Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Variation of constants formula for hyperbolic systems

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submitted: 27. Feb. 2007

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No. 1212
Berlin 2007

2000 *Mathematics Subject Classification.* Primary 35L90, 37L05, 35B30, 58D25 Secondary 37L50, 34K20, 34K19 .

Key words and phrases. Semilinear hyperbolic systems, variations of constants formula, sun star calculus, smooth dependence on data, linarized stability, semigroup theory.

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Abstract

A smooth variation of constants formula for semilinear hyperbolic systems is established using a suitable Banach space X of continuous functions together with its sun dual space $X^{\odot *}$. It is shown that mild solutions of this variation of constants formula generate a smooth semiflow in X. This proves that the stability of stationary states for the nonlinear flow is determined by the stability of the linearized semigroup.

1 Introduction

This paper is concerned with the variation of constants formula for semilinear hyperbolic systems of the following type:

For $0 < x < l$ and $t > 0$

$$\text{(SH)} \begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = K(x)\frac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + H(x,u(t,x),v(t,x)), \\ u(t,0) = E\,v(t,0), \quad v(t,l) = Du(t,l) \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), \end{cases}$$

where $u(t,x) \in \mathbb{R}^{n_1}$, $v(t,x) \in \mathbb{R}^{n_2}$, $K(x) = \operatorname{diag}\,(k_i(x))_{1\le i\le n}$ is a diagonal matrix of functions $k_i \in C^1([0,l],\mathbb{R})$, $k_i(x) < 0$ for $i = 1,\dots,n_1$ and $k_i(x) > 0$ for $i = n_1+1,\dots,n = n_1+n_2$, and $D = (d_{ji})_{\substack{1\le i\le n_1 \\ 1\le j\le n_2}}$, $E = (e_{ij})_{\substack{1\le i\le n_1 \\ 1\le j\le n_2}}$ are matrices.

We consider the prototype system

$$(\mathrm{H}_0) \begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = K(x)\frac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix}, \\ u(t,0) = E\,v(t,0), \quad v(t,l) = Du(t,l) \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x). \end{cases}$$

Let $T(t)$ denote the semigroup for (H_0), obtained by integrating along characteristics (Proposition 2.1).

The nonlinearity $H :]0,l[\times \mathbb{R}^n \to \mathbb{R}^n$, $H = H(x,z)$, $x \in]0,l[$, $z = (u,v) \in \mathbb{R}^n := \mathbb{R}^{n_1+n_2}$, generates a Nemytskij oprator

$$\mathfrak{H}\,(u,v)\,(x) := H(x,u(x),v(x))$$

defined on a suitable function space

$$X \subset \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \mid u :]0,l[\to \mathbb{R}^{n_1},\ v :]0,l[\to \mathbb{R}^{n_2} \right\}.$$

Formally the variation of constants formula for (SH) reads

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = T(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t T(t-s)\mathfrak{H}(u(s), v(s))\, ds. \tag{1}$$

Now the question arises which choice of Banach space X is best suited for (1). It is tempting to take the Hilbert space $L^2(]0,l[\,, \mathbb{R}^n)$. The semigroup $T(t)$ is strongly continuous on L^2. Since the Nemytskij operator $\mathfrak{H}$ is not defined on L^2, a standard procedure then is to truncate the nonlinearity H so that $\mathfrak{H} : L^2 \to L^2$ becomes well defined and globally Lipschitz. However, $\mathfrak{H}$ will not be Fréchet differentiable due to the (rather surprising) fact that $\mathfrak{H} : L^2 \to L^2$ is differentiable at some $(u,v) \in L^2(]0,l[\,, \mathbb{R}^n)$ if and only if for almost all $x \in]0,l[$ the function $z \mapsto H(x,z)$ is affine [3]. Our aim is to establish a smooth variation of constants formula (1) for a large class of nonlinearities H (as appearing in applications) which can be used to prove principle of linearized stability and smooth center manifold theorem.

We choose the phase space

$$X := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in C([0,l], \mathbb{R}^{n_1+n_2}) \mid u(0) = Ev(0),\ v(l) = Du(l) \right\}.$$

$T(t)$ is a strongly continuous semigroup on X, but the Nemytskij operator $\mathfrak{H}$ maps out of X for almost any choice of H. The main idea is to embed X into a larger space $X^{\odot *}$ (called X sun star) which is defined in terms of a combination of properties of the space X and the semigroup T. On $X^{\odot *}$ there acts the sun star semigroup $T^{\odot *}(t) : X^{\odot *} \to X^{\odot *}$ which is an extension of T onto $X^{\odot *}$. Using this extension we arrive at the following

Definition 1.1 (Variation of constants formula). *Let $T > 0$. The pair $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T], X)$ is called a mild (or weak) solution to* (SH) *if*

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = T(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t T^{\odot *}(t-s)\mathfrak{H}(u(s), v(s))\, ds.$$

This variation of constants formula in terms of the spaces X and $X^{\odot *}$ has the following main advantages: First the Nemytskij operator $\mathfrak{H}$ is a smooth map from X into $X^{\odot *}$. And second the weak star integral $\int_0^t T^{\odot *}(t-s)\mathfrak{H}(u(s), v(s))\, ds$ is norm continuous with values in X (Lemma 2.6) so that we get back from the extended space $X^{\odot *}$ into the phase space X. This in a certain sense allows to treat semilinear hyperbolic systems like ordinary differential equations. By applying the implicit function theorem it follows that mild solutions to (SH) generate a smooth Frechet differentiable semiflow in X (section 3, theorem 3.7). In particular this implies that the stability of stationary states for the nonlinear hyperbolic system (SH) is determined by the stability of the linearized semigroup. Moreover, our variation of constants formula can be used to prove smooth center manifold theorem for (SH) [5].

We calculate representations of $X^{\odot}$ and $X^{\odot *}$ for hyperbolic systems with static (section 2) and dynamic (section 4) boundary conditions. For static reflection boundary conditions we prove that $X^{\odot *}$ is isomorphic to $L^{\infty}(]0,l[\,,\mathbb{R}^n)$ and that the canonical injection $j : X \to X^{\odot *}$, which is defined by the pairing on X^*, $\langle jx, x^{\odot}\rangle := \langle x^{\odot}, x\rangle_{X^*}$, is simply the inclusion of X in L^{∞}. For dynamic boundary conditions we prove that $X^{\odot *}$ is isomorphic to $L^{\infty}(]0,l[\,,\mathbb{R}^n) \times \mathbb{R}^{n_2}$ and one has to use the identification $X \simeq j(X) \subset X^{\odot *}$ in the variation of constants formula (Def. 4.5). In section 3 we apply the results and prove unique local existence of mild solutions, the smooh semiflow generation (linearization theorem) which, by a standard argument, implies theorem 3.9 on linearized stability. This theorem does not include spectrum (of the generator) determined growth of the linearized semigroup. For this a spectral mapping problem has to be solved. How the spectrum of the semigroup is related to the spectrum of its generator is not obvious since (SH) is hyperbolic. It is addressed in two other articles, see [6, 5].

The author would like to thank Julia Ehrt for reading the manuscript.

2 Variation of Constants Formula

The semigroup $T(t)$ to (H_0) shifts the components of u to the right according to the characteristic speeds $k_i(x)$, $1 \leq i \leq n_1$, where u is reflected at $x = l$ via the matrix D to v, the components of v are shifted to the left according to the characteristic speeds k_{n_1+i}, $1 \leq i \leq n_2$, and reflected at $x = 0$ to u via E.

