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Destabilization patterns in large regular networks

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Abstract

We describe a generic mechanism for the destabilization in large regular networks of identical coupled oscillators. Based on a reduction method for the spectral problem, we first present a criterion for this type of destabilization. Then, we investigate the related bifurcation scenario, showing the existence of a large number of coexisting periodic solutions with different frequencies, spatial patterns, and stability properties. Even for unidirectional coupling this can be understood in analogy to the well-known Eckhaus scenario for diffusive systems.

Networks of coupled oscillators have received a lot of attention over the last decade [9]. Their study can contribute to the understanding of fundamental dynamical features in coupled systems of many kinds, ranging from atoms or neurons to lasers, living organisms [12, 6], or even electrical power stations. The central question is to understand, how specific properties of the individual behavior and the coupling architecture can give rise to the emergence of new collective phenomena. The synchronization and desynchronization are examples of such collective phenomena, which has been extensively investigated (see [8] and refs. therein). A particularly challenging task is to deal with networks, containing a large number of oscillators [5]. As in the well-known Kuramoto system [4], a description of the behavior of a large network has to be based on structural properties of the network that are independent on the actual size.

In this paper we analyze the behavior of large networks of identical coupled oscillators with a certain regular structure, as for example a ring with unidirectional coupling. Similar to the Master Stability Function of Pecora and Carroll [7], we reduce the stability analysis of the large coupled system to the level of complexity of one single oscillator. In contrast to the approach of Pecora and Carroll, we determine not only the stability boundary, but describe the whole bifurcation scenario. It turns out that in a large network with certain structural properties an oscillatory instability of an individual oscillator induces a destabilization of the whole network leading to the emergence of a large number of coexisting periodic solutions with different frequencies, spatial patterns, and stability properties. As a paradigm for this scenario, we analyze in detail the behavior of a unidirectionally coupled ring of Stuart-Landau oscillators. The observed scenario is similar to the Eckhaus scenario, which is a well-known destabilization mechanism in the context of spatially extended systems with diffusion. The remarkable fact that it can also occur in systems with convective (unidirectional) coupling can be explained by the structure of the spectrum of the linearized system. In the limit of a large number of oscillators, it turns out to be organized by curves being densely filled with eigenvalues. In this way, we are able to show

