implication*

$$\left.\begin{array}{c} \langle K_\varepsilon u_\varepsilon, u_\varepsilon \rangle \to \langle K_0 u_0, u_0 \rangle \\ u_\varepsilon \rightharpoonup u_0 \end{array}\right\} \implies \|G_\varepsilon u_0 - u_\varepsilon\|_V \to 0.$$

Proof: We use the uniform coercivity and find

$$\begin{aligned} c\|G_\varepsilon u_0 - u_\varepsilon\|^2 &\leq \langle K_\varepsilon(G_\varepsilon u_0 - u_\varepsilon), G_\varepsilon u_0 - u_\varepsilon\rangle \\ &= \langle K_\varepsilon G_\varepsilon u_0, G_\varepsilon u_0\rangle - 2\langle K_\varepsilon G_\varepsilon u_0, u_\varepsilon\rangle + \langle K_\varepsilon u_\varepsilon, u_\varepsilon\rangle \\ &\to \langle K_0 u_0, u_0\rangle - 2\langle K_0 u_0, u_0\rangle + \langle K_0 u_0, u_0\rangle = 0, \end{aligned}$$

where we used $K_\varepsilon G_\varepsilon u_0 \to K_0 u_0$ together with $G_\varepsilon u_0 \rightharpoonup u_0$ and $u_\varepsilon \rightharpoonup u_0$. As $c > 0$ is independent of ε, the proof is finished. ■

We now state a strong convergence result for linear Hamiltonian systems. A corresponding result is valid for linear mechanical systems. If in addition to the weak or weak* convergence of the solutions z_ε we also have the convergence of the energies to the energy of the limiting solution, then the convergence statement can be improved considerably.

Theorem 2.15 *Let $Z, Z_\varepsilon, L_\varepsilon, \Omega_\varepsilon$ be as in the previous section and assume that $H_\varepsilon = Q_{L_\varepsilon}$. Moreover, assume that a family $(G_\varepsilon)_{\varepsilon>0}$ of joint recovery operators as in (2.29) exists. Let $z_\varepsilon : \mathbb{R} \to Z$, $\varepsilon \in [0,1]$, be weak solutions of the Hamiltonian system (2.26) such that $z_\varepsilon \overset{*}{\rightharpoonup} z$ in $\mathrm{L}^\infty(\mathbb{R}, Z)$ and $H_\varepsilon(z_\varepsilon(t_0)) \to H_0(z(t_0))$ for some $t_0 \in \mathbb{R}$ (and hence all $t \in \mathbb{R}$). Then, for a.a. $t \in \mathbb{R}$ we have*

$$z_\varepsilon(t) \rightharpoonup z(t) \quad \textit{and} \quad \|G_\varepsilon z(t) - z_\varepsilon(t)\|_Z \to 0.$$

Proof: We use Lemma 2.14 and the energy conservation $H_\varepsilon(z_\varepsilon(t_0)) = H_\varepsilon(z_\varepsilon(t))$ for all $t \in \mathbb{R}$ and $\varepsilon \in [0,1]$. However, to apply Lemma 2.14 we need to show $z_\varepsilon(t_0) \rightharpoonup z(t_0)$. For this, we use $z_\varepsilon \overset{*}{\rightharpoonup} z$ and $G_\varepsilon z \overset{*}{\rightharpoonup} z$ in $\mathrm{L}^\infty(\mathbb{R}, Z)$. Moreover, we have

$$\begin{aligned} c\|G_\varepsilon z(t) - z_\varepsilon(t)\|^2 &\leq \langle L_\varepsilon(G_\varepsilon z(t) - z_\varepsilon(t)), G_\varepsilon z(t) - z_\varepsilon(t)\rangle \\ &= \langle L_\varepsilon G_\varepsilon z(t), G_\varepsilon z(t) - 2z_\varepsilon(t)\rangle + 2H_\varepsilon(z_\varepsilon(t)). \end{aligned}$$

Using $H_\varepsilon(z_\varepsilon(t)) = H_\varepsilon(z_\varepsilon(t_0)) \to H_0(z(t_0))$ and $L_\varepsilon G_\varepsilon z(t) \to L_0 z(t)$ for all $t \in \mathbb{R}$ we find after integration over $[t_1, t_2]$ that

$$\begin{aligned} & c\int_{t_1}^{t_2} \|G_\varepsilon z(t) - z_\varepsilon(t)\|^2 \,\mathrm{d}t \leq \int_{t_1}^{t_2} \langle L_\varepsilon G_\varepsilon z(t), G_\varepsilon z(t) - 2z_\varepsilon(t)\rangle \,\mathrm{d}t + 2(t_2 - t_1)H_\varepsilon(z_\varepsilon(t_0)) \\ & \to \int_{t_1}^{t_2} \langle L_0 z(t), z(t) - 2z(t)\rangle \,\mathrm{d}t + 2(t_2 - t_1)H_0(z(t_0)) \\ & \quad - \int_{t_1}^{t_2} 2H_0(z(t)) \,\mathrm{d}t + 2(t_2 - t_1)H_0(z(t_0)) = 0 \end{aligned}$$

This implies that, choosing a subsequence, we have $G_\varepsilon z(t) - z_\varepsilon(t) \to 0$ a.e. in $\mathbb{R}$. Using $G_\varepsilon z(t) \rightharpoonup z(t)$ this implies $z_\varepsilon(t) \rightharpoonup z(t)$ a.e. in $\mathbb{R}$. Since the limit $z_\varepsilon \overset{*}{\rightharpoonup} z$ is unique, the result holds without choosing a subsequence. ■

3 Applications to wave equations

3.1 Homogenization and Γ-convergence

We consider the situation of fast oscillating coefficients in functionals. In principle the result seems to be well known, however, mostly the assumptions on the coefficients are more restrictive. We consider an open domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary and set $Y = (\mathbb{R}/_{\mathbb{Z}})^d$ for the unit torus of dimension d. We assume

$$a \in \mathrm{L}^\infty(\Omega \times Y; \mathbb{R}^{m\times m}_{\mathrm{sym}}) \;\text{ and }\; \exists\, \alpha > 0 \;\forall \xi \in \mathbb{R}^m: \; a(x,y)\xi \cdot \xi \geq \alpha|\xi|^2 \text{ a.e. in } \Omega \times Y. \tag{3.1}$$

The coefficient functions a_ε are then defined via

$$a_\varepsilon(x) = ⨍_{w\in C_\varepsilon(x)} a(w, \tfrac{1}{\varepsilon}x)\,\mathrm{d}w \quad \text{where } C_\varepsilon(x) = \varepsilon([\tfrac{1}{\varepsilon}x] + [0,1)^d). \tag{3.2}$$

Here, $⨍$ means the average, $[\cdot]$ denotes the componentwise application of the Gauß bracket, and $\frac{1}{\varepsilon}x$ as second argument of a is understood modulo 1 in each component.

Proposition 3.1 *For a and a_ε satisfying* (3.1) *and* (3.2) *we define*

$$a_*(x) = \big(\textstyle\int_Y a(x,y)^{-1}\,\mathrm{d}y\big)^{-1} \quad \text{and} \quad a^*(x) = \textstyle\int_Y a(x,y)\,\mathrm{d}y$$

as well as the following functionals on $\mathrm{L}^2(\Omega;\mathbb{R}^m)$:

$$\Phi_\varepsilon(u) = \textstyle\int_\Omega a_\varepsilon(x)u(x)\cdot u(x)\,\mathrm{d}x,$$
$$\Phi_*(u) = \textstyle\int_\Omega a_*(x)u(x)\cdot u(x)\,\mathrm{d}x, \quad \Phi^*(u) = \textstyle\int_\Omega a^*(x)u(x)\cdot u(x)\,\mathrm{d}x$$

Then the following holds true:

(a) If $u_\varepsilon \to u$ (strongly) in $\mathrm{L}^2(\Omega)$, then $\Phi_\varepsilon(u_\varepsilon) \to \Phi^*(u)$.