We have the following formula for the semigroup $T(t)$, obtained by integrating along characteristics:

Proposition 2.1. *Let $\Gamma_i^t : [0,l] \to \mathbb{R}$ be the solution map to the characteristic equation*

$$\frac{d}{dt}\gamma_i(t) = -k_i(\gamma_i(t)),$$

i.e. $\Gamma_i^t x$ is the solution $\gamma_i(t)$ with initial condition $\gamma_i(0) = x$ (the maximal time t depends on x). Let $t^* := \min\left\{t \in \mathbb{R} \mid \exists_{1\leq i\leq n_1}\Gamma_i^t(0) = l \ \vee\ \exists_{n_1+1\leq j\leq n}\Gamma_j^t(l) = 0\right\}$. Then for $0 \leq t \leq t^*$ the semigroup $T(t)$ to (H_0) is expressed by the formula

$$T(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix},$$

where for $1 \le i \le n_1$

$$u_i(x) = u_{0i}(a), \textit{ if } \Gamma_i^t(a) = x,$$
$$u_i(x) = \sum_{j=1}^{n_2} e_{ij} v_{0j}(a_j), \textit{ if } \Gamma_i^{t_0}(0) = x,\ t_1 = t - t_0 > 0,\ \Gamma_j^{t_1}(a_j) = 0,\ 1 \le j \le n_2,$$

and for $1 \le j \le n_2$

$$v_j(x) = v_{0j}(a), \textit{ if } \Gamma_{n_1+j}^t(a) = x,$$
$$v_j(x) = \sum_{i=1}^{n_1} d_{ji} u_{0i}(a_i), \textit{ if } \Gamma_{n_1+j}^{t_0}(l) = x,\ t_1 = t - t_0 > 0,\ \Gamma_i^{t_1}(a_i) = l,\ 1 \le i \le n_1.$$

We calculate the space $X^{\odot}$.

Let A denote the infinitesimal generator of the semigroup $T(t)$,

$$A = K \cdot \partial_x, \quad \mathcal{D}(A) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in X \mid A \begin{pmatrix} u \\ v \end{pmatrix} \in X \right\}.$$

Let $T^*(t) : X^* \to X^*$ be the adjoint semigroup, i.e. $T^*(t)x^* := x^* T(t)$ for $x^* \in X^*$. $T^*(t)$ is not necessarily strongly continuous, it is continuous with respect to the weak star topology, i.e. $t \mapsto \langle T^*(t)x^*, x\rangle$ is continuous for all $x \in X$. The "sun" space $X^{\odot}$ is precisely the subspace on which T^* is strongly continuous, i.e.,

$$X^{\odot} := \{x^* \in X^* \mid \lim_{t \downarrow 0} \|T^*(t)x^* - x^*\| = 0\}. \tag{2}$$

One has

$$X^{\odot} = \overline{\mathcal{D}(A^*)}, \tag{3}$$

where A^* is the adjoint of the generator A of the semigroup T [2, 8].

By Riesz representation theorem to each functional Λ on $C([0,l], \mathbb{R}^n)$ there corresponds a unique row vector of Radon measures $\mu : \mathfrak{B} \to \mathbb{R}^n$, defined on the σ Algebra of Borel sets $\mathfrak{B}$, so that

$$\Lambda(f) = \int_0^l d\mu\, f \quad \text{for } f \in C([0,l], \mathbb{R}^n).$$

Moreover, there is a one to one correspondence of Radon measures μ_η and equivalence classes $[\eta]$ of functions of bounded variation (written as row vectors), which are right continuous on $]0,l[$, expressed by

$$\eta(b) - \eta(a) = \mu_\eta\left(]a,b]\right), \quad a, b \in \left]0, l\right]. \tag{4}$$

According to (4) two functions are equivalent if their difference is constant on $[0,l]$.

Due to the boundary conditions incorporated in the space X the dual space X^* consists of all bounded Radon measures (μ,ν), $\mu : \mathfrak{B} \to \mathbb{R}^{n_1}$ and $\nu : \mathfrak{B} \to \mathbb{R}^{n_2}$, such that μ is not atomic at $\{0\}$ and ν is not atomic at $\{l\}$:

$$X^* = \Big\{([\rho],[\sigma]) \in BV([0,l],\mathbb{R}^n) \mid \rho,\sigma \text{ are right continuous on } \,]0,l[\,, \qquad (5)$$
$$\rho \text{ is continuous in } 0,\ \sigma \text{ is continuous in } l\Big\}.$$

Let

$$K_u(x) := \operatorname{diag}\,(k_i(x))_{1\le i\le n_1}\,, \quad K_v(x) := \operatorname{diag}\,(k_i(x))_{n_1+1\le i\le n}\,.$$

Theorem 2.2. $([\zeta],[\eta]) \in \mathcal{D}(A^*)$ *and* $A^*([\zeta],[\eta]) = ([\rho],[\sigma])$ *if and only if there exist representatives* $\tilde\rho \in [\rho]$, $\tilde\sigma \in [\sigma]$ *which satisfy*

$$\tilde\rho(l) + \tilde\sigma(l)D = 0, \quad (\tilde\rho(0)E + \tilde\sigma(0))\,D = 0$$

and

$$[\zeta(\cdot)] = \left[-\int_0^{\cdot} \tilde\rho(x)K_u^{-1}(x)\,dx\right], \quad [\eta(\cdot)] = \left[\int_0^{\cdot} (\tilde\rho(0)E + \tilde\sigma(0) - \tilde\sigma(x))\,K_v^{-1}(x)\,dx\right]. \qquad (6)$$

Proof. Suppose $([\zeta],[\eta]) \in \mathcal{D}(A^*)$ and $A^*([\zeta],[\eta]) = ([\rho],[\sigma])$. This means that

$$\begin{aligned}\forall (u,v)\in\mathcal{D}(A):\ \left\langle ([\zeta],[\eta]), A\begin{pmatrix}u\\v\end{pmatrix}\right\rangle &= \int_0^l d\mu_\zeta\, K_u\,\partial_x u + \int_0^l d\mu_\eta K_v \partial_x v \qquad (7)\\ &= \int_0^l d\mu_\rho u + \int_0^l d\mu_\sigma v\\ &= (\rho(l)+\sigma(l)D)\,u(l) - (\rho(0)E+\sigma(0))\,v(0)\\ &\quad - \int_0^l \rho\,\partial_x u\,dx - \int_0^l \sigma\,\partial_x v\,dx.\end{aligned}$$

By choosing an appropriate pair of representatives we can assume that

$$(\rho(l) + \sigma(l)D) = 0. \qquad (8)$$

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of smooth and bounded maps from $[0,l]$ into $\mathbb{R}^{n_1}$ so that

$$K_u\partial_x u_n \longrightarrow_{n\to\infty} \chi_{[s_1,t_1]} e_i^u,$$
$$u_n(0) = \partial_x u_n(0) = \partial_x u_n(l) = 0.$$

where $0 < s_1 \le t_1 < l$, $\chi_{[s_1,t_1]}$ denotes the characteristic function to the interval $[s_1,t_1]$ and e_i^u is the i-th canonical basis in $\mathbb{R}^{n_1}$. Further choose smooth $v_n : [0,l] \to \mathbb{R}^{n_2}$ with

$$K_v\partial_x v_n \longrightarrow_{n\to\infty} \chi_{[s_2,t_2]} c_n,$$
$$v_n(0) = \partial_x v_n(0) = \partial_x v_n(l) = 0,$$

where the column vector $c_n \in \mathbb{R}^{n_2}$ is determined by the boundary condition $v_n(l) = Du_n(l)$. Then

$$c_n \longrightarrow_{n\to\infty} \left(\int_{s_2}^{t_2} K_v^{-1}(x)\,dx\right)^{-1} D \int_{s_1}^{t_1} K_u^{-1}(x)\,dx\, e_i^u =: c.$$

