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Elliptic model problems including mixed boundary conditions and material heterogeneities

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Abstract

We present model problems in three dimensions, where the operator $-\nabla \cdot \mu\nabla$ maps the Sobolev space $W^{1,p}_\Gamma(\Omega)$ isomorphically onto $W^{-1,p}_\Gamma(\Omega)$ for a $p > 3$. The emphasis is here on the case where different boundary conditions meet material heterogeneities.

Résumé

Cet article présente des situations modèles en trois dimensions, dans lesquelles l'opérateur $-\nabla \cdot \mu\nabla$ est un isomorphisme de $W^{1,p}_\Gamma(\Omega)$ sur $W^{-1,p}_\Gamma(\Omega)$ pour un $p > 3$. On s'intéresse notamment au cas où des conditions au bord mixtes Dirichlet/Neumann sont combinées avec des sauts du coefficient μ.

1 Introduction

Many elliptic problems originating from science, engineering, and technology exhibit mixed boundary conditions and non-smooth material parameters, see [1] and the references cited there. For instance, in the simulation of operation and fabrication of semiconductor devices one is regularly confronted with heterogeneous materials in the volume and on the boundary (contacts), see [43], while dealing with elliptic and parabolic equations as mathematical models, see [16]. However, not much is known concerning maximal regularity for elliptic operators which include mixed boundary conditions. Moreover, most of this is restricted to Hilbert space scales, see e.g. [41], [40], [24], [11], [7]. Unfortunately, the Hilbert space $H^{3/2}$ is a principle threshold for mixed elliptic, second order problems at least in the case when the Dirichlet and Neumann boundary part meet on smooth parts of the boundary, see [45], see also [41]. Thus, within this scale one cannot expect (simultaneously) an embedding of the domains of these operators in L^∞ (or even in C^α) in case of three or more space dimensions. But exactly this is desirable in view of nonlinear, in particular quasilinear problems, see [39], [25], [37].

Concerning optimal regularity in non-Hilbert spaces there are the results of [44], [45] [5], [21], [10], [19], [20]; for the pure Dirichlet or pure Neumann case see [29] and [52], respectively. Gröger proved in [21] that under only L^∞ (and ellipticity) assumptions on the coefficient function μ, the Lipschitz property of the domain Ω and very weak assumptions on the Dirichlet boundary part $\partial\Omega \setminus \Gamma$ the operator

$$-\nabla \cdot \mu \nabla \,:\, W^{1,p}_\Gamma(\Omega) \to W^{-1,p}_\Gamma(\Omega) \tag{1.1}$$

is a topological isomorphism for a certain $p > 2$ ($W^{1,p}_\Gamma(\Omega)$ denoting the subspace of $W^{1,p}(\Omega)$ including a trace zero condition on the Dirichlet boundary part $\partial\Omega \setminus \Gamma$, and $W^{-1,p}_\Gamma(\Omega)$ the dual of $W^{1,p'}_\Gamma(\Omega)$). This result has found numerous applications within the treatment of applied problems. Nevertheless, it is well known that under these general assumptions one can only expect that p exceeds 2 arbitrarily little. This is the reason why the applications of [21] remained restricted to two dimensional problems. Because the demand for three dimensional modelling and simulation steadily increases, the question arises under which assumptions the isomorphism property of (1.1) can be obtained for a $p > 3$ and, in particular, whether this is true with mixed boundary conditions. Dauge proved in [10] that if the domain is a convex polyhedron and the border between Dirichlet and Neumann boundary part consists of (finitely many) line segments, then the Laplacian provides a topological isomorphism between $W^{1,p}_\Gamma$ and $W^{-1,p}_\Gamma$ for some $p > 3$. In this paper we generalise this to prototypical situations where mixed boundary conditions and heterogeneous, anisotropic coefficient functions occur simultaneously. Thus, this calculus allows for jumps in the normal component of the gradient of solutions across internal interfaces.

This means, e.g. in electrostatics, that the jump in the normal component of the electric field $\nu_+ \cdot \varepsilon\nabla\varphi - \nu_- \cdot \varepsilon\nabla\varphi$ across a prescribed interface equals the surface charge density on the interface, and this surface charge density is represented by a distribution on the underlying domain Ω.

In view of an adequate localisation principle, see [21], the geometric constellations we investigate may be viewed as local constituents of rather complex global settings.

Since the knowledge of the singularity of solutions is crucial for the efficiency of numerical methods, there exist of course several numerical approaches to determine singular exponents of concrete anisotropic problems, see [34], [9], [48] and the references therein. For a more general numerical approach to heterogeneous elliptic problems see for instance [2], [26], [8], [50] and the references cited there.

In detail, our results are as follows:

Theorem 1.1. *Let $\Lambda \subset \mathbb{R}^2$ be an (open) triangle, let further P be the center of one of its sides and Υ the (open) leg between P and one of its neighbouring vertices. Define $\Pi \overset{\text{def}}{=} \Lambda \times]-1,1[$ and the boundary part Σ as $\Upsilon \times]-1,1[$. Suppose Ξ to be a plane within $\mathbb{R}^3$ that intersects $\{P\} \times]-1,1[$ in exactly one point. Assume that the elliptic coefficient function μ takes its values in the set of real, symmetric, positive definite 3×3 matrices and is constant on both components of $\Pi \setminus \Xi$. Then there is a $p > 3$ such that*

$$-\nabla \cdot \mu\nabla \,:\, W^{1,p}_{\Sigma}(\Pi) \to W^{-1,p}_{\Sigma}(\Pi) \tag{1.2}$$

is a topological isomorphism.

Theorem 1.2. *Let $\Lambda \subset \mathbb{R}^2$ be an (open) triangle, Υ be one of its (open) sides or $\partial\Lambda \setminus \Upsilon$ one of its (closed) sides. Define $\Pi \overset{\text{def}}{=} \Lambda \times]-1,1[$ and the boundary part Σ as $\Upsilon \times]-1,1[$. Let further Ξ be a plane the intersection of which with the boundary of Σ consists of exactly two points. Assume that the elliptic coefficient function μ takes its values in the set of real, symmetric, positive definite 3×3 matrices and is constant on both components of $\Pi \setminus \Xi$. Then there is a $p > 3$ such that* (1.2) *is a topological isomorphism.*

Corollary 1.3. *Let Λ, Υ and Π be as in Theorem 1.2. Let Σ be $\Upsilon \times]-1,1[$ combined with the ground plate or/and the upper plate. Let further Ξ be a plane as in Theorem 1.2 which does neither touch the upper/lower plate and let μ be as in Theorem 1.2. Then the conclusion of Theorem 1.2 also holds.*

Remark 1.4. The supposition that the plane Ξ has only a finite intersection with edges where the Dirichlet boundary part meets with the complementing boundary part is crucial, see Remark 8.5 below.

Remark 1.5. Let us further mention that $\Pi \cup \Sigma$ in Theorem 1.1 can be taken as Gröger's third model set, see [21], and thus Theorem 1.1 can be viewed as a regularity assertion for Gröger's third model constellation if the coefficient function has a discontinuity along a plane.

Operators of type (1.1) — which may be seen as the principal part of the (Dirichlet)-homogenization of an elliptic operator — are of fundamental significance in many application areas. This is the case not only in mechanics (see [34, Ch. IV/V]), thermodynamics (see [47]), and electrodynamics (see [46]) of heterogeneous media, but also in mining, multiphase flow, mathematical biology (see [15], [6]), and semiconductor device simulation (see [43], [16], [18]), in particular quantum electronics (see [51],[4], [32],[49], [50], [35]).

The nonhomogeneous coefficient function μ represents varying material properties as the context requires. It may be thermal conductivity in a heat equation (see [47, §21]), or dielectric permittivity in a Poisson equation, or diffusivity in a transport equation (see for instance [43, §2.2] for carrier continuity equations), or effective electron mass in a Schrödinger equation (see [32]).

Let us emphasise that the matrices which constitute the coefficient function μ may be not diagonal and, in particular, not multiples of the identity, see [1] and [34, Ch. IV/V]. This is motivated by the applications, for instance in heat conduction, see [47, §21.B]. On the other hand anisotropic coefficients are absolutely necessary in view of (local) deformation and transformation of the domain in the localisation procedure, see [21]. It should be noted that in case of an essentially anisotropic coefficient matrix μ the generic properties of the elliptic operator differ dramatically from the case of a scalar coefficient, see [12, Remark 5.1], [13, §5], and [42, Ch. 5].

The problems under consideration in Theorems 1.1 and 1.2 may be reduced via some subtle transformation and reflection techniques to corresponding Dirichlet boundary value problems, which were treated in [37]. The crucial point is that a deep idea of Maz'ya [37] permits to restrict the investigation to the edge singularities as far as is concerned the integrability of the gradient of the solution up to an index $p > 3$. This heavily rests on the a priori known Hölder continuity of the solution, see [33, Ch. III.14]. It would be a hard graft to determine the singularities at the vertices, when material heterogeneities and different boundary conditions meet, see [10]. But, here fortunately, one can confine oneself to investigating the edge singularities.

2 Notations

Throughout this paper, $\Omega \subset \mathbb{R}^d$ always denotes a bounded Lipschitz domain (see [23] for definition) and $\Gamma \subset \partial\Omega$ is an open part of its boundary. $W^{1,p}(\Omega)$ denotes the (complex) Sobolev space on Ω consisting of those $L^p(\Omega)$ functions whose first order distributional derivatives also belong to $L^p(\Omega)$ (see [23] or [36]). (Note that

Ω enjoys the extension property for $W^{1,p}(\Omega)$ in view of being a bounded Lipschitz domain, see [17, Thm. 3.10] or [36, Ch. 1.1.16]. Thus, $W^{1,p}(\Omega)$ is identical with the completion of the set $\{v|_\Omega : v \in C^\infty(\mathbb{R}^3)\}$ with respect to the norm $\|v\|_{W^{1,p}} \stackrel{\text{def}}{=} \left(\int_\Omega \left(|\nabla v| + |v|\right)^p \, d\mathrm{x}\right)^{1/p}$). We use the symbol $W^{1,p}_\Gamma(\Omega)$ for the closure of

$$\left\{v|_\Omega : v \in C^\infty(\mathbb{R}^3), \operatorname{supp} v \cap (\partial\Omega \setminus \Gamma) = \emptyset\right\}$$

in $W^{1,p}(\Omega)$. If $\Gamma = \emptyset$ we write as usual $W^{1,p}_0(\Omega)$ instead of $W^{1,p}_\emptyset(\Omega)$. $W^{-1,p'}_\Gamma(\Omega)$ denotes the dual to $W^{1,p}_\Gamma(\Omega)$ and $W^{-1,p'}(\Omega)$ denotes the dual to $W^{1,p}_0(\Omega)$, when $\frac{1}{p} + \frac{1}{p'} = 1$ holds. If Ω is understood, then we sometimes abbreviate $W^{\pm 1,p}_\Gamma$, $W^{1,p}_0$ and $W^{-1,p}$, respectively. $\langle \cdot, \cdot \rangle_X$ always indicates the duality between a Banach space and its dual; in case of $X = \mathbb{C}^d$ we mostly write $\langle \cdot, \cdot \rangle$. If ω is a Lebesgue measurable, essentially bounded function on Ω taking its values in the set of real, symmetric $d \times d$ matrices, then we define $-\nabla \cdot \omega \nabla : W^{1,2}_\Gamma(\Omega) \to W^{-1,2}_\Gamma(\Omega)$ by

$$\langle -\nabla \cdot \omega \nabla v, w \rangle_{W^{-1,2}_\Gamma} \stackrel{\text{def}}{=} \int_\Omega \langle \omega \nabla v, \nabla w \rangle \, d\mathrm{x} \, ; \quad v, w \in W^{1,2}_\Gamma(\Omega). \tag{2.1}$$

The maximal restriction of $-\nabla \cdot \omega \nabla$ to any of the spaces $W^{-1,p}_\Gamma(\Omega)$ $(p > 2)$ we will denote by the same symbol. Finally, we define for any two complex numbers σ, λ

$$\sigma^\lambda \stackrel{\text{def}}{=} \exp(\lambda \log |\sigma| + i\lambda \arg \sigma) \, , \quad \arg \sigma \in]-\pi, \pi] \, ; \tag{2.2}$$

and for $\iota, \vartheta \in]-\pi, \pi]$ with $\iota < \vartheta$ we define the sector

$$K^\vartheta_\iota \stackrel{\text{def}}{=} \{(r\cos\theta, r\sin\theta) : r > 0, \theta \in]\iota, \vartheta[\}.$$

3 Preliminaries

In this section we first recall the optimal regularity result from [37] for heterogeneous Dirichlet problems on polyhedral domains and explain how to identify the occurring edge singularities.

