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An Inverse Problem for Fluid-Solid Interaction

J. Elschner[2], G.C. Hsiao[1], and A. Rathsfeld[2]

[1] *Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716, U.S.A.*

[2] *Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, 10117 Berlin, Germany*

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

Any acoustic plane wave incident to an elastic obstacle results in a scattered field with a corresponding far field pattern. Mathematically, the scattered field is the solution of a transmission problem coupling the reduced elastodynamic equations over the domain occupied by the obstacle with the Helmholtz equation in the exterior. The far field pattern is obtained applying an integral operator to the scattered field function restricted to a simple smooth surface surrounding the obstacle. The subject of our paper is the inverse problem, where the shape of the elastic body represented by a parametrization of its boundary is to be reconstructed from a finite number of measured far field patterns.

We define a family of objective functionals depending on a non-negative regularization parameter such that, for regularization parameter zero, the shape of the sought elastic obstacle is a minimizer of the functional. For any positive regularization parameter, there exists a regularized solution minimizing the functional. Moreover, for the regularization parameter tending to zero, these regularized solutions converge to the solution of the inverse problem provided the latter is uniquely determined by the given far field patterns. The whole approach is based on the variational form of the partial differential operators involved. Hence, numerical approximations can be found applying finite element discretization. Note that, though the transmission problem in its weak formulation may have non-unique solutions for domains with so-called Jones frequencies, the scattered field and its far field pattern is unique and depend continuously on the shape of the obstacle.

1 Introduction

If an elastic body is surrounded by a fluid and if an acoustic plane wave is incident, then an elastic wave is incited inside of the body, and the acoustic wave in the fluid is scattered. This phenomenon is modelled by a transmission problem for the displacement amplitude and the acoustic pressure. The displacement amplitude satisfies the reduced elastodynamic equations inside the body, and the acoustic pressure is a solution of the Helmholtz equation in the domain exterior to it. On the boundary of the body the traction of the displacement amplitude points into the normal direction and is equal to the acoustic pressure from the outside. Moreover, the normal component of the displacement is proportional to the normal derivative of the pressure over the boundary of the body. Finally, the scattered field satisfies the Sommerfeld radiation condition at infinity.

The Fredholm theory for the variational equations corresponding to such a fluid-solid interaction is well established and several variants of finite element and boundary element methods have been proposed for the numerical solution (cf. e.g. [14, 6, 15, 10]). Note that the homogeneous transmission problem may have eigensolutions for special shapes of the elastic obstacle and for special values of the frequency (cf. e.g. [6, 17]). As usual, the scattered acoustic field has the typical behaviour of outgoing Helmholtz solutions characterized by the so-called far field pattern. This pattern can be expressed by an integral operator applied to the scattered field (cf. e.g. [3]). Measuring the acoustic field at points of large distance to the elastic body, the far field pattern can be detected.

Similarly to inverse problems for the scattering of acoustic waves by sound hard and soft obstacles, inverse problems for the fluid-solid interaction can be formulated. For instance, suppose the shape of an elastic obstacle is unknown, but the far field patterns of scattered waves, resulting from certain incident plane waves, are known. The inverse problem is to recover the shape of the elastic scatterer from the measured far field patterns. To our knowledge, uniqueness of solutions for the inverse problem with a finite number of given far field patterns have not been investigated yet. For the uniqueness in the case of known far fields for all incident directions we refer to [16]. Treating the numerical solution, the number of given far field patterns is finite, and all the methods developed for the reconstruction in the case of sound hard and soft obstacles (cf. e.g. [3]) should have counterparts for the case of elastic scatterers.

In the present paper, we formulate the mathematical model of the solid-fluid interaction in Section 2. We introduce the variational formulations of the transmission problem for the reduced elastodynamic and the Helmholtz equations. The unbounded exterior domain is truncated by a boundary integral equation method. Following [10], we prove that the sesqui-linear variational form satisfies a Gårding's inequality. If a technical condition for the boundary integral operator is satisfied, then the variational equation has a unique acoustic field solution. This is true even in the exceptional case where the elastic wave is not unique. In this case, the variational equation cannot be solved directly. Instead, the variational equation should be solved for a slightly modified frequency. We prove that this commonly known approximation is correct, i.e., that the acoustic field solutions for the modified frequencies converge to the true solution if the perturbed frequencies tend to the correct value.

In Section 3 we introduce the inverse problem. We restrict the class of obstacles to starlike domains with boundary parametrizations from a Sobolev space. Our inverse problem is ill-posed, and its solution requires a regularization. Including such a regularization, we propose three different reformulations of the inverse problem in form of optimization problems. These reformulations differ essentially in the number of unknown optimization parameters after discretization. We analyze the reformulation with the smallest number of optimization parameters. For this, the objective functional is the least squares deviation of the measured far field patterns from those corresponding to the obstacle which is to be optimized. Of course, we add the scaled square norm of the boundary parametrizations for regularization. On the other hand, we prove that the scattered acoustic field depends continuously on the shape of the obstacles even if the incited elastic wave is not unique. Consequently, the objective functional of the reformulated inverse problem is continuous, and, for any regularization parameter, there exists a regularized solution, i.e. a minimizer of the regularized objective functional. In case the involved far field patterns determine the obstacle uniquely, the regularized solutions converge to this unique obstacle whenever the regularization parameter tends to zero. For this convergence, we can even admit measurement errors in the size of the regularization parameter.

We conclude the paper with some details for the numerical solution. In Section 4 we derive a formula for the directional derivative of the scattered acoustic field with respect to the parametrization of the obstacle boundary. The gradient computation is based on the solution of the variational equation of the transmission problem with modified right-hand sides. Thus this gradient formula is efficient if the discretized variational equation, i.e. the finite element system, is solved by a direct solver which may be adapted to systems with sparse coefficient matrices (cf. [20]). In Section 5 we discuss the numerical solution of the optimization problems. The parametrizations of the obstacle boundaries are reduced to finite series of trigonometric functions resp. spherical harmonics. Having formulas for the gradients at our disposal, we suggest the Gauß-Newton method (cf. [18]) for the numerical computation of the minimizers. The computations for a simple two-dimensional scatterer presented in Section 7 confirm the theoretical results of the paper.

2 Direct Problem

The direct problem is the following. Suppose a bounded elastic body is given occupying the bounded domain Ω in the two- or three-dimensional space $\mathbb{R}^d$ (d=2 or d=3). This body is surrounded by a homogenous compressible inviscid fluid filling the complementary space $\mathbb{R}^d \setminus \Omega$. If an acoustic time-harmonic wave with the scalar pressure field $p^{inc}(x)e^{\mathbf{i}\omega t}$ is incident on the elastic obstacle in Ω, then this wave is scattered, i.e. the amplitude p of the total pressure $p(x)^{\mathbf{i}\omega t}$ is the sum of the incoming p^{inc} and the scattered wave amplitude p^s. The interaction with the elastic body is controlled by the incited elastic wave described by the three-dimensional displacement amplitude u. Clearly, u satisfies the reduced

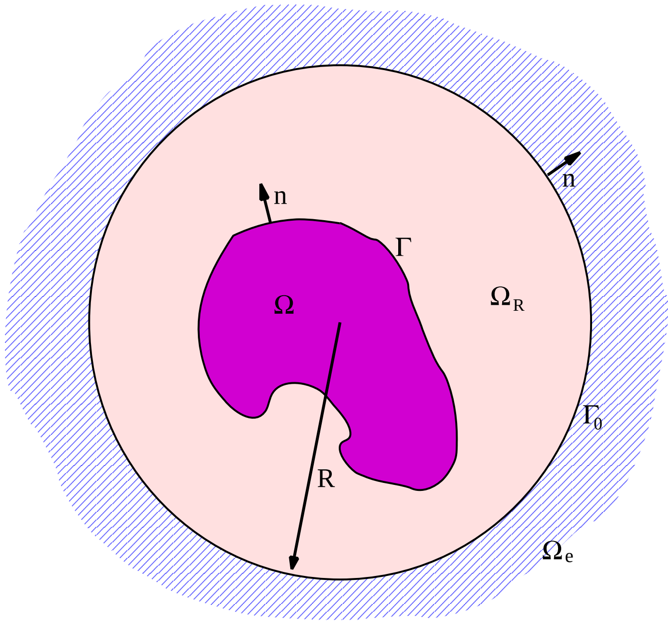

Figure 1: Domains.

elastodynamic equations

$$\begin{aligned} \Delta^{*} u(x) + \varrho\,\omega^2 u(x) &= 0, \quad x \in \Omega, \\ \Delta^{*} u(x) &:= \mu\,\Delta u(x) + (\lambda+\mu)\,\nabla\,[\nabla \cdot u(x)], \end{aligned} \tag{2.1}$$

with the density ϱ of the elastic body and with the Lamé constants λ and μ such that $\lambda + \frac{2}{d}\mu > 0$ and $\mu > 0$. The pressures p, p^s and p^{inc} are solutions of the Helmholtz equation in $\mathbb{R}^d \setminus \mathrm{cl}(\Omega)$. However, for the numerical treatment, we truncate $\mathbb{R}^d \setminus \mathrm{cl}(\Omega)$. We suppose the origin is in Ω and, for a large R greater than the radius of Ω, we introduce the annular domain $\Omega_R := \{x \in \mathbb{R}^d \setminus \mathrm{cl}(\Omega) : \ |x| < R\}$ (cf. Figure 1). The Helmholtz equation for p^s, e.g., takes the form

$$\Delta p^s(x) + {k_w}^2 p^s(x) = 0, \quad x \in \Omega_R \tag{2.2}$$

with ${k_w}^2 = \omega^2/c^2$ and c the speed of sound in the fluid. The equations (2.1) and (2.2) are coupled by transmission conditions on the boundary values of u and p on the boundary $\Gamma = \partial\Omega$ of Ω, and the truncation of $\mathbb{R}^d \setminus \mathrm{cl}(\Omega)$ to Ω_R is to be modelled by a non-local boundary condition on the outer boundary $\Gamma_0 := \partial\Omega_R \setminus \Gamma$ of Ω_R (cf. (2.3)). To define the non-local boundary condition, we denote the exterior normals at the curves/surfaces Γ and Γ_R by n (cf. Figure 1) and we introduce the boundary traces

$$\begin{gathered} u^- := u|_\Gamma, \quad p^+ := p^s|_\Gamma, \quad p^- := p^s|_{\Gamma_0}, \\ \tfrac{\partial p^+}{\partial n} := n \cdot [\nabla\, p^s]|_\Gamma, \quad \sigma := \tfrac{\partial p^-}{\partial n} := n \cdot [\nabla\, p^s]|_{\Gamma_0}, \\ t^- := t^-(u) := 2\mu \tfrac{\partial u}{\partial n}\Big|_\Gamma + \lambda[\nabla \cdot u]\,n\Big|_\Gamma + \mu \begin{cases} \begin{pmatrix} n_2(\partial_{x_1} u_2 - \partial_{x_2} u_1) \\ n_1(\partial_{x_2} u_1 - \partial_{x_1} u_2) \end{pmatrix}\Bigg|_\Gamma & \text{if } d=2 \\ n \times [\nabla \times u]\Big|_\Gamma & \text{if } d=3 \end{cases} . \end{gathered}$$

Using boundary integral operators to describe the Dirichlet-to-Neumann mapping for the continuations to infinity, the truncation condition on Γ_0 takes the form (cf. e.g. [12])

$$\begin{aligned} V_{\Gamma_0}\sigma + \left(\frac{1}{2}I - K_{\Gamma_0}\right)p^- &= 0, \quad \text{on } \Gamma_0, \\ K_{\Gamma_0}p^-(x) &:= \int_{\Gamma_0} \frac{\partial E(x,y)}{\partial n(y)} p^-(y)\mathrm{d}_{\Gamma_0}y, \\ V_{\Gamma_0}\sigma(x) &:= \int_{\Gamma_0} E(x,y)\sigma(y)\mathrm{d}_{\Gamma_0}y, \\ E(x,y) &:= E_{k_w}(x,y) := \begin{cases} \frac{\mathbf{i}}{4}H_0^{(1)}(k_w|x-y|) & \text{if } d=2 \\ \dfrac{1}{4\pi}\dfrac{e^{\mathbf{i}k_w|x-y|}}{|x-y|} & \text{if } d=3 \end{cases} \end{aligned} \tag{2.3}$$

with $H_0^{(1)}$ the Hankel function of the first kind and of order 0. If ϱ_f stands for the fluid density, then the boundary conditions on the boundary Γ can be described as

$$t^-(x) = -\left\{p^+(x) + p^{inc}(x)\right\} n(x), \quad x \in \Gamma, \tag{2.4}$$

$$u^-(x)\cdot n(x) = \frac{1}{\varrho_f\omega^2}\left\{\frac{\partial p^+(x)}{\partial n} + \frac{\partial p^{inc}(x)}{\partial n}\right\}, \quad x \in \Gamma. \tag{2.5}$$

Altogether, the incited displacement u and the scattered scalar pressure field p^s are the solutions of the elliptic partial differential equations (2.1) and (2.2) with the boundary condition (2.3) and the transmission conditions (2.5) and (2.4).

Of course, the scattered field p^s extends to the exterior domain $\Omega_e := \mathbb{R}^d \setminus \mathrm{cl}(\Omega \cup \Omega_R)$ as the solution of the exterior boundary value problem including the Sommerfeld radiation condition (note that σ and p^- are interrelated by (2.3))

$$\begin{aligned} \Delta p^s(x) + {k_w}^2 p^s(x) &= 0, \quad x \in \Omega_e, \\ p^s(x) &= p^-(x), \quad x \in \Gamma_0, \\ \frac{\partial p^s(x)}{\partial n} &= \sigma(x), \quad x \in \Gamma_0, \\ \frac{x}{|x|}\cdot\nabla p^s(x) - \mathbf{i}k_w p^s(x) &= o\left(|x|^{-(d-1)/2}\right), \quad |x| \longrightarrow \infty. \end{aligned}$$

It is well known that this exterior solution has a so-called far field pattern p^∞, i.e. asymptotically the relation

$$p^s(x) = \frac{e^{\mathbf{i}k_w|x|}}{|x|^{(d-1)/2}}\, p^\infty\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|^{(d+1)/2}}\right), \quad |x| \longrightarrow \infty, \tag{2.6}$$

$$p^\infty(\hat{x}) = \begin{cases} -\dfrac{e^{\mathbf{i}\pi/4}}{\sqrt{8\pi k_w}} \displaystyle\int_{\Gamma_0} \Big\{p^-(y)[\mathbf{i}k_w\, \hat{x}\cdot n(y)] + \sigma(y)\Big\} e^{-\mathbf{i}k_w\hat{x}\cdot y}\mathrm{d}_{\Gamma_0}y & \text{if } d=2 \\ -\dfrac{1}{4\pi}\displaystyle\int_{\Gamma_0} \Big\{p^-(y)[\mathbf{i}k_w\, \hat{x}\cdot n(y)] + \sigma(y)\Big\} e^{-\mathbf{i}k_w\hat{x}\cdot y}\mathrm{d}_{\Gamma_0}y & \text{if } d=3, \end{cases} \tag{2.7}$$

$$\hat{x} \in \mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \ |x| = 1\}$$

holds (cf. e.g. [3]). We shortly write $p^\infty = \mathcal{H}(p^-, \sigma)$ for the representation of the far field pattern in formula (2.7).

The transmission and boundary value problem (2.1)-(2.5) can be reformulated in a weak form. This standard variational equation for the unknown solution vector (u, p, σ) with $u \in [H^1(\Omega)]^d$, $p \in H^1(\Omega_R)$, and $\sigma \in H^{-1/2}(\Gamma_0)$ takes the form

$$\mathcal{B}\Big((u,p,\sigma)^\top,(v,q,\chi)^\top\Big) \;=\; \mathcal{R}\Big(p^{inc},(v,q,\chi)^\top\Big) \;:=\; \begin{pmatrix} -\int_\Gamma p^{inc} n\cdot\overline{v} \\ -\int_\Gamma \dfrac{\partial p^{inc}}{\partial n}\,\overline{q} \\ 0 \end{pmatrix} \tag{2.8}$$

valid for any $v \in [H^1(\Omega)]^d$, $q \in H^1(\Omega_R)$, and $\chi \in H^{-1/2}(\Gamma_0)$. The sesqui-linear form $\mathcal{B}$ is given by

$$\begin{aligned}
\mathcal{B}\Big((u,p,\sigma)^\top,(v,q,\chi)^\top\Big) &:= \begin{pmatrix} a\left((u,p,\sigma)^\top,(v,q,\chi)^\top\right) \\ b\left((u,p,\sigma)^\top,(v,q,\chi)^\top\right) \\ c\left((u,p,\sigma)^\top,(v,q,\chi)^\top\right) \end{pmatrix}, \\
a\left((u,p,\sigma)^\top,(v,q,\chi)^\top\right) &:= \int\limits_\Omega \Big\{\lambda\,\nabla\cdot u\,\overline{\nabla\cdot v} + \frac{\mu}{2}\sum_{i,j=1}^{d}\left[\partial_i u_j\overline{\partial_j v_i} + \partial_i u_j\overline{\partial_i v_j}\,\right] - \varrho\omega^2 u\cdot\overline{v}\Big\} \\
&\quad + \int\limits_\Gamma p^+\, n\cdot\overline{v}\,, \\
b\left((u,p,\sigma)^\top,(v,q,\chi)^\top\right) &:= \int\limits_{\Omega_R} \Big\{\nabla p\cdot\overline{\nabla q} - {k_w}^2 p\overline{q}\Big\} + \varrho_f\omega^2\int\limits_\Gamma u^-\cdot n\,\overline{q} - \int\limits_{\Gamma_0}\sigma\,\overline{q}\,, \\
c\left((u,p,\sigma)^\top,(v,q,\chi)^\top\right) &:= \int\limits_{\Gamma_0}\Big\{V_{\Gamma_0}\sigma + \Big(\frac{1}{2}I - K_{\Gamma_0}\Big)p^-\Big\}\overline{\chi}
\end{aligned}$$

Following [10], the basic results for the direct problems can be summarized as follows.

Theorem 2.1. *The sesqui-linear form $\mathcal{B}((u,p,\sigma)^\top,(v,q,\chi)^\top)$ defined in* (2.8) *satisfies a Gårding's inequality in the form*

$$\begin{aligned}
\Re e\ \mathbf{B}(v,q,\chi;v,q,\chi) \;\ge\; & \alpha\Big\{||v||^2_{[H^1(\Omega)]^d} + ||q||^2_{H(\Omega_R)} + ||\chi||^2_{H^{-1/2}(\Gamma_0)}\Big\} \\
& -c_0\Big\{||v||^2_{[H^{1-\varepsilon}(\Omega)]^d} + ||q||^2_{H^{1-\varepsilon}(\Omega_R)} + ||\chi||^2_{H^{-1/2-\varepsilon}(\Gamma_0)}\Big\}
\end{aligned}$$

for all $(v,q,\chi) \in [H^1(\Omega)]^d \times H^1(\Omega_R) \times H^{-1/2}(\Gamma_0)$, where $\varepsilon > 0$ is a small number, $\alpha > 0$ and $c_0 = c_0(\varepsilon) \ge 0$ are constants, and

$$\mathbf{B}(v,q,\chi;v,q,\chi) := a\left((v,q,\chi)^\top,(v,q,\chi)^\top\right) + b\left((v,q,\chi)^\top,(v,q,\chi)^\top\right) + 2\ c\left((v,q,\chi)^\top,(v,q,\chi)^\top\right).$$

Theorem 2.1 implies that the Fredholm alternative is applicable to (2.8). In order to ensure the existence of the solution, we need the

Theorem 2.2. *If* (a) *the boundary Γ and the material constants (μ,λ,ρ) are such that there are no nontrivial solutions of*

$$\Delta^* u_0(x) + \rho\omega^2 u_0(x) = 0,\ x\in\Omega, \qquad t^-(u_0)(x) = 0,\ x\in\Gamma, \qquad u_0^-(x)\cdot n = 0,\ x\in\Gamma, \tag{2.9}$$

and (b) $-{k_w}^2$ *is not an eigenvalue of the interior Dirichlet problem for the Helmholtz equation inside Γ_0, then the corresponding homogeneous problem of* (2.8) *has only the trivial solution $(u,p,\chi) = (0,0,0) \in [H^1(\Omega)]^d \times H^1(\Omega_R) \times H^{-1/2}(\Gamma_0)$ and* (2.8) *is uniquely solvable. If condition* (b) *holds but the solution of the homogeneous equation* (2.9) *is not unique, then the special form of the right-hand side in* (2.8) *guarantees the existence of a solution (u,p,χ). In this case, the components p and χ are unique.*

Theorem 2.3. *Suppose $\omega = \omega_0$ is a frequency such that the homogeneous problem* (2.9) *has a nontrivial solution. Then there is a small $\varepsilon > 0$ such that* (2.8) *has a unique solution for any frequency $\omega \ne \omega_0$ with $|\omega-\omega_0| < \varepsilon$. If condition* (b) *holds for all frequencies ω with $|\omega-\omega_0| < \varepsilon$ and if the solution of* (2.8) *is denoted by $(u^\omega,p^\omega,\sigma^\omega)$, then the unique solution components p^ω and σ^ω depend continuously on ω for ω satisfying $|\omega-\omega_0| < \varepsilon$. In particular, $\lim_{\omega\to\omega_0} p^\omega = p^{\omega_0}$ and $\lim_{\omega\to\omega_0}\sigma^\omega = \sigma^{\omega_0}$.*

Nontrivial solutions of (2.9) are often referred to as Jones modes, and the associated frequencies are called Jones frequencies (cf. [15, 11, 6] resp. [17] for divergence free Jones modes and formulas for spherical modes). It is known that Jones frequencies exist for spheres and other axisymmetric bodies (cf. [15, 11] and the references therein), whereas Jones modes are not supported by "almost every" elastic body of "arbitrary" shape (cf. [8]). If $\omega = \omega_0$ is a Jones frequency, then the solutions p and σ can be approximated by a regularization scheme based on Theorem 2.3. Solving (2.8) for a slightly perturbed frequency ω', provides us with an approximate solution (p, σ) close to the solution of (2.8) for the exact frequency ω.

