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Local limit theorems for ladder moments

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Abstract. Let $S_0 = 0, \{S_n\}_{n\geq 1}$ be a random walk generated by a sequence of i.i.d. random variables $X_1, X_2, ...$ and let $\tau^- := \min\{n \geq 1 : \ S_n \leq 0\}$ and $\tau^+ := \min\{n \geq 1 : \ S_n > 0\}$. Assuming that the distribution of X_1 belongs to the domain of attraction of an α-stable law, $\alpha \neq 1$, we study the asymptotic behavior of $\mathbb{P}(\tau^{\pm} = n)$ as $n \to \infty$.

1. Introduction and main result

Let $X, X_1, X_2, ...$ be a sequence of independent identically distributed random variables. Denote $S_0 = 0, S_n = X_1 + X_2 + ... + X_n$. We assume that

$$\sum_{n=1}^{\infty} \frac{1}{n}\mathbb{P}(S_n > 0) = \sum_{n=1}^{\infty} \frac{1}{n}\mathbb{P}(S_n \leq 0) = \infty.$$

This condition means that $\{S_n\}_{n\geq 0}$ is an oscillating random walk, and, in particular, the stopping moments

$$\tau^- := \min\{n \geq 1 : \ S_n \leq 0\} \quad \text{and} \quad \tau^+ := \min\{n \geq 1 : \ S_n > 0\}$$

are well-defined proper random variables. Furthermore, it follows from the Wiener-Hopf factorization (see, for example, [3, Theorem 8.9.1, p. 376]) that for all $z \in (0, 1)$,

$$1 - \mathbb{E}z^{\tau^-} = \exp\left\{-\sum_{n=1}^{\infty} \frac{z^n}{n}\mathbb{P}(S_n \leq 0)\right\} \tag{1}$$

and

$$1 - \mathbb{E}z^{\tau^+} = \exp\left\{-\sum_{n=1}^{\infty} \frac{z^n}{n}\mathbb{P}(S_n > 0)\right\}. \tag{2}$$

Rogozin [15] has shown that the Spitzer condition

$$n^{-1}\sum_{k=1}^{n} \mathbb{P}\left(S_k > 0\right) \to \rho \in (0, 1) \quad \text{as } n \to \infty \tag{3}$$

holds if and only if τ^+ belongs to the domain of attraction of a spectrally positive stable law with parameter ρ. Since (1) and (2) imply the equality

$$(1 - \mathbb{E}z^{\tau^+})(1 - \mathbb{E}z^{\tau^-}) = 1 - z \quad \text{for all } z \in (0, 1),$$

one can deduce from the Rogozin result that (3) holds if and only if there exists a function $l(n)$ slowly varying at infinity such that, as $n \to \infty$,

$$\mathbb{P}\left(\tau^- > n\right) \sim \frac{l(n)}{n^{1-\rho}}, \quad \mathbb{P}\left(\tau^+ > n\right) \sim \frac{1}{\Gamma(\rho)\Gamma(1-\rho)n^{\rho}l(n)}. \tag{4}$$

Doney [11] proved that the Spitzer condition is equivalent to

$$\mathbb{P}\left(S_n > 0\right) \to \rho \in (0, 1) \quad \text{as } n \to \infty. \tag{5}$$

Therefore, both relations in (4) are valid under condition (5).

To get a more detailed information about the asymptotic properties of $l(x)$ it is necessary to impose additional hypotheses on the distribution of X. Rogozin [15] has shown that $l(x)$ is asymptotically a constant if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n}\Big(\mathbb{P}\left(S_n > 0\right) - \rho\Big) < \infty. \tag{6}$$

It follows from the Spitzer-Rósen theorem (see [3, Theorem 8.9.23, p. 382]) that if $\mathbb{E}X^2$ is finite, then (6) holds with $\rho = 1/2$, and, consequently,

$$\mathbb{P}(\tau^{\pm} > n) \sim \frac{C^{\pm}}{n^{1/2}} \quad \text{as } n \to \infty, \tag{7}$$

where $C^{\pm}$ are positive constants. If $\mathbb{E}X^2 = \infty$ much less is known about the form of $l(x)$. For instance, if the distribution of X is symmetric, then, clearly,

$$\left|\mathbb{P}\left(S_n > 0\right) - \frac{1}{2}\right| = \frac{1}{2}\mathbb{P}\left(S_n = 0\right). \tag{8}$$

Furthermore, according to [14, Theorem III.9, p. 49], there exists $C > 0$ such that for all $n \geq 1$,

$$\mathbb{P}\left(S_n = 0\right) \leq \frac{C}{\sqrt{n}}.$$

By this estimate and (8) we conclude that (6) holds with $\rho = 1/2$. Thus, (7) is valid for all symmetric random walks. Assuming that $\mathbb{P}(X > x) = \left(x^{\alpha} l_0(x)\right)^{-1},\ x > 0$, with $1 < \alpha < 2$ and $l_0(x)$ slowly varying at infinity, Doney [8] established for a number of cases relationships between the asymptotic behavior of $l_0(x)$ and $l(x)$ at infinity.

The aim of the present paper is to study the asymptotic behavior of the probabilities $\mathbb{P}\left(\tau^{\pm} = n\right)$ as $n \to \infty$.

We assume throughout that the distribution of X is either non-lattice or arithmetic with span $h > 0$, i.e. the h is the maximal number such that the support of the distribution of X is contained in the set $\{kh,\ k = 0, \pm 1, \pm 2, ...\}$.

Let

$$\mathcal{A} := \{0 < \alpha < 1;\ |\beta| < 1\} \cup \{1 < \alpha < 2; |\beta| \leq 1\} \cup \{\alpha = 2, \beta = 0\}$$

be a subset in $\mathbb{R}^2$. For $(\alpha, \beta) \in \mathcal{A}$ we write $X \in \mathcal{D}(\alpha, \beta)$ if the distribution of X belongs to the domain of attraction of a stable law with characteristic function

$$\Psi(t) := \exp\left\{-c|t|^{\alpha}\left(1 - i\beta\frac{t}{|t|}\tan\frac{\pi\alpha}{2}\right)\right\}, c > 0, \tag{9}$$

and, in addition, $\mathbb{E}X = 0$ if $1 < \alpha \leq 2$. One can show (see, for instance, [16]) that if $X \in \mathcal{D}(\alpha, \beta)$, then condition (5) holds with

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha}\arctan\left(\beta\tan\frac{\pi\alpha}{2}\right) \in (0, 1). \tag{10}$$

Here is our main result.

Theorem 1. *Assume* $X \in \mathcal{D}(\alpha,\beta)$*. If* $\alpha \le 2$ *and* $\beta < 1$*, then, as* $n \to \infty$*,*

$$\mathbb{P}\left(\tau^- = n\right) = (1-\rho)\frac{l(n)}{n^{2-\rho}}(1+o(1)). \tag{11}$$

In the case $\{1 < \alpha < 2,\ \beta = 1\}$ *equality* (11) *remains valid under the additional hypothesis*

$$\int_1^\infty \frac{F(-x)}{x(1-F(x))}dx < \infty. \tag{12}$$

Denote $T^- := \min\{n \ge 1 : S_n < 0\}$ and set

$$\Omega(z) = \exp\left\{\sum_{n=1}^\infty \frac{z^n}{n}\mathbb{P}(S_n = 0)\right\} =: \sum_{k=0}^\infty \omega_k z^k. \tag{13}$$

The next statement relates the asymptotic behavior of $\mathbb{P}(\tau^- = n)$ and $\mathbb{P}(T^- = n)$.

Theorem 2. *If* (11) *holds, then*

$$\lim_{n\to\infty} \frac{\mathbb{P}(T^- = n)}{\mathbb{P}(\tau^- = n)} = \Omega(1).$$

Applying Theorems 1 and 2 to the random walk $\{-S_n\}_{n\ge 0}$, one can easily find an asymptotic representation for $\mathbb{P}(\tau^+ = n)$:

Theorem 3. *Assume* $X \in \mathcal{D}(\alpha,\beta)$*. If* $\alpha \le 2$ *and* $\beta > -1$*, then, as* $n \to \infty$*,*

$$\mathbb{P}\left(\tau^+ = n\right) = \frac{\rho}{\Gamma(\rho)\Gamma(1-\rho)n^{1+\rho}l(n)}(1+o(1)). \tag{14}$$

In the case $\{1 < \alpha < 2,\ \beta = -1\}$ *equality* (14) *remains valid under the additional hypothesis*

$$\int_1^\infty \frac{1-F(x)}{xF(-x)}dx < \infty. \tag{15}$$

In some special cases the asymptotic behavior of $\mathbb{P}(\tau^\pm = n)$ as $n \to \infty$ is already known from the literature. Eppel [12] proved that if $\mathbb{E}X = 0$ and $\mathbb{E}X^2$ is finite, then

$$\mathbb{P}\left(\tau^\pm = n\right) \sim \frac{C^\pm}{n^{3/2}}. \tag{16}$$

Observe that in this case $\mathbb{E}X^2 < \infty$ implies $X \in \mathcal{D}(2,0)$.

