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Mathematical results on existence for viscoelastodynamic problems with unilateral constraints

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Abstract

We consider a damped wave equation

$$u_{tt} - \Delta u - \alpha \Delta u_t = f,\ x \in (-\infty, 0] \times \mathbb{R}^{d-1},\ t > 0,\ \alpha > 0,$$

with unilateral boundary conditions

$$u(0,\cdot) \geq 0,\ (u_{x_1} + \alpha u_{x_1 t})(0,\cdot) \geq 0,\ \big(u(u_{x_1} + \alpha u_{x_1 t})\big)(0,\cdot) = 0.$$

We study also the evolution of a Kelvin-Voigt material:

$$\rho u_{i,tt} = \partial_j \sigma^0_{ij}(u) + \partial_j \sigma^1_{ij}(u_t) + f_i,\ x \in (-\infty, 0] \times \mathbb{R}^{d-1},\ t > 0,$$

with boundary conditions on $\{0\} \times \mathbb{R}^{d-1} \times [0, \infty)$

$$u_1 \leq 0,\ \sigma^0_{11}(u) + \sigma^1_{11}(u_t) \leq 0,\ u_1\big(\sigma^0_{11}(u) + \sigma^1_{11}(u_t)\big) = 0,$$
$$\sigma^0_{12}(u) + \sigma^1_{12}(u_t) = 0 \quad \text{and} \quad \sigma^0_{13}(u) + \sigma^1_{13}(u_t) = 0.$$

Under appropriate regularity assumptions on the initial data, both problems possess a weak solution which is obtained as the limit of a sequence of penalized problems; the functional properties of all the traces are precisely identified through Fourier analysis, and this enables us to infer the existence of a strong solution.

1 Introduction and notations

This paper aims to give some new mathematical results on existence for a damped wave equation with an obstacle and for full viscoelasticity in the particular case of a Kelvin-Voigt material with unilateral boundary conditions.

We consider in Section 2 a damped wave equation taking place in a half-space, with an obstacle at the boundary. Let $u(x,t)$ be the displacement at time t of the material point of spatial coordinate $x = (x_1, x') \in (-\infty, 0] \times \mathbb{R}^{d-1}$ at rest. Let $f(x_1, x', t)$ denote a density of external forces, depending on space and time. Define $\Omega = (-\infty, 0] \times \mathbb{R}^{d-1}$ and let α be a positive number. The mathematical problem is formulated as follows:

$$u_{tt} - \Delta u - \alpha \Delta u_t = f,\ x \in \Omega,\ t > 0, \tag{1.1}$$

with Cauchy initial data

$$u(\cdot, 0) = u_0 \quad \text{and} \quad u_t(\cdot, 0) = u_1, \tag{1.2}$$

and Signorini boundary conditions at $x_1 = 0$, $t > 0$,

$$0 \leq u \perp u_{x_1} + \alpha u_{x_1 t} \geq 0. \tag{1.3}$$

The orthogonality has the natural meaning: an appropriate duality product between two terms of relation vanishes.

We suppose that the initial position u_0 belongs to the Sobolev space $H^2(\Omega)$ and satisfies the compatibility condition $u_0(0,\cdot,\cdot) \geq 0$, the initial velocity u_1 belongs to $H^1(\Omega)$ and the density of forces f belongs to $L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))$. The choice of a function f defined for all non negative time is justified by the use of a Fourier transform in the later part of the article. This is not significant restriction as we can always extend f by 0 if it is defined only for finite times.

Let K be the convex set:

$$K = \{v \in H^1_{\mathrm{loc}}(\Omega \times [0,\infty)) : \nabla v_t \in L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega)), v|_{\{0\}\times\mathbb{R}^{d-1}} \geq 0\}.$$

This unusual convex set has been devised in order to write a weak formulation of our problem. Since we expect to find a scalar product $(\nabla u_t, \nabla w)$, we require ∇u_t to be square integrable. Thus, the weak formulation associated to (1.1)-(1.3) is obtained by multiplying (1.1) by $v - u$, $v \in K$ and by integrating formally over $\Omega \times (0, \tau)$. Then, we get:

$$\begin{cases} \text{Find } u \in K \text{ such that for all } v \in K \text{ and for all } \tau \in (0,\infty), \\ \displaystyle\int_\Omega (u_t(v-u))\big|_0^\tau \, dx - \int_0^\tau \int_\Omega u_t(v_t - u_t) \, dx \, dt \\ \displaystyle + \int_0^\tau \int_\Omega (\nabla u + \alpha \nabla u_t)(\nabla v - \nabla u) \, dx \, dt \geq \int_0^\tau \int_\Omega f(v-u) \, dx \, dt. \end{cases} \tag{1.4}$$

We treat also in Section 3 the evolution of a Kelvin-Voigt material (see [1]) occupying a half-space, satisfying Signorini conditions at the boundary and Cauchy data at $t = 0$. We make the assumptions of small deformations. Let $\varepsilon_{ij}(u) = (u_{j,x_i} + u_{i,x_j})/2$ be the strain tensor and let there be given two Hooke tensors, a^n_{ijkl}, $n = 0, 1$. We define the two stress tensors σ^n_{ij} corresponding respectively to the elastic and the viscous part of the stress:

$$\sigma^n_{ij}(u) = a^n_{ijkl}\varepsilon_{kl}(u); \tag{1.5}$$

here, we have used the summation convention on repeated indices. The displacement field u satisfies the system

$$\rho u_{i,tt} = \sigma^0_{ij,x_j}(u) + \sigma^1_{ij,x_j}(u_t) + f_i, \; x \in \Omega, \; t > 0. \tag{1.6}$$

The initial data are given by

$$u(\cdot, 0) = v_0 \quad \text{and} \quad u_t(\cdot, 0) = v_1. \tag{1.7}$$

The components of the unit external normal are δ_{1j} (δ is the Kronecker), and a basis of tangential vectors can be taken as $\tau_j = \delta_{2j}$, and $\tau'_j = \delta_{3j}$. Denote by

$\Sigma = \{0\} \times \mathbb{R}^{d-1}$ the boundary of Ω. Then, the boundary conditions on $\Sigma \times [0, \infty)$ are

$$0 \geq u_1 \perp \sigma^0_{11}(u) + \sigma^1_{11}(u_t) \leq 0, \tag{1.8a}$$
$$\sigma^0_{12}(u)+\sigma^1_{12}(u_t) = 0 \quad \text{and} \quad \sigma^0_{13}(u) + \sigma^1_{13}(u_t) = 0. \tag{1.8b}$$

One of the main results of Section 3 is to make (1.8a) precise and to justify the use of duality here.

In order to simplify the problem, we have considered an homogeneous and isotropic material; then, the Hooke tensors a^n_{ijkl} are defined with the help of Lamé constants λ^n and μ^n:

$$a^n_{ijkl} = \lambda^n \delta_{ij}\delta_{kl} + 2\mu^n \delta_{ik}\delta_{jl},\ n = 0, 1.$$

We define two elasticity operators A^n by

$$A^n u = a^n_{ijkl}\partial_j \varepsilon_{kl}(u),\ n = 0, 1.$$

Then, the problem (1.6)-(1.8) can be rewritten as follows:

$$\rho u_{tt} - A^0 u - A^1 u_t = f,\ x \in \Omega,\ t > 0, \tag{1.9a}$$
$$0 \geq u_1 \perp \left(\sigma^0_{11}(u) + \sigma^1_{11}(u_t)\right) \leq 0 \quad \text{on} \quad \Sigma \times [0, \infty), \tag{1.9b}$$
$$\sigma^0_{12}(u) + \sigma^1_{12}(u_t) = 0 \quad \text{and} \quad \sigma^0_{13}(u) + \sigma^1_{13}(u_t) = 0 \quad \text{on} \quad \Sigma \times [0, \infty), \tag{1.9c}$$
$$u(\cdot, 0) = v_0 \quad \text{and} \quad u_t(\cdot, 0) = v_1. \tag{1.9d}$$

Let us describe now the functional hypotheses on the data; if X is a space of scalar functions, the bold-face notation $\mathbf{X}$ denotes systematically the space X^d. For the final result, we require v_0 to belong to $\mathbf{H}^{5/2}(\Omega)$, v_1 to $\mathbf{H}^{3/2}(\Omega)$ and f to $\mathbf{H}^1_{\text{loc}}([0, \infty); L^2(\Omega))$. The initial data must satisfy the compatibility condition $(v_0)_1(0, x') \leq 0$ for all $x' \in \Sigma$. Let K be the convex set defined by:

$$K = \{v \in \mathbf{H}^1(\Omega \times (0, \tau)) : \nabla v_t \in \mathbf{L}^2(\Omega \times (0, \tau)), v(0, \cdot) \leq 0\}.$$

Define two bilinear forms by

$$a^0(u, v) = \int_\Omega a^0_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v)\, dx \quad \text{and} \quad a^1(u, v) = \int_\Omega a^1_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v)\, dx.$$

We obtain a weak formulation of the problem (1.9) as follows: we multiply (1.9a) by $v - u$, $v \in K$ and we integrate formally the result over $\Omega \times (0, \tau)$; we obtain then the variational inequality:

$$\begin{cases} \text{Find } u \in K \text{ such that for all } v \in K \text{ and for all } \tau \in (0, \infty), \\ \displaystyle\int_0^\tau \int_\Omega \rho u_{tt} \cdot (v - u)\, dx\, dt + \int_0^\tau a^0(u, u - v)\, dt \\ \displaystyle + \int_0^\tau a^1(u_t, v - u)\, dt \geq \int_0^\tau \int_\Omega f \cdot (v - u)\, dx\, dt. \end{cases} \tag{1.10}$$

The existence result for (1.1)-(1.3) is easily established by the penalty method, and was already proved by Jarušek et al [4] in the case of distributed constraints.

Jarušek has also proved in [3] an existence result for (1.9), in a much more general and complicated case, since it allows for contact, a given friction at the boundary, a nonlinear constitutive law for viscoelasticity and a general geometry. However, the boundary conditions must be understood in the sense of duality, since this is the sense in which his traces are defined.

In the present paper, for both problems, we penalize the obstacle constraint, we construct a solution of the penalized problem, and we show the existence of a weak solution by passing to the limit with respect to the penalty parameter. Then, under appropriate regularity conditions on the data, we prove that the penalized solution has traces, which can be estimated, and therefore, the limiting weak solution that we obtained is a strong solution. Observe that nothing is known about uniqueness. These two problems are treated in the same article, because they are quite close. Proofs for the second problem are shortened, when very close proofs for the first one. Nevertheless, there are substantial differences in detail, since the second problem is much more complicated than the first one. In particular, the bulk of the proof in Section 3 consists in obtaining a solution of a linear system through Fourier-Laplace transform, and then to estimate this solution in anisotropic Sobolev spaces.

2 The damped wave equation with Signorini boundary conditions

2.1 The penalized problem

We approximate (1.1)-(1.3) by the penalty method. This means that we replace the rigid constraint (1.3) by a very stiff response. When the constraint is active, the response is linear, and it vanishes when the constraint is not active. More precisely, letting $r^- = -\min(r, 0)$, we replace u by u^ϵ, which satisfies

$$u^\epsilon_{tt} - \Delta u^\epsilon - \alpha \Delta u^\epsilon_t = f,\ x \in \Omega,\ t > 0, \tag{2.1}$$

with initial data

$$u^\epsilon(\cdot, 0) = u_0 \quad \text{and} \quad u^\epsilon_t(\cdot, 0) = u_1, \tag{2.2}$$

and boundary condition

$$(u^\epsilon_{x_1} + \alpha u^\epsilon_{x_1 t})(0, \cdot, \cdot) = (u^\epsilon(0, \cdot, \cdot))^-/\epsilon. \tag{2.3}$$

Define the following sets:

$$Q_\tau = \Omega \times (0, \tau) \quad \text{and} \quad I_\tau = \Sigma \times (0, \tau),\ \forall \tau \in (0, \infty). \tag{2.4}$$

Theorem 2.1 *Let $W_{\mathrm{loc}} = \{u \in H^1_{\mathrm{loc}}([0,\infty) \times \Omega) : \nabla u_t \in L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))\}$. Assume that u_0 and u_1 belong to $H^1(\Omega)$, f belongs to $L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))$; then for every $\epsilon > 0$ there exists a unique weak solution $u^\epsilon \in W_{\mathrm{loc}}$ of the problem* (2.1)-(2.3) *such that*

$$u^\epsilon \in L^\infty_{\mathrm{loc}}([0,\infty); H^1(\Omega)), \tag{2.5a}$$
$$u^\epsilon_t \in L^2_{\mathrm{loc}}([0,\infty); H^1(\Omega)), \tag{2.5b}$$
$$u^\epsilon_{tt} \in L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega)), \tag{2.5c}$$

and for every $\tau \in (0,T)$ and for all $v \in W_{\mathrm{loc}}$, the following variational equality is satisfied:

$$\begin{aligned}\int_\Omega ((u^\epsilon_t v)(\cdot,\tau) - (u_1 v)(\cdot,0))\,dx - \int_{Q_\tau} u^\epsilon_t v_t\,dx\,dt + \int_{Q_\tau} \nabla u^\epsilon \nabla v\,dx\,dt \\ +\alpha \int_{Q_\tau} \nabla u^\epsilon_t \nabla v\,dx\,dt - \frac{1}{\epsilon}\int_{I_\tau} (u^\epsilon)^- v\,dx'\,dt = \int_{Q_\tau} f v\,dx\,dt.\end{aligned} \tag{2.6}$$

Proof. The Theorem is proved by the standard Galerkin method and the reader can see for example [2] or the appendix of [4]. □

2.2 A priori estimates

We establish here estimates up to the boundary and interior estimates which, later, will enable us to infer the existence of a weak solution to (1.1)-(1.3).