(b) In the weak topology of $\mathrm{L}^2(\Omega)$ we have $\Phi_\varepsilon \xrightarrow{\Gamma} \Phi_*$. A family of recovery operators is given by $G_\varepsilon : u \mapsto (a_\varepsilon)^{-1}a_*u$.

(c) Define $\Psi_\varepsilon : \mathrm{H}^1_0((0,l);\mathbb{R}^m) \to \mathbb{R}; v \mapsto \Phi_\varepsilon(v')$ and $\Psi_*(v) = \Phi_*(v')$, then $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_*$ in the weak topology of $\mathrm{H}^1_0((0,l);\mathbb{R}^m)$. A family of recovery operators is given by

$$\widehat{G}_\varepsilon : \mathrm{H}^1_0((0,l);\mathbb{R}^m) \to \mathrm{H}^1_0((0,l);\mathbb{R}^m);$$
$$(\widehat{G}_\varepsilon u)(x) = \textstyle\int_0^x (a_\varepsilon(y))^{-1}a(y)u'(y)\,\mathrm{d}y - \frac{x}{l}\int_0^l (a_\varepsilon(y))^{-1}a(y)u'(y)\,\mathrm{d}y.$$

Proof: Note that the functionals Φ_ε, Φ_*, and Φ^* are uniformly coercive and bounded, i.e., there exists $C > 0$ such that for all $\varepsilon > 0$ and all $u \in \mathrm{L}^2(\Omega;\mathbb{R}^m)$ we have $\frac{1}{C}\|u\|_2^2 \leq \Phi_\varepsilon(u) \leq \|u\|_2^2$. This implies uniform continuity:

$$\forall\, \varepsilon > 0 \;\forall\, u,v \in \mathrm{L}^2(\Omega;\mathbb{R}^m): \quad |\Phi_\varepsilon(u) - \Phi_\varepsilon(v)| \leq C\big(\|u\|_2 + \|v\|_2\big)\|u-v\|_2. \tag{3.3}$$

ad (a). Using (3.3) it is sufficient to show the statement for constant sequences $u_\varepsilon = u$. Moreover, it is sufficient to show the result for a dense subset like $\mathrm{C}^\infty_\mathrm{c}(\Omega;\mathbb{R}^m)$. Set $N_\varepsilon = \{\, n\in\mathbb{Z}^d \mid \varepsilon(n+[0,1)^d)\subset\Omega\,\}$ and $y^\varepsilon_n = \varepsilon(n+(\frac12,\dots,\frac12))$ such that $\varepsilon(n+[0,1)^d) = C_\varepsilon(y^\varepsilon_n)$. With this define $\Omega_\varepsilon = \cup_{n\in N_\varepsilon} C_\varepsilon(y^\varepsilon_n)$, then $\Omega_\varepsilon\subset\Omega$ and $\mathrm{vol}(\Omega\backslash\Omega_\varepsilon)\leq C\varepsilon$, since Ω is bounded and has a Lipschitz boundary. We have

$$\big|\Phi_\varepsilon(u) - \textstyle\int_{\Omega_\varepsilon} a_\varepsilon(x)u(x)\cdot u(x)\,\mathrm{d}x\big| \leq \mathrm{vol}\big(\Omega\setminus\Omega_\varepsilon\big)\|a_\varepsilon\|_\infty\|u\|^2_\infty \leq C\varepsilon.$$

The same result holds, when a_ε is replaced by a^*. Hence, it suffices to estimate the integrals over Ω_ε. For this define the piecewise constant approximation

$$a^*_\varepsilon(x) = ⨍_{w\in C_\varepsilon(x)} a^*(w)\,\mathrm{d}x \quad \text{if } C_\varepsilon(x)\subset\Omega_\varepsilon.$$

The classical result for the density of Lebesgue points of a^* shows that $a^*_\varepsilon(x)\to a^*(x)$ a.e. in Ω. Hence, we have $\Phi^*_\varepsilon(u)\to\Phi^*(u)$, where $\Phi^*_\varepsilon(u) = \int_\Omega a^*_\varepsilon u\cdot u\,\mathrm{d}x$. The remaining difference is estimated as follows

$$\begin{aligned}&\big|\textstyle\int_{\Omega_\varepsilon}\big[a_\varepsilon(x){-}a^*(x)\big]u(x)\cdot u(x)\,\mathrm{d}x\big| \leq \sum_{n\in N_\varepsilon}\Big|\int_{C_\varepsilon(y^\varepsilon_n)} a_\varepsilon(x){-}a^*_\varepsilon(y^\varepsilon_n)\big]u(x)\cdot u(x)\,\mathrm{d}x\Big|\\ &\leq \sum_{n\in N_\varepsilon}\Big|\int_{C_\varepsilon(y^\varepsilon_n)}\big[a_\varepsilon(x){-}a^*_\varepsilon(y^\varepsilon_n)\big]u(y^\varepsilon_n)\cdot u(y^\varepsilon_n)\,\mathrm{d}x\Big| + \mathrm{vol}(C_\varepsilon(y^\varepsilon_n))2\|a\|_\infty\|u\|_\infty\varepsilon\sqrt{d}\|\nabla u\|_\infty\end{aligned}$$

Using $\int_{C_\varepsilon(y^\varepsilon_n)} a_\varepsilon(x)\,\mathrm{d}x = \int_{C_\varepsilon(y^\varepsilon_n)\times Y} a(w,y)\,\mathrm{d}w\,\mathrm{d}y = \mathrm{vol}(C_\varepsilon(y^\varepsilon_n))a^*_\varepsilon(y^\varepsilon_n)$ the first term vanishes and then $\sum_{n\in N_\varepsilon}\mathrm{vol}(C_\varepsilon(y^\varepsilon_n)) = \mathrm{vol}(\Omega_\varepsilon)$ gives the desired convergence result.

ad (b). We first argue as in the proof of part (a) to show that for all u in $\mathrm{L}^2(\Omega)$ we have $G_\varepsilon u = (a_\varepsilon)^{-1}a_*u \rightharpoonup u$ for $\varepsilon\to0$. It suffices to consider smooth u and v with $a_*u, v\in\mathrm{C}^\infty_\mathrm{c}(\Omega;\mathbb{R}^m)$ and to show $\langle G_\varepsilon u,v\rangle\to\langle u,v\rangle$. As above, consider the average of $(a_\varepsilon)^{-1}$ over $C_\varepsilon(y^\varepsilon_n)$, namely

$$b_\varepsilon(x) = ⨍_{C_\varepsilon(x)} a_\varepsilon(z)^{-1}\,\mathrm{d}z = ⨍_{C_\varepsilon(x)}\big(⨍_{C_\varepsilon(z)} a(w,\tfrac{1}{\varepsilon}z)\,\mathrm{d}w\big)^{-1}\mathrm{d}z = \textstyle\int_Y\big(⨍_{C_\varepsilon(x)} a(w,y)\,\mathrm{d}w\big)^{-1}\mathrm{d}y.$$

Since a is measurable and bounded from above and below, we can use the density of the Lebesgue points and the continuity of the inversion to conclude that $b_\varepsilon(x)\to\int_Y a(x,y)^{-1}\,\mathrm{d}y = a_*(x)^{-1}$ for a.e. $x\in\Omega$. This proves $G_\varepsilon u\rightharpoonup u$. Moreover, choosing $v=u$ we have

$$\Phi_\varepsilon(G_\varepsilon u) = \langle a_\varepsilon G_\varepsilon u, G_\varepsilon u\rangle = \langle a_*u,(a_\varepsilon)^{-1}a_*u\rangle \to \langle a_*u,u\rangle = \Phi_*(u).$$