Asymptotic representation (16) is valid for all continuous symmetric (implying $\rho = 1/2$ in (5)) random walks (see [13, Chapter XII, Section 7]). Note that the restriction $X \in \mathcal{D}(\alpha,\beta)$ is superfluous in this situation.

Recently Borovkov [2] has shown that if (3) is valid and

$$n^{1-\rho}\Big(\mathbb{P}(S_n > 0) - \rho\Big) \to const \in (-\infty,\infty) \quad \text{as } n \to \infty, \tag{17}$$

then (11) holds with $\ell(n) \equiv const \in (0,\infty)$. Proving the mentioned result Borovkov does not assume that the distribution of X is taken from the domain of attraction of a stable law. However, he gives no explanations how one can check the validity of (17) in the general situation.

Let $\chi^+ := S_{\tau^+}$ be the ascending ladder height. Alili and Doney [1, Remark 1, p. 98] have shown that (14) holds if $\mathbb{E}\chi^+$ is finite. By Theorem 3 of [9] the assumption $\mathbb{E}\chi^+ < \infty$ is equivalent to (15), i.e. for the case $\{1 < \alpha < 2, \beta = -1\}$ our Theorem 3 is (implicitly) contained in [1] . Alili and Doney analyzed the distribution of τ^+ only. Clearly, one can easily derive the statement of our Theorem 1 for the case $\{1 < \alpha < 2, \beta = 1\}$ from their result (for instance, applying Theorem 2). However, for these spectrally one-sided cases we present an alternative proof, which clarifies the "typical" behavior of the random walk on the events $\{\tau^\pm = n\}$. See Section 3.2 and Section 5 for more details.

2. Auxiliary results

2.1. **Notation.** In what follows we denote by $C, C_1, C_2, ...$ finite positive constants which may be *different* from formula to formula and by $l(x), l_1(x), l_2(x)...$ functions slowly varying at infinity which are, as a rule, *fixed.*

For $x \geq 0$ let

$$B_n(x) := \mathbb{P}\left(S_n \in (0, x]; \tau^- > n\right),$$
$$b_n(x) := B_n(x+1) - B_n(x) = \mathbb{P}\left(S_n \in (x, x+1]; \tau^- > n\right).$$

Introduce the renewal function

$$H(x) := 1 + \sum_{k=1}^{\infty} \mathbb{P}\left(\chi_1^+ + ... + \chi_k^+ \leq x\right), \ x \geq 0, \ H(x) = 0, x < 0,$$

where $\{\chi_i^+\}_{i\geq 1}$ is a sequence of i.i.d. random variables distributed the same as χ^+. Observe that by the duality principle for random walks for $x \geq 0$

$$\begin{aligned} 1 + \sum_{j=1}^{\infty} B_j(x) &= 1 + \sum_{j=1}^{\infty} \mathbb{P}\left(S_j \in (0, x]; \tau^- > j\right) \\ &= 1 + \sum_{j=1}^{\infty} \mathbb{P}\left(S_j \in (0, x]; S_j > S_0, S_j > S_1, ..., S_j > S_{j-1}\right) \\ &= H(x). \end{aligned} \tag{18}$$

In the sequel we deal rather often with slowly varying functions and, following Doney [9], say that a slowly varying function $l^*(x)$ is an α-conjugate of a slowly varying function $l^{**}(x)$ when the following relations are valid

$$y \sim x^\alpha l^*(x) \text{ as } x \to \infty \text{ if and only if } x \sim y^{1/\alpha} l^{**}(y).$$

It is known that if $X \in \mathcal{D}(\alpha, \beta)$ with $\alpha \in (0, 2)$, and $F(x) := \mathbb{P}(X \leq x)$, then

$$1 - F(x) + F(-x) \sim \frac{1}{x^\alpha l_0(x)} \quad \text{as } x \to \infty, \tag{19}$$

where $l_0(x)$ is a function slowly varying at infinity. Besides, for $\alpha \in (0, 2)$,

$$\frac{F(-x)}{1 - F(x) + F(-x)} \to q, \quad \frac{1 - F(x)}{1 - F(x) + F(-x)} \to p \quad \text{as } x \to \infty, \tag{20}$$

with $p+q=1$ and $\beta=p-q$ in (9). Let $\{c_n\}_{n\geq 1}$ be a sequence specified by the relation

$$c_n := \inf\left\{x\geq 0 : 1-F(x)+F(-x)\leq n^{-1}\right\}. \tag{21}$$

In view of (19) this sequence is regularly varying at infinity with index α^{-1}, i.e.

$$c_n = n^{1/\alpha} l_1(n), \tag{22}$$

where $l_1(x)$ is a slowly varying function being an α-conjugate of $l_0(x)$:

$$c_n^\alpha l_0(c_n) \sim n \quad \text{as } n\to\infty. \tag{23}$$

Moreover,

$$\frac{S_n}{c_n} \xrightarrow{d} Y_\alpha \quad \text{as } n\to\infty,$$

where Y_α is a random variable obeying an $\alpha-$stable law.

For the case $\alpha=2$ the normalizing sequence $\{c_n\}_{n\geq 1}$ requires a special description. Let $V(x)=\int_{-x}^{x} y^2 dF(x)$ be the truncated variance of X. Clearly, $\liminf_{x\to\infty} V(x) > 0$ for every nondegenerate random variable X. Furthermore, it is known ([13], Chapter XVII, Section 5) that $X\in\mathcal{D}(2,0)$ if and only if $V(x)$ varies slowly at infinity. In this case the normalizing sequence c_n satisfies

$$\frac{V(c_n)}{c_n^2} \sim \frac{C}{n} \quad \text{as } n\to\infty. \tag{24}$$

The last relation means that (22) holds with $\alpha=2$ and $l_1(x)$ is a 2-conjugate of $1/V(x)$. Besides,

$$\lim_{x\to\infty} \frac{x^2(1-F(x)+F(-x))}{V(x)} = 0. \tag{25}$$

2.2. **Basic lemmas.** Now we formulate a number of results concerning the distributions of the random variables τ^-,τ^+ and χ^+. Recall that a random variable ζ is called relatively stable if there exists a nonrandom sequence $d_n\to\infty$ as $n\to\infty$ such that

$$\frac{1}{d_n}\sum_{k=1}^{n}\zeta_k \xrightarrow{p} 1 \text{ as } n\to\infty,$$

where $\zeta_k \stackrel{d}{=} \zeta,\ k=1,2,...$ and are independent.

Lemma 4. *(see [15] and [10, Theorem 9]) Assume $X\in\mathcal{D}(\alpha,\beta)$. Then, as $x\to\infty$,*

$$\mathbb{P}\left(\chi^+ > x\right) \sim \frac{1}{x^{\alpha\rho} l_2(x)} \text{ if } \alpha\rho<1, \tag{26}$$

and χ^+ is relatively stable if $\alpha\rho=1$.

Lemma 5. *Suppose $X\in\mathcal{D}(\alpha,\beta)$. If $\alpha\rho<1$, then, as $x\to\infty$,*

$$H(x) \sim \frac{x^{\alpha\rho} l_2(x)}{\Gamma(1-\alpha\rho)\Gamma(1+\alpha\rho)}. \tag{27}$$

If $\alpha\rho=1$, then, as $x\to\infty$,

$$H(x)\sim x l_3(x), \tag{28}$$

where

$$l_3(x) := \left(\int_0^x \mathbb{P}\left(\chi^+ > y\right) dy \right)^{-1}, \quad x > 0.$$

In addition, there exists a constant $C > 0$ such that in both cases

$$H(c_n) \le C n^{\rho} l(n) \quad \textit{for all } n \ge 1. \tag{29}$$

Proof. If $\alpha\rho < 1$, then by [13, Chapter XIV, formula (3.4)]

$$H(x) \sim \frac{1}{\Gamma(1-\alpha\rho)\Gamma(1+\alpha\rho)} \frac{1}{\mathbb{P}(\chi^+ > x)} \quad \text{as } x \to \infty.$$

Hence, recalling (26), we obtain (27).

If $\alpha\rho = 1$, then (28) follows from Theorem 2 in [15].