The simple-layer operator V_{Γ_0} defines an isomorphism

$$V_{\Gamma_0} : H^{-1/2}(\Gamma_0) \to H^{1/2}(\Gamma_0),$$

if $-{k_w}^2$ is not an eigenvalue (cf. e.g. [13] and [9] for explicit formulas of the eigenvalues of V_{Γ_0}). If condition (a) holds but (b) is violated, then (2.8) with vanishing right-hand side has a non-trivial solution. In this case the sesqui-linear form $\mathcal{B}$ should be modified by replacing the integral equation (2.3) with an indirect approach based on the linear combination of simple- and double-layer potential (cf. e.g. [1]) or with an equation including the Dirichlet-to-Neumann mapping, which can easily be evaluated using the series expansions into spherical harmonics. For ease of reading, the proofs of the Theorems 2.1, 2.2 and 2.3 will be presented in the Appendix. The proofs of the Theorems 2.1 and 2.2 are essentially contained in [10].

We denote the solution operator corresponding to the equation with sesqui-linear form (2.8) by

$$S_\Gamma : H^1_{loc}(\mathbb{R}^d \setminus \Omega) \ni p^{inc} \mapsto (u, p, \sigma)^\top \in [H^1(\Omega)]^d \times H^1(\Omega_R) \times H^{-1/2}(\Gamma_0)\,,$$

$$S_\Gamma(p^{inc}) = \begin{pmatrix} S^u_\Gamma(p^{inc}) \\ S^p_\Gamma(p^{inc}) \\ S^\sigma_\Gamma(p^{inc}) \end{pmatrix} = \begin{pmatrix} u \\ p \\ \sigma \end{pmatrix}.$$

Using this, we introduce the restricted solution operators $A^\sigma_\Gamma : H^1_{loc}(\mathbb{R}^d \setminus \Omega) \to H^{-1/2}(\Gamma_0)$ and $A^p_\Gamma : H^1_{loc}(\mathbb{R}^d \setminus \Omega) \to H^{1/2}(\Gamma_0)$ by setting

$$A^\sigma_\Gamma(p^{inc}) \;:=\; S^\sigma_\Gamma(p^{inc}), \quad A^p_\Gamma(p^{inc}) \;:=\; S^p_\Gamma(p^{inc})\big|_{\Gamma_0}. \tag{2.10}$$

3 Inverse Problem

The inverse problem is the following. Given a finite number of incident fields p^{inc}_k, $k = 1, 2, \ldots, K$ and the corresponding far field patterns p^∞_k (cf. (2.6)) measured for the scattered fields incited by the incident acoustic waves, we seek the unknown curve/surface Γ located in the interior of Γ_0 such that the p^∞_k coincide with the far field patterns corresponding to the fields scattered by the elastic body bounded by Γ. In other words we seek Γ such that

$$p^\infty_k \;=\; \mathcal{H}\Big(A^p_\Gamma(p^{inc}_k), A^\sigma_\Gamma(p^{inc}_k)\Big), \quad k = 1, \ldots, K. \tag{3.1}$$

The inverse problem can be reformulated in various ways as an optimization problem. We shall introduce three different objective functionals, the optimizations of which provide us with approximate solutions of the inverse problem.

Firstly, we fix the class of curves/surfaces Γ in which the solution is sought. We suppose that Γ is starlike, i.e. Γ can be represented as $\Gamma = \{\mathbf{r}(\hat{x})\hat{x} : \hat{x} \in \mathbb{S}^{d-1}\}$ with a continuously differentiable function $\mathbf{r} : \mathbb{S}^{d-1} \to \mathbb{R}$. For a curve/surface inside of Γ_R, we have to require $0 < \mathbf{r}(\hat{x}) < R$ for all $\hat{x} \in \mathbb{S}^{d-1}$. To get rid of these constraints, we change the representation to

$$\Gamma \;=\; \Gamma^{\mathbf{r}} \;:=\; \big\{\mathbf{r}_R(\hat{x})\hat{x} : \hat{x} \in \mathbb{S}^{d-1}\big\}, \quad \mathbf{r}_R(\hat{x}) \;:=\; \frac{R}{2} + \frac{R}{\pi}\arctan\big(\mathbf{r}(\hat{x})\big) \tag{3.2}$$

with an arbitrary continuously differentiable function $\mathbf{r}$. In the subsequent numerical schemes we shall look for an approximation of $\mathbf{r}$ as a truncated series expansion into spherical harmonics. Therefore, we restrict the class of functions to the Sobolev space $H^\delta(\mathbb{S}^{d-1})$, the norm of which can be easily expressed in terms of the coefficients of the spherical harmonics. Moreover, to avoid slow approximation of corners, vertices, and edges, we suppose $\mathbf{r} \in H^\delta(\mathbb{S}^{d-1})$ with a fixed $\delta > (d+1)/2$. The embedding of $H^\delta(\mathbb{S}^{d-1})$ in $C^1(\mathbb{S}^{d-1})$ will later be crucial in the proof of a continuous dependence of the objective functional with respect to the curve/surface (cf. Lemma 3.1).

Next, we fix a constant $\alpha > 0$ and a small regularization parameter $\gamma \geq 0$. Depending on α and γ, the first objective functional is defined by

$$\begin{aligned}\mathcal{J}_1(\mathbf{r}, p_1, p_2, \dots, p_K, \sigma_1, \sigma_2, \dots, \sigma_K; \gamma) := \sum_{k=1}^{K} \Big\{ & \|\mathcal{H}(p_k, \sigma_k) - p_k^\infty\|^2_{L^2(\mathbb{S}^{d-1})} + \alpha \left\|p_k - A^p_{\Gamma^{\mathbf{r}}}(p_k^{inc})\right\|^2_{H^{1/2}(\Gamma_0)} \\ & + \alpha \left\|\sigma_k - A^\sigma_{\Gamma^{\mathbf{r}}}(p_k^{inc})\right\|^2_{H^{-1/2}(\Gamma_0)} + \gamma \|p_k\|^2_{H^{1/2}(\Gamma_0)} \\ & + \gamma \|\sigma_k\|^2_{H^{-1/2}(\Gamma_0)} \Big\} + \gamma \|\mathbf{r}\|^2_{H^\delta(\mathbb{S}^{d-1})} \end{aligned} \tag{3.3}$$

with $\mathbf{r} \in H^\delta(\mathbb{S}^{d-1})$ and with $p_k \in H^{1/2}(\Gamma_0)$ resp. $\sigma_k \in H^{-1/2}(\Gamma_0)$ for $k = 1, \dots K$. Since the number α is fixed in the subsequent considerations the dependence on α is suppressed in the argument list of the functional. For the objective functional in (3.3), we consider the optimization problem of finding a solution $(\mathbf{r}^\gamma, p_1^\gamma, p_2^\gamma, \dots, p_K^\gamma, \sigma_1^\gamma, \sigma_2^\gamma, \dots, \sigma_K^\gamma)$ such that

$$\begin{aligned}&\mathcal{J}_1(\mathbf{r}^\gamma, p_1^\gamma, p_2^\gamma, \dots, p_K^\gamma, \sigma_1^\gamma, \sigma_2^\gamma, \dots, \sigma_K^\gamma; \gamma) = \\ &\qquad \inf_{\substack{\mathbf{r} \in H^\delta(\mathbb{S}^{d-1}) \\ p_k \in H^{1/2}(\Gamma_0), k=1,\dots,K \\ \sigma_k \in H^{-1/2}(\Gamma_0), k=1,\dots,K}} \mathcal{J}_1(\mathbf{r}, p_1, p_2, \dots, p_K, \sigma_1, \sigma_2, \dots, \sigma_K; \gamma).\end{aligned} \tag{3.4}$$

Obviously, the inverse problem (3.1) has a solution Γ generated by a $H^\delta(\mathbb{S}^{d-1})$ function through (3.2) if and only if the optimization problem (3.4) with $\gamma = 0$ has a solution and the minimal value of the objective functional $\mathcal{J}_1(\dots; 0)$ is zero. Indeed, the solution of the inverse problem is $\Gamma = \Gamma^{\mathbf{r}^0}$ and the functions p_k^0 and σ_k^0 are the traces on Γ_0 of the scattered pressure and its normal derivative, respectively, incited by the incoming wave p_k^{inc}. For the numerical solution of the optimization problem (3.4) with $\gamma = 0$, we have to take into account that the mapping defined by the right-hand side of (3.1) is severely ill-posed (cf. the representation of $\mathcal{H}$ in (2.7) as an integral operator with smooth kernel function). Therefore, we have introduced the regularization terms with coefficient γ in (3.3). Choosing a suitably small $\gamma > 0$, the numerical solution of (3.4) yields a regularized approximation which is much closer to the exact solution of (3.4) with $\gamma = 0$ than the direct numerical solution of (3.4) with $\gamma = 0$.

Unfortunately, the optimization problem (3.4) has a large number of unknowns to be optimized. Indeed, for each measured far field pattern p_k^∞, there appear two extra unknowns p_k and σ_k. In order to reduce this to only one extra unknown, we represent the pressure p in the domain Ω_e exterior to Γ_0 by a simple-layer potential with a density φ_k and get (cf. [12, 3])

$$\begin{aligned} p_k(x) &= \int_{\Gamma_0} E(x,y)\varphi_k(y)\mathrm{d}_{\Gamma_0}y, \quad x \in \Omega_e, \\ p_k(x) &= V_{\Gamma_0}\varphi_k(x), \quad x \in \Gamma_0, \\ p_k^\infty(\hat{x}) &= \mathcal{H}\varphi_k(\hat{x}) := \mathcal{H}(0, -\varphi_k)(\hat{x}), \quad \hat{x} \in \mathbb{S}^{d-1}. \end{aligned}$$

Using this notation, the second objective functional can be defined as

$$\begin{aligned}\mathcal{J}_2(\mathbf{r}, \varphi_1, \varphi_2, \dots, \varphi_K; \gamma) := \sum_{k=1}^{K} \Big\{ & \|\mathcal{H}\varphi_k - p_k^\infty\|^2_{L^2(\mathbb{S}^{d-1})} + \alpha \left\|A^p_{\Gamma^{\mathbf{r}}}(p_k^{inc}) - V_{\Gamma_0}\varphi_k\right\|^2_{L^2(\Gamma_0)} \\ & + \gamma \|\varphi_k\|^2_{L^2(\Gamma_0)} \Big\} + \gamma \|\mathbf{r}\|^2_{H^\delta(\mathbb{S}^{d-1})} \end{aligned} \tag{3.5}$$

and the corresponding optimization problem is to find a solution vector $(\mathbf{r}^\gamma, \varphi_1^\gamma, \varphi^\gamma, \dots, \varphi_K^\gamma)$ such that

$$\mathcal{J}_2(\mathbf{r}^\gamma, \varphi_1^\gamma, \varphi_2^\gamma, \dots, \varphi_K^\gamma; \gamma) \;=\; \inf_{\substack{\mathbf{r}\in H^\delta(\mathbb{S}^{d-1})\\ \varphi_k\in H^{-1/2}(\Gamma_0),k=1,\dots,K}} \mathcal{J}_2(\mathbf{r}, \varphi_1, \varphi_2, \dots, \varphi_K; \gamma). \tag{3.6}$$

Similarly to the first optimization problem (3.4), the inverse problem (3.1) has a solution Γ generated by a $H^\delta(\mathbb{S}^{d-1})$ function through (3.2) if and only if the optimization problem (3.6) with $\gamma = 0$ has a solution and if the minimal value of the objective functional $\mathcal{J}_2(\dots; 0)$ is zero. Again, for the numerical solution of the optimization problem (3.4) with $\gamma = 0$, we have introduced the regularization terms with coefficient γ in (3.5).

A drastic reduction in the number of unknown functions for the optimization can be reached, if we eliminate the boundary traces completely. This leads us to the third objective functional

$$\mathcal{J}(\mathbf{r}; \gamma) \;:=\; \mathcal{J}_3(\mathbf{r}; \gamma) := \sum_{k=1}^{K} \left\|\mathcal{H}\big(A^p_{\Gamma^{\mathbf{r}}}(p_k^{inc}), A^\sigma_{\Gamma^{\mathbf{r}}}(p_k^{inc})\big) - p_k^\infty\right\|^2_{L^2(\mathbb{S}^{d-1})} + \gamma\,\|\mathbf{r}\|^2_{H^\delta(\mathbb{S}^{d-1})}, \tag{3.7}$$

and the corresponding optimization problem is to find a solution $\mathbf{r}^\gamma$ such that

$$\mathcal{J}(\mathbf{r}^\gamma; \gamma) \;=\; \inf_{\mathbf{r}\in H^\delta(\mathbb{S}^{d-1})} \mathcal{J}(\mathbf{r}; \gamma). \tag{3.8}$$

The equivalence of (3.8) with $\gamma = 0$ and the inverse problem (3.1) is obvious. Again, the parameter γ is introduced to regularize the numerical solution of (3.8) with $\gamma = 0$. In the following, we restrict our considerations to the optimization of this third objective functional. Note, however, that the first two functionals are based on a splitting of the operator into a linear ill-posed far field operator and an almost well-posed non-linear operator, which can be a starting point to design and analyse fast numerical two-step algorithms (cf. e.g. [7, 2, 3]).

For an analysis of the optimization problem (3.8) with $\gamma > 0$ and of its relation to (3.8) with $\gamma = 0$, the essential point is the continuity of the objective functional in (3.7). We get

Lemma 3.1. *Suppose condition* (b) *of Theorem 2.2 is satisfied for fixed* ω. *In accordance with Theorem 2.2, there is a unique solution of the weak transmission-boundary value problem* (2.8) *for any curve/surface* $\Gamma^{\mathbf{r}}$ *with* $\mathbf{r} \in H^\delta(\mathbb{S}^{d-1})$. *Then, for any fixed right-hand side function* $p^{inc} \in H^1_{loc}(\mathbb{R}^d \setminus \{0\})$, *the following mappings are continuous:*

$$H^\delta(\mathbb{S}^{d-1}) \ni \mathbf{r} \;\mapsto\; A^p_{\Gamma^{\mathbf{r}}}(p^{inc}) \in H^{1/2}(\Gamma_0)$$
$$H^\delta(\mathbb{S}^{d-1}) \ni \mathbf{r} \;\mapsto\; A^\sigma_{\Gamma^{\mathbf{r}}}(p^{inc}) \in H^{-1/2}(\Gamma_0)$$

Consequently, the mapping $H^\delta(\mathbb{S}^{d-1}) \ni \mathbf{r} \mapsto \mathcal{J}(\mathbf{r}, \gamma) \in L^2(\mathbb{S}^d)$ is continuous for all $\gamma \geq 0$, too. The assumption with condition (b) of Theorem 2.2 is only technical and can be circumvented as mentioned after Theorem 2.2. For ease of reading, the proof of the lemma will be presented in the Appendix.

Corollary 3.2. *Suppose condition* (b) *of Theorem 2.2 is satisfied for fixed* ω. *Then, for any fixed* $\gamma > 0$, *there exists a minimizer* $\mathbf{r}^\gamma$ *for the optimization problem* (3.8).

Proof. We conclude that $\|\mathbf{r}\| > \{\mathcal{J}(0,\gamma)/\gamma\}^{1/2}$ implies $\mathcal{J}(\mathbf{r},\gamma) \geq \gamma\|\mathbf{r}\|^2 > \mathcal{J}(0,\gamma) \geq \inf_{\mathbf{r}} \mathcal{J}(\mathbf{r},\gamma)$. Consequently, we get

$$\inf_{\mathbf{r}\in H^\delta(\mathbb{S}^{d-1})} \mathcal{J}(\mathbf{r},\gamma) \;=\; \inf_{\substack{\mathbf{r}\in H^\delta(\mathbb{S}^{d-1}):\\ \|\mathbf{r}\|\leq\{\mathcal{J}(0,\gamma)/\gamma\}^{1/2}}} \mathcal{J}(\mathbf{r},\gamma).$$

However, the set of all $\mathbf{r} \in H^\delta(\mathbb{S}^{d-1})$ with $\|\mathbf{r}\|_{H^\delta(\mathbb{S}^{d-1})} \leq \{\mathcal{J}(0,\gamma)/\gamma\}^{1/2}$ is compact with respect to the weak topology. From any sequence $\mathbf{r}_n$, $n = 1, 2, \dots$ with $\|\mathbf{r}_n\| \leq \{\mathcal{J}(0,\gamma)/\gamma\}^{1/2}$ and $\mathcal{J}(\mathbf{r}_n; \gamma) \leq$

$\inf_{\mathbf{r}\in H^\delta} \mathcal{J}(\mathbf{r};\gamma)+1/n$, we can choose a weakly convergent subsequence. For $\varepsilon>0$ with $\delta-\varepsilon>(d+1)/2$, the last subsequence has a subsequence $\mathbf{r}'_n$ converging in the space $H^{\delta-\varepsilon}(\mathbb{S}^{d-1})$. Indeed, $H^\delta(\mathbb{S}^{d-1})$ is compactly embedded in $H^{\delta-\varepsilon}(\mathbb{S}^{d-1})$. By Lemma 3.1 the mapping $\mathcal{J}(\mathbf{r};\gamma)$ is continuous with respect to $\mathbf{r}\in H^{\delta-\varepsilon}(\mathbb{S}^{d-1})$. In other words, the limit $\mathbf{r}'$ of the subsequence in $\mathbf{r}'_n\in H^{\delta-\varepsilon}(\mathbb{S}^{d-1})$ satisfies $\mathcal{J}(\mathbf{r}';\gamma)\le \inf_{\mathbf{r}\in H^\delta}\mathcal{J}(\mathbf{r};\gamma)$. On the other hand, $\mathbf{r}'$ coincides with the weak limit which proves $\mathbf{r}'\in H^\delta(\mathbb{S}^{d-1})$. □

Now the question arises whether the minimizers $\mathbf{r}^\gamma$ converge to a minimizer of the optimization problem (3.8) for $\gamma=0$, i.e., to a solution of the inverse problem (3.1). However, before we formulate the corresponding theorem, we modify the optimization problem slightly. Suppose we do not have the exact far field patterns p_k^∞. Instead, we have noisy measurement data $p_k^{\infty,\gamma,no}$ for the optimization with regularization parameter γ. For the asymptotic analysis, we suppose that the noisy data converges to zero for γ tending to zero. More precisely, we suppose there is a constant $c>0$ with

$$\sum_{k=1}^K \|p_k^\infty - p_k^{\infty,\gamma,no}\|^2_{L^2(\mathbb{S}^{d-1})} \le c\,\gamma. \tag{3.9}$$

If we replace the exact data by the measurement data in the objective functional (3.7), we arrive at the following functional and the corresponding modified optimization problem.

$$\mathcal{J}_{no}(\mathbf{r};\gamma) := \sum_{k=1}^K \left\|\mathcal{H}\big(A^p_{\Gamma^\mathbf{r}}(p_k^{inc}),A^\sigma_{\Gamma^\mathbf{r}}(p_k^{inc})\big)-p_k^{\infty,\gamma,no}\right\|^2_{L^2(\mathbb{S}^{d-1})}+\gamma\,\|\mathbf{r}\|^2_{H^\delta(\mathbb{S}^{d-1})}\,, \tag{3.10}$$

$$\mathcal{J}_{no}(\mathbf{r}^{\gamma,no};\gamma) = \inf_{\mathbf{r}\in H^\delta(\mathbb{S}^{d-1})} \mathcal{J}_{no}(\mathbf{r};\gamma). \tag{3.11}$$

Clearly, problem (3.8) is a special case of (3.11). The existence of minimizers $\mathbf{r}^{\gamma,no}$ for (3.11) is guaranteed by Corollary 3.2.