Definition 3.1. Let numbers $\theta_0 < \theta_1 < \ldots < \theta_n \leq \theta_0 + 2\pi$ be given and, additionally, real, positive definite 2×2-matrices $\rho^1, \ldots, \rho^n$. We introduce on $]\theta_0, \theta_n[\setminus \{\theta_1, \ldots, \theta_{n-1}\}$ coefficient functions b_0, b_1, b_2 the restrictions of which to the interval $]\theta_j, \theta_{j+1}[$ are given by

$$\begin{aligned}
b_0(\theta) &= \rho^j_{11} \cos^2\theta + 2\rho^j_{12} \sin\theta\cos\theta + \rho^j_{22} \sin^2\theta \, , \\
b_1(\theta) &= (\rho^j_{22} - \rho^j_{11}) \sin\theta\cos\theta + \rho^j_{12}(\cos^2\theta - \sin^2\theta) \, , \\
b_2(\theta) &= \rho^j_{11} \sin^2\theta - 2\rho^j_{12} \sin\theta\cos\theta + \rho^j_{22} \cos^2\theta \, .
\end{aligned} \tag{3.1}$$

If $\theta_n \neq \theta_0 + 2\pi$, then we define the space $\mathcal{H}$ as $W_0^{1,2}]\theta_0, \theta_n[$, else as the periodic Soblev space $W^{1,2}(]\theta_0, \theta_n[) \cap \{\psi : \psi(\theta_0) = \psi(\theta_n)\}$ (which clearly may be identified with the Sobolev space $W^{1,2}(S^1)$ on the unit circle S^1). For every $\lambda \in \mathbb{C}$ we define the quadratic form $\mathfrak{t}_\lambda$ on $\mathcal{H}$ by

$$\mathfrak{t}_\lambda[\psi] \stackrel{\text{def}}{=} \int_{\theta_0}^{\theta_n} b_2\, \psi' \,\overline{\psi'} + \lambda b_1 \psi\, \overline{\psi'} - \lambda b_1 \psi' \,\overline{\psi} - \lambda^2 b_0 \psi \overline{\psi}\, d\theta \tag{3.2}$$

and $\mathcal{A}_\lambda$ as the operator which is induced by $\mathfrak{t}_\lambda$ on $L^2]\theta_0, \theta_n[$.

Remark 3.2. It is easy to check that $b_2 \geq \frac{\rho_{11}^j \rho_{22}^j - (\rho_{12}^j)^2}{\rho_{22}^j}$. From this it is straightforward to see that each form $\mathfrak{t}_\lambda$ is sectorial, what is also true for $\mathcal{A}_\lambda$, see [30], Ch. VI.

Definition 3.3. Let $\Omega \subset \mathbb{R}^3$ be a polyhedron which, additionally, is a Lipschitz domain and $\{\Omega_k\}_k$ a (finite, disjoint) polyhedral partition of Ω. Let μ be a matrix function on Ω which is constant on each Ω_k and takes real, symmetric, positive definite 3×3 matrices as values. Take any edge E of any of the Ω_k's and consider an arbitrary inner point P of this edge. Choose a new orthogonal coordinate system (x, y, z) with origin at the point P such that the direction of E coincides with the z-axis. We denote by $\mathcal{O}_E$ the corresponding orthogonal transformation matrix and by $\mu_{E,P}$ the piecewise constant matrix function which coincides in a neighbourhood of P with $\mathcal{O}_E \mu(\mathcal{O}_E^{-1}(x + P))\mathcal{O}_E^{-1}$ and which satisfies

$$\mu_{E,P}(tx, ty, z) = \mu_{E,P}(x, y, 0), \quad \text{for all}\, (x, y, z) \in \mathbb{R}^3,\, t > 0. \tag{3.3}$$

By $\mu_E(\cdot, \cdot)$ we denote the upper left 2×2 block of $\mu_{E,P}(\cdot, \cdot, 0)$.

Remark 3.4. There exist angles $\theta_0 < \theta_1 < \ldots < \theta_n \leq \theta_0 + 2\pi$, such that μ_E is constant on each of the sectors $K_{\theta_j}^{\theta_{j+1}}$ and takes real, symmetric, positive definite matrices as values. Note that $\theta_n = \theta_0 + 2\pi$ if μ_E corresponds to an interior edge E, otherwise μ_E is given on an infinite sector $K_{\theta_0}^{\theta_n}$ which coincides near P with the intersection of (the transformed) Ω with the x-y-plane.

Definition 3.5. We call an edge E of Ω a geometric edge if $E \subset \partial\Omega$ and all inner points of E belong to the closure of exactly one sub-polyhedron Ω_k. Further, we say that E is a bimaterial outer edge if $E \subset \partial\Omega$ and the function μ_E takes exactly two different values.

We proceed by quoting the central linear regularity result [37, Thm. 2.3], by means of which our regularity results will be deduced:

Proposition 3.6. *Let Ω, $\{\Omega_k\}_k$ and μ as in Definition 3.3. For any edge E let μ_E be the 2×2 matrix valued function on $K_{\theta_0}^{\theta_n}$ in the sense of Definition 3.3. If for every edge E the thus induced operators $\mathcal{A}_\lambda$ on $L^2]\theta_0, \theta_n[$ have a trivial kernel for all λ with $\Re\lambda \in]0, \frac{1}{3} + \epsilon[$ ($\epsilon > 0$ arbitrarily small), then there is a $p > 3$ such that*

$$-\nabla \cdot \mu\nabla : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega) \tag{3.4}$$

is a topological isomorphism.

Remark 3.7. Unfortunately, there are some errors in the paper [37]. First, the assertion of [37, Theorem 2.3] that the exponent p can be taken from the interval $[2, 2/(1-\widehat{\lambda}_\Upsilon)[$ is erroneous, since we have overlooked the assumptions of [37, Theorem 2.4]. The correct formulation of the linear regularity result proved in [37] is given in Proposition 3.6 above. We also found that the signs in formulas for the coefficients of certain generalized Sturm-Liouville equations are not correct, in detail: in [37, p. 240] we have used the wrong sign in the formula for the Mellin transform $\widetilde{r\partial_r u} = -\lambda\tilde{u}$, which has to be replaced by $\widetilde{r\partial_r u} = \lambda\tilde{u}$. Therefore the formulas [37, (3.33)] for the sesquilinear form $a(u, v; \lambda)$ and [37, (3.32)] for the corresponding differential problem differ in sign from the correct formulas (3.2) and (3.5), (3.6). The correctness of the other considerations given in [37] is not affected by this.

Thus, the question arises how to find the parameters λ for which the operator $\mathcal{A}_\lambda$ has only a trivial kernel. One proceeds as follows: standard arguments show that any function u from the kernel of the operator $\mathcal{A}_\lambda$ obeys the differential equation

$$(b_2u')' + \lambda(b_1u)' + \lambda b_1u' + \lambda^2 b_0 u = 0 \tag{3.5}$$

on each of the intervals $]\theta_j, \theta_{j+1}[$ and, additionally, in every point $\theta \in \{\theta_1, \dots, \theta_{n-1}\}$ the transmission conditions

$$[u]_\theta = 0\ ,\ [b_2u' + \lambda b_1 u]_\theta = 0 \tag{3.6}$$

have to be satisfied. (As usual, $[w]_\theta$ stands for $\lim_{\vartheta\searrow\theta} w(\vartheta) - \lim_{\vartheta\nearrow\theta} w(\vartheta)$). In order to find the critical parameters λ, one employs the elementary solutions of the differential equation (3.5) on each of the subintervals $]\theta_j, \theta_{j+1}[$

$$\theta \mapsto e^{-i\lambda\theta}(\alpha e^{2i\theta} + 1)^\lambda\,, \quad \theta \mapsto e^{i\lambda\theta}(\bar{\alpha}e^{-2i\theta} + 1)^\lambda$$

which were announced in the pioneering paper [9] (see also [37, Ch. 3.6] for further details). The complex number $\alpha = \alpha_j$ is determined by the matrix

$$m = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \rho_{11}^j & \rho_{12}^j \\ \rho_{12}^j & \rho_{22}^j \end{pmatrix}$$

as

$$\alpha \stackrel{\text{def}}{=} \frac{i(m_{22} - D_m^{1/2}) - m_{12}}{i(m_{22} + D_m^{1/2}) + m_{12}}, \tag{3.7}$$

where D_m denotes the determinant of the matrix m.

Remark 3.8. Because m_{22} is positive, α necessarily satisfies $0 \le |\alpha| < 1$. Moreover, if

$$\begin{pmatrix} \check{m}_{11} & \check{m}_{12} \\ \check{m}_{12} & \check{m}_{22} \end{pmatrix} = \begin{pmatrix} m_{11} & -m_{12} \\ -m_{12} & m_{22} \end{pmatrix},$$

then $\check{\alpha} = \bar{\alpha}$.

Making on any interval $]\theta_j, \theta_{j+1}[$ an ansatz

$$u_j(\theta) \stackrel{\text{def}}{=} c_{j,+} e^{-i\lambda\theta}(\alpha_j e^{2i\theta} + 1)^\lambda + c_{j,-} e^{i\lambda\theta}(\bar{\alpha}_j e^{-2i\theta} + 1)^\lambda, \tag{3.8}$$

these functions automatically satisfy (3.5), while the boundary conditions together with the transmission conditions (3.6) for $\theta = \theta_j$ ($j \in \{1, \dots, n\}$) lead to a $2n \times 2n$ homogeneous linear system for the coefficients $c_{j,+}, c_{j,-}$. The usual criterion for the (nontrivial) solvability of this system gives the characteristic equation of the problem (3.5, 3.6) and allows (in principle) to determine the critical values λ — or at least to give estimates for the real part of them. In the next chapters we will do this for all edges resulting from our problems.

4 Auxiliary Results

Lemma 4.1. *In the terminology from above let λ with $\Re\lambda \in]0,1[$ be a number such that there exists a (nontrivial) function $v_\lambda \in \mathcal{H}$ from the kernel of $\mathcal{A}_\lambda$, see Definition 3.1. Let ω_{13}, ω_{23} and ω_{33} be real valued, bounded, measurable functions on $K_{\theta_0}^{\theta_n}$ and define the coefficient function ω on $\mathcal{K}_{\theta_0}^{\theta_n} \stackrel{\text{def}}{=} K_{\theta_0}^{\theta_n} \times \mathbb{R}$ by*

$$\omega(x,y,z) \stackrel{\text{def}}{=} \begin{pmatrix} \rho_{11}^j & \rho_{12}^j & \omega_{13}(x,y) \\ \rho_{12}^j & \rho_{22}^j & \omega_{23}(x,y) \\ \omega_{13}(x,y) & \omega_{23}(x,y) & \omega_{33}(x,y) \end{pmatrix}, \quad \text{if} \quad (x,y) \in K_{\theta_j}^{\theta_{j+1}}. \tag{4.1}$$

Then there is a compactly supported element $f \in W^{-1,6}(\mathcal{K}_{\theta_0}^{\theta_n})$ such that the — also compactly supported — variational solution $v \in W_0^{1,2}(\mathcal{K}_{\theta_0}^{\theta_n})$ of $\nabla \cdot \omega \nabla v = f$ on $\mathcal{K}_{\theta_0}^{\theta_n}$ does not belong to $W_0^{1,\frac{2}{1-\Re\lambda}}(\mathcal{K}_{\theta_0}^{\theta_n})$.