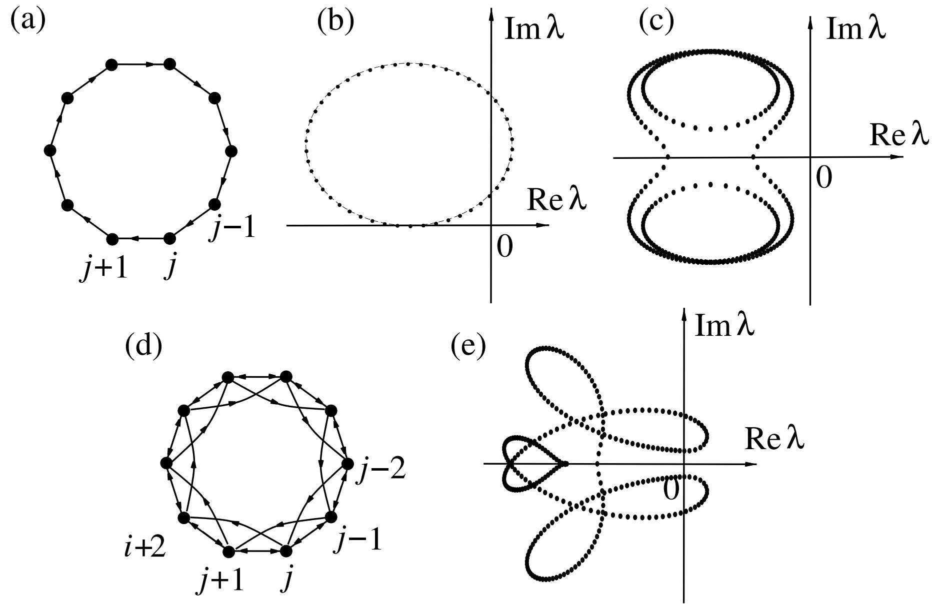

Figure 1: (a) Ring of unidirectionally coupled oscillators. (b) Eigenvalues of the steady state for the ring of 50 unidirectionally coupled SL oscillators. (c) Eigenvalues for the ring of 150 oscillators determined by (2) and (4). (d) More complicated network, corresponding to (2) with nonvanishing A, B_1 as in (4) and $B_{-1} = B_2 = \mathrm{Id}$. (e) Eigenvalues of the steady state for the network from Fig. (d).

that the observed phenomenon is of a generic nature and can be observed in a large class of coupled systems where similar spectral conditions are met.

Stability analysis for a network with ring structure. A general oscillator network with ring structure can be written as follows.

$$\dot{u_j} = Au_j + \sum_m B_m u_{j+m} + H_j(\mathbf{u}), \tag{1}$$

with the oscillator number $j = 1, \dots, N$ and numbers of coupled modes m taken modulo N. Here, $\mathbf{u} = (u_1, \dots, u_N)^T$, $u_j \in R^n$, are state variables of j-th oscillator with steady state in the origin. For convenience, the linear terms are separated and the functions H_j contain only terms of order two or higher. A and B_m are square matrices of size n, describing the linear part of the oscillator itself and of the coupling to the m-th neighbor to the right, respectively. Following Pecora and Carroll [7], we can write the linearized problem in a block diagonalized form

$$\dot{\xi_j} = \left[A + \sum_m \gamma_j^m B_m \right] \xi_j. \tag{2}$$

where $\gamma_j = e^{\frac{2\pi i}{N}j}$ for $j = 1 \ldots N$ are the eigenvalues of the $N \times N$ coupling matrix

$$G = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

appearing as G^m in the coupling to the m-th neighbor. Note that we violate here the condition $\sum_j G_{ij} = 0$, which was used in [7] to distinguish between the synchronous mode and the asynchronous modes. Using the fact that for $N \to \infty$ the roots of unity γ_j are filling densely the unit circle in the complex plane, we replace the discrete eigenvalues γ_j by the continuous parameter $e^{i\varphi}$. In this way, we obtain from (2) the characteristic equation

$$\chi(\lambda, \varphi) := \det \left(\lambda \cdot \mathrm{Id} - \mathrm{A} - \sum_m e^{im\varphi} B_m \right) = 0 \qquad (3)$$

for eigenvalues of the stationary state of large coupled system. This equation is independent on the number of nodes N and determines n closed curves $\lambda(\varphi)$ in the complex plane which become densely filled with eigenvalues for increasing N. We will call these curves pseudo-continuous spectrum (PCS), since there is a strong analogy with the pseudo-continuous spectrum of delay-differential equations [11] with large delay. Figure 1 shows some examples of networks and numerically computed eigenvalues where one can observe the curves of PCS: for the unidirectionally coupled ring of Stuart-Landau oscillators (a) we obtain the spectrum (b), and for the coupling matrices

$$A = \begin{bmatrix} -2 & 1 \\ -1 & -1.5 \end{bmatrix}, \; B_1 = \begin{bmatrix} -1 & 0.3 \\ -0.3 & 1 \end{bmatrix} \qquad (4)$$

the spectrum (c). For the more complicated network (d) we obtain the spectrum (e), using A, B_1 as above and additionally $B_{-1} = B_2 = \mathrm{Id}$.