It remains to show the liminf estimate. For this, we use the identity

$$\Phi_\varepsilon(u_\varepsilon) = \Phi_\varepsilon(u_\varepsilon{-}G_\varepsilon u_0) + 2\langle a_\varepsilon G_\varepsilon u_0, u_\varepsilon\rangle - \Phi_\varepsilon(G_\varepsilon u_0).$$

Now $u_\varepsilon\rightharpoonup u_0$ implies that the two last terms converge to $2\langle a_*u_0,u_0\rangle - \Phi_*(u_0) = \Phi_0(u_0)$. Since the first term on the right-hand side is non-negative, the desired estimate follows.

ad (c). The result follows by applying part (b) to the derivative of the functions in $\mathrm{H}^1((0,l);\mathbb{R}^m)$. In particular, note that

$$(\widehat{G}_\varepsilon u)'(x) = a_\varepsilon(x)^{-1}a_*(x)u'(x) - ⨍_0^l a_\varepsilon(y)^{-1}a_*(y)u'(y)\,\mathrm{d}y = (G_\varepsilon u')(x) - ⨍_0^l G_\varepsilon u'(y)\,\mathrm{d}y.$$

Using $\fint_0^l u'(y)\,\mathrm{d}y = u(l) - u(0) = 0$ we easily find $(\widehat{G}_\varepsilon u)' \rightharpoonup u'$ in $\mathrm{L}^2((0,l);\mathbb{R}^m)$. Together with the boundary conditions this implies $\widehat{G}_\varepsilon u \rightharpoonup u$ in $\mathrm{H}^1((0,l);\mathbb{R}^m)$.

The convergence $\Psi_\varepsilon(\widehat{G}_\varepsilon u_0) \to \Psi_*(u_0)$ is now a direct consequence of Part (b). The liminf estimate follows exactly as in (b). Thus, $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_*$ is established. ∎

3.2 Lagrangian wave equation

In this section we show how the abstract results of Section 2.4 apply to semilinear wave equations with oscillatory coefficients. The emphasis here is on the fact that we are able to allow for general coefficients of L^∞ type. The same holds true for the nonlinearity of lower order. For simplicity we only treat the one-dimensional case, since only for this case we have available the Γ-convergence result for the derivative in Proposition 3.1(c). We expect that the analogous result also holds in higher dimensions when the nonlinearity has sufficiently slow growth.

By $Y = \mathbb{S}^1 = \mathbb{R}/_{\mathbb{Z}}$ we denote the microscopic periodicity interval and by $\Lambda = (0,l)$ the macroscopic physical domain. Consider density and stiffness matrices

$$\begin{aligned} &\rho, a \in \mathrm{L}^\infty(\Lambda \times Y; \mathbb{R}^{m\times m}_{\mathrm{sym}}) \text{ such that,} \\ &\exists\, \alpha, r > 0\ \forall \xi \in \mathbb{R}^m\ \forall (x,y) \in \Lambda \times Y:\ a(x,y)\xi\cdot\xi \geq \alpha|\xi|^2, \quad \rho(x,y)\xi\cdot\xi \geq r|\xi|^2. \end{aligned} \tag{3.4}$$

Moreover, consider a potential $F : \Lambda \times Y \times \mathbb{R}^m \to \mathbb{R}$ such that

$$F \in \mathrm{L}^\infty(\Lambda \times Y; \mathrm{C}^1_{\mathrm{loc}}(\mathbb{R}^m)), \quad F(x,y,u) \geq 0. \tag{3.5}$$

For $\varepsilon > 0$ we let $C_\varepsilon(x) = (\varepsilon[\frac{x}{\varepsilon}], \varepsilon[\frac{x}{\varepsilon}]{+}\varepsilon) \cap \Lambda$, define the oscillatory functions

$$\rho_\varepsilon(x) = \fint_{C_\varepsilon(x)} \rho(w, \frac{x}{\varepsilon})\,\mathrm{d}w, \quad a_\varepsilon(x) = \fint_{C_\varepsilon(x)} a(w, \frac{x}{\varepsilon})\,\mathrm{d}w, \quad F_\varepsilon(x,u) = \fint_{C_\varepsilon(x)} F(w, \frac{x}{\varepsilon}, u)\,\mathrm{d}w,$$

and consider the hyperbolic systems

$$\rho_\varepsilon(x) u_{tt}(t,x) = \frac{\partial}{\partial x}\Big(a_\varepsilon(x) u_x(t,x)\Big) - \mathrm{D}_u F_\varepsilon(x, u(t,x)). \tag{3.6}$$

Our aim is to show that the solutions of this problem converge to solutions of the homogenized problem

$$\rho^*(x) u_{tt}(t,x) = \frac{\partial}{\partial x}\Big(a_*(x) u_x(t,x)\Big) - \mathrm{D}_u F^*(x, u(t,x)), \tag{3.7}$$

where the effective quantities are given by

$$\rho^*(x) = \int_Y \rho(x,y)\,\mathrm{d}y, \quad a^*(x) = \Big(\int_Y a(x,y)^{-1}\,\mathrm{d}y\Big)^{-1}, \quad F^*(x,u) = \int_Y F(x,y,u)\,\mathrm{d}y. \tag{3.8}$$

The following result will be a direct application of the abstract results in Section 2.4. As Hilbert spaces we choose $V = V_\varepsilon = \mathrm{H}^1_0(\Lambda;\mathbb{R}^m)$ and $X = \mathrm{L}^2(\Lambda;\mathbb{R}^m)$. The total energy potential $\Phi_\varepsilon : V \to \mathbb{R}$ and the kinetic energy $\mathcal{K}_\varepsilon$ read

$$\Phi_\varepsilon(v) = \int_\Lambda \tfrac{1}{2}a_\varepsilon(x)u'(x)\cdot u'(x) + F_\varepsilon(x,u(x))\,\mathrm{d}x \quad\text{and}\quad \mathcal{K}_\varepsilon(v) = \int_\Lambda \tfrac{1}{2}\rho_\varepsilon(x)v(x)\cdot v(x)\,\mathrm{d}x.$$

Theorem 3.2 *Take any family $(u_\varepsilon)_{\varepsilon>0}$ of weak solutions $u_\varepsilon \in \mathrm{C}^0(\mathbb{R};V_\varepsilon)\cap \mathrm{C}^1(\mathbb{R};X)$ of* (3.6) *which is uniformly bounded in energy. Assume that for a subsequence we have*

$$\forall\, t\in\mathbb{R}:\; u_{\varepsilon_k}(t) \rightharpoonup u(t) \quad \text{and} \quad \dot u_{\varepsilon_k} \overset{*}{\rightharpoonup} \dot u \quad \text{in } \mathrm{L}^\infty(\mathbb{R};X).$$

Then, u is a solution of the homogenized problem (3.7).

Moreover, if for some time t we have additionally $(u_\varepsilon(t),\dot u_\varepsilon(t)) \rightharpoonup (u(t),\dot u(t))$ in $V^*\times V^*$, then this convergence holds true for all $t\in\mathbb{R}$.

Remark 3.3 *We emphasize that the Γ-limit of the Lagrangian energy functional $E_\varepsilon = \Phi_\varepsilon + \mathcal{K}_\varepsilon$ in the weak topology of $V\times X$ (which is the natural topology) is* **not** *the limit energy. This is only true if we use the weak topology in $V\times V$, i.e, strong convergence of the velocities in $\mathrm{L}^2(\Lambda;\mathbb{R}^m)$.*

Proof: It is easy to see that $\Phi_\varepsilon \in \mathrm{C}^1(V,\mathbb{R})$ with $\mathrm{D}\Phi_\varepsilon(u) = -\frac{\partial}{\partial x}(a_\varepsilon u') + \mathrm{D}_u F_\varepsilon(\cdot,u)$ and that (2.18) is satisfied. In particular, we note that V is compactly embedded into $\mathrm{C}^0(\overline{\Lambda};\mathbb{R}^m)$ and, hence, into X.