Proof. It is not hard to calculate that the function ψ_0 given by

$$\psi_0(x,y) = (x^2 + y^2)^{\lambda/2} v_\lambda(\arg(x+iy)) \tag{4.2}$$

belongs to $W_{\text{loc}}^{1,p}(K_{\theta_0}^{\theta_n})$ if $p \in [2, \frac{2}{1-\Re\lambda}[$ but not to $W_{\text{loc}}^{1,\frac{2}{1-\Re\lambda}}(K_{\theta_0}^{\theta_n})$. (Recall that v_λ does not vanish identically on $]\theta_0, \theta_n[$.) By construction of $\mathcal{A}_\lambda$, the function ψ_0 satisfies

$$-\nabla \cdot \rho \nabla \psi_0 = 0. \tag{4.3}$$

in the distributional sense, see [37]. We define now the function ψ by $\psi(x,y,z) \stackrel{\text{def}}{=} \psi_0(x,y)$ and notice that ψ belongs to $W^{1,p}_{\text{loc}}(\mathcal{K}^{\theta_n}_{\theta_0})$ for $p \in [2, \frac{2}{1-\Re\lambda}[$, but not to $W^{1,\frac{2}{1-\Re\lambda}}_{\text{loc}}(\mathcal{K}^{\theta_n}_{\theta_0})$. Suppose $\varphi = \varphi_1 \otimes \varphi_2$ with $\varphi_1 \in C_0^\infty(K^{\theta_n}_{\theta_0})$ and $\varphi_2 \in C_0^\infty(\mathbb{R})$, then

$$\int_{\mathcal{K}^{\theta_n}_{\theta_0}} \langle \omega \nabla \psi, \nabla \varphi \rangle_{\mathbb{R}^3} \, dx\, dy\, dz = \int_{\mathbb{R}} \int_{K^{\theta_n}_{\theta_0}} \langle \rho \nabla \psi_0, \nabla \varphi_1 \rangle_{\mathbb{R}^2} dx\, dy\; \varphi_2(z)\, dz$$
$$+ \int_{K^{\theta_n}_{\theta_0}} \Big(\omega_{13} \frac{\partial \psi_0}{\partial x} + \omega_{23} \frac{\partial \psi_0}{\partial y}\Big) \varphi_1 \, dx\, dy \int_{\mathbb{R}} \frac{\partial \varphi_2}{\partial z}\, dz. \quad (4.4)$$

The first addend vanishes by (4.3) and the second by $\varphi_2 \in C_0^\infty(\mathbb{R})$. The set of φ's with the above tensor product structure is total in $C_0^\infty(\mathcal{K}^{\theta_n}_{\theta_0})$, therefore (4.4) is also zero for any φ from this latter space. Let η be a function from $C_0^\infty(\mathbb{R}^3)$ which equals 1 in a neighbourhood of $0 \in \mathbb{R}^3$ and which vanishes outside a ball B. Then one calculates for any $\varphi \in C_0^\infty(\mathcal{K}^{\theta_n}_{\theta_0})$

$$\int_{\mathcal{K}^{\theta_n}_{\theta_0}} \langle \omega \nabla(\eta\psi), \nabla \varphi \rangle \, d\mathrm{x} = - \int_{\mathcal{K}^{\theta_n}_{\theta_0}} \varphi \, \langle \omega \nabla \psi, \nabla \eta \rangle \, d\mathrm{x}$$
$$+ \int_{\mathcal{K}^{\theta_n}_{\theta_0}} \psi \langle \omega \nabla \eta, \nabla \varphi \rangle \, d\mathrm{x} + \int_{\mathcal{K}^{\theta_n}_{\theta_0}} \langle \omega \nabla \psi, \nabla(\eta\varphi) \rangle \, d\mathrm{x}. \quad (4.5)$$

$\int_{\mathcal{K}^{\theta_n}_{\theta_0}} \langle \omega \nabla \psi, \nabla(\eta\varphi) \rangle \, d\mathrm{x}$ vanishes because (4.4) always is zero if $\varphi \in C_0^\infty(\mathcal{K}^{\theta_n}_{\theta_0})$. On the other hand, it is not hard to see that the other two addends on the right hand side define — in their dependence on φ — continuous linear forms on $W_0^{1,6/5}(\mathcal{K}^{\theta_n}_{\theta_0})$, namely: the property $\psi \in W^{1,2}_{\text{loc}}(\mathcal{K}^{\theta_n}_{\theta_0})$ and the compact support property of η imply $\langle \omega \nabla \psi, \nabla \eta \rangle \in L^2(\mathcal{K}^{\theta_n}_{\theta_0})$. One combines this with the embedding $W_0^{1,6/5}(\mathcal{K}^{\theta_n}_{\theta_0}) \hookrightarrow L^2(\mathcal{K}^{\theta_n}_{\theta_0})$; thus it becomes clear for the first addend from the right hand side of (4.5). Concerning the second addend, one easily estimates

$$\Big| \int_{\mathcal{K}^{\theta_n}_{\theta_0}} \psi \langle \omega \nabla \eta, \nabla \varphi \rangle \, d\mathrm{x} \Big| \le \|\omega \nabla \eta\|_{L^\infty(\mathcal{K}^{\theta_n}_{\theta_0})} \|\psi\|_{L^6(B \cap \mathcal{K}^{\theta_n}_{\theta_0})} \|\varphi\|_{W_0^{1,6/5}(\mathcal{K}^{\theta_n}_{\theta_0})}$$
$$\le \|\omega \nabla \eta\|_{L^\infty(\mathcal{K}^{\theta_n}_{\theta_0})} \|\psi\|_{W^{1,2}(B \cap \mathcal{K}^{\theta_n}_{\theta_0})} \|\varphi\|_{W_0^{1,6/5}(\mathcal{K}^{\theta_n}_{\theta_0})}.$$

Thus, setting $\upsilon \stackrel{\text{def}}{=} \eta\psi$, one obtains the assertion. □

Remark 4.2. If $\theta_n = \theta_0 + 2\pi$, then $K^{\theta_n}_{\theta_0} = \mathbb{R}^2, \mathcal{K}^{\theta_n}_{\theta_0} = \mathbb{R}^3$ and, hence, $W_0^{1,p}(\mathcal{K}^{\theta_n}_{\theta_0}) = W^{1,p}(\mathbb{R}^3)$.

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and Γ be an open subset of its boundary. Assume that ϕ is a mapping from a neighbourhood of Ω into*

$\mathbb{R}^d$ which is additionally bi-Lipschitz. Let us denote $\phi(\Omega) = \Omega_\bullet$ and $\phi(\Gamma) = \Gamma_\bullet$. Then

i) For any $p \in]1,\infty[$ ϕ induces a linear, topological isomorphism

$$\Psi_p : W^{1,p}_{\Gamma_\bullet}(\Omega_\bullet) \to W^{1,p}_{\Gamma}(\Omega)$$

which is given by $(\Psi_p f)(x) = f(\phi(x)) = (f \circ \phi)(x)$.

ii) $\Psi^*_{p'}$ is a linear, topological isomorphism between $W^{-1,p}_{\Gamma}(\Omega)$ and $W^{-1,p}_{\Gamma_\bullet}(\Omega_\bullet)$.

iii) If ω is a bounded measurable function on Ω, taking its values in the set of $d \times d-$ matrices, then

$$\Psi^*_{p'}\nabla \cdot \omega \nabla \Psi_p = \nabla \cdot \omega_\bullet \nabla \tag{4.6}$$

with

$$\omega_\bullet(\mathrm{y}) = (D\phi)(\phi^{-1}(\mathrm{y}))\omega(\phi^{-1}(\mathrm{y}))\big(D\phi\big)^T(\phi^{-1}(\mathrm{y}))\frac{1}{\big|\det(D\phi)(\phi^{-1}\mathrm{y})\big|}. \tag{4.7}$$

$D\phi$ denotes the Jacobian of ϕ and $\det(D\phi)$ the corresponding determinant).

If, in particular, $-\nabla \cdot \omega \nabla : W^{1,p}_{\Gamma}(\Omega) \to W^{-1,p}_{\Gamma}(\Omega)$ is a topological isomorphism, then $-\nabla \cdot \omega_\bullet \nabla : W^{1,p}_{\Gamma_\bullet}(\Omega_\bullet) \to W^{-1,p}_{\Gamma_\bullet}(\Omega_\bullet)$ also is (and vice versa).

Proof. The proof of i) is contained in [22, Thm. 2.10)]. ii) follows from i) by duality. We prove iii): For $f \in W^{1,p}_{\Gamma_\bullet}(\Omega_\bullet)$, $g \in W^{1,p'}_{\Gamma_\bullet}(\Omega_\bullet)$ we get by the change of variables formula:

$$\begin{aligned}
&\langle -\Psi^*_{p'}\nabla \cdot \omega\nabla(\Psi_p f), g\rangle_{W^{-1,p}_{\Gamma_\bullet}(\Omega_\bullet)} = \langle -\nabla \cdot \omega \nabla(\Psi_p f), \Psi_{p'} g\rangle_{W^{-1,p}_{\Gamma}(\Omega)} \\
&= \langle -\nabla \cdot \omega\nabla(f \circ \phi), g \circ \phi\rangle_{W^{1,p'}_{\Gamma}(\Omega)} \\
&= \int_\Omega \langle \omega(\mathrm{x})\nabla(f\circ\phi)(\mathrm{x}), \nabla(g\circ\phi)(\mathrm{x})\rangle \, d\mathrm{x} \\
&= \int_\Omega \langle \omega(\mathrm{x})\big(D\phi\big)^T(\mathrm{x})(\nabla f)(\phi(\mathrm{x})), \big(D\phi\big)^T(\mathrm{x})(\nabla g)(\phi(\mathrm{x}))\rangle \, d\mathrm{x} \\
&= \int_\Omega \langle (D\phi)(\mathrm{x})\omega(\mathrm{x})(D\phi)^T(\mathrm{x})(\nabla f)(\phi(\mathrm{x})), (\nabla g)(\phi(\mathrm{x}))\rangle \frac{|\det(D\phi)(\mathrm{x})|}{|\det(D\phi)(\mathrm{x})|} \, dx \\
&= \int_{\Omega_\bullet} \langle (D\phi)(\phi^{-1}(\mathrm{y}))\omega(\phi^{-1}(\mathrm{y}))\frac{(D\phi)^T(\phi^{-1}(\mathrm{y}))}{|\det(D\phi)(\phi^{-1}\mathrm{y})|}\nabla f(\mathrm{y}), \nabla g(\mathrm{y})\rangle \, d\mathrm{y} \\
&= \Big\langle -\nabla \cdot \big((D\phi)(\phi^{-1}(\cdot))\omega(\phi^{-1}(\cdot))\frac{(D\phi)^T(\phi^{-1}(\cdot))}{|\det(D\phi)(\phi^{-1}(\cdot))|}\nabla f\big), g\Big\rangle_{W^{-1,p}_{\Gamma_\bullet}(\Omega_\bullet)}.
\end{aligned}$$

The essential point is that ϕ — as a Lipschitz continuous function — is differentiable almost everywhere and its (weak) derivative is essentially bounded (see [14, Ch. 4.2.3]). The last assertion follows from i), ii) and (4.6). ☐

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, convex, polygonal domain and Γ be an open subset of $\partial\Omega$ such that $\bar{\Omega} \cap \{(x,0,z) : x,z \in \mathbb{R}\} = \bar{\Gamma}$. Let for any $\mathrm{x} = (x,y,z)$ the symbol x_- denote the element $(x,-y,z)$ and define $\hat{\Omega}$ as the interior of*

$$\Omega \cup \{\mathrm{x} : \mathrm{x}_- \in \Omega\} \cup \bar{\Gamma}.$$

If ω is a bounded, measurable function on Ω taking its values in the set of real, symmetric 3×3-matrices, then we define

$$\hat{\omega}(\mathrm{x}) \stackrel{\text{def}}{=} \begin{cases} \omega(\mathrm{x}), & \text{if } \mathrm{x} \in \Omega, \\ \begin{pmatrix} \omega_{11}(\mathrm{x}_-) & -\omega_{12}(\mathrm{x}_-) & \omega_{13}(\mathrm{x}_-) \\ -\omega_{12}(\mathrm{x}_-) & \omega_{22}(\mathrm{x}_-) & -\omega_{23}(\mathrm{x}_-) \\ \omega_{13}(\mathrm{x}_-) & -\omega_{23}(\mathrm{x}_-) & \omega_{33}(\mathrm{x}_-) \end{pmatrix}, & \text{if } \mathrm{x}_- \in \Omega. \end{cases} \tag{4.8}$$

i) If $\psi \in W^{1,2}_\Gamma(\Omega)$ satisfies the equation $-\nabla \cdot \omega \nabla \psi = f \in W^{-1,2}_\Gamma(\Omega)$, then the equation $-\nabla \cdot \hat{\omega} \nabla \hat{\psi} = \hat{f} \in W^{-1,2}(\hat{\Omega})$ holds for $\hat{\psi}$ with

$$\hat{\psi}(\mathrm{x}) = \begin{cases} \psi(\mathrm{x}), & \text{if } \mathrm{x} \in \Omega, \\ \psi(\mathrm{x}_-), & \text{if } \mathrm{x}_- \in \Omega \end{cases}$$

and $\hat{f}$ defined by $\langle \hat{f}, \varphi \rangle_{W^{-1,2}(\hat{\Omega})} \stackrel{\text{def}}{=} \frac{1}{2} \langle f, \varphi|_\Omega + \varphi_-|_\Omega \rangle_{W^{1,2}_\Gamma(\Omega)}$. (For $\varphi \in W^{1,1}_0(\hat{\Omega})$ the function φ_- is defined by $\varphi_-(\mathrm{x}) \stackrel{\text{def}}{=} \varphi(\mathrm{x}_-)$.)

ii) Moreover, if $f \in W^{-1,p}_\Gamma(\Omega)$, then $\hat{f} \in W^{-1,p}(\hat{\Omega})$; and if $-\nabla \cdot \hat{\omega} \nabla : W^{1,p}_0(\hat{\Omega}) \to W^{-1,p}(\hat{\Omega})$ is a topological isomorphism, then $-\nabla \cdot \omega \nabla : W^{1,p}_\Gamma(\Omega) \to W^{-1,p}_\Gamma(\Omega)$ also is.