Lemma 2.2 *Assume that f belongs to $L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))$, u_0 to $H^1(\Omega)$ and u_1 to $L^2(\Omega)$. Then independently of $\epsilon > 0$, u^ϵ_t, ∇u^ϵ are bounded in $L^\infty_{\mathrm{loc}}([0,\infty); L^2(\Omega))$, ∇u^ϵ_t is bounded in $L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))$ and $(u^\epsilon(0,\cdot,\cdot))^-/\sqrt{\epsilon}$ is bounded in $L^\infty_{\mathrm{loc}}([0,\infty);$ $L^2(\mathbb{R}^-))$.*

Proof. These estimates are simply an application of the Gronwall lemma to the energy identity. We multiply (2.1) by u^ϵ_t and we integrate this expression over Q_τ to get

$$\int_{Q_\tau} u^\epsilon_{tt} u^\epsilon_t\,dx\,dt - \int_{Q_\tau} \Delta u^\epsilon u^\epsilon_t\,dx\,dt - \alpha\int_{Q_\tau} \Delta u^\epsilon_t u^\epsilon_t\,dx\,dt = \int_{Q_\tau} f u^\epsilon_t\,dx\,dt.$$

We integrate the first integral in time in the above relation, we use Green's formula for the second and the third one, and with the help of the boundary conditions (2.3), we obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega \left(|u^\epsilon_t(\cdot,\tau)|^2 + |\nabla u^\epsilon(\cdot,\tau)|^2\right) dx + \alpha\int_{Q_\tau} |\nabla u^\epsilon_t|^2\,dx\,dt \\ +\frac{1}{2\epsilon}\int_\Sigma \left((u^\epsilon(0,\cdot,\cdot))^-\right)^2\Big|_0^\tau\,dx' = \int_{Q_\tau} f u^\epsilon_t\,dx\,dt + \frac{1}{2}\int_\Omega \left(|\nabla u_0|^2 + |u_1|^2\right) dx.\end{aligned}$$

We may deduce from a classical Gronwall lemma that u_t^ϵ and ∇u^ϵ are bounded in $L^\infty_{\text{loc}}([0,\infty);L^2(\Omega))$, ∇u_t^ϵ is bounded in $L^2_{\text{loc}}([0,\infty);L^2(\Omega))$ and $(u^\epsilon(0,\cdot,\cdot))^-/\sqrt{\epsilon}$ is bounded in $L^\infty_{\text{loc}}([0,\infty);L^2(\mathbb{R}^{d-1}))$ independently of $\epsilon>0$. □

Remark 2.3 *If we suppose that f vanishes for t large then, independently of $\epsilon>0$, u_t^ϵ and ∇u^ϵ are bounded in $L^\infty([0,\infty);L^2(\Omega))$, ∇u_t^ϵ is bounded in $L^2([0,\infty);L^2(\Omega))$. These properties can be proved using the arguments given in the proof of Lemma 2.2, with the origin of time moved to T if $f(\cdot,t)$ vanishes for $t\ge T$; since the integral involving f vanishes, the conclusion is clear.*

Lemma 2.4 *Assume the hypotheses of Lemma 2.2. Then for all non negative, continuously differentiable and compactly supported ψ on $\mathbb{R}^{d-1}$ and for all $\tau\in[0,T]$,*

$$\int_{I_\tau}\frac{(u^\epsilon)^-}{\epsilon}\psi\,dx'\,dt$$

is bounded independently of $\epsilon>0$. In particular, $(u^\varepsilon(0,\cdot,\cdot))^-/\varepsilon$ is a bounded measure on I_τ.

Proof. Let ϕ be a continuous function with compact support; we multiply (2.1) by ϕ and we integrate over Q_τ; thanks to the boundary conditions (2.3) and Green's formula, we obtain

$$\int_\Omega \phi u_t^\epsilon(\cdot,t)\big|_0^\tau\,dx+\int_{Q_\tau}\nabla\phi(\nabla u^\epsilon+\alpha\nabla u_t^\epsilon)\,dx\,dt-\frac{1}{\epsilon}\int_{I_\tau}(u^\epsilon)^-\phi\,dx'\,dt=\int_{Q_\tau}\phi f\,dx\,dt.$$

Since the product $|zy|$ can be estimated by $|z|^2/2+|y|^2/2$, we get the following inequality:

$$\begin{aligned}\frac{1}{\epsilon}\int_{I_\tau}(u^\epsilon)^-\phi\,dx'\,dt &\le \frac{1}{2}\int_\Omega\left(|u_t^\epsilon(\cdot,\tau)|^2+|u_1|^2\right)dx+\int_\Omega|\phi|^2\,dx\\ &+\int_{Q_\tau}|\nabla\phi\nabla u^\epsilon|\,dx\,dt+\alpha\int_{Q_\tau}|\nabla\phi\nabla u_t^\epsilon|\,dx\,dt+\int_{Q_\tau}|\phi f|\,dx\,dt.\end{aligned}\tag{2.7}$$

The right hand side of (2.7) is bounded since f belongs to $L^2_{\text{loc}}([0,\infty);L^2(\Omega))$, u_1 to $L^2(\Omega)$ and u_t^ϵ, ∇u^ϵ and ∇u_t^ϵ belong to $L^2_{\text{loc}}([0,\infty);L^2(\Omega))$. Moreover $(u^\epsilon(0,\cdot,\cdot))^-$ is non negative; if the trace ψ of ϕ over Σ is non negative, the inequality is clear. The last statement of the theorem is obtained by a classical approximation argument. Write $\mu^\epsilon=(u^\epsilon(0,\cdot,\cdot))^-/\epsilon$. Let ψ_n be an increasing sequence of non negative, continuously differentiable and compactly supported functions on Σ, which are at most equal to ψ. Then the integrals of ψ_n against μ^ϵ converge to the integral of $\lim_n\psi_n$ against μ^ϵ, so that the integral of any non negative, continuous and compactly supported function against μ^ϵ is non negative, and this is precisely the definition of a non negative measure on Σ. □

Lemma 2.5 *Assume the hypotheses of Lemma 2.2, and suppose moreover that u_0 belongs to $H^2(\Omega)$. Then independently of $\epsilon > 0$, Δu^ϵ is bounded in the space $L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))$.*

Proof. Once again we use energy techniques, but now we multiply relation (2.1) by Δu^ϵ and we integrate over Q_τ:

$$\int_{Q_\tau} u^\epsilon_{tt}\Delta u^\epsilon\,dx\,dt - \int_{Q_\tau}|\Delta u^\epsilon|^2\,dx\,dt - \alpha\int_{Q_\tau}\Delta u^\epsilon_t\Delta u^\epsilon\,dx\,dt = \int_{Q_\tau} f\Delta u^\epsilon\,dx\,dt. \quad (2.8)$$

We integrate by parts the first integral in (2.8) first in time, then in space; we use Green's formula several times, and since the third integral in the left hand side of (2.8) contains a total time derivative, we obtain

$$\begin{aligned}&\int_\Omega \big((u^\epsilon_t\Delta u^\epsilon)(\cdot,t)\big)\big|_0^\tau dx - \int_{I_\tau}(u^\epsilon_t u^\epsilon_{x_1t})(0,\cdot)\,dx'\,dt + \int_{Q_\tau}|\nabla u^\epsilon_t|^2\,dx\,dt\\ &\quad - \int_{Q_\tau}|\Delta u^\epsilon|^2\,dx\,dt - \frac{\alpha}{2}\int_\Omega|\Delta u^\epsilon(\cdot,t)|^2\big|_0^\tau\,dx = \int_{Q_\tau} f\Delta u^\epsilon\,dx\,dt.\end{aligned} \quad (2.9)$$

According to the boundary condition (2.3), (2.9) becomes

$$\begin{aligned}&\int_{Q_\tau}|\Delta u^\epsilon|^2\,dx\,dt + \frac{\alpha}{2}\int_\Omega|\Delta u^\epsilon(\cdot,\tau)|^2\,dx = \frac{\alpha}{2}\int_\Omega|\Delta u_0|^2\,dx\\ &+\frac{1}{2\alpha\epsilon}\int_{I_\tau}\big((u^\epsilon)^-\big)^2\,dx'\,dt - \frac{1}{\alpha}\int_{I_\tau}(u^\epsilon_t u^\epsilon_{x_1})\,dx'\,dt + \int_\Omega(u^\epsilon_t\Delta u^\epsilon)(\cdot,\tau)\,dx\\ &-\int_{Q_\tau} f\Delta u^\epsilon\,dx\,dt - \int_\Omega u_1\Delta u_0\,dx + \int_{Q_\tau}|\nabla u^\epsilon_t|^2\,dx\,dt.\end{aligned} \quad (2.10)$$

In order to estimate the left hand side of (2.10), we organize the terms of its right hand side into different groups. The initial data terms

$$\frac{\alpha}{2}\int_\Omega|\Delta u_0|^2\,dx \quad\text{and}\quad -\int_\Omega u_1\Delta u_0\,dx$$

are bounded thanks to our assumptions on u_0 and u_1. The terms

$$\frac{1}{2\alpha\epsilon}\int_{I_\tau}\big((u^\epsilon)^-\big)^2\,dx'\,dt \quad\text{and}\quad \int_{Q_\tau}|\nabla u^\epsilon_t|^2\,dx\,dt$$

are bounded independently of ϵ thanks to Lemma 2.2. We estimate the remaining terms with the help of the inequality $xy \le \gamma|x|^2/2 + |y|^2/(2\gamma)$ for all $\gamma > 0$ and all real x and y. Therefore,

$$\begin{aligned}\int_\Omega(u^\epsilon_t\Delta u^\epsilon)(\cdot,\tau)\,dx &\le \frac{\gamma_1}{2}\int_\Omega|\Delta u^\epsilon(\cdot,\tau)|^2\,dx + \frac{1}{2\gamma_1}\int_\Omega|u^\epsilon_t(\cdot,\tau)|^2\,dx\\ \int_{Q_\tau} f\Delta u^\epsilon\,dx\,dt &\le \frac{1}{2\gamma_2}\int_{Q_\tau}|f|^2\,dx\,dt + \frac{\gamma_2}{2}\int_{Q_\tau}|\Delta u^\epsilon|^2\,dx\,dt\end{aligned}$$

and we will choose γ_1 and γ_2 later. The boundary term is estimated as

$$\int_{I_\tau} |u_t^\epsilon u_{x_1}^\epsilon|\, dx'\, dt \le \frac{1}{2\gamma_3} \int_{I_\tau} |u_t^\epsilon|^2\, dx'\, dt + \frac{\gamma_3}{2} \int_{I_\tau} |u_{x_1}^\epsilon|^2\, dx'\, dt. \tag{2.11}$$

If w and w_{x_1} belong to $L^2(\Omega)$, we have the classical estimate

$$\int_\Sigma |w(0,\cdot)|^2\, dx' \le C \int_\Omega \left(|w|^2 + |w_{x_1}|^2\right) dx,$$

which we apply to the right hand side of (2.11), getting thus

$$\begin{aligned}\int_{I_\tau} |u_t^\epsilon u_{x_1}^\epsilon|\, dx'\, dt \le & \frac{C}{2\gamma_3} \int_{Q_\tau} \left(|u_t^\epsilon|^2 + |u_{x_1 t}|^2\right) dx\, dt \\ & + \frac{C\gamma_3}{2} \int_{Q_\tau} \left(|u_{x_1}^\epsilon|^2 + |u_{x_1 x_1}^\epsilon|^2\right) dx\, dt.\end{aligned}$$

We use now the ellipticity of Δ: there exists a constant C_1 such that for all w in $H^2(\Omega)$,

$$\int_\Omega |w_{x_1 x_1}|^2\, dx \le C_1 \int_\Omega \left(|w|^2 + |\Delta w|^2\right) dx.$$

We gather all these estimates and we infer from (2.10) the following inequality:

$$\begin{aligned}\int_{Q_\tau} |\Delta u^\epsilon|^2\, dx\, dt + \frac{\alpha}{2} \int_\Omega |\Delta u^\epsilon(\cdot,\tau)|^2\, dx \le C_0 + \frac{\gamma_1}{2} \int_\Omega |\Delta u^\epsilon(\cdot,\tau)|^2\, dx \\ + \frac{1}{2\gamma_1} \int_\Omega |u_t^\epsilon(\cdot,\tau)|^2\, dx + \frac{1}{2\gamma_2} \int_{Q_\tau} |f|^2\, dx\, dt + \frac{\gamma_2}{2} \int_{Q_\tau} |\Delta u^\epsilon|^2\, dx\, dt \\ + \frac{C}{2\gamma_3} \int_{Q_\tau} |u_t^\epsilon|^2\, dx\, dt + \frac{C}{2\gamma_3} \int_{Q_\tau} |u_{x_1 t}^\epsilon|^2\, dx\, dt + \frac{C\gamma_3}{2} \int_{Q_\tau} |u_{x_1}^\epsilon|^2\, dx\, dt \\ + \frac{CC_1\gamma_3}{2} \int_{Q_\tau} |u^\epsilon|^2\, dx\, dt + \frac{CC_1\gamma_3}{2} \int_{Q_\tau} |\Delta u^\epsilon|^2\, dx\, dt.\end{aligned} \tag{2.12}$$

Now we choose the γ_i's: it suffices to have the inequalities

$$\gamma_1 < \alpha \quad \text{and} \quad \gamma_2/2 + CC_1\gamma_3/2 < 1,$$

and the conclusion is clear. □

Remark 2.6 *If we suppose that f vanishes for $t \ge T$, then, independently of $\epsilon > 0$, we have the estimate*

$$\int_{Q_\tau} |\Delta u^\epsilon|^2\, dx\, dt \le C(1+\tau).$$

This property is proved by moving the origin of times to T, and by studying carefully (2.12) *with the help of Remark 2.3.*

Let us turn now to interior estimates.