Theorem 3.3. *Suppose condition* (b) *of Theorem 2.2 is satisfied for fixed* ω. *Then we have:*

i) Suppose $\mathbf{r}^{\gamma,no}$ *is a set of minimizers for* (3.11). *Then the minimal functional values* $\mathcal{J}_{no}(\mathbf{r}^{\gamma,no};\gamma)$ *tend to* $\inf_{\mathbf{r}\in H^\delta(\mathbb{S}^{d-1})}\mathcal{J}(\mathbf{r};0)$ *for* $\gamma\to 0$.

ii) Suppose the far field patterns p_k^∞ *are the exact patterns for a fixed solution* $\mathbf{r}^*$ *of the inverse problem* (3.1), *i.e.,* $\mathcal{J}(\mathbf{r}^*,0)=0$. *Then, for* $\delta>\delta-\varepsilon>(d+1)/2$ *and for any set of minimizers* $\mathbf{r}^{\gamma,no}$, *there exists a subsequence* $\mathbf{r}^{\gamma_n,no}$ *converging weakly in* $H^\delta(\mathbb{S}^{d-1})$ *and strongly in* $H^{\delta-\varepsilon}(\mathbb{S}^{d-1})$ *to a solution* $\mathbf{r}^{**}$ *of* (3.8) *with* $\gamma=0$ *and, therewith, to a solution of the inverse problem* (3.1).

iii) If, additionally to the assumptions of ii), the solution $\mathbf{r}^*$ *of the inverse problem* (3.1) *is unique, then we even get that* $\mathbf{r}^{\gamma,no}$ *tends to* $\mathbf{r}^*$ *weakly in* $H^\delta(\mathbb{S}^{d-1})$ *and strongly in* $H^{\delta-\varepsilon}(\mathbb{S}^{d-1})$.

Remark 3.4. *To our knowledge, there are no uniqueness results available for measurements corresponding to a finite set of incident directions. However, uniqueness is proved if the far fields coincide for all incident directions (cf.* [16]*).*

Proof. Take an arbitrary $\mathbf{r}$. From the definition of the functionals and the minimizers as well as from (3.9), we conclude that $\mathcal{J}_{no}(\mathbf{r}^{\gamma,no};\gamma)\le\mathcal{J}_{no}(\mathbf{r};\gamma)\le\mathcal{J}(\mathbf{r};0)+c\cdot\gamma+\gamma\|\mathbf{r}\|^2$. This implies the relation $\limsup_{\gamma\to0}\mathcal{J}_{no}(\mathbf{r}^{\gamma,no};\gamma)\le\mathcal{J}(\mathbf{r};0)$. In other words, $\limsup_{\gamma\to0}\mathcal{J}_{no}(\mathbf{r}^{\gamma,no};\gamma)\le\inf_\mathbf{r}\mathcal{J}(\mathbf{r};0)$. On the other hand, $\mathcal{J}_{no}(\mathbf{r}^{\gamma,no};\gamma)\ge\mathcal{J}(\mathbf{r}^{\gamma,no};0)-c\cdot\gamma\ge\inf_\mathbf{r}\mathcal{J}(\mathbf{r};0)-c\cdot\gamma$ implies $\liminf_{\gamma\to0}\mathcal{J}_{no}(\mathbf{r}^{\gamma,no};\gamma)\ge\inf_\mathbf{r}\mathcal{J}(\mathbf{r};0)$ and assertion i) follows.

If $\mathcal{J}(\mathbf{r}^*;0)=0$, then the minimizer $\mathbf{r}^{\gamma,no}$ of the optimization problem in (3.11) satisfies the estimate $\gamma\|\mathbf{r}^{\gamma,no}\|^2\le\mathcal{J}_{no}(\mathbf{r}^{\gamma,no};\gamma)\le\mathcal{J}_{no}(\mathbf{r}^*;\gamma)\le\mathcal{J}(\mathbf{r}^*;0)+\gamma\|\mathbf{r}^*\|^2+c\cdot\gamma=\gamma\|\mathbf{r}^*\|^2+c\cdot\gamma$. Hence, there is a constant c_0 such that $\|\mathbf{r}^{\gamma,no}\|\le c_0$. Similarly to the proof of Corollary 3.2, there exists a subsequence $\mathbf{r}^{\gamma_n,no}$ which converges weakly in $H^\delta(\mathbb{S}^{d-1})$ and strongly in $H^{\delta-\varepsilon}(\mathbb{S}^{d-1})$ to a limit $\mathbf{r}^{**}\in H^\delta(\mathbb{S}^{d-1})$. From the lower bound $\delta-\varepsilon>(d+1)/2$ and from Lemma 3.1 applied to the $H^{\delta-\varepsilon}(\mathbb{S}^{d-1})$ convergent subsequence, we get $\mathcal{J}(\mathbf{r}^{**};0)\le\lim\mathcal{J}(\mathbf{r}^{\gamma_n,no};0)\le\limsup\mathcal{J}_{no}(\mathbf{r}^{\gamma_n,no};\gamma_n)$. Together with assertion i) we obtain ii).

Finally, if the optimal solution is unique, then all the limits $\mathbf{r}^{**}$ from assertion ii) coincide with $\mathbf{r}^*$. However, any sequence such that each subsequence contains a subsequence convergent to the same limit is convergent. This implies iii). □

4 Gradient Computation

In this section we derive formulas for the gradients of the solution operator S_Γ with respect to Γ in order to enable the application of gradient based optimization methods for (3.8). More precisely, we suppose the curve/surface $\Gamma = \Gamma^{\mathbf{r}_{00}}, \mathbf{r}_{00} \in H^\delta(\mathbb{S}^{d-1})$ is fixed such that the frequency ω is not a Jones mode for $\Omega^{\mathbf{r}_{00}}$. We denote the corresponding solutions of (2.8) by u_0, p_0, and σ_0. If a second function $\mathbf{r}_D \in H^\delta(\mathbb{S}^{d-1})$ is fixed, then we define $\mathbf{r}_h := \mathbf{r}_{00} + h\mathbf{r}_D$ and denote the corresponding solutions of (2.8) by u_h, p_h, and σ_h. The derivatives of u_0, p_0, and σ_0 with respect to $\Gamma^{\mathbf{r}_{00}}$ in the direction of $\mathbf{r}_D$ are the limits

$$\tilde{u}_{der} := \lim_{h\to 0} \frac{u_h - u_0}{h}\,, \quad \tilde{p}_{der} := \lim_{h\to 0} \frac{p_h - p_0}{h}\,, \quad \tilde{\sigma}_{der} := \lim_{h\to 0} \frac{\sigma_h - \sigma_0}{h}\,. \tag{4.1}$$

If the space of all $\mathbf{r}$ is restricted to the span of a finite number of basis functions including $\mathbf{r}_D$, then the derivatives (4.1) are just the components of the gradient of the solution operator S_Γ computed at $\Gamma = \Gamma^{\mathbf{r}_{00}}$.

In view of the definition of the radial function $\mathbf{r}_R$ by formula (3.2), we get

$$\begin{aligned}
\mathbf{r}_{h,R}(\hat{x}) &:= \frac{R}{2} + \frac{R}{\pi} \arctan\big(\mathbf{r}_{00}(\hat{x}) + h\mathbf{r}_D(\hat{x})\big) \\
&= \frac{R}{2} + \frac{R}{\pi} \arctan\big(\mathbf{r}_{00}(\hat{x})\big) + \frac{R}{\pi} \frac{1}{1 + [\mathbf{r}_{00}(\hat{x})]^2}\, h\,\mathbf{r}_D(\hat{x}) + \mathcal{O}\left(h^2\right) \\
&= \mathbf{r}_0(\hat{x}) + h\mathbf{r}_d(\hat{x}) + \mathcal{O}\left(h^2\right), \quad \mathbf{r}_0(\hat{x}) := \frac{R}{2} + \frac{R}{\pi} \arctan\big(\mathbf{r}_{00}(\hat{x})\big), \\
&\qquad\qquad \mathbf{r}_d(\hat{x}) := \frac{R}{\pi} \frac{1}{1 + [\mathbf{r}_{00}(\hat{x})]^2}\, \mathbf{r}_D(\hat{x}).
\end{aligned}$$

Consequently, the derivative of u_0, p_0, and σ_0 with respect to $\mathbf{r}$ taken at $\mathbf{r} = \mathbf{r}_{00}$ in the direction of $\mathbf{r}_D$ is equal to the derivative of u_0, p_0, and σ_0 with respect to $\mathbf{r}_R$ taken at $\mathbf{r}_R = \mathbf{r}_0$ in the direction of $\mathbf{r}_d$. We only have to compute the last. To simplify the notation in the forthcoming formulas, we write $\Gamma^{\mathbf{r}_R}$ for $\Gamma^{\mathbf{r}}$ and $\Omega^{\mathbf{r}_R}$ for $\Omega^{\mathbf{r}}$. Fortunately, the objective functional (3.7) contains only the traces on Γ_0. Thus we only have to compute the restricted derivatives $\tilde{p}_{der}|_{\Gamma_0}$ and $\tilde{\sigma}_{der}$. This enables us to employ the material derivative which allows us to work on the fixed domains Ω and Ω_R defined with $\Gamma = \Gamma^{\mathbf{r}_0}$. We shall show that the derivatives are the solutions of the variational equation (2.8) with a new right-hand side.

To define the material derivatives, we choose a Lipschitz continuous function ψ defined on the closure of $\Omega \cup \Omega_R$. This ψ is to vanish over the outer boundary Γ_0 of Ω_R. More precisely, setting $\hat{x} := x/|x|$ we choose

$$\psi(x) \quad := \quad \frac{|x|^3\big(R - |x|\big)}{[\mathbf{r}_0(\hat{x})]^3\big(R - \mathbf{r}_0(\hat{x})\big)}\, \mathbf{r}_d(\hat{x})\, \hat{x}\,. \tag{4.2}$$

Now, for any small $h > 0$, define the automorphism Φ_h of the closure of $\Omega \cup \Omega_R$ by $\Phi_h(x) := x + h\psi(x)$. Of course, the transformed boundary $\Gamma_h := \Phi_h(\Gamma)$ is nothing else than $\Gamma^{\mathbf{r}_h}$ and tends to Γ for $h \to 0$. Suppose Ω_h is the interior of Γ_h and Ω_{Rh} is the domain enclosed by Γ_h and Γ_0. We denote by $\mathcal{B}_h$, a_h, b_h, and c_h the sesqui-linear forms of (2.8) with Γ replaced by Γ_h, Ω by Ω_h, and Ω_R by Ω_{Rh}, respectively.

Suppose (u_h, p_h, σ_h) is the solution for Γ_h, i.e. of

$$\mathcal{B}_h\Big((u_h,p_h,\sigma_h)^\top,(v,q,\chi)^\top\Big) = \mathcal{R}_h\Big(p^{inc},(v,q,\chi)^\top\Big) := \begin{pmatrix} -\int_{\Gamma_h} p^{inc}\, n\cdot\overline{v} \\ -\int_{\Gamma_h} \dfrac{\partial p^{inc}}{\partial n}\,\overline{q} \\ 0 \end{pmatrix} \tag{4.3}$$

$$\forall (v,q,\chi) \in [H^1(\Omega_h)]^d \times H^1(\Omega_{Rh}) \times H^{-1/2}(\Gamma_0).$$

With these functions (u_h, p_h, σ_h) we look for the derivatives (4.1). However, on the boundary Γ_0 we have $\Phi_h(x) = x$. Hence, if we look for the derivative $\tilde{p}_{der}$ close to Γ_0 and for $\tilde{\sigma}_{der}$ on Γ_0, it is enough to look for the limits

$$u_{der} := \lim_{h\to 0} \frac{u_h\circ\phi_h - u_0\circ\phi_0}{h}\,, \quad p_{der} := \lim_{h\to 0}\frac{p_h\circ\phi_h - p_0\circ\phi_0}{h}\,, \tag{4.4}$$

$$\sigma_{der} := \lim_{h\to 0}\frac{\sigma_h\circ\phi_h - \sigma_0\circ\phi_0}{h} = \tilde{\sigma}_{der}\,. \tag{4.5}$$

Note that $u_h \circ \phi_h$ lives on Ω and $p_h \circ \phi_h$ on Ω_R, which makes the computation easier.

To get a series expansion of $p_h\circ\phi_h$ into powers of h, we first have to find the corresponding expansions of the sesqui-linear form $\mathcal{B}_h((u\circ\phi_h^{-1}, p\circ\phi_h^{-1}, \sigma_h\circ\phi_h^{-1})^\top, (v\circ\phi_h^{-1}, q\circ\phi_h^{-1}, \chi\circ\phi_h^{-1})^\top)$ and of the right-hand side $\mathcal{R}_h(p^{inc}\circ\phi_h^{-1}, (v\circ\phi_h^{-1}, q\circ\phi_h^{-1}, \chi\circ\phi_h^{-1})^\top)$. This is based on the substitution $y = \Phi_h(x)$ and the following formulas. For dimension $d = 3$ we get

$$\Phi_h'(x) = \begin{pmatrix} 1+h\,\partial_1\psi_1(x) & h\,\partial_2\psi_1(x) & h\,\partial_3\psi_1(x) \\ h\,\partial_1\psi_2(x) & 1+h\,\partial_2\psi_2(x) & h\,\partial_3\psi_2(x) \\ h\,\partial_1\psi_3(x) & h\,\partial_2\psi_3(x) & 1+h\,\partial_3\psi_3(x) \end{pmatrix} \tag{4.6}$$

$$J(x) := \det\left(\Phi_h'(x)\right) = 1 + hJ_1(x) + h^2J_2(x) + \mathcal{O}\big(h^3\big) \tag{4.7}$$

$$J_1(x) := \partial_1\psi_1(x) + \partial_2\psi_2(x) + \partial_3\psi_3(x)$$

$$\begin{aligned} J_2(x) := {} & \partial_1\psi_1(x)\partial_2\psi_2(x) - \partial_1\psi_2(x)\partial_2\psi_1(x) + \partial_1\psi_1(x)\partial_3\psi_3(x) - \partial_1\psi_3(x)\partial_3\psi_1(x) \\ & + \partial_2\psi_2(x)\partial_3\psi_3(x) - \partial_2\psi_3(x)\partial_3\psi_2(x) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial y_1} = {} & \frac{1+h[\partial_2\psi_2(x)+\partial_3\psi_3(x)] + h^2[\partial_2\psi_2(x)\partial_3\psi_3(x) - \partial_2\psi_3(x)\partial_3\psi_2(x)]}{J(x)}\frac{\partial}{\partial x_1} \\ & - \frac{h\,\partial_1\psi_2(x) + h^2[\partial_1\psi_2(x)\partial_3\psi_3(x) - \partial_3\psi_2(x)\partial_1\psi_3(x)]}{J(x)}\frac{\partial}{\partial x_2} \\ & - \frac{h\,\partial_1\psi_3(x) - h^2[\partial_1\psi_2(x)\partial_2\psi_3(x) - \partial_1\psi_3(x)\partial_2\psi_2(x)]}{J(x)}\frac{\partial}{\partial x_3} \end{aligned} \tag{4.8}$$

$$\begin{aligned} \frac{\partial}{\partial y_2} = {} & -\frac{h\,\partial_2\psi_1(x) + h^2[\partial_2\psi_1(x)\partial_3\psi_3(x) - \partial_3\psi_1(x)\partial_2\psi_3(x)]}{J(x)}\frac{\partial}{\partial x_1} \\ & \frac{1+h[\partial_1\psi_1(x)+\partial_3\psi_3(x)] + h^2[\partial_1\psi_1(x)\partial_3\psi_3(x) - \partial_1\psi_3(x)\partial_3\psi_1(x)]}{J(x)}\frac{\partial}{\partial x_2} \\ & - \frac{h\,\partial_2\psi_3(x) - h^2[\partial_2\psi_1(x)\partial_1\psi_3(x) - \partial_2\psi_3(x)\partial_1\psi_1(x)]}{J(x)}\frac{\partial}{\partial x_3} \end{aligned} \tag{4.9}$$

$$\begin{aligned} \frac{\partial}{\partial y_3} = {} & -\frac{h\,\partial_3\psi_1(x) - h^2[\partial_3\psi_2(x)\partial_2\psi_1(x) - \partial_3\psi_1(x)\partial_2\psi_2(x)]}{J(x)}\frac{\partial}{\partial x_1} \\ & - \frac{h\,\partial_3\psi_2(x) - h^2[\partial_3\psi_2(x)\partial_1\psi_1(x) - \partial_1\psi_2(x)\partial_3\psi_1(x)]}{J(x)}\frac{\partial}{\partial x_2} \\ & \frac{1+h[\partial_2\psi_2(x)+\partial_1\psi_1(x)] + h^2[\partial_2\psi_2(x)\partial_1\psi_1(x) - \partial_2\psi_1(x)\partial_1\psi_2(x)]}{J(x)}\frac{\partial}{\partial x_3} \end{aligned} \tag{4.10}$$

Using (4.7), we write (4.8)-(4.10) in the form

$$\frac{\partial}{\partial y_i} = \frac{\partial}{\partial x_i} + h\sum_{j=1}^{d} c_{i,j}\frac{\partial}{\partial x_j} + \mathcal{O}\left(h^2\right) , \quad i = 1,\dots,d \,. \tag{4.11}$$

Similarly, for dimension $d = 2$, we have

$$\Phi_h'(x) = \begin{pmatrix} 1 + h\,\partial_1\psi_1(x) & h\,\partial_2\psi_1(x) \\ h\,\partial_1\psi_2(x) & 1 + h\,\partial_2\psi_2(x) \end{pmatrix} \tag{4.12}$$

$$J(x) := \det\left(\Phi_h'(x)\right) = 1 + hJ_1(x) + h^2 J_2(x) \tag{4.13}$$

$$J_1(x) := \partial_1\psi_1(x) + \partial_2\psi_2(x)$$
$$J_2(x) := \partial_1\psi_1(x)\partial_2\psi_2(x) - \partial_1\psi_2(x)\partial_2\psi_1(x)$$

$$\frac{\partial}{\partial y_1} = \frac{1 + h\,\partial_2\psi_2(x)}{J(x)}\frac{\partial}{\partial x_1} - \frac{h\,\partial_1\psi_2(x)}{J(x)}\frac{\partial}{\partial x_2} \tag{4.14}$$

$$\frac{\partial}{\partial y_2} = -\frac{h\,\partial_2\psi_1(x)}{J(x)}\frac{\partial}{\partial x_1} + \frac{1 + h\,\partial_1\psi_1(x)}{J(x)}\frac{\partial}{\partial x_2} \tag{4.15}$$

Again, using (4.13), we write (4.14)-(4.15) in the form (4.11). For the integrals over the domains, we have

$$\int\limits_{\Omega_{Rh}} F(\Phi_h^{-1}(y))\mathrm{d}y = \int\limits_{\Omega_R} F(x)J(x)\mathrm{d}x \,, \quad \int\limits_{\Omega_h} F(\Phi_h^{-1}(y))\mathrm{d}y = \int\limits_{\Omega} F(x)J(x)\mathrm{d}x \,. \tag{4.16}$$

If $d = 3$ and if $\gamma :\, D \to \Gamma$ is a parametrization of the two-dimensional boundary surface Γ, then

$$\begin{aligned}
&\int\limits_{\Gamma_h} \frac{\partial}{\partial n}[F\circ\Phi_h^{-1}](y)G(\Phi_h^{-1}(y))\mathrm{d}_{\Gamma_h}y \\
&= \int\limits_D \frac{\partial_{t_1}\Phi_h(\gamma(t))\times\partial_{t_2}\Phi_h(\gamma(t))}{\left|\partial_{t_1}\Phi_h(\gamma(t))\times\partial_{t_2}\Phi_h(\gamma(t))\right|}\cdot\left[[\Phi_h^{-1}]'^{\top}(\Phi_h(\gamma(t)))\,\nabla F(\gamma(t))\right]\, G(\gamma(t))\left|\partial_{t_1}\Phi_h(\gamma(t))\times\partial_{t_2}\Phi_h(\gamma(t))\right|\mathrm{d}t \\
&= \int\limits_D [\Phi_h'(\gamma(t))\partial_{t_1}\gamma(t)\times\Phi_h'(\gamma(t))\partial_{t_2}\gamma(t)]\cdot\left[[\Phi_h'(\gamma(t))^{\top}]^{-1}\nabla F(\gamma(t))\right]\, G(\gamma(t))\mathrm{d}t \\
&= \int\limits_D [\partial_{t_1}\gamma(t)\times\partial_{t_2}\gamma(t)]\cdot[\nabla F(\gamma(t))]\, G(\gamma(t))\mathrm{d}t \\
&\quad + h\int\limits_D [\psi'(\gamma(t))\partial_{t_1}\gamma(t)\times\partial_{t_2}\gamma(t)]\cdot\nabla F(\gamma(t))\, G(\gamma(t))\mathrm{d}t \\
&\quad + h\int\limits_D [\partial_{t_1}\gamma(t)\times\psi'(\gamma(t))\partial_{t_2}\gamma(t)]\cdot\nabla F(\gamma(t))\, G(\gamma(t))\mathrm{d}t \\
&\quad - h\int\limits_D [\partial_{t_1}\gamma(t)\times\partial_{t_2}\gamma(t)]\cdot\left[\psi'(\gamma(t))^{\top}\nabla F(\gamma(t))\right]\, G(\gamma(t))\mathrm{d}t \;+\; \mathcal{O}(h^2) \\
&= \int\limits_\Gamma \frac{\partial F(x)}{\partial n}G(x)\mathrm{d}_\Gamma x \;+\; h\int\limits_\Gamma \frac{\psi'(\gamma(t))\partial_{t_1}\gamma(t)\times\partial_{t_2}\gamma(t)+\partial_{t_1}\gamma(t)\times\psi'(\gamma(t))\partial_{t_2}\gamma(t)}{|\partial_{t_1}\gamma(t)\times\partial_{t_2}\gamma(t)|}\cdot\nabla F(x)\, G(x)\mathrm{d}_\Gamma x \\
&\quad - h\int\limits_\Gamma n\cdot\left[\psi'(x)^{\top}\nabla F(x)\right]\, G(x)\mathrm{d}_\Gamma x \;+\; \mathcal{O}(h^2)\,.
\end{aligned}$$