The limiting space V_0 equals V and the limiting quantities are defined via ρ^*, a_* and F^* in a similar manner. For the recovery operator $G_\varepsilon : V\to V$ we choose $\widehat G_\varepsilon$ as defined in Proposition 3.1(c). It remains to verify condition (2.19). The condition (ii) there means

$$u_\varepsilon \rightharpoonup u_0 \text{ in } V = \mathrm{H}^1(\Lambda;\mathbb{R}^m) \quad\Longrightarrow\quad \widehat G^*_\varepsilon \rho_\varepsilon u_\varepsilon \rightharpoonup \rho^* u_0 \text{ in } V^* = \mathrm{H}^{-1}(\Lambda;\mathbb{R}^m). \tag{3.9}$$

To verify this, note that we have $u_\varepsilon \to u_0$ in X and as in the proof of Proposition 3.1 we conclude $\rho_\varepsilon u_\varepsilon \to \rho^* u_0$ in X (arithmetic mean). Applying $\langle\cdot,v\rangle$ to $\widehat G^*_\varepsilon\rho_\varepsilon u_\varepsilon$, using duality as well as $\widehat G_\varepsilon v \rightharpoonup v$ in V, the desired result follows.

For condition (2.19)(i) we decompose

$$\langle \widehat G^*_\varepsilon \mathrm{D}\Phi_\varepsilon(u_\varepsilon),v\rangle = \int_\Lambda -(a_\varepsilon u_\varepsilon')'\,\widehat G_\varepsilon v\,\mathrm{d}x + \int_\Lambda \mathrm{D}_u F_\varepsilon(x,u_\varepsilon(x))\widehat G_\varepsilon v(x)\,\mathrm{d}x.$$

The first term converges to $\langle a_* u_0' v'\rangle$ by Proposition 3.1. For the second term we again use the compact embedding of V into $\mathrm{C}^0(\overline{\Lambda};\mathbb{R}^m)$ giving $u_\varepsilon \to u_0$ and $\widehat G_\varepsilon v \to v$ uniformly in $\overline{\Lambda}$. Thus, we conclude $\int_\Lambda \mathrm{D}_u F_\varepsilon(x,u_\varepsilon(x))\widehat G_\varepsilon v(x)\,\mathrm{d}x \to \int_\Lambda \mathrm{D}_u F^*(x,u_0(x))v(x)\,\mathrm{d}x$, where again the oscillations of F_ε in x are simply averaged out. ∎

3.3 Hamiltonian wave equation

For the Hamiltonian case we restrict to the linear case by assuming $F \equiv 0$. For a general matrix-valued function $b(x,y) \in \mathbb{R}^{m\times m}$ with $b, b^{-1} \in \mathrm{L}^\infty(\Lambda \times Y; \mathbb{R}^{m\times m})$ we define b_ε as in (3.2). With this we introduce the velocity variable v and the Hamiltonian H_ε via

$$u_t = b_\varepsilon v \quad \text{and} \quad H_\varepsilon(u,v) = \frac{1}{2}\int_\Lambda b_\varepsilon^\mathsf{T} \rho_\varepsilon b_\varepsilon v \cdot v + a_\varepsilon u' \cdot u' \,\mathrm{d}x.$$

We keep b_ε general at this moment to be able to explore all possibilities that are compatible with our method.

The underlying space is $Z = V \times X = \mathrm{H}^1_0(\Lambda; \mathbb{R}^m) \times \mathrm{L}^2(\Lambda; \mathbb{R}^m)$ and the corresponding symplectic structure reads $\Omega_\varepsilon = \left(\begin{smallmatrix} 0 & -\rho_\varepsilon b_\varepsilon \\ b_\varepsilon^\mathsf{T} \rho_\varepsilon & 0 \end{smallmatrix}\right)$ and is to be considered as a mapping from Z into $Z^* = \mathrm{H}^{-1}(\Lambda; \mathbb{R}^m) \times \mathrm{L}^2(\Lambda; \mathbb{R}^m)$.

For the recovery operator $G_\varepsilon : Z \to Z$ we may assume the diagonal form $G_\varepsilon = \left(\begin{smallmatrix} \widehat{G}_\varepsilon & 0 \\ 0 & \widetilde{G}_\varepsilon \end{smallmatrix}\right)$ with $\widehat{G}_\varepsilon$ from Proposition 3.1(c). The second component $\widetilde{G}_\varepsilon$ has to be chosen such that the joint recovery conditions (2.29) hold. Letting $A_\varepsilon : Z \to Z^*; \binom{u}{v} \mapsto \binom{-(a_\varepsilon u')'}{b_\varepsilon^\mathsf{T}\rho_\varepsilon b_\varepsilon v}$ and using Lemma 2.2 this is equivalent to showing $\Omega_\varepsilon G_\varepsilon \binom{u}{v} \to \Omega_0 \binom{u}{v}$ and $A_\varepsilon G_\varepsilon \binom{u}{v} \to A_0 \binom{u}{v}$ for all $\binom{u}{v} \in Z$. Since A_0 and Ω_0 must have the form $A_0\binom{u}{v} = \binom{-(a_* u')'}{rv}$ and $\Omega_0 \binom{u}{v} = \binom{-\mu v}{\mu^\mathsf{T} u}$, we have to satisfy

$$\forall\, v \in \mathrm{L}^2(\Lambda; \mathbb{R}^m): \ b_\varepsilon^\mathsf{T} \rho_\varepsilon b_\varepsilon \widetilde{G}_\varepsilon v \to rv \text{ and } \widetilde{G}_\varepsilon v \rightharpoonup v \text{ in } \mathrm{L}^2(\Lambda;\mathbb{R}^m), \tag{3.10a}$$

$$\forall\, v \in \mathrm{L}^2(\Lambda; \mathbb{R}^m): \ \rho_\varepsilon b_\varepsilon \widetilde{G}_\varepsilon v \to \mu v \text{ in } \mathrm{H}^{-1}(\Lambda;\mathbb{R}^m), \tag{3.10b}$$

$$\forall\, u \in \mathrm{H}^1(\Lambda; \mathbb{R}^m): \ b_\varepsilon^\mathsf{T} \rho_\varepsilon \widehat{G}_\varepsilon u \to \mu^\mathsf{T} u \text{ in } \mathrm{L}^2(\Lambda;\mathbb{R}^m). \tag{3.10c}$$

In relation (3.10c) we have $\widehat{G}_\varepsilon u \to u$ (strongly) in $\mathrm{L}^2(\Lambda;\mathbb{R}^m)$, hence we must choose b_ε such that $b_\varepsilon^\mathsf{T} \rho_\varepsilon \widehat{u} \to \mu^\mathsf{T} \widehat{u}$ for all $\widetilde{u}$. Thus, we are forced to take $b_\varepsilon = \rho_\varepsilon^{-1}\mu$ (where the slight generalization $b_\varepsilon = \rho_\varepsilon^{-1}\mu_\varepsilon$ with $\mu_\varepsilon v \to \mu v$ would also be possible). Inserting this into the first condition of (3.10a) we see that $\widetilde{G}_\varepsilon$ must be chosen such that

$$\widetilde{G}_\varepsilon v - \mu^{-1}\rho_\varepsilon \mu^{-\mathsf{T}} r v \to 0 \text{ in } \mathrm{L}^2(\Lambda;\mathbb{R}^m).$$

Together with $\widetilde{G}_\varepsilon v \rightharpoonup v$ and $\rho_\varepsilon \widetilde{v} \rightharpoonup \rho^* \widetilde{v}$ this implies

$$r = \mu^\mathsf{T} (\rho^*)^{-1} \mu \quad \text{and} \quad \widetilde{G}_\varepsilon v = \mu^{-1} \rho_\varepsilon \mu^{-\mathsf{T}} r v,$$

again neglecting a slight generalization, where r might depend on ε. Finally, condition (3.10b) follows since $\rho_\varepsilon b_\varepsilon \widetilde{G}_\varepsilon v = \mu \widetilde{G}_\varepsilon v$ converges to μv weakly in $\mathrm{L}^2(\Lambda;\mathbb{R}^m)$, which is compactly embedded into $\mathrm{H}^{-1}(\Lambda;\mathbb{R}^m)$.

