Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Temporal decorrelation for branching random walks with state dependent branching rate

Matthias Birkner[1]

submitted: January 23, 2007

[1] Weierstrass Institute
for Applied Analysis and Stochastics
Mohrenstr. 39, 10117 Berlin, Germany
birkner@wias-berlin.de

No. 1202
Berlin 2007

2000 *Mathematics Subject Classification.* 60K35.

Key words and phrases. State dependent branching, temporal decorrelation.

.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider branching random walks in $d \geq 3$ with a Lipschitz branching rate function and show that the system, starting either in a Poisson field or in equilibrium, decorrelates over long time horizons, and employ this to obtain an ergodic theorem. We use coupling and a stochastic representation of the Palm distribution.

1 Introduction and result

We study critical branching random walks with state dependent branching rate. Informally, this is a system of particles which perform independent random walks on $\mathbb{Z}^d$ according to some random walk kernel a. Additionally, while there are k particles at a given site, a branching event occurs at rate $\sigma(k)$, in such an event, one particle dies and is replaced by a random number of offspring. $\sigma : \mathbb{Z}_+ \to \mathbb{R}_+$ is the branching rate function, $\sigma(k) = \text{const.} \times k$ corresponds to the classical case of independently branching random walks. See Chapter 2 of [1] for a formal construction and background. Here, we focus on a particular subclass: Our standing assumptions are

$$a \text{ is shift-invariant and } \textit{symmetric}, \ a\text{-random walk is } \textit{transient}, \tag{1.1}$$

$$\text{the branching law is critical binary,} \tag{1.2}$$

$$\text{the branching rate function } \sigma \text{ is Lipschitz: } |\sigma(m) - \sigma(n)| \leq C_\sigma |m-n|. \tag{1.3}$$

Note that (1.3) in particular implies that σ grows at most linearly, $\sigma(k) \leq C_\sigma k$ for all k (as $\sigma(0) = 0$ by definition). Assuming finite second moments for the jumps of an a-random walk, (1.1) enforces $d \geq 3$.

We will write $\xi_x(t)$ for the number of particles at site $x \in \mathbb{Z}^d$ at time t in the system of state dependent branching random walks. Let $\mathcal{H}_\vartheta$ denote the Poisson field with constant intensity $\vartheta \geq 0$ on $\mathbb{Z}^d$, $(S_t)_{t \geq 0}$ the semigroup of state dependent branching random walks on $\mathbb{Z}^d$ as constructed in Section 2.2 of [1], $\Lambda_\vartheta = \lim_{t\to\infty} \mathcal{H}_\vartheta S_t$ (limit in the vague topology) is the equilibrium constructed in Proposition 3, Chapter 2 of [1]. For $x \in \mathbb{Z}^d$, denote by θ^x the shift operator, i.e. for spatial configurations ξ let $(\theta^x \xi)_y = \xi_{y+x}$. We write $a_t(x,y)$ for the transition probabilities of a-random walk.

The following ergodic theorem is the main result of this paper.

Theorem 1. *Let $f, g : \mathbb{N}_0^{\mathbb{Z}^d} \to \mathbb{R}$ be local functions satisfying the following 'linear' growth condition:*

$$\begin{aligned} &\exists\, C_1, C_2 \geq 0 \text{ and } A \subset \mathbb{Z}^d, |A| < \infty \ \text{ such that } \ f(\eta), g(\eta) \ \text{ depend only on } (\eta_x)_{x\in A} \\ &\qquad\text{and} \quad |f(\eta)|, |g(\eta)| \leq C_1 + C_2 \textstyle\sum_{x\in A} \eta_x. \end{aligned} \tag{1.4}$$

Then we have for any $\vartheta > 0$

$$\lim_{s,t\to\infty} \mathbb{E}_{\mathcal{H}_\vartheta}\left[f(\xi(s)) g(\theta^z \xi(s+t))\right] = \int f \, d\Lambda_\vartheta \int g \, d\Lambda_\vartheta \tag{1.5}$$

uniformly in $z \in \mathbb{Z}^d$, in particular

$$\lim_{t\to\infty} \mathbb{E}_{\Lambda_\vartheta}\left[f(\xi(0)) g(\xi(t))\right] = \int f \, d\Lambda_\vartheta \int g \, d\Lambda_\vartheta. \tag{1.6}$$

AMS (2000) subject classification: 60K35
Key words: State dependent branching, temporal decorrelation

Hence, starting either from $\mathcal{H}_\vartheta$ or from Λ_ϑ, we have

$$\frac{1}{t}\int_0^t f(\xi_s)\,ds \underset{t\to\infty}{\longrightarrow} \int f(\xi)\,\Lambda_\theta(d\xi) \quad \textit{in } L_2 \textit{ (and in } L_1\textit{)}. \tag{1.7}$$

Remark 1. *In the situation of a* linear *test function f, under the weak additional assumption that (note that (1.1) essentially enforces $\delta \geq 1$)*

$$a_t(0,0) \leq C\big(1 \wedge t^{-\delta}\big) \quad \textit{for some } \delta > 1,\ C > 0, \tag{1.8}$$

(1.7) can easily be proved to hold a.s. as well, i.e. starting from $\mathscr{L}(\xi(0)) \in \{\mathcal{H}_\vartheta, \Lambda_\vartheta\}$ we then have for all $x \in \mathbb{Z}^d$

$$\frac{1}{t}\int_0^t \xi_x(s)\,ds \underset{t\to\infty}{\longrightarrow} \vartheta \quad \textit{almost surely.} \tag{1.9}$$

Remark 2. (1.7) is well known in the classical case $\sigma(k) = \text{const.} \times k$ corresponding to independently branching random walks, (see e.g. [5], or the references in [2] after Prop. 2.3), but note that in our scenario, there is no infinite divisibility, and no explicit calculations with Laplace transforms are feasible. Ergodic theorems are also well known for many interacting particle systems where the number of possible states per site is finite, cf e.g. [3].

In [2], limit theorems for the fluctuations of the occupation time $\int_0^t \xi_0(s)\,ds$ around its mean are studied. For this, Thm. 1 is required as a building block.

Assuming $a_t(0,0) \sim \text{cst.} \times t^{-\delta}$ (with $\delta > 1$ by (1.1)), we see from the proof of Remark 1 that in the case of linear f (where an 'almost explicit' calculation is feasible) that the error in (1.5) is of the order $t^{1-\delta}$. It seems natural to conjecture that this true for more general test functions as well. Presently, we are lacking explicit bounds on the success probabilities of the couplings employed below in order to give a proof.