Lemma 2.7 *Assume the hypotheses of Lemma 2.5. Then for all $\beta > 0$, u^ϵ_{tt} and Δu^ϵ_t are bounded in the space $L^2_{\rm loc}([0,\infty); L^2((-\infty,-\beta)\times\Sigma))$, independently of $\epsilon > 0$.*

Proof. The idea of the proof is twofold: we multiply u^ϵ by a truncation function $\varphi \in C_0^\infty(\mathbb{R})$, and we define $v^\epsilon = \varphi u^\epsilon$; we will observe that $w^\epsilon = v^\epsilon_t$ satisfies a heat equation, whose right hand side will be estimated thanks to the previous lemmas. Let us go now into details.

Let φ be a truncation function which is equal to 1 if $x \le -\beta$ and to 0 if $x \ge -\beta/2$, $\beta \ge 0$. Then, we multiply u^ϵ by φ which enables us to forget about the strongly non linear boundary conditions. Define

$$v^\epsilon(x_1,\cdot,\cdot) = \varphi(x_1)u^\epsilon(x_1,\cdot,\cdot). \tag{2.13}$$

The derivatives of v^ϵ are given by:

$$v^\epsilon_{tt} = \varphi u^\epsilon_{tt}, \tag{2.14a}$$
$$\Delta v^\epsilon = \varphi\Delta u^\epsilon + 2\varphi_{x_1}\nabla u^\epsilon + \varphi_{x_1x_1}u^\epsilon, \tag{2.14b}$$
$$\Delta v^\epsilon_t = \varphi\Delta u^\epsilon_t + 2\varphi_{x_1}\nabla u^\epsilon_t + \varphi_{x_1x_1}u^\epsilon_t. \tag{2.14c}$$

Observe that thanks to relations (2.1) and (2.14), we have

$$v^\epsilon_{tt} - \Delta v^\epsilon - \alpha\Delta v^\epsilon_t = \widetilde{g}^\epsilon, \tag{2.15}$$

where $\widetilde{g}^\epsilon = \varphi f - 2\varphi_{x_1}(\nabla u^\epsilon + \alpha\nabla u^\epsilon_t) - \varphi_{x_1x_1}(u^\epsilon + \alpha u^\epsilon_t)$. Since f, u^ϵ_t, ∇u^ϵ, ∇u^ϵ_t and u^ϵ are bounded in $L^2_{\rm loc}([0,\infty); L^2(\Omega))$, $\widetilde{g}^\epsilon$ is bounded in $L^2_{\rm loc}([0,\infty); L^2(\Omega))$. Let us define

$$w^\epsilon = v^\epsilon_t \quad \text{and} \quad g^\epsilon = \widetilde{g}^\epsilon + \Delta v^\epsilon. \tag{2.16}$$

Substituting (2.16) in (2.15), we obtain

$$w^\epsilon_t - \alpha\Delta w^\epsilon = g^\epsilon. \tag{2.17}$$

Let us prove now that w^ϵ_t is bounded in $L^2_{\rm loc}([0,\infty); L^2(\Omega))$. For this purpose, we multiply (2.17) by w^ϵ_t; we integrate this expression over Ω,

$$\int_\Omega |w^\epsilon_t|^2\,dx - \alpha\int_\Omega \Delta w^\epsilon w^\epsilon_t\,dx = \int_\Omega g^\epsilon\, w^\epsilon_t\,dx.$$

We use Green's formula in the second term on the left hand side of the above expression, getting thus the following inequality:

$$\int_\Omega |w^\epsilon_t|^2\,dx + \alpha\int_\Omega \nabla w^\epsilon_t\nabla w^\epsilon\,dx = \int_\Omega g^\epsilon w^\epsilon_t\,dx. \tag{2.18}$$

We integrate (2.18) over $(0,\tau)$ and we observe that the product $|g^\epsilon w^\epsilon_t|$ can be estimated by $|g^\epsilon|^2/2 + |w^\epsilon_t|^2/2$ and we obtain

$$\begin{aligned}&\int_{Q_\tau} |w^\epsilon_t|^2\,dx\,dt + \alpha\int_\Omega |\nabla w^\epsilon(\cdot,\tau)|^2\,dx \\ &\le \alpha\int_\Omega |\nabla w^\epsilon(\cdot,0)|^2\,dx + \int_{Q_\tau} |g^\epsilon|^2\,dx\,dt.\end{aligned} \tag{2.19}$$

Since u_1 belongs to $H^1(\Omega)$ and φ belongs to $C_0^\infty(\mathbb{R})$, $\nabla w^\epsilon(\cdot,0) = \varphi_{x_1}u_1 + \varphi\nabla u_1$ is bounded in $L^2(\Omega)$. Moreover g^ϵ is bounded in the space $L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega))$ because Δv^ϵ and $\widetilde{g}^\epsilon$ are bounded in $L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega))$. Therefore (2.13), (2.16) and (2.19) enable us to deduce that u^ϵ_{tt} is bounded in $L^2_{\mathrm{loc}}([0,\infty);L^2((-\infty,-\beta)\times\Sigma))$. We use analogous arguments to show that Δu^ϵ_t is bounded in $L^2_{\mathrm{loc}}([0,\infty);L^2((-\infty,-\beta)\times\Sigma))$. We multiply (2.17) by Δw^ϵ, we integrate over Q_τ and thanks to Green's formula, we obtain

$$-\frac{1}{2}\int_\Omega |\nabla w^\epsilon|^2\Big|_0^\tau\,dx - \alpha\int_{Q_\tau}|\Delta w^\epsilon|^2\,dx\,dt = \int_{Q_\tau} g^\epsilon\Delta w^\epsilon\,dx\,dt. \tag{2.20}$$

Therefore the product $|g^\epsilon\Delta w^\epsilon|$ can be estimated by $|g^\epsilon|^2/2\gamma + \gamma|\Delta w^\epsilon|^2/2$, and if we choose $\gamma\in(0,2\alpha)$, we obtain the following inequality:

$$\left(\alpha-\frac{\gamma}{2}\right)\int_{Q_\tau}|\Delta w^\epsilon|^2\,dx\,dt \le \frac{1}{2\gamma}\int_{Q_\tau}|g^\epsilon|^2\,dx\,dt + \frac{1}{2}\int_\Omega|\nabla w^\epsilon(\cdot,0)|^2\,dx. \tag{2.21}$$

Since g^ϵ and $\nabla w^\epsilon(\cdot,0)$ are respectively bounded in $L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega))$ and $L^2(\Omega)$, according to (2.13), (2.16) and (2.21), we infer that Δu^ϵ_t is bounded in the space $L^2_{\mathrm{loc}}([0,\infty);L^2((-\infty,-\beta)\times\Sigma))$. $\square$

2.3 Existence of a weak solution

In this section, we show that it is possible to pass to the limit in the variational formulation of the penalized problem, to obtain a weak solution of (1.1)-(1.3). There is a small subtlety due to unboundedness of Ω.

Theorem 2.8 *Assume the hypotheses of Lemma 2.5. Then there exists a solution of the variational inequality* (1.4)*; this solution can be obtained as a limit of a subsequence of the penalty approximation defined by* (2.1)-(2.3).

Proof. Let v belong to K and φ be a function belonging to $C_0^\infty(\bar\Omega\times[0,\infty))$, which takes its values in $[0,1]$. Multiplying (2.1) by $(v-u^\epsilon)\varphi$ and integrating over Q_τ and then observing that

$$\int_{I_\tau}\left((u^\epsilon)^-\varphi(v-u^\epsilon)\right)dx'\,dt = \int_{I_\tau}\left(\left((u^\epsilon)^-\right)^2\varphi\right)dx'\,dt + \int_{I_\tau}\left((u^\epsilon)^-\varphi v\right)dx'\,dt$$

is non negative, we may deduce the following inequality:

$$\begin{aligned}&\int_\Omega u^\epsilon_t\varphi(v-u^\epsilon)\Big|_0^\tau\,dx - \int_{Q_\tau}u^\epsilon_t(\varphi(v-u^\epsilon))_t\,dx\,dt\\ &+\int_{Q_\tau}(\nabla u^\epsilon+\alpha\nabla u^\epsilon_t)\nabla(\varphi(v-u^\epsilon))dx\,dt \ge \int_{Q_\tau} f\varphi(v-u^\epsilon)\,dx\,dt.\end{aligned} \tag{2.22}$$

We infer from Lemmas 2.2, 2.4 and 2.5 that it is possible to extract a subsequence, still denoted by u^ϵ, such that

$$u^\epsilon \rightharpoonup u \quad \text{in} \quad L^2_{\text{loc}}([0,\infty); L^2(\Omega)) \quad \text{weak} \quad *, \tag{2.23a}$$

$$u^\epsilon_t \rightharpoonup u_t \quad \text{in} \quad L^\infty_{\text{loc}}([0,\infty); L^2(\Omega)) \quad \text{weak} \quad *, \tag{2.23b}$$

$$\nabla u^\epsilon \rightharpoonup \nabla u \quad \text{in} \quad L^2_{\text{loc}}([0,\infty); L^2(\Omega)) \quad \text{weak} \quad *, \tag{2.23c}$$

$$\Delta u^\epsilon \rightharpoonup \Delta u \quad \text{in} \quad L^\infty_{\text{loc}}([0,\infty); L^2(\Omega)) \quad \text{weak} \quad *, \tag{2.23d}$$

$$\nabla u^\epsilon_t \rightharpoonup \nabla u_t \quad \text{in} \quad L^2_{\text{loc}}([0,\infty); L^2(\Omega)) \quad \text{weak} \quad *. \tag{2.23e}$$

Define the set: $Q_R = \{x : x_1 < 0, |x'| \le R\} \times [0, R]$. Thanks to the classical compactness properties of injections of Sobolev spaces on bounded open sets, we see for all $R > 0$, the restrictions of u^ϵ and ∇u^ϵ to Q_R converge strongly to their respective limits in $L^2(Q_R)$; therefore, we can pass to the limit in all the terms of (2.22) except possibly the first two terms.

Let us prove that u_t is continuous from $[0,\infty)$ to $L^2(\Omega)$ equipped with the weak topology: we infer from the estimates of Lemma 2.7 that for all $\beta > 0$, u^ϵ_{tt} restricted to $x_1 < -\beta$ is bounded in $L^2_{\text{loc}}([0,\infty); L^2((-\infty, -\beta] \times \Sigma))$; therefore it is plain that u^ϵ_t converges to a function u_t whose restriction to $x_1 < -\beta$ is continuous from $[0,\infty)$ to $L^2((-\infty,-\beta] \times \Sigma)$. Let $t_j \in [0,\infty)$ be a sequence converging to $t_\infty < \infty$; as u_t belongs to $L^2_{\text{loc}}([0,\infty); L^2(\Omega))$, we may extract a subsequence, still denoted by t_j, such that

$$u^\epsilon_t(\cdot, t_j) \rightharpoonup z \quad \text{in} \quad L^2(\Omega) \quad \text{weak}.$$

But since for all $\beta > 0$,

$$u^\epsilon_t(\cdot, t_j) 1_{\{x_1 < -\beta\}} \to u_t(\cdot, t_\infty) 1_{\{x_1 < -\beta\}} \quad \text{in} \quad L^2(\Omega) \quad \text{weak},$$

we see that z must coincide with $u_t(\cdot, t_\infty)$, and that all the sequence converges strongly to $u_t(\cdot, t_\infty)$; this proves that u_t is continuous from $[0,\infty)$ to $L^2(\Omega)$ weak.