The notion of the PCS allows the following main conclusions: (i) At a destabilization, the critical part of the spectrum is given by an arc of the PCS crossing the imaginary axis, cf. Fig. 2. (ii) Since the PCS crossing the imaginary axis brings immediately a large number of eigenvalues to the right half-plane, classical bifurcation theory is not suitable to describe the bifurcation scenario exhaustively. Instead, the destabilization in large coupled systems should be described in terms of instabilities of spatially extended systems. Moreover, the situation depicted in Fig. 2 can be considered as a generic scenario for destabilization in large coupled systems. In order to demonstrate the behavior of a system with such properties, we analyze now in detail the example of a unidirectionally coupled ring of Stuart-Landau oscillators.

Coupled Stuart-Landau (SL) oscillators. In the system

$$\dot{z}_j = (\alpha + i\beta) z_j - z_j \left| z_j \right|^2 + e^{i\psi} z_{j-1}, \qquad (5)$$

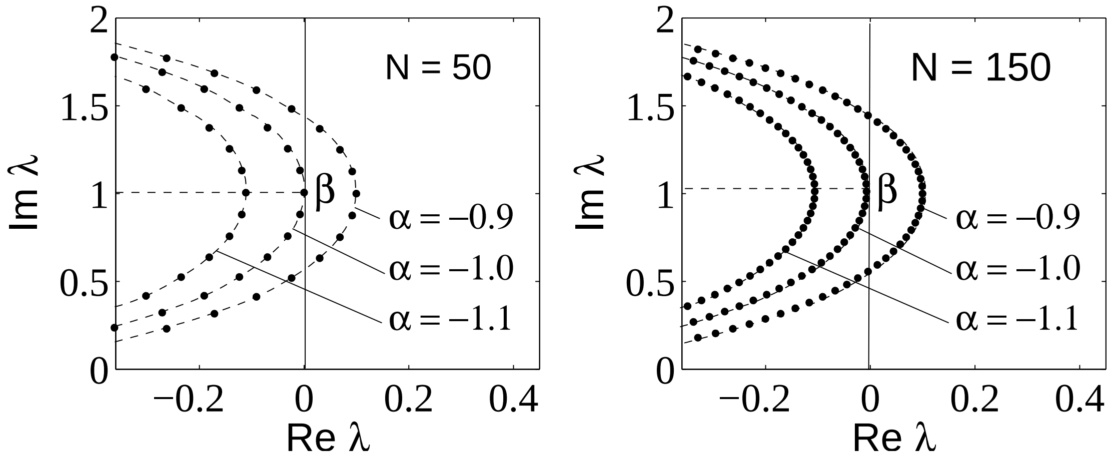

Figure 2: Critical part of the spectrum for unidirectionally coupled SL oscillators with changing control parameter: stable ($\alpha = -1.1$), at bifurcation ($\alpha = -1$), unstable ($\alpha = -0.9$). For increased N (right panel), the PCS curves are unchanged, but more densely filled with eigenvalues. ($\beta = 1$)

$j = 1 \dots N$, each single oscillator undergoes a supercritical Hopf bifurcation for increasing α at $\alpha = 0$ where a stable limit cycle with frequency β emerges. These single oscillators are coupled in a unidirectional ring structure. The coupling strength can be rescaled to unity and the coupling phase ψ remains as an additional parameter. Using the general formula (3), we obtain the eigenvalues of the steady state

$$\lambda_k = \alpha + i\beta + e^{i(\psi+\varphi_k)}, \;\; \varphi_k = 2\pi k/N.$$

They are located on the circle of PCS with the center at $\alpha + i\beta$ and radius 1, cf. Fig. 1(b). By increasing the control parameter α, the PCS shifts to the right and the steady state becomes unstable at $\alpha = -1$. Figure 2 shows the critical part of the spectrum. Note, that the destabilization threshold has been decreased due the mutual coupling ($\alpha = 0$ for the uncoupled oscillators).