Let us demonstrate the validity of (29). We know from [15] (see also [7]) that $\tau^+ \in \mathcal{D}(\rho, 1)$ under the conditions of the lemma and, in addition, $\chi^+ \in \mathcal{D}(\alpha\rho, 1)$ if $\alpha\rho < 1$. This means, in particular, that for sequences $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$ specified by

$$\mathbb{P}(\tau^+ > a_n) \sim \frac{1}{n} \quad \text{and} \quad \mathbb{P}(\chi^+ > b_n) \sim \frac{1}{n} \quad \text{as } n \to \infty, \tag{30}$$

and vectors $\{(\tau_k^+, \chi_k^+)\}_{k\ge1}$, being independent copies of (τ^+, χ^+), we have

$$\frac{1}{a_n} \sum_{k=1}^n \tau_k^+ \xrightarrow{d} Y_\rho \quad \text{and} \quad \frac{1}{b_n} \sum_{k=1}^n \chi_k^+ \xrightarrow{d} Y_{\alpha\rho} \quad \text{as } n \to \infty. \tag{31}$$

Moreover, it was established by Doney (see Lemma in [10], p. 358) that

$$b_n \sim C c_{[a_n]} \text{ as } n \to \infty, \tag{32}$$

where $[x]$ stands for the integer part of x. Therefore, $c_n \sim C b_{[a^{-1}(n)]}$, where, with a slight abuse of notation, $a^{-1}(n)$ is the inverse function to a_n. Hence, on account of (30),

$$\begin{aligned} \mathbb{P}(\chi^+ > c_n) &\sim C_1 \mathbb{P}(\chi^+ > b_{[a^{-1}(n)]}) \sim \frac{C_1}{a^{-1}(n)} \\ &\sim C_2 \mathbb{P}(\tau^+ > a_{[a^{-1}(n)]}) \sim C_3 \mathbb{P}(\tau^+ > n) \sim \frac{C_4}{n^{\rho} l(n)}. \end{aligned} \tag{33}$$

This proves (29) for $\alpha\rho < 1$.

If $\alpha\rho = 1$, then, instead of the second equivalence in (30), one should define b_n by

$$\frac{1}{b_n} \int_0^{b_n} \mathbb{P}(\chi^+ > y) dy \sim \frac{1}{n} \quad \text{as } n \to \infty$$

(see [15, p. 595]). In this case the second convergence in (31) transforms to

$$\frac{1}{b_n} \sum_{k=1}^n \chi_k^+ \xrightarrow{p} 1 \quad \text{as } n \to \infty,$$

while (33) should be changed to

$$\frac{1}{c_n}\int_0^{c_n}\mathbb{P}(\chi^+>y)dy\sim\frac{C_1}{b_{[a^{-1}(n)]}}\int_0^{b_{[a^{-1}(n)]}}\mathbb{P}(\chi^+>y)dy\sim\frac{C_1}{a^{-1}(n)}$$
$$\sim C_1\mathbb{P}(\tau^+>a_{[a^{-1}(n)]})\sim C_2\mathbb{P}(\tau^+>n)\sim\frac{C_3}{n^\rho l(n)}.$$

The lemma is proved. □

The next result is a part of Corollary 3 in [9].

Lemma 6. *Assume* $X\in\mathcal{D}(\alpha,1)$ *with* $1<\alpha<2$ *(implying* $\rho=1-\alpha^{-1}$*). Then*

$$\mathbb{P}(\tau^->n)\sim\frac{C}{c_n}\sim\frac{C}{n^{1/\alpha}l_1(n)}\qquad \text{as } n\to\infty$$

if and only if

$$\int_1^\infty\frac{F(-x)}{x(1-F(x))}dx<\infty.$$

Now we prove a useful result which may be viewed as a statement concerning “small“ deviations of S_n on the set $\{\tau^->n\}$.

Let h be the span and $g_{\alpha,\beta}(x)$ be the density of a stable distribution with parameters α and β in (9) (we agree to consider $h=0$ for non-lattice distributions). For a set A taken from the Borel σ-algebra on $(0,\infty)$ denote

$$\mu(A)=g_{\alpha,\beta}(0)\int_A H(x-h)\nu(dx),$$

where ν is the counting measure on $\{h,2h,3h,\ldots\}$ in the arithmetic case and the Lebesgue measure on $(0,\infty)$ in the non-lattice case.

Lemma 7. *Suppose* $X\in\mathcal{D}(\alpha,\beta)$*. Then*

$$\lim_{n\to\infty}nc_n\mathbb{P}(S_n\in A;\,\tau^->n)=\mu(A)\tag{34}$$

for any A *taken from the Borel* σ*-algebra on* $(0,\infty)$*.*

Proof. Assume first that the distribution of X is non-lattice. Using the Stone local limit theorem (see, for instance, [3, Section 8.4, p. 351]) it is not difficult to show that for $\lambda>0$,

$$\lim_{n\to\infty}c_n\mathbb{E}\left(e^{-\lambda S_n};\,S_n>0\right)=g_{\alpha,\beta}(0)\int_0^\infty e^{-\lambda y}dy=\frac{g_{\alpha,\beta}(0)}{\lambda}.\tag{35}$$

Set

$$G(\lambda):=\sum_{n=1}^\infty\frac{\mathbb{E}\left(e^{-\lambda S_n};\,S_n>0\right)}{n}\tag{36}$$

and specify a sequence of measures

$$\mu_n(dx):=nc_n\mathbb{P}(S_n\in dx;\,\tau^->n),\quad n\ge1.$$

Since $\{c_n\}_{n\geq 1}$ varies regularly and (35) is valid, applying Theorem 2 from [6] to the equality

$$\sum_{n=0}^{\infty} z^n \mathbb{E}\left(e^{-\lambda S_n};\, \tau^- > n\right) = \exp\left\{\sum_{n=1}^{\infty} \frac{z^n}{n}\mathbb{E}\left(e^{-\lambda S_n};\, S_n > 0\right)\right\} \tag{37}$$

shows that for all $\lambda > 0$,

$$\begin{aligned}\lim_{n\to\infty} nc_n \mathbb{E}\left(e^{-\lambda S_n};\, \tau^- > n\right) &= \lim_{n\to\infty}\int_0^{\infty} e^{-\lambda x}\mu_n(dx) \\ &= \frac{g_{\alpha,\beta}(0)}{\lambda}\exp\left\{G(\lambda)\right\}.\end{aligned} \tag{38}$$

It follows from (37) that

$$\begin{aligned}\frac{g_{\alpha,\beta}(0)}{\lambda}\exp\left\{G(\lambda)\right\} &= \frac{g_{\alpha,\beta}(0)}{\lambda}\Big(1+\sum_{k=1}^{\infty}\mathbb{E}\left(e^{-\lambda S_k};\, \tau^- > k\right)\Big) \\ &= \frac{g_{\alpha,\beta}(0)}{\lambda} + \frac{g_{\alpha,\beta}(0)}{\lambda}\int_0^{\infty} e^{-\lambda x}\left(\sum_{k=1}^{\infty}\mathbb{P}\left(S_k\in dx;\, \tau^- > k\right)\right) \\ &= \frac{g_{\alpha,\beta}(0)}{\lambda} + \frac{g_{\alpha,\beta}(0)}{\lambda}\int_0^{\infty} e^{-\lambda x}\left(\sum_{j=1}^{\infty}\mathbb{P}(\chi_1^+ + \ldots + \chi_j^+ \in dx)\right),\end{aligned}$$

where at the last step we have used the duality principle. Integrating by parts and recalling the definition of $H(x)$, we get

$$\begin{aligned}\frac{g_{\alpha,\beta}(0)}{\lambda}\exp\left\{G(\lambda)\right\} &= \frac{g_{\alpha,\beta}(0)}{\lambda} + g_{\alpha,\beta}(0)\int_0^{\infty} e^{-\lambda x}(H(x)-1)dx \\ &= g_{\alpha,\beta}(0)\int_0^{\infty} e^{-\lambda x}H(x)dx.\end{aligned} \tag{39}$$

Combining (38) and (39) and using the continuity theorem for Laplace transforms, we obtain (34) for non-lattice distributions.