Thus, we have explored the possible ways to transform the linear wave equation into a Hamiltonian systems in such way that a family of joint recovery operators exists. The essential freedom we have is the choice of $\mu : \Lambda \to \mathbb{R}^{m\times m}$ such that $\mu, \mu^{-1} \in \mathrm{L}^\infty(\Lambda; \mathbb{R}^{m\times m})$. We define the symplectic form $\Omega = \left(\begin{smallmatrix} 0 & -\mu \\ \mu^\mathsf{T} & 0 \end{smallmatrix}\right)$ and the Hamiltonians H_ε, $\varepsilon > 0$, and H_0 via

$$H_\varepsilon(u,v) = \tfrac{1}{2} \int_\Lambda \mu^\mathsf{T} \rho_\varepsilon^{-1} \mu v \cdot v + a_\varepsilon u' \cdot u' \,\mathrm{d}x \quad \text{and} \quad H_0(u,v) = \tfrac{1}{2}\int_\Lambda \mu^\mathsf{T} (\rho^*)^{-1} \mu v \cdot v + a_* u' \cdot u' \,\mathrm{d}x.$$

Theorem 3.4 *Let $\rho, a \in \mathrm{L}^\infty(\Lambda \times Y; \mathbb{R}^{m\times m})$ and $\rho_\varepsilon, a_\varepsilon, \mu$ be as defined above. For $\varepsilon > 0$ let $z_\varepsilon = (u_\varepsilon, v_\varepsilon) : \mathbb{R} \to V \times Z$ be weak solutions of the Hamiltonian system*

$$\mu^\mathsf{T} \dot{u}_\varepsilon = \mathrm{D}_v H_\varepsilon(u_\varepsilon, v_\varepsilon), \quad -\mu \dot{v}_\varepsilon = \mathrm{D}_v H_\varepsilon(u_\varepsilon, v_\varepsilon),$$

and assume that $z_\varepsilon \overset{*}{\rightharpoonup} z = (u, v)$ in $\mathrm{L}^\infty(\mathbb{R}; V \times X)$. Then, z is a solution of the effective Hamiltonian system

$$\mu^\mathsf{T} \dot{u} = \mathrm{D}_v H_0(u, v), \quad -\mu \dot{v} = \mathrm{D}_v H_0(u, v),$$

In particular, the homogenized problem is given by the effective Hamiltonian H_0 that is the Γ limit of H_ε for $\varepsilon \to 0$ in the weak topology of the natural energy space $V \times X$.

While the above results have potential for generalization into the multi-dimensional case, we now treat a particular simple Hamiltonian form, which arises by using the momentum $p = \rho_\varepsilon \dot{u}$ and the strain $w = u'$. The wave equation $\rho_\varepsilon \ddot{u} = (a_\varepsilon u')'$ can be rewritten as the system

$$\left.\begin{array}{l} \dot{q} = (a_\varepsilon w)' \\ \dot{w} = (\rho_\varepsilon^{-1} p)' \end{array}\right\} \iff \begin{pmatrix} 0 & \partial_x^{-1} \\ \partial_x^{-1} & 0 \end{pmatrix} \begin{pmatrix} \dot{w} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} a_\varepsilon w \\ \rho_\varepsilon^{-1} p \end{pmatrix} = \mathrm{D}H_\varepsilon(w, p),$$

where $H_\varepsilon(w,p) = \frac{1}{2}\int_\Lambda a_\varepsilon w \cdot w + \rho_\varepsilon^{-1} p \cdot p \,\mathrm{d}x$. Now the relevant Hilbert space is

$$Z_0 = X_0 \times X_0 \quad \text{with } X_0 = \{\, w \in \mathrm{L}^2(\Lambda; \mathbb{R}^m) \mid \textstyle\int_\Lambda w(x)\,\mathrm{d}x = 0 \,\}.$$

On the space X_0 the operator ∂_x^{-1} can be defined by $\big(\partial_x^{-1} u\big)(x) = \int_\Lambda K(x,\xi) u(\xi)\,\mathrm{d}\xi$ with $K(x,\xi) = (x-\xi)/l + \mathrm{sign}(x-l)/2$. Since K satisfies $K(x,\xi) = -K(\xi,x)$ the operator ∂_x^{-1} is skew symmetric, which implies that Ω is a symplectic form. From the above it is again clear, that the effective Hamiltonian H_0 is obtained as the Γ-limit, namely $H_0(w,p) = \frac{1}{2}\int_\Lambda a_* w' \cdot w' + (\rho^*)^{-1} p \cdot p \,\mathrm{d}x$.

4 Discrete lattice models

In this section we want to apply the abstract theory for the passage from microscopic discrete systems to macroscopic continuum models. While the macroscopic system will be a system of wave equations as discussed above, the microscopic system is an infinite lattice of mass points subjected to Newton's law according to a background potential $\Psi_{\gamma,0}$ and interaction potentials $\Psi_{\gamma,\beta}$:

$$M_\gamma \ddot{u}_\gamma = -\mathrm{D}\Psi_\gamma(u_\gamma) + \sum_{0<|\beta|\le R} \mathrm{D}\Psi_{\gamma,\beta}(u_{\gamma+\beta} - u_\gamma) - \mathrm{D}\Psi_{\gamma,\beta}(u_\gamma - u_{\gamma-\beta}), \quad \gamma \in \mathbb{Z}^d. \tag{4.1}$$

Here, $u_\gamma \in \mathbb{R}^m$ denotes the vector of all displacement of atoms in the cell associated with the lattice site $\gamma \in \mathbb{Z}^d$. We write $\mathbf{u} = (u_\gamma)_\gamma \in \ell^2(\mathbb{Z}^d; \mathbb{R}^m)$ and $\mathbf{v} = \dot{\mathbf{u}} = (\dot{u}_\gamma)_\gamma \in \ell^2(\mathbb{Z}^d; \mathbb{R}^m)$ for vector of displacements and velocities, respectively. The system is mechanical system with kinetic and potential energies

$$\mathcal{K}(\dot{\mathbf{u}}) = \sum_{\gamma\in\mathbb{Z}^d} \frac{1}{2} M_\gamma \dot{u}_\gamma \cdot u_\gamma \quad \text{and} \quad \Phi(\mathbf{u}) = \sum_{\gamma\in\mathbb{Z}^d} \Big(\Psi_{\gamma,0}(u_\gamma) + \sum_{0<|\beta|\le R} \Psi_{\gamma,\beta}(u_\gamma - u_{\gamma+\beta}) \Big). \tag{4.2}$$

4.1 Embedding of lattices into continua

The main technique of treating the multiscale passage is to embed the discrete system into the the continuous space $Z = V \times X$ with

$$V = \mathrm{H}^1(\mathbb{R}^d;\mathbb{R}^m) \quad \text{and} \quad X = \mathrm{L}^2(\mathbb{R}^d;\mathbb{R}^m).$$

However, the embedding has to be such that the dynamics of the discrete model is exactly represented in the continuous counterpart in suitable closed subspaces V_ε and X_ε. Moreover, we want to be able to find exact formulas for the energies $\mathcal{K}_\varepsilon(\mathbf{v}) = \frac{1}{2}\langle \mathbf{M}_\varepsilon \mathbf{v}, \mathbf{v}\rangle$ and $\Phi_\varepsilon : V_\varepsilon \to \mathbb{R}$ and for the induced symplectic structure Ω_ε.