In Section 2, we recall some results on state dependent branching random walks, the proof of Thm. 1 is given in Section 3.

2 Preliminaries

We recall some results from [1] that will be required in the following. A convenient state space for the process (following Liggett & Spitzer, cf [4]) will be

$$\mathfrak{X} = \Big\{\mu \in \mathbb{Z}_+^{\mathbb{Z}^d} \,:\, |\mu|_\gamma := \textstyle\sum_{x\in\mathbb{Z}^d} \gamma_x \mu(x) < \infty\Big\},$$

where $\gamma : \mathbb{Z}^d \to \mathbb{R}_+$ is a weight function satisfying $\sum_y a(x,y)\gamma_y \leq M\gamma_x$ for some M.

Note that for given initial condition $\xi_0 \in \mathfrak{X}$, the process $(\xi_t)_{t\geq 0}$ can be obtained as a strong solution to the following system of Poisson-process driven stochastic integral equations:

$$\begin{aligned}
\xi_t(x) \quad = \quad & \xi_0(x) + \sum_{y\neq x}\Big[\int_{[0,t]\times\mathbb{N}} \mathbf{1}(\xi_{s-}(y) \geq n)\bar{N}^{y,x}(ds\,dn) \\
& \qquad\qquad - \int_{[0,t]\times\mathbb{N}} \mathbf{1}(\xi_{s-}(x) \geq n)\bar{N}^{x,y}(ds\,dn)\Big] \\
& + \int_{[0,t]\times\mathbb{N}\times[0,1]} \mathbf{1}\Big(\xi_{s-}(x) \geq n, \frac{\sigma(\xi_{s-}(x))}{C_\sigma \xi_{s-}(x)} \geq u\Big)\bar{N}^{x,+}(ds\,dn\,du) \\
& - \int_{[0,t]\times\mathbb{N}\times[0,1]} \mathbf{1}\Big(\xi_{s-}(x) \geq n, \frac{\sigma(\xi_{s-}(x))}{C_\sigma \xi_{s-}(x)} \geq u\Big)\bar{N}^{x,-}(ds\,dn\,du),
\end{aligned} \tag{2.1}$$

where $\bar{N}^{x,y}$, $x \neq y$, are independent Poisson processes on $[0,\infty) \times \mathbb{N}$ and $\bar{N}^{x,+}, \bar{N}^{x,-}$, $x \in \mathbb{Z}^d$, are independent Poisson processes on $[0,\infty) \times \mathbb{N} \times [0,1]$, all independent of ξ_0. $\bar{N}^{x,y}$ has intensity measure $a(x,y)dt \otimes d\ell$, $\bar{N}^{x,+}$, $\bar{N}^{x,-}$ have intensity measure $(C_\sigma/2)dt \otimes d\ell \otimes du$ (dt, du refer to Lebesgue measure, ℓ is counting measure). See [1], Section 2.2.

Furthermore, (2.1) can be used to obtain an explicit coupling of two versions ξ, $\tilde{\xi}$ of the process with different initial conditions, in which the two coordinates try to take as many steps in unison as possible. Let $\xi_0, \tilde{\xi}_0 \in \mathfrak{X} \times \mathfrak{X}$, and solve (2.1) for these two initial conditions with the same driving Poisson processes. The pair $(\xi_t, \tilde{\xi}_t)_{t\geq 0}$ is then a Markov process on $\mathfrak{X} \times \mathfrak{X}$ with generator given by (we write $a_+ := a \vee 0$)

$$\begin{aligned} \mathscr{G}^{(2)} f(\xi,\tilde{\xi}) \quad = \quad & \sum_{x,y\in\mathbb{Z}^d} a(x,y) \Big\{ \xi(x) \wedge \tilde{\xi}(x) \big(f(\xi^{x,y}, \tilde{\xi}^{x,y}) - f(\xi,\tilde{\xi}) \big) \\ & \qquad + (\xi(x) - \tilde{\xi}(x))_+ \big(f(\xi^{x,y}, \tilde{\xi}) - f(\xi,\tilde{\xi}) \big) \\ & \qquad + (\tilde{\xi}(x) - \xi(x))_+ \big(f(\xi, \tilde{\xi}^{x,y}) - f(\xi,\tilde{\xi}) \big) \Big\} \\ & + \frac{1}{2} \sum_{x\in\mathbb{Z}^d} \Big\{ \big(\sigma(\xi(x)) \wedge \sigma(\tilde{\xi}(x)) \big) \big(f(\xi^{x,+}, \tilde{\xi}^{x,+}) + f(\xi^{x,-}, \tilde{\xi}^{x,-}) - 2f(\xi,\tilde{\xi}) \big) \\ & \qquad + \big(\sigma(\xi(x)) - \sigma(\tilde{\xi}(x)) \big)_+ \big(f(\xi^{x,+}, \tilde{\xi}) + f(\xi^{x,-}, \tilde{\xi}) - 2f(\xi,\tilde{\xi}) \big) \\ & \qquad + \big(\sigma(\tilde{\xi}(x)) - \sigma(\xi(x)) \big)_+ \big(f(\xi, \tilde{\xi}^{x,+}) + f(\xi, \tilde{\xi}^{x,-}) - 2f(\xi,\tilde{\xi}) \big) \Big\}, \end{aligned} \tag{2.2}$$

for functions f which are Lipschitz with respect to the norm $|\xi|_\gamma + |\tilde{\xi}|_\gamma$, see [1], Prop. 2 (and note that because of the linear growth of σ guaranteed by (1.3), there is no need to restrict to square-summable configurations). We write $\xi^{x,+}$ for the configuration ξ with an additional particle at x, $\xi^{x,-}$ for the ξ with one particle removed from x, and $\xi^{x,y}$ for the configuration in which a particle is moved from x to y. Applying $\mathscr{G}^{(2)}$ to the functions $f_x(\xi,\tilde{\xi}) := |\xi_x - \tilde{\xi}_x|$ we obtain

$$\mathscr{G}^{(2)} f_x(\xi,\tilde{\xi}) \leq \sum_y a(y,x) \big(f_y(\xi,\tilde{\xi}) - f_x(\xi,\tilde{\xi}) \big),$$

and hence the coupled pair satisfies the estimate (see e.g. [3], Thm. 2.15)

$$\mathbb{E}\, |\xi_t(x) - \tilde{\xi}_t(x)| \leq \sum_y a_t^T(x,y) \mathbb{E}\, |\xi_0(x) - \tilde{\xi}_0(x)|, \tag{2.3}$$

where $a_t^T(x,y)$ are the transition probabilities of a random walk with jump rate matrix a^T, the transpose of a ($a^T = a$ by our symmetry assumption).