In the arithmetic case we have by the Gnedenko local limit theorem

$$\lim_{n\to\infty} c_n\mathbb{E}\left(e^{-\lambda S_n};\, S_n > 0\right) = g_{\alpha,\beta}(0)\sum_{k=1}^{\infty} e^{-\lambda hk} = \frac{g_{\alpha,\beta}(0)e^{-\lambda h}}{1-e^{-\lambda h}}. \tag{40}$$

Proceeding as by the derivation of (39), we obtain

$$\begin{aligned}\frac{g_{\alpha,\beta}(0)e^{-\lambda h}}{1-e^{-\lambda h}}\exp\left\{G(\lambda)\right\} &= \frac{g_{\alpha,\beta}(0)e^{-\lambda h}}{1-e^{-\lambda h}}\Big(1+\sum_{k=1}^{\infty}\mathbb{E}\left(e^{-\lambda S_k};\, \tau^- > k\right)\Big) \\ &= \frac{g_{\alpha,\beta}(0)e^{-\lambda h}}{1-e^{-\lambda h}} + \frac{g_{\alpha,\beta}(0)e^{-\lambda h}}{1-e^{-\lambda h}}\sum_{j=1}^{\infty} e^{-\lambda hj}\left(H(hj)-H(hj-h)\right) \\ &= g_{\alpha,\beta}(0)e^{-\lambda h}\sum_{j=0}^{\infty} e^{-\lambda hj}H(hj) = g_{\alpha,\beta}(0)\sum_{k=1}^{\infty} e^{-\lambda hk}H(hk-h).\end{aligned}$$

This, together with (40), finishes the proof of the lemma. □

Lemma 8. *Under the conditions of Theorem 1 for any* $\alpha \in (0,2)$ *there exists* $C>0$ *such that for all* $y>0$ *and all* $n \geq 1$*,*

$$b_n(y) \leq \frac{C}{c_n}\frac{l(n)}{n^{1-\rho}} \tag{41}$$

and

$$B_n(y) \leq \frac{C\,(y+1)}{c_n}\frac{l(n)}{n^{1-\rho}}. \tag{42}$$

Proof. For $n=1$ the statement of the lemma is obvious. Let $\{S_n^*\}_{n\geq 0}$ be a random walk distributed as $\{S_n\}_{n\geq 0}$ and independent of it. One can easily check that for each $n \geq 2$,

$$\begin{aligned}
b_n(y) &= \mathbb{P}\left(y < S_n \leq y+1; \tau^- > n\right)\\
&= \int_0^\infty \mathbb{P}\Big(y - S_{[n/2]} < S_n - S_{[n/2]} \leq y+1-S_{[n/2]}; S_{[n/2]} \in dz; \tau^- > n\Big)\\
&\leq \int_0^\infty \mathbb{P}\Big(y - z < S^*_{n-[n/2]} \leq y+1-z; S_{[n/2]} \in dz; \tau^- > [n/2]\Big)\\
&\leq \mathbb{P}\Big(\tau^- > [n/2]\Big) \sup_z \mathbb{P}\Big(z < S^*_{n-[n/2]} \leq z+1\Big).
\end{aligned} \tag{43}$$

Since the density of any α-stable law is bounded, it follows from the Gnedenko and Stone local limit theorems that if the distribution of X is either arithmetic or non-lattice, then there exists a constant $C>0$ such that for all $n \geq 1$ and all $z \geq 0$,

$$\mathbb{P}\left(S_n \in (z, z+\Delta]\right) \leq \frac{C\Delta}{c_n}. \tag{44}$$

Hence it follows, in particular, that, for any $z>0$,

$$\mathbb{P}\left(S_n \in (0,z]\right) \leq \frac{C(z+1)}{c_n}. \tag{45}$$

Substituting (44) into (43), and recalling (22) and properties of regularly varying functions, we get (41). Estimate (42) follows from (41) by summation. □

Lemma 9. *Under the conditions of Theorem 1 for any* $\alpha \in (1,2]$ *there exists* $C>0$ *such that for all* $n \geq 1$ *and all* $x>0$*,*

$$b_n(x) \leq C\left(\frac{H(x+1)}{nc_n} + \frac{l(n)}{n^{1-\rho}}\frac{x+1}{c_n^2}\right) \tag{46}$$

and

$$B_n(x) \leq C\left(\frac{(x+1)\,H(x+1)}{nc_n} + \frac{l(n)}{n^{1-\rho}}\frac{(x+1)^2}{c_n^2}\right). \tag{47}$$

Proof. According to formula (5) in [12],

$$nB_n(x) = \mathbb{P}\left(S_n \in (0,x]\right) + \sum_{k=1}^{n-1}\int_0^x B_{n-k}(x-y)\mathbb{P}\left(S_k \in dy\right). \tag{48}$$

Hence we get

$$nb_n(x) = \mathbb{P}(S_n \in (x, x+1]) + \sum_{k=1}^{n-1} \int_0^x b_{n-k}(x-y) \mathbb{P}(S_k \in dy)$$
$$+ \sum_{k=1}^{n-1} \int_x^{x+1} B_{n-k}(x+1-y) \mathbb{P}(S_k \in dy). \quad (49)$$

Using (41), (45), (22), the inequality $1/\alpha < 1$ and properties of slowly varying functions, we deduce

$$\begin{aligned}\sum_{k=1}^{[n/2]} \int_0^x b_{n-k}(x-y) \mathbb{P}(S_k \in dy) &\le C \sum_{k=1}^{[n/2]} \frac{l(n-k)}{c_{n-k}(n-k)^{1-\rho}} \mathbb{P}(S_k \in [0,x]) \\ &\le C_1 (x+1) \sum_{k=1}^{[n/2]} \frac{1}{c_k} \frac{l(n-k)}{c_{n-k}(n-k)^{1-\rho}} \\ &\le C_2 \frac{x+1}{c_n} \frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} \frac{1}{c_k} \\ &\le C_3 (x+1) \frac{n^\rho l(n)}{c_n^2}. \end{aligned} \quad (50)$$

On the other hand, in view of (44) and monotonicity of $B_k(x)$ in x we conclude (assuming that x is integer without loss of generality and letting $B_k(-1) = 0$ and $H(-1) = 0$) that

$$\begin{aligned}&\sum_{k=[n/2]+1}^{n} \int_0^x b_{n-k}(x-y) \mathbb{P}(S_k \in dy) \\ &\le \sum_{k=[n/2]+1}^{n} \sum_{j=0}^{x} (B_{n-k}(x-j+1) - B_{n-k}(x-j-1)) \mathbb{P}(S_k \in (j, j+1]) \\ &\le \sum_{k=[n/2]+1}^{n} \sum_{j=0}^{x} (B_{n-k}(x-j+1) - B_{n-k}(x-j-1)) \frac{C}{c_k} \\ &\le \frac{C}{c_n} \sum_{j=0}^{x} \sum_{k=0}^{\infty} (B_k(x-j+1) - B_k(x-j-1)) \\ &= \frac{C}{c_n} \sum_{j=0}^{x} (H(x-j+1) - H(x-j-1)) \\ &\le \frac{C}{c_n} (H(x) + H(x+1)) \le \frac{2C}{c_n} H(x+1),\end{aligned}$$

where for the intermediate equality we have used (18). This gives

$$\sum_{k=[n/2]+1}^{n} \int_0^x b_{n-k}(x-y) \mathbb{P}(S_k \in dy) \le \frac{C}{c_n} H(x+1). \quad (51)$$

Since $x \mapsto B_n(x)$ increases for every n,

$$\sum_{k=1}^{n-1} \int_x^{x+1} B_{n-k}(x+1-y)\mathbb{P}\left(S_k \in dy\right) \le \sum_{k=1}^{n-1} B_{n-k}(1)\mathbb{P}(S_k \in (x, x+1]). \tag{52}$$

Further, in view of (42) and (44) we have

$$\sum_{k=1}^{[n/2]} B_{n-k}(1)\mathbb{P}(S_k \in (x, x+1]) \le \frac{C_1}{c_n}\frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} \frac{1}{c_k} \le \frac{C_2 n^\rho l(n)}{c_n^2}. \tag{53}$$

Using (44) once again yields

$$\sum_{k=[n/2]+1}^{n-1} B_{n-k}(1)\mathbb{P}(S_k \in (x, x+1]) \le \frac{C}{c_n} \sum_{k=[n/2]+1}^{n-1} B_{n-k}(1) \le \frac{C}{c_n} H(1). \tag{54}$$

Substituting (53) and (54) into the right hand side of (52), we obtain the upper bound

$$\sum_{k=1}^{n-1} \int_x^{x+1} B_{n-k}(x+1-y)\mathbb{P}\left(S_k \in dy\right) \le C\Big(\frac{n^\rho l(n)}{c_n^2} + \frac{1}{c_n}\Big). \tag{55}$$

Combining (50), (51), (55), (44) and (49) proves (46). Observing that $H(x)$ is nondecreasing and integrating (46), we get estimate (47). □

To prove Theorem 1 in the case $\alpha = 2$ we need the following technical lemma which may be known from the literature.