By construction $(u_n, v_n) \in \mathcal{D}(A)$ and by inserting (u_n, v_n) into (7), using the normalization (8), and passing $n \to \infty$ we get

$$(\zeta(t_1) - \zeta(s_1))\, e_i^u + (\eta(t_2) - \eta(s_2))\, c = -\int_{s_1}^{t_1} \rho(x) K_u^{-1}(x)\, dx e_i^u - \int_{s_2}^{t_2} \sigma(x) K_v^{-1}(x)\, dx c.$$

Hence

$$\forall_{0\le s_1\le t_1<l} : \zeta(t_1) - \zeta(s_1) = -\int_{s_1}^{t_1} \rho(x) K_u^{-1}(x)\, dx. \tag{9}$$

Since we use the representation (5) ζ is continuous at $x = 0$, but it may have a jump at $x = l$. Using (9) and (8) we get from (7) $\forall (u, v) \in \mathcal{D}(A)$:

$$\int_0^l d\mu_\eta K_v \partial_x v = -\left(\zeta(l) - \zeta(l-)\right) K_u(l)\partial_x u(l) - \left(\rho(0)E + \sigma(0)\right) v(0) - \int_0^l \sigma\, \partial_x v\, dx. \tag{10}$$

Choose $(u_n, v_n) \in \mathcal{D}(A)$ so that $\partial_x u_n(l) = 0$, $K_v \partial_x v_n \rightarrow_{n\to\infty} e_j^v \chi_{[s_1,t_1]}$, where $e_j^v \in \mathbb{R}^{n_2}$ is the j-th unit vector in $\mathbb{R}^{n_2}$ and $0 \le s_1 \le t_1 < l$. Clearly, such exist with $v_n(l) \rightarrow_{n\to\infty} 0$, $v_n(0) \rightarrow_{n\to\infty} -\int_{s_1}^{t_1} K_v^{-1}(x)\, dx\, e_j^v$. Putting such (u_n, v_n) into (10) and passing to the limit we get, since η is right continuous in l,

$$\forall_{0\le s_1\le t_1\le l} : \eta(t_1) - \eta(s_1) = \int_{s_1}^{t_1} \left(\rho(0)E + \sigma(0) - \sigma(x)\right) K_v^{-1}(x)\, dx. \tag{11}$$

From (11) and (10) we get

$$\forall_{(u,v)\in\mathcal{D}(A)} : \left(\zeta(l) - \zeta(l-)\right) K_u(l)\partial_x u(l) + \left(\rho(0)E + \sigma(0)\right) Du(l) = 0.$$

Hence

$$\zeta(l) = \zeta(l-),$$

so (9) holds for all $s_1, t_1 \in [0, l]$, and

$$\left(\rho(0)E + \sigma(0)\right) D = 0.$$

$\square$

For $[\zeta]$ and $[\eta]$ satisfying (6) we have for the total variation of the measures

$$\|\mu_\zeta\|_{Var} = \|\tilde{\rho} K_u^{-1}\|_{L^1([0,l])},$$
$$\|\mu_\eta\|_{Var} = \|\left(\rho(0)E + \sigma(0) - \tilde{\sigma}\right) K_v^{-1}\|_{L^1([0,l])}.$$

Hence Theorem 2.2 and (3) yield that $X^\odot$ is just the space of Radon measures who have an L^1 density:

$$X^\odot = \left\{ (\mu,\nu) \in X^* \mid \mu(A) = \int_A \rho(x)\,dx, \nu(A) = \int_A \sigma(x)\,dx,\ (\rho,\sigma) \in L^1([0,l],\mathbb{R}^n), A \in \mathfrak{B} \right\},$$

where $\|\mu\|_{Var} = \|\rho\|_{L^1}$, $\|\nu\|_{Var} = \|\sigma\|_{L^1}$. We have proven

Theorem 2.3. *$X^\odot$ is isomorphic to $L^1([0,l],\mathbb{R}^n)$. For $(\rho,\sigma) \in L^1([0,l],\mathbb{R}^n)$ and $\begin{pmatrix} u \\ v \end{pmatrix} \in X$ the dual pairing is given by*

$$\langle (\rho,\sigma), \begin{pmatrix} u \\ v \end{pmatrix} \rangle_{X^*} = \int_0^l \rho(x)u(x)\,dx + \int_0^l \sigma(x)v(x)\,dx.$$

It follows that $X^{\odot *} := (X^\odot)^*$ can be indentified with $L^\infty([0,l],\mathbb{R}^n)$. Using this identification the canonical inclusion

$$j : X \to X^{\odot *}, \quad \langle jx, x^\odot \rangle := \langle x^\odot, x \rangle_{X^*},$$

becomes the standard inclusion of X into $L^\infty([0,l],\mathbb{R}^n)$. Moreover, it is straightforward to calculate $T^\odot = T^*_{|X^\odot}$ and $T^{\odot *} = (T^\odot)^*$ on L^1 and L^∞, respectively. Indeed, for $\begin{pmatrix} u \\ v \end{pmatrix} \in X$ let $T(t)\begin{pmatrix} u \\ v \end{pmatrix} =: \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}$. Then for $(\zeta,\eta) \in L^1([0,l],\mathbb{R}^n) \simeq X^\odot$ we have

$$\begin{aligned} \langle T^\odot(\zeta,\eta), \begin{pmatrix} u \\ v \end{pmatrix} \rangle &= \langle (\zeta,\eta), \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} \rangle = \int_0^l \zeta(x)\hat{u}(x)\,dx + \int_0^l \eta(x)\hat{v}(x)\,dx \\ &= \int_0^l \check{\zeta}(x)u(x)\,dx + \int_0^l \check{\eta}(x)v(x)\,dx, \end{aligned}$$

where $\check{\zeta}$ is obtained by shifting ζ to the left with characteristic speeds $-K_u$ and extending at $x = l$ with ηD, the whole multiplied by a Jacobian. $\check{\eta}$ is obtained by shifting η to the right with characteristic speeds $-K_v$ and extending at $x = 0$ with ζE, the whole multiplied by a Jacobian. Using this notation

$$T^\odot(\zeta,\eta) = (\check{\zeta},\check{\eta}).$$

For $\begin{pmatrix} f \\ g \end{pmatrix} \in L^\infty([0,l],\mathbb{R}^n) \simeq X^{\odot *}$ and $(\zeta,\eta) \in L^1([0,l],\mathbb{R}^n) \simeq X^\odot$ we have

$$\begin{aligned} \langle T^{\odot *}\begin{pmatrix} f \\ g \end{pmatrix}, (\zeta,\eta) \rangle &= \langle (\check{\zeta},\check{\eta}), \begin{pmatrix} f \\ g \end{pmatrix} \rangle = \int_0^l \check{\zeta}(x)f(x)\,dx + \int_0^l \check{\eta}(x)g(x)\,dx \\ &= \int_0^l \zeta(x)\hat{f}(x)\,dx + \int_0^l \eta(x)\hat{g}(x)\,dx. \end{aligned}$$

Here $\hat{f}$ and $\hat{g}$ are defined as $\hat{u}$ and $\hat{v}$ in an almost everywhere sense on L^∞. So

$$T^{\odot *}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \hat{f} \\ \hat{g} \end{pmatrix}.$$

We have proven

Theorem 2.4 (Representation of $X^{\odot *}$ and $T^{\odot *}$)**.** *$X^{\odot *}$ is isomorphic to $L^\infty([0,l],\mathbb{R}^n)$. A formula for $T^{\odot *}$ on $X^{\odot *}$ is given as for T in Proposition 2.1. Hence $T^{\odot *}$ is just the extension of T to $L^\infty([0,l],\mathbb{R}^n)$.*

From the formula in Proposition 2.1 it follows that $t \mapsto T^{\odot *}z$ ($z \in X^{\odot *}$) is strongly continuous if and only if $z \in X$. This means that T is $\odot$-reflexive on X:

$$X^{\odot\odot} = j(X) \simeq X.$$

Further note

Remark 2.5. *$T^{\odot *} : X^{\odot *} \to X^{\odot *}$ is weak star continuous, but not Bochner measurable.*

We have the following important

Lemma 2.6. *Let $f : [0,T] \to X^{\odot *}$ be norm continuous. Then the weak-star integral*

$$t \mapsto \int_0^t T^{\odot *}(t-s)f(s)\,ds \tag{12}$$

is norm continuous and takes values in X.