Bifurcating multiple periodic solutions. We describe now the bifurcation scenario for increasing α. At $\alpha = -1$, the PCS crosses the imaginary axis (see Fig. 2) and in a sequence of Hopf bifurcations a set of unstable eigenvalues appears. We analyze now the bifurcating periodic solutions and their stability. Due to the symmetry properties of system (5), the bifurcating periodic solutions are rotating waves of the form $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_N)^T$, $\bar{z}_j = Ae^{i(\omega t + j2\pi k/N)}$, where ω is the temporal frequency and every oscillator is phase-shifted by $2\pi k/N$ with respect to the neighboring one. Substituting $\bar{\mathbf{z}}$ into (5), we obtain the equation for unknown real parameters A and ω:

$$i\omega = \alpha + i\beta - A^2 + e^{i(\psi+2\pi k/N)},$$

which has N solutions

$$A_k = \sqrt{\alpha + \cos(\psi + 2\pi k/N)}, \quad (6)$$
$$\omega_k = \beta + \sin(\psi + 2\pi k/N), \quad k = 0, \ldots, N \quad (7)$$

corresponding to periodic solutions $\overline{z}^{(k)}$ with the components $\overline{z}_j^{(k)} = A_k e^{i(\omega_k t + j2\pi k/N)}$. Each of them originates at a Hopf bifurcation at

$$\alpha_k = -\cos(\psi + 2\pi k/N), \quad (8)$$

where $A_k = 0$. Figure 3(I) shows the amplitudes A_k of the bifurcating periodic solutions versus α. The frequencies ω_k of the periodic solutions do not depend on α, therefore the branches are straight lines in the (ω, α) bifurcation diagram in Fig. 3(II). In this diagram, the periodic orbits appear along the circle $\alpha^2 + (\omega - \beta)^2 = 1$ (line H).

The stability analysis of the bifurcating periodic solutions (6-7) can be performed analytically. Omitting technical details, we arrive to the following characteristic equation

$$\left[\Lambda(\varphi_l) + A_k^2 + \cos\left(\psi + 2\pi k/N\right)\left(1 - e^{i\varphi_l}\right)\right]^2 \quad (9)$$
$$= A_k^4 - \sin^2\left(\psi + 2\pi k/N\right)\left(1 - e^{i\varphi_l}\right)^2, \quad (10)$$

which determines N Floquet exponents $\Lambda(\varphi_l)$, $l = 0, \ldots, N-1$ of the periodic solutions. Similarly to the steady state, the Floquet exponents Λ of periodic solutions appear in the form of pseudo-continuous spectrum $\Lambda(\varphi)$, which can be found from (9-10) assuming φ to be a continuous parameter. Deeper analysis of Eqs. (9-10) reveals additional details of the bifurcation scenario, which are presented in Fig 3.

Let us first consider this scenario in more details for the case $\psi = 0$. The role of non-vanishing ψ will be discussed later. The periodic solution with $k = 0$, $A = \sqrt{\alpha + 1}$, $\omega_0 = \beta$ is born stable at $\alpha = -1$ and remains stable for $\alpha > -1$. All other periodic solutions with $k \neq 0$ appear unstable at some$\alpha = \alpha_k > -1$. There is a principal difference between the periodic solutions, which emerge for $-1 < \alpha < 0$ (we will call them *modes*), and those for $0 < \alpha < 1$ (*antimodes*). With increasing α, all modes become eventually stable via a series of subsequent torus bifurcations (cf. Fig. 3). The number of these torus bifurcations again grows with $N \to \infty$. However, the threshold where the modes become stable is independent on the number of oscillators and given by

$$\alpha = 3A^2/2 - \sqrt{1 + A^4/4} \quad (11)$$

in the (α, A)-plane and

$$\alpha = \left(2(\omega - \beta)^2 - 1\right) / \sqrt{1 - (\omega - \beta)^2} \quad (12)$$

in the (α, ω)-plane (line E Fig. 3). Motivated by the similarity to the Eckhaus destabilization mechanism for PDEs of reaction-diffusion type equations in large or unbounded domains [2, 10] and delay equations [11] with large delay, we call this stability boundary