Lemma 10. *Let $w(n)$ be a monotone increasing function. If, for some $\gamma > 0$, there exist slowly varying functions $l^*(n)$ and $l^{**}(n)$ such that, as $n \to \infty$,*

$$\sum_{k=n}^{\infty} \frac{w(k)}{k^{\gamma+1} l^*(k)} \sim \frac{1}{n^\gamma l^{**}(n)},$$

then, as $n \to \infty$,

$$w(n) \sim \gamma \frac{l^*(n)}{l^{**}(n)}.$$

Proof. Let, for this lemma only, $r_i(n), n = 1, 2, ...; i = 1, 2, 3, 4$ be sequences of real numbers vanishing as $n \to \infty$. For $\Delta \in (0, 1)$ we have by monotonicity of $w(n)$ and properties of slowly varying functions

$$\begin{aligned} w([\Delta n]) \sum_{k=[\Delta n]}^{n} \frac{1}{k^{\gamma+1} l^*(k)} &= w([\Delta n]) \frac{1 + r_2(n)}{\gamma n^\gamma l^*(n)} \left(\Delta^{-\gamma} - 1\right) \\ &\le \sum_{k=[\Delta n]}^{n} \frac{w(k)}{k^{\gamma+1} l^*(k)} = \frac{1 + r_1(n)}{n^\gamma l^{**}(n)} \left(\Delta^{-\gamma} - 1\right) \\ &\le w(n) \sum_{k=[\Delta n]}^{n} \frac{1}{k^{\gamma+1} l^*(k)} \\ &= w(n) \frac{1 + r_2(n)}{\gamma n^\gamma l^*(n)} \left(\Delta^{-\gamma} - 1\right). \end{aligned}$$

Hence it follows that

$$w([\Delta n]) \le \frac{1+r_1(n)}{1+r_2(n)} \frac{\gamma l^*(n)}{l^{**}(n)} \le w(n)$$

and, therefore,

$$\frac{1+r_1(n)}{1+r_2(n)} \frac{\gamma l^*(n)}{l^{**}(n)} \le w(n) \le \frac{1+r_3([n\Delta^{-1}])}{1+r_4([n\Delta^{-1}])} \frac{\gamma l^*([n\Delta^{-1}])}{l^{**}([n\Delta^{-1}])}.$$

Since l^* and l^{**} are slowly varying functions, we get

$$\lim_{n\to\infty} \frac{w(n) l^{**}(n)}{\gamma l^*(n)} = 1,$$

as desired. □

Remark 11. By the same arguments one can show that if $w(x)$ is a monotone increasing function and, for some $\gamma > 0$, there exist slowly varying functions $l^*(x)$ and $l^{**}(x)$ such that, as $x \to \infty$,

$$\int_x^\infty \frac{w(y)dy}{y^{\gamma+1} l^*(y)} \sim \frac{1}{x^\gamma l^{**}(x)},$$

then, as $x \to \infty$,

$$w(x) \sim \gamma \frac{l^*(x)}{l^{**}(x)}.$$

3. Proof of Theorem 1

3.1. **Proof of Theorem 1 for** $\{0 < \alpha < 2,\ \beta < 1\} \cap \{\alpha \ne 1\}$**.** For a fixed $\varepsilon \in (0,1)$ write

$$\begin{aligned}
\mathbb{P}\left(\tau^- = n\right) &= \mathbb{P}\left(S_n \le 0; \tau^- > n-1\right) \\
&= \int_0^\infty \mathbb{P}\left(X_n \le -y\right) \mathbb{P}\left(S_{n-1} \in dy; \tau^- > n-1\right) \\
&= \int_0^{\varepsilon c_n} \mathbb{P}\left(X \le -y\right) \mathbb{P}\left(S_{n-1} \in dy; \tau^- > n-1\right) \\
&\quad + \int_\varepsilon^\infty \mathbb{P}\left(X \le -y c_n\right) \mathbb{P}\left(S_{n-1} \in c_n dy; \tau^- > n-1\right).
\end{aligned}$$

We evaluate the last two integrals separately.

We know from (19) and (20) that if $X \in \mathcal{D}(\alpha, \beta)$ with $0 < \alpha < 2$ and $\beta < 1$, then, for a $q \in (0, 1]$,

$$\mathbb{P}\left(X \le -y\right) \sim \frac{q}{y^\alpha l_0(y)} \quad \text{as } y \to \infty, \tag{56}$$

and, according to our construction,

$$\mathbb{P}\left(X \le -c_n\right) \sim q n^{-1} \quad \text{as } n \to \infty.$$

Moreover, for any $\varepsilon > 0$,

$$\frac{\mathbb{P}\left(X \le -y c_n\right)}{\mathbb{P}\left(X \le -c_n\right)} \to y^{-\alpha} \quad \text{as } n \to \infty, \tag{57}$$

uniformly in $y\in(\varepsilon,\infty)$. On the other hand, if $M_\alpha^+(t), 0\le t\le 1$, is the Levy meander of order $\alpha\neq 1$ and the conditions of Theorem 1 are valid, then (see [10])

$$\left\{\frac{S_n}{c_n}\,\middle|\tau^->n\right\}\overset{d}{\to} M_\alpha^+:=M_\alpha^+(1)\quad\text{as } n\to\infty. \tag{58}$$

We show that

$$\int_0^\infty \frac{\mathbb{P}\left(M_\alpha^+\in dy\right)}{y^\alpha}<\infty. \tag{59}$$

Indeed, if this is not the case, for any N one can find $\varepsilon_N\in(0,1)$ such that

$$\int_{\varepsilon_N}^{1/\varepsilon_N} \frac{\mathbb{P}\left(M_\alpha^+\in dy\right)}{y^\alpha}>2N.$$

This yields

$$\begin{aligned}&\lim_{n\to\infty}\int_{\varepsilon_N}^{1/\varepsilon_N}\frac{\mathbb{P}\left(X\le -yc_n\right)}{\mathbb{P}\left(X\le -c_n\right)}\mathbb{P}\left(\frac{S_{n-1}}{c_n}\in dy\,|\,\tau^->n-1\right)\\&=\int_{\varepsilon_N}^{1/\varepsilon_N}\frac{\mathbb{P}\left(M_\alpha^+\in dy\right)}{y^\alpha}>2N.\end{aligned}$$

By (4) we have, as $n\to\infty$,

$$\begin{aligned}\frac{2l(n)}{n^{1-\rho}}&\ge\mathbb{P}\left(\tau^->n\right)=\sum_{k=n+1}^\infty\mathbb{P}\left(\tau^-=k\right)\\&\ge\sum_{k=n+1}^\infty\mathbb{P}\left(X_k\le -c_k\right)\mathbb{P}\left(\tau^->k-1\right)\times\\&\qquad\qquad\int_{\varepsilon_N}^{1/\varepsilon_N}\frac{\mathbb{P}\left(X_k\le -yc_k\right)}{\mathbb{P}\left(X_k\le -c_k\right)}\mathbb{P}\left(\frac{S_{k-1}}{c_k}\in dy\,|\,\tau^->k-1\right)\\&\ge N\sum_{k=n+1}^\infty\mathbb{P}\left(X_k\le -c_k\right)\mathbb{P}\left(\tau^->k-1\right)\sim N\sum_{k=n+1}^\infty\frac{ql(k)}{k^{2-\rho}}\sim\frac{N}{1-\rho}\frac{ql(n)}{n^{1-\rho}},\end{aligned}$$

leading to a contradiction for $N>2(1-\rho)q^{-1}$. Thus, (59) is established.

It easily follows from (57) and (58) that, as $n\to\infty$,

$$\begin{aligned}&\int_\varepsilon^\infty\mathbb{P}\left(X\le -yc_n\right)\mathbb{P}\left(S_{n-1}\in c_ndy;\tau^->n-1\right)\\&=\mathbb{P}\left(X\le -c_n\right)\mathbb{P}\left(\tau^->n-1\right)\int_\varepsilon^\infty\frac{\mathbb{P}\left(X\le -yc_n\right)}{\mathbb{P}\left(X\le -c_n\right)}\mathbb{P}\left(\frac{S_{n-1}}{c_n}\in dy\,|\,\tau^->n-1\right)\\&\sim\frac{ql(n)}{n^{2-\rho}}\int_\varepsilon^\infty\frac{\mathbb{P}\left(X\le -yc_n\right)}{\mathbb{P}\left(X\le -c_n\right)}\mathbb{P}\left(\frac{S_{n-1}}{c_n}\in dy\,|\,\tau^->n-1\right)\\&\sim\frac{ql(n)}{n^{2-\rho}}\int_\varepsilon^\infty\frac{\mathbb{P}\left(M_\alpha^+\in dy\right)}{y^\alpha}.\end{aligned} \tag{60}$$

Taking into account (59), we obtain

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{n^{2-\rho}}{ql(n)}\int_{\varepsilon}^{\infty}\mathbb{P}\left(X\le -yc_n\right)\mathbb{P}\left(S_{n-1}\in c_n dy;\tau^->n-1\right)$$
$$=\int_0^{\infty}\frac{\mathbb{P}\left(M_\alpha^+\in dy\right)}{y^\alpha}. \tag{61}$$

To complete the proof of Theorem 1 it remains to demonstrate that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{n^{2-\rho}}{l(n)}\int_0^{\varepsilon c_n}\mathbb{P}\left(X\le -y\right)\mathbb{P}\left(S_{n-1}\in dy;\tau^->n-1\right)=0. \tag{62}$$

To this aim we observe that

$$\int_0^{\varepsilon c_n}\mathbb{P}\left(X\le -y\right)\mathbb{P}\left(S_{n-1}\in dy;\tau^->n-1\right)$$
$$\le\sum_{j=0}^{[\varepsilon c_n]+1}\mathbb{P}\left(X\le -j\right)b_{n-1}(j)=:R(\varepsilon c_n)$$

and evaluate $R(\varepsilon c_n)$ separately for the following three cases:

(i) $0<\alpha<1$, $|\beta|<1$;

(ii) $1<\alpha<2$, $|\beta|<1$;

(iii) $1<\alpha<2$, $\beta=-1$.