Proof. i) It is known that $\hat{\psi}$ belongs to $W^{1,p}_0(\hat{\Omega})$, see [17, Lemma 3.4]. Thus, i) is obtained by the definitions of $\hat{\psi}, \hat{f}, -\nabla \cdot \omega \nabla, -\nabla \cdot \hat{\omega} \nabla$ and straightforward calculations, based on Proposition 4.3 when applied to the transformation $\mathrm{x} \mapsto \mathrm{x}_-$.

ii) The operator $f \mapsto \hat{f}$ is the adjoint to $\varphi \mapsto \frac{1}{2}(\varphi|_\Omega + \varphi_-|_\Omega)$. The latter maps each $W^{1,p}_0(\hat{\Omega})$ continuously into $W^{1,p}_\Gamma(\Omega)$ for any $p \in]1, \infty[$. The last statement is then implied by the preceding ones and the definition of $\hat{\psi}$. □

Remark 4.5. The proposition is mutatis mutandis true for the reflection at other planes.

In the sequel we will transform our model problems which include mixed boundary conditions to the case of Dirichlet conditions – which are imposed in Proposition 3.6. In essence, this happens via a linear transformation leading to a peculiar triangle, a bi-Lipschitz transformation and a reflection argument. All of this is carried out in the next chapter.

5 Proof of Theorem 1.1

5.1 Transformation of the problem

Proposition 4.3 allows us in a first step to reduce the case of an arbitrary triangle Λ to that one where Λ is the triangle with the vertices $(1,-1)$, $(-1,1)$, $(1/2,1/2)$ and, additionally, Υ is the line segment between $(0,0)$ and $(1,-1)$. Namely, first one shifts the triangle such that P becomes the origin. Let P_1 denote the vertex where (the shifted) Υ ends and P_2 the vertex which does not touch $-\Upsilon$. We now transform $\mathbb{R}^2$ under the linear mapping which assigns P_1 to $(1,-1)$ and P_2 to $(1/2,1/2)$. Extending this mapping to $\mathbb{R}^3$ by letting the z-component invariant, one obtains the special geometric constellation stated above. Clearly, the transformed plane Ξ maintains the properties demanded in the suppositions of Theorem 1.1. In particular, we denote the point, where the (transformed) plane intersects the z-axis, by $P_\diamond$. In a natural sense we may speak of an upper half space $\mathcal{G}_u$ and a lower half space $\mathcal{G}_l$ (each on one side of the intersecting plane Ξ), where the coefficient function μ takes the values

$$\mu^+ = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \text{ on } \mathcal{G}_u, \qquad \mu^- = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix} \text{ on } \mathcal{G}_l.$$

We transform the problem via the bi-Lipschitz transformation

$$\phi \overset{\text{def}}{=} \begin{cases} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{on} \quad \{(x,y,z) : y > x\} \\ \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{on} \quad \{(x,y,z) : y \le x\}\,. \end{cases} \tag{5.1}$$

(Please notice that the determinants of both matrices in (5.1) equal 1). $\phi(\Lambda)$ is again a triangle — denoted by Λ_ϕ — and has now the vertices $(0,0)$, $(0,\sqrt{2})$, $(\sqrt{2},0)$, while the new domain is $\Pi_\phi = \Lambda_\phi \times]-1,1[$. Υ equals the subinterval $]0,\sqrt{2}[$ of the x-axis. The image Ξ_ϕ of $\Pi \cap \Xi$ consists of two triangles having one common edge $E_\phi \subset \{(x,x,z) : x > 0,\ z \in \mathbb{R}\}$. (Of course, if Ξ was orthogonal to the z-axis, then both triangles are also orthogonal to the z-axis.) Clearly, the Neumann boundary part is now the rectangle with the vertices $(0,0,-1)$, $(0,0,1)$, $(\sqrt{2},0,-1)$, $(\sqrt{2},0,1)$. The transformed matrix (see Proposition 4.3)

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}^T \tag{5.2}$$

is calculated as

$$\begin{pmatrix} \frac{a_{11}+2a_{12}+a_{22}}{2} & a_{12}+a_{22} & \frac{a_{13}+a_{23}}{\sqrt{2}} \\ a_{12}+a_{22} & 2a_{22} & \sqrt{2}a_{23} \\ \frac{a_{13}+a_{23}}{\sqrt{2}} & \sqrt{2}a_{23} & a_{33} \end{pmatrix}, \tag{5.3}$$

while the transformed matrix

$$\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}^T \tag{5.4}$$

is calculated as

$$\begin{pmatrix} 2a_{11} & a_{12}+a_{11} & \sqrt{2}a_{13} \\ a_{12}+a_{11} & \frac{a_{11}+2a_{12}+a_{22}}{2} & \frac{a_{13}+a_{23}}{\sqrt{2}} \\ \sqrt{2}a_{13} & \frac{a_{13}+a_{23}}{\sqrt{2}} & a_{33} \end{pmatrix} \tag{5.5}$$

(and analogously for the matrix b). We reflect the problem at the x-z-plane in the spirit of Proposition 4.4 and obtain a new triangle $\hat{\Lambda}$ with the vertices $(0,\sqrt{2})$, $(\sqrt{2},0)$, $(0,-\sqrt{2})$, a new domain $\hat{\Pi} \stackrel{\text{def}}{=} \hat{\Lambda}\times]-1,1[$ and the coefficient function $\hat{\mu}$ on $\hat{\Pi}$ is defined as in (4.8). Thus, we end up with a Dirichlet problem on $\hat{\Pi}$. By Proposition 4.4 it suffices to show that

$$-\nabla\cdot\hat{\mu}\nabla : W_0^{1,p}(\hat{\Pi}) \to W^{-1,p}(\hat{\Pi})$$

is a topological isomorphism for a $p>3$. For this, however, we may apply Proposition 3.6: we are done if we are able to show that for all edges E the induced operators $\mathcal{A}_\lambda$ have a trivial kernel for all λ with $\Re\lambda \in]0,\frac{1}{3}+\epsilon[$ ($\epsilon>0$ arbitrarily small). The occurring edges E are the following:

- geometric edges,
- bimaterial outer edges,
- the edges E_z^+ and E_z^- lying between $P_\diamond$ and $(0,0,1)$, or between $P_\diamond$ and $(0,0,-1)$, respectively,
- the edge E_{xz}, which is the intersection of the (transformed) Ξ with the x-z-plane,
- E_ϕ and the reflected E_ϕ.

5.2 Reformulation of the transmission conditions

The aim of this subsection is to express the transmission conditions for the ansatz functions in a condensed manner in terms of $\alpha_j, \alpha_{j+1}, \theta_j$.

Lemma 5.1. *Let α be defined by (3.7), and*

$$m = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \rho^j_{11} & \rho^j_{12} \\ \rho^j_{12} & \rho^j_{22} \end{pmatrix},$$

$$u(\theta) \stackrel{\text{def}}{=} c_+ e^{-i\lambda\theta}(\alpha e^{2i\theta}+1)^\lambda + c_- e^{i\lambda\theta}(\bar{\alpha}e^{-2i\theta}+1)^\lambda, \tag{5.6}$$

where c_+, c_- are arbitrary complex constants. Further, let b_1, b_2 be defined as in (3.1). Then

$$b_2(\theta)u'(\theta) + \lambda b_1(\theta)u(\theta) = -iD_m^{1/2}\lambda[c_+e^{-i\lambda\theta}(\alpha e^{2i\theta}+1)^\lambda - c_-e^{i\lambda\theta}(\bar{\alpha}e^{-2i\theta}+1)^\lambda], \tag{5.7}$$

where D_m again denotes the determinant of the matrix m.

Proof. First, one easily verifies

$$u'(\theta) = c_+e^{-i\lambda\theta}(\alpha e^{2i\theta}+1)^\lambda\lambda(-i)\frac{1-\alpha e^{2i\theta}}{1+\alpha e^{2i\theta}} + c_-e^{i\lambda\theta}(\bar{\alpha}e^{-2i\theta}+1)^\lambda i\lambda\frac{1-\bar{\alpha}e^{-2i\theta}}{1+\bar{\alpha}e^{-2i\theta}}. \tag{5.8}$$

Next we want to prove

$$-b_2(\theta)i\frac{1-\alpha e^{2i\theta}}{1+\alpha e^{2i\theta}} + b_1(\theta) = -iD_\rho^{1/2}. \tag{5.9}$$

For this we calculate

$$i\frac{1-\alpha e^{2i\theta}}{1+\alpha e^{2i\theta}} = i\frac{e^{-2i\theta}-\alpha}{e^{-2i\theta}+\alpha} \tag{5.10}$$

and abbreviate the denominator $i(m_{22}+D_m^{1/2})+m_{12}$ of α by N_α. One has

$$e^{-2i\theta}-\alpha = \big(e^{-2i\theta}[i(m_{22}+D_m^{1/2})+m_{12}] - i(m_{22}-D_m^{1/2})+m_{12}\big)/N_\alpha$$

and

$$e^{-2i\theta}+\alpha = \big(e^{-2i\theta}[i(m_{22}+D_m^{1/2})+m_{12}] + i(m_{22}-D_m^{1/2})-m_{12}\big)/N_\alpha,$$

what leads to

$$i\frac{1-\alpha e^{2i\theta}}{1+\alpha e^{2i\theta}} = i\frac{m_{12}(e^{-2i\theta}+1)+m_{22}i(e^{-2i\theta}-1)+D_m^{1/2}i(e^{-2i\theta}+1)}{m_{12}(e^{-2i\theta}-1)+m_{22}i(e^{-2i\theta}+1)+D_m^{1/2}i(e^{-2i\theta}-1)}. \tag{5.11}$$

We augment the last fraction by $\frac{\sin\theta}{e^{-2i\theta}-1}$; exploiting the equation

$$\frac{e^{-2i\theta}+1}{e^{-2i\theta}-1}\sin\theta = \frac{e^{-i\theta}(e^{-i\theta}+e^{i\theta})}{e^{-i\theta}(e^{-i\theta}-e^{i\theta})}\sin\theta = \frac{\cos\theta}{-i} = i\cos\theta,$$

the right hand side of (5.11) becomes

$$
\begin{aligned}
i\frac{im_{12}\cos\theta + im_{22}\sin\theta - D_m^{1/2}\cos\theta}{m_{12}\sin\theta - m_{22}\cos\theta + iD_m^{1/2}\sin\theta} &= \frac{-m_{12}\cos\theta - m_{22}\sin\theta - iD_m^{1/2}\cos\theta}{m_{12}\sin\theta - m_{22}\cos\theta + iD_m^{1/2}\sin\theta} \\
&= \frac{(-m_{12}\cos\theta - m_{22}\sin\theta - iD_m^{1/2}\cos\theta)(m_{12}\sin\theta - m_{22}\cos\theta - iD_m^{1/2}\sin\theta)}{(m_{12}\sin\theta - m_{22}\cos\theta)^2 + D_m\sin^2\theta} \\
&= \frac{(m_{22}-m_{11})\cos\theta\sin\theta + m_{12}(\cos^2\theta - \sin^2\theta) + iD_m^{1/2}}{m_{11}\sin^2\theta - 2m_{12}\cos\theta\sin\theta + m_{22}\cos^2\theta} = \frac{b_1(\theta) + iD_m^{1/2}}{b_2(\theta)}. \quad (5.12)
\end{aligned}
$$

Thus, (5.9) holds true. By complex conjugation one obtains from (5.9)

$$
b_2(\theta)i\frac{1-\bar{\alpha}e^{-2i\theta}}{1+\bar{\alpha}e^{-2i\theta}} + b_1(\theta) = iD_m^{1/2}. \quad (5.13)
$$

(5.9) and (5.13) together with (5.8) give the assertion (5.7). □

Corollary 5.2. *Let u be the function on $[\theta_0, \theta_n]$ which coincides on $]\theta_j, \theta_{j+1}[$ with u_j defined in* (3.8).