Observe that strong solutions of (2.1) provide in fact a simultaneous coupling for arbitrarily many initial conditions, with an analogous expression for the joint generator $\mathscr{G}^{(n)}$ of an n-tuple of solutions.

Denote the Palm distribution of $\mathcal{H}_\vartheta S_t$ with respect to $x \in \mathbb{Z}^d$ by $\hat{\mathcal{H}}_\vartheta^{(x,t)}$, i.e.

$$\int \varphi(\xi)\, \hat{\mathcal{H}}_\vartheta^{(x,t)}(d\xi) = \frac{1}{\vartheta} \mathbb{E}_{\mathcal{H}_\vartheta} \Big[\xi_x(t) \varphi(\xi(t)) \Big]. \tag{2.4}$$

Let us recall the stochastic representation for the Palm distribution given in Section 2.5 of [1] for the case considered here. Fix $T > 0$, $z \in \mathbb{Z}^d$, let $\alpha = (\alpha_t)_{0\leq t\leq T}$ be a càdlàg path in $\mathbb{Z}^d$ with $\alpha_T = z$. One can interpret the Palm distribution as the configuration seen around an individual 'sampled randomly from the current population' (which is sometimes called 'ego', see e.g. [6]). With this in mind, we give a construction of the 'rest of the population' (i.e. the so-called reduced Palm distribution), conditional on the space-time ancestral line of ego, sampled at z at time T, being α. In order to do this consider a particle system $\tilde{\xi}_t^{T,\alpha}$, $0 \leq t \leq T$, starting from $\mathscr{L}(\tilde{\xi}_0^{T,\alpha}) = \mathcal{H}_\vartheta$, which evolves as follows: Particles move independently according to the kernel a, at time t, particles at $x \neq \alpha_{t-}$ branch critical binary at rate $\sigma(\tilde{\xi}^{T,\alpha}_{t-}(x))/\tilde{\xi}^{T,\alpha}_{t-}(x)$ per particle as usual, while at $x = \alpha_{t-}$,

$$\text{one particle dies at rate } \sigma\big(\tilde{\xi}^{T,\alpha}_{t-}(x)+1\big)\frac{\tilde{\xi}^{T,\alpha}_{t-}(x)}{2(\tilde{\xi}^{T,\alpha}_{t-}(x)+1)}, \text{ and}$$

$$\text{one particle is born at rate } \sigma\big(\tilde{\xi}^{T,\alpha}_{t-}(x)+1\big)\frac{\tilde{\xi}^{T,\alpha}_{t-}(x)+2}{2(\tilde{\xi}^{T,\alpha}_{t-}(x)+1)}.$$

More formally, $(\tilde{\xi}^{T,\alpha}_t)_{0\le t\le T}$ is an $\mathfrak{X}$-valued time inhomogeneous Markov process with generators given by

$$\begin{aligned} \mathscr{G}^{T,\alpha}_t f(\eta) &= \sum_{x,y\in\mathbb{Z}^d} \eta(x)a(x,y)\Big(f(\eta^{x,y})-f(\eta)\Big) \\ &\quad + \sum_{x\neq\alpha_{t-}} \frac{1}{2}\sigma\big(\eta(x)\big)\Big\{f(\eta^{x,+})+f(\eta^{x,-})-2f(\eta)\Big\} \\ &\quad + \sigma\big(\eta(\alpha_{t-})+1\big)\frac{\eta(\alpha_{t-})}{2(\eta(\alpha_{t-})+1)}\Big(f(\eta^{\alpha_{t-},-})-f(\eta)\Big) \\ &\quad + \sigma\big(\eta(\alpha_{t-})+1\big)\frac{\eta(\alpha_{t-})+2}{2(\eta(\alpha_{t-})+1)}\Big(f(\eta^{\alpha_{t-},+})-f(\eta)\Big) \end{aligned} \tag{2.5}$$

(in the sense that

$$f\big(\tilde{\xi}^{T,\alpha}_t\big) - f\big(\tilde{\xi}^{T,\alpha}_0\big) - \int_0^t \mathscr{G}^{T,\alpha}_s f\big(\tilde{\xi}^{T,\alpha}_s\big)\,ds$$

is a martingale for f Lipschitz). For an interpretation of the last two terms in (2.5) note that there are $\tilde{\xi}^{T,\alpha}_{t-}(\alpha_{t-})+1$ particles at site α_{t-} immediately before a potential jump at time t, including the ancestor of 'ego'. Given that a branching event occurs, it will involve this ancestor with probability $1/(\tilde{\xi}^{T,\alpha}_{t-}(\alpha_{t-})+1)$, and in this case, it must necessarily be a birth.

One can construct $\tilde{\xi}^{T,\alpha}$ using driving Poisson processes analogously to (2.1), we refrain from writing out the details.

Let $(Y_t)_{0\le t\le T}$ be a random walk with jump rate matrix $a^T(x,y) = a(y,x)$, starting from $Y_0 = z$, denote the distribution of the time-reversed path $(Y_{(T-t)-})_{0\le t\le T}$ by π^z. We obtain from Proposition 5 in [1] that

$$\hat{\mathcal{H}}^{(z,T)}_\vartheta = \int \mathscr{L}\big(\tilde{\xi}^{T,\alpha}_T + \delta_z\big)\,\pi^z(d\alpha). \tag{2.6}$$

In words, the space-time ancestral line of the sampled individual at z is an a^T-random walk path, starting from z backwards in time. Read forwards in time, it is the path α in the construction above.