Lemma 2.6 is known for general sun reflexive semigroups [2, Lemma 2.1, p.54]. We give a simple alternative proof which does not use sun reflexivity:

Proof. Let $T_2(t)$ be the extension of $T(t)$ onto the space $L^2([0,l],\mathbb{R}^n)$, defined as in Proposition 2.1. Then $T_2(t)$ is strongly continuous. Further let A_2 denote its infinitesimal generator with domain $\mathcal{D}(A_2) = \{(u,v) \in W^{1,2}([0,l],\mathbb{R}^n) \mid u(0) = Ev(0), \quad v(l) = Du(l)\}$.

Since $f : [0,T] \to X^{\odot *}$ is norm continuous, there exists a sequence $f_k : [0,T] \to X^{\odot *}$ of smooth maps so that $f_k \to f$ uniformly on $[0,T]$ with respect to the norm of $X^{\odot *}$. By standard semigroup theory we have

$$\int_0^\cdot T^{\odot *}(\cdot - s)f_k(s)\,ds = \int_0^\cdot T_2(\cdot - s)f_k(s)\,ds \in C([0,T],\mathcal{D}(A_2)) \hookrightarrow C([0,T],X).$$

Since we have for $t \in [0,T]$

$$\left\| \int_0^t T^{\odot *}(t-s)\left(f(s) - f_k(s)\right)\,ds \right\|_{X^{\odot *}} \le \sup_{r\in[0,T]} \|T^{\odot *}(r)\|_{\mathcal{L}(X^{\odot *})} T \|f - f_k\|_{C([0,T],X^{\odot *})}$$

it follows that (12) belongs to $C([0,T],X)$. $\square$

Lemma 2.6 leads us to Definition 1.1.

3 Smooth Semiflow

In this section we use Definition 1.1 and prove local existence, uniqueness and Fréchet differentiability of the solution map in X. We show the existence of a smooth semiflow on X for (SH). Due to Lemma 2.6 and Lemma 3.2 the proofs work straightforwardly as in the theory of ODEs.

We need the following assumptions on H:

Definition 3.1. *We say that the nonlinearity $H :]0,l[\times\mathbb{R}^n \to \mathbb{R}^n$, $H = H(x,z)$, $x \in]0,l[$, $z = (u,v) \in \mathbb{R}^n$, satisfies the C^k Carathéodory condition, $k \geq 1$, if:*

- *For a.a. $x \in]0,l[$ $H(x,\cdot) \in C^k(\mathbb{R}^n)$ and $H(\cdot,z)$ is measurable for all $z \in \mathbb{R}^n$.*
- *For all compact $\mathcal{K} \subset \mathbb{R}^n$ there exists a constant $M > 0$ such that $\left\|\frac{\partial^i H(x,z)}{\partial z^i}\right\| \leq M$ for $0 \leq i \leq k$, all $z \in \mathcal{K}$ and a.a. $x \in]0,l[$.*
- *For all compact $\mathcal{K} \subset \mathbb{R}^n$ and $\epsilon > 0$ there exists a $\delta > 0$ such that for all $z_1 \in \mathcal{K}$, $z_2 \in \mathbb{R}^n$ with $\|z_1 - z_2\| < \delta$ and a.a. $x \in]0,l[$ we have $\left\|\frac{\partial^k H(x,z_1)}{\partial z^k} - \frac{\partial^k H(x,z_2)}{\partial z^k}\right\| < \epsilon$.*

One of the main advantages of using the space X together with its sun dual $X^{\odot *}$ is based on the following simple

Lemma 3.2. *If H satisfies the C^k Carathéodory condition then the Nemytskij operator $\mathfrak{H}\,(u,v)\,(x) := H(x,u(x),v(x))$ is a C^k smooth map from X into $X^{\odot *}$.*

Theorem 3.3 (Unique local existence)**.** *For any $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in X$ there exists a $\delta > 0$, depending only on $\|\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\|_X$, $\mathfrak{H}$ and $T(t)$, such that* (SH) *has a unique weak solution $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,\delta],X)$ with $u(0) = u_0$, $v(0) = v_0$.*

Proof. Let $0 < \delta < 1$. By Lemma 2.6 the map

$$\mathcal{G}\begin{pmatrix} u \\ v \end{pmatrix}(t) := T(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t T^{\odot *}(t-s)\mathfrak{H}(u(s),v(s))\,ds \tag{13}$$

maps $C([0,\delta],X)$ into itself. Define the closed subspace

$$B_\delta := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in C([0,\delta],X) \mid \text{for } t \in [0,\delta] \ \left\| \begin{pmatrix} u \\ v \end{pmatrix}(t) - T(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right\|_X \leq 1 \right\}.$$

Since $\mathfrak{H}$ is locally Lipschitz on X with values in $X^{\odot *}$ there exists $L > 0$, depending only on $\left\| \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right\|_X$, $\mathfrak{H}$ and $T(t)$ such that if $z_1, z_2 \in B_\delta$ then

$$\|\mathcal{G}(z_1)(t) - \mathcal{G}(z_2)(t)\|_X \leq \delta L \|z_1 - z_2\|_{C([0,\delta],X)} . \tag{14}$$

Because $\mathfrak{H}$ is locally bounded there exists $M > 0$ such that for $z \in B_\delta$

$$\begin{aligned} \left\| \mathcal{G} \begin{pmatrix} u \\ v \end{pmatrix} (t) - T(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right\|_X &\leq \left\| \int_0^t T^{\odot *}(t-s) \mathfrak{H}(u(s), v(s)) \, ds \right\|_{X^{\odot *}} \\ &\leq M\delta \qquad \text{for } t \in [0, \delta] . \end{aligned} \tag{15}$$

Therefore (14) and (15) yield that for sufficiently small $\delta > 0$ $\mathcal{G}$ maps B_δ into itself and becomes a contraction. By Banachs contraction mapping theorem $\mathcal{G}$ has a fixed point in B_δ. ☐

Theorem 3.4 (Regularity)**.** *Let* $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T], X)$ *be a weak solution to* (SH) *with initial data* $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in W^{1,p}(]0,l[\,,\mathbb{R}^n)$*,* $p \in]1, \infty[$*. Then* $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T], W^{1,p}(]0,l[\,,\mathbb{R}^n)) \cap C^1([0,T], L^p(]0,l[\,,\mathbb{R}^n))$ *and* (SH) *holds in a classical sense.*