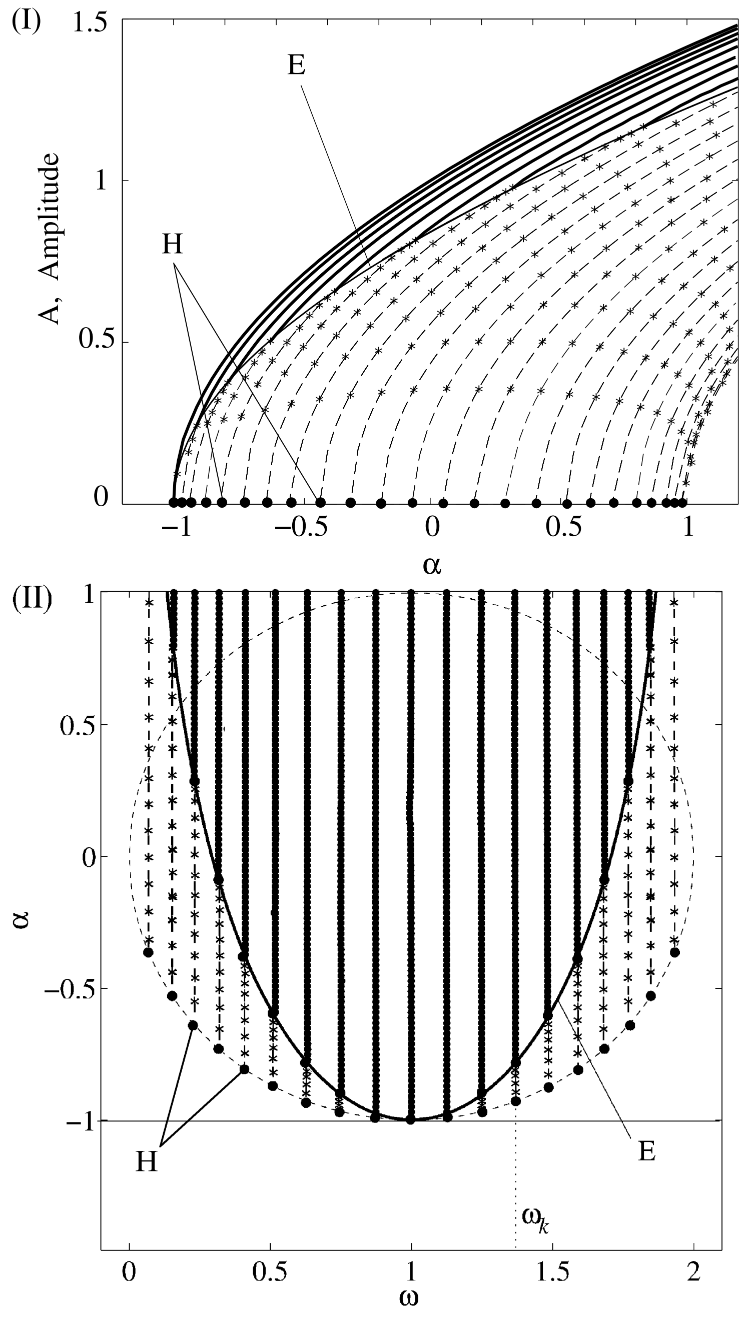

Figure 3: Bifurcation diagram for 50 coupled SL oscillators: Amplitudes (I) and frequencies (II) versus parameterα. Branches of periodic solutions emerge unstable (dashed lines) at Hopf bifurcations (H), undergo a sequence of torus bifurcations (crosses) until they are stable (bold lines) after the Eckhaus-line (E).