3 Proofs

Proof of Remark 1. By shift-invariance, it suffices to consider $x=0$. Let $T_t := \int_0^t \xi_0(s)\,ds$, and $\mathscr{L}(\xi(0)) \in \{\mathcal{H}_\vartheta, \Lambda_\vartheta\}$. By Lemma 4 from [1], we have $\mathbb{E}\,T_t = \vartheta t$, and by Lemma 3.3 from [2], we find

$$\begin{aligned} \mathrm{Var}_{\mathcal{H}_\vartheta} T_t &= 2\int_0^t du \int_u^t dv\,\Big(\mathbb{E}_{\mathcal{H}_\vartheta}\big[\xi_0(u)\xi_0(v)\big]-\vartheta^2\Big) \\ &= 2\int_0^t du\int_u^t dv\,\Big\{\vartheta a_{v-u}(0,0) + \int_0^u a_{v-u+2r}(0,0)\mathbb{E}_{\mathcal{H}_\vartheta}\big[\sigma(\xi_0(u-r))\big]\,dr\Big\} \\ &= O(t^{3-\delta}) \end{aligned}$$

by (1.8) and (1.3). Similarly, we find $\mathrm{Var}_{\Lambda_\vartheta} T_t = O(t^{3-\delta})$. As $\delta > 1$ by assumption, we can proceed as on p. 399 of [5], imitating Etemadi's proof of the strong law of large numbers, to obtain (1.9). $\square$

Proof of Thm. 1. Let us first treat the case f, g bounded. Furthermore, by passing to $\bar{f}(\eta) := f(\eta) - f(\underline{0})$ if necessary (where $\underline{0}$ denotes the empty configuration), we can assume that

$$|f(\eta)| \le C_2 \sum_{x \in A} \eta_x. \tag{3.1}$$

Fix $s \ge 0$, $x, z \in \mathbb{Z}^d$ for the moment. Consider the coupled pair $(\tilde{\xi}(t), \xi(t))_{t \ge 0}$ with joint generator $\mathscr{G}^{(2)}$ given by (2.2), starting from $\mathscr{L}\big((\tilde{\xi}(0), \xi(0))\big) = \hat{\mathcal{H}}_\vartheta^{(x,s)} \otimes \mathcal{H}_\vartheta$. Put

$$\mathscr{A}_{\epsilon,t,z} := \Big\{ \eta \in \mathfrak{X} : \mathbb{P}\big(\tilde{\xi}_y(t) = \xi_y(t) \,\forall y \in A + z \,\big|\, \tilde{\xi}(0) = \eta\big) \ge 1 - \epsilon \Big\}. \tag{3.2}$$

$\mathbb{P}$ refers to the joint probability space on which the coupling is defined and $\mathbb{P}(\cdot | \tilde{\xi}(0) = \eta)$ to a regular version of the conditional distribution given $\tilde{\xi}(0)$ (recall that $\mathfrak{X}^2$ is Polish). Because the initial condition is a product measure, we have $\mathscr{L}(\xi(t)) = \mathscr{L}(\xi(t) | \tilde{\xi}(0) = \eta)$ independent of η. Note that the definition of $\mathscr{A}_{\epsilon,t,z}$ does not depend on (x,s), and that $\eta \in \mathscr{A}_{\epsilon,t}$ implies

$$\begin{aligned}
&\big| S_t g(\theta^z \eta) - \int g \, d\Lambda_\vartheta \big| \\
\le\ & \big| S_t g(\theta^z \eta) - \int S_t g(\theta^z \eta') \, \mathcal{H}_\vartheta(d\eta') \big| + \big| \int S_t g(\eta') \, \mathcal{H}_\vartheta(d\eta') - \int g \, d\Lambda_\vartheta \big| \\
=\ & \big| \mathbb{E}\big[g(\theta^z \tilde{\xi}(t)) - g(\theta^z \xi(t)) \big| \tilde{\xi}(0) = \eta \big] \big| + \big| \int S_t g(\eta') \, \mathcal{H}_\vartheta(d\eta') - \int g \, d\Lambda_\vartheta \big| \\
\le\ & \epsilon ||g||_\infty + \big| \int S_t g(\eta') \, \mathcal{H}_\vartheta(d\eta') - \int g \, d\Lambda_\vartheta \big| \le \epsilon \big(||g||_\infty + 1 \big)
\end{aligned} \tag{3.3}$$

if t is large enough (where we use [1], Prop. 3 in the last line). By Lemma 2 we have

$$\lim_{t \to \infty} \sum_{y \in A + z} \mathbb{E}_{\hat{\mathcal{H}}_\vartheta^{(x,s)} \otimes \mathcal{H}_\vartheta} \big[|\tilde{\xi}_y(t) - \xi_y(t)| \big] = 0 \tag{3.4}$$

uniformly in x, z, s (observe that $a_t(x,y) \le a_t(0,0)$ to obtain uniformity), hence in particular

$$\lim_{t \to \infty} \inf_{s \ge 0} \int_{\mathfrak{X}} \mathbb{P}\big(\tilde{\xi}_y(t) = \xi_y(t) \,\forall y \in A + z \,\big|\, \tilde{\xi}(0) = \eta \big) \, \hat{\mathcal{H}}_\vartheta^{(x,s)}(d\eta) = 1, \tag{3.5}$$

uniformly in x, z, hence

$$\sup_{x,z \in \mathbb{Z}^d} \sup_{s \ge 0} \hat{\mathcal{H}}_\vartheta^{(x,s)} \big(\mathscr{A}^c_{\epsilon,t,z} \big) \le \epsilon \tag{3.6}$$

whenever $t \ge t_0 \,(= t_0(\epsilon))$. (Let $0 \le \phi \le 1$, μ a probability measure such that $\int \phi \, d\mu \ge 1 - \epsilon^2$, then $1 - \epsilon^2 \le 1 - \epsilon + \epsilon \mu(\{x : \phi(x) \ge 1 - \epsilon\})$, hence $\mu(\{x : \phi(x) \ge 1 - \epsilon\}) \ge 1 - \epsilon$.) Define

$$\tilde{f}(\eta) := \frac{f(\eta)}{\sum_{x \in A} \eta_x}. \tag{3.7}$$

By (3.1), this is a bounded local function (if $\sum_{x \in A} \eta_x = 0$, $\tilde{f}(\eta)$ takes some arbitrary value). We have

$$\begin{aligned}
&\mathbb{E}_{\mathcal{H}_\vartheta} \big[f(\xi(s)) g(\theta^z \xi(s+t)) \big] = \mathbb{E}_{\mathcal{H}_\vartheta} \big[f(\xi(s)) S_t g(\theta^z \xi(s)) \big] = \sum_{x \in A} \mathbb{E}_{\mathcal{H}_\vartheta} \big[\xi_x(s) \tilde{f}(\xi(s)) S_t g(\theta^z \xi(s)) \big] \\
=\ & \vartheta \sum_{x \in A} \int \tilde{f}(\eta) S_t g(\eta) \, \hat{\mathcal{H}}_\vartheta^{(x,s)}(d\eta) = \int g \, d\Lambda_\vartheta \times \vartheta \sum_{x \in A} \int \tilde{f}(\eta) \, \hat{\mathcal{H}}_\vartheta^{(x,s)}(d\eta) + R(s,t) \\
=\ & \int g \, d\Lambda_\vartheta \times \mathbb{E}_{\mathcal{H}_\vartheta} \big[f(\xi(s)) \big] + R(s,t),
\end{aligned}$$

where the remainder term satisfies

$$|R(s,t)| \le 2\epsilon\, ||\tilde{f}||_\infty \big(||g||_\infty + 1\big) \quad \text{for } s \ge 0, t \ge t_0 \tag{3.8}$$

by (3.6). This together with the fact that $\mathbb{E}_{\mathcal{H}_\vartheta}\big[f(\xi(s))\big] \to \int f\, d\Lambda_\vartheta$ as $s \to \infty$ proves (1.5) in the bounded case.