Let us prove now that $u^\epsilon_t(\cdot, t)$ converges weakly to $u_t(\cdot, t)$ for all $t > 0$: let γ be an arbitrary positive number; let z belong to $L^2(\Omega)$; denote by C_1 is an upper bound for $|u^\epsilon_t|_{L^\infty([0,T]; L^2(\Omega))}$ with T fixed. We choose β so small that

$$\left(\int_{-\beta < x_1 < 0} |z|^2 \, dx \right)^{1/2} \le \frac{\gamma}{4 C_1};$$

then, for $t \in [0, T]$

$$\begin{aligned} \left| \int_\Omega (u^\epsilon_t(\cdot,t) - u_t(\cdot,t)) z \, dx \right| &\le \left| \int_{x_1 < -\beta} (u^\epsilon_t(\cdot,t) - u_t(\cdot,t)) z \, dx \right| \\ &+ \left(\int_{-\beta < x_1 < 0} |z|^2 \, dx \right)^{1/2} \left(\int_{-\beta < x_1 < 0} |u^\epsilon_t(\cdot,t) - u_t(\cdot,t)|^2 \, dx \right)^{1/2}. \end{aligned} \tag{2.24}$$

By definition of C_1, the second term on the right hand side of (2.24) is estimated by $C_1 \gamma / 2 C_1 = \gamma/2$. As $u^\epsilon_t|_{(-\infty,-\beta) \times I_T}$ is bounded in $H^1((-\infty,-\beta) \times I_T)$, we see that

$$\int_{-\infty}^{-\beta} \int_\Sigma u^\epsilon_t z \, dx \quad \text{converges} \quad \text{to} \quad \int_{-\infty}^{-\beta} \int_\Sigma u_t z \, dx$$

uniformly with respect to $t \in [0,T]$. It suffices therefore to choose ϵ so small that the first term on the right hand side of (2.24) is estimated by $\gamma/2$. This proves that the convergence of $\int_\Omega u_t^\epsilon z\,dx$ to $\int_\Omega u_t z\,dx$ is uniform on compact sets in time. In particular, as ϵ tends to 0, it is plain that for all $\tau > 0$,

$$\int_\Omega u_t^\epsilon \varphi(v - u^\epsilon)\,dx \to \int_\Omega u_t \varphi(v-u)\,dx.$$

Let us turn now to the term

$$\int_{Q_\tau} u_t^\epsilon(\varphi_t(v-u^\epsilon) + \varphi(v_t - u_t^\epsilon))\,dx\,dt.$$

It is clear that

$$\int_{Q_\tau} u_t^\epsilon(\varphi_t(v-u^\epsilon) + \varphi v_t)\,dx\,dt \to \int_{Q_\tau} u_t(\varphi_t(v-u) + \varphi v_t)\,dx\,dt.$$

There remains to prove the convergence

$$\int_{Q_\tau} |u_t^\epsilon|^2 \varphi\,dx\,dt \to \int_{Q_\tau} |u_t|^2\varphi\,dx\,dt.$$

We observe that

$$\int_{Q_\tau} |u_t^\epsilon - u_t|^2\varphi\,dx\,dt \le \int_0^\tau \int_{x_1 \le -\beta} |u_t^\epsilon - u_t|^2\varphi\,dx\,dt + \int_0^\tau \int_{-\beta \le x_1 \le 0} |u_t^\epsilon - u_t|^2 \varphi\,dx\,dt.$$

Let γ be any positive number. We infer from the estimates over $|u_t^\epsilon|_{L^2(I_\tau)}$ and $|\nabla u_t^\epsilon|_{L^2(Q_\tau)}$ that there exists a constant C_2 independent from ϵ such that

$$|u_t^\epsilon(x_1,\cdot,\cdot)|_{L^2(\Sigma\times(0,\tau))} \le C_2.$$

Therefore,

$$\int_0^\tau \int_{-\beta\le x_1\le 0} |u_t^\epsilon - u_t|^2\varphi\,dx\,dt \le C_2^2\beta.$$

We choose β so small that $C_2^2\beta \le \gamma/2$; then we know from the estimates of Lemmas 2.5 and 2.7 that the restriction of u^ϵ to $\{x_1 < -\beta\}$ intersected with a ball containing the support of φ is bounded in H^2 of that set; therefore, for ϵ small enough,

$$\int_{Q_\tau} |u_t^\epsilon - u_t|^2 \varphi\,dx\,dt \le \frac{\gamma}{2},$$

and the convergence of the first two terms of (2.22) is proved.

We observe now that since u, u_t, ∇u and ∇u_t belong to $L^2_{\text{loc}}([0,\infty); L^2(\Omega))$, we may replace φ by φ_R in the variational inequality where φ_R is equal to 1 over the set Q_R and vanishes outside of Q_{R+1}. It is plain that as $R \to \infty$ all the terms in (2.22) converge to their limit; thus we have proved the existence of the desired weak solution. $\square$

Remark 2.9 *Nothing is known about uniqueness.*

2.4 Auxiliary results on the damped wave equation with Dirichlet boundary conditions

We establish *a priori* estimates on the damped wave equation with Dirichlet boundary conditions. These estimates will enable us to give some properties on the trace spaces which we use in the next subsection.

Lemma 2.10 *Assume u_0 belongs to $H^{5/2}(\Omega)$; then, there exists a function $z \in H^3(\Omega \times [0,\infty))$ with compact support in t such that the trace of z on Σ is equal to u_0.*

Proof. We extend u_0 into a function belonging to $H^{5/2}(\mathbb{R}^d)$: as the boundary of Ω is smooth, this extension is a consequence of classical results on Sobolev spaces. Then there exists a function Z belonging to $H^3(\mathbb{R}^d \times [0,\infty))$ whose trace is u_0. It suffices now to select a cutoff function $\varphi \in C^\infty([0,\infty))$ which is equal to 1 on $[0,1]$ and to 0 on $[2,\infty)$, and to define z as the restriction of φZ to $\Omega \times [0,\infty)$. $\square$

Lemma 2.11 *Assume u_0 belongs to $H^{5/2}(\Omega)$, u_1 belongs to $H^1(\Omega)$ and f belongs to $L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))$. Define z as in Lemma 2.10 and let $\bar{u}$ be the solution of* (1.1) *with the initial data* (1.2) *and boundary condition $\bar{u}(0,\cdot,\cdot) = z(0,\cdot,\cdot)$. Then the trace $\bar{g} = -(\bar{u}_{x_1} + \alpha \bar{u}_{x_1 t})(0,\cdot,\cdot)$ is well defined and belongs to the space $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^{d-1}))$. Moreover, if f is compactly supported in time,*

$$\int_0^\tau |\bar{g}(\cdot,t)|^2_{L^2(\Sigma)}\, dt$$

increases at most polynomially with respect to τ.

Proof. The function $\zeta = \bar{u} - z$ satisfies the equation

$$\zeta_{tt} - \Delta\zeta - \alpha\Delta\zeta_t = F,\ x \in \Omega,\ t > 0, \tag{2.25}$$

where $F = f - z_{tt} + \Delta z + \alpha \Delta z_t$, with initial data

$$\zeta(\cdot,0) = 0 \quad \text{and} \quad \zeta_t(\cdot,0) = u_1,$$

and the Dirichlet boundary condition $\zeta(0,\cdot,\cdot) = 0$. If we multiply (2.25) by ζ_t and integrate, and if we suppose that f and F are compactly supported in time, we may easily deduce that ζ_t, $\nabla\zeta$ are bounded in $L^\infty_{\mathrm{loc}}([0,\infty); L^2(\Omega))$ and $\nabla\zeta_t$ is bounded in $L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))$. In order to get more information, we multiply (2.25) by $\Delta\zeta_t$; since the boundary term vanishes, we get immediately the identity

$$\alpha \int_{Q_\tau} |\Delta\zeta_t|^2\, dx\, dt + \frac{1}{2}\int_\Omega |\Delta\zeta(\cdot,\tau)|^2\, dx + \int_\Omega |\nabla\zeta_t(\cdot,\tau)|^2\, dx$$
$$= \int_\Omega |\nabla\zeta_t(\cdot,0)|^2\, dx - \int_{Q_\tau} F\Delta\zeta_t\, dx\, dt.$$

We remark that the product $F\Delta\zeta_t$ can be estimated by $\alpha|\Delta\zeta_t|^2/2+|F|^2/(2\alpha)$ then $\Delta\zeta_t$ is bounded in $L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega))$, $\Delta\zeta$ and $\nabla\zeta_t$ are bounded in $L^\infty_{\mathrm{loc}}([0,\infty);L^2(\Omega)$). In particular, if the support in time of F is bounded, $\Delta\zeta_t$ is bounded in $L^2([0,\infty);$ $L^2(\Omega))$ and $\Delta\zeta$ and $\nabla\zeta_t$ are bounded in $L^\infty([0,\infty);L^2(\Omega))$. Therefore we may deduce that $\zeta_{x_1t}(0,\cdot,\cdot)$ and $\zeta_{x_1}(0,\cdot,\cdot)$ belong respectively to $L^2_{\mathrm{loc}}([0,\infty);H^{1/2}(\mathbb{R}^{d-1}))$ and to $L^\infty_{\mathrm{loc}}([0,\infty);H^{1/2}(\mathbb{R}^{d-1}))$, and if the support in time of f is bounded, the local character of these spaces may be removed. □

2.5 Regularity of the trace

We characterize the trace spaces using Fourier analysis and we prove that u is a strong solution of (1.1)-(1.3). Here, we mean by strong solution that all the traces can be defined.

Let ν be a positive number. Denote by $v^\epsilon=\exp(-\nu t)(u^\epsilon-\bar{u})$ a solution of

$$(\nu+\partial_t)^2v^\epsilon-(1+\alpha(\nu+\partial_t))\Delta v^\epsilon=0,\ x\in\Omega,\ t>0, \tag{2.26a}$$
$$(1+\alpha(\nu+\partial_t))v^\epsilon_{x_1}(0,\cdot,\cdot)=e^{-\nu t}\bar{g}-(v^\epsilon(0,\cdot,\cdot)+e^{-\nu t}\bar{u}(0,\cdot,\cdot))^-/\epsilon, \tag{2.26b}$$
$$v^\epsilon(\cdot,t)=0 \quad\text{and}\quad v^\epsilon_t(\cdot,t)=0. \tag{2.26c}$$

We denote by $\xi=(\xi_2,...,\xi_d)^T$ and ω respectively the dual variables to $x'=(x_2,...,$ $x_d)^T$ and t. The Fourier transform of $u(0,x',t)$is $\widehat{u}(0,\xi,\omega)$. where the convention for the Fourier transform is

$$\widehat{u}(0,\xi,\omega)=\int_{\mathbb{R}^d}e^{-i(\xi\cdot x'+\omega t)}u(0,x',t)\,dx'\,dt.$$

Then $u(0,x',t)$ belongs to the Sobolev space $H^{a,b}_{\mathrm{loc}}(\mathbb{R}^{d-1}\times[0,\infty))$, $(a,b)\in\mathbb{R}^2$, iff $|\xi|^a\widehat{u}(0,\xi,\omega)$ and $|\omega|^b\widehat{u}(0,\xi,\omega)$ belong to $L^2(\mathbb{R}^d)$.

We apply a partial Fourier transform in the tangential variable to (2.26a), and we get the following differential equation:

$$\widehat{v}^\epsilon_{x_1x_1}=\left(|\xi|^2+\frac{(\nu+i\omega)^2}{1+\alpha(\nu+i\omega)}\right)\widehat{v}^\epsilon. \tag{2.27}$$

Define $\widehat{\lambda}$ to be the causal determination of the square root of $|\xi|^2+(\nu+i\omega)^2/(1+\alpha(\nu+i\omega))$:

$$\widehat{\lambda}(\xi,\omega)=\sqrt{|\xi|^2+\frac{(\nu+i\omega)^2}{1+\alpha(\nu+i\omega)}};$$

thus $\widehat{\lambda}$ is holomorphic in the lower half-plane $\Im(\omega)<0$ and $\Re\widehat{\lambda}\geq 0$ for $\Im(\omega)=0$. The general solution of (2.27) is given by $\widehat{a}^\epsilon e^{\widehat{\lambda}x_1}+\widehat{b}^\epsilon e^{-\widehat{\lambda}x_1}$; since we performed a Fourier transform on v^ϵ, we assumed implicitly that v^ϵ and $\widehat{v}^\epsilon$ are tempered respectively in (x',t) and (ξ,ω). We remark that the term $\widehat{b}^\epsilon e^{-\widehat{\lambda}x_1}$ can be tempered only if

$\widehat{b}^\epsilon$ decays at infinity very fast, and since this must be true for all x_1, it implies that $\widehat{b}^\epsilon$ vanishes, the proof is similar to the one given in [6]; we deduce that the solution of (2.27) is $\widehat{a}^\epsilon e^{\widehat{\lambda} x_1}$. In particular,

$$((1+\alpha(\nu+\partial_t))v^\epsilon_{x_1}\widehat{)}(0,\xi,\omega) = \widehat{\lambda}_1 \widehat{v}^\epsilon(0,\xi,\omega), \tag{2.28}$$

where $\widehat{\lambda}_1 = (1+\alpha(\nu+i\omega))\widehat{\lambda}$. Define

$$g(x',t) = e^{-\nu t}\bar{g}(x',t) \quad \text{and} \quad h(x',t) = e^{-\nu t}\bar{u}(0,x',t)$$

If we let $w^\epsilon(x',t)$ be the trace $v^\epsilon(0,x',t)$, (2.26) can be written now

$$\lambda_1 * w^\epsilon = g + (w^\epsilon + h)^-/\epsilon, \tag{2.29}$$

where w^ϵ vanishes for all $t \le 0$.

Remark 2.12 *It is clear that $\widehat{\lambda}$ is a holomorphic function in $\Im(\omega) < 0$ and thus we may deduce that λ_1 is a causal distribution.*

Lemma 2.13 *Let u^ϵ be the solution of* (2.1)-(2.3). *Then we may extract a subsequence, still denoted by u^ϵ such that*

$$u^\epsilon(0,\cdot,\cdot) \rightharpoonup u(0,\cdot,\cdot) \quad \text{weakly} \quad \text{in} \quad H^{1/2,5/4}_{\mathrm{loc}}(\mathbb{R}^{d-1}\times[0,\infty)).$$

Moreover u is a strong solution of (1.1)-(1.3).