Proof. Let $T_p(t)$ be the extension of $T(t)$ onto the space $L^p([0,l], \mathbb{R}^n)$, defined as in Proposition 2.1. $T_p(t)$ is strongly continuous. Further let A_p denote its infinitesimal generator with domain $\mathcal{D}(A_p) = W^{1,p}(]0,l[\,,\mathbb{R}^n) \cap X$. Let $h > 0$ and $0 \leq t < t + h \leq T$

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} (t+h) - \begin{pmatrix} u \\ v \end{pmatrix} (t) = \; & (T(h) - I)\, T(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \\ & + \int_0^t T^{\odot *}(t-s) \left(\mathfrak{H}(u(s+h), v(s+h)) - \mathfrak{H}(u(s), v(s)) \right) ds \\ & + \int_0^h T^{\odot *}(t+h-s) \mathfrak{H}(u(s), v(s)) \, ds \end{aligned}$$

Because $\begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T], X)$ there exists $L > 0$ so that

$$\begin{aligned} & \|\mathfrak{H}(u(s+h), v(s+h)) - \mathfrak{H}(u(s), v(s))\|_{L^p([0,l],\mathbb{R}^n)} \\ \leq & L \|(u(s+h), v(s+h)) - (u(s), v(s))\|_{L^p([0,l],\mathbb{R}^n)} . \end{aligned} \tag{16}$$

Hence

$$\begin{aligned} & \left\| \int_0^t T^{\odot *}(t-s) \left(\mathfrak{H}(u(s+h), v(s+h)) - \mathfrak{H}(u(s), v(s)) \right) ds \right\|_{L^p([0,l],\mathbb{R}^n)} \\ = & \left\| \int_0^t T_p(t-s) \left(\mathfrak{H}(u(s+h), v(s+h)) - \mathfrak{H}(u(s), v(s)) \right) ds \right\|_{L^p([0,l],\mathbb{R}^n)} \\ \leq & L \sup_{r \in [0,T]} \|T_p(r)\| \int_0^t \|(u(s+h), v(s+h)) - (u(s), v(s))\|_{L^p([0,l],\mathbb{R}^n)} \, ds. \end{aligned}$$

Moreover,

$$\begin{aligned}\left\|(T(h)-I)\,T(t)\begin{pmatrix}u_0\\v_0\end{pmatrix}\right\|_{L^p([0,l],\mathbb{R}^n)} &= \left\|\int_0^h T_p(s)T_p(t)A_p\begin{pmatrix}u_0\\v_0\end{pmatrix}\right\|_{L^p([0,l],\mathbb{R}^n)}\\ &\le h\sup_{r\in[0,T]}\|T_p(r)\|\left\|A_p\begin{pmatrix}u_0\\v_0\end{pmatrix}\right\|_{L^p([0,l],\mathbb{R}^n)},\end{aligned}$$

and

$$\begin{aligned}\left\|\int_0^h T^{\odot *}(t+h-s)\mathfrak{H}(u(s),v(s))\,ds\right\|_{L^p([0,l],\mathbb{R}^n)} &\le \left\|\int_0^h T^{\odot *}(t+h-s)\mathfrak{H}(u(s),v(s))\,ds\right\|_{X^{\odot *}}\\ &\le h\sup_{r\in[0,T]}\left\|T^{\odot *}(r)\right\|\,\|\mathfrak{H}(u,v)\|_{C([0,T],X^{\odot *})}.\end{aligned}$$

Hence there exists $c>0$ so that

$$\left\|\begin{pmatrix}u\\v\end{pmatrix}(t+h)-\begin{pmatrix}u\\v\end{pmatrix}(t)\right\|_{L^p} \le hc + c\int_0^t \left\|\begin{pmatrix}u\\v\end{pmatrix}(s+h)-\begin{pmatrix}u\\v\end{pmatrix}(s)\right\|_{L^p} ds.$$

Applying Gronwalls Lemma yields that $t\mapsto \begin{pmatrix}u\\v\end{pmatrix}(t)$ is Lipschitz from $[0,T]$ into L^p. By (16) $t\mapsto \mathfrak{H}(u(t),v(t))$ is Lipschitz from $[0,T]$ into L^p. Since $1<p<\infty$ the space L^p is reflexive, so we have

$$\mathfrak{H}(u,v)\in W^{1,\infty}([0,T],L^p([0,l],\mathbb{R}^n)).$$

Then by standard semigroup theory [1, Proposition 4.1.6, p.51] we get for

$$z(t) := \int_0^t T^{\odot *}(t-s)\mathfrak{H}(u(s),v(s))\,ds = \int_0^t T_p(t-s)\mathfrak{H}(u(s),v(s))\,ds$$

that

$$z\in C([0,T],\mathcal{D}(A_p))\cap C^1([0,T],L^p(]0,l[\,,\mathbb{R}^n)) \quad \text{and} \quad \frac{d}{dt}z(t) = A_p z(t) + \mathfrak{H}(z(t)).$$

□

For $z_0\in X$ let $\omega=\omega(z_0)\in\,]0,\infty]$ denote the maximal time up to which the solution with initial data z_0 exists, i.e.

$$\omega(z_0) := \sup\{t\in\mathbb{R} \mid \text{there exists a mild solution } z\in C([0,t],X) \text{ with } z(0)=z_0\}.$$

The following is a standard consequence of Theorem 3.3

Theorem 3.5. *For any $z_0\in X$ either*
ı) $\omega(z_0)=\infty$
or
ıı) $\omega(z_0)<\infty$ and $\lim_{t\uparrow\omega(z_0)}\|z(t)\|_X=\infty$, where $z:[0,\omega(z_0)[\,\to X$ denotes the weak solution with $z(0)=z_0$.

Theorem 3.6. *Let $z \in C([0,T],X)$ be a weak solution of* (SH) *up to T. Then there exists a neighborhood U of $z(0)$ in X such that for all $y_0 \in U$ there is a weak solution $y \in C([0,T],X)$ of* (SH) *satisfying $y(0) = y_0$.*
There exists a constant $c > 0$ such that for all $y_0 \in U$

$$\|z(t) - y(t)\|_X \leq c \|z(0) - y_0\|_X .$$

Proof. Let $M := \sup_{0\leq r\leq T} \|T(r)\| + 1$ and L be a Lipschitz constant for $\mathfrak{H}$ on the Ball $\left\{x \in X \mid \|x\|_X \leq 1 + \sup_{0\leq r\leq T} \|z(r)\|_X\right\}$. Then for $t \in [0,T]$ such that $\sup_{0\leq r\leq t} \|y(r)\|_X \leq 1 + \sup_{0\leq r\leq T} \|z(r)\|_X$ we have

$$\|y(t) - z(t)\|_X \leq M \|y_0 - z_0\|_X + \int_0^t ML \|y(s) - z(s)\|_X \, ds.$$

Gronwall's inequality yields

$$\|y(t) - z(t)\|_X \leq Me^{LMT} \|y_0 - z_0\|_X . \tag{17}$$

Set

$$U := \left\{y \in X \mid \|y - z_0\|_X < M^{-1}e^{-LMT}\right\}.$$

Hence for $y_0 \in U$ and $t \in [0,T]$ $\|y(t)\| < 1 + \|z(t)\|$ and (17) holds. ☐