In the general case, we approximate f and g by

$$f_M(\eta) := \big(f(\eta) \wedge M\big) \vee (-M), \quad g_M(\eta) := \big(g(\eta) \wedge M\big) \vee (-M), \quad M > 0.$$

and control the error terms using Cauchy-Schwarz, using that $\sup_{x,t} \mathbb{E}_{\mathcal{H}_\vartheta}\big[\xi_x(t)^2\big] < \infty$ by [1], Lemma 5. Here are some details: We know from the above that for any M

$$\Big|\mathbb{E}_{\mathcal{H}_\vartheta}\big[f_M(\xi(s)) g_M(\xi(s+t))\big] - \int f_M\, d\Lambda_\vartheta \int g_M\, d\Lambda_\vartheta\Big| \le \delta$$

whenever $s, t \ge R\,(= R(\delta, M))$. In order to control the error introduced by the cut-off we note that by (1.4)

$$\begin{aligned}
&\Big|\mathbb{E}_{\mathcal{H}_\vartheta}\big[f_M(\xi(s)) g_M(\xi(s+t)) - f(\xi(s)) g(\xi(s+t))\big]\Big| \\
&\le \ \mathbb{E}_{\mathcal{H}_\vartheta}\Big[\big(C_1 + C_2 \sum_{x\in A} \xi_x(s)\big)\big(C_1 + C_2 \sum_{x\in A} \xi_x(s+t)\big)\mathbf{1}\big(C_1 + C_2 \sum_{x\in A} \xi_x(s) > M\big)\Big] \\
&\quad + \mathbb{E}_{\mathcal{H}_\vartheta}\Big[\big(C_1 + C_2 \sum_{x\in A} \xi_x(s)\big)\big(C_1 + C_2 \sum_{x\in A} \xi_x(s+t)\big)\mathbf{1}\big(C_1 + C_2 \sum_{x\in A} \xi_x(s+t) > M\big)\Big] \\
&\le \ \bigg(\mathbb{E}_{\mathcal{H}_\vartheta}\Big[\big(C_1 + C_2 \sum_{x\in A} \xi_x(s)\big)^2 \mathbf{1}\big(C_1 + C_2 \sum_{x\in A} \xi_x(s) > M\big)\Big]\bigg)^{1/2} \\
&\quad \times \bigg(\mathbb{E}_{\mathcal{H}_\vartheta}\Big[\big(C_1 + C_2 \sum_{x\in A} \xi_x(s+t)\big)^2\Big]\bigg)^{1/2} \\
&\quad + \bigg(\mathbb{E}_{\mathcal{H}_\vartheta}\Big[\big(C_1 + C_2 \sum_{x\in A} \xi_x(s)\big)^2\Big]\bigg)^{1/2} \\
&\quad \times \bigg(\mathbb{E}_{\mathcal{H}_\vartheta}\Big[\big(C_1 + C_2 \sum_{x\in A} \xi_x(s+t)\big)^2 \mathbf{1}\big(C_1 + C_2 \sum_{x\in A} \xi_x(s+t) > M\big)\Big]\bigg)^{1/2},
\end{aligned}$$

which converges to 0 as $M \to \infty$ uniformly in $s, t \ge 0$ because the family $\{\xi_x(t)^2 : x \in \mathbb{Z}^d, t \ge 0\}$ is uniformly integrable under $\mathcal{H}_\vartheta$. For this note e.g. that

$$\sup_{x,t} \mathbb{E}_{\mathcal{H}_\vartheta}\big[\xi_x(t)^3\big] < \infty. \tag{3.9}$$

To obtain (3.9), one can either use the comparison result in Thm. 1 of [1] and compare ξ with a classical system of independent critically branching random walks with branching rate function $C_\sigma \times k$, for which global boundedness of all moments is known, or carry through the program sketched in Remark 5 of [1].

This proves (1.5). (1.6) follows from this by taking $s \to \infty$ first. Finally, (1.7) is straightforward from (1.5) resp. (1.6). $\square$

Lemma 1. *For $x \in \mathbb{Z}^d$, $T, t \ge 0$ there is a coupling of $\hat{\mathcal{H}}_\vartheta^{(x,T)} S_t$ and $\mathcal{H}_\vartheta S_{T+t}$, i.e. an $\mathfrak{X}^2$-valued random variable $(\eta^{(1)}, \eta^{(2)})$ such that $\mathscr{L}(\eta^{(1)}) = \hat{\mathcal{H}}_\vartheta^{(x,T)} S_t$, $\mathscr{L}(\eta^{(2)}) = \mathcal{H}_\vartheta S_{T+t}$ and*

$$\forall\, y \in \mathbb{Z}^d \ :\ \mathbb{E}\,\big|\eta_y^{(1)} - \eta_y^{(2)}\big| \le a_t(x,y) + \frac{C_\sigma}{2} \int_t^\infty a_u(x,y)\, du. \tag{3.10}$$

By the assumed transience of a-random walk, the right-hand side converges to 0 as $t \to \infty$.

Note that combined with Prop. 3 in [1] this implies in particular $\hat{\mathcal{H}}_\vartheta^{(x,T)} S_t \Rightarrow \Lambda_\vartheta$ as $t \to \infty$, and in fact that this convergence is uniform in (x, T).

Proof. The idea behind the proof is as follows: between time 0 and time T, we use the stochastic representation of $\hat{\mathcal{H}}^{(x,T)}$ given by (2.6) and couple it with a second process that follows the 'ordinary' dynamics, both use the same initial condition (with distribution $\mathcal{H}_\vartheta$). Then we use the standard coupling described by $\mathscr{G}^{(2)}$ between time T and time $T+t$.