Proof. Formally, we multiply (2.29) by $\alpha(\nu w^\epsilon + w^\epsilon_t) + w^\epsilon$, and we estimate the pseudodifferential term in the Fourier variable; we obtain

$$\begin{aligned} &\frac{1}{(2\pi)^d}\Re\int_{\mathbb{R}^d}\widehat{\lambda}_1\widehat{w}^\epsilon\overline{(1+\alpha(\nu+i\omega))\widehat{w}^\epsilon}\,d\omega\,d\xi \\ &\qquad = \frac{1}{(2\pi)^d}\Re\int_{\mathbb{R}^d}\widehat{g}\overline{(1+\alpha(\nu+i\omega))\widehat{w}^\epsilon}\,d\omega\,d\xi \\ &+\frac{1}{\epsilon}\int_0^\infty\int_{\mathbb{R}^{d-1}}(w^\epsilon+h)^-(1+\alpha(\nu+\partial_t))w^\epsilon\,dx'\,dt. \end{aligned} \tag{2.30}$$

Since $(u^\epsilon(0,\cdot,\cdot))^-/\sqrt{\epsilon}$ is bounded in the space $L^\infty_{\mathrm{loc}}([0,\infty);L^2(\mathbb{R}^{d-1}))$, the absolute value of the second integral in the right hand side of (2.30) is bounded and we infer that there exists $C_1>0$ such that

$$\Re\int_{\mathbb{R}^d}\widehat{\lambda}_1|\widehat{w}^\epsilon|^2\overline{(1+\alpha(\nu+i\omega))}\,d\omega\,d\xi \le C_1 + \Re\int_{\mathbb{R}^d}\widehat{g}\overline{(1+\alpha(\nu+i\omega))\widehat{w}^\epsilon}\,d\omega\,d\xi. \tag{2.31}$$

On the other hand, we have

$$\Re\widehat{\lambda}^2 = |\xi|^2 + \frac{\nu^2(1+\alpha\nu)+(-1+\alpha\nu)\omega^2}{|1+\alpha(\nu+i\omega)|^2} \quad \text{and} \quad \Im\widehat{\lambda}^2 = \frac{2\nu\omega+\alpha\omega(\nu^2+\omega^2)}{|1+\alpha(\nu+i\omega)|^2}.$$

We may choose ν such that $\nu\alpha = 1$; we get then

$$\Re\widehat{\lambda}^2 = |\xi|^2 + \frac{2}{\alpha^2|2+i\alpha\omega|^2} \quad \text{and} \quad \Im\widehat{\lambda}^2 = \frac{\omega(3+\alpha^2\omega^2)}{\alpha|2+i\alpha\omega|^2}. \tag{2.32}$$

Therefore we infer that

$$\arg\widehat{\lambda} = \frac{1}{2}\arctan\left(\frac{|\xi|^2|2+i\alpha\omega|^2+2}{\alpha\omega(3+\alpha^2\omega^2)}\right).$$

According to (2.32), $\arg\widehat{\lambda}$ belongs to $[0,\pi/4]$ and since $\widehat{\lambda}$ is never equal to zero, we get for $|\xi|+|\omega| \gg 1$ the following inequality:

$$\Re\widehat{\lambda} \geq C\big(1+|\xi|+\sqrt{|\omega|}\big). \tag{2.33}$$

Therefore, we obtain

$$C\int_{\mathbb{R}^d}|2+i\alpha\omega|^2\big(1+|\xi|+\sqrt{|\omega|}\big)|\widehat{w}^\epsilon|^2\,d\omega\,d\xi \leq C_1 + \int_{\mathbb{R}^d}|2+i\alpha\omega||\widehat{g}||\widehat{w}^\epsilon|\,d\omega\,d\xi.$$

We estimate the product $|zy|$ by $|z|^2/(2\gamma)+\gamma|y|^2/2$, $\gamma>0$, we see that

$$\begin{aligned}\Big(C-\frac{\gamma}{2}\Big)\int_{\mathbb{R}^d}&|2+i\alpha\omega|^2\big(1+|\xi|+\sqrt{|\omega|}\big)|\widehat{w}^\epsilon|^2\,d\omega\,d\xi\\ &\leq C_1+\frac{1}{2\gamma}\int_{\mathbb{R}^d}\frac{|\widehat{g}|^2}{1+|\xi|+\sqrt{|\omega|}}\,d\omega\,d\xi.\end{aligned} \tag{2.34}$$

We choose γ such that $\gamma<2C$, since g belongs to $L^2([0,\infty);H^{1/2}(\mathbb{R}^{d-1}))$ then it is easy to deduce from (2.34) that $u^\epsilon(0,\cdot,\cdot)$ is bounded in $H^{1/2,5/4}_{\mathrm{loc}}(\mathbb{R}^{d-1}\times[0,\infty))$. Moreover it is clear that $(u_{x_1}+\alpha u_{x_1t})(0,\cdot,\cdot)$ is bounded in the space $H^{-1/2,-1/4}_{\mathrm{loc}}(\mathbb{R}^{d-1}\times[0,\infty))$. Therefore all the traces are defined and we may deduce that u is a strong solution of (2.1)-(2.3). $\square$

Remark 2.14 *We have been unable to establish that the energy loss is purely viscous as in the case of the one-dimensional viscously damped wave equation on the half-line and with unilateral boundary conditions [6, 7].*

3 The evolution of a Kelvin-Voigt material with Signorini boundary conditions

As for the damped wave equation with unilateral boundary conditions, *a priori* estimates on the penalized problem and care relative due to the unboundedness of Ω enable us to pass to the limit in the penalized variational formulation and to deduce the existence of a solution to (1.9). Korn's inequality plays here an important rôle. If we denote by $\bar{u}$ the solution of (1.9a) with initial data (1.9d) and Dirichlet boundary data at $x_1 = 0$ then we establish that the trace $-(a^0_{11kl}\varepsilon_{kl}(\bar{u}) + a^1_{11kl}\varepsilon_{kl}(\bar{u}_t))|_{\Sigma\times[0,\infty)}$ increases exponentially with time in $L^2_{\text{loc}}(\Sigma \times [0,\infty))$and not polynomially as in the case of the damped wave equation with Dirichlet boundary conditions studied in Subsection 2.4. We determine the trace spaces using analogous techniques already developed in Section 2.5 but here we perform a Fourier transform in the tangential variables (x_2, x_3, t) and a Laplace transform in x_1.

3.1 The penalized problem

We approximate (1.9) as in Section 2.1. More precisely, let $r^+ = \max(r, 0)$, we replace u by u^ε which is solution of the following penalized problem:

$$\rho u^\epsilon_{tt} - A^0 u^\epsilon - A^1 u^\epsilon_t = f,\ x \in \Omega,\ t > 0, \tag{3.1}$$

with initial data

$$u^\epsilon(\cdot, 0) = v_0 \quad \text{and} \quad u^\epsilon_t(\cdot, 0) = v_1, \tag{3.2}$$

and boundary conditions

$$a^0_{11kl}\varepsilon_{kl}(u^\epsilon) + a^1_{11kl}\varepsilon_{kl}(u^\epsilon_t) = -(u^\epsilon_1)^+/\epsilon, \tag{3.3a}$$

$$a^0_{12kl}\varepsilon_{kl}(u^\epsilon) + a^1_{12kl}\varepsilon_{kl}(u^\epsilon_t) = 0 \quad \text{and} \quad a^0_{13kl}\varepsilon_{kl}(u^\epsilon) + a^1_{13kl}\varepsilon_{kl}(u^\epsilon_t) = 0. \tag{3.3b}$$

Recall that Q_τ and I_τ were defined by (2.4).

Theorem 3.1 *Let $W = \{u \in \mathbf{H}^1_{\text{loc}}([0,\infty) \times \Omega) : \nabla u_t \in \mathbf{L}^2_{\text{loc}}([0,\infty); L^2(\Omega))\}$. Then for each $\epsilon > 0$ there exists a unique weak solution $u^\epsilon \in W$ of the problem* (3.1)-(3.3) *such that*

$$u^\epsilon \in \mathbf{L}^\infty_{\text{loc}}([0,\infty); H^1(\Omega)),$$
$$u^\epsilon_t \in \mathbf{L}^2_{\text{loc}}([0,\infty); H^1(\Omega)),$$
$$u^\epsilon_{tt} \in \mathbf{L}^2_{\text{loc}}([0,\infty); L^2(\Omega)),$$

and for every $\tau \in (0,T)$ and for all $v \in W$, the following variational equality is satisfied:

$$\int_{Q_\tau} \rho u^\epsilon_{tt} \cdot v \, dx\, dt + \int_0^\tau (a^0(u^\epsilon, v) + a^1(u^\epsilon_t, v))\, dt + \int_{I_\tau} \frac{(u^\epsilon_1)^+}{\epsilon} v_1 \, dx'\, dt \geq \int_{Q_\tau} f \cdot v \, dx\, dt. \tag{3.4}$$

Proof. We leave the verification of the proof to the reader as it is analogous to the one developed in [3]. $\square$

3.2 Estimates on the penalized solution

We establish *a priori* estimates which are essential to prove the existence of a weak solution to (3.1)-(3.3). These estimates are obtained thanks to the techniques already developed in Section 2.2 for the damped wave equation and to Korn's inequality.

Lemma 3.2 *Assume that f belongs to $\mathbf{L}^2_{\mathrm{loc}}\big([0,\infty);L^2(\Omega)\big)$, v_0 to $\mathbf{H}^1(\Omega)$ and v_1 to $\mathbf{L}^2(\Omega)$. Then independently of $\epsilon>0$, u^ϵ_t and ∇u^ϵ are bounded in $\mathbf{L}^\infty_{\mathrm{loc}}([0,\infty);L^2(\Omega))$, ∇u^ϵ_t is bounded in $\mathbf{L}^2_{\mathrm{loc}}([0,\infty);L^2(\Omega))$ and $(u^\epsilon_1(0,\cdot,\cdot))^+/\sqrt{\epsilon}$ is bounded in the space $L^\infty_{\mathrm{loc}}([0,\infty);L^2(\mathbb{R}^{d-1}))$.*

Proof. These estimates are a simple application of the Gronwall lemma to the energy estimate. We multiply (3.1) by u^ε_t and integrate this expression over Q_τ to get

$$\begin{aligned}\frac{1}{2}\int_\Omega\big(\rho|u^\epsilon_t|^2+a^0_{ijkl}\varepsilon_{ij}(u^\epsilon)\varepsilon_{kl}(u^\epsilon)\big)\big|_0^\tau\,dx+\int_{Q_\tau}a^1_{ijkl}\varepsilon_{ij}(u^\epsilon_t)\varepsilon_{kl}(u^\epsilon_t)\,dx\,dt\\ +\frac{1}{2\epsilon}\int_\Sigma\big((u^\epsilon_1)^+\big)^2\big|_0^\tau\,dx'=\int_{Q_\tau}f\cdot u^\epsilon\,dx\,dt.\end{aligned}\tag{3.5}$$

According to Korn's inequality, it is possible to infer that there exist two positive constants C_1 and C_2 such that

$$\int_\Omega a^n_{ijkl}\varepsilon_{kl}(z)\varepsilon_{ij}(z)\,dz\geq C_1\int_\Omega|\nabla z|^2\,dz-C_2\int_\Omega|z|^2\,dz,\ n=0,1.$$

As fu^ϵ_t can be estimated by $|f|^2/(2\gamma)+\gamma|u^\epsilon_t|^2/2$, $\gamma>0$, and using the above inequality, we deduce from (3.5) that

$$\begin{aligned}&\frac{1}{2}\int_\Omega\big(\rho|u^\epsilon_t|^2+C_1|\nabla u^\epsilon|^2\big)(\cdot,\tau)\,dx+C_1\int_{Q_\tau}|\nabla u^\epsilon_t|^2\,dx\,dt\\ &+\frac{1}{2\epsilon}\int_\Sigma\big((u^\epsilon_1)^+\big)\big|_0^\tau\,dx'\leq\frac{C_2}{2}\int_\Omega|u^\epsilon(\cdot,\tau)|^2\,dx+\Big(C_2+\frac{\gamma}{2}\Big)\int_{Q_\tau}|u^\epsilon_t|^2\,dx\,dt\\ &+\frac{1}{2\gamma}\int_{Q_\tau}|f|^2\,dx\,dt+\frac{1}{2}\int_\Omega\big(\rho|v_1|^2+a^0_{ijkl}\varepsilon_{ij}(v_0)\varepsilon_{kl}(v_0)\big)\,dx.\end{aligned}$$

A classical Gronwall lemma enables us to deduce that u^ϵ_t and ∇u^ϵ are bounded in the space $\mathbf{L}^\infty_{\mathrm{loc}}([0,\infty);L^2(\Omega))$, ∇u^ϵ_t is bounded in $\mathbf{L}^2_{\mathrm{loc}}([0,\infty);L^2(\Omega))$ and $(u^\epsilon_1(0,\cdot,\cdot))^+/\sqrt{\epsilon}$ is bounded in $L^\infty_{\mathrm{loc}}([0,\infty);L^2(\mathbb{R}^{d-1}))$. □

Remark 3.3 *If we suppose that f vanishes for large t then independently of $\epsilon>0$,*

$$\operatorname*{ess\,sup}_{0\leq t\leq T}|u^\epsilon(\cdot,t)|_{H^1}\leq C(1+T)$$

and

$$\Big(\int_0^T|u^\epsilon_t(\cdot,t)|^2_{H^1}\,dt\Big)^{1/2}\leq C(1+T).$$

These properties can be proved using the arguments given in the proof of Lemma 3.2, with the origin of time moved to T if f vanishes for $t \geq T$; since the integral involving f vanishes, the conclusion is clear.