i) Assume first $j \in \{1, \ldots, n-1\}$ and let D_j and D_{j+1} denote the determinants of the matrices

$$
\begin{pmatrix} \rho_{11}^j & \rho_{12}^j \\ \rho_{12}^j & \rho_{22}^j \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \rho_{11}^{j+1} & \rho_{12}^{j+1} \\ \rho_{12}^{j+1} & \rho_{22}^{j+1} \end{pmatrix},
$$

respectively. If we abbreviate $\alpha \overset{\text{def}}{=} \alpha_j$ and $\beta \overset{\text{def}}{=} \alpha_{j+1}$, then the transmission conditions in the point $\theta = \theta_j$,

$$
[u]_\theta = [b_2u' + \lambda b_1 u]_\theta = 0 \quad (5.14)
$$

express as

$$
\begin{aligned}
c_{j,+}e^{-i\lambda\theta}(\alpha e^{2i\theta}+1)^\lambda &+ c_{j,-}e^{i\lambda\theta}(\bar{\alpha}e^{-2i\theta}+1)^\lambda \\
&= c_{j+1,+}e^{-i\lambda\theta}(\beta e^{2i\theta}+1)^\lambda + c_{j+1,-}e^{i\lambda\theta}(\bar{\beta}e^{-2i\theta}+1)^\lambda \quad (5.15)
\end{aligned}
$$

and

$$
\begin{aligned}
D_j^{1/2}[c_{j,+}e^{-i\lambda\theta}&(\alpha e^{2i\theta}+1)^\lambda - c_{j,-}e^{i\lambda\theta}(\bar{\alpha}e^{-2i\theta}+1)^\lambda] \\
&= D_{j+1}^{1/2}[c_{j+1,+}e^{-i\lambda\theta}(\beta e^{2i\theta}+1)^\lambda - c_{j+1,-}e^{i\lambda\theta}(\bar{\beta}e^{-2i\theta}+1)^\lambda], \quad (5.16)
\end{aligned}
$$

respectively. Thus, in case of $D_j = D_{j+1}$ for (5.14) *it is necessary and sufficient that*

$$
c_{j,+}(\alpha e^{2i\theta}+1)^\lambda = c_{j+1,+}(\beta e^{2i\theta}+1)^\lambda \quad (5.17)
$$

and

$$
c_{j,-}(\bar{\alpha}e^{-2i\theta}+1)^\lambda = c_{j+1,-}(\bar{\beta}e^{-2i\theta}+1)^\lambda \quad (5.18)
$$

hold.

ii) Assume now $\theta_n = \theta_0 + 2\pi$. Then the corresponding transmission conditions in θ_0 express as the following two equations:

$$\begin{aligned} c_{1,+}e^{-i\lambda\theta_0}(\alpha_1 e^{2i\theta_0}+1)^\lambda + c_{1,-}e^{i\lambda\theta_0}(\bar\alpha_1 e^{-2i\theta_0}+1)^\lambda \\ = c_{n,+}e^{-i\lambda\theta_n}(\alpha_n e^{2i\theta_0}+1)^\lambda + c_{n,-}e^{i\lambda\theta_n}(\bar\alpha_n e^{-2i\theta_0}+1)^\lambda \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} D_1^{1/2}[c_{1,+}e^{-i\lambda\theta_0}(\alpha_1 e^{2i\theta_0}+1)^\lambda - c_{1,-}e^{i\lambda\theta_0}(\bar\alpha_1 e^{-2i\theta_0}+1)^\lambda] \\ = D_n^{1/2}[c_{n,+}e^{-i\lambda\theta_n}(\alpha_n e^{2i\theta_0}+1)^\lambda - c_{n,-}e^{i\lambda\theta_n}(\bar\alpha_n e^{-2i\theta_0}+1)^\lambda], \end{aligned} \tag{5.20}$$

respectively.

5.3 Discussion of the edge singularities

For geometric edges and bimaterial outer edges we show in the Appendix that the operators $\mathcal{A}_\lambda$ have a trivial kernel if $\Re\lambda \in]0, 1/2]$.

Next we consider the edges E_z^+ and E_z^-: starting with E_z^+, one has to deal with the coefficient matrices

$$\hat m \overset{\text{def}}{=} \begin{pmatrix} \frac{a_{11}+2a_{12}+a_{22}}{2} & -a_{12}-a_{22} \\ -a_{12}-a_{22} & 2a_{22} \end{pmatrix} \qquad \text{if } \theta \in]-\pi/2, -\pi/4[$$

$$\hat o \overset{\text{def}}{=} \begin{pmatrix} 2a_{11} & -a_{12}-a_{11} \\ -a_{12}-a_{11} & \frac{a_{11}+2a_{12}+a_{22}}{2} \end{pmatrix} \qquad \text{if } \theta \in]-\pi/4, 0[$$

$$o \overset{\text{def}}{=} \begin{pmatrix} 2a_{11} & a_{12}+a_{11} \\ a_{12}+a_{11} & \frac{a_{11}+2a_{12}+a_{22}}{2} \end{pmatrix} \qquad \text{if } \theta \in]0, \pi/4[$$

$$m \overset{\text{def}}{=} \begin{pmatrix} \frac{a_{11}+2a_{12}+a_{22}}{2} & a_{12}+a_{22} \\ a_{12}+a_{22} & 2a_{22} \end{pmatrix} \qquad \text{if } \theta \in]\pi/4, \pi/2[.$$

Thus, one has to consider the ansatz functions (see Remark 3.8)

$$u \overset{\text{def}}{=} \begin{cases} \hat w \overset{\text{def}}{=} \hat c_+ e^{-i\lambda\cdot}(\bar\alpha e^{2i\cdot}+1)^\lambda + \hat c_- e^{i\lambda\cdot}(\alpha e^{-2i\cdot}+1)^\lambda & \text{on }]-\pi/2, -\pi/4[\\ \hat v \overset{\text{def}}{=} \hat d_+ e^{-i\lambda\cdot}(\bar\beta e^{2i\cdot}+1)^\lambda + \hat d_- e^{i\lambda\cdot}(\beta e^{-2i\cdot}+1)^\lambda & \text{on }]-\pi/4, 0[\\ v \overset{\text{def}}{=} d_+ e^{-i\lambda\cdot}(\beta e^{2i\cdot}+1)^\lambda + d_- e^{i\lambda\cdot}(\bar\beta e^{-2i\cdot}+1)^\lambda & \text{on }]0, \pi/4[\\ w \overset{\text{def}}{=} c_+ e^{-i\lambda\cdot}(\alpha e^{2i\cdot}+1)^\lambda + c_- e^{i\lambda\cdot}(\bar\alpha e^{-2i\cdot}+1)^\lambda & \text{on }]\pi/4, \pi/2[, \end{cases}$$

with α defined by (3.7) (and β analogously from the entries of the matrix o.) Please notice that the determinants of the matrices m, $\hat{m}$, o, $\hat{o}$ all equal the determinant of the matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix},$$

the value of which we denote by D in this proof. Taking this into account, the transmission conditions in $\theta = -\pi/4$ read in view of (5.17)/(5.18)

$$\hat{c}_+(1 - i\bar{\alpha})^\lambda = \hat{d}_+(1 - i\bar{\beta})^\lambda \tag{5.21}$$

and

$$\hat{c}_-(1 + i\alpha)^\lambda = \hat{d}_-(1 + i\beta)^\lambda. \tag{5.22}$$

Analogously, the transmission conditions in 0 equivalently express as

$$\hat{d}_+(1 + \bar{\beta})^\lambda = d_+(1 + \beta)^\lambda \tag{5.23}$$

and

$$\hat{d}_-(1 + \beta)^\lambda = d_-(1 + \bar{\beta})^\lambda, \tag{5.24}$$

while those in $\pi/4$ can be written as

$$d_+(1 + i\beta)^\lambda = c_+(1 + i\alpha)^\lambda \tag{5.25}$$

and

$$d_-(1 - i\bar{\beta})^\lambda = c_-(1 - i\bar{\alpha})^\lambda. \tag{5.26}$$

The boundary condition $u(\pi/2) = w(\pi/2) = 0$ leads to

$$c_+e^{-i\lambda\pi/2}(1 - \alpha)^\lambda + c_-e^{i\lambda\pi/2}(1 - \bar{\alpha})^\lambda = 0, \tag{5.27}$$

or, in other words,

$$c_+ = -c_-e^{i\lambda\pi}\frac{(1 - \bar{\alpha})^\lambda}{(1 - \alpha)^\lambda}, \tag{5.28}$$

while the boundary condition $u(-\pi/2) = \hat{w}(-\pi/2) = 0$ gives

$$\hat{c}_+e^{i\lambda\pi/2}(1 - \bar{\alpha})^\lambda + \hat{c}_-e^{-i\lambda\pi/2}(1 - \alpha)^\lambda = 0,$$

or, alternatively,

$$\hat{c}_- = -\hat{c}_+e^{i\lambda\pi}\frac{(1 - \bar{\alpha})^\lambda}{(1 - \alpha)^\lambda}. \tag{5.29}$$

Combining (5.28), (5.26), (5.24), (5.22), (5.29), (5.21), (5.23), (5.25), one ends up with the characteristic equation for λ:

$$\begin{aligned} e^{2i\lambda\pi}&\frac{(1 - \bar{\alpha})^\lambda}{(1 - \alpha)^\lambda}\frac{(1 - i\bar{\beta})^\lambda}{(1 - i\bar{\alpha})^\lambda}\frac{(1 + \beta)^\lambda}{(1 + \bar{\beta})^\lambda}\frac{(1 + i\alpha)^\lambda}{(1 + i\beta)^\lambda}\frac{(1 - \bar{\alpha})^\lambda}{(1 - \alpha)^\lambda}\frac{(1 - i\bar{\beta})^\lambda}{(1 - i\bar{\alpha})^\lambda}\frac{(1 + \beta)^\lambda}{(1 + \bar{\beta})^\lambda}\frac{(1 + i\alpha)^\lambda}{(1 + i\beta)^\lambda} \\ &= e^{2i\lambda\pi}\Big(\frac{(1 + i\alpha)^\lambda}{(1 - \alpha)^\lambda}\frac{(1 - \bar{\alpha})^\lambda}{(1 - i\bar{\alpha})^\lambda}\Big)^2\Big(\frac{(1 + \beta)^\lambda}{(1 + i\beta)^\lambda}\frac{(1 - i\bar{\beta})^\lambda}{(1 + \bar{\beta})^\lambda}\Big)^2 = 1. \end{aligned} \tag{5.30}$$

(Let us remark that c_+ cannot vanish unless also the other coefficients vanish.) Moreover, we notice that all the terms $1+i\alpha$, $\frac{1}{1-\alpha}$, $1-\bar{\alpha}$, $\frac{1}{1-i\bar{\alpha}}$, $1+\beta$, $\frac{1}{1+i\beta}$, $1-i\bar{\beta}$, $\frac{1}{1+\bar{\beta}}$ have positive real part because $|\alpha|, |\beta| < 1$. Hence, we have

$$\frac{(1+i\alpha)^\lambda}{(1-\alpha)^\lambda} = \Big(\frac{1+i\alpha}{1-\alpha}\Big)^\lambda \quad \text{and} \quad \frac{(1-\bar{\alpha})^\lambda}{(1-i\bar{\alpha})^\lambda} = \Big(\frac{1-\bar{\alpha}}{1-i\bar{\alpha}}\Big)^\lambda, \tag{5.31}$$

as well as

$$\frac{(1+\beta)^\lambda}{(1+i\beta)^\lambda} = \Big(\frac{1+\beta}{1+i\beta}\Big)^\lambda \quad \text{and} \quad \frac{(1-i\bar{\beta})^\lambda}{(1+\bar{\beta})^\lambda} = \Big(\frac{1-i\bar{\beta}}{1+\bar{\beta}}\Big)^\lambda, \tag{5.32}$$

if $\Re\lambda \leq 1$. Further, observing the relations

$$\overline{\Big(\frac{1+i\alpha}{1-\alpha}\Big)} = \frac{1}{\frac{1-\bar{\alpha}}{1-i\bar{\alpha}}} \quad \text{and} \quad \overline{\Big(\frac{1+\beta}{1+i\beta}\Big)} = \frac{1}{\frac{1-i\bar{\beta}}{1+\bar{\beta}}} \tag{5.33}$$

and putting $\sigma = \arg \dfrac{1+i\alpha}{1-\alpha}$ and $\kappa = \arg \dfrac{1+\beta}{1+i\beta}$, this altogether enables us to rewrite (5.30) as

$$e^{2i\lambda(\pi+2(\sigma+\kappa))} = 1. \tag{5.34}$$

It is obvious that all λ satisfying (5.34) must be real. Our claim is now: $\sigma+\kappa$ equals $\pi/2$ or $-3\pi/2$. For this, we mention that, by definition, $\sigma, \kappa \in]-\pi, \pi]$; thus the claim is true, if we can show

$$\frac{1+i\alpha}{1-\alpha}\frac{1+\beta}{1+i\beta} = i. \tag{5.35}$$

This we will do now: exploiting the definitions of α, β we get

$$\begin{aligned}
\frac{1+i\alpha}{1-\alpha} &= \frac{1}{2}\frac{D^{1/2}+m_{12}-m_{22}+i\big(D^{1/2}+m_{22}-m_{12}\big)}{m_{12}+iD^{1/2}} \\
&= \frac{1}{2m_{11}}\Big(D^{1/2}+m_{11}-m_{12}+i\big(D^{1/2}-m_{11}+m_{12}\big)\Big) \\
&= \frac{1}{2m_{11}}\Big(D^{1/2}+\frac{a_{11}-a_{22}}{2}+i\big(D^{1/2}+\frac{a_{22}-a_{11}}{2}\big)\Big).
\end{aligned}$$

Analogously, we calculate:

$$\begin{aligned}
\frac{1+\beta}{1+i\beta} &= \frac{2io_{22}}{D^{1/2}+o_{12}-o_{22}+i(D^{1/2}+o_{22}-o_{12})} \\
&= \frac{2io_{22}}{D^{1/2}+\frac{a_{11}-a_{22}}{2}+i(D^{1/2}+\frac{a_{22}-a_{11}}{2})}.
\end{aligned}$$