For a given path $\alpha = (\alpha_s)_{0\le s\le T}$ with $\alpha_T = x$ consider the family of generators $\mathscr{G}_s^{T,\alpha,(2)}$, $(0 \le s \le T)$ acting on (Lipschitz) functions f on $\mathfrak{X}^2$,

$$\begin{aligned}
\mathscr{G}_s^{T,\alpha,(2)} f(\xi,\tilde\xi) \;=\; & \sum_{x,y\in\mathbb{Z}^d} a(x,y)\Big\{\xi(x)\wedge\tilde\xi(x)\big(f(\xi^{x,y},\tilde\xi^{x,y}) - f(\xi,\tilde\xi)\big) \\
& \qquad + (\xi(x)-\tilde\xi(x))_+\big(f(\xi^{x,y},\tilde\xi) - f(\xi,\tilde\xi)\big) \\
& \qquad + (\tilde\xi(x)-\xi(x))_+\big(f(\xi,\tilde\xi^{x,y}) - f(\xi,\tilde\xi)\big)\Big\} \\
& + \frac{1}{2}\sum_{\alpha_s\neq x\in\mathbb{Z}^d} \Big\{\big(\sigma(\xi(x))\wedge\sigma(\tilde\xi(x))\big)\big(f(\xi^{x,+},\tilde\xi^{x,+}) + f(\xi^{x,-},\tilde\xi^{x,-}) - 2f(\xi,\tilde\xi)\big) \\
& \qquad + \big(\sigma(\xi(x))-\sigma(\tilde\xi(x))\big)_+\big(f(\xi^{x,+},\tilde\xi) + f(\xi^{x,-},\tilde\xi) - 2f(\xi,\tilde\xi)\big) \\
& \qquad + \big(\sigma(\tilde\xi(x))-\sigma(\xi(x))\big)_+\big(f(\xi,\tilde\xi^{x,+}) + f(\xi,\tilde\xi^{x,-}) - 2f(\xi,\tilde\xi)\big)\Big\} \\
& + \Big\{\sigma\big(\xi(\alpha_s)+1\big)\frac{\xi(\alpha_s)+2}{2\xi(\alpha_s)+2}\wedge\frac{\sigma\big(\tilde\xi(\alpha_s)\big)}{2}\Big\}\Big(f(\xi^{x,+},\tilde\xi^{x,+}) - f(\xi,\tilde\xi)\Big) \\
& + \Big(\sigma\big(\xi(\alpha_s)+1\big)\frac{\xi(\alpha_s)+2}{2\xi(\alpha_s)+2} - \frac{\sigma\big(\tilde\xi(\alpha_s)\big)}{2}\Big)_+\Big(f(\xi^{x,+},\tilde\xi) - f(\xi,\tilde\xi)\Big) \\
& + \Big(\frac{\sigma\big(\tilde\xi(\alpha_s)\big)}{2} - \sigma\big(\xi(\alpha_s)+1\big)\frac{\xi(\alpha_s)+2}{2\xi(\alpha_s)+2}\Big)_+\Big(f(\xi,\tilde\xi^{x,+}) - f(\xi,\tilde\xi)\Big) \\
& + \Big\{\sigma\big(\xi(\alpha_s)+1\big)\frac{\xi(\alpha_s)}{2\xi(\alpha_s)+2}\wedge\frac{\sigma\big(\tilde\xi(\alpha_s)\big)}{2}\Big\}\Big(f(\xi^{x,-},\tilde\xi^{x,-}) - f(\xi,\tilde\xi)\Big) \\
& + \Big(\sigma\big(\xi(\alpha_s)+1\big)\frac{\xi(\alpha_s)}{2\xi(\alpha_s)+2} - \frac{\sigma\big(\tilde\xi(\alpha_s)\big)}{2}\Big)_+\Big(f(\xi^{x,-},\tilde\xi) - f(\xi,\tilde\xi)\Big) \\
& + \Big(\frac{\sigma\big(\tilde\xi(\alpha_s)\big)}{2} - \sigma\big(\xi(\alpha_s)+1\big)\frac{\xi(\alpha_s)}{2\xi(\alpha_s)+2}\Big)_+\Big(f(\xi,\tilde\xi^{x,-}) - f(\xi,\tilde\xi)\Big).
\end{aligned}$$

One checks readily that on functions f depending only on the first coordinate, $\mathscr{G}_s^{T,\alpha,(2)}$ acts like $\mathscr{G}_s^{T,\alpha}$, while on functions depending only on the second coordinate, it acts like the generator $\mathscr{G}$ of (S_t). Furthermore, for $f_z(\xi,\tilde\xi) := |\xi_z - \tilde\xi_z|$ we have

$$\mathscr{G}_s^{T,\alpha,(2)} f_z(\xi,\tilde\xi) \le \sum_y a(y,z)\big(f_y(\xi,\tilde\xi) - f_z(\xi,\tilde\xi)\big) + C_\sigma \mathbf{1}_{\{\alpha_s\}}(z). \tag{3.11}$$

In order to see this note that the 'motion' part of $\mathscr{G}_s^{T,\alpha,(2)}$ acts like that of $\mathscr{G}^{(2)}$, and the part refering to branching events away from α_s vanishes. The part refering to branching events at site α_s obviously vanishes when $z \neq \alpha_s$, e.g. a straightforward case-by-case analysis (decomposing according to the sign of $\xi_z - \tilde\xi_z$) shows that it is bounded by C_σ, the Lipschitz constant of σ, when $z = \alpha_s$.

Now consider the process $(\xi_s,\tilde\xi_s)_{s\le T}$ starting from $\xi_0 = \tilde\xi_0$ with $\mathscr{L}(\xi_0) = \mathcal{H}_\vartheta$, which evolves from time $s=0$ to time $s=T$ according to the time-inhomogeneous process described by the $\mathscr{G}_s^{T,\alpha,(2)}$. The function $g(z,s) := \mathbb{E}\,|\xi_s(z) - \tilde\xi_s(z)|$ satisfies $g(\cdot,0)\equiv 0$ and

$$\frac{\partial}{\partial s} g(z,s) = \sum_y a(y,z)\big(g(y,s) - g(z,s)\big) + C_\sigma \mathbf{1}_{\{\alpha_s\}}(z)$$

by (3.11), hence (cf e.g. [3], Thm. 2.15)

$$g(z,T) = C_\sigma \int_0^T a_{T-s}^T(z,\alpha_s)\,ds.$$

Now averaging α with respect to π^x (see the discussion above (2.6)) in this construction and recording the configuration of the coupled pair at time T, we obtain an $(\mathfrak{X} \times \mathfrak{X})$-valued random variable $(\eta, \tilde{\eta})$ such that $\mathscr{L}(\eta) = \hat{\mathcal{H}}_\vartheta^{(x,T)}$, $\mathscr{L}(\tilde{\eta}) = \mathcal{H}_\vartheta S_T$, and

$$\mathbb{E}\,|\eta_z - \tilde{\eta}_z| \le \delta_{xz} + C_\sigma \int \int_0^T a_{T-s}^T(z, \alpha_s)\, ds\, \pi^x(d\alpha) = \delta_{xz} + C_\sigma \int_0^T a_{2u}(z,x)\, du \tag{3.12}$$

Now plug the pair $(\eta, \tilde{\eta})$ into the coupling provided by $\mathscr{G}^{(2)}$ and let this run for a time interval of length t. Combining (3.12) and (2.3) yields (3.10).