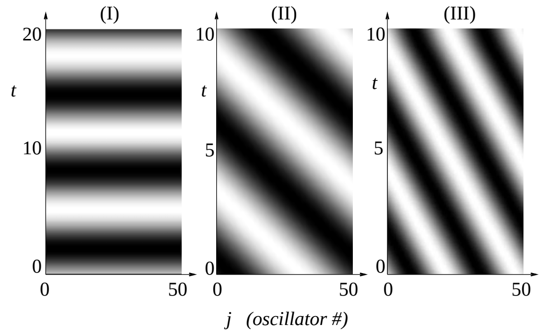

Figure 4: Spatial patterns for some of the bifurcating stable time-periodic solutions (6-7). (I) – homogeneous pattern for $k = 0$. (II) - period-1 pattern for $k = 1$. (III) - period-2 pattern for $k = 2$. Other parameters: $\psi = 0$, $\beta = 1$, and $\alpha = -0.9$

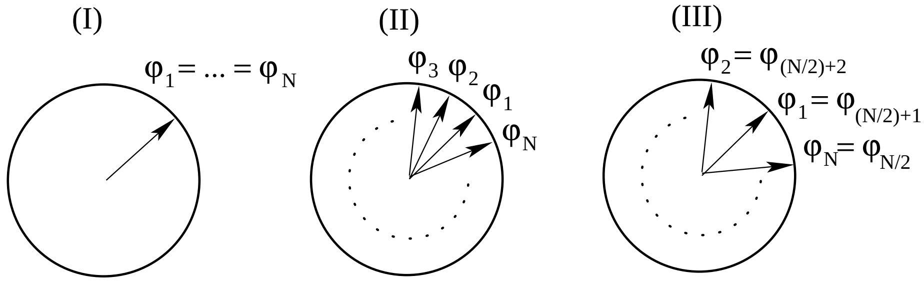

Figure 5: Representation of bifurcating solutions (6-7) in terms of oscillator phases, see also Fig. 4.

E Eckhaus line. The remarkable feature of this bifurcation scenario is that there appears a number proportional to N of coexistent periodic solutions, interacting in a complex competition-cooperation scenario.

Appearance of diffusive spatial patterns. The oscillator number j plays here the role of the space. Recall that the j-th component of the k-th periodic solution is given as $\overline{z}_j^{(k)} = A_k e^{i(\omega_k t + j2\pi k/N)}$. For $k = 0$, this is a completely synchronized solution with $\overline{z}_1^{(0)} = \cdots = \overline{z}_N^{(0)}$, producing a spatially homogeneous pattern, cf. Fig. 4(I). The solutions with $k = 1$ and $k = 2$ produce the patterns given in Fig. 4(II,III). For $k = \frac{N}{2}$, neighboring nodes oscillate in anti-phase. All such patterns corresponding to modes become eventually stable with increasing α. Figure 5 gives an alternative view on the discussed periodic solutions. The existence of such periodic patterns is well-known for systems with diffusion or diffusive coupling [1, 3, 10]. Our result shows the remarkable fact, that such patterns appear generically in a system of convectively coupled oscillators.

The role of ψ. The coupling phase ψ plays a decisive role for the relation of spatial patterns and time frequencies: In the case $\psi = 0$, the most unstable eigenvalue is $\lambda_0 = \alpha + i\beta + 1$ and corresponds to $k = 0$. Therefore, the synchronous periodic solution bifurcates first with increasing α and is most stable. Next the solution with $k = 1$ appears, and so