(i). In view of (41), (19) and properties of regularly varying functions with index $\alpha\in(0,1)$ we have

$$\begin{aligned}R(\varepsilon c_n)&\le C\frac{1}{c_n}\frac{l(n)}{n^{1-\rho}}\sum_{j=0}^{[\varepsilon c_n]+1}\mathbb{P}\left(X\le -j\right)\\&\le C_1\frac{1}{c_n}\frac{l(n)}{n^{1-\rho}}\varepsilon c_n\mathbb{P}\left(X\le -\varepsilon c_n\right)\\&\le C_2\frac{l(n)}{n^{1-\rho}}\varepsilon^{1-\alpha}\frac{l_0(c_n)}{l_0(\varepsilon c_n)}\mathbb{P}\left(X\le -c_n\right)\\&\le C_3\frac{l(n)}{n^{2-\rho}}\varepsilon^{1-\alpha}\frac{l_0(c_n)}{l_0(\varepsilon c_n)}\le C_4\frac{l(n)}{n^{2-\rho}}\varepsilon^{1-\alpha-\delta}\end{aligned} \tag{63}$$

for any fixed $\delta\in(0,1-\alpha)$ and all sufficiently large n. At the last step we have used the fact that for every slowly varying function $l^*(x)$ and every $\delta>0$ there exists a constant C_δ such that

$$\frac{l^*(x)}{l^*(ax)}\le C_\delta\max\{a^\delta,a^{-\delta}\}\quad\text{for all } a,x>0. \tag{64}$$

(ii) In view of (46), equivalences (27), (19), and estimate (64) with any fixed $\delta \in (0, \min\{2-\alpha, 1-\alpha(1-\rho)\})$, we have for all sufficiently large n,

$$\begin{aligned}
R(\varepsilon c_n) &\le C \sum_{j=1}^{[\varepsilon c_n]+1} \frac{1}{j^\alpha l_0(j)} \left(\frac{j^{\alpha\rho} l_2(j)+1}{n c_n} + \frac{l(n)}{n^{1-\rho}} \frac{j+1}{c_n^2} \right) \\
&\le C_1 \frac{1}{n c_n} \sum_{j=1}^{[\varepsilon c_n]+1} \frac{l_2(j)}{j^{\alpha(1-\rho)} l_0(j)} + C \frac{l(n)}{n^{1-\rho}} \frac{1}{c_n^2} \sum_{j=1}^{[\varepsilon c_n]+1} \frac{1}{j^{\alpha-1} l_0(j)} \\
&\le C_2 \frac{1}{n c_n} (\varepsilon c_n)^{1-\alpha(1-\rho)} \frac{l_2(\varepsilon c_n)}{l_0(\varepsilon c_n)} + C_3 \frac{l(n)}{n^{1-\rho}} \frac{1}{c_n^2} \frac{(\varepsilon c_n)^{2-\alpha}}{l_0(\varepsilon c_n)} \\
&\le C_4 \frac{1}{n c_n} (\varepsilon c_n)^{1-\alpha(1-\rho)-\delta} + C_5 \frac{l(n)}{n^{1-\rho}} \frac{1}{c_n^2} (\varepsilon c_n)^{2-\alpha-\delta}.
\end{aligned}$$

Hence on account of (22) we conclude that

$$\begin{aligned}
R(\varepsilon c_n) &\le C_4 \frac{\varepsilon^{1-\alpha(1-\rho)-\delta}}{n c_n^{\alpha(1-\rho)+\delta}} + C_5 \frac{\varepsilon^{2-\alpha-\delta} l(n)}{n^{1-\rho}} \frac{1}{c_n^{\alpha+\delta}} \\
&\le C_6 \frac{l(n)}{n^{2-\rho}} \left(\varepsilon^{1-\alpha(1-\rho)-\delta} + \varepsilon^{2-\alpha-\delta} \right). \qquad (65)
\end{aligned}$$

(iii). It follows from (10) that if $\beta = -1$, then $\alpha\rho = 1$. By Lemma 5, $H(x) \le C x l_3(x)$. Combining this estimate with (46), we get

$$b_n(j) \le C \left(\frac{j l_3(j)+1}{n c_n} + \frac{l(n)}{n^{1-\rho}} \frac{j+1}{c_n^2} \right).$$

Recalling (56) and using (64) once again, we obtain for any fixed $\delta \in (0, 2-\alpha)$ and all $n \ge n(\delta)$,

$$\begin{aligned}
R_n(\varepsilon c_n) &\le C \sum_{j=0}^{[\varepsilon c_n]+1} \mathbb{P}(X \le -j) \left(\frac{j l_3(j)+1}{n c_n} + \frac{l(n)}{n^{1-\rho}} \frac{j+1}{c_n^2} \right) \\
&\le C_1 (\varepsilon c_n)^{2-\alpha} \left(\frac{1}{n c_n} \frac{l_3(\varepsilon c_n)}{l_0(\varepsilon c_n)} + \frac{l(n)}{n^{1-\rho}} \frac{1}{c_n^2 l_0(\varepsilon c_n)} \right) \\
&\le C_2 \varepsilon^{2-\alpha-\delta} \left(\frac{1}{n} \frac{c_n l_3(c_n)}{c_n^\alpha l_0(c_n)} + \frac{l(n)}{n^{1-\rho}} \frac{1}{c_n^\alpha l_0(c_n)} \right) \\
&\le C_3 \varepsilon^{2-\alpha-\delta} \frac{l(n)}{n^{2-\rho}}, \qquad (66)
\end{aligned}$$

where the inequalities $H(c_n) \le C c_n l_3(c_n) \le C n^\rho l(n)$ have been used for the last step.

Estimates (63) – (66) imply (62). Combining (61) with (62) leads to

$$\mathbb{P}(\tau^- = n) \sim \frac{q l(n)}{n^{2-\rho}} \int_0^\infty \frac{\mathbb{P}(M_\alpha^+ \in dy)}{y^\alpha} = \frac{q l(n)}{n^{2-\rho}} \mathbb{E}\left(M_\alpha^+\right)^{-\alpha}. \qquad (67)$$

Summation over n gives

$$\mathbb{P}\left(\tau^- > n\right) = \sum_{k=n+1}^{\infty} \mathbb{P}\left(\tau^- = k\right) \sim \frac{q}{1-\rho}\frac{l(n)}{n^{1-\rho}}\mathbb{E}\left(M_\alpha^+\right)^{-\alpha}.$$

Comparing this with (4), we get an interesting identity

$$\mathbb{E}\left(M_\alpha^+\right)^{-\alpha} = (1-\rho)/q \tag{68}$$

which, in view of (67), completes the proof of Theorem 1 for $\{0 < \alpha < 2, \beta < 1\} \cap \{\alpha \neq 1\}$.

Remark 12. One can check that the proof of Theorem 1 for $\{0 < \alpha < 2, \beta < 1\} \cap \{\alpha \neq 1\}$ does not use the fact that in the lattice case the distribution of X is arithmetic.

3.2. **Proof of Theorem 1 for $\{1 < \alpha < 2, \beta = 1\}$.** In view of (10) the assumption $\beta = 1$ implies $q = 0$ in (20) and $\rho = 1 - 1/\alpha$. We fix an integer $N > 1$ and, for $c_n > N$, write

$$\begin{aligned}\mathbb{P}\left(\tau^- = n\right) = &\int_0^N \mathbb{P}\left(X \le -y\right)\mathbb{P}\left(S_{n-1} \in dy; \tau^- > n-1\right)\\ &+ \int_N^{c_n} \mathbb{P}\left(X \le -y\right)\mathbb{P}\left(S_{n-1} \in dy; \tau^- > n-1\right)\\ &+ \int_{c_n}^{\infty} \mathbb{P}\left(X \le -y\right)\mathbb{P}\left(S_{n-1} \in dy; \tau^- > n-1\right)\\ =: &\, I_1(N,n) + I_2(N,c_n) + I_3(c_n).\end{aligned}$$

Our aim is to show that the last two integrals divided by $n^{-1/\alpha-1}l(n)$ vanish as first $n \to \infty$ and then $N \to \infty$, while

$$\lim_{N\to\infty}\lim_{n\to\infty}\frac{n^{1+1/\alpha}}{l(n)}I_1(N,n) = 1/\alpha = 1-\rho. \tag{69}$$