Taking into account $o_{22} = m_{11}$, this gives (5.35). Hence, the transcendental equation (5.34) for λ reads in any case as $e^{4i\pi\lambda} = 1$. Trivially, the smallest positive λ possible is $\lambda_0 = 1/2$. Thus, the edge E_z^+ meets the preconditions of Proposition 3.6. The

considerations for the edge E_z^- are the same, word by word.
Next we consider the edge E_{xz}, lying in the x-z-plane. The coefficient matrices belonging to its neighbouring sectors are

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \frac{a_{11}+2a_{12}+a_{22}}{2} & a_{12}+a_{22} & \frac{a_{13}+a_{23}}{\sqrt{2}} \\ a_{12}+a_{22} & 2a_{22} & \sqrt{2}a_{23} \\ \frac{a_{13}+a_{23}}{\sqrt{2}} & \sqrt{2}a_{23} & a_{33} \end{pmatrix}, \tag{5.36}$$

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \frac{b_{11}+2b_{12}+b_{22}}{2} & b_{12}+b_{22} & \frac{b_{13}+b_{23}}{\sqrt{2}} \\ b_{12}+b_{22} & 2b_{22} & \sqrt{2}b_{23} \\ \frac{b_{13}+b_{23}}{\sqrt{2}} & \sqrt{2}b_{23} & b_{33} \end{pmatrix}, \tag{5.37}$$

if $y > 0$ and their reflected counterparts

$$\hat{Q} = \begin{pmatrix} q_{11} & -q_{12} & q_{13} \\ -q_{12} & q_{22} & -q_{23} \\ q_{13} & -q_{23} & q_{33} \end{pmatrix} \quad \text{and} \quad \hat{R} = \begin{pmatrix} r_{11} & -r_{12} & r_{13} \\ -r_{12} & r_{22} & -r_{23} \\ r_{13} & -r_{23} & r_{33} \end{pmatrix},$$

if $y < 0$, (see (5.2), (5.4) and Proposition 4.4). According to Proposition 3.6 one has to perform a rotation in the x-z-plane which moves the edge E_{xz} to the z-axis. This means, one has to consider the matrices

$$\begin{pmatrix} \cos\varsigma & 0 & -\sin\varsigma \\ 0 & 1 & 0 \\ \sin\varsigma & 0 & \cos\varsigma \end{pmatrix} M \begin{pmatrix} \cos\varsigma & 0 & \sin\varsigma \\ 0 & 1 & 0 \\ -\sin\varsigma & 0 & \cos\varsigma \end{pmatrix},$$

M taken as $Q, R, \hat{Q}, \hat{R}$, respectively and ς being the angle between the edge E_{xz} and the z-axis. A straightforward calculation shows that the resulting upper 2×2 blocks look alike

$$\begin{gathered} \begin{pmatrix} s_{11} & -s_{12} \\ -s_{12} & s_{22} \end{pmatrix} \ \text{if } -\theta \in]\zeta, \pi[, \qquad \begin{pmatrix} t_{11} & -t_{12} \\ -t_{12} & t_{22} \end{pmatrix} \ \text{if } -\theta \in]0, \zeta[, \\ \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} \ \text{if } \theta \in]0, \zeta[, \qquad \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \ \text{if } \theta \in]\zeta, \pi[. \end{gathered} \tag{5.38}$$

Hence (see Remark 3.8), the corresponding numbers α_1, α_2, α_3, α_4 are related by $\alpha_1 = \overline{\alpha}_4$ and $\alpha_2 = \overline{\alpha}_3$. In the sequel we employ the numbers α for α_3 and β for α_4. In this notation we show:

Lemma 5.3. *Assume the existence of complex numbers $c_+, c_-, d_+, d_-, \hat{c}_+, \hat{c}_-, \hat{d}_+, \hat{d}_-$ (at least one of them nonzero) such that*

$$u \stackrel{\text{def}}{=} \begin{cases} \hat{v} \stackrel{\text{def}}{=} \hat{d}_+ e^{-i\lambda\cdot}(\bar{\beta}e^{2i\cdot}+1)^\lambda + \hat{d}_- e^{i\lambda\cdot}(\beta e^{-2i\cdot}+1)^\lambda & \text{on }]-\pi, -\zeta[\\ \hat{w} \stackrel{\text{def}}{=} \hat{c}_+ e^{-i\lambda\cdot}(\bar{\alpha}e^{2i\cdot}+1)^\lambda + \hat{c}_- e^{i\lambda\cdot}(\alpha e^{-2i\cdot}+1)^\lambda & \text{on }]-\zeta, 0[\\ w \stackrel{\text{def}}{=} c_+ e^{-i\lambda\cdot}(\alpha e^{2i\cdot}+1)^\lambda + c_- e^{i\lambda\cdot}(\bar{\alpha}e^{-2i\cdot}+1)^\lambda & \text{on }]0, \zeta[\\ v \stackrel{\text{def}}{=} d_+ e^{-i\lambda\cdot}(\beta e^{2i\cdot}+1)^\lambda + d_- e^{i\lambda\cdot}(\bar{\beta}e^{-2i\cdot}+1)^\lambda & \text{on }]\zeta, \pi[. \end{cases}$$

obeys the transmission conditions in $-\pi, -\zeta, 0, \zeta$. Then $\Re\lambda \notin]0, \frac{1}{2}]$.

Proof. The transmission condition $[b_2u' + \lambda b_1 u]_0 = 0$ together with Corollary 5.2 (see in particular (5.16)) implies

$$\hat{c}_+(\bar{\alpha}+1)^\lambda - \hat{c}_-(\alpha+1)^\lambda = c_+(\alpha+1)^\lambda - c_-(\bar{\alpha}+1)^\lambda. \tag{5.39}$$

On the other hand, the transmission condition for $b_2u' + \lambda b_1 u$ in $-\pi/\pi$ (see (5.20)) gives

$$\hat{d}_+ e^{i\lambda\pi}(\bar{\beta}+1)^\lambda - \hat{d}_- e^{-i\lambda\pi}(\beta+1)^\lambda = d_+ e^{-i\lambda\pi}(\beta+1)^\lambda - d_- e^{i\lambda\pi}(\bar{\beta}+1)^\lambda. \tag{5.40}$$

Let us first consider the case where

$$\hat{c}_+ = c_-, \quad \hat{c}_- = c_+, \quad \hat{d}_+ = d_-, \quad \hat{d}_- = d_+. \tag{5.41}$$

Inserting these relations in (5.39) and (5.40) one obtains that both sides of (5.39) and (5.40) in fact have to vanish. But this means in view of Lemma 5.1 nothing else but

$$b_2(\theta)u'(\theta) + \lambda b_1(\theta)u(\theta) = 0 \quad \text{for} \quad \theta = 0, \pi.$$

Thus, the restriction of u to the interval $]0,\pi[$ leads to a bimaterial problem including a Neumann condition on both interval ends. Then $\Re\lambda \notin]0,1/2]$, see Theorem 8.2 below.

Assume now that (5.41) is not satisfied. Then we introduce the function

$$\tilde{u} \stackrel{\text{def}}{=} \begin{cases} \tilde{w} \stackrel{\text{def}}{=} \hat{c}_- e^{-i\lambda\cdot}(\alpha e^{2i\cdot}+1)^\lambda + \hat{c}_+ e^{i\lambda\cdot}(\bar{\alpha}e^{-2i\cdot}+1)^\lambda & \text{on }]0,\zeta[\\ \tilde{v} \stackrel{\text{def}}{=} \hat{d}_- e^{-i\lambda\cdot}(\beta e^{2i\cdot}+1)^\lambda + \hat{d}_+ e^{i\lambda\cdot}(\bar{\beta}e^{-2i\cdot}+1)^\lambda & \text{on }]\zeta,\pi[\end{cases} \tag{5.42}$$

on $[0,\pi]$ and consider the function

$$\begin{aligned} u_\star &\stackrel{\text{def}}{=} u|_{[0,\pi]} - \tilde{u} \\ &= \begin{cases} (c_+ - \hat{c}_-)e^{-i\lambda\cdot}(\alpha e^{2i\cdot}+1)^\lambda + (c_- - \hat{c}_+)e^{i\lambda\cdot}(\bar{\alpha}e^{-2i\cdot}+1)^\lambda & \text{on }]0,\zeta[\\ (d_+ - \hat{d}_-)e^{-i\lambda\cdot}(\beta e^{2i\cdot}+1)^\lambda + (d_- - \hat{d}_+)e^{i\lambda\cdot}(\bar{\beta}e^{-2i\cdot}+1)^\lambda & \text{on }]\zeta,\pi[. \end{cases} \end{aligned}$$

It is straightforward to verify that the condition $[u]_0 = 0$ implies $u_\star(0) = 0$ and the periodicity condition in $-\pi/\pi$ yields $u_\star(\pi) = 0$. Next we intend to show the transmission conditions $[u_\star]_\zeta = [b_2(\theta)u_\star' + \lambda b_1(\theta)u_\star]_\zeta = 0$. Because we already know by supposition $[u]_\zeta = [b_2(\theta)u' + \lambda b_1(\theta)u]_\zeta = 0$ it remains to show $[\tilde{u}]_\zeta = [b_2(\theta)\tilde{u}' + \lambda b_1(\theta)\tilde{u}]_\zeta = 0$. One easily verifies $[\tilde{u}]_\zeta = [u]_{-\zeta}$, and the latter is zero by supposition. Finally, by Lemma 5.1 we have

$$\begin{aligned} [b_2(\theta)\tilde{u}' + \lambda b_1(\theta)\tilde{u}]_\zeta &= \big(b_2\tilde{w}' + \lambda b_1\tilde{w}\big)|_\zeta - \big(b_2\tilde{v}' + \lambda b_1\tilde{v}\big)|_\zeta \\ &= -i\lambda D_\alpha\big(\hat{c}_- e^{-i\lambda\zeta}(\alpha e^{2i\zeta}+1)^\lambda - \hat{c}_+ e^{i\lambda\zeta}(\bar{\alpha}e^{-2i\zeta}+1)^\lambda\big) \\ &\quad + i\lambda D_\beta\big(\hat{d}_- e^{-i\lambda\zeta}(\beta e^{2i\zeta}+1)^\lambda - \hat{d}_+ e^{i\lambda\zeta}(\bar{\beta}e^{-2i\zeta}+1)^\lambda\big) \\ &= -\big(b_2\hat{w}' + \lambda b_1\hat{w}\big)|_{-\zeta} + \big(b_2\hat{v}' + \lambda b_1\hat{v}\big)|_{-\zeta} \\ &= [b_2u' + \lambda b_1 u]_{-\zeta}. \end{aligned}$$

But the right hand side of this equation is zero in view of the transmission condition $[b_2u' + \lambda b_1 u]_{-\zeta} = 0$. Thus, this second case leads to a bimaterial Dirichlet problem, for which also Theorem 8.2 gives $\Re\lambda \notin]0, 1/2]$. $\square$

It remains to consider the edge E_ϕ (and its reflected counterpart). Let first $t \in \mathbb{R}$ be a number such that $(0,0,t) + E_\phi$ has its endpoint in $\mathbf{0} \in \mathbb{R}^3$ and $\mathcal{O}$ be a rotation of the plane $\{(x,x,z) : x, z \in \mathbb{R}\}$ which transforms $\big((0,0,t) + E_\phi\big)$ to the z-axis. Suppose that for one λ with $\Re\lambda \in]0, 1/2]$ there is a (nontrivial) function v_λ from the kernel of the resulting operator $\mathcal{A}_\lambda$. If one takes the coefficient function defined in (4.1) as

$$\omega(x,y,z) \stackrel{\text{def}}{=} \begin{pmatrix} \rho^j_{11} & \rho^j_{12} & \rho^j_{13} \\ \rho^j_{12} & \rho^j_{22} & \rho^j_{23} \\ \rho^j_{13} & \rho^j_{23} & \rho^j_{33} \end{pmatrix} \quad \text{if} \quad (x,y) \in K^{\theta_{j+1}}_{\theta_j}, \tag{5.43}$$

then, by Lemma 4.1, there is a compactly supported element $f \in W^{-1,6}(\mathbb{R}^3)$ such that the — also compactly supported — variational solution $v \in W^{1,2}(\mathbb{R}^3)$ of $-\nabla \cdot \omega \nabla v = f$ does not belong to $W^{1,4}(\mathbb{R}^3)$. Because the support of v is compact, it can then (the more) not belong to $W^{1,6}(\mathbb{R}^3)$. Now we revoke the transformations $\mathcal{O}$, the shift $(0,0,t)$ and ϕ. Applying Proposition 4.3, one obtains a $f_\bullet \in W^{-1,6}(\mathbb{R}^3)$ and a $v_\bullet \in W^{1,2}(\mathbb{R}^3) \setminus W^{1,4}(\mathbb{R}^3)$ satisfying $-\nabla \cdot \omega_\bullet \nabla v_\bullet = f_\bullet$, or, equivalently, $-\nabla \cdot \omega_\bullet \nabla v_\bullet + v_\bullet = f_\bullet + v_\bullet \in W^{-1,6}(\mathbb{R}^3)$. It is not hard to see that the matrix valued function $\omega_\bullet$ takes above Ξ the matrix μ^+ (see pg. 9) as value and below Ξ the matrix μ^-, see Definition 3.3 and, in particular, (3.3). But a result of [13, Thm. 3.11], see also [3, Ch. 4.5], says that

$$-\nabla \cdot \omega_\bullet \nabla + 1 : W^{1,p}(\mathbb{R}^3) \to W^{-1,p}(\mathbb{R}^3)$$

is a topological isomorphism for any $p \in]1, \infty[$. This contradicts the above supposition. The proof for the reflected E_ϕ runs along the same lines; thus the proof of Theorem 1.1 is complete.