For $\varepsilon > 0$ we define the embedding operator

$$\widehat{E}_\varepsilon : \begin{cases} \ell^2(\mathbb{Z}^d) & \to & \mathrm{H}^1(\mathbb{R}^d), \\ \mathbf{u} = (u_\gamma)_\gamma & \mapsto & \Big[x \mapsto \sum\limits_{\gamma\in\mathbb{Z}^d} u_\gamma \widehat{H}\big(\tfrac{1}{\varepsilon}x{-}\gamma\big)\Big], \end{cases}$$

where $\widehat{H} \in \mathrm{W}^{1,\infty}(\mathbb{R}^d)$ is the piecewise affine interpolation between the values $\widehat{H}(y) = 1$ for $y \in [-1/4, 1/4]^d$ and $\widehat{H}(y) = 0$ for $y \notin [-3/4, 3/4]^d$. The embedding into $\mathrm{L}^2(\mathbb{R}^d)$ is done in a similar spirit, namely

$$\overline{E}_\varepsilon : \begin{cases} \ell^2(\mathbb{Z}^d) & \to & \mathrm{L}^2(\mathbb{R}^d), \\ \mathbf{p} = (p_\gamma)_\gamma & \mapsto & \Big[x \mapsto 2^d \sum\limits_{\gamma\in\mathbb{Z}^d} p_\gamma \overline{H}\big(\tfrac{1}{\varepsilon}x{-}\gamma\big)\Big], \end{cases}$$

where $\overline{H}(y) = 1$ for $y \in [-1/4, 1/4]^d$ and 0 otherwise.

The normalization constants were chosen such that for $U \in \mathrm{C}^1_\mathrm{c}(\mathbb{R}^d)$ and $\mathbf{u}_\varepsilon = (U(\varepsilon\gamma))_\gamma$ we have $\widehat{E}_\varepsilon \mathbf{u}_\varepsilon \rightharpoonup U$ in $\mathrm{H}^1(\mathbb{R}^d)$ and $\overline{E}_\varepsilon \mathbf{u}_\varepsilon \rightharpoonup U$ in $\mathrm{L}^2(\mathbb{R}^d)$, which corresponds in a natural way to our relation $x = \varepsilon\gamma$ between the microscopic and the macroscopic scale. Note however, that the norms scale with ε, namely $2^d\|\mathbf{p}\|^2_{\ell^2} = \varepsilon^d \|\overline{E}_\varepsilon \mathbf{p}\|^2_{\mathrm{L}^2}$ and $\|\widehat{E}_\varepsilon \mathbf{u}\|_{\mathrm{L}^2} \approx \varepsilon^d \|\mathbf{u}\|^2_{\ell^2}$. The construction of $\widehat{H}$ and $\overline{H}$ was done such that the symplectic form in the discrete system has a particularly simple representation in $\mathrm{L}^2(\mathbb{R}^d;\mathbb{R}^m)$ after the embedding, namely

$$\langle \mathbf{x}, \widetilde{\mathbf{p}}\rangle - \langle \widetilde{\mathbf{x}}, \mathbf{p}\rangle = \frac{1}{\varepsilon^d}\int_{\mathbb{R}^d} \big[(\widehat{E}_\varepsilon \mathbf{x})(y)\cdot(\overline{E}_\varepsilon \widetilde{\mathbf{p}})(y) - (\widehat{E}_\varepsilon \widetilde{\mathbf{x}})(y)\cdot(\overline{E}_\varepsilon \mathbf{p})(y)\big]\,\mathrm{d}y. \tag{4.3}$$

Thus, up to a normalization constant we find the canonical symplectic form of the continuous problem in the cotangent bundle of $\mathrm{L}^2(\mathbb{R};\mathbb{R}^m)$.

4.2 Transformation of the energies and equation

Moreover, we are able to write the kinetic and potential energies in terms of the embeddings. For simplicity, we restrict ourselves in the sequel to the one-dimensional case as we did in Section 3, since we will rely on some results from there. We will also restrict to the case of nearest-neighbor interaction with a quadratic potential $\Psi_{\gamma,1}$. We expect that the analysis can be generalized using suitable elaborate notation, see e.g., [Mie06c].

We assume that the chain is microscopically periodic with a period $N \in \mathbb{N}$ and that the coefficients may vary macroscopically as well in a L^∞ manner. For this purpose we use the functions m, a, and ψ, satisfying

$$\begin{aligned} &m, a \in \mathrm{L}^\infty(\mathbb{R}/_{N\mathbb{Z}} \times \mathbb{R}; \mathbb{R}^{m\times m}) \text{ and } \psi \in \mathrm{L}^\infty(\mathbb{R}/_{N\mathbb{Z}} \times \mathbb{R}; \mathrm{C}^1_{\mathrm{loc}}(\mathbb{R}^m)),\\ &\exists\, \alpha > 0\ \forall\, (\eta, x) \in \mathbb{R}/_{N\mathbb{Z}} \times \mathbb{R}\ \forall\, \xi \in \mathbb{R}^m :\\ &\qquad \min\big\{m(\eta, x)\xi\cdot\xi,\ a(\eta, x)\xi\cdot\xi,\ \psi(\eta, x, \xi)\big\} \geq \alpha|\xi|^2. \end{aligned} \tag{4.4}$$

We assume that the functions m, a and ψ are piecewise constant in the first variable, namely $m(\eta, x) = m(\gamma, x)$ for $\gamma \in \mathbb{Z}/_{N\mathbb{Z}}$ and $|\eta-\gamma| < 1/2$. As in Section 3.2 (cf. (3.2)) we denote with $m_\varepsilon, a_\varepsilon$, and ψ_ε the piecewise averages over the small cells $C_\varepsilon(x)$, namely

$$m_\varepsilon(x) = ⨍_{C_\varepsilon(x)} m(x/\varepsilon, y)\,\mathrm{d}y \quad \text{with } C_\varepsilon(x) = \varepsilon\big(\big[\tfrac{x}{\varepsilon N}\big] + N\big) + [0, \varepsilon N]$$

and similarly for a_ε and ψ_ε. With this we define the discrete functions as

$$M_\gamma = m_\varepsilon(\varepsilon\gamma), \quad \Psi_{\gamma,0}(u) = \varepsilon^2\psi_\varepsilon(\varepsilon\gamma, u), \quad \Psi_{\gamma,1}(u) = \frac{1}{2}a_\varepsilon(\varepsilon\gamma)u\cdot u.$$

Relying heavily on the piecewise affine nature of our embedding operators the discrete energies (4.2) take the form

$$\begin{aligned} \widehat{\mathcal{K}}_\varepsilon(\mathbf{p}) &= \textstyle\sum_{\gamma\in\mathbb{Z}} \frac{1}{2}M_\gamma^{-1}p_\gamma\cdot p_\gamma = \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \frac{1}{2}m_\varepsilon(x)^{-1}(\overline{E}_\varepsilon p)(x)\cdot(\overline{E}_\varepsilon p)(x)\,\mathrm{d}x,\\ \widehat{\Phi}_\varepsilon(\mathbf{u}) &= \textstyle\sum_{\gamma\in\mathbb{Z}} \Big(\Psi_{\gamma,1}(u_{\gamma+1}-u_\gamma) + \Psi_{\gamma,0}(u_\gamma)\Big)\\ &= \textstyle\frac{1}{\varepsilon^3}\int_{\mathbb{R}} \frac{1}{2}\big[a_\varepsilon(x)(\partial_x\widehat{E}_\varepsilon\mathbf{u})(x)\cdot(\partial_x\widehat{E}_\varepsilon\mathbf{u})(x)\big] + F_\varepsilon(x, \widehat{E}_\varepsilon\mathbf{u})(x))\,\mathrm{d}x, \end{aligned}$$

where $F_\varepsilon(x, u) = 2\overline{H}_{\mathrm{per}}(\frac{1}{\varepsilon}x)\psi_\varepsilon(x, u)$ with $\overline{H}_{\mathrm{per}}(y) = \sum_{\gamma\in\mathbb{Z}} \overline{H}(y-\gamma)$. For the nonlinearity we used that $\widehat{E}_\varepsilon\mathbf{u}$ is constant on the small intervals $(\varepsilon(\gamma-1/4), \varepsilon(\gamma+1/4))$.