Suppose there exists a weak solution $z \in C([0,T],X)$ of (SH). Then according to Theorem 3.6 there exists an open neighborhood U of $z(0)$ in X so that we can define a solution map

$$S^t : U \to X, \quad S^t(y_0) := y(t) \quad (t \in [0,T]). \tag{18}$$

Theorem 3.7 (Smooth semiflow property)**.** *For each $t \in [0,T]$ the map $S^t : U \to X$ is C^k smooth. The map $(t,u) \mapsto S^t u$ is continuous from $[0,T] \times U$ into X. The total derivative DS^t satisfies the equation*

$$\begin{pmatrix}\tilde{h}_u(t)\\ \tilde{h}_v(t)\end{pmatrix} = DS^t \begin{pmatrix}h_u\\ h_v\end{pmatrix}, \; \begin{pmatrix}\tilde{h}_u(t)\\ \tilde{h}_v(t)\end{pmatrix} = T(t) \begin{pmatrix}h_u\\ h_v\end{pmatrix} + \int_0^t T^{\odot *}(t-s) D\mathfrak{H}(u(s),v(s)) \begin{pmatrix}\tilde{h}_u(s)\\ \tilde{h}_v(s)\end{pmatrix} ds.$$

Proof. For $z = \begin{pmatrix}u\\ v\end{pmatrix} \in C([0,T],X)$ and initial data $z_0 = \begin{pmatrix}u_0\\ v_0\end{pmatrix} \in X$ the operator $\mathcal{G}(z)$, defined in formula (13), maps $C([0,T],X)$ into itself (Lemma 2.6). To emphasize the dependence on z_0 we write $\mathcal{G}(z,z_0)$. Define

$$\mathcal{F} : C([0,T],X) \times X \to C([0,T],X), \quad (\mathcal{F}(z,z_0))(t) := (\mathcal{G}(z,z_0))(t) - z(t).$$

By assumption for each $z_0 \in U$ the equation $\mathcal{F}(z,z_0) = 0$, $z \in C([0,T],X)$, has a unique solution $z = \gamma(z_0)$.

$\mathcal{G}$ is C^k from $C([0,T],X)\times X$ into $C([0,T],X)$ and we have for $h_j \in C([0,T],X)$, $1 \le j \le k$, $t \in [0,T]$

$$\left(\frac{\partial^j \mathcal{G}}{\partial z^j}(z,z_0)h_1 \dots h_j\right)(t) = \int_0^t T^{\odot *}(t-s) D^j \mathfrak{H}(z(s))(h_i(s))_{1\le i\le j}\, ds. \tag{19}$$

Indeed, for $j=1$ we have

$$\begin{aligned} I &:= \mathcal{G}(z+h_1,z_0)(t) - \mathcal{G}(z,z_0)(t) - \int_0^t T^{\odot *}(t-s) D\mathfrak{H}(z(s))h_1(s)\, ds \\ &= \int_0^t T^{\odot *}(t-s)\Big[-D\mathfrak{H}(z(s))h_1(s) + \mathfrak{H}(z(s)+h_1(s)) - \mathfrak{H}(z(s))\Big]\, ds \\ &= \int_0^t T^{\odot *}(t-s) \int_0^1 \Big[-D\mathfrak{H}(z(s)) + D\mathfrak{H}(z(s)+\theta h_1(s))\Big] h_1(s)\, d\theta\, ds \end{aligned}$$

Therefore, by the uniform continuity of the derivative of H on compact sets, see Definition 3.1, we have

$$\begin{aligned} \frac{\|I\|_{C([0,T],X)}}{\|h_1\|_{C([0,T],X)}} &\le \int_0^t \left\|T^{\odot *}(t-s)\right\|_{\mathcal{L}(X^{\odot *})} \int_0^1 \|-D\mathfrak{H}(z(s)) + D\mathfrak{H}(z(s)+\theta h_1(s))\|_{\mathcal{L}(X,X^{\odot *})}\, d\theta\, ds \\ &\overset{\|h_1\|_{C([0,T],X)}\downarrow 0}{\longrightarrow} 0. \end{aligned}$$

By induction one obtains (19) for $1 \le j \le k$.

A generalization of Banachs fixed point theorem yields that $\frac{\partial \mathcal{F}}{\partial z}$ is an isomorphism from $C([0,T],X)$ onto itself: Indeed, assume $w \in C([0,T],X)$ is given. Then for $h \in C([0,T],X)$ the equation $\frac{\partial \mathcal{F}}{\partial z}(z,z_0)h = w$ is equivalent to $\mathcal{P}h = h$, where $\mathcal{P} : C([0,T],X) \to C([0,T],X)$,

$$(\mathcal{P}h)(t) = \int_0^t T^{\odot *}(t-s) D\mathfrak{H}(z(s))h(s)\, ds - w(t).$$

There exists a constant $M > 0$, depending only on $T(t), \mathfrak{H}, z$, so that for $h_1, h_2 \in C([0,T],X)$

$$\|\mathcal{P}h_1(t) - \mathcal{P}h_2(t)\|_X \le Mt\, \|h_1 - h_2\|_{C([0,T],X)}\,.$$

Proceeding with $\mathcal{P}^2 = \mathcal{P}\circ\mathcal{P}$ we get $\|(P^2h_1)(t) - (P^2h_2)(t)\|_X \le \frac{(Mt)^2}{2}\|h_1 - h_2\|_{C([0,T],X)}$. By induction

$$\left\|\mathcal{P}^i h_1 - \mathcal{P}^i h_2\right\|_{C([0,T],X)} \le \frac{(MT)^i}{i!}\|h_1 - h_2\|_{C([0,T],X)}\,.$$

Thus for i sufficiently large $\mathcal{P}^i$ is a contraction on $C([0,T],X)$.
By applying the implicit function theorem we get that γ is a C^k smooth map from U into $C([0,T],X)$. $\square$

Remark 3.8. *The map $x \in U \to S^{\cdot}x \in C([0,T],X)$ is C^k smooth.*

The formula in Theorem 3.7 means that the linearized flow $\begin{pmatrix} \tilde{h}_u(t) \\ \tilde{h}_v(t) \end{pmatrix} = DS^t \begin{pmatrix} h_u \\ h_v \end{pmatrix}$ is the weak solution of the linearized system

$$\text{(LH)} \begin{cases} \frac{\partial}{\partial t} \begin{pmatrix} \tilde{h}_u(t,x) \\ \tilde{h}_v(t,x) \end{pmatrix} = K(x) \frac{\partial}{\partial x} \begin{pmatrix} \tilde{h}_u(t,x) \\ \tilde{h}_v(t,x) \end{pmatrix} + DH(x, u(t,x), v(t,x)) \begin{pmatrix} \tilde{h}_u(t,x) \\ \tilde{h}_v(t,x) \end{pmatrix}, \\ \tilde{h}_u(t,0) = E\,\tilde{h}_v(t,0), \quad \tilde{h}_v(t,l) = D\tilde{h}_u(t,l) \\ \tilde{h}_u(0,x) = h_u(x),\ \tilde{h}_v(0,x) = h_v(x). \end{cases}$$

Let z_0 be a stationary state of the flow, i.e.

$$S^t z_0 = z_0 \quad \text{for } t \geq 0.$$

Then the linearized flow $DS^t(z_0)$ is a C_0 semigroup on X.

A standard consequence of Theorem 3.7 (see [7, Theorem 11.22]) is

Theorem 3.9 (Linearized Stability)**.** *Let z_0 be a stationary state. Suppose that the linearized semigroup $DS^t(z_0)$ is exponentially stable, i.e. there exist constants $\omega_0 > 0$ and $M > 0$ so that*

$$\left\| DS^t(z_0) \right\|_{\mathcal{L}(X)} \leq M e^{-\omega_0 t} \quad \textit{for } t \geq 0.$$

Then z_0 is exponentially stable for the nonlinear flow S^t: There exists a neighbourhood U of z_0 in X and constants $N > 0$, $\omega \in \,]0, \omega_0[$ so that for all $z \in U$

$$\omega(z) = \infty \quad \textit{and } \|z(t) - z_0\| \leq N e^{-\omega t} \textit{ for } t \geq 0.$$

The natural question arises how to determine the stability of the semigroup $DS^t(z_0)$. A standard method is to estimate the spectral bound of the operator

$$K(x)\frac{\partial}{\partial x} + D_{(u,v)}H(x, u(t,x), v(t,x)) \quad + \quad \text{boundary reflection conditions}$$

in the complexification of the space X. That this works is not obvious due to hyperbolicity of (LH).