□

Lemma 2. *For $x \in \mathbb{Z}^d$, $T, t \ge 0$ there is a coupling $(\eta^{(1)}, \eta^{(2)})$ of $\hat{\mathcal{H}}_\vartheta^{(x,T)} S_t$ and $\mathcal{H}_\vartheta S_t$ such that*

$$\forall\, y \in \mathbb{Z}^d \;:\; \mathbb{E}\,\big|\eta_y^{(1)} - \eta_y^{(2)}\big| \le a_t(x,y) + \frac{C_\sigma}{2} \int_t^\infty a_u(x,y)\, du + r(t), \tag{3.13}$$

where $r(t) := \sup_{u \ge 0} \mathbb{E}_{\mathcal{H}_\vartheta S_u \otimes \mathcal{H}_\vartheta} \big|\tilde{\xi}_0(t) - \xi_0(t)\big| \longrightarrow 0$ as $t \to \infty$. Note that this implies

$$\lim_{t\to\infty} \mathbb{E}_{\hat{\mathcal{H}}_\vartheta^{(x,T)} \otimes \mathcal{H}_\vartheta} \big[|\tilde{\xi}_z(t) - \xi_z(t)|\big] = 0 \quad \textit{uniformly in } T \ge 0 \textit{ and } x, z \in \mathbb{Z}^d.$$

Proof. This is in principle a variation on Lemma 7 from [1] with the little complication that $\hat{\mathcal{H}}_\vartheta^{(x,T)}$ is not shift-invariant. Still, in view of Lemma 1, $\hat{\mathcal{H}}_\vartheta^{(x,T)} S_t$ is 'almost' shift invariant when t is large.
First we check that

$$\lim_{t\to\infty} \sup_{u \ge 0} \mathbb{E}_{\mathcal{H}_\vartheta S_u \otimes \mathcal{H}_\vartheta} \big[|\tilde{\xi}_z(t) - \xi_z(t)|\big] = 0. \tag{3.14}$$

If this were not the case, we could find $\epsilon^* > 0$ and sequences $(u_n) \subset \mathbb{R}_+$, $(t_n) \subset \mathbb{R}_+$ such that $t_n \to \infty$ and

$$\mathbb{E}_{\mathcal{H}_\vartheta S_{u_n} \otimes \mathcal{H}_\vartheta} \big[|\tilde{\xi}_z(t_n) - \xi_z(t_n)|\big] \ge \epsilon^* \quad \text{for all } n.$$

We can find a subsequence $(u_{n(k)})_k$ such that $u_{n(k)} \to \bar{u} \in \overline{\mathbb{R}}_+$ as $k \to \infty$. As the function

$$t \mapsto \mathbb{E}\big[|\tilde{\xi}_z(t) - \xi_z(t)|\big]$$

is non-increasing for any choice of initial conditions (see [1], p. 27), this implies

$$\liminf_{k\to\infty} \mathbb{E}_{\mathcal{H}_\vartheta S_{u_{n(k)}} \otimes \mathcal{H}_\vartheta} \big[|\tilde{\xi}_z(t) - \xi_z(t)|\big] \ge \epsilon^*$$

for any $t \in \mathbb{R}_+$. On the other hand we obtain from Lemma 3 that

$$\lim_{k\to\infty} \mathbb{E}_{\mathcal{H}_\vartheta S_{u_{n(k)}} \otimes \mathcal{H}_\vartheta} \big[|\tilde{\xi}_z(t) - \xi_z(t)|\big] = \mathbb{E}_{\mathcal{H}_\vartheta S_{\bar{u}} \otimes \mathcal{H}_\vartheta} \big[|\tilde{\xi}_z(t) - \xi_z(t)|\big]$$

(when $\bar{u} = \infty$, '$\mathcal{H}_\vartheta S_{\bar{u}}$' means Λ_ϑ). Thus if (3.14) failed, we would have $\liminf_{t\to\infty} \mathbb{E}_{\mathcal{H}_\vartheta S_{\bar{u}} \otimes \mathcal{H}_\vartheta} \big[|\tilde{\xi}_z(t) - \xi_z(t)|\big] > 0$ in contradiction to Lemma 7 from [1].

The idea of the proof is now to compare three versions of our process: The pair $(\xi^{(1)}, \xi^{(2)})$ starts from the coupling considered in Lemma 1 at time T, so in particular $\mathscr{L}(\xi^{(1)}(0)) = \hat{\mathcal{H}}_\vartheta^{(x,T)}$, $\mathscr{L}(\xi^{(2)}(0)) = \mathcal{H}_\vartheta S_T$. $\xi^{(3)}$ starts independently from $(\xi^{(1)}, \xi^{(2)})$ with $\mathscr{L}(\xi^{(3)}(0)) = \mathcal{H}_\vartheta$. The joint dynamics of $\big(\xi^{(1)}(t), \xi^{(2)}(t), \xi^{(3)}(t)\big)_{t \ge 0}$ is a 'trivariate' version of the 'obvious' coupling, e.g. realised by simultaneously solving (2.1) with these three initial conditions. Note that each pair $\big(\xi^{(i)}(t), \xi^{(j)}(t)\big)_{t\geq 0}$ $(i \neq j)$ evolves according to the pair-coupling generator $\mathscr{G}^{(2)}$. Thus we find

$$\begin{aligned}
&\mathbb{E}_{\hat{\mathcal{H}}_\vartheta^{(x,s)}\otimes\mathcal{H}_\vartheta}\big[|\tilde{\xi}_z(t) - \xi_z(t)|\big] \\
&= \mathbb{E}\big[|\xi_z^{(1)}(t) - \xi_z^{(3)}(t)|\big] \\
&\leq \mathbb{E}\big[|\xi_z^{(1)}(t) - \xi_z^{(2)}(t)|\big] + \mathbb{E}\big[|\xi_z^{(2)}(t) - \xi_z^{(3)}(t)|\big] \\
&= \mathbb{E}\big[|\xi_z^{(1)}(t) - \xi_z^{(2)}(t)|\big] + \mathbb{E}_{\mathcal{H}_\vartheta S_s\otimes\mathcal{H}_\vartheta}\big[|\tilde{\xi}_z(t) - \xi_z(t)|\big] \\
&\leq a_t(x,z) + \frac{C_\sigma}{2}\int_t^\infty a_v(x,z)\,dv + \sup_{s\geq 0}\mathbb{E}_{\mathcal{H}_\vartheta S_s\otimes\mathcal{H}_\vartheta}\big[|\tilde{\xi}_z(t) - \xi_z(t)|\big]
\end{aligned}$$

by (3.10). This concludes the proof in view of (3.14). □

The last lemma verifies a continuity property of the coupled process with respect to the initial conditions.