In particular, our construction guarantees that the discrete lattice system

$$m_\varepsilon(\varepsilon\gamma)\ddot{u}_\gamma = -\varepsilon^2\mathrm{D}_u\psi_\varepsilon(x, u_\gamma) + a_\varepsilon(\varepsilon\gamma)(u_{\gamma+1}-u_\gamma) + a_\varepsilon(\varepsilon(\gamma+1))(u_{\gamma-1}-u_\gamma),\ \gamma\in\mathbb{Z}, \tag{4.5}$$

is equivalent to the Hamiltonian system on $Z_\varepsilon = V_\varepsilon \times X_\varepsilon$ with Hamiltonian $\mathcal{H}_\varepsilon$ and symplectic structure Ω_ε given by

$$\begin{aligned} &V_\varepsilon = \widehat{E}_\varepsilon\ell^2(\mathbb{Z}; \mathbb{R}^m) \subset \mathrm{H}^1(\mathbb{R}; \mathbb{R}^m), \quad X_\varepsilon = \overline{E}_\varepsilon\ell^2(\mathbb{Z}; \mathbb{R}^m) \subset \mathrm{L}^2(\mathbb{R}; \mathbb{R}^m),\\ &\mathcal{H}_\varepsilon(u, p) = \mathcal{K}_\varepsilon(u) + \Phi_\varepsilon(u) \text{ with } \mathcal{K}_\varepsilon(\overline{E}_\varepsilon\mathbf{p}) = \varepsilon\widehat{\mathcal{K}}_\varepsilon(\mathbf{p}) \text{ and } \Phi_\varepsilon(\widehat{E}_\varepsilon\mathbf{u}) = \varepsilon^3\widehat{\Phi}_\varepsilon(\mathbf{u}),\\ &\langle\Omega\tbinom{u}{p}, \tbinom{\widetilde{u}}{\widetilde{p}}\rangle = \textstyle\int_{\mathbb{R}} u\cdot\widetilde{p} - \widetilde{u}\cdot p\,\mathrm{d}x. \end{aligned}$$

The different rescaling in terms of ε for the kinetic energy, the potential energy and the symplectic form arise from the fact that we also rescale the time by defining a macroscopic time $\tau = \varepsilon t$ by letting $u(\tau) = \widehat{E}_\varepsilon\mathbf{u}(\tau/\varepsilon)$ and $p = \varepsilon\overline{E}_\varepsilon\mathbf{p}(\tau/\varepsilon)$, cf. [Mie06c, GHM06b] for more details. The resulting Hamiltonian system reads

$$\begin{pmatrix} 0 & -I\\ I & 0 \end{pmatrix}\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}\tau}u\\ \frac{\mathrm{d}}{\mathrm{d}\tau}p \end{pmatrix} = \Omega\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}\tau}u\\ \frac{\mathrm{d}}{\mathrm{d}\tau}p \end{pmatrix} = \begin{pmatrix} \mathrm{D}\Phi_\varepsilon(u)\\ \mathrm{D}\mathcal{K}_\varepsilon(p) \end{pmatrix} = \mathrm{D}\mathcal{H}_\varepsilon(u, p) \subset V_\varepsilon^* \times X_\varepsilon^*. \tag{4.6}$$

4.3 Passage to the limit

We are now able to pass to the limit in the problem (4.6) by using our abstract theory together with the analysis for the wave equations in Section 3.

For this we need to construct recovery operators $\widehat{G}_\varepsilon : V = \mathrm{H}^1(\mathbb{R};\mathbb{R}^m) \to V_\varepsilon$ for the potential energy and recovery operators $\widetilde{G}_\varepsilon : X = \mathrm{L}^2(\mathbb{R};\mathbb{R}^m) \to X_\varepsilon$ for the kinetic energy (i.e., (2.29)(ii) holds) such that additionally the symplectic form passes to the limit in the sense of (2.29)(i). Here this means

$$\begin{array}{lcl} V_\varepsilon \ni u_\varepsilon \rightharpoonup u_0 \in V_0 = V \text{ in } V & \Longrightarrow & \widetilde{G}_\varepsilon^* u_\varepsilon \rightharpoonup u_0 \text{ in } X = \mathrm{L}^2(\mathbb{R};\mathbb{R}^m), \\ X_\varepsilon \ni p_\varepsilon \rightharpoonup p_0 \in X_0 = X \text{ in } X & \Longrightarrow & \widehat{G}_\varepsilon^* p_\varepsilon \rightharpoonup p_0 \text{ in } V^* = \mathrm{H}^{-1}(\mathbb{R};\mathbb{R}^m). \end{array} \tag{4.7}$$

Note that any recovery operators $\widehat{G}_\varepsilon$ and $\widetilde{G}_\varepsilon$ provide weak convergence of $\widehat{G}_\varepsilon v_0$ and $\widetilde{G}_\varepsilon p_0$ in the better spaces V and X, respectively. However, this does not imply (4.7). Nevertheless, we show in the following result that the canonical recovery operators associated with the potential and the kinetic energies, respectively, do fulfill these conditions.

Lemma 4.1 *With the functions $m, a,$ and ψ from* (4.4) *we have the limits*

$$\begin{array}{l} \Phi_0 = \Gamma\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon : u \mapsto \int_{\mathbb{R}} \frac{1}{2} a_*(x) u'(x)\cdot u'(x) + \psi^*(x, u(x))\,\mathrm{d}x \\ \text{and } \mathcal{K}_0(p) = \int_{\mathbb{R}} \frac{1}{2} m^*(x)^{-1} p(x)\cdot p(x)\,\mathrm{d}x, \end{array}$$

where the effective functions $m^*, a_*,$ and ψ^* are given by

$$\begin{array}{l} m^*(x) = \frac{1}{N}\sum_{\gamma=1}^N m(\gamma, x) = \fint_{[0,N]} m(\eta, x)\,\mathrm{d}\eta, \\ a_*(x) = \big(\frac{1}{N}\sum_{\gamma=1}^N a(\gamma, x)^{-1}\big)^{-1} = \big(\fint_{[0,N]} a(\eta, x)^{-1}\,\mathrm{d}\eta\big)^{-1}, \\ \psi^*(x,u) = \frac{1}{N}\sum_{\gamma=1}^N \psi(\gamma, x, u) = \fint_{[0,N]} \psi(\eta, x, u)\,\mathrm{d}\eta. \end{array}$$

Moreover, the canonical recovery operators $(\widehat{G}_\varepsilon)_\varepsilon$ and $(\widetilde{G}_\varepsilon)_\varepsilon$ constructed as in Proposition 2.5 satisfy (4.7).

Proof: We first convince ourselves that the given formulas are the associated Γ-limits, $\Phi_\varepsilon \xrightarrow{\Gamma} \Phi_0$ in V and $\mathcal{K}_\varepsilon \xrightarrow{\Gamma} \mathcal{K}_0$ in X. For this we simply interprete Φ_ε and $\mathcal{K}_\varepsilon$ as special cases of the functionals considered in Proposition 3.1. This needs a generalization as we now allow for the value $+\infty$ under the integrand. For instance we implement the condition $p_\varepsilon \in X_\varepsilon = \overline{E}_\varepsilon \ell^2(\mathbb{R};\mathbb{R}^m)$ by allowing $p_\varepsilon \in X$ but defining $\mathcal{K}_\varepsilon$ via $\int_{\mathbb{R}} k_\varepsilon(x, p(x))\,\mathrm{d}x$ with $k_\varepsilon(x,p) = \frac{1}{2} m_\varepsilon(x)^{-1} p\cdot p$ for $x \in (-\frac{\varepsilon}{4}, \frac{\varepsilon}{4})$ mod ε and $k_\varepsilon(x,p) = +\infty$ otherwise. Taking the harmonic mean the values $+\infty$ turn into 0, the average is well defined, and we obtain the desired results. We assume that $\widetilde{G}_\varepsilon : X \to X_\varepsilon \subset X$ is given via Proposition 2.5 when applied to $\mathcal{K}_\varepsilon$. For the construction of $\widehat{G}_\varepsilon$ we use the auxiliary quadratic form

$$Q_\varepsilon(u) = \int_{\mathbb{R}} \tfrac{1}{2} a_\varepsilon(x) u'(x)\cdot u'(x) + \tfrac{\kappa^2}{2}|u(x)|^2\,\mathrm{d}x \quad \text{for } u \in V_\varepsilon \quad \text{and } \infty \text{ otherwise,}$$

where κ is an arbitrary, fixed number. Since the leading term is identical to that of Φ_ε it is easy to see that the recovery sequence $\widehat{G}_\varepsilon$ for Q_ε is a recovery sequence of the nonquadratic Φ_ε as well.