This problem is addressed in [6, 5].

4 Dynamic boundary conditions

In this section we briefly consider systems with dynamic boundary conditions:

For $0 < x < l$ and $t > 0$

$$
(\mathrm{SHD}) \begin{cases} \frac{\partial}{\partial t} \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = K(x) \frac{\partial}{\partial x} \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + H(x, u(t,x), v(t,x)), \\ \frac{d}{dt} [v(t,l) - Du(t,l)] = F(u(t,\cdot), v(t,\cdot)), \\ u(t,0) = E\, v(t,0), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), \end{cases}
$$

where $F : C([0,l], \mathbb{R}^n) \to \mathbb{R}^{n_2}$ is C^k smooth with bounded and uniformly continuous derivatives on bounded sets.

Remark 4.1. *If we cancel the u equation and put $H = 0$ then* (SHD) *simplifies to a retarded functional differential. By choosing $n_1 = n_2$, $E = I_{n_1}$ and $H = 0$* (SHD) *becomes a functional differential equation of neutral type [4, chapter 3].*

We consider the prototype problem

$$
(\mathrm{HD}_0) \begin{cases} \frac{\partial}{\partial t} \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = K(x) \frac{\partial}{\partial x} \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix}, \\ \frac{d}{dt} [v(t,l) - Du(t,l)] = 0, \\ u(t,0) = E\, v(t,0), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x). \end{cases}
$$

For the phase space we choose

$$
X := \{(u,v) \in C([0,l], \mathbb{R}^n) \mid u(0) = Ev(0)\}.
$$

As in Proposition 2.1 we can calculate the semigroup $T(t)$ for (HD_0): v_0 is shifted to the left, reflected at $x = 0$ via E to u, and extended at $x = l$ by D times u plus the constant vector $v_0(l) - Du_0(l)$ (so that the solution stays continuous). The generator for $T(t)$ is given by $A \begin{pmatrix} u \\ v \end{pmatrix} = K\partial_x \begin{pmatrix} u \\ v \end{pmatrix}$ with domain

$$
\begin{aligned} \mathcal{D}(A) = \Big\{ &(u,v) \in C^1([0,l], \mathbb{R}^n) \mid u(0) = Ev(0),\ K_u(0)\partial_x u(0) = EK_v(0)\partial_x v(0), \\ &K_v(l)\partial_x v(l) = DK_u(l)\partial_x u(l) \Big\}. \end{aligned}
$$

We use the representation

$$
\begin{aligned} X^* = \Big\{ &([\rho],[\sigma]) \in BV([0,l], \mathbb{R}^n) \mid \rho, \sigma \text{ are right continuous on }]0,l[\,, \\ &\rho \text{ is continuous in } 0 \Big\}. \end{aligned} \tag{20}
$$

Theorem 4.2. *$([\zeta],[\eta]) \in \mathcal{D}(A^*)$ and $A^*([\zeta],[\eta]) = ([\rho],[\sigma])$ if and only if there exist representatives $\tilde\rho \in [\rho]$, $\tilde\sigma \in [\sigma]$ which satisfy*

$$\tilde\rho(l) = 0,\ \tilde\sigma(l) = 0,\ \tilde\rho(0)E + \tilde\sigma(0) = 0$$

and there exists a row vector $c \in \mathbb{R}^{n_2}$ so that

$$[\zeta(\cdot)] = \left[t \mapsto \begin{cases} \int_t^l \tilde\rho(x) K_u^{-1}(x)\,dx, & 0 \le t < l \\ -cD, & t = l \end{cases}\right], \tag{21}$$
$$[\eta(\cdot)] = \left[t \mapsto \begin{cases} \int_t^l \tilde\sigma(x) K_v^{-1}(x)\,dx, & 0 \le t < l \\ c, & t = l \end{cases}\right].$$

Proof. Suppose $([\zeta],[\eta]) \in \mathcal{D}(A^*)$ and $A^*([\zeta],[\eta]) = ([\rho],[\sigma])$. This means that

$$\begin{aligned} \forall (u,v) \in \mathcal{D}(A):\ \left\langle ([\zeta],[\eta]), A\begin{pmatrix} u \\ v\end{pmatrix}\right\rangle &= \int_0^l d\mu_\zeta\, K_u\,\partial_x u + \int_0^l d\mu_\eta K_v \partial_x v \\ &= \int_0^l d\mu_\rho u + \int_0^l d\mu_\sigma v \\ &= \rho(l)u(l) + \sigma(l)v(l) - (\rho(0)E + \sigma(0))\,v(0) \\ &\quad - \int_0^l \rho\,\partial_x u\,dx - \int_0^l \sigma\,\partial_x v\,dx. \end{aligned} \tag{22}$$

We normalize the representatives ρ and σ such that

$$\rho(l) = 0 \quad \text{and} \quad \sigma(l) = 0. \tag{23}$$

For fixed $0 < s_1 \le t_1 < l$ we can choose a bounded sequence $(u_n, 0) \in \mathcal{D}(A)$ such that $K_u \partial_x u_n \to_{n\to\infty} \chi_{[s_1,t_1]} e_j$ and insert it into (22). By passing to the limit we get

$$\forall_{s_1,t_1 \in [0,l[} : \zeta(t_1) - \zeta(s_1) = -\int_{s_1}^{t_1} \rho(x) K_u^{-1}(x)\,dx. \tag{24}$$

Note that we have chosen a representation of X^* where ζ is continuous at $x = 0$.

Inserting (23) and (24) into (22) we get since ζ is allowed to have a jump at $x = l$

$$(\zeta(l) - \zeta(l-))\,K_u(l)\partial_x u(l) + \int_0^l d\mu_\eta K_v \partial_x v = -(\rho(0)E + \sigma(0))\,v(0) - \int_0^l \sigma\,\partial_x v\,dx.$$

By taking a sequence $(u_n, v_n) \in \mathcal{D}(A)$ such that $v_n(0) = 0$, $\partial_x u_n(l) = 0$ and $K_v \partial_x v_n \to_{n\to\infty} \chi_{[s_1,t_1]}$, where $0 \le s_1 \le t_1 < l$, we get

$$\forall_{0 \le s_1 \le t_1 < l} : \eta(t_1) - \eta(s_1) = -\int_{s_1}^{t_1} \sigma(x) K_v^{-1}(x)\,dx. \tag{25}$$

So η is continuous in $x = 0$ but may have a jump at $x = l$. Inserting (25), (23) and (24) into (22) yields

$$\forall (u, v) \in \mathcal{D}(A) : ((\zeta(l) - \zeta(l-)) + (\eta(l) - \eta(l-)) D) K_u(l)\partial_x u(l) + (\rho(0)E + \sigma(0)) v(0) = 0.$$

Hence

$$(\zeta(l) - \zeta(l-)) + (\eta(l) - \eta(l-)) D = 0 \quad \text{and} \quad \rho(0)E + \sigma(0) = 0.$$

□

From Theorem 4.2 and (3) we get

$$X^{\odot} = \Big\{(\mu, \nu) \in X^* \mid \mu(A) = \int_A \rho(x)\, dx - cD\delta_l,\ \nu(A) = \int_A \sigma(x)\, dx + c\delta_l, \\ (\rho, \sigma) \in L^1([0, l], \mathbb{R}^n),\ c \in \mathbb{R}^{n_2}\Big\}.$$

Here $\delta_l : \mathfrak{B} \to \mathbb{R}$ is the dirac mass on $\{l\}$ and c, ρ, σ are written as row vectors.