To start with, recall that according to Lemma 4 under our conditions

$$\mathbb{P}(\chi_+ > x) \sim \frac{1}{x^{\alpha-1}l_2(x)} \quad \text{as } x \to \infty.$$

Moreover, it was shown by Doney [9, Corollary 3] that (12) is equivalent to the relation $l_2(x) \sim Cl_0(x)$ as $x \to \infty$. Then Lemma 9 gives the upper bound

$$b_n(x) \le C\Big(\frac{x^{\alpha-1}l_0(x)}{nc_n} + \frac{l(n)x}{n^{1-\rho}c_n^2}\Big) \quad \text{for all } x \ge 1.$$

Besides, Lemma 6, (22) and (4) imply exisence of a constant $K > 0$ such that

$$c_n \sim \frac{n^{1-\rho}}{Kl(n)} \quad \text{as } n \to \infty. \tag{70}$$

This equivalence justifies the inequality

$$b_n(x) \le C\frac{l(n)}{n^{2-\rho}}\Big(x^{\alpha-1}l_0(x) + \frac{nx}{c_n^2}\Big) \quad \text{for all } x \ge 1. \tag{71}$$

As a result, we have for $c_n > N > 1$ the estimate

$$I_2(N, c_n) \le \sum_{j=N}^{[c_n]+1} \mathbb{P}(X \le -j) b_{n-1}(j)$$
$$\le C \frac{l(n)}{n^{2-\rho}} \Big(\sum_{j=N}^{[c_n]+1} j^{\alpha-1} l_0(j) \mathbb{P}(X \le -j) + \frac{n}{c_n^2} \sum_{j=N}^{[c_n]+1} j \mathbb{P}(X \le -j) \Big). \qquad (72)$$

It easily follows from (12) and (20) with $p = 1$ and $q = 0$, that

$$\sum_{j=N}^{[c_n]+1} j^{\alpha-1} l_0(j) \mathbb{P}(X \le -j) \le C \sum_{j=N}^{[c_n]+1} \frac{1}{j} \frac{\mathbb{P}(X \le -j)}{\mathbb{P}(X \ge j)} \to 0 \qquad (73)$$

as first $n \to \infty$ and than $N \to \infty$.

Further, recalling that $\mathbb{P}(X \le -j) = o(\mathbb{P}(X \ge j))$ as $j \to \infty$, we obtain by (23) and (20), for sufficiently large n and a function $r(N) \to 0$ as $N \to \infty$:

$$\sum_{j=N}^{[c_n]+1} j \mathbb{P}(X \le -j) \le r(N) \sum_{j=N}^{[c_n]+1} j \mathbb{P}(X \ge j)$$
$$\le C r(N) \sum_{j=N}^{[c_n]+1} \frac{1}{j^{\alpha-1} l_0(j)} \le C_1 r(N) \frac{c_n^{2-\alpha}}{l_0(c_n)}$$
$$\le C_2 r(N) \frac{c_n^2}{n}. \qquad (74)$$

Combining (72), (73) and (74), we conclude that

$$\lim_{N\to\infty} \limsup_{n\to\infty} \frac{n^{1+1/\alpha}}{l(n)} I_2(N, c_n) = 0. \qquad (75)$$

To establish a similar result for $I_3(c_n)$, observe that if $\beta = 1$, then, by (20) and (21),

$$\mathbb{P}(X \le -c_n) = o(\mathbb{P}(X \ge c_n)) = o(1/n) \quad \text{as } n \to \infty,$$

and, therefore,

$$I_3(c_n) \le \mathbb{P}(X \le -c_n) \mathbb{P}(\tau^- > n) = o\Big(\frac{l(n)}{n^{2-\rho}}\Big) \quad \text{as } n \to \infty. \qquad (76)$$

Applying Lemma 7 and recalling (70), we have

$$\lim_{n\to\infty} \frac{n^{1+1/\alpha}}{l(n)} I_1(N, n) = \lim_{n\to\infty} K n c_n I_1(N, n) = K \int_0^N \mathbf{P}(X \le -x) \mu(dx). \qquad (77)$$

In view of (71),

$$\mu((x, x+1]) = \lim_{n\to\infty} n c_n b_n(x) \le C x^{\alpha-1} \ell_0(x).$$

From this, taking into account conditions (73) and (12), we get

$$\int_0^\infty \mathbf{P}(X \le -x) \mu(dx) < \infty.$$

Hence we conclude that

$$\lim_{N\to\infty}\lim_{n\to\infty}\frac{n^{1+1/\alpha}}{l(n)}I_1(N,n)=K\int_0^\infty \mathbf{P}(X\le -x)\mu(dx). \tag{78}$$

Combining (75), (76) and (78) yields, as $n\to\infty$,

$$\mathbb{P}(\tau^-=n)\sim\frac{Kl(n)}{n^{1+1/\alpha}}\int_0^\infty \mathbf{P}(X\le -x)\mu(dx)\sim\frac{1}{nc_n}\int_0^\infty \mathbf{P}(X\le -x)\mu(dx)\ . \tag{79}$$

Comparing this formula with the tail behavior of τ^- given by (4) leads to the equalities

$$K\int_0^\infty \mathbf{P}(X\le -x)\mu(dx)=1-\rho=1/\alpha. \tag{80}$$

This justifies (69), finishing the proof of our theorem for $1<\alpha<2,\beta=1$.

3.3. **Proof of Theorem 1 for** $\{\alpha=2,\beta=0\}$**.** Consider first the case of arithmetic distributions and assume for simplicity that $h=1$ from now on. In this case we write

$$\begin{aligned}\mathbb{P}(\tau^-=n)&=\sum_{j=1}^{\infty}\mathbb{P}(X\le -j)\mathbb{P}(S_{n-1}=j;\tau^->n-1)\\&=\Delta_1(c_n)+\Delta_2(c_n),\end{aligned}$$

where

$$\Delta_1(c_n):=\sum_{j=1}^{[c_n]}\mathbb{P}(X\le -j)\mathbb{P}(S_{n-1}=j;\tau^->n-1),$$

$$\Delta_2(c_n):=\sum_{j=[c_n]+1}^{\infty}\mathbb{P}(X\le -j)\mathbb{P}(S_{n-1}=j;\tau^->n-1).$$

Recall that if $\alpha=2$ then $\rho=1/2$. In view of (24), (25) and (4)

$$\begin{aligned}\Delta_2(c_n)&\le\mathbb{P}(X\le -c_n)\mathbb{P}(\tau^->n-1)\\&=o\left(\frac{1}{n}\frac{l(n)}{n^{1/2}}\right)\ =\ o\left(\frac{l(n)}{n^{3/2}}\right)\ \text{ as }\ n\to\infty.\end{aligned}$$

To evaluate $\Delta_1(c_n)$ denote $g_{2,0}(x)=(\sqrt{2\pi})^{-1}\exp\{-x^2/2\},\ x\in(-\infty,\infty)$, the density of the standard normal law and set

$$w(n):=\sum_{j=1}^{[c_n]}g_{2,0}\left(\frac{j}{c_n}\right)\mathbb{P}(X\le -j)H(j-1).$$

By formula (3.15) in [5], as $n\to\infty$,

$$\mathbb{P}(S_{n-1}=j;\tau^->n-1)\sim\frac{H(j-1)}{n}\mathbb{P}(S_{n-1}=j)\sim\frac{H(j-1)}{nc_n}g_{2,0}\left(\frac{j}{c_n}\right)$$

uniformly in $j\in[1,c_n]$. This gives

$$\Delta_1(c_n)=\frac{1+r(n)}{nc_n}w(n), \tag{81}$$

where $r(n)\to 0$ as $n\to\infty$. As a result we obtain

$$\mathbb{P}(\tau^-=n)=\frac{1+r(n)}{nc_n}w(n)+o\left(\frac{l(n)}{n^{3/2}}\right). \tag{82}$$

Hence it follows that, as $n\to\infty$,

$$\begin{aligned}\frac{l(n)}{n^{1/2}}\sim\mathbb{P}(\tau^->n)&=\sum_{k=n+1}^{\infty}\left(\frac{1+r(k)}{kc_k}w(k)+o\left(\frac{l(k)}{k^{3/2}}\right)\right)\\&=(1+r_1(n))\sum_{k=n+1}^{\infty}\frac{w(k)}{kc_k}+o\left(\frac{l(n)}{n^{1/2}}\right),\end{aligned}$$

where $r_1(n)\to 0$ as $n\to\infty$. Since $w(n)$ is monotone increasing in n, and $c_n\sim n^{1/2}l_1(n)$ as $n\to\infty$, Lemma 10 with $\gamma=1-\rho=1/2$ yields after obvious transformations

$$\frac{w(n)}{nc_n}\sim\frac{1}{2}\frac{l(n)}{n^{3/2}}\quad\text{as } n\to\infty, \tag{83}$$

which, on account of (82) finishes the proof of (11) for $\{\alpha=2,\beta=0\}$ in the arithmetic case. To establish the same result for non-lattice distributions one should apply the respective statements in [4].