Second, we derive condition (4.7). Consider any family $(u_\varepsilon)_\varepsilon$ with $u_\varepsilon \rightharpoonup u_0$ in V. As $\widetilde{G}^*_\varepsilon u_\varepsilon$ is bounded in X, it suffices to test with a dense set of $w \in X$. We choose any $w \in \mathrm{C}^0_\mathrm{c}(\mathbb{R};\mathbb{R}^m)$. Let $\mathrm{sppt}(w) \subset [-R{+}1, R{-}1]$ for some $R>0$. Then, $\mathrm{sppt}(\widetilde{G}_\varepsilon w) \subset [-R,R]$ and $u_\varepsilon|_{[-R,R]} \to u_0|_{[-R,R]}$ in $\mathrm{L}^2([-R,R];\mathbb{R}^m)$, and we find

$$\langle \widetilde{G}^*_\varepsilon u_\varepsilon, w\rangle = \textstyle\int_{-R}^R u_\varepsilon\,(\widetilde{G}^*_\varepsilon w)\,\mathrm{d}x \to \int_{-R}^R u_0 w\,\mathrm{d}x = \langle u, w\rangle,$$

which is the first line in (4.7).

For $\widehat{G}_\varepsilon$ we argue similarly by using $\mathrm{C}^1_\mathrm{c}(\mathbb{R};\mathbb{R}^m)$ as a dense set in V. Now, $\widehat{G}_\varepsilon v$ will not have compact support, sub satisfy a uniform bound $|\widehat{G}_\varepsilon v(x)| \geq C\mathrm{e}^{-\kappa|x|}$. Moreover, $\widehat{G}_\varepsilon v \rightharpoonup v$ in V implies strong L^2-convergence on compact intervals $[-R,R]$. For a family $(p_\varepsilon)_\varepsilon$ with $p_\varepsilon \rightharpoonup p_0$ in X we can estimate as follows

$$\begin{aligned}&|\langle \widehat{G}^*_\varepsilon p_\varepsilon, v\rangle - \langle p_0, v\rangle| = |\langle p_\varepsilon, \widehat{G}_\varepsilon v\rangle - \langle p_0, v\rangle|\\ &\leq \textstyle\int_{|x|>R}\big(|p_\varepsilon|{+}|p_0|\big)C\mathrm{e}^{-\kappa|x|}\,\mathrm{d}x + \big|\int_{|x|<R} p_\varepsilon\cdot\widehat{G}_\varepsilon v{-}p_0\cdot v\,\mathrm{d}x\big|.\end{aligned}$$

The first term can be estimated by $\sup_{\varepsilon\in[0,1]}\|p_\varepsilon\|_X 2C\mathrm{e}^{-\kappa R}/\sqrt{\kappa}$ and, thus, can be made small independently of ε by choosing R big enough. Then, keeping R fixed the second term tends to 0 for $\varepsilon \to 0$ because of weak convergence of p_ε and strong convergence of $\widehat{G}_\varepsilon v$ in $\mathrm{L}^2([-R,R];\mathbb{R}^m)$. Thus, the second condition in (4.7) is established as well. ■

We summary the finding in the main result as follows.

Theorem 4.2 *Let $m, a,$ and ψ be given as in (4.4). Consider a family $(\mathbf{u}_\varepsilon)_\varepsilon$ of solutions in $\mathrm{C}^2(\mathbb{R};\ell^2(\mathbb{Z};\mathbb{R}^m))$ such that*

$$(\widehat{E}_\varepsilon\mathbf{u}(\tfrac{\cdot}{\varepsilon}), \overline{E}_\varepsilon\mathbf{M}_\varepsilon\varepsilon\dot{\mathbf{u}}(\tfrac{\cdot}{\varepsilon})) \overset{*}{\rightharpoonup} (u,p) \text{ in } \mathrm{L}^\infty(\mathbb{R};\mathrm{H}^1(\mathbb{R};\mathbb{R}^m)\times\mathrm{L}^2(\mathbb{R};\mathbb{R}^m)).$$

Then, (u,p) is a solution of the effective, macroscopic wave equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}u(\tau,x) = m^*(x)^{-1}p(\tau,x), \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}p(\tau,x) = \partial_x\big(a_*(x)\partial_x u(\tau,x)\big) - \mathrm{D}_u\psi^*(x,u)$$

with the effective Hamiltonian $\int_\mathbb{R}\frac{1}{2}(m^*)^{-1}p\cdot p + \frac{1}{2}a_*u'\cdot u' + \psi^*(\cdot,u)\,\mathrm{d}x$. Moreover, if for some $\tau\in\mathbb{R}$ we have $(\widehat{E}_\varepsilon\mathbf{u}(\tau/\varepsilon), \overline{E}_\varepsilon\mathbf{M}_\varepsilon\varepsilon\dot{\mathbf{u}}(\tau/\varepsilon)) \rightharpoonup (u(\tau),p(\tau))$, then the same holds for all $\tau\in\mathbb{R}$.

A Appendix

Lemma A.1 *Let Y be a reflexive or separable Banach space. Then, $y_n \to y$ is equivalent to the property that for all sequences $(\eta_n)_{n\in\mathbb{N}}$ in Y^* with $\eta_n \overset{*}{\rightharpoonup} \eta$ we have $\langle y_n, \eta_n\rangle \to \langle y, \eta\rangle$.*

Proof: The implication "$\Rightarrow$" follows by the triangle inequality via $\langle \eta_n, y_n \rangle = \langle \eta_n, y \rangle + \langle \eta_n, y_n - y \rangle \to \langle \eta, y \rangle + 0$, since $y_n - y \to 0$ and $(\eta_n)_{n\in\mathbb{N}}$ is bounded due to weak* convergence.

For the opposite implication first note that taking $\eta_n \equiv \eta$ implies $y_n \rightharpoonup y$. Second, we use that there exists $\eta_n \in Y^*$ such that $\|\eta_n\|_* = 1$ and $\langle y_n - y, \eta_n \rangle = \|y_n - y\| = \delta_n$. Now choose a subsequence such that $\lim_{k\to\infty} \delta_{n_k} = \limsup_{n\to\infty} \delta_n$. Choosing a further subsequence if necessary, we may assume $\eta_{n_k} \overset{*}{\rightharpoonup} \eta$ by using the property of Y. We define the sequence $(\eta_n)_{n\in\mathbb{N}}$ as $\eta_n = \eta_{n_k}$ if $n = n_k$ for some k and as $\eta_n = \eta$ else. Then $\eta_n \overset{*}{\rightharpoonup} \eta$ and we have

$$\delta_{n_k} = \|y_{n_k} - y\| = \langle y_{n_k}, \eta_{n_k} \rangle - \langle y, \eta_{n_k} \rangle \to \langle y, \eta \rangle - \langle y, \eta \rangle = 0.$$

As $\limsup \|y_n - y\| = \lim_{k\to\infty} \delta_{n_k} = 0$ the strong convergence is proved. ∎

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