Lemma 3.4 *Assume that f belongs to $\mathbf{L}^2_{\text{loc}}([0,\infty);L^2(\Omega))$, v_0 to $\mathbf{H}^1(\Omega)$ and v_1 to $\mathbf{L}^2(\Omega)$. Then independently of $\epsilon > 0$, the trace $(u_1^\epsilon(0,\cdot,\cdot))^+/\epsilon$ is bounded in the space of measures on I_T.*

Proof. Let φ be a cut-off function, which belongs to $\mathbf{C}^1(\mathbb{R}^{d-1})$, is equal to 1 in the sphere of center 0 and radius $R > 0$ and vanishes outside of a sphere of radius $R+1$. We multiply (3.1) by φ and we integrate over Q_τ; due to the boundary conditions (3.3), we obtain

$$\int_\Omega \rho u_t^\epsilon \cdot \varphi\Big|_0^\tau dx + \frac{1}{\epsilon}\int_{I_\tau}(u_1^\epsilon)^+\varphi_1\, dx'\, dt + \int_{Q_\tau}\sigma^0_{ij}(u^\epsilon)\varepsilon_{ij}(\varphi)\, dx\, dt + \int_{Q_\tau}\sigma^1_{ij}(u_t^\epsilon)\varepsilon_{ij}(\varphi)\, dx\, dt = \int_{Q_\tau} f\cdot\varphi\, dx\, dt.$$

As the product $|zy|$ can be estimated by $|z|^2/2 + |y|^2/2$, we get the following inequality:

$$\begin{aligned}\frac{1}{\epsilon}\int_{I_\tau}(u_1^\epsilon)^+\varphi_1\, dx'\, dt \leq \frac{\rho}{2}\int_\Omega\left(|u_t^\epsilon(\cdot,\tau)|^2 + |v_1|^2\right) dx + \rho\int_\Omega|\varphi|^2\, dx \\ + \int_{Q_\tau}\left|\left(\sigma^0_{ij}(u^\epsilon) + \sigma^1_{ij}(u^\epsilon_t)\right)\varepsilon_{ij}(\varphi)\right| dx\, dt + \int_{Q_\tau}|f\cdot\varphi|\, dx\, dt.\end{aligned} \tag{3.6}$$

We may deduce that the right hand side of (3.6) is bounded using the Lemma 3.2. Since $(u_1^\epsilon(0,\cdot,\cdot))^+$ is non negative, the conclusion is clear. □

Lemma 3.5 *Assume that f, v_0 and v_1 belong respectively to $\mathbf{L}^2_{\text{loc}}([0,\infty);L^2(\Omega))$, $\mathbf{H}^2(\Omega)$, $\mathbf{L}^2(\Omega)$. Then independently of $\epsilon > 0$, A^0u^ϵ and A^1u^ϵ are bounded in $\mathbf{L}^\infty_{\text{loc}}([0,\infty), L^2(\Omega))$.*

Proof. Once again we use energy techniques, but now we multiply relation (3.1) by A^1u^ϵ and we integrate over Q_τ, we obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega|A^1u^\epsilon(\cdot,\tau)|^2\, dx = \frac{1}{2}\int_\Omega|A^1v_0|^2\, dx + \int_{Q_\tau}\rho u^\epsilon_{tt}\cdot(A^1u^\epsilon)\, dx\, dt \\ - \int_{Q_\tau}(A^0u^\epsilon)\cdot(A^1u^\epsilon)\, dx\, dt - \int_{Q_\tau} f\cdot(A^1u^\epsilon)\, dx\, dt.\end{aligned} \tag{3.7}$$

We observe that

$$\begin{aligned}\int_{Q_\tau}\rho u^\epsilon_{tt}\cdot(A^1u^\epsilon)\, dx\, dt = \rho\int_\Omega u^\epsilon_t\cdot(A^1u^\epsilon)\Big|_0^\tau dx \\ -\rho\int_{I_\tau}u^\epsilon_{1,t}\sigma^1_{1j}(u^\epsilon)\, dx'\, dt + \int_{Q_\tau}a^1_{ijkl}\varepsilon_{ij}(u^\epsilon_t)\varepsilon_{kl}(u^\epsilon_t)\, dx\, dt.\end{aligned} \tag{3.8}$$

Carrying (3.8) into (3.7) and using the boundary conditions (3.3), we obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega |A^1u^\epsilon(\cdot,\tau)|^2\,dx &= \frac{1}{2}\int_\Omega |A^1v_0|^2\,dx - \int_{Q_\tau}(A^0u^\epsilon)\cdot(A^1u^\epsilon)\,dx\,dt \\ -\int_{Q_\tau} f\cdot(A^1u^\epsilon)\,dx\,dt &+ \rho\int_\Omega u_t^\epsilon\cdot(A^1u^\epsilon)\big|_0^\tau\,dx + \frac{\rho}{\epsilon}\int_{I_\tau} u_{1,t}^\epsilon(u_1^\epsilon)^+\,dx'\,dt \\ &+ \rho\int_{I_\tau} u_{1,t}^\epsilon\sigma_{1j}^0(u^\epsilon)\,dx'\,dt + \int_{Q_\tau} a_{ijkl}^1\varepsilon_{ij}(u_t^\epsilon)\varepsilon_{kl}(u_t^\epsilon)\,dx\,dt.\end{aligned} \tag{3.9}$$

On the other hand, we observe that

$$\int_{I_\tau}|\sigma_{1j}^0(u^\epsilon)|^2\,dx'\,dt \le C\left(\int_{Q_\tau}|u^\epsilon|^2\,dx\,dt + \int_{Q_\tau}|A^1u^\epsilon|^2\,dx\,dt\right), \tag{3.10}$$

and for all v belonging to $\mathbf{H}^1(\Omega)$ and A^1v belonging to $\mathbf{L}^2(\Omega)$, we get

$$|A^0v|_{L^2(\Omega)} \le C|v|_{L^2(\Omega)} + |A^1v|_{L^2(\Omega)}. \tag{3.11}$$

Define

$$F(t) = \int_\Omega |A^1u^\epsilon(\cdot,t)|^2\,dx. \tag{3.12}$$

According to (3.10)-(3.12) and since $u_t^\epsilon\cdot(A^1u^\epsilon)$ can be estimated by $|u_t^\epsilon|^2/(2\gamma) + \gamma|A^1u^\epsilon|^2/2$, $\gamma > 0$, it is possible to infer from (3.9) the following inequality:

$$\begin{aligned}\left(\frac{1}{2}-\frac{\rho\gamma}{2}\right)F(\tau) &\le \frac{1}{2}F(0) + (2+C)\int_0^\tau F(t)\,dt + \frac{1}{2}\int_{Q_\tau}|f|^2\,dx\,dt \\ +\frac{\rho}{2\gamma}\int_\Omega |u_t^\epsilon(\cdot,\tau)|^2\,dx &+ \rho\int_\Omega |v_1\cdot(A^1v_0)|\,dx + \int_{Q_\tau} a_{ijkl}^1\varepsilon_{ij}(u_t^\epsilon)\varepsilon_{kl}(u_t^\epsilon)\,dx\,dt \\ &+\frac{\rho}{2\varepsilon}\int_\Sigma \left(u_1^\epsilon(\cdot,\tau)\right)^+\,dx' + (1+C)\int_{Q_\tau}|u^\epsilon|^2\,dx\,dt + \int_{I_\tau}|u_t^\epsilon|^2\,dx'\,dt.\end{aligned}$$

If we choose γ such that $\rho\gamma < 1$, we may infer using Lemma 3.2 and a classical Gronwall inequality that F is bounded in $L^\infty_{\mathrm{loc}}([0,\infty))$. This proves the Lemma. □

Remark 3.6 *If we suppose that f vanishes for t large, then, independently of $\epsilon > 0$, A^0u^ϵ and A^1u^ϵ are polynomially increasing. These properties can be proved using the arguments given in Remark 3.3.*

Let us turn now to interior estimates.

Lemma 3.7 *Assume that f belongs to $\mathbf{L}^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))$, v_0 to $\mathbf{H}^2(\Omega)$, v_1 to $\mathbf{L}^2(\Omega)$. Then for all $\beta > 0$, u_{tt}^ϵ, $A^1u_t^\epsilon$ are bounded in $\mathbf{L}^2([0,\infty); L^2((-\infty,-\beta)\times\Sigma))$, independently of $\epsilon > 0$.*

Proof. As for the proof of Lemma 2.7, we use here a truncation function which enables us to forget about the strongly nonlinear boundary conditions. More precisely, we multiply u^ϵ by a cutoff function $\varphi(x_1) \in C^\infty([0,\infty))$ which is equal to 0 on $x_1 \le -\beta$ and to 1 on $x_1 \ge -\beta/2$, $\beta > 0$. Define

$$v^\epsilon(x_1,\cdot,\cdot) = \varphi(x_1)u^\epsilon(x_1,\cdot,\cdot). \tag{3.13}$$

The derivatives of v^ϵ are given by:

$$\begin{aligned} v^\epsilon_{tt} &= \varphi u^\epsilon_{tt}, && (3.14a)\\ \varepsilon_{kl,x_j}(v^\epsilon) &= \varphi\varepsilon_{kl,x_j}(u^\epsilon) + 2\varphi_{x_1}\varepsilon_{kl}(u^\epsilon) + \varphi_{x_1x_1}u^\epsilon_k, && (3.14b)\\ \varepsilon_{kl,x_j}(v^\epsilon_t) &= \varphi\varepsilon_{kl,x_j}(u^\epsilon_t) + 2\varphi_{x_1}\varepsilon_{kl}(u^\epsilon_t) + \varphi_{x_1x_1}u^\epsilon_{k,t}. && (3.14c) \end{aligned}$$

Notice that thanks to relations (3.1) and (3.14), we have

$$v^\epsilon_{tt} - A^0v^\epsilon - A^1v^\epsilon = \widetilde{g}^\epsilon, \tag{3.15}$$

where $\widetilde{g}^\varepsilon_i = \varphi f_i - 2\varphi_{x_1}(a^0_{ijkl}\varepsilon_{kl}(u^\epsilon) + a^1_{ijkl}\varepsilon_{kl}(u^\epsilon_t)) - \varphi_{x_1x_1}(a^0_{ijkl}u^\epsilon_k + a^1_{ijkl}u^\epsilon_{k,t})$. Thanks to Lemma 3.2, we deduce that $\widetilde{g}^\varepsilon$ is bounded in $\mathbf{L}^2_{\text{loc}}([0,\infty);L^2(\Omega))$. Define

$$w^\epsilon = v^\epsilon_t \quad \text{and} \quad g^\epsilon = \widetilde{g}^\epsilon + A^0v^\epsilon. \tag{3.16}$$

We substitute (3.16) in (3.15), we obtain

$$w^\epsilon_t - A^1w^\epsilon = g^\epsilon. \tag{3.17}$$

We will prove that w^ε_t is bounded in $\mathbf{L}^2_{\text{loc}}([0,\infty);L^2(\Omega))$. For this purpose, we multiply (3.17) by w^ε_t; we integrate this expression over Q_τ, we obtain

$$\int_{Q_\tau}|w^\epsilon_t|^2\,dx\,dt - \int_{Q_\tau}(A^1w^\epsilon)\cdot w^\epsilon_t\,dx\,dt = \int_{Q_\tau} g^\epsilon\cdot w^\epsilon_t\,dx\,dt.$$

As

$$\int_{Q_\tau}(A^1w^\epsilon)\cdot w^\epsilon_t\,dx\,dt = -\frac{1}{2}\int_\Omega a^1_{ijkl}\varepsilon_{ij}(w^\epsilon)\varepsilon_{kl}(w^\epsilon)\Big|_0^\tau\,dx, \tag{3.18}$$

we infer that

$$\begin{aligned} &\int_{Q_\tau}|w^\epsilon_t|^2\,dx\,dt + \frac{1}{2}\int_\Omega a^1_{ijkl}\varepsilon_{ij}(w^\epsilon)\varepsilon_{kl}(w^\epsilon)\Big|_{t=\tau}\,dx\\ &= \frac{1}{2}\int_\Omega a^1_{ijkl}\varepsilon_{ij}(w^\epsilon)\varepsilon_{kl}(w^\epsilon)\Big|_{t=0}\,dx + \int_{Q_\tau} g^\epsilon\cdot w^\epsilon_t\,dx\,dt. \end{aligned} \tag{3.19}$$

According to Korn's inequality, we infer that there exists C_1 and C_2 such that

$$\int_\Omega a^1_{ijkl}\varepsilon_{ij}(w^\epsilon)\varepsilon_{kl}(w^\epsilon)\,dx \ge C_1\int_\Omega|\nabla w^\epsilon|^2\,dx - C_2\int_\Omega|w^\epsilon|^2\,dx. \tag{3.20}$$

Carrying the above inequality into (3.19) and observing that $g^\epsilon \cdot w_t^\epsilon$ can be estimated by $|g^\epsilon|^2/2 + |w_t^\epsilon|^2/2$, we get

$$\begin{aligned}\int_{Q_\tau} |w_t^\epsilon|^2 \, dx \, dt + C_1 \int_\Omega |\nabla w^\epsilon(\cdot,\tau)|^2 \, dx \leq \int_\Omega a^1_{ijkl} \varepsilon_{ij}(w^\epsilon) \varepsilon_{kl}(w^\epsilon)\big|_{t=0} \, dx \\ + C_2 \int_\Omega |w^\epsilon(\cdot,\tau)|^2 \, dx + \int_{Q_\tau} |g^\epsilon|^2 \, dx \, dt.\end{aligned} \tag{3.21}$$

As v_0 belongs to $\mathbf{H}^2(\Omega)$, v_1 belongs to $\mathbf{H}^1(\Omega)$, φ belongs to $C_0^\infty(\mathbb{R})$, g^ε is bounded in $\mathbf{L}^2_{\text{loc}}([0,\infty); L^2(\Omega))$, we infer that the right hand side of (3.21) is bounded. Therefore using identities (3.13) and (3.16), it is possible to deduce that u_{tt}^ε is bounded in $\mathbf{L}^2_{\text{loc}}([0,\infty); L^2((-\infty,-\beta) \times \Sigma))$.