Because $\|\mu\|_{Var} = |cD| + \|\rho\|_{L^1([0,l],\mathbb{R}^{n_1})}$ and $\|\nu\|_{Var} = |c| + \|\sigma\|_{L^1([0,l],\mathbb{R}^{n_2})}$ we obtain that $X^{\odot}$ is isomorphic to $L^1([0, l], \mathbb{R}^n) \times \mathbb{R}^{n_2}$,

$$X^{\odot} \simeq L^1([0, l], \mathbb{R}^n) \times \mathbb{R}^{n_2}.$$

We describe how $T^{\odot}(t)$ acts on $X^{\odot}$. Let $(\rho, \sigma, c) \in X^{\odot} \simeq L^1([0, l], \mathbb{R}^n) \times \mathbb{R}^{n_2}$. For $\begin{pmatrix} u \\ v \end{pmatrix} \in X$ denote $\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = T(t) \begin{pmatrix} u \\ v \end{pmatrix}$. We have

$$\begin{aligned}
\langle T^{\odot}(t)(\rho, \sigma, c), \begin{pmatrix} u \\ v \end{pmatrix}\rangle &= \langle (\rho, \sigma, c), \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}\rangle \\
&= \int_0^l \rho(x)\hat{u}(x)\, dx + \int_0^l \sigma(x)\hat{v}(x)\, dx + c\,(\hat{v}(l) - D\hat{u}(l)) \\
&= \int_0^l \check{\rho}(x)u(x)\, dx + \int_0^l \check{\sigma}(x)v(x)\, dx + \left(c + \int_{\gamma(t)}^l \sigma(x)\, dx\right)(v(l) - Du(l)).
\end{aligned}$$

Hence

$$T^{\odot}(t)(\rho, \sigma, c) = (\check{\rho}, \check{\sigma}, c + \int_{\gamma(t)}^l \sigma(x)\, dx).$$

Here $\check{\rho}$ is obtained from ρ by left translation using the characteristic speeds $-K_u$ and extending at $x = l$ with $\sigma \cdot D$, multiplied by a Jacobian. $\check{\sigma}$ is obtained by right translation of σ with speeds $-K_v$ and extending at $x = 0$ with $\rho \cdot E$, multiplied by a Jacobian. $\gamma(t)$ is a $\mathbb{R}^{n_2}$ vector where each component is the distance which each point $x = l$ has traveled to the left during time t using the characteristic speed K_v.

We use the standard identification

$$X^{\odot *} \simeq L^\infty([0,l],\mathbb{R}^n) \times \mathbb{R}^{n_2}$$

and calculate $T^{\odot *}$ on $L^\infty([0,l],\mathbb{R}^n) \times \mathbb{R}^{n_2}$. For each column vector $\begin{pmatrix} f \\ g \\ d \end{pmatrix} \in L^\infty \times \mathbb{R}^{n_2}$ and row vector $(\rho, \sigma, c) \in L^1([0,l],\mathbb{R}^n) \times \mathbb{R}^{n_2} \simeq X^\odot$ we have

$$\begin{aligned}
\langle T^{\odot *}(t) \begin{pmatrix} f \\ g \\ d \end{pmatrix}, (\rho, \sigma, c)\rangle &= \langle \begin{pmatrix} f \\ g \\ d \end{pmatrix}, (\check{\rho}, \check{\sigma}, c + \int_{\gamma(t)}^l \sigma(x)\, dx)\rangle \\
&= \int_0^l \check{\rho} f(x)\, dx + \int_0^l \check{\sigma} g(x)\, dx + \int_{\gamma(t)}^l \sigma(x)\, dx \cdot d + c \cdot d \\
&= \int_0^l \rho(x) \hat{f}(x)\, dx + \int_0^l \sigma(x) (\widehat{g,d})(x)\, dx + c \cdot d.
\end{aligned}$$

Now $(\widehat{g,d})(x)$ means left translation of g with characteristic speeds K_v extending at $x = l$ with $D \cdot f + d$. $\hat{f}(x)$ means right translation of f with characteristic speeds K_u extended at $x = 0$ with $E \cdot g$.

Theorem 4.3. *For sufficiently small $t \geq 0$*

$$T^{\odot *}(t) \begin{pmatrix} f \\ g \\ d \end{pmatrix} = \begin{pmatrix} \hat{f} \\ (\widehat{g,d}) \\ d \end{pmatrix}.$$

The canonical injection $j : X \to X^{\odot *}$ takes the form

$$j \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \\ v(l) - Du(l) \end{pmatrix}.$$

The extended semigroup $T^{\odot *}$ is like T just that there is an additional freedom in extending the left traveling solution $v(t,x)$ at $x = l$ by the formula

$$v(t,l) = Du(t,l) + d, \quad d \in \mathbb{R}^{n_2},$$

in a possibly discontinuous manner. If $d = v(l) - Du(l)$ then the solutions stay in X, or in other words $j \circ T(t) = T^{\odot *} \circ j$.

According to (4.3) the map $t \mapsto T^{\odot *}(t) \begin{pmatrix} f \\ g \\ d \end{pmatrix}$ is strongly continuous if and only if $\begin{pmatrix} f \\ g \\ d \end{pmatrix} \in j(X)$. Hence

$$X^{\odot\odot} = j(X) \simeq X.$$

Therefore Lemma 2.6 holds true. Moreover we have

Lemma 4.4. *The map*

$$\begin{pmatrix} u \\ v \end{pmatrix} \in X \mapsto \begin{pmatrix} \mathfrak{H}(u,v) \\ F(u,v) \end{pmatrix} \in X^{\odot *}$$

is C^k smooth.

In the following definition we need to identify X with $j(X) \subset X^{\odot *}$.

Definition 4.5 (Variation of constants formula)**.** *Let $T > 0$. The pair $(u,v) \in C([0,T], X)$ is called a mild (or weak) solution to* (SHD) *if*

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = T(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t T^{\odot *}(t-s) \begin{pmatrix} \mathfrak{H}(u(s), v(s)) \\ F(u(s), v(s)) \end{pmatrix} ds.$$

The results of section 3 transfer to hyperbolic systems with dynamic boundary conditions (SHD).

References

[1] T. Cazenave and A. Haraux, *An introduction to semilinear evolution equations*, Oxford: Clarendon Press, 1998, Oxford Lecture Series in Mathematics and its Applications 13.

[2] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H.-P. Walther, *Delay equations*, Springer New York, 1995.

[3] M. A. Krasnoselskij et al., *Integral operators in spaces of summable functions*, Noordhoff International Publishing, 1976.

[4] M. Lichtner, *Exponential dichotomy and smooth invariant center manifolds for semilinear hyperbolic systems*, PhD thesis (2006), http://dochost.rz.hu-berlin.de/browsing/dissertationen/.

[5] ______, *Principle of linearized stability and center manifold theorem for semilinear hyperbolic systems*, WIAS preprint 1155 (2006).

[6] ______, *Spectral mapping theorem for linear hyperbolic systems*, WIAS preprint 1150 (2006), 13.

[7] Joel Smoller, *Shock waves and reaction-diffusion equations*, Springer New York, 1994.

[8] Jan van Neerven, *The adjoint of a semigroup of linear operators*, Lecture Notes in Mathematics 1529, Springer Berlin Heidelberg, 1992.