6 Proof of Theorem 1.2 and of Corollary 1.3

First we consider the case where Υ is one side of the triangle Λ. Modulo an affine transformation in $\mathbb{R}^2$ we may focus on the case where Υ is identical with the interval $]0,1[$ on the x-axis, see Proposition 4.3. We reflect Π symmetrically at the x-z-plane and obtain a domain $\hat{\Pi}$ and a reflected coefficient function $\hat{\mu}$. The resulting boundary conditions are then homogeneous Dirichlet on all $\partial\hat{\Pi}$. By Proposition 4.4 it is sufficient to show that

$$-\nabla \cdot \hat{\mu} \nabla : W^{1,p}_0(\hat{\Pi}) \to W^{-1,p}(\hat{\Pi})$$

is a topological isomorphism for a $p > 3$. Of course, we will again apply Proposition 3.6 and have, hence, to discuss the edge singularities. The occurring edges are:

i) geometric edges,

ii) bimaterial outer edges,

iii) the intersection of the x-z-plane with Ξ, in particular, the parts of the z-axis below and above the intersection point with Ξ is a bimaterial outer edge.

For all these edges we already know that the corresponding operators $\mathcal{A}_\lambda$ have a trivial kernel provided $\Re\lambda \in]0, 1/2]$; namely: the claim for geometric edges and bimaterial outer edges is shown in te next chapter (see Theorem 8.1 and Theorem 8.2) while the situation of iii) is exactly the same as treated in Lemma 5.3.

Let us now regard the second case: modulo an affine transformation in $\mathbb{R}^2$ we may restrict ourself to the case where Υ is the union of the interval $]0, 1[$ on the x-axis and the interval $[0, 1[$ on the y-axis. Again we reflect the problem at the x-z-plane, but afterwards a second time at the y-z-plane. Thus, we end up with a Dirichlet problem on $\breve{\Pi} \stackrel{\text{def}}{=} V \times]-1, 1[$, where $V \subset \mathbb{R}^2$ is the square with the vertices $(0, 1), (1, 0), (0, -1), (-1, 0)$. Denoting the new coefficient function by $\breve{\mu}$, it suffices by Proposition 4.4 to show that

$$-\nabla \cdot \breve{\mu} \nabla : W_0^{1,p}(\breve{\Pi}) \to W^{-1,p}(\breve{\Pi})$$

is a topological isomorphism for a $p > 3$. According to Proposition 3.6, it remains to show that for every edge E the kernels of the corresponding operators $\mathcal{A}_\lambda$ are trivial if $\Re\lambda \in]0, 1/3 + \epsilon[$ (ϵ arbirarily small). If $(0, 0, t)$ is the intersection point of Ξ with the z-axis, then the occurring edges are:

i) geometric edges,

ii) bimaterial outer edges,

iii) $\{(0, 0, s) : s \in]-1, t[\}$,

iv) $\{(0, 0, s) : s \in]t, 1[\}$,

v) the intersection of the x-z-plane with Ξ,

vi) the intersection of the x-z-plane with the reflected Ξ,

vii) the intersection of the y-z-plane with Ξ,

viii) the intersection of the y-z-plane with the reflected Ξ.

The geometric and bimaterial outer edges are treated in the Appendix. iii), iv), v), vi) lead again to a constellation (5.38), which was treated in Lemma 5.3. This is also true for vii) and viii), but requires here an additional moment's thought: let us denote the value of the coefficient function μ above Ξ by μ^+ and below Ξ by μ^-. Concerning vii), the 'reflected matrices' then equal

$$\begin{pmatrix} \mu_{11}^+ & -\mu_{12}^+ & -\mu_{13}^+ \\ -\mu_{12}^+ & \mu_{22}^+ & \mu_{23}^+ \\ -\mu_{13}^+ & \mu_{23}^+ & \mu_{33}^+ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mu_{11}^- & -\mu_{12}^- & -\mu_{13}^- \\ -\mu_{12}^- & \mu_{22}^- & \mu_{23}^- \\ -\mu_{13}^- & \mu_{23}^- & \mu_{33}^- \end{pmatrix}.$$

We perform now a rotation within the x-y-plane which transforms the (positive) y-axis into the (positive) x-axis and the (positive) x-axis into the negative y-axis; clearly the transformed edge lies then in the x-z-plane. One obtains the transformed coefficient matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_{11}^+ & \mu_{12}^+ & \mu_{13}^+ \\ \mu_{12}^+ & \mu_{22}^+ & \mu_{23}^+ \\ \mu_{13}^+ & \mu_{23}^+ & \mu_{33}^+ \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mu_{22}^+ & -\mu_{12}^+ & \mu_{23}^+ \\ -\mu_{12}^+ & \mu_{11}^+ & -\mu_{13}^+ \\ \mu_{23}^+ & -\mu_{13}^+ & \mu_{33}^+ \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_{11}^- & \mu_{12}^- & \mu_{13}^- \\ \mu_{12}^- & \mu_{22}^- & \mu_{23}^- \\ \mu_{13}^- & \mu_{23}^- & \mu_{33}^- \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mu_{22}^- & -\mu_{12}^- & \mu_{23}^- \\ -\mu_{12}^- & \mu_{11}^- & -\mu_{13}^- \\ \mu_{23}^- & -\mu_{13}^- & \mu_{33}^- \end{pmatrix}$$

while the reflected matrices transform as follows:

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_{11}^+ & -\mu_{12}^+ & -\mu_{13}^+ \\ -\mu_{12}^+ & \mu_{22}^+ & \mu_{23}^+ \\ -\mu_{13}^+ & \mu_{23}^+ & \mu_{33}^+ \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mu_{22}^+ & \mu_{12}^+ & \mu_{23}^+ \\ \mu_{12}^+ & \mu_{11}^+ & \mu_{13}^+ \\ \mu_{23}^+ & \mu_{13}^+ & \mu_{33}^+ \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_{11}^- & -\mu_{12}^- & -\mu_{13}^- \\ -\mu_{12}^- & \mu_{22}^- & \mu_{23}^- \\ -\mu_{13}^- & \mu_{23}^- & \mu_{33}^- \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mu_{22}^- & \mu_{12}^- & \mu_{23}^- \\ \mu_{12}^- & \mu_{11}^- & \mu_{13}^- \\ \mu_{23}^- & \mu_{13}^- & \mu_{33}^- \end{pmatrix}$$

Thus, from this point on we are in the same situation as in the discussion for the edge E_{xz} (see pg. 15) and everything runs completely the same way. viii) is analogous to vii).

We come to the proof of Corollary 1.3: because we demanded that the plane Ξ should not touch the upper plate nor the ground plate it is possible to divide the problem by a suitable partition of unity into one which affects the upper (lower) part and is separated from Ξ and one which contains $\Pi \cap \Xi$ but has only a Dirichlet condition on its upper (ground) plate. The latter is already treated in Theorem 1.2. The first can be reflected at the upper (ground) plate and one ends up again with the setting which is treated in Theorem 1.2.

7 Concluding remarks

The results of this paper easily carry over to problems with Robin boundary conditions. Indeed, one can prove that if ϖ is the surface measure on $\partial\Omega$ and $\varkappa \in L^\infty(\Gamma, d\varpi)$, then the linear map $T : W^{1,p}_\Gamma(\Pi) \to W^{-1,p}_\Gamma(\Pi)$ given by

$$\langle T\psi, \varphi\rangle_{W^{-1,p}_\Gamma} = \int_\Gamma \varkappa\, \psi\, \varphi\, d\varpi$$

(and representing the Robin boundary condition) is infinitesimally small with respect to the operator $\nabla \cdot \mu \nabla$. Thus, the domains of both operators are the same by classical perturbation theory, see [30, Ch. IV.1].

The reader has possibly asked himself why the results are deduced from [37] and why the concept of that paper does not work for boundary conditions which are not Dirichlet. One problem consists in finding an adequate energy space in case of edges on Neumann boundary parts which, additionally, has to be in correspondence with the properties of the Mellin transform. Our attempts to find such an energy space have failed up to now.

In principle it is possible to generalize our results to the case where not only one plane intersects the domain, but severals do. In order to classify the singularities stemming from the additional inner edges (where the planes meet) one can apply the result [12, Thm. 2.5]. We have not carried out this here only for technical simplicity, see also [31].

8 Appendix: The transcendental equation for geometric edges and bimaterial outer edges

It is the aim of this chapter to discuss the edge singularities for geometric edges and bimaterial outer edges; precisely, we intend to show the following two theorems:

Theorem 8.1. *For any geometric edge E the kernels of the associated operators $\mathcal{A}_\lambda$ are trivial in each of the following two cases:*
a) the opening angle $\theta_1 - \theta_0$ is not larger than π and $\Re\lambda \in]0, 1[$
b) $\theta_1 - \theta_0 \in]\pi, 2\pi[$ and $\Re\lambda \in]0, 1/2]$.

Theorem 8.2. *Let $K^{\theta_1}_{\theta_0}, K^{\theta_2}_{\theta_1}$ be two neighbouring sectors in $\mathbb{R}^2$ with $\theta_1 - \theta_0$, $\theta_2 - \theta_1 \le \pi$ and $\theta_2 - \theta_0 < 2\pi$. Let ρ^1, ρ^2 be two real, positive definite 2×2 matrices corresponding to the sectors $K^{\theta_1}_{\theta_0}, K^{\theta_2}_{\theta_1}$. Let $\mathfrak{t}_\lambda$ be the form defined in (3.2) either on $H^1_0(\theta_0, \theta_2)$ or on $H^1(\theta_0, \theta_2)$. Then there is an $\epsilon > 0$ such that the kernel of the corresponding operator $\mathcal{A}_\lambda$ (see Definition 3.1) is trivial if $\Re\lambda \in]0, 1/2 + \epsilon]$.*

We will prove the theorems in several steps, starting with the following

Lemma 8.3. *Let $\alpha \in \mathbb{C}$ with $|\alpha| < 1$, and define for $\gamma \in]-\pi, \pi]$ the number*

$$\sigma \stackrel{\text{def}}{=} \arg \frac{\alpha e^{-2i\gamma} + 1}{\alpha + 1} \in]-\pi, \pi] \,.$$

Then either $\gamma, \gamma + \sigma \in]-\pi, 0[$ or $\gamma = \sigma = 0$ or $\gamma, \gamma + \sigma \in]0, \pi[$, or $\gamma = \gamma + \sigma = \pi$.