We will show that $A^1 w^\epsilon$ is bounded in $\mathbf{L}^2_{\text{loc}}([0,\infty); L^2(\Omega))$ using an analogous method. We multiply (3.17) by $A^1 w^\epsilon$, we integrate over Q_τ, we obtain

$$\int_{Q_\tau} w_t^\epsilon \cdot (A^1 w^\epsilon) \, dx \, dt - \int_{Q_\tau} |A^1 w^\epsilon|^2 \, dx \, dt = \int_{Q_\tau} g^\epsilon \cdot (A^1 w^\epsilon) \, dx \, dt. \tag{3.22}$$

Carrying (3.18) and (3.20) into (3.22) and $g^\epsilon \cdot (A^1 w^\epsilon)$ being estimated by $|g^\epsilon|^2/2 + |A^1 w^\epsilon|^2/2$, we obtain

$$\begin{aligned}\int_{Q_\tau} |A^1 w^\epsilon|^2 \, dx \, dt + C_1 \int_\Omega |\nabla w^\epsilon(\cdot,\tau)|^2 \, dx \leq \int_\Omega a^1_{ijkl} \varepsilon_{ij}(w^\epsilon) \varepsilon_{kl}(w^\epsilon)\big|_{t=0} \, dx \\ + C_2 \int_\Omega |w^\epsilon(\cdot,\tau)|^2 \, dx + \int_{Q_\tau} |g^\epsilon|^2 \, dx \, dt.\end{aligned} \tag{3.23}$$

Thanks to (3.13) and (3.16), we may deduce from (3.23) that $A^1 u_t^\epsilon$ is bounded in $\mathbf{L}^2_{\text{loc}}([0,\infty); L^2((-\infty,-\beta) \times \Sigma))$. $\square$

3.3 Existence of a weak solution

Thanks to the estimates obtained in Section 3.2, we are able to pass to the limit in the variational formulation associated to the penalized problem (3.1)-(3.3). Therefore it a routine to deduce that there exists a solution to (1.9).

Because Ω is an unbounded set, the proof will be technical but similar to the one developed in Section 2.3.

Theorem 3.8 *Assume that f belongs to $\mathbf{L}^2_{\text{loc}}([0,\infty); L^2(\Omega))$, v_0 to $\mathbf{H}^2(\Omega)$, v_1 to $\mathbf{L}^2(\Omega)$. Then there exists a solution to the variational inequality* (1.10)*; this solution is the limit of a subsequence of the penalty approximation defined by* (3.1)-(3.3).

Proof. Let $\varphi \in C_0^\infty(\bar{\Omega} \times [0,\infty))$ be a function which takes its values between 0 and 1. We suppose here that v belongs to K. Multiplying (3.1) by $(v - u^\epsilon)\varphi$ and integrating over Q_τ and

$$\int_{I_\tau} (u_1^\epsilon)^+ (\varphi(v_1 - u_1^\epsilon)) \, dx' \, dt = -\int_{I_\tau} \left((u_1^\epsilon)^+\right)^2 \varphi \, dx' \, dt + \int_{I_\tau} (u_1^\epsilon)^+ \varphi v_1 \, dx' \, dt$$

being negative, then we get the following inequality:

$$\begin{aligned}\int_\Omega \rho u_t^\epsilon \cdot (\varphi(v-u^\epsilon))|_0^\tau \, dx - \int_{Q_\tau} \rho u_t^\epsilon \cdot (\varphi(v-u^\epsilon))_t \, dx \, dt \\ + \int_{Q_\tau} (a_{ijkl}^0 \varepsilon_{kl}(u^\epsilon)\varepsilon_{ij}(u^\epsilon) + a_{ijkl}^1 \varepsilon_{kl}(u_t^\epsilon)\varepsilon_{ij}(u^\epsilon))(\varphi(v_i - u_i^\epsilon)) \, dx \, dt \\ \geq \int_{Q_\tau} f \cdot \big(\varphi(v-u^\epsilon)\big) \, dx \, dt.\end{aligned} \tag{3.24}$$

We may deduce from Lemmas 3.2 and 3.5 that there exists a subsequence, still denoted by u^ϵ, such that

$$u^\varepsilon \rightharpoonup u \quad \text{in} \quad \mathbf{L}^2_{\text{loc}}([0,\infty); L^2(\Omega)) \quad \text{weak} \quad *, \tag{3.25a}$$

$$u_t^\varepsilon \rightharpoonup u_t \quad \text{in} \quad \mathbf{L}^\infty_{\text{loc}}([0,\infty); L^2(\Omega)) \quad \text{weak} \quad *, \tag{3.25b}$$

$$\nabla u^\varepsilon \rightharpoonup \nabla u \quad \text{in} \quad \mathbf{L}^2_{\text{loc}}([0,\infty); L^2(\Omega)) \quad \text{weak} \quad *, \tag{3.25c}$$

$$A^n u^\varepsilon \rightharpoonup A^n u \quad \text{in} \quad \mathbf{L}^\infty_{\text{loc}}([0,\infty); L^2(\Omega)) \quad \text{weak} \quad *, \quad n = 0,1, \tag{3.25d}$$

$$\nabla u_t^\varepsilon \rightharpoonup \nabla u_t \quad \text{in} \quad \mathbf{L}^2_{\text{loc}}([0,\infty); L^2(\Omega)) \quad \text{weak} \quad * . \tag{3.25e}$$

Thanks to the classical compactness properties of Sobolev spaces injections on bounded open sets, we see that for all $R > 0$, the restrictions of u^ε and $a_{ijkl}^n \varepsilon_{kl}(u^\epsilon)$, $n = 0,1$, to $Q_R = \{x : x_1 < 0, |x'| \leq R\} \times [0,R]$ (a set which has already been defined in the Section 2.3), converge strongly to their respective limits in $\mathbf{L}^2(Q_R)$. On the other hand, using the same techniques as those of Section 2.3, we may prove that u^ϵ converges strongly to u in $\mathbf{L}^2_{\text{loc}}([0,\infty); L^2(\Omega))$. The complete proof can be found in [5], p. 113-115.

We observe now that since u and u_t belong to $\mathbf{L}^2_{\text{loc}}([0,\infty); L^2(\Omega))$, we may replace φ by φ_R in the variational inequality where φ_R is equal to 1 over the set Q_R and vanishes outside of Q_{R+1}. When R tends to infinity all the terms in (3.24) converge to their limit; thus we have proved the existence of a weak solution. □

Remark 3.9 *As for the damped wave equation with Signorini boundary conditions, the uniqueness is still an open problem.*

3.4 Preliminary results

In this section, we establish estimates on the problem (1.9a) with initial data (1.9d) and the Dirichlet boundary condition which enable us to charaterize the trace spaces in the next section.

Lemma 3.10 *Assume v_0 and v_1 belong respectively to $\mathbf{H}^{5/2}(\Omega)$ and $\mathbf{H}^{3/2}(\Omega)$; then, there exists a function with compact support in t such that the trace of z and z_t on Σ are respectively v_0 and v_1.*

Proof. We extend v_0 and v_1 into functions belonging respectively to $\mathbf{H}^{5/2}(\mathbb{R}^d)$ and $\mathbf{H}^{3/2}(\mathbb{R}^d)$. Then there exists a function Z belonging to $\mathbf{H}^3(\mathbb{R}^d \times [0,\infty))$ such that $Z|_{\mathbb{R}^d\times\{0\}} = v_0$ and $Z_t|_{\mathbb{R}^d\times\{0\}} = v_1$. We select a cutoff function $\varphi \in C^\infty([0,\infty))$ which is equal to 1 on $[0,1]$ and to 0 on $[2,\infty)$, and we define z as the restriction of $\varphi(x)Z(x,t)$ to $\Omega \times [0,\infty)$. □

Lemma 3.11 *Assume v_0 belongs to $\mathbf{H}^{5/2}(\Omega)$, v_1 belongs to $\mathbf{H}^1(\Omega)$ and f belongs to $\mathbf{L}^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))$. Define z as in Lemma 3.10 and let $\bar{u}$ be the solution of* (1.9a) *with initial data* (1.9d) *and boundary condition $\bar{u}(0,\cdot,\cdot) = z(0,\cdot,\cdot)$. Then the trace $\bar{g} = -(a^0_{11kl}\varepsilon_{kl}(\bar{u}) + a^1_{11kl}\varepsilon_{kl}(\bar{u}_t))\big|_{\Sigma\times[0,\infty)}$ is well defined and belongs to the space $L^2_{\mathrm{loc}}([0,\infty); L^2(\Sigma))$. Moreover, there exists $K > 0$ such that $e^{-Kt}\bar{g} \in L^2(\Sigma \times [0,\infty))$.*

Proof. Let $\zeta = \bar{u} - z$ be the solution of the following problem:

$$\rho\zeta_{tt} - A^0\zeta - A^1\zeta_t = F,\ x \in \Omega,\ t > 0, \tag{3.26}$$

where $F = f - \rho\ddot{z} + A^0 z + A^1 z_t$ with initial data $\zeta(\cdot,0) = \zeta_t(\cdot,0) = 0$ and boundary condition $\zeta(0,\cdot,\cdot) = 0$. Multiplying (3.26) by ζ_t and integrating over Q_τ, Korn's inequality enables us to deduce that ζ_t and $\nabla\zeta$ are bounded in $\mathbf{L}^\infty_{\mathrm{loc}}([0,\infty); L^2(\Omega))$, $\nabla\zeta_t$ is bounded in $\mathbf{L}^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))$. If we multiply (3.26) by $A^1\zeta$, we may deduce that $A^0\zeta$ and $A^1\zeta$ are bounded in $\mathbf{L}^\infty_{\mathrm{loc}}([0,\infty); L^2(\Omega))$, arguing as in the proof of Lemma 3.5. On the other hand, we have

$$\int_{Q_\tau} \zeta_{tt} \cdot (A^1\zeta_t)\,dx\,dt = -\frac{1}{2}\int_\Omega a^1_{ijkl}\varepsilon_{kl}(\zeta_t)\varepsilon_{ij}(\zeta_t)\big|_0^\tau\,dx.$$

Therefore, we multiply (3.26) by $A^1\zeta_t$, we integrate over Q_τ, and thanks to the above identity, we get

$$\begin{aligned}&\frac{\rho}{2}\int_\Omega a^1_{ijkl}\varepsilon_{kl}(\zeta_{tt})\varepsilon_{ij}(\zeta_t)\big|_{t=\tau}\,dx + \int_{Q_\tau}|A^1\zeta_t|^2\,dx\,dt\\ &+\frac{1}{2}\int_\Omega (A^0\zeta)\cdot(A^1\zeta)\big|_0^\tau\,dx = \frac{\rho}{2}\int_\Omega a^1_{ijkl}\varepsilon_{kl}(\zeta_t)\varepsilon_{ij}(\zeta_t)\big|_{t=0}\,dx\\ &-\int_{Q_\tau} F\cdot(A^1\zeta_t)\,dx\,dt.\end{aligned}$$

According to Gronwall's lemma, there exists $K > 0$ such that

$$\int_{Q_\tau}|A^1\zeta_t|^2\,dx\,dt \le Ce^{K\tau}\left(|F|^2_{L^2(0,\tau;L^2(\Omega))} + |\xi_t(\cdot,0)|^2_{H^1(\Omega)} + |\xi|^2_{L^2(0,\tau;L^2(\Omega))}\right).$$

The Lemma is now clear. □

3.5 The trace spaces

We proceed as in Section 2.5. A Fourier-Laplace transform and Lemma 3.11 enable us to infer that all the traces can be defined. Therefore it is plain that a weak solution of (1.9) is also a strong one.