Here the matrix $\psi'(x)$ is the Jacobian matrix $(\partial\psi_m(x)/\partial x_n)_{m,n=1}^d$. Obviously, the value of the numerator $\psi'(\gamma(t))\partial_{t_1}\gamma(t)\times\partial_{t_2}\gamma(t)+\partial_{t_1}\gamma(t)\times\psi'(\gamma(t))\partial_{t_2}\gamma(t)$ remains unchanged if $\partial_{t_2}\gamma(t)$ is replaced by the vector

$\partial_{t_2}\gamma(t)+\lambda\partial_{t_1}\gamma(t)$. Hence, without loss of generality we may assume that $\partial_{t_2}\gamma(t)$ and $\partial_{t_1}\gamma(t)$ are orthogonal in the second integral of the last right-hand side. Similarly, a scaling of $\partial_{t_1}\gamma(t)$ and $\partial_{t_2}\gamma(t)$ does not change the value of $\psi'(\gamma(t))\partial_{t_1}\gamma(t)\times\partial_{t_2}\gamma(t)+\partial_{t_1}\gamma(t)\times\psi'(\gamma(t))\partial_{t_2}\gamma(t)$ divided by $|\partial_{t_1}\gamma(t)\times\partial_{t_2}\gamma(t)|$. So we may suppose that $\partial_{t_1}\gamma(t)$ is a tangential vector τ of unit length and $\partial_{t_2}\gamma(t)=n\times\tau$ with the normal vector n. We get that $\psi'(\gamma(t))\partial_{t_1}\gamma(t)\times\partial_{t_2}\gamma(t)+\partial_{t_1}\gamma(t)\times\psi'(\gamma(t))\partial_{t_2}\gamma(t)$ divided by $|\partial_{t_1}\gamma(t)\times\partial_{t_2}\gamma(t)|$ is equal to $\psi'(\gamma(t))\tau\times[n\times\tau]+\tau\times\psi'(\gamma(t))[n\times\tau]$, which is equal

$$\begin{aligned}\psi'(\gamma(t))\tau\times[n\times\tau]+\tau\times\psi'(\gamma(t))[n\times\tau] &= \Big\{\tau\cdot\psi'(\gamma(t))\tau\;\tau+n\cdot\psi'(\gamma(t))\tau\;n\Big\}\times[n\times\tau]+\tau\times\\ &\qquad \Big\{[n\times\tau]\cdot\psi'(\gamma(t))[n\times\tau]\;[n\times\tau]+n\cdot\psi'(\gamma(t))[n\times\tau]\;n\Big\}\\ &= \Big\{\tau\cdot\psi'(\gamma(t))\tau+[n\times\tau]\cdot\psi'(\gamma(t))[n\times\tau]\Big\}n\\ &\qquad -\tau\cdot\psi'(\gamma(t))^\top n\;\tau-[n\times\tau]\cdot\psi'(\gamma(t))^\top n\;[n\times\tau]\\ &= \Big\{\mathrm{trace}(\psi'(\gamma(t)))-n\cdot\psi'(\gamma(t))n\Big\}n\\ &\qquad -\Big\{\psi'(\gamma(t))^\top n-n\cdot\psi'(\gamma(t))^\top n\;n\Big\}\\ &= \mathrm{trace}(\psi'(\gamma(t)))n-\cdot\psi'(\gamma(t))^\top n\,.\end{aligned}$$

In other words,

$$\begin{aligned}\int_{\Gamma_h}\frac{\partial}{\partial n}[F\circ\Phi_h^{-1}](y)G(\Phi_h^{-1}(y))\mathrm{d}_{\Gamma_h}y &= \int_\Gamma\frac{\partial F(x)}{\partial n}G(x)\mathrm{d}_\Gamma x\;+\;h\int_\Gamma\mathrm{trace}(\psi'(x))\frac{\partial F(x)}{\partial n}G(x)\mathrm{d}_\Gamma x\\ &\quad -h\int_\Gamma n\cdot\big\{\big[\psi'(x)^\top+\psi'(x)\big]\nabla F(x)\big\}\;G(x)\mathrm{d}_\Gamma x\;+\;\mathcal{O}(h^2).\end{aligned}$$

Similarly, we obtain

$$\int_{\Gamma_h}n\cdot w\circ\Phi_h^{-1}(y)\mathrm{d}_{\Gamma_h}y \;=\; \int_\Gamma n\cdot w(x)\mathrm{d}_\Gamma x\;+\;h\int_\Gamma\big[\mathrm{trace}(\psi'(x))n-\psi'(x)^\top n\big]\cdot w(x)\mathrm{d}_\Gamma x\,.$$

If $d=2$ and if $\gamma:\,D\to\Gamma$ is a parametrization of the one-dimensional boundary curve Γ, then we introduce the rotation matrix

$$\mathbf{R}\;:=\;\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

and conclude

$$\begin{aligned}&\int_{\Gamma_h}\frac{\partial}{\partial n}[F\circ\Phi_h^{-1}](y)G(\Phi_h^{-1}(y))\mathrm{d}_{\Gamma_h}y\\ &=\int_D\frac{\mathbf{R}\Phi_h'(\gamma(t))\partial_t\gamma(t)}{\Big|\Phi_h'(\gamma(t))\partial_t\gamma(t)\Big|}\cdot\Big[\big[\Phi_h^{-1}\big]'^\top\Big(\Phi_h(\gamma(t))\Big)\nabla F(\gamma(t))\Big]\;G(\gamma(t))\Big|\Phi_h'(\gamma(t))\partial_t\gamma(t)\Big|\mathrm{d}t\\ &=\int_D[\mathbf{R}\Phi_h'(\gamma(t))\partial_t\gamma(t)]\cdot\Big[\big[\Phi_h'(\gamma(t))^\top\big]^{-1}\nabla F(\gamma(t))\Big]\;G(\gamma(t))\mathrm{d}t\\ &=\int_D[\mathbf{R}\partial_t\gamma(t)]\cdot[\nabla F(\gamma(t))]\;G(\gamma(t))\mathrm{d}t\\ &\qquad -h\int_D[\mathbf{R}\partial_t\gamma(t)]\cdot\big[\psi'(\gamma(t))^\top\nabla F(\gamma(t))\big]\;G(\gamma(t))\mathrm{d}t\\ &\qquad +h\int_D[\mathbf{R}\psi'(\gamma(t))\partial_t\gamma(t)]\cdot[\nabla F(\gamma(t))]\;G(\gamma(t))\mathrm{d}t\;+\mathcal{O}(h^2)\end{aligned}$$

$$
\begin{aligned}
&= \int\limits_\Gamma \frac{\partial F(x)}{\partial n} G(x)\mathrm{d}_\Gamma x \;-\; h \int\limits_\Gamma n\cdot\left[\psi'(x)^\top \nabla F(x)\right]\, G(x)\mathrm{d}_\Gamma x \\
&\qquad + h \int\limits_\Gamma \left[\mathbf{R}\psi'(x)\mathbf{R}^\top n\right]\cdot \nabla F(x)\; G(x)\mathrm{d}_\Gamma x \;+ \mathcal{O}(h^2)\,. \\
&= \int\limits_\Gamma \frac{\partial F(x)}{\partial n}\, G(x)\mathrm{d}_\Gamma x \;+\; h \int\limits_\Gamma \mathrm{trace}(\psi'(x))\frac{\partial F(x)}{\partial n}\, G(x)\mathrm{d}_\Gamma x \\
&\qquad - h \int\limits_\Gamma n\cdot\left\{\left[\psi'(x)^\top + \psi'(x)\right]\nabla F(x)\right\}\, G(x)\mathrm{d}_\Gamma x \;+\; \mathcal{O}(h^2).
\end{aligned}
$$

Similarly, we obtain

$$
\int\limits_{\Gamma_h} n\cdot w\circ\Phi_h^{-1}(y)\mathrm{d}_{\Gamma_h} y \;\;=\;\; \int\limits_\Gamma n\cdot w(x)\mathrm{d}_\Gamma x \;+\; h \int\limits_\Gamma \left[\mathrm{trace}(\psi'(x))n - \psi'(x)^\top n\right]\cdot w(x)\mathrm{d}_\Gamma x\,.
$$

To get the series expansions of $\mathcal{B}_h((u\circ\phi_h^{-1}, p\circ\phi_h^{-1}, \sigma\circ\phi_h^{-1})^\top, (v\circ\phi_h^{-1}, q\circ\phi_h^{-1}, \chi\circ\phi_h^{-1})^\top)$ into powers of h, we substitute the variable of integration $y=\Phi_h(x)$ and apply the previous formulas.

$$
\begin{aligned}
a_h\left(\begin{pmatrix} u\circ\phi_h^{-1}\\ p\circ\phi_h^{-1}\\ \sigma\circ\phi_h^{-1}\end{pmatrix}, \begin{pmatrix} v\circ\phi_h^{-1}\\ q\circ\phi_h^{-1}\\ \chi\circ\phi_h^{-1}\end{pmatrix}\right) &= \int\limits_{\Omega_h} \Bigg\{\lambda\,\nabla_y\cdot[u\circ\phi_h^{-1}]\,\overline{\nabla_y\cdot[v\circ\phi_h^{-1}]} \\
&\quad + \frac{\mu}{2}\sum_{i,j=1}^d \left[\partial_{y_i}[u_j\circ\phi_h^{-1}]\overline{\partial_{y_j}[v_i\circ\phi_h^{-1}]} + \partial_{y_i}[u_j\circ\phi_h^{-1}]\overline{\partial_{y_i}[v_j\circ\phi_h^{-1}]}\right] \\
&\quad - \varrho\omega^2[u\circ\phi_h^{-1}]\cdot\overline{[v\circ\phi_h^{-1}]}\Bigg\} + \int\limits_{\Gamma_h}[p\circ\phi_h^{-1}]\,n\cdot\overline{[v\circ\phi_h^{-1}]}
\end{aligned}
$$

$$
\begin{aligned}
&= \int\limits_\Omega \Bigg\{\lambda\,\nabla_x\cdot u\,\overline{\nabla_x\cdot v} + \lambda h J_1\,\nabla_x\cdot u\,\overline{\nabla_x\cdot v} + h\lambda\Big\{\Big[\sum_{i,j=1}^d c_{i,j}\partial_{x_j}u_i\Big]\overline{\nabla_x\cdot v} + \nabla_x\cdot u\overline{\Big[\sum_{i,j=1}^d c_{i,j}\partial_{x_j}v_i\Big]}\Big\} \\
&\quad + \frac{\mu}{2}\sum_{i,j=1}^d\left[\partial_{x_i}u_j\overline{\partial_{x_j}v_i} + \partial_{x_i}u_j\overline{\partial_{x_i}v_j}\right] + \frac{\mu h J_1}{2}\sum_{i,j=1}^d\left[\partial_{x_i}u_j\overline{\partial_{x_j}v_i} + \partial_{x_i}u_j\overline{\partial_{x_i}v_j}\right] \\
&\quad + \frac{\mu h}{2}\sum_{i,j,k=1}^d\left[c_{ik}\partial_{x_k}u_j\overline{\partial_{x_j}v_i} + \partial_{x_i}u_j\overline{c_{jk}\partial_{x_k}v_i} + c_{ik}\partial_{x_k}u_j\overline{\partial_{x_i}v_j} + \partial_{x_i}u_j\overline{c_{i,k}\partial_{x_k}v_j}\right] \\
&\quad - \varrho\omega^2 u\cdot\overline{v} - hJ_1\varrho\omega^2 u\cdot\overline{v} + \mathcal{O}\left(h^2\right)\Bigg\} + \int\limits_\Gamma\Big\{p\,n\cdot\overline{v} + h\,p\left[\mathrm{trace}(\psi')n - [\psi']^\top n\right]\cdot\overline{v} + o(h)\Big\},
\end{aligned}
$$

$$b_h\left(\left(\begin{array}{c} u\circ\phi_h^{-1}\\ p\circ\phi_h^{-1}\\ \sigma\circ\phi_h^{-1}\end{array}\right),\left(\begin{array}{c} v\circ\phi_h^{-1}\\ q\circ\phi_h^{-1}\\ \chi\circ\phi_h^{-1}\end{array}\right)\right) = \int\limits_{\Omega_{Rh}}\left\{\nabla_y\left[p\circ\phi_h^{-1}\right]\cdot\overline{\nabla_y\left[q\circ\phi_h^{-1}\right]} - {k_w}^2[p\circ\phi_h^{-1}]\overline{[q\circ\phi_h^{-1}]}\right\}$$
$$+\varrho_f\omega^2\int\limits_{\Gamma_h}[u\circ\phi_h^{-1}]\cdot n\,\overline{[q\circ\phi_h^{-1}]} - \int\limits_{\Gamma_0}\sigma\,\overline{q}$$
$$= \int\limits_{\Omega_R}\Big\{\nabla_x p\cdot\overline{\nabla_x q} + hJ_1\nabla_x p\cdot\overline{\nabla_x q} +$$
$$h\sum_{i,j}\left[\partial_{x_i}p\,\overline{c_{i,j}\partial_{x_j}q} + c_{i,j}\partial_{x_j}p\,\overline{\partial_{x_i}q}\,\right] - {k_w}^2p\overline{q} - hJ_1{k_w}^2p\overline{q} + \mathcal{O}\left(h^2\right)\Big\}$$
$$+\varrho_f\omega^2\int\limits_{\Gamma}\left\{u\cdot n\,\overline{q} + h\left[\mathrm{trace}(\psi')n - [\psi']^\top n\right]\cdot u\,\overline{q} + o(h)\right\} - \int\limits_{\Gamma_0}\sigma\,\overline{q}.$$

In other words, we conclude

$$\mathcal{B}_h\left(\left(\begin{array}{c} u\circ\phi_h^{-1}\\ p\circ\phi_h^{-1}\\ \sigma\circ\phi_h^{-1}\end{array}\right),\left(\begin{array}{c} v\circ\phi_h^{-1}\\ q\circ\phi_h^{-1}\\ \chi\circ\phi_h^{-1}\end{array}\right)\right) = \mathcal{B}\left(\left(\begin{array}{c} u\\ p\\ \sigma\end{array}\right),\left(\begin{array}{c} v\\ q\\ \chi\end{array}\right)\right) + h\,\mathcal{B}_1\left(\left(\begin{array}{c} u\\ p\\ \sigma\end{array}\right),\left(\begin{array}{c} v\\ q\\ \chi\end{array}\right)\right)$$
$$+\mathcal{B}_2^h\left(\left(\begin{array}{c} u\\ p\\ \sigma\end{array}\right),\left(\begin{array}{c} v\\ q\\ \chi\end{array}\right)\right), \tag{4.17}$$

where we have set

$$\mathcal{B}_1\left(\left(\begin{array}{c} u\\ p\\ \sigma\end{array}\right),\left(\begin{array}{c} v\\ q\\ \chi\end{array}\right)\right) := \left(\begin{array}{c} a_1\left((u,p)^\top,(v,q)^\top\right)\\ b_1\left((u,p)^\top,(v,q)^\top\right)\\ 0\end{array}\right), \tag{4.18}$$

$$a_1\left((u,p)^\top,(v,q)^\top\right) := \int\limits_{\Omega}\Bigg\{J_1\lambda\,\nabla_x\cdot u\,\overline{\nabla_x\cdot v} + \lambda\Big\{\Big[\sum_{i,j=1}^d c_{i,j}\partial_{x_j}u_i\Big]\overline{\nabla_x\cdot v} + \nabla_x\cdot u\,\overline{\Big[\sum_{i,j=1}^d c_{i,j}\partial_{x_j}v_i\Big]}\Big\}$$
$$+\frac{\mu J_1}{2}\sum_{i,j=1}^d\left[\partial_{x_i}u_j\overline{\partial_{x_j}v_i} + \partial_{x_i}u_j\overline{\partial_{x_i}v_j}\right]$$
$$+\frac{\mu}{2}\sum_{i,j,k=1}^d\Big[c_{ik}\partial_{x_k}u_j\overline{\partial_{x_j}v_i} + \partial_{x_i}u_j\overline{c_{i,k}\partial_{x_k}v_j} + c_{ik}\partial_{x_k}u_j\overline{\partial_{x_i}v_j} \tag{4.19}$$
$$+\partial_{x_i}u_j\overline{c_{j,k}\partial_{x_k}v_i}\Big] - J_1\varrho\omega^2u\cdot\overline{v}\Bigg\}$$
$$+\int\limits_{\Gamma}\left\{p\left[\mathrm{trace}(\psi')n - [\psi']^\top n\right]\cdot\overline{v}\right\},$$

$$b_1\left((u,p)^\top,(v,q)^\top\right) := \int\limits_{\Omega_R}\left\{J_1\nabla_x p\cdot\overline{\nabla_x q} + \sum_{i,j}\left[\partial_{x_i}p\,\overline{c_{i,j}\partial_{x_j}q} + c_{i,j}\partial_{x_j}p\,\overline{\partial_{x_i}q}\right] - J_1{k_w}^2p\overline{q}\right\}$$
$$+\varrho_f\omega^2\int\limits_{\Gamma}\left\{\left[\mathrm{trace}(\psi')n - [\psi']^\top n\right]\cdot u\,\overline{q}\right\},$$

with the remainder sesqui-linear forms $\mathcal{B}_1$ and $\mathcal{B}_2^h$ such that the estimates

$$\left|\mathcal{B}_2^h\left((u,p,\sigma)^\top,(v,q,\chi)^\top\right)\right| \leq o(h)\sqrt{\|u\|^2_{[H^1(\Omega)]^d}+\|p\|^2_{H^1(\Omega_R)}}\sqrt{\|v\|^2_{[H^1(\Omega)]^d}+\|q\|^2_{H^1(\Omega_R)}}, \quad (4.20)$$

$$\left|\mathcal{B}_1\left((u,p,\sigma)^\top,(v,q,\chi)^\top\right)\right| \leq c\sqrt{\|u\|^2_{[H^1(\Omega)]^d}+\|p\|^2_{H^1(\Omega_R)}}\sqrt{\|v\|^2_{[H^1(\Omega)]^d}+\|q\|^2_{H^1(\Omega_R)}} \quad (4.21)$$

hold for a positive constant c independent of h and for any $u,v\in[H^1(\Omega)]^d$ and $p,q\in H^1(\Omega_R)$.