Proof. The cases $\gamma = 0$ and $\gamma = \pi$ are straightforward. In the remaining cases one has

$$\begin{aligned} e^{i(\gamma+\sigma)} &= e^{i\gamma} \frac{\alpha e^{-2i\gamma} + 1}{\alpha + 1} \frac{|\alpha + 1|}{|\alpha e^{-2i\gamma} + 1|} \\ &= \frac{(\alpha e^{-i\gamma} + e^{i\gamma})(\overline{\alpha} + 1)}{|\alpha + 1||\alpha e^{-2i\gamma} + 1|} = \frac{|\alpha|^2 e^{-i\gamma} + e^{i\gamma} + 2\Re\big(\alpha e^{-i\gamma}\big)}{|\alpha + 1||\alpha e^{-2i\gamma} + 1|}. \end{aligned}$$

Thus, the imaginary part of $e^{i(\gamma+\sigma)}$ equals $\frac{(1-|\alpha|^2)\sin\gamma}{|1+\alpha||\alpha e^{-2i\gamma}+1|}$, and its sign depends in an obvious way only on γ . □

It follows the proof of Theorem 8.1; without loss of generality we may assume $\theta_1 = \pi$. Again exploiting the ansatz functions (3.8), the Dirichlet conditions in $\theta_0, \theta_1 \in]-\pi, \pi]$ read

$$c_+ e^{-i\lambda\theta_0} (\alpha e^{2i\theta_0} + 1)^\lambda + c_- e^{i\lambda\theta_0} (\bar{\alpha} e^{-2i\theta_0} + 1)^\lambda = 0 \tag{8.1}$$
$$c_+ e^{-i\lambda\pi} (\alpha + 1)^\lambda + c_- e^{i\lambda\pi} (\bar{\alpha} + 1)^\lambda = 0. \tag{8.2}$$

These equations are nontrivially solvable in c_+, c_- iff

$$\begin{aligned} 1 &= e^{-2i\lambda\pi} e^{2i\lambda\theta_0} \frac{(\bar{\alpha} e^{-2i\theta_0} + 1)^\lambda}{(\bar{\alpha} + 1)^\lambda} \frac{(\alpha + 1)^\lambda}{(\alpha e^{2i\theta_0} + 1)^\lambda} \\ &= e^{-2i\lambda\pi} e^{2i\lambda\theta_0} \Big(\frac{\bar{\alpha} e^{-2i\theta_0} + 1}{\bar{\alpha} + 1}\Big)^\lambda \Big(\frac{\alpha + 1}{\alpha e^{2i\theta_0} + 1}\Big)^\lambda, \end{aligned} \tag{8.3}$$

compare the considerations in §5.3, in particular (5.31). Putting $\nu \stackrel{\text{def}}{=} \arg \frac{\bar{\alpha} e^{-2i\theta_0} + 1}{\bar{\alpha} + 1}$, we may write (8.3) as $e^{2i\lambda(\theta_0 + \nu - \pi)} = 1$. Obviously, λ must be real and $|\bar{\alpha}| = |\alpha| < 1$. Hence, in case a), where $\theta_0 \in [0, \pi[$, we obtain $\theta_0 + \nu \in [0, \pi]$ by Lemma 8.3, which excludes $\Re\lambda \in]0, 1[$. If $\theta_0 \in]-\pi, 0[$, then, by Lemma 8.3, we have $\theta_0 + \nu \in]-\pi, 0[$, which shows the assertion in case b).

Concerning Theorem 8.2, we may apply a rotation (corresponding to a shift in the angle space) and thus reduce the general case to that one where $\theta_0 = -\gamma$, $\theta_1 = 0$ and $\theta_2 = \delta$. Again using the ansatz functions (3.8) we are getting the following equations expressing the transmission conditions in 0, see Corollary 5.2

$$c_+(\alpha+1)^\lambda + c_-(\bar\alpha+1)^\lambda = d_+(\beta+1)^\lambda + d_-(\bar\beta+1)^\lambda \tag{8.4}$$

and

$$D_m^{1/2}[c_+(\alpha+1)^\lambda - c_-(\bar\alpha+1)^\lambda] = D_o^{1/2}[d_+(\beta+1)^\lambda - d_-(\bar\beta+1)^\lambda]. \tag{8.5}$$

We define

$$\epsilon_\gamma \overset{\text{def}}{=} \begin{cases} -1 & \text{if Dirichlet in } \gamma \\ 1 & \text{if Neumann in } \gamma \end{cases}$$

and analogously for δ. In this convention, (see Lemma 5.1) the boundary condition in $-\gamma$ yields

$$c_+e^{i\lambda\gamma}(\alpha e^{-2i\gamma}+1)^\lambda - \epsilon_\gamma c_- e^{-i\lambda\gamma}(\bar\alpha e^{2i\gamma}+1)^\lambda = 0$$

or, what is the same,

$$c_- = \epsilon_\gamma c_+ e^{2i\lambda\gamma}\frac{(\alpha e^{-2i\gamma}+1)^\lambda}{(\bar\alpha e^{2i\gamma}+1)^\lambda}. \tag{8.6}$$

On the other hand, the corresponding boundary condition in δ implies

$$d_+e^{-i\lambda\delta}(\beta e^{2i\delta}+1)^\lambda - \epsilon_\delta d_- e^{i\lambda\delta}(\bar\beta e^{-2i\delta}+1)^\lambda = 0$$

or, equivalently,

$$d_+ = \epsilon_\delta d_- e^{2i\lambda\delta}\frac{(\bar\beta e^{-2i\delta}+1)^\lambda}{(\beta e^{2i\delta}+1)^\lambda}. \tag{8.7}$$

We insert (8.6) and (8.7) in (8.4) and (8.5) and obtain

$$c[(\alpha+1)^\lambda + \epsilon_\gamma e^{2i\lambda\gamma}\frac{(\alpha e^{-2i\gamma}+1)^\lambda}{(\bar\alpha e^{2i\gamma}+1)^\lambda}(\bar\alpha+1)^\lambda] - d[\epsilon_\delta e^{2i\lambda\delta}\frac{(\bar\beta e^{-2i\delta}+1)^\lambda}{(\beta e^{2i\delta}+1)^\lambda}(\beta+1)^\lambda + (\bar\beta+1)^\lambda] = 0 \tag{8.8}$$

and

$$D_m^{1/2}c[(\alpha+1)^\lambda - \epsilon_\gamma e^{2i\lambda\gamma}\frac{(\alpha e^{-2i\gamma}+1)^\lambda}{(\bar\alpha e^{2i\gamma}+1)^\lambda}(\bar\alpha+1)^\lambda] + D_o^{1/2}d[(\bar\beta+1)^\lambda - \epsilon_\delta e^{2i\lambda\delta}\frac{(\bar\beta e^{-2i\delta}+1)^\lambda}{(\beta e^{2i\delta}+1)^\lambda}(\beta+1)^\lambda] = 0 \tag{8.9}$$

for $c = c_+$ and $d = d_-$. (8.8), (8.9) are nontrivially solvable iff

$$\begin{aligned} &D_o^{1/2}\Big[1+\epsilon_\gamma e^{2i\lambda\gamma}\frac{(\alpha e^{-2i\gamma}+1)^\lambda}{(\alpha+1)^\lambda}\frac{(\bar\alpha+1)^\lambda}{(\bar\alpha e^{2i\gamma}+1)^\lambda}\Big]\Big[1-\epsilon_\delta e^{2i\lambda\delta}\frac{(\bar\beta e^{-2i\delta}+1)^\lambda}{(\bar\beta+1)^\lambda}\frac{(\beta+1)^\lambda}{(\beta e^{2i\delta}+1)^\lambda}\Big] \\ &+ D_m^{1/2}\Big[1+\epsilon_\delta e^{2i\lambda\delta}\frac{(\bar\beta e^{-2i\delta}+1)^\lambda}{(\bar\beta+1)^\lambda}\frac{(\beta+1)^\lambda}{(\beta e^{2i\delta}+1)^\lambda}\Big]\Big[1-\epsilon_\gamma e^{2i\lambda\gamma}\frac{(\alpha e^{-2i\gamma}+1)^\lambda}{(1+\alpha)^\lambda}\frac{(\bar\alpha+1)^\lambda}{(\bar\alpha e^{2i\gamma}+1)^\lambda}\Big] \\ &= 0. \end{aligned} \tag{8.10}$$

Putting

$$\sigma = \arg \frac{\alpha e^{-2i\gamma} + 1}{\alpha + 1}, \qquad \kappa = \arg \frac{\bar{\beta} e^{-2i\delta} + 1}{\bar{\beta} + 1},$$

and arguing as in (5.31)–(5.33), this altogether enables us to rewrite (8.10) as

$$D_o^{1/2}\Big[1 + \epsilon_\gamma e^{2i\lambda(\gamma+\sigma)}\Big]\Big[1 - \epsilon_\delta e^{2i\lambda(\delta+\kappa)}\Big] + D_m^{1/2}\Big[1 + \epsilon_\delta e^{2i\lambda(\delta+\kappa)}\Big]\Big[1 - \epsilon_\gamma e^{2i\lambda(\gamma+\sigma)}\Big] = 0,$$

or, what is the same,

$$\begin{aligned} D_o^{1/2}\Big[e^{-i\lambda(\gamma+\sigma)} + \epsilon_\gamma e^{i\lambda(\gamma+\sigma)}\Big]\Big[e^{-i\lambda(\delta+\kappa)} - \epsilon_\delta e^{i\lambda(\delta+\kappa)}\Big] + \\ + D_m^{1/2}\Big[e^{-i\lambda(\delta+\kappa)} + \epsilon_\delta e^{i\lambda(\delta+\kappa)}\Big]\Big[e^{-i\lambda(\gamma+\sigma)} - \epsilon_\gamma e^{i\lambda(\gamma+\sigma)}\Big] = 0. \end{aligned} \tag{8.11}$$

This means that in the pure Dirichlet case (with $\epsilon_\gamma = \epsilon_\delta = -1$) (8.10) can be written equivalently as

$$D_o^{1/2} \sin \lambda(\gamma + \sigma)\, \cos \lambda(\delta + \kappa) + D_m^{1/2} \cos \lambda(\gamma + \sigma)\, \sin \lambda(\delta + \kappa) = 0 \tag{8.12}$$

and in the pure Neumann case (with $\epsilon_\gamma = \epsilon_\delta = 1$) as

$$D_o^{1/2} \cos \lambda(\gamma + \sigma)\, \sin \lambda(\delta + \kappa) + D_m^{1/2} \sin \lambda(\gamma + \sigma)\, \cos \lambda(\delta + \kappa) = 0. \tag{8.13}$$

Because D_m and D_o are arbitrary positive constants it suffices to focus the following discussion on (8.12).

Lemma 8.4. *If $\gamma, \delta \le \pi$ with $\gamma + \delta < 2\pi$, then any solution λ of (8.12) satisfies $\Re\lambda \notin]0, 1/2 + \epsilon]$ for an $\epsilon > 0$.*

Proof. Since, by Lemma 8.3, $\sin \lambda(\gamma + \sigma) \neq 0$ and $\sin \lambda(\delta + \kappa) \neq 0$, if $0 < \Re\lambda < 1$, we can rewrite (8.12) as

$$D_o^{1/2} \cot \lambda(\delta + \kappa) + D_m^{1/2} \cot \lambda(\gamma + \sigma) = 0\,. \tag{8.14}$$

Note that

$$\Re \cot(\xi + i\eta) = \frac{(\cosh^2 \eta - \sinh^2 \eta) \sin \xi \cos \xi}{(\sin \xi \cosh \eta)^2 + (\cos \xi \sinh \eta)^2} = \frac{\sin 2\xi}{2(\sin^2 \xi + \sinh^2 \eta)}\,,$$

hence, with $\lambda = \vartheta + i\nu$, the real parts of (8.14) have the form

$$\frac{D_o^{1/2} \sin 2\vartheta(\delta + \kappa)}{\sin^2 \vartheta(\delta + \kappa) + \sinh^2 \nu(\delta + \kappa)} + \frac{D_m^{1/2} \sin 2\vartheta(\gamma + \sigma)}{\sin^2 \vartheta(\gamma + \sigma) + \sinh^2 \nu(\gamma + \sigma)} = 0\,. \tag{8.15}$$

If $0 < \vartheta \le 1/2$, then Lemma 8.3 shows that $0 < 2\vartheta(\delta + \kappa), 2\vartheta(\gamma + \sigma) \le \pi$, and therefore both terms on the left hand side of (8.15) are non-negative. Due to $\gamma + \delta < 2\pi$, at most one of them may be zero. This proves the assertion. ◻

Remark 8.5. If one has in $-\gamma$ a Dirichlet condition and in δ a Neumann condition (what means $-\epsilon_\gamma = \epsilon_\delta = 1$), then (8.11) reads as

$$-D_o^{1/2} \sin\bigl(\lambda(\gamma+\sigma)\bigr) \sin\bigl(\lambda(\delta+\kappa)\bigr) + D_m^{1/2} \cos\bigl(\lambda(\gamma+\sigma)\bigr) \cos\bigl(\lambda(\delta+\kappa)\bigr) = 0. \quad (8.16)$$

If we again suppose $\gamma, \delta \in]0, \pi[$, then we may divide (8.16) by $\sin\bigl(\lambda(\gamma+\sigma)\bigr) \sin\bigl(\lambda(\delta+\kappa)\bigr)$ (provided $\Re\lambda \in]0,1[$) and obtain the equivalent condition

$$\cot\bigl(\lambda(\gamma+\sigma)\bigr) \cot\bigl(\lambda(\delta+\kappa)\bigr) = \frac{D_o^{1/2}}{D_m^{1/2}}. \quad (8.17)$$

It is not hard to see that there are parameter configurations $\gamma, \delta, \alpha, \beta, D_o, D_m$ such that (8.17) is fulfilled for λ with arbitrarily small (positive) real part; see also [38], where the case of scalar multiples of the Laplacian already was treated.

Remark 8.6. In fact, the results of Theorem 8.1 and Theorem 8.2 are already proved in [12] (see Lemma 2.9 and Lemma 2.5) by completely different methods and based on the results of Il'yin [27], [28]. Our intention was here to give a proof which is straight forward and self-contained.

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