Let us remark first that the problem (3.3) can be written under an equivalent form: let us extend by 0 for $t \le 0$ the difference $v^\epsilon = e^{-\nu t}(u^\epsilon - \bar{u})$; then it satisfies

$$\begin{aligned}&\rho(\nu+\partial_t)^2 v_i^\epsilon - ((\lambda^0+\mu^0)+(\lambda^1+\nu^1)(\nu+\partial_t))\operatorname{div} v^\epsilon\\ &-(\mu^0+\mu^1(\nu+\partial_t))\Delta v_i^\epsilon = 0, \quad x\in\Omega, \quad t>0,\end{aligned} \tag{3.27}$$

with boundary conditions at $\{x_1 = 0\}$

$$(\mu^0+\nu\mu^1)(v^\epsilon_{j,x_1}+v^\epsilon_{1,x_j})+\mu^1(v^\epsilon_{j,x_1t}+v^\epsilon_{1,x_jt}) = 0,\ j=2,3, \tag{3.28a}$$

$$\begin{aligned}(\lambda^0+\lambda^1(\nu+\partial_t))\operatorname{div} v^\epsilon + 2(\mu^0+\mu^1(\nu+\partial_t))v^\epsilon_{x_1}&\\ = e^{-\nu t}\bar{g} - \frac{\left(v_1^\epsilon - e^{-\nu t}\bar{u}\right)^+}{\epsilon}&,\end{aligned} \tag{3.28b}$$

and with initial data

$$v^\epsilon(\cdot,0)=0 \quad\text{and}\quad v_t^\epsilon(\cdot,0)=0. \tag{3.29}$$

If v^ϵ is a tempered distribution, we may perform a Fourier transform in the tangential variable (x', t) and a Laplace transform in x_1. Denoting by ξ and ω the dual variables of x' and t and by η the dual variable of x_1, we are led to the system:

$$\begin{aligned}&\rho(\nu+i\omega)^2\widetilde{v}^\epsilon - ((\lambda^0+\mu^0)+(\lambda^1+\mu^1)(\nu+i\omega))\begin{pmatrix}\eta\\ i\xi\end{pmatrix}(\eta, i\xi^T)\widetilde{v}^\epsilon\\ &\quad+(\mu^0+\mu^1(\nu+i\omega))(|\xi|^2-\eta^2)\widetilde{v}^\epsilon = 0.\end{aligned} \tag{3.30}$$

Equation (3.30) is a linear system of equations; we seek its eigenvalues η_i and its eigenvectors ϕ_i:

$$\eta_1^2 = |\xi|^2 + \frac{\rho(\nu+i\omega)^2}{\mu^0+\mu^1(\nu+i\omega)} \quad\text{and}\quad \phi_1 = \begin{pmatrix}0\\ i\xi^\perp\end{pmatrix}, \tag{3.31a}$$

$$\eta_2^2 = |\xi|^2 + \frac{\rho(\nu+i\omega)^2}{\mu^0+\mu^1(\nu+i\omega)} \quad\text{and}\quad \phi_2 = \begin{pmatrix}1\\ 0\end{pmatrix}, \tag{3.31b}$$

$$\eta_3^2 = |\xi|^2 + \frac{\rho(\nu+i\omega)^2}{\lambda^0+2\mu^0+(\lambda^1+2\mu^1)(\nu+i\omega)} \quad\text{and}\quad \phi_3 = \begin{pmatrix}\eta_3\\ i\xi\end{pmatrix}, \tag{3.31c}$$

where $\xi^\perp$ is obtained from ξ by a rotation of $\pi/2$. We choose η_i to be the causal determination of the square root of η_i^2. Let us denote by $\widehat{v}^\epsilon$ the partial Fourier transform of v^ϵ with respect to the tangential variables. As v^ϵ and $\widetilde{v}^\epsilon$ are tempered distributions, $\widehat{v}^\epsilon$ is also tempered; therefore, it can only include factors of the form $e^{\eta_i x_1}$, and thus, it must be of the form

$$\widehat{v}^\epsilon(x_1,\xi,\omega) = \sum_{i=1}^{3}\theta_i(\xi,\omega)\phi_i e^{\eta_i x_1}. \tag{3.32}$$

Our goal now is to determine θ_i. Define $v^\epsilon = (v_1^\epsilon, (v^\epsilon)')$. If we apply a partial Fourier transform in the tangential variable to the boundary condition (3.28a), we obtain

$$(\widehat{v}^\epsilon)'_{x_1}(0,\xi,\omega) = -i\xi\widehat{v}_1^\epsilon(0,\xi,\omega). \tag{3.33}$$

Carrying (3.32) into (3.33), we infer that at $x_1 = 0$,

$$i\xi^\perp\eta_2\theta_1 + i\xi\eta_3\theta_3 = -i\xi(\theta_2 + \eta_3\theta_3),$$

thus it is clear that $\theta_1 = 0$ and $\theta_2 = -2\eta_3\theta_3$. Furthermore relation (3.32) taken at $x_1 = 0$ enables us to deduce that $\theta_3 = -\widehat{v}_1^\epsilon(0,\xi,\omega)/\eta_3$. Finally, we obtain

$$\widehat{v}^\epsilon(x_1,\xi,\omega) = 2\widehat{v}_1^\epsilon(0,\xi,\omega)\phi_2 e^{\eta_2 x_1} - \widehat{v}_1^\epsilon(0,\xi,\omega)\phi_3 e^{\eta_3 x_1}/\eta_3. \tag{3.34}$$

At last using (3.34), the left hand side of (3.28b) can be written as a product of convolution: if we perform a Fourier transform of the left hand side of (3.28b) and since

$$\widehat{v}_{1,x_1}^\epsilon(0,\xi,\omega) = (2\eta_2 - \eta_3)\widehat{v}_1^\epsilon(0,\xi,\omega) \quad \text{and} \quad (\widehat{v}^\epsilon)'(0,\xi,\omega) = -i\xi\widehat{v}_1^\epsilon(0,\xi,\omega)/\eta_3,$$

we obtain

$$\left((\lambda^0 + \lambda^1(\nu+\partial_t))\operatorname{div} v^\epsilon + 2(\mu^0 + \mu^1(\nu+\partial_t))v_{1,x_1}^\epsilon\right)\widehat{\ }(0,\xi,\omega) = \widehat{b}\widehat{v}_1^\epsilon(0,\xi,\omega)$$

where

$$\widehat{b} = (\lambda^0 + 2\mu^0 + (\lambda^1 + 2\mu^1)(\nu + i\omega))(2\eta_2 - \eta_3) + (\lambda^0 + \lambda^1(\nu+i\omega))|\xi|^2/\eta_3.$$

Let $w^\epsilon(x',t)$ be the trace $v^\epsilon(0,x',t)$; then (3.28b) can be written now

$$b * w_1^\epsilon = e^{-\nu t}\bar{g} - \frac{\left(w_1^\epsilon - e^{-\nu t}\bar{u}_1(0,\cdot,\cdot)\right)^+}{\epsilon}. \tag{3.35}$$

Lemma 3.12 *Let $u^\epsilon = (u_1^\epsilon, u_2^\epsilon, u_3^\epsilon)^T$ be the solution of* (3.1)-(3.3a). *Then we may extract a subsequence, still denoted by u_1^ϵ, such that*

$$u_1^\epsilon(0,\cdot,\cdot) \rightharpoonup u_1(0,\cdot,\cdot) \quad \text{weakly} \quad \text{in} \quad H^{1/2,5/4}_{\text{loc}}(\mathbb{R}^{d-1}\times[0,\infty)).$$

Moreover u is a strong solution of (1.9).

Proof. We denote by $\widehat{\psi}$ and $\widehat{g}$ the respective Fourier transforms of $\psi = \lambda^0 + 2\mu^0 + (\lambda^1 + 2\mu^1)(\nu + \partial_t)$ and $g = e^{-\nu t}\bar{g}$. Multiplying (3.35) by ψw_1^ϵ and using Plancherel identity, we obtain

$$\frac{1}{(2\pi)^d}\Re\int_{\mathbb{R}^d}\overline{\widehat{\psi}}\,\widehat{b}|\widehat{w}_1^\epsilon|^2\,d\xi\,d\omega = \frac{1}{(2\pi)^d}\Re\int_{\mathbb{R}^d}\widehat{g}\overline{\widehat{\psi}\widehat{w}_1^\epsilon}\,d\xi\,d\omega$$
$$-\int_0^\infty\int_{\mathbb{R}^{d-1}}\frac{\left(w_1^\epsilon - e^{-\nu t}\bar{u}_1(0,\cdot,\cdot)\right)^+}{\epsilon}\psi w_1^\epsilon\,dx'\,dt.$$

According to Cauchy-Schwarz's inequality and since $(u_1^\epsilon(0,\cdot,\cdot))^+/\sqrt{\epsilon}$ is bounded in $L^\infty_{\rm loc}([0,\infty);L^2(\mathbb{R}^{d-1}))$, the absolute value of the second integral on the right hand side of the above inequality is bounded by C_1, therefore we get

$$\frac{1}{(2\pi)^d}\Re\int_{\mathbb{R}^d}\overline{\widehat{\psi}}\,\widehat{b}|\widehat{w}_1^\epsilon|^2\,d\xi\,d\omega \le C_1+\frac{1}{(2\pi)^d}\Re\int_{\mathbb{R}^d}\widehat{g}\overline{\widehat{\psi}\widehat{w}_1^\epsilon}\,d\xi\,d\omega. \tag{3.36}$$

Define

$$\kappa=\frac{-\rho(\nu+i\omega)^2-2|\xi|^2(\mu^0+(\nu+i\omega)\mu^1)}{-\rho(\nu+i\omega)^2-|\xi|^2\big(\lambda^0+2\mu^0+(\nu+i\omega)(\lambda^1+2\mu^1)\big)},$$
$$x_0=\sqrt{\frac{2\rho(\lambda^0+\nu\lambda^1)}{4\lambda^1\mu^1+(\lambda^1)^2}}.$$

Then $\widehat{b}=\widehat{\psi}(2\eta_2-\kappa\eta_3)$ and we remark also that it is sufficient to find a function h which depends on ξ and ω such that $\Re(2\eta_2-\kappa\eta_3)\ge|h|$. If we assume that $|\xi|+|\omega|\gg 1$, we have two cases to consider according to the values taken by $|\xi|$. We suppose first that $|\xi|^2+2\rho(\nu\mu^1-\mu^0)/(\mu^1)^2\ge 0$; then η_2 can be approximated by $\widetilde{\eta}_2$ defined as follows:

$$|\widetilde{\eta}_2^2|^2=\left(|\xi|^2+\frac{\rho(\nu\mu^1-\mu^0)}{(\mu^1)^2}\right)^2+\left(\frac{\rho\mu^1\omega}{(\mu^1)^2}\right)^2.$$

Therefore it is easy to deduce that $|\widetilde{\eta}_2^2|^2\ge|\xi|^4/4+\rho^2\omega^2/(\mu^1)^2$ and then, in the case $|\xi|^2+2\rho(\nu\mu^1-\mu^0)/(\mu^1)^2\ge 0$, we obtain the following estimate:

$$\Re\eta_2\ge\cos\big(\pi/4\big)|\eta_2|\ge\frac{1}{\sqrt{2}}\left(\frac{|\xi|^4}{4}+\frac{\rho^2\omega^2}{(\mu^1)^2}\right)^{1/4}. \tag{3.37}$$

In the other case, we suppose $|\xi|^2+2\rho(\nu\mu^1-\mu^0)/(\mu^1)^2\le 0$; then, it is plain that

$$|\Re\eta_2^2|\le\frac{3\rho(\nu\mu^1+\mu^0)}{(\mu^1)^2}\quad\text{and}\quad|\Im\eta_2^2|\ge\frac{\rho\mu^1|\omega|}{(\mu^1+\nu\mu^0)^2+(\mu^0)^2},$$

which implies that there exists $C>0$ such that

$$|\mathrm{arc\,cotan}\eta_2^2|\le\frac{3(\nu\mu^1+\mu^0)\big((\mu^1+\nu\mu^0)^2+(\mu^0)^2\big)}{(\mu^1)^3|\omega|}\le\frac{C}{|\omega|}.$$

We deduce from the above inequality and from $|\eta_2|^2\ge C|\omega|$ that $|\arg\eta_2^2|\le\pi/2+C/|\omega|$ and thus $\cos(\arg\eta_2)\ge 1/2$. In the case $|\xi|^2+2\rho(\nu\mu^1-\mu^0)/(\mu^1)^2\le 0$, we get

$$\Re\eta_2\ge C\sqrt{|\omega|}/2. \tag{3.38}$$

Therefore in both cases, we infer from (3.37) and (3.38) that there exists $M>0$ such that

$$\Re\eta_2\ge M\left(\omega^2+|\xi|^4\right)^{1/4}. \tag{3.39}$$

Furthermore, there exists $C > 0$ such that $|\kappa|^2 \leq 1 + C1_{\{|\xi|\leq x_0\}}/|\omega|^2$ and for $|\xi|$ large enough, $|\eta_2| \geq |\eta_3|$. Then (3.39) enables us to deduce

$$\Re(2\eta_2 - \kappa\eta_3) \geq M \left(\omega^2 + |\xi|^4\right)^{1/4}. \tag{3.40}$$

Carrying (3.40) into (3.36), we obtain

$$M \int_{\mathbb{R}^d} |\widehat{\psi}|^2 \left(\omega^2 + |\xi|^4\right)^{1/4} |\widehat{w}_1^\varepsilon|^2 \, d\xi \, d\omega \leq C_1 + \int_{\mathbb{R}^d} |\widehat{g}||\widehat{\psi}||\widehat{w}_1^\epsilon| \, d\xi \, d\omega. \tag{3.41}$$

We estimate the product zy by $|z|^2/(2\gamma) + \gamma|y|^2/2$, $\gamma > 0$, we see that

$$\left(M - \frac{\gamma}{2}\right) \int_{\mathbb{R}^d} |\omega|^2 \left(\omega^2 + |\xi|^4\right)^{1/4} |\widehat{w}_1^\varepsilon|^2 \, d\xi \, d\omega \leq C_1 + \frac{1}{2\gamma} \int_{\mathbb{R}^d} \frac{|\widehat{g}|^2}{\left(\omega^2 + |\xi|^4\right)^{1/4}} \, d\xi \, d\omega.$$

We choose γ such that $\gamma < 2M$. On the other hand, $e^{-Kt}\bar{g}(\cdot, t)$ is bounded in $L^2(\Sigma \times [0, \infty))$, so that $g(\cdot, t)$ is bounded in $L^2(\Sigma \times [0, \infty))$ if we choose $\nu > K$. Therefore u_1^ϵ is bounded in $H^{1/2,5/4}_{\mathrm{loc}}(\Sigma \times [0, \infty))$. In particular, $(\lambda^0 + \lambda^1(\nu + \partial_t)) \operatorname{div} v^\epsilon + 2(\mu^0 + \mu^1(\nu + \partial_t))v^\epsilon_{x_1}$ is bounded in $H^{-1/2,-1/4}_{\mathrm{loc}}(\Sigma \times [0, \infty))$. We conclude that u is a strong solution of (1.9) because all the traces can be defined. $\square$

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