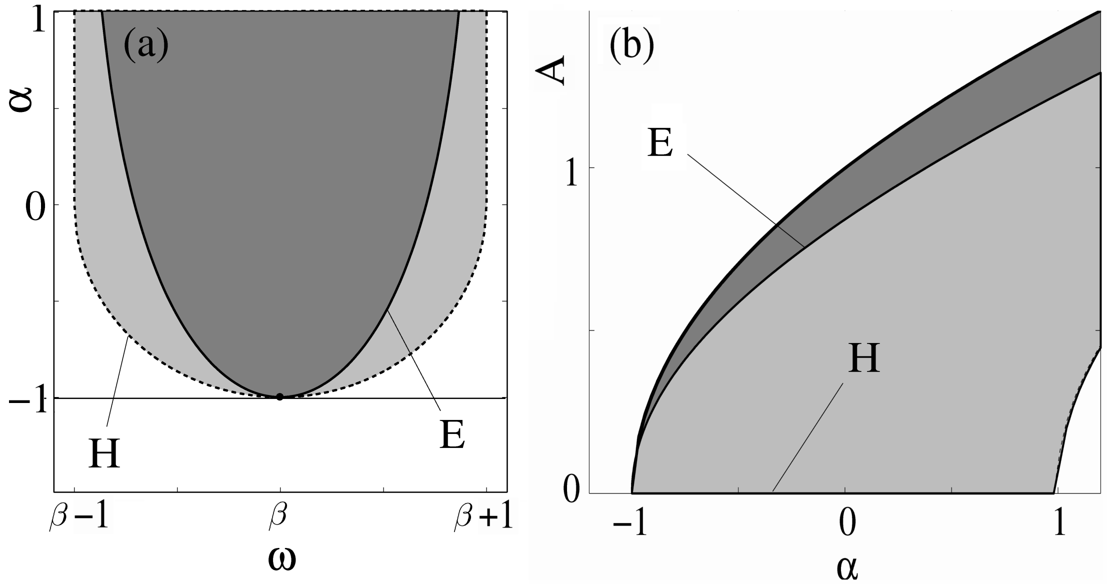

Figure 6: Bifurcation scenario independent on the number of the network nodes. C – appearance of multiple periodic solutions via the bifurcation of the PCS. E – Eckhaus line, at which these periodic orbits become stable. Gray: region of existence of periodic orbits. Dark gray: stability region for multiple periodic orbits.

on, cf. Fig. 4. However, invoking the coupling phase ψ, we note that there is no reason for this specific role of the synchronous mode. Indeed, the first periodic orbit to appear is that with $2\pi k/N \approx -\psi$ and neighboring oscillators are phase-shifted by the value $\approx \psi$. The spatial period of this solution is close to $\psi N/2\pi$. In particular for $\psi = \pi$, the anti-phase synchronized state appears to be the most stable regime.

Universal description of the bifurcation. Independent on the number N of nodes in the network, we can identify the following universal features of the destabilization mechanism: (i) The PCS allows to identify the destabilization threshold, based on a reduced spectral problem. (ii) At the Hopf-line H, cf. Eq. (8), Figs. 3, 6, multiple periodic orbits emerge. (iii) The Eckhaus line E, cf. Eqs. (11), (12), Figs. 3, and 6, gives the stability boundary for the multiple periodic orbits. With increasing number of nodes, only the number of periodic orbits changes proportionally to N, and the corresponding regions in Fig. 3 become more and more densely filled with coexisting states. The main qualitative features survive as indicated in Fig. 6: for α increasing from -1 to 0, the frequency band between $\beta - 1$ and $\beta + 1$ is densely filled with periodic solutions (left panel) and those with the highest amplitudes (b) and frequency closest to β (a) are stable.

Conclusions. This Letter reveals a generic mechanism for the destabilization of large networks. First, we show how to calculate efficiently the spectrum in large networks, and characterize the spectral situation leading to the described destabilization phenomenon. Then we use a simple example to reveal the scenario in detail. Based on our analysis, the following conclusions can be drawn: (i) Multistability is an inherent feature of large coupled systems. (ii) Even for identical oscillators with a fixed intrinsic frequency, in a large network this frequency can split up into a quasi-continuous frequency band of periodic solutions. (iii) The different frequencies come along with spatial patterns of different period. (iv) Among these quasi-continuum of available periodic states, there is a competition-cooperation mechanism with a universal stability boundary which can be understood in analogy to the well-known Eckhaus scenario.

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