To get the series expansions of $\mathcal{R}_h(p^{inc}\circ\phi_h^{-1},(v\circ\phi_h^{-1},q\circ\phi_h^{-1},\chi\circ\phi_h^{-1})^\top)$ into powers of h, we substitute the variable of integration $y=\Phi_h(x)$ and obtain

$$-\int_{\Gamma_h}\left[p^{inc}\circ\phi_h^{-1}\right]\overline{n\cdot v\circ\phi_h^{-1}} = -\int_\Gamma\left\{p^{inc}\,\overline{n\cdot v}+h\,p^{inc}[\mathrm{trace}(\psi')n-[\psi']^\top n]\cdot\overline{v}+o(h)\right\},$$

$$-\int_{\Gamma_h}\frac{\partial\left[p^{inc}\circ\phi_h^{-1}\right]}{\partial n}\overline{q\circ\phi_h^{-1}} = -\int_\Gamma\left\{\frac{\partial p^{inc}}{\partial n}\overline{q}+h\big[\mathrm{trace}(\psi')n-\psi'n-[\psi']^\top n\big]\cdot\nabla p^{inc}\,\overline{q}+o(h)\right\}.$$

In other words, we conclude

$$\mathcal{R}_h\left(p^{inc}\circ\phi_h^{-1},(v\circ\phi_h^{-1},q\circ\phi_h^{-1},\chi\circ\phi_h^{-1})^\top\right) = \mathcal{R}\left(p^{inc},(v,q,\chi)^\top\right)+h\,\mathcal{R}_1\left(p^{inc},(v,q,\chi)^\top\right)+\mathcal{R}_2^h\left(p^{inc},(v,q,\chi)^\top\right),$$

$$\mathcal{R}_1\left(p^{inc},(v,q,\chi)^\top\right) := \begin{pmatrix}-\int_\Gamma p^{inc}[\mathrm{trace}(\psi')n-[\psi']^\top n]\cdot\overline{v}\\ -\int_\Gamma\big[\mathrm{trace}(\psi')n-\psi'n-[\psi']^\top n\big]\cdot\nabla p^{inc}\,\overline{q}\\ 0\end{pmatrix}, \quad (4.22)$$

where the remainder form $\mathcal{R}_2^h$ satisfies the estimate

$$\left|\mathcal{R}_2^h\left(p^{inc},(v,q,\chi)^\top\right)\right| \leq o(h)\,\|p^{inc}\|_{H^1(\Omega_R)}\sqrt{\|v\|^2_{[H^1(\Omega)]^d}+\|q\|^2_{H^1(\Omega_R)}}. \quad (4.23)$$

Now we are ready to derive the formulas for the derivatives. To simplify the notation, we write $w_0:=(u_0,p_0,\sigma_0)$ for the solution of (2.8), $w_h:=(u_h,p_h,\sigma_h)$ for the solution of (4.3), $t:=(v,q,\chi)$ for the vector of test functions, and $w_{der}:=(u_{der},p_{der},\sigma_{der})$ for the derivatives of (4.4) and (4.5). From (4.3), i.e., from

$$\mathcal{B}_h\big([w_h\circ\Phi_h]\circ\Phi_h^{-1},[t\circ\Phi_h]\circ\Phi_h^{-1}\big)=\mathcal{R}_h\big([p^{inc}\circ\Phi_h]\circ\Phi_h^{-1},[t\circ\Phi_h]\circ\Phi_h^{-1}\big)$$

and from (4.17), and (4.22), we conclude that, for any t,

$$\begin{aligned}&\mathcal{B}\big([w_h\circ\Phi_h],[t\circ\Phi_h]\big)+h\,\mathcal{B}_1\big([w_h\circ\Phi_h],[t\circ\Phi_h]\big)+\mathcal{B}_2^h\big([w_h\circ\Phi_h],[t\circ\Phi_h]\big)\\ &\quad=\mathcal{R}\big([p^{inc}\circ\Phi_h],[t\circ\Phi_h]\big)+h\,\mathcal{R}_1\big([p^{inc}\circ\Phi_h],[t\circ\Phi_h]\big)+\mathcal{R}_2^h\big([p^{inc}\circ\Phi_h],[t\circ\Phi_h]\big).\end{aligned}$$

On the other hand, if p^{inc} is a locally smooth function, then $\Phi_h(x)=x+h\psi(x)$ implies the asymptotic expansion $[p^{inc}\circ\Phi_h]=p^{inc}+h\psi\cdot\nabla p^{inc}+\mathcal{O}(h^2)$. Substituting this and using (2.8) with $u=u_0$, $p=p_0$, and $\chi=\chi_0$, we continue

$$\begin{aligned}&\mathcal{B}\big([w_h\circ\Phi_h],[t\circ\Phi_h]\big)+h\,\mathcal{B}_1\big([w_h\circ\Phi_h],[t\circ\Phi_h]\big)+o(h)\\ &\quad=\mathcal{R}\big(p^{inc},[t\circ\Phi_h]\big)+h\,\mathcal{R}\big(\psi\cdot\nabla p^{inc},[t\circ\Phi_h]\big)+h\,\mathcal{R}_1\big([p^{inc}\circ\Phi_h],[t\circ\Phi_h]\big)+o(h)\\ &\quad=\mathcal{B}\big(w_0,[t\circ\Phi_h]\big)+h\,\mathcal{R}\big(\psi\cdot\nabla p^{inc},[t\circ\Phi_h]\big)+h\,\mathcal{R}_1\big([p^{inc}\circ\Phi_h],[t\circ\Phi_h]\big)+o(h).\end{aligned}$$

From the proof of Lemma 3.1, we know that w_h is continuous and bounded with respect to h. Comparing the left-hand and the right-hand side in the last chain of equalities, we obtain $[w_h\circ\Phi_h]\to w_0$ for $h\to 0$. Substituting the test function $t\circ\Phi_h$ by t, the last equation yields

$$\mathcal{B}\Big(\frac{[w_h\circ\Phi_h]-w_0}{h},t\Big) = \mathcal{R}\big(\psi\cdot\nabla p^{inc},t\big)+\mathcal{R}_1\big([p^{inc}\circ\Phi_h],t\big)-\mathcal{B}_1\big([w_h\circ\Phi_h],t\big)+o(1).$$

Passing to the limit for $h \to 0$, we get

$$\mathcal{B}\big(w_{der}, t\big) \;=\; \mathcal{R}\big(\psi \cdot \nabla p^{inc}, t\big) + \mathcal{R}_1\big(p^{inc}, t\big) - \mathcal{B}_1\big(w_0, t\big) \tag{4.24}$$

for any test function vector $t = (v, q, \chi)$. In other words, the derivatives $w_{der} := (u_{der}, p_{der}, \sigma_{der})$ are the solution functions of a system with the same sesqui-linear form $\mathcal{B}$ as in the direct problem but with a different right-hand side defined by (4.18) and (4.22).

5 Numerical Solution of the Inverse Problem

For a numerical computation, the infinite dimensional space $H^\delta(\mathbb{S}^{d-1})$ must be replaced by a finite dimensional subspace. We use the orthonormal set of trigonometric resp. spherical harmonic basis functions $\varphi_n,\ n = 1, \dots, N$ spanning this subspace. If $d = 2$, then N should be an odd number and

$$\varphi_n\left(e^{\mathbf{i}2\pi t}\right) \;:=\; \begin{cases} (1+k^2)^{-\delta/2}\ \cos\left(2\pi k t\right) & \text{if } k = \left|n - \frac{N+1}{2}\right| \text{ and } n = 1, \dots, \frac{N+1}{2}, \\ (1+k^2)^{-\delta/2}\ \sin\left(2\pi k t\right) & \text{if } k = n - \frac{N+1}{2} \text{ and } n = 1 + \frac{N+1}{2}, \dots, N. \end{cases}$$

If $d = 3$, then N should be of the form $N = J(J+2)+1$. For any n with $(j-1)(j+1)+1 < n \le j(j+2)+1$, we choose $m = n - j(j+1) - 1$ and set

$$\varphi_n\left(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta\right) := (1+|j|^2)^{-\delta/2}\sqrt{\frac{2j+1}{4\pi}\frac{(j-|m|)!}{(j+|m|)!}}\,P_j^{|m|}(\cos\theta)\begin{cases}\sin\left(m\varphi\right) & \text{if } m>0\\ \cos\left(|m|\varphi\right) & \text{else,}\end{cases}$$

$$P_j^{|m|}(t) := (1-t^2)^{|m|/2}\frac{\mathrm{d}^{|m|}P_j(t)}{\mathrm{d}t^{|m|}}, \quad P_j(\cos\theta) \;:=\; \sin^j(\theta).$$

We introduce the finite vector $s = (s_n)_{n=1}^N$ of unknown coefficients and represent the unknown function $\mathbf{r}$ as the linear combination

$$\mathbf{r} \;=\; \mathbf{r}(s) \;:=\; \sum_{n=1}^{N} s_n\varphi_n. \tag{5.1}$$

The solution $\mathbf{r}^\gamma$ of (3.8) is sought in the form (5.1) and the optimization (3.8) reduces to the problem of finding $s^\gamma \in \mathbb{R}^N$ such that (cf. (3.7))

$$\mathcal{J}(\mathbf{r}(s^\gamma);\gamma) \;=\; \inf_{s\in\mathbb{R}^N}\ \mathcal{J}(\mathbf{r}(s);\gamma). \tag{5.2}$$

The right choice of the regularization parameter γ is a difficult problem. An accurate determination of the degree of ill-posedness seems to be impossible. However, a general method like the L-curve method could be applied (cf. e.g. [5]). On the other hand, if an obstacle of a certain class is to be determined, then a typical example of this class can be chosen, and the corresponding far field pattern can be simulated with a very accurate computation by the finite-element method for the direct problem. Using these data, the regularization parameter can be fitted to guarantee the best reconstruction of the typical example. Heuristically, this regularization parameter should be a good choice for the class of obstacles.

If γ is fixed, then (5.2) is a finite dimensional non-linear optimization problem for a differentiable objective function and without constraints. Methods like the conjugate gradient algorithm (cf. e.g. [18]) can be employed. However, the function evaluation is time consuming, and in each iteration step a line search is to be performed with several function evaluations. Moreover, the gradient based optimization schemes are local in nature, i.e., the limits of these iterative procedures may be local minima instead of the global minimum. Restarts from different initial solutions or a good initial guess obtained by stochastic optimization algorithms may improve the reliability of the outcome.

To avoid the expensive line search in the gradient based methods, we recommend to apply the Gauß-Newton method or the Levenberg-Marquardt scheme without line search (cf. e.g. [18]). Indeed, the least squares form of the objective functional suggests the application of the Gauß-Newton method. More precisely, if we introduce the vector $\mathcal{G}^{meas} := (p_1^\infty, p_2^\infty, \dots, p_K^\infty, 0)$ and the operator mapping

$$\mathcal{G} : \mathbb{R}^N \longrightarrow [L^2(\mathbb{S}^{d-1})]^K \times H^\delta(\mathbb{S}^{d-1}), \quad \mathcal{G}(s) := \begin{pmatrix} \mathcal{H}\big(A^p_{\Gamma^{\mathbf{r}}(s)}(p_1^{inc}), A^\sigma_{\Gamma^{\mathbf{r}}(s)}(p_1^{inc})\big) \\ \mathcal{H}\big(A^p_{\Gamma^{\mathbf{r}}(s)}(p_2^{inc}), A^\sigma_{\Gamma^{\mathbf{r}}(s)}(p_2^{inc})\big) \\ \dots \\ \mathcal{H}\big(A^p_{\Gamma^{\mathbf{r}}(s)}(p_K^{inc}), A^\sigma_{\Gamma^{\mathbf{r}}(s)}(p_K^{inc})\big) \\ \sqrt{\gamma}\mathbf{r}(s) \end{pmatrix},$$

then the functional $\mathcal{J}(\mathbf{r}(s), \gamma)$ is equal to $\|\mathcal{G}(s) - \mathcal{G}^{meas}\|_*^2$ where

$$\|(p_1^\infty, p_2^\infty, \dots, p_K^\infty, \mathbf{r})^\top\|_* \quad := \quad \sqrt{\sum_{k=1}^K \|p_k^\infty\|^2_{L^2(\mathbb{S}^{d-1})} + \|\mathbf{r}\|^2_{H^\delta(\mathbb{S}^{d-1})}}\,.$$

The optimization according to (5.2) amounts in solving the operator equation $\mathcal{G}(s) = \mathcal{G}^{meas}$. The Gauß-Newton iteration starts with an initial vector s^0, and, for each s^j, the next iterate s^{j+1} is obtained by

$$\begin{aligned} \mathcal{G}(s^{j+1}) = \mathcal{G}\big(s^j + [s^{j+1} - s^j]\big) \quad &\sim \quad \mathcal{G}(s^j) + \nabla\mathcal{G}(s^j)[s^{j+1} - s^j] = \mathcal{G}^{meas}, \\ \nabla\mathcal{G}(s^j)[s^{j+1} - s^j] \quad &= \quad \mathcal{G}^{meas} - \mathcal{G}(s^j), \\ s^{j+1} \quad &= \quad s^j + \big[\nabla\mathcal{G}(s^j)^* \nabla\mathcal{G}(s^j)\big]^{-1} \nabla\mathcal{G}(s^j)^* \big[\mathcal{G}^{meas} - \mathcal{G}(s^j)\big]. \end{aligned}$$

Roughly speaking (cf. [18]), this method converges quadratically if $\mathcal{G}(s) = \mathcal{G}^{meas}$ has a solution. It converges at least linearly if the minimal value $\|\mathcal{G}(s) - \mathcal{G}^{meas}\|$ is not too large. Unfortunately, due to the regularization and discretization, $\mathcal{G}(s) = \mathcal{G}^{meas}$ may have no solution.

Finally, we present a formula for the Jacobian $\nabla\mathcal{G}(s)$ which is needed for the Gauß-Newton method or for other gradient based optimization routines. We get

$$\begin{aligned} \nabla\mathcal{G}(s) \quad &= \quad \big(\partial_{s_n}\mathcal{G}(s)\big)_{n=1}^N, \\ \partial_{s_n}\mathcal{G}(s) \quad &= \quad \begin{pmatrix} \mathcal{H}\big(\partial_{s_n}[A^p_{\Gamma^{\mathbf{r}}(s)}(p_1^{inc})], \partial_{s_n}[A^\sigma_{\Gamma^{\mathbf{r}}(s)}(p_1^{inc})]\big) \\ \mathcal{H}\big(\partial_{s_n}[A^p_{\Gamma^{\mathbf{r}}(s)}(p_2^{inc})], \partial_{s_n}[A^\sigma_{\Gamma^{\mathbf{r}}(s)}(p_2^{inc})]\big) \\ \dots \\ \mathcal{H}\big(\partial_{s_n}[A^p_{\Gamma^{\mathbf{r}}(s)}(p_K^{inc})], \partial_{s_n}[A^\sigma_{\Gamma^{\mathbf{r}}(s)}(p_K^{inc})]\big) \\ \sqrt{\gamma}\varphi_n(s) \end{pmatrix}. \end{aligned}$$

The derivatives $\partial_{s_n}[A^p_{\Gamma^{\mathbf{r}}(s)}(p_k^{inc})]$ and $\partial_{s_n}[A^\sigma_{\Gamma^{\mathbf{r}}(s)}(p_k^{inc})]$ are nothing else than the derivatives $p_{der}|_\Gamma$ and σ_{der} (cf. (4.1), (4.4), and (4.5)) with $\mathbf{r}_0 = \mathbf{r}(s)$ and $\mathbf{r}_d = \varphi_n$. These derivatives can be computed from the variational equation (4.24).

6 Appendix

6.1 Proof of Theorem 2.1

By the definition of $\mathcal{B}$ in (2.8), we have

$$\begin{aligned} \mathbf{B}(v, q, \chi; v, q, \chi) \quad &= \quad a\left((v,q,\chi)^\top, (v,q,\chi)^\top\right) + b\left((v,q,\chi)^\top, (v,q,\chi)^\top\right) + 2\, c\left((v,q,\chi)^\top, (v,q,\chi)^\top\right) \\ &= \quad I_\Omega + II_{\Omega_R} + III_{\Gamma_0} + R_\Gamma + R_{\Gamma_0}, \end{aligned}$$

where

$$
\begin{aligned}
I_\Omega &= \int\limits_\Omega \Big\{\lambda|\nabla\cdot v|^2 + \frac{\mu}{2}\sum_{i,j=1}^d \Big[|\partial_i v_j|^2 + \partial_i v_j\overline{\partial_j v_i}\Big] - \rho\omega^2|v|^2\Big\}\mathrm{d}x, \\
II_{\Omega_R} &= \int\limits_{\Omega_R} \Big\{|\nabla q|^2 - {k_w}^2|q|^2\Big\}\mathrm{d}x, \\
III_{\Gamma_0} &= 2\int\limits_{\Gamma_0} (V_{\Gamma_0}\chi)\bar\chi \mathrm{d}s, \\
R_\Gamma &= \int\limits_\Gamma \Big\{q^+ n\cdot\overline{v^-} + \rho_f\omega^2(v^-\cdot n)\overline{q^+}\Big\}\mathrm{d}s, \\
R_{\Gamma_0} &= \int\limits_{\Gamma_0}\Big(q^-\bar\chi - \chi\overline{q^-}\Big)\mathrm{d}s - 2\int\limits_{\Gamma_0} K_{\Gamma_0}q^-\,\bar\chi\,\mathrm{d}s.
\end{aligned}
$$

By using Korn's inequality (cf. e.g. [4]), we obtain

$$\Re\mathrm{e}\ I_\Omega \ge \alpha_\Omega ||v||^2_{[H^1(\Omega)]^d} - c_\Omega||v||^2_{[H^0(\Omega)]^2}.$$

Similarly, we have

$$\Re\mathrm{e}\ II_{\Omega_R} \ge \alpha_{\Omega_R}||q||^2_{H^1(\Omega_R)} - c_{\Omega_R}||q||^2_{H^0(\Omega_R)}.$$

As is well-known, the simple-layer boundary integral operator V_{Γ_0} is continuous from $H^{-1/2}(\Gamma_0)$ into $H^{1/2}(\Gamma_0)$ (cf. e.g. [13]), and V_{Γ_0} is $(H^{-1/2}(\Gamma_0), H^{1/2}(\Gamma_0))$-coercive, i.e.,

$$\Re\mathrm{e}\ \langle V_{\Gamma_0}\chi,\chi\rangle_{\Gamma_0} \ge \alpha_{\Gamma_0}||\chi||^2_{H^{-1/2}(\Gamma_0)} - c_{\Gamma_0}||\chi||^2_{H^{-1}(\Gamma_0)}.$$

The double-layer boundary integral operator K_{Γ_0} is continuous from $H^{1/2}(\Gamma_0)$ into $H^{3/2}(\Gamma_0)$, and hence is compact on $H^{1/2}(\Gamma_0)$. This implies that

$$|\langle K_{\Gamma_0}q^-,\chi\rangle_{\Gamma_0}| \le ||K_{\Gamma_0}q^-||_{H^{3/2-\varepsilon_1}(\Gamma_0)}||\chi||_{H^{-3/2+\varepsilon_1}(\Gamma_0)}$$

for $q\in H^1(\Omega_R)$ and $\chi\in H^{-1/2}(\Gamma_0)$ and for $0<\varepsilon_1<1$. Consequently,

$$
\begin{aligned}
\Bigg|\int\limits_{\Gamma_0} K_{\Gamma_0}q^-\,\bar\chi\,\mathrm{d}s\Bigg| &\le \alpha_{\Gamma_0}||q||_{H^{1/2-\varepsilon_1}(\Gamma_0)}||\chi||_{H^{-1/2-(1-\varepsilon_1)}(\Gamma_0)} \\
&\le \frac{c_{\Gamma_0}}{2}\Big\{||q||^2_{H^{1-\varepsilon}(\Omega_R)} + ||\chi||^2_{H^{-1/2-\varepsilon}(\Gamma_0)}\Big\}
\end{aligned}
$$

with $\varepsilon = \min\{\varepsilon_1, 1-\varepsilon_1\}$. Thus,

$$\Re\mathrm{e}\ R_{\Gamma_0} \ge -c_{\Gamma_0}\Big\{||q||^2_{H^{1-\varepsilon}(\Omega_R)} + ||\chi||^2_{H^{-1/2-\varepsilon}(\Gamma_0)}\Big\},$$

since the first term in R_{Γ_0} has real part zero.

In the same manner, one can show that R_Γ defines a compact term. More precisely, we see that, for $1/2-\varepsilon>0$,

$$
\begin{aligned}
\Bigg|\int\limits_\Gamma q^+ n\cdot\overline{v^-}\mathrm{d}s\Bigg| &= |\langle q, n\cdot\bar v\rangle_\Gamma| \ \le\ ||q||_{H^0(\Gamma)}||v||_{[H^0(\Gamma)]^d} \ \le\ c_\Gamma||q||_{H^{1/2-\varepsilon}(\Gamma)}||u||^2_{[H^{1/2-\varepsilon}(\Gamma)]^d} \\
&\le\ c_\Gamma||q||^2_{H^{1-\varepsilon}(\Omega_R)} + ||u||^2_{[H^{1-\varepsilon}(\Gamma)]^d}.
\end{aligned}
$$

Collecting all these terms yields the desired estimates. This completes the proof of Theorem 2.1.

We remark that from the estimates, we may conclude that there exists a compact sesqui-linear form $\mathbf{C}(\cdot,\cdot)$ such that

$$\Re\mathrm{e}\,\Big\{\mathbf{B}(v,q,\chi;v,q,\chi)+\mathbf{C}(v,q,\chi)\Big\}\quad\geq\alpha\Big\{||v||^2_{[H^1(\Omega)]^d}+||q||^2_{H^1(\Omega_R)}+||\chi||^2_{H^{-1/2}(\Gamma_0)}\Big\}$$

for all $(v,q,\chi)^\top\in[H^1(\Omega)]^d\times H^1(\Omega_R)\times H^{-1/2}(\Gamma_0)$.