Lemma 3. *Consider the coupled process $(\tilde{\xi}(t), \xi(t))_{t\geq 0}$ with generator $\mathscr{G}^{(2)}$. Let initial conditions $\mu_n, \mu \in \mathfrak{X}\times\mathfrak{X}$ be given and assume that*

$$\mu_n \Rightarrow \mu \quad \textit{as } n\to\infty, \textit{ where } \mathfrak{X}\times\mathfrak{X} \subset \mathcal{N}(\mathbb{Z}^d)\times\mathcal{N}(\mathbb{Z}^d) \textit{ is equipped with the vague topology}$$

and furthermore $\sup_n \sup_{x\in\mathbb{Z}^d} \int |\tilde{\xi}_x|^2 + |\xi_x|^2\, d\mu_n < \infty$. *Then we have for any $z \in \mathbb{Z}^d$, $t \geq 0$*

$$\mathbb{E}_{\mu_n}\Big[\big|\tilde{\xi}_z(t) - \xi_z(t)\big|\Big] \to \mathbb{E}_\mu\Big[\big|\tilde{\xi}_z(t) - \xi_z(t)\big|\Big] \quad \textit{as } n\to\infty.$$

Proof. For given finite $A \subset \mathbb{Z}^d$ and $\varepsilon > 0$ we can use Skorohod coupling to find for each sufficiently large n a random element $\big(\tilde{\xi}(0), \xi(0), \tilde{\xi}'(0), \xi'(0)\big)$ with values in $(\mathcal{N}(\mathbb{Z}^d))^4$ such that $\mathscr{L}\big((\tilde{\xi}(0), \xi(0)\big) = \mu_n$, $\mathscr{L}\big((\tilde{\xi}'(0), \xi'(0)\big) = \mu$ and the event

$$\mathcal{E}(A) = \big\{\tilde{\xi}_x(0) = \tilde{\xi}'_x(0), \xi_x(0) = \xi'_x(0)\,\forall x\in A\big\}$$

has $\mathbb{P}(\mathcal{E}(A)) \geq 1-\varepsilon$. Denote the simultaneous solution of (2.1) with these initial conditions by $\big(\tilde{\xi}(t), \xi(t), \tilde{\xi}'(t), \xi'(t)\big)_{t\geq 0}$. This allows to estimate (for n sufficiently large)

$$\begin{aligned}
&\left|\mathbb{E}_{\mu_n}\Big[\big|\tilde{\xi}_z(t) - \xi_z(t)\big|\Big] - \mathbb{E}_\mu\Big[\big|\tilde{\xi}_z(t) - \xi_z(t)\big|\Big]\right| \\
&= \left|\mathbb{E}\Big[\big|\tilde{\xi}_z(t) - \xi_z(t)\big| - \big|\tilde{\xi}'_z(t) - \xi'_z(t)\big|\Big]\right| \\
&\leq \mathbb{E}\Big[\big|\tilde{\xi}_z(t) - \tilde{\xi}'_z(t)\big| + \big|\xi_z(t) - \xi'_z(t)\big|\Big] \\
&\leq \sum_y a^T(z,y)\mathbb{E}\Big[\big|\tilde{\xi}_y(0) - \tilde{\xi}'_y(0)\big| + \big|\xi_y(0) - \xi'_y(0)\big|\Big] \\
&\leq 2M_1\sum_{y\in A^c} a_t^T(z,y) + \sum_{y\in A} a_t^T(z,y)\mathbb{E}\Big[\mathbf{1}_{\mathcal{E}(A)^c}\big|\tilde{\xi}_y(0) - \tilde{\xi}'_y(0)\big| + \mathbf{1}_{\mathcal{E}(A)^c}\big|\xi_y(0) - \xi'_y(0)\big|\Big] \\
&\leq 2M_1\sum_{y\in A^c} a_t^T(z,y) + 2\sqrt{\varepsilon}M_2^{1/2},
\end{aligned}$$

where $M_1 := \sup_{x,n}\int|\tilde{\xi}|^2 + |\xi|^2\,\mu_n(d\xi)$, $M_2 := \sup_{x,n}\int|\tilde{\xi}|^2 + |\xi|^2\,\mu_n(d\xi) < \infty$, we used (2.3) in the fourth line, and the Cauchy-Schwarz inequality in the last line. The last line can be made arbitrarily small by choosing A large, ε small. □

Acknowledgement The author would like to thank Iljana Zähle and Jochen Blath for various helpful discussions on this subject.

References

[1] Matthias Birkner, Particle systems with locally dependent branching: long-time behaviour, genealogy and critical parameters, PhD thesis, Johann Wolfgang Goethe-Universität Frankfurt, 2003. `http://publikationen.stub.uni-frankfurt.de/volltexte/2003/314/`

[2] Matthias Birkner, Iljana Zähle, A functional CLT for the occupation time of state-dependent branching random walk, to appear in *Ann. Probab.*, 2007.

[3] Thomas M. Liggett, *Interacting particle systems*, Springer-Verlag, 1985.

[4] Thomas M. Liggett and Frank Spitzer, Ergodic theorems for coupled random walks and other systems with locally interacting components, *Z. Wahrsch. Verw. Gebiete*, **56** (4), 443–468, 1981.

[5] Sylvie Méléard, Sylvie Roelly, An ergodic result for critical spatial branching processes, *Stochastic analysis and related topics (Silivri, 1990)*, 333–341, Progr. Probab., **31**, Birkhäuser, 1992.

[6] Andreas Stöckl, Anton Wakolbinger, On clan-recurrence and -transience in time stationary branching Brownian particle systems, *Measure-valued processes, stochastic partial differential equations, and interacting systems (Montreal, PQ, 1992)*, 213–219, CRM Proc. Lecture Notes, **5**, Amer. Math. Soc., 1994.