6.2 Proof of Theorem 2.2

Let $(u_0,p_0,\sigma_0)^\top\in[H^1(\Omega)]^d\times H^1(\Omega_R)\times H^{-1/2}(\Gamma_0)$ be a solution of the corresponding homogeneous equation of (2.8), i.e.,

$$\mathcal{B}\Big((u_0,p_0,\sigma_0)^\top,(v,q,\chi)^\top\Big)=(0,0,0)^\top$$

holds for any $(v,q,\chi)^\top\in[H^1(\Omega)]^d\times H^1(\Omega_R)\times H^{-1/2}(\Gamma_0)$. We extend the solution p_0 to the exterior domain Ω_e and consider the exterior Dirichlet problem defined by

$$\Delta p_1(x)+{k_w}^2p_1(x)=0,\ x\in\Omega_e\qquad p_1|_{\Gamma_0}=p_0^-\in H^{1/2}(\Gamma_0)$$

together with the radiation condition for p_1 at infinity. This exterior Dirichlet problem has a unique solution, which admits the representation

$$p_1(x)=\int\limits_{\Gamma_0}\Big\{\frac{\partial}{\partial n_y}E_{k_w}(x,y)p_1^+(y)-E_{k_w}(x,y)\frac{\partial}{\partial n_y}p_1^+(y)\Big\}\mathrm{d}s_y.$$

Here p_1^+ and $\frac{\partial}{\partial n}p_1^+$ are the Cauchy data of p_1 on Γ_0 satisfying the boundary integral equation

$$V_{\Gamma_0}\Big(\frac{\partial}{\partial n}p_1^+\Big)+\Big(\frac{1}{2}I-K_{\Gamma_0}\Big)p_0^-=0$$

on Γ_0, where we have substituted p_1^+ by p_0^- on Γ_0. Now, the weak formulation of the boundary integral equation reads

$$\Big\langle V_{\Gamma_0}\frac{\partial}{\partial n}p_1^+,\chi\Big\rangle_{\Gamma_0}+\Big\langle\Big(\frac{1}{2}I-K_{\Gamma_0}\Big)p_0^-,\chi\Big\rangle_{\Gamma_0}=0$$

for all $\chi\in H^{-1/2}(\Gamma_0)$. A comparison with the equation $c((u_0,p_0,\sigma_0)^\top,(v,q,\chi)^\top)=0$ gives

$$\Big\langle V_{\Gamma_0}\Big(\frac{\partial}{\partial n}p_1^+-\sigma_0\Big),\chi\Big\rangle_{\Gamma_0}=0$$

for all $\chi\in H^{-1/2}(\Gamma_0)$. Hence if V_{Γ_0} is invertible, this implies that $\frac{\partial}{\partial n}p_1^+\equiv\sigma_0$ in $H^{-1/2}(\Gamma_0)$. Thus, $(u_0,p)\in[H^1(\Omega)]^d\times H^1_{loc}(\mathbb{R}^d\setminus\mathrm{cl}\,(\Omega))$ with

$$p(x):=\begin{cases}p_0(x)&\text{if }x\in\Omega_R\cup\Gamma_0\\p_1(x)&\text{if }x\in\Omega_e\end{cases}$$

will be a solution of the homogeneous transmission problem for the Lamé equations in Ω, the Helmholtz equation in $\mathbb{R}^d\setminus\Omega$, and the transmission conditions $t^-(u_0)=-n\partial_np$ and $n\cdot u_0=p/(\varrho_f\omega^2)$ on Γ. The latter has only the trivial solution $(0,0)$, provided ω is not a Jones frequency (cf. [15, 10, 14, 6]).

If ω is a Jones frequency, then the solution is just $p\equiv0$ and u is equal to a Jones mode, i.e., u_0 is a solution of the Lamé equations on Ω with both, the traction $t^-(u_0)$ and the normal component $n\cdot u_0$ of the displacement, vanishing on Γ. Consequently, the solution component p_0 is zero, and the solution of the sesqui-linear system (2.8) is unique if it exits. So it remains to prove that a solution exists. However, due to Theorem 2.1, the Fredholm alternative holds for (2.8), and the existence of a solution follows, if we can prove that the right-hand side of (2.8) vanishes for all solutions $(v,q,\chi)^\top=(v_0,q_0,\chi_0)^\top$ of the homogeneous adjoint equation.

Suppose $(v_0, q_0, \chi_0)^\top \in [H^1(\Omega)]^d \times H^1(\Omega_R) \times H^{-1/2}(\Gamma_0)$ solves the homogeneous adjoint equation, i.e.,

$$\mathbf{B}(u, p, \sigma; v_0, q_0, \chi_0) \quad = \quad 0. \tag{6.1}$$

holds for any $(u, p, \sigma)^\top \in [H^1(\Omega)]^d \times H^1(\Omega_R) \times H^{-1/2}(\Gamma_0)$. Then (6.1) for all $(u, p, \sigma)^\top = (u, 0, 0)^\top$ with u vanishing in a neighbourhood of Γ, yields

$$\Delta^* v_0(x) + \varrho\,\omega^2 v_0(x) \quad = \quad 0, \quad x \in \Omega. \tag{6.2}$$

Similarly, (6.1) for all $(u, p, \sigma)^\top = (0, p, 0)^\top$ with p vanishing in a neighbourhood of Γ and Γ_0 implies

$$\Delta q_0(x) + {k_w}^2 q_0(x) \quad = \quad 0, \quad x \in \Omega_R. \tag{6.3}$$

The formula $V^*_{\Gamma_0} = V_{-k_w,\Gamma_0}$, including V_{-k_w,Γ_0} the single layer operator V_{Γ_0} with k_w replaced by $-k_w$, and choosing $(u, p, \sigma)^\top = (0, 0, \sigma)^\top$ in (6.1) provides us with

$$q_0|_{\Gamma_0} \quad = \quad 2\,V_{-k_w,\Gamma_0}\chi_0. \tag{6.4}$$

Choosing $(u, p, \sigma)^\top = (0, p, 0)^\top$ in (6.1) with p vanishing close to Γ and applying Green's formula for the integrals over Ω_R, we arrive at

$$\partial_n q_0|_{\Gamma_0} \quad = \quad \left(-I + 2\,K^*_{\Gamma_0}\right)\chi_0 \;=\; \left(-I + 2\,K'_{-k_w,\Gamma_0}\right)\chi_0, \tag{6.5}$$

where K'_{-k_w,Γ_0} is the transposed double layer operator on Γ_0 corresponding to the wave number $-k_w$. Choosing $(u, p, \sigma)^\top = (0, p, 0)^\top$ in (6.1) with p vanishing close to Γ_0 and using Green's formula, we obtain

$$n \cdot v_0|_\Gamma \quad = \quad \partial_n q_0|_\Gamma. \tag{6.6}$$

Finally, choosing $(u, p, \sigma)^\top = (0, p, 0)^\top$ in (6.1) with p vanishing close to Γ_0 and using Green's formula for the integrals over Ω, we obtain

$$t(v_0)|_\Gamma \quad = \quad -\varrho_f \omega^2 q_0\, n|_\Gamma. \tag{6.7}$$

Now we define $\tilde{u} := v_0$ and set

$$\tilde{p}(x) \quad := \quad \varrho_f \omega^2 \cdot \begin{cases} q_0(x) & \text{if } x \in \Omega_R \cup \Gamma_0 \\ \int_{\Gamma_0} E_{-k_w}(x, y)\chi_0(y)\mathrm{d}y & \text{if } x \in \Omega. \end{cases} \tag{6.8}$$

Note that $\tilde{p}$ is a continuous extension of q_0 as a solution of the exterior Helmholtz equation with radiation condition since the boundary data of q_0 and $\int_{\Gamma_0} E_{-k_w}(x, y)\chi_0(y)\mathrm{d}y$ coincide on Γ_0 by (6.4) and (6.5). From (6.2), (6.3), (6.6), and (6.7), we observe that the function pair $(\tilde{u}, \tilde{p})$ is a solution of the homogeneous transmission problem

$$\begin{aligned} \Delta^* \tilde{u}(x) + \varrho\,\omega^2 \tilde{u}(x) &= 0, \quad x \in \Omega, \\ \Delta \tilde{p}(x) + (-k_w)^2 \tilde{p}(x) &= 0, \quad x \in \mathbb{R}^d \setminus \Omega, \\ t^-(\tilde{u}(x)) &= -n\tilde{p}(x), \quad x \in \Gamma, \\ n \cdot \tilde{u}(x) &= \frac{1}{\varrho_f \omega^2}\partial_n \tilde{p}(x), \quad x \in \Gamma, \end{aligned}$$

and the corresponding Sommerfeld radiation condition. As mentioned above, this problem has a non-trivial solution only if ω is a Jones frequency. In this case, the solution satisfies $\tilde{p} \equiv 0$, and $\tilde{u}$ is equal to a solution of the Lamé equations on Ω with both, the traction $t^-(\tilde{u})$ and the normal component $n \cdot \tilde{u}$ of the displacement, vanishing on Γ. In other words, we conclude $n \cdot v_0|_\Gamma \equiv 0$, $q_0 \equiv 0$ and $V_{\Gamma_0}\chi_0 \equiv 0$, i.e., $\chi_0 \equiv 0$. Obviously, the right-hand side of (2.8) vanishes for $(v, q, \chi)^\top = (v_0, q_0, \chi_0)^\top$. This completes the proof.

6.3 Proof of Theorem 2.3

Suppose $\mathcal{M} \subseteq \mathbb{R}$ is the open set of all frequencies ω such that condition (b) of Theorem 2.2 is satisfied. Then the solution of (2.8) depending on the frequency parameter $\omega \in \mathcal{M}$ is the solution of an equation with an operator function depending on ω. All the operators are Fredholm with index zero due to Theorem 2.1. The set of parameters ω such that the operator is not invertible is either discrete (i.e., countable with no finite accumulation point) or the whole set $\mathcal{M}$. Due to the proof of Theorem 2.2, each nontrivial solution of the homogeneous equation corresponds to a nontrivial solution of (2.9). According to [14], there exists only a discrete set of frequencies with nontrivial traction free solutions of the Lamé equations. In other words, the set of parameters ω such that the operator is not invertible is discrete, and, for any ω_0, there is a small neighbourhood $\mathcal{N}_\varepsilon := \{\omega : \ |\omega - \omega_0| < \varepsilon\} \subseteq \mathcal{M}$ such that the operator is invertible for $\omega \in \mathcal{N}_\varepsilon \setminus \{\omega_0\}$. In particular, the solution is unique for $\omega \in \mathcal{N}_\varepsilon \setminus \{\omega_0\}$.

We continue the proof with an invariance relation for the Lamé-Helmholtz operator. The solution will be found in this invariant subspace. More precisely, by C_ω and C^*_ω we denote the operators corresponding to the sesqui-linear form $\mathcal{B}$, i.e.

$$\begin{aligned} C_\omega, C^*_\omega :\ [H^1(\Omega)]^d \times H^1(\Omega_R) \times H^{-1/2}(\Gamma_0) &\longrightarrow [H^{-1}(\Omega)]^d \times H^{-1}(\Omega_R) \times H^{1/2}(\Gamma_0), \\ \mathcal{B}\Big((u,p,\sigma)^\top, (v,q,\chi)^\top\Big) \ = \ \Big\langle C_\omega (u,p,\sigma)^\top, (v,q,\chi)^\top \Big\rangle &\ = \ \Big\langle (u,p,\sigma)^\top, C^*_\omega (v,q,\chi)^\top \Big\rangle. \end{aligned}$$

Using the connection between differential operator and sesqui-linear form, we get

$$\begin{aligned} C_\omega(u,p,\sigma)^\top \ &= \ \begin{pmatrix} \Delta^* u + t(u)\,\delta_\Gamma + \varrho\,\omega^2\, u + p\, n \delta_\Gamma \\ \Delta p - \partial_n p\,\delta_\Gamma + \partial_n p\,\delta_{\Gamma_0} + \frac{\omega^2}{c^2} p + \varrho_f \omega^2 u\cdot n\,\delta_\Gamma - \sigma\delta_{\Gamma_0} \\ V_{\Gamma_0}\sigma + \Big(\frac{1}{2}I - K_{\Gamma_0}\Big) p \end{pmatrix} \\ &= \ C_{\omega_0}(u,p,\sigma)^\top + E_{\omega_0,\omega}(u,p,\sigma)^\top + [\omega^2-\omega_0^2]\ D\,(u,p,\sigma)^\top, \end{aligned} \tag{6.9}$$

$$D\,(u,p,\sigma)^\top \ := \ \begin{pmatrix} \varrho\, u \\ \frac{1}{c^2} p + \varrho_f u \cdot n\,\delta_\Gamma \\ 0 \end{pmatrix}, \tag{6.10}$$

$$E_{\omega_0,\omega}(u,p,\sigma)^\top \ := \ \begin{pmatrix} 0 \\ 0 \\ V^\omega_{\Gamma_0}\sigma + \Big(\frac{1}{2}I - K^\omega_{\Gamma_0}\Big)p - V^{\omega_0}_{\Gamma_0}\sigma - \Big(\frac{1}{2}I - K^{\omega_0}_{\Gamma_0}\Big)p \end{pmatrix}. \tag{6.11}$$

Here the generalized functions $\Delta^* u + t(u)\delta_\Gamma \in [H^{-1}(\Omega)]^d$ and $\Delta p - \partial_n p\,\delta_\Gamma + \partial_n p\,\delta_{\Gamma_0} \in H^{-1}(\Omega_R)$ are defined by the corresponding bilinear forms, i.e. by the formulas

$$\begin{aligned} \big\langle \Delta^* u + t(u)\delta_\Gamma, v \big\rangle \ &:= \ \int_\Omega \Big\{ \lambda \nabla\cdot u\, \overline{\nabla \cdot v} + \frac{\mu}{2} \sum_{i,j=1}^d \big[\partial_i u_j\, \overline{\partial_j v_i} + \partial_i u_j\, \overline{\partial_i v_j}\,\big] \Big\}, \quad \forall v \in [H^1(\Omega)]^d, \\ \big\langle \Delta p - \partial_n p\,\delta_\Gamma + \partial_n p\,\delta_{\Gamma_0}, q \big\rangle \ &:= \ \int_{\Omega_R} \nabla\, p\, \overline{\nabla\, q}, \quad \forall q \in H^1(\Omega_R). \end{aligned}$$

We consider the subspaces $[\tilde{H}^1(\Omega)]^d \times H^1(\Omega_R) \times H^{-1/2}(\Gamma_0)$ and $[\tilde{H}^{-1}(\Omega)]^d \times H^{-1}(\Omega_R) \times H^{1/2}(\Gamma_0)$ with $[\tilde{H}^{\pm 1}(\Omega)]^d$ defined as the set of all $u \in [H^{\pm 1}(\Omega)]^d$ which are L^2 orthogonal to the solutions u^J of (2.9) for $\omega = \omega_0$. Moreover, we introduce the bounded L^2 projections $P^\pm$ of $[H^{\pm 1}(\Omega)]^d \times H^{\pm 1}(\Omega_R) \times H^{\mp 1/2}(\Gamma_0)$ onto $[\tilde{H}^{\pm 1}(\Omega)]^d \times H^{\pm 1}(\Omega_R) \times H^{\mp 1/2}(\Gamma_0)$ by

$$P^\pm(u,p,\sigma)^\top \ := \ \left(u - \sum_{j=1}^{j_*} \Big\langle u, u^J_j \Big\rangle_{L^2(\Omega)} u^J_j \ , \ p \ , \ \sigma \right)^\top, \tag{6.12}$$

where $u^J_j \in [H^1(\Omega)]^d,\ j = 1, 2, \ldots, j_*$ is an L^2 orthonormal basis in the space of homogeneous solutions for (2.9). From the proof of Theorem 2.2 we know that the kernel functions of C_{ω_0} and $C^*_{\omega_0}$ are of the form $(u_J, 0, 0)$ with u_J a solution of (2.9). Consequently, C_{ω_0} maps $[\tilde{H}^1(\Omega)]^d \times H^1(\Omega_R) \times H^{-1/2}(\Gamma_0)$ onto $[\tilde{H}^{-1}(\Omega)]^d \times H^{-1}(\Omega_R) \times H^{1/2}(\Gamma_0)$ and is invertible. In other words $P^- C_{\omega_0}|_{\mathrm{im}\, P^+}$ is invertible. Due to (6.9), even C_ω maps $[\tilde{H}^1(\Omega)]^d \times H^1(\Omega_R) \times H^{-1/2}(\Gamma_0)$ into $[\tilde{H}^{-1}(\Omega)]^d \times H^{-1}(\Omega_R) \times H^{1/2}(\Gamma_0)$, i.e. we get the invariance formula $(I - P^-)C_\omega P^+ = 0$. On the other hand, any function from $\mathrm{im}\,(I - P^+)$ is of the form $(u^J, 0, 0)^\top$ with u^J a solution of (2.9), and $C_{\omega_0}(u^J, 0, 0)^\top = (0, 0, 0)^\top$. From (6.9), we even get $C_\omega(u^J, 0, 0)^\top = (\varrho[\omega^2 - \omega_0^2]u^J, 0, 0)^\top$. Consequently, the second invariance relation $P^- C_\omega(I - P^+) = 0$ holds.

As mentioned in the proof of Theorem 2.2, the right-hand side of (2.8) is orthogonal to the kernel of the adjoint operator $C^*_{\omega_0}$, i.e. the right-hand side $(u', p', \sigma')^\top$ of the equivalent operator equation $C_\omega(u^\omega, p^\omega, \sigma^\omega)^\top = (u', p', \sigma')^\top$ is in the space $\mathrm{im}\, P^- = [\tilde{H}^{-1}(\Omega)]^d \times H^{-1}(\Omega_R) \times H^{1/2}(\Gamma_0)$. In other words, the solution of

$$P^- C_\omega|_{\mathrm{im}\, P^+}\, (u^\omega, p^\omega, \sigma^\omega)^\top = P^-(u', p', \sigma')^\top \tag{6.13}$$

in the space $\mathrm{im}\, P^+$ is a solution of $C_\omega(u^\omega, p^\omega, \sigma^\omega)^\top = (u', p', \sigma')^\top$ due to $(I - P^-)C_\omega P^+ = 0$. However, the operator $P^- C_\omega|_{\mathrm{im}\, P^+}$ is continuous with respect to ω and invertible for $\omega = \omega_0$. By a Neumann series argument, $P^- C_\omega|_{\mathrm{im}\, P^+}$ is invertible and $[P^- C_\omega|_{\mathrm{im}\, P^+}]^{-1}$ is continuous in a small neighbourhood $\{\omega : \; |\omega - \omega_0| < \varepsilon\}$. Finally, the solution $(u^\omega, p^\omega, \sigma^\omega)^\top$ of (6.13) and, equivalently of (2.8), is continuous with respect to ω, too.

6.4 Proof of Lemma 3.1

We fix the frequency $\omega = \omega_0$, choose a parametrization $\mathbf{r} = \mathbf{r}_0$, and prove the continuity with respect to $\mathbf{r}$ at $\mathbf{r}_0$. If ω_0 is not a Jones frequency of the domain $\Omega^{\mathbf{r}_0}$, then the inverse operators C_ω^{-1} are continuous in $\mathbf{r}$ near $\mathbf{r}_0$ (cf. the following transformation technique) and the result follows easily. Thus we may suppose that ω_0 is a Jones frequency. For this case, the idea is to adapt the arguments in the proof of Theorem 2.3 to show that, for any fixed $\mathbf{r}$, there is an analytic family of solutions to a modified boundary value problem depending on the frequency ω. Representing the solution at $\omega = \omega_0$ by a Cauchy integral over this family, we obtain the lemma in the case of a Jones frequency.

Since the domains Ω and Ω_R in Section 2 depend on $\Gamma = \Gamma^{\mathbf{r}}$, we write $\Omega = \Omega^{\mathbf{r}}$ and $\Omega_R = \Omega_R^{\mathbf{r}}$. We define $\widetilde{\Omega} := \{x \in \mathbb{R}^d : \; |x| < R/2\}$ and $\widetilde{\Omega}_R := \{x \in \mathbb{R}^d : \; R/2 < |x| < R\}$ and introduce the transformation $T^{\mathbf{r}} : \mathrm{cl}(\widetilde{\Omega} \cup \widetilde{\Omega}_R) \to \mathrm{cl}(\Omega^{\mathbf{r}} \cup \Omega^{\mathbf{r}}_R)$ and its inverse $[T^{\mathbf{r}}]^{-1}$ by

$$y = T^{\mathbf{r}}(x) \;:=\; \hat{x} \cdot \begin{cases} \dfrac{2\mathbf{r}(\hat{x})|x|}{R} & \text{if } 0 \le |x| \le \frac{R}{2} \\ \mathbf{r}(\hat{x}) + \dfrac{2(R - \mathbf{r}(\hat{x}))(|x| - R/2)}{R} & \text{if } \frac{R}{2} \le |x| \le R, \end{cases} \qquad \hat{x} := \frac{x}{|x|},$$

$$x = [T^{\mathbf{r}}]^{-1}(y) \;:=\; \hat{y} \cdot \begin{cases} \dfrac{R|y|}{2\mathbf{r}(\hat{y})} & \text{if } 0 \le |y| \le \mathbf{r}(\hat{y}) \\ R\dfrac{\frac{1}{2}(R + |y|) - \mathbf{r}(\hat{y})}{R - \mathbf{r}(\hat{y})} & \text{if } \mathbf{r}(\hat{y}) \le |y| \le R, \end{cases} \qquad \hat{y} := \frac{y}{|y|},$$

Clearly, these transformations are continuously differentiable on the open sets $\Omega^{\mathbf{r}} \cup \Omega^{\mathbf{r}}_R$ and $\widetilde{\Omega} \cup \widetilde{\Omega}_R$, respectively. They are only Lipschitz continuous on $\mathrm{cl}(\Omega^{\mathbf{r}} \cup \Omega^{\mathbf{r}}_R)$ and $\mathrm{cl}(\widetilde{\Omega} \cup \widetilde{\Omega}_R)$, respectively. The pull back $F(x) = f(T^{\mathbf{r}}(x))$ of a function f differentiable on $\Omega^{\mathbf{r}} \cup \Omega^{\mathbf{r}}_R$ is differentiable on $\widetilde{\Omega} \cup \widetilde{\Omega}_R$ and satisfies

$$[\partial_{y_i} f](T^{\mathbf{r}}(x)) \;=\; \sum_{k=1}^{d} \frac{\partial [T^{\mathbf{r}}]^{-1}_k\big(T^{\mathbf{r}}(x)\big)}{\partial y_i} \partial_{x_k} F(x), \quad i = 1, \ldots, d. \tag{6.14}$$

Hence, substituting $y = T^{\mathbf{r}}(x)$ into the sesqui-linear form $\mathcal{B}$ of (2.8) over $\Omega^{\mathbf{r}} \cup \Omega^{\mathbf{r}}_R$, we arrive at a sesqui-linear form $\mathcal{B}^{\mathbf{r}}$ defined on $[H^1(\widetilde{\Omega})]^d \times H^1(\widetilde{\Omega}_R) \times H^{-1/2}(\Gamma_0)$. The coefficient functions for the functions u, v, p, q, σ, and χ and their partial derivatives in the domain and curve/surface integrals of $\mathcal{B}^{\mathbf{r}}$ stem from

(6.14) and the Jacobians of the integral transformations. In other words, all these are simple expressions of the transformation functions of $T^{\mathbf{r}}$ and of their first order derivatives. Consequently, the sesqui-linear form $\mathcal{B}^{\mathbf{r}}$ depends continuously on $\mathbf{r} \in H^{\delta}(\mathbb{S}^{d-1}) \subset C^1(\mathbb{S}^{d-1})$, and so do the operators $C^{\mathbf{r}}_{\omega}$, $[C^{\mathbf{r}}_{\omega}]^*$, and $D^{\mathbf{r}}$. In the following, whenever we speak about convergence for $\mathbf{r} \to \mathbf{r}_0$ of entities defined over $\Omega^{\mathbf{r}}$, $\Omega^{\mathbf{r}}_R$, and $\Gamma^{\mathbf{r}}$ to those defined over $\Omega^{\mathbf{r}_0}$, $\Omega^{\mathbf{r}_0}_R$, and $\Gamma^{\mathbf{r}_0}$, we have in mind the convergence of the corresponding entities defined over the standard domains $\widetilde{\Omega}$, $\widetilde{\Omega}^{\mathbf{r}_0}_R$, and the boundary curve of $\widetilde{\Omega}$.

Now suppose the curve $\mathbf{r}_0$ is fixed such that the considered frequency ω_0 is a Jones frequency for $\Omega^{\mathbf{r}_0}$. To analyze the operator $C^{\mathbf{r}}_{\omega_0}$, we modify the operator $C^{\mathbf{r}}_{\omega}$ by fixing the frequency in some places. We define the new operator $\widetilde{C}^{\mathbf{r}}_{\omega} : \ [H^1(\Omega^{\mathbf{r}})]^d \times H^1(\Omega^{\mathbf{r}}_R) \times H^{-1/2}(\Gamma_0) \ \longrightarrow \ [H^{-1}(\Omega^{\mathbf{r}})]^d \times H^{-1}(\Omega^{\mathbf{r}}_R) \times H^{1/2}(\Gamma_0)$ by

$$\widetilde{C}^{\mathbf{r}}_{\omega}(u,p,\sigma)^{\top} \quad := \quad \begin{pmatrix} \Delta^{*} u + t(u)\,\delta_{\Gamma^{\mathbf{r}}} + \varrho\,\omega^2\, u + p\, n\delta_{\Gamma^{\mathbf{r}}} \\ \Delta p - \partial_n p\,\delta_{\Gamma^{\mathbf{r}}} + \partial_n p\,\delta_{\Gamma_0} + \frac{\omega_0^2}{c^2} p + \varrho_f \omega_0^2 u \cdot n\,\delta_{\Gamma^{\mathbf{r}}} - \sigma\delta_{\Gamma_0} \\ V^{\omega_0/c}_{\Gamma_0}\sigma + \left(\frac{1}{2} I - K^{\omega_0/c}_{\Gamma_0}\right) p \end{pmatrix}.$$

Clearly, $\widetilde{C}^{\mathbf{r}}_{\omega_0} = C^{\mathbf{r}}_{\omega_0}$. We can eliminate the unknowns p and σ introducing the operator $G^{\mathbf{r}}_{\omega} : \ [H^1(\Omega^{\mathbf{r}})]^d \longrightarrow [H^{-1}(\Omega^{\mathbf{r}})]^d$ by

$$\begin{aligned} G^{\mathbf{r}}_{\omega} u \quad &:= \quad \Delta^{*} u + t(u)\,\delta_{\Gamma^{\mathbf{r}}} + \varrho\,\omega^2\, u + \varrho_f\,\omega_0^2 \left[N_t D^{\omega_0/c}(u \cdot n)\, n \right] \delta_{\Gamma^{\mathbf{r}}} \ , \\ & N_t D^{\omega_0/c}(g) := p|_{\Gamma^{\mathbf{r}}}, \quad \Delta p(x) + \frac{\omega_0^2}{c^2} p(x) = 0, \quad x \in \mathbb{R}^d \setminus \mathrm{cl}(\Omega), \\ & \qquad \partial_n p(x) = g(x), \quad x \in \Gamma^{\mathbf{r}}, \\ & \qquad \frac{x}{|x|} \cdot \nabla p(x) - \mathbf{i}\frac{\omega_0}{c} p(x) = o\left(|x|^{-(d-1)/2}\right), \quad |x| \longrightarrow \infty. \end{aligned}$$

If $\tilde{p}^{\mathbf{r}}_{inc}$ depending on the right-hand side function p^{inc} is the solution of the exterior problem for the Helmholtz equation

$$\begin{aligned} \Delta \tilde{p}^{\mathbf{r}}_{inc}(x) + \frac{\omega_0^2}{c^2} \tilde{p}^{\mathbf{r}}_{inc}(x) \quad &= \quad 0, \quad x \in \mathbb{R}^d \setminus \mathrm{cl}(\Omega), \\ \partial_n \tilde{p}^{\mathbf{r}}_{inc}(x) \quad &= \quad -\partial_n p^{inc}(x), \quad x \in \Gamma^{\mathbf{r}}, \\ \frac{x}{|x|} \cdot \nabla \tilde{p}^{\mathbf{r}}_{inc}(x) - \mathbf{i}\frac{\omega_0}{c} \tilde{p}^{\mathbf{r}}_{inc}(x) \quad &= \quad o\left(|x|^{-(d-1)/2}\right), \quad |x| \longrightarrow \infty, \end{aligned}$$

then the equation

$$G^{\mathbf{r}}_{\omega} u \quad = \quad p^{\mathbf{r}}_{rhs}, \quad p^{\mathbf{r}}_{rhs} \ := \ -[p^{inc} + \tilde{p}^{\mathbf{r}}_{inc}] n \delta_{\Gamma^{\mathbf{r}}} \tag{6.15}$$

is equivalent to

$$\widetilde{C}^{\mathbf{r}}_{\omega} \begin{pmatrix} u \\ p \\ \sigma \end{pmatrix} \quad = \quad \begin{pmatrix} -p^{inc}\, n\, \delta_{\Gamma^{\mathbf{r}}} \\ -\partial_n p^{inc} \delta_{\Gamma^{\mathbf{r}}} \\ 0 \end{pmatrix}. \tag{6.16}$$

Note that $p^{\mathbf{r}}_{rhs}$ is orthogonal to the Jones modes corresponding to $\Omega^{\mathbf{r}}$ and ω_0 (if there exist any). In particular, $(u,p,\sigma)^{\top}$ solves (6.16) if and only if $(u, p - \tilde{p}^{\mathbf{r}}_{inc}, \sigma - \partial_n \tilde{p}^{\mathbf{r}}_{inc}|_{\Gamma_0})^{\top}$ solves

$$\widetilde{C}^{\mathbf{r}}_{\omega} \begin{pmatrix} u \\ p - \tilde{p}^{\mathbf{r}}_{inc} \\ \sigma - \partial_n \tilde{p}^{\mathbf{r}}_{inc}|_{\Gamma_0} \end{pmatrix} \quad = \quad \begin{pmatrix} p^{\mathbf{r}}_{rhs} \\ 0 \\ 0 \end{pmatrix}$$

or equivalently if u solves (6.15) and p and σ are defined as

$$\begin{aligned}
\Delta p(x) + \frac{\omega_0^2}{c^2} p(x) &= 0, \quad x \in \mathbb{R}^d \setminus \mathrm{cl}(\Omega), \\
\partial_n p(x) &= \partial_n \tilde{p}^{\mathbf{r}}_{inc}(x) + \varrho_f \omega_0^2 u(x) \cdot n(x), \quad x \in \Gamma^{\mathbf{r}} \\
\frac{x}{|x|} \cdot \nabla p(x) - \mathbf{i}\frac{\omega_0}{c} p(x) &= o\left(|x|^{-(d-1)/2}\right), \quad |x| \longrightarrow \infty, \qquad (6.17) \\
\sigma &:= \partial_n p|_{\Gamma_0}.
\end{aligned}$$

If u is given and p and σ are defined by the above equations, then we write $p^{\mathbf{r}}_u := p$ and $\sigma^{\mathbf{r}}_u := \sigma$.

Clearly, $G^{\mathbf{r}_0}_\omega$ is a compact perturbation of the Lamé wave operator ($G^{\mathbf{r}_0}_\omega$ with $\omega_0 = 0$). Hence it is Fredholm with index zero. Using the last equivalence, we conclude that $G^{\mathbf{r}_0}_\omega u = 0$ implies $\widetilde{C}^{\mathbf{r}_0}_\omega (u, p^{\mathbf{r}_0}_u, \sigma^{\mathbf{r}_0}_u)^\top = 0$. For real ω, the proof of Theorem 2.2 implies $p^{\mathbf{r}_0}_u = 0$, $\sigma^{\mathbf{r}_0}_u = 0$, and that u is either zero or a Jones mode. On the other hand, the set of Jones frequencies for a fixed domain is countable. Consequently, there is an ω_* such that $G^{\mathbf{r}_0}_{\omega_*}$ is invertible, too. For $\mathbf{r}$ close to $\mathbf{r}_0$, $G^{\mathbf{r}_0}_{\omega_*}$ is invertible, too. In other words, the equation $G^{\mathbf{r}}_\omega u = p^{\mathbf{r}}_{rhs}$ is equivalent to

$$\left[G^{\mathbf{r}}_{\omega_*} + \varrho[\omega^2 - \omega_*^2] J\right] u = p^{\mathbf{r}}_{rhs}, \qquad (6.18)$$

$$\left[\lambda I - \left[G^{\mathbf{r}}_{\omega_*}\right]^{-1} J\right] u = \lambda \left[G^{\mathbf{r}}_{\omega_*}\right]^{-1} p^{\mathbf{r}}_{rhs}, \quad \lambda := \lambda(\omega) := -\varrho^{-1}[\omega^2 - \omega_*^2]^{-1}, \qquad (6.19)$$

where J stands for the embedding of $[H^1(\Omega^{\mathbf{r}})]^d$ into $[H^{-1}(\Omega^{\mathbf{r}})]^d$. If a nonzero solution u exists for the homogeneous equation (6.18), then this is an eigenfunction of the compact operator $[G^{\mathbf{r}}_{\omega_*}]^{-1} J$. We choose a simple closed curve $\Theta \subseteq \mathbb{C}$ around $\lambda_0 := \lambda(\omega_0)$ such that the only eigenvalue of $[G^{\mathbf{r}_0}_{\omega_*}]^{-1} J$ inside and on Θ is λ_0 and that no eigenvalues of $[G^{\mathbf{r}}_{\omega_*}]^{-1} J$ are located on Θ for $\mathbf{r}$ close to $\mathbf{r}_0$. From the proof of Theorem 2.3 applied to the operators $G^{\mathbf{r}}_\omega$, we conclude the existence of a possibly nonunique solution

$$u^{\mathbf{r}}_\lambda := \left[G^{\mathbf{r}}_\omega\right]^{-1} p^{\mathbf{r}}_{rhs} = \left\{\lambda I - \left[G^{\mathbf{r}}_{\omega_*}\right]^{-1} J\right\}^{-1} \lambda \left[G^{\mathbf{r}}_{\omega_*}\right]^{-1} p^{\mathbf{r}}_{rhs}$$

with $\lambda = \lambda(\omega)$ or equivalently $\omega^2 := \omega(\lambda)^2 := -\varrho^{-1}\lambda^{-1} + \omega_*^2$. Note that $p^{\mathbf{r}}_{rhs}$ is orthogonal to the Jones modes corresponding to $\Omega^{\mathbf{r}}$ and ω_0 (if there exist any). This solution $u^{\mathbf{r}}_\lambda$ exists for any λ close to λ_0 and depends analytically on λ (cf. end of proof of Theorem 2.3). Hence,

$$u^{\mathbf{r}}_{\lambda_0} = \frac{1}{\pi \mathbf{i}} \int_\Theta \frac{1}{(\lambda - \lambda_0)} \left[G^{\mathbf{r}}_{\omega(\lambda)}\right]^{-1} p^{\mathbf{r}}_{rhs} \, \mathrm{d}_\Theta \lambda.$$

Obviously, the function under the integral is continuous with respect to λ over Θ, uniformly bounded with respect to $\mathbf{r}$, and converges pointwise on Θ for $\mathbf{r} \to \mathbf{r}_0$, i.e. by our convention,

$$\left[\left[G^{\mathbf{r}}_{\omega(\lambda)}\right]^{-1} p^{\mathbf{r}}_{rhs}\right] \circ T^{\mathbf{r}} \longrightarrow \left[\left[G^{\mathbf{r}_0}_{\omega(\lambda)}\right]^{-1} p^{\mathbf{r}_0}_{rhs}\right] \circ T^{\mathbf{r}_0}.$$

In other words, the mapping $\mathbf{r} \mapsto u^{\mathbf{r}}_{\lambda_0}$ is $[H^1]^d$ continuous at $\mathbf{r}_0$. Finally, the functions $A^p_{\Gamma^{\mathbf{r}}}(p^{inc}) = p^{\mathbf{r}}_{u^{\mathbf{r}}_{\lambda_0}}$ and $A^\sigma_{\Gamma^{\mathbf{r}}}(p^{inc}) = \sigma^{\mathbf{r}}_{u^{\mathbf{r}}_{\lambda_0}}$ depend continuously on $\mathbf{r}$, too.

7 Numerical Tests

7.1 An Example for the Direct Problem

The first step of the solution for the inverse problem is to implement a solver for the direct problem. For example, we have chosen the constants for the transmission and boundary value problem (2.1)-(2.5) such that $k_w = \varrho\,\omega^2 = \varrho_f\,\omega^2 = \lambda = 1$ and $\mu = 0.5$. We have implemented a piecewise linear finite element method (FEM) coupled with boundary elements based on the variational equation (2.8). The FEM

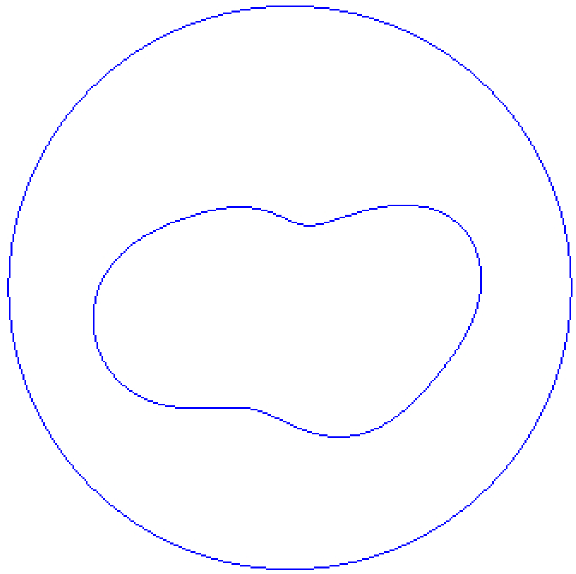

Figure 2: Boundary curve Γ of solid included in circle Γ_0.

grid is generated by `Netgen` (cf. [19]). The integrals of the boundary integral operators are discretized by Simpson's rule. In particular, the weakly singular integrals are computed over finer quadrature partitions. Assuming that the restriction of the FEM grid to the circle Γ_0 is uniform, the discretization of the boundary integral operator leads to circulant matrices such that non-optimal quadratures do not affect the overall computation time. Finally, the linear system of the FEM is solved by the direct solver `Pardiso` which is adapted to sparse matrices (cf. [20]).

In our test example, the radius R of Γ_0 is set to 6 and the boundary curve Γ of the solid is defined by (3.2) and (cf. (5.1) and set $\delta = 2$)

$$\mathbf{r}\left(e^{\mathbf{i}2\pi t}\right) \quad := \quad -\frac{\sin(2\pi t)}{(1+1^2)} + 3\,\frac{\cos(4\pi t)}{(1+2^2)} + 0.2\,\frac{\sin(6\pi t)}{(1+3^2)} + 0.5\,\frac{\cos(8\pi t)}{(1+4^2)} + 3\,\frac{\sin(8\pi t)}{(1+4^2)}\,. \tag{7.1}$$

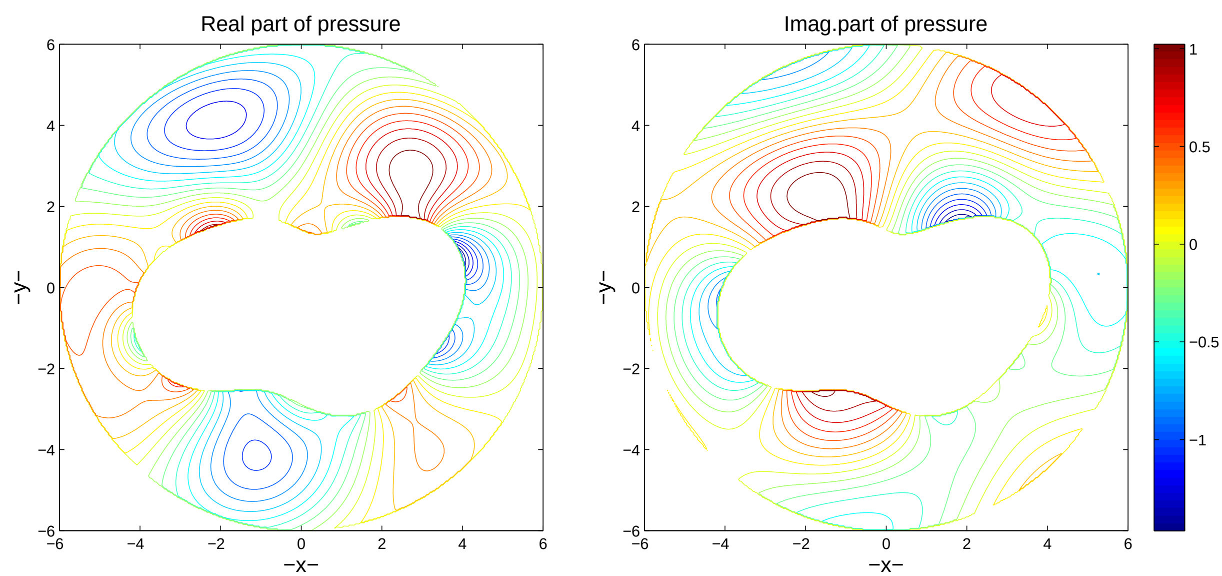

Figure 3: Scattered pressure field p^s over annular domain Ω_R.

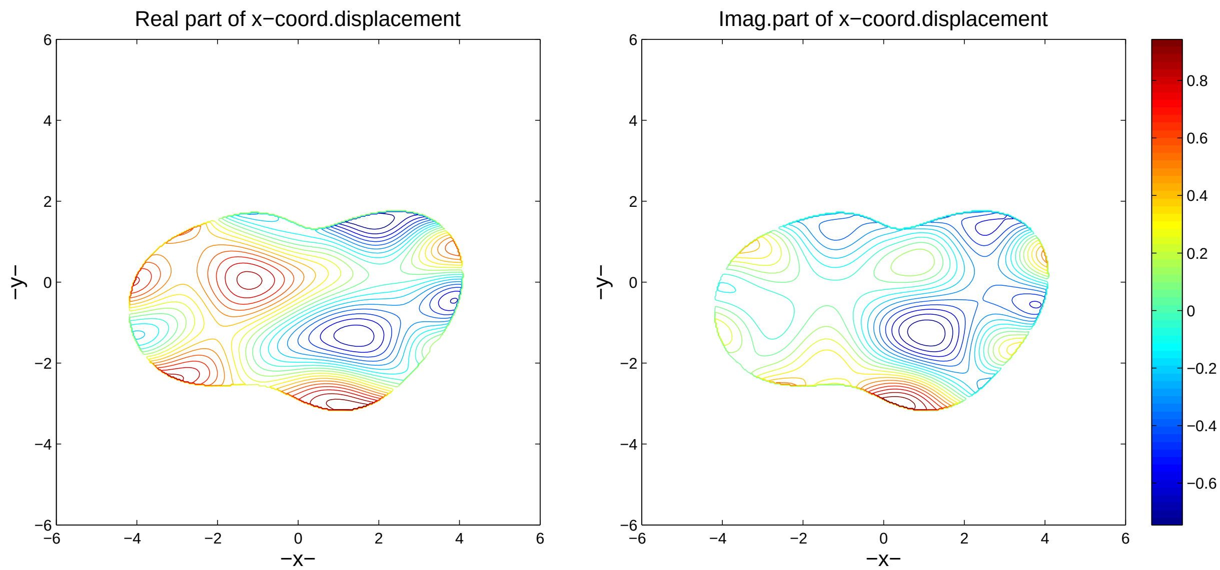

Figure 4: x_1-component of displacement field u over domain Ω.

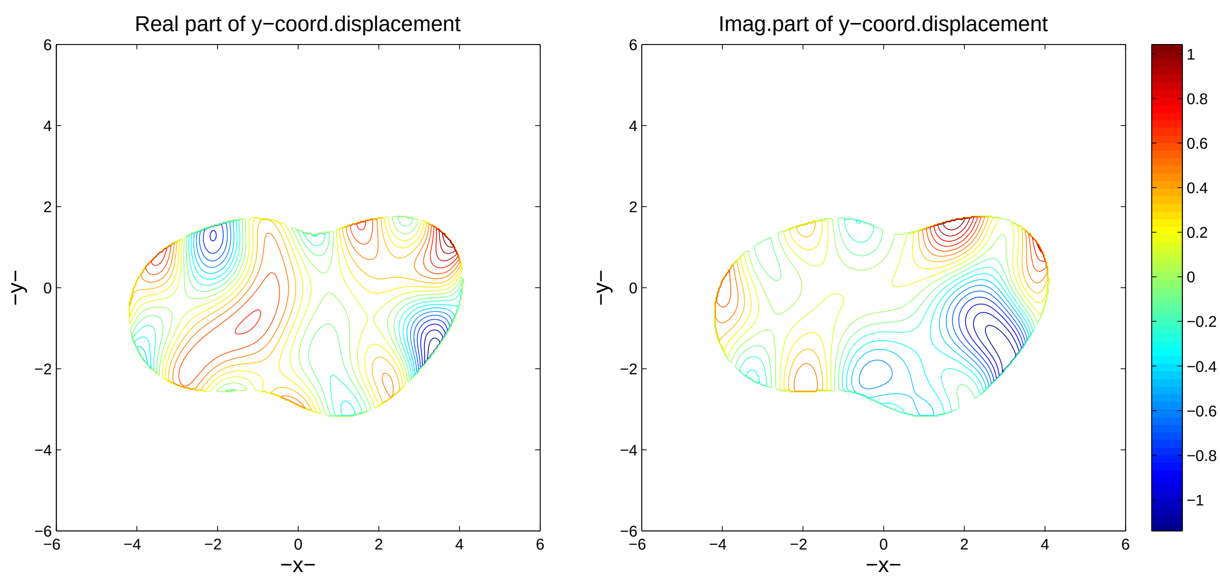

Figure 5: x_2-component of displacement field u over domain Ω.

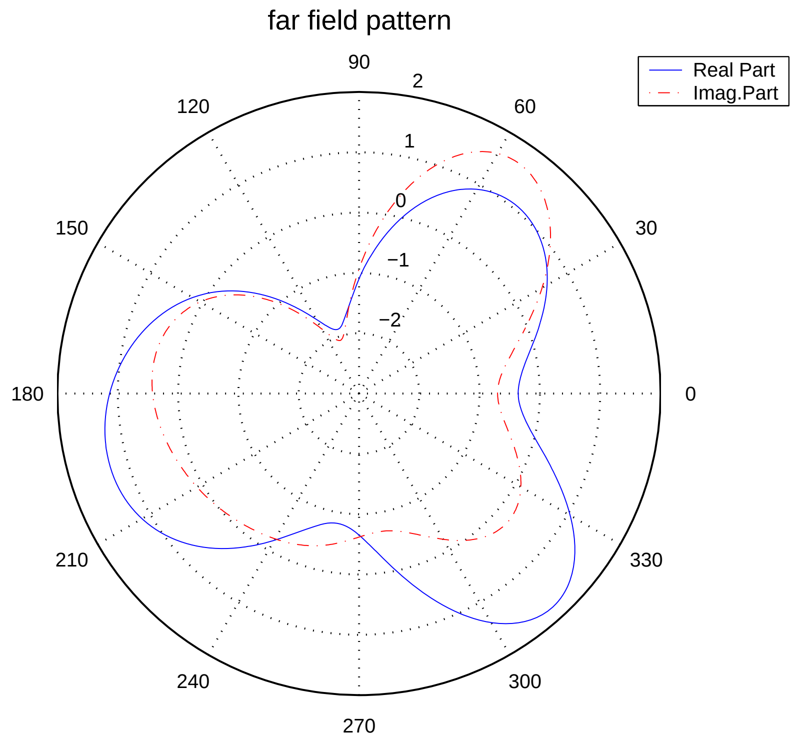

Figure 6: Far field pattern p^{∞} of scattered field p.

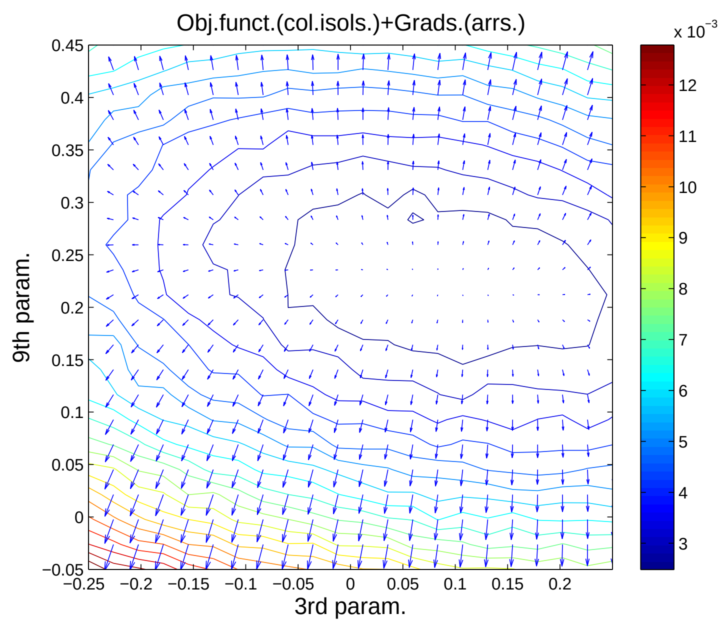

Figure 7: Objective functional $\mathcal{J}$ depending on the two parameters s_3 and s_9. The other parameters s_n are fixed to the values of the exact solution.

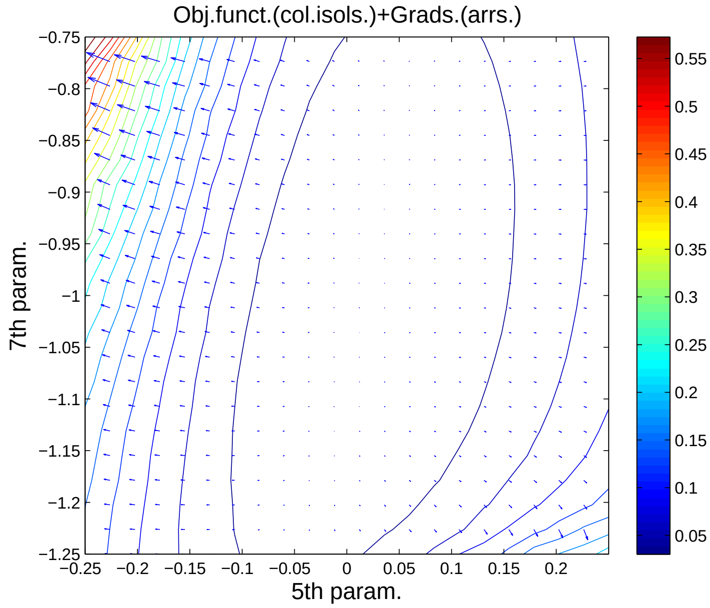

Figure 8: Objective functional $\mathcal{J}$ depending on the two parameters s_5 and s_7. The other parameters s_n are fixed to the values of the exact solution.

The corresponding curves Γ and Γ_0 are shown in Figure 2. Figures 3 to 5 exhibit the scattered pressure field p^s over Ω_R and the components of the displacement field u over Ω resulting from the incident field $p^{inc}(x) := e^{\mathbf{i}x_1}$. Finally, the far field pattern p^∞ of p (cf. (2.6)) is shown in Figure 6.

7.2 A Test for the Inverse Problem

In order to check the algorithm for the inverse problem of Section 3, we try to reconstruct the shape of Ω used in the last subsection (cf. formula (7.1) and Figure 2). More precisely, we take the far field data (cf. Figure 6) at the eighty directions $e^{\mathbf{i}2\pi k/80}$, $k = 0, 1, \ldots, 79$, measured for the incident wave $p^{inc}(x) := e^{\mathbf{i}2\pi x_1}$. Minimizing the functional $\mathcal{J}$ (cf. (3.7)) with the L^2 norm replaced by a quadrature approximation over the eighty points, we try to find the geometry determined by the radial function $\mathbf{r} = \mathbf{r}^{exact}$. Note that the far field values are generated by FEM on a discretization level (mesh size $h = 0.03125$) higher than the FEM level used for the inverse algorithm.

In accordance with Section 5, we determine $N = 11$ Fourier coefficients s_n of the radial function (cf. (5.1)). For our first tests, we set the regularization parameter γ to zero. Clearly, the operators $A^p_{\Gamma\mathbf{r}}$ and $A^\sigma_{\Gamma\mathbf{r}}$ in (3.7) are approximated by the FEM (mesh size $h = 0.5$, 0.25, 0.125, 0.0625). Two-dimensional sections of the graphs of the objective functional are shown in Figures 7 - 9. Beside the isolines marking the values of the objective functional, the arrows indicate the numerical gradients which are determined by the method described in Section 4. Figure 9 suggests that the far field pattern is extremely sensitive to s_6, the coefficient of the constant term in the Fourier series expansion (cf. (5.1)). To avoid troubles with algorithms for optimization problems with badly scaled parameters, we have replaced the parameter s_6 by the internal parameter $s_6' = 5s_6$. Indeed, without this scaling the Gauß-Newton algorithm (over FEM level $h = 0.0625$) does not converge properly. For the scaled version, however, the algorithm converges.

Now the geometry is reconstructed by the Gauß-Newton scheme of Section 5. The initial solution is the circle of radius 3, i.e. all the Fourier coefficients s_n^0 vanish. To evaluate the reconstruction quality of our algorithm, we introduce the maximum error of the radial function $\mathbf{r}_R^j := R/2 + R/\pi \arctan(\mathbf{r}^j)$

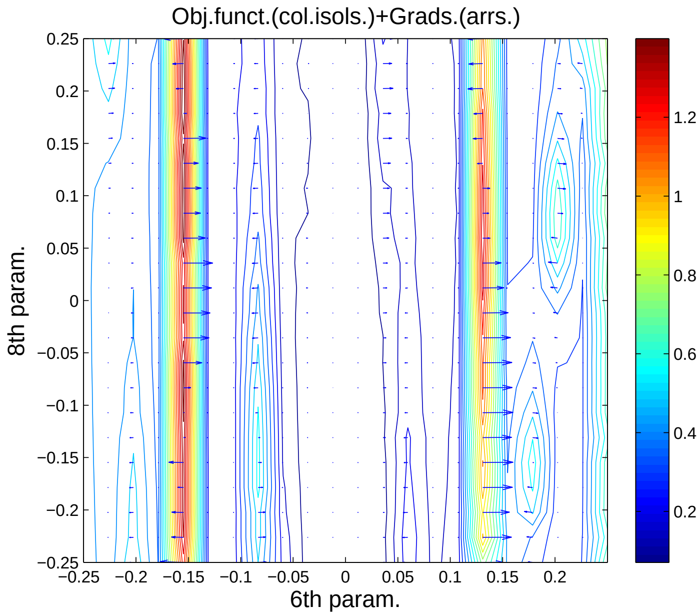

Figure 9: Objective functional $\mathcal{J}$ depending on the two parameters s_6 and s_8. The other parameters s_n are fixed to the values of the exact solution.

corresponding to the jth iterate s^j (cf. Section 5) by

$$me \;:=\; \sup_{0\le t\le 2\pi} \left| \mathbf{r}_R^j\left(e^{\mathbf{i}2\pi t}\right) - \mathbf{r}_R^{exact}\left(e^{\mathbf{i}2\pi t}\right) \right|, \quad \mathbf{r}_R^{exact}(\hat{x}) \;:=\; \frac{R}{2} + \frac{R}{\pi}\arctan\left(\mathbf{r}^{exact}(\hat{x})\right),$$

$$\mathbf{r}^{exact}(\hat{x}) \;:=\; \sum_{n=1}^{N} s_n^{exact}\varphi_n(\hat{x})\,.$$

The values of me and some of the reconstructed Fourier coefficients s_n are presented in Table 1. If the mesh size h of the FEM grid tends to zero, then the reconstructed radial functions tend to the exact solution. The index $j = j_{min}$ in Table 1 is the index of the iteration step corresponding to the minimal value of the objective function. Note that the objective functional decreases during the Gauß-Newton iteration until it reaches a level where it oscillates slightly around an almost minimal value. Figures 10 and 11 exhibit the reconstructed geometry for the levels $h = 0.5$ and $h = 0.25$. For smaller h, the difference to the reconstructed geometry in Figure 2 is not visible anymore. The evolution of the geometry for the iteration on level $h = 0.125$ is shown in Figure 12.

h	$j = j_{min}$	s_7^j	s_8^j	s_9^j	$\mathcal{J}(\mathbf{r}^j, 0)$	me
0.5	12	-0.85149	-0.15550	-0.31742	0.017704	0.6004
0.25	16	-0.99807	0.08073	0.31065	0.007246	0.2112
0.125	15	-0.96986	0.01926	0.17444	0.000457	0.0519
0.0625	22	-0.99213	0.00434	0.19237	0.000018	0.0129
exact		-1.	0.	0.2		

Table 1: Convergence of the reconstruction for mesh size h of the FEM grid tending to zero.

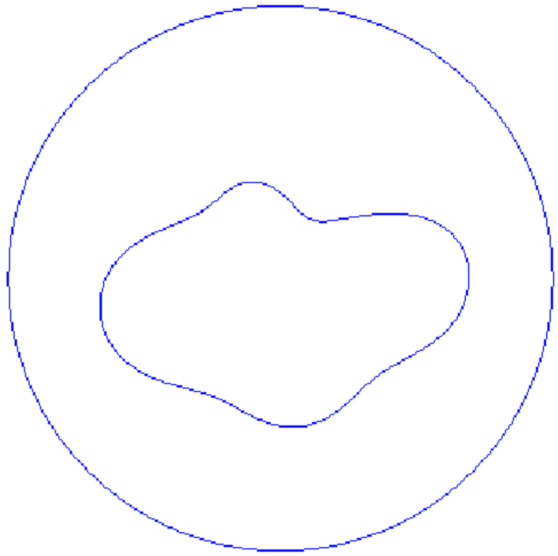

Figure 10: Reconstructed geometry for level $h = 0.5$.

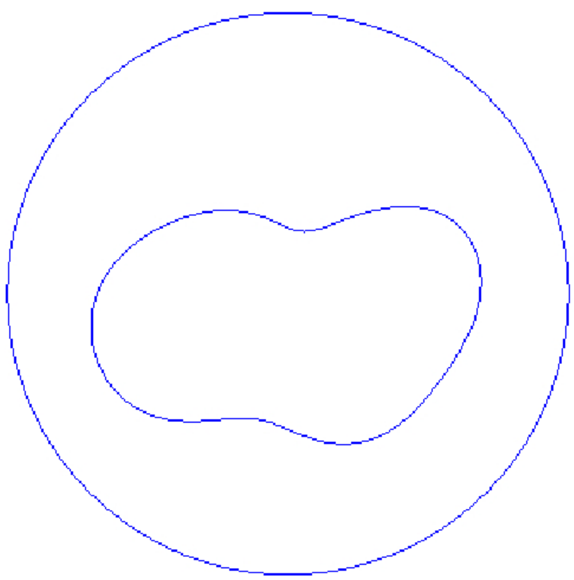

Figure 11: Reconstructed geometry for level $h = 0.25$.

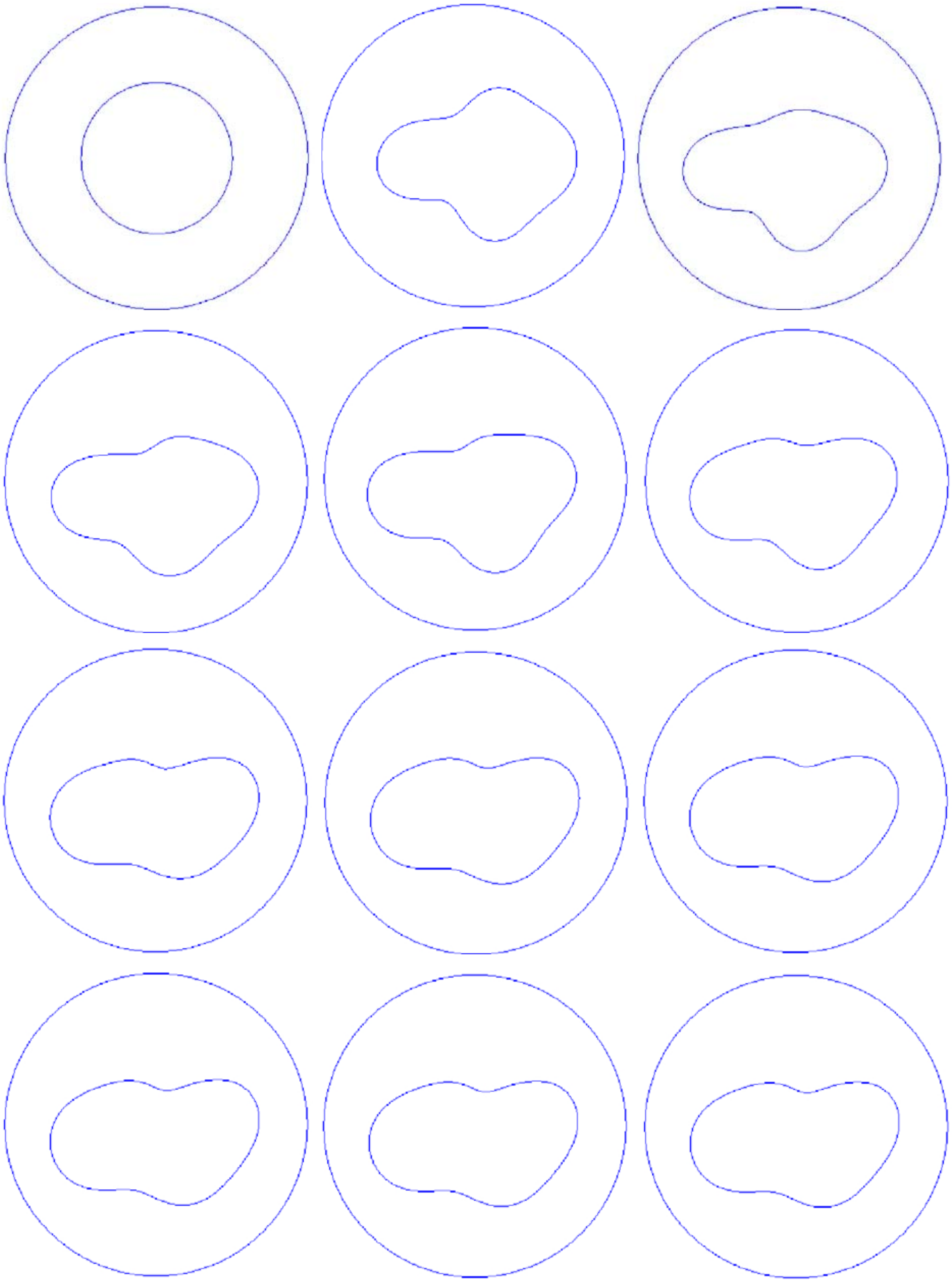

Figure 12: Evolution of geometry during iteration for level $h = 0.125$.

So far we have assumed exact measurement data, and the regularization parameter γ has been set to zero. Next, for the discretization level $h = 0.125$, we have added a stochastic error to each of the eighty measured far field values. These errors have been determined as the product of an error level ε times a random number between minus one and plus one generated by the c-code function drand48. Surprisingly, we have observed good reconstruction results even with regularization parameter $\gamma = 0$. For $\varepsilon = 0.16$, we still have an error me of 0.05. Even for $\varepsilon = 0.49$, the error me is 0.07. Probably, because of the huge number of measured far field data, regularization is needed only for degrees of freedom N much larger than eleven.

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