4. Proof of Theorem 2

Applying (2) to the random walk $\{-S_n\}_{n\geq 0}$, we have

$$1-\mathbb{E}z^{T^-}=\exp\left\{-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathbb{P}(S_n<0)\right\}.$$

Recalling (13) and (1) we obtain

$$1-\mathbb{E}z^{T^-}=\left(1-\mathbb{E}z^{\tau^-}\right)\Omega(z). \tag{84}$$

On account of $\mathbb{P}(\tau^-=0)=0$, equality (84) implies

$$\mathbb{P}(T^-=n)=\sum_{k=1}^{n}\mathbb{P}(\tau^-=k)\omega_{n-k}-\omega_n,\ n\geq 1. \tag{85}$$

Suppose first that the distribution of X is arithmetic. By the Gnedenko local theorem we get for this case

$$\frac{1}{n}\mathbb{P}(S_n=0)=\frac{g_{\alpha,\beta}(0)}{nc_n}(1+o(1))\quad\text{as } n\to\infty.$$

This representation and Theorem 2 in [6] provide existence of a constant $C>0$ such that

$$\omega_n=\frac{C}{nc_n}(1+o(1))\quad\text{as } n\to\infty.$$

Using this equality and (11) in (85) and recalling that $\mathbb{P}(\tau^-<\infty)=1$, we obtain

$$\mathbb{P}(T^-=n)=\Omega(1)\mathbb{P}(\tau^-=n)(1+o(1))+o((nc_n)^{-1})\quad\text{as } n\to\infty.$$

Observing that $\mathbb{P}(\tau^-=n)\geq C/nc_n$, we get the desired statement for the arithmetic case.

If the distribution of X is non-lattice, then there exists a constant $r \in (0,1)$ such that $\mathbb{P}(S_n = 0) \le r^n$ for all $n \ge 1$ (we may choose r as the total mass of the lattice component of the distribution of X). Consequently, $\omega_n \le r^n$ for all $n \ge 1$. From this estimate and (85) we see that the statement of Theorem 2 is valid in the non-lattice case as well.

5. Discussion and concluding remarks

We see by (1) that the distribution of τ^- is completely specified by the sequence $\{\mathbb{P}(S_n > 0)\}_{n\ge 1}$. As we have mentioned in the introduction, the validity of condition (5) is sufficient to reveal the asymptotic behavior of $\mathbb{P}(\tau^- > n)$ as $n \to \infty$. Thus, in view of (4), nonformal arguments based on the plausible smoothness of $l(n)$ immediately give the desired answer

$$\begin{aligned}\mathbb{P}(\tau^- = n) &= \mathbb{P}(\tau^- > n-1) - \mathbb{P}(\tau^- > n)\\ &= \frac{l(n-1)}{(n-1)^{1-\rho}} - \frac{l(n)}{n^{1-\rho}} \approx l(n)\left(\frac{1}{(n-1)^{1-\rho}} - \frac{1}{n^{1-\rho}}\right)\\ &\approx \frac{(1-\rho)l(n)}{n^{2-\rho}} \sim \frac{1-\rho}{n}\mathbb{P}(\tau^- > n)\end{aligned}$$

under the Doney condition only. In the present paper we failed to achieve such a generality. However, it is worth to be mentioned that the Doney condition, being formally weaker than the conditions of Theorem 1, requires in the general case the knowledge of the behavior of the whole sequence $\{\mathbb{P}(S_n > 0)\}_{n\ge 1}$, while the assumptions of Theorem 1 concern a single summand only. Of course, imposing a stronger condition makes our life easier and allows us to give, in a sense, a constructive proof showing what happens in reality at the distant moment τ^- of the first jump of the random walk in question below zero. Indeed, our arguments for the case $\{0 < \alpha < 2,\ \beta < 1\} \cap \{\alpha \ne 1\}$ demonstrate (compare (56), (57), and (60)) that for any $x_2 > x_1 > 0$,

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{P}(S_{n-1} \in (c_n x_1, c_n x_2] | \tau^- = n)\\ &\quad= \lim_{n\to\infty} \frac{\mathbb{P}(\tau^- > n-1)}{\mathbb{P}(\tau^- = n)} \int_{x_1}^{x_2} \mathbb{P}(X < -yc_n)\mathbb{P}(S_{n-1} \in c_n dy | \tau^- > n-1)\\ &\quad= \lim_{n\to\infty} \frac{\mathbb{P}(\tau^- > n-1)q}{\mathbb{P}(\tau^- = n)n} \int_{x_1}^{x_2} \frac{\mathbb{P}(X < -yc_n)}{\mathbb{P}(X < -c_n)}\mathbb{P}(S_{n-1} \in c_n dy | \tau^- > n-1)\\ &\quad= \frac{q}{1-\rho}\int_{x_1}^{x_2} \frac{\mathbb{P}(M_\alpha^+ \in dy)}{y^\alpha}.\end{aligned}$$

In view of (68) this means that the contribution of the trajectories of the random walk satisfying $S_{n-1}c_n^{-1} \to 0$ or $S_{n-1}c_n^{-1} \to \infty$ as $n \to \infty$ to the event $\{\tau^- = n\}$ is negligibly small in probability. A typicaltrajectory looks in this case as follows: it is located over the level zero up to moment $n-1$ with $S_{n-1} \in (\varepsilon c_n\ , \varepsilon^{-1} c_n)$ for sufficiently small $\varepsilon > 0$ and at moment $\tau^- = n$ the trajectory makes a big negative jump $X_n < -S_{n-1}$ of order $O(c_n)$.

On the other hand, if $\{1 < \alpha < 2,\ \beta = 1\}$ and condition (12) holds, then (compare (34), (77), (79), and (80)) for any $N_2 > N_1 > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}(S_{n-1} \in (N_1, N_2] | \tau^- = n) \\
&\qquad = \lim_{n\to\infty} \frac{1}{\mathbb{P}(\tau^- = n)} \int_{N_1}^{N_2} \mathbb{P}(X < -y)\mathbb{P}(S_{n-1} \in dy; \tau^- > n-1) \\
&\qquad = \lim_{n\to\infty} K\alpha n c_n \int_{N_1}^{N_2} \mathbb{P}(X < -y)\mathbb{P}(S_{n-1} \in dy; \tau^- > n-1) \\
&\qquad = K\alpha \int_{N_1}^{N_2} \mathbb{P}(X < -y)\mu(dy).
\end{aligned}$$

Thus, the main contribution to $\mathbb{P}(\tau^- = n)$ is given in this case by the trajectories located over the level zero up to moment $n-1$ with $S_{n-1} \in [0, N]$ for sufficiently big N and with not "too big" jump $X_n < -S_{n-1}$ of order $O(1)$.

Unfortunately, our approach to investigate the behavior of $\mathbb{P}(\tau^- = n)$ in the case $\alpha = 2$ is pure analytical and does not allow us to extract typical trajectories without further restrictions on the distribution of X. However, we can still deduce from our proof some properties of the random walk conditioned on $\{\tau^- = n\}$. Observe that, for any fixed $\varepsilon > 0$, the trajectories with $S_{n-1} > \varepsilon c_n$ give no essential contribution to $\mathbb{P}(\tau^- = n)$. Indeed, it follows from (81) and (83) that $\Delta_1(\varepsilon c_n) \sim \Delta_1(c_n)$ as $n \to \infty$ for every fixed ε. This, along with the estimate from above for $\Delta_2(c_n)$, gives the claimed property. Furthermore, one can easily verify that if $\sum_{j=1}^{\infty} \mathbb{P}(X \le -j)H(j) = \infty$, then for every $N \ge 1$,

$$\sum_{j=1}^{N} \mathbb{P}(X \le -j)\mathbb{P}(S_{n-1} = j; \tau^- > n-1) = o\left(\frac{l(n)}{n^{3/2}}\right) \quad \text{as } n \to \infty,$$

i.e. the contribution of the trajectories with $S_{n-1} = O(1)$ to $\mathbb{P}(\tau^- = n)$ is negligible small. As a result we see that $S_{n-1} \to \infty$ but $S_{n-1} = o(c_n)$ for all "typical" trajectories meeting the condition $\{\tau^- = n\}$. Thus, under the conditions of Theorem 1 we have for $\alpha = 2$ a kind of "continuous transition" between the two strategies that take place for the case $\alpha < 2$. We note, for completeness, that if $\sum_{j=1}^{\infty} \mathbb{P}(X \le -j)H(j)$ is finite, then the typical behavior of the trajectories is similar to that for the case $\{0 < \alpha < 2,\ \beta = 1\}$.

Unfortunately, the methods of the present paper do not work for $\alpha = 1$, and we leave the problem on the asymptotic behavior of $\mathbb{P}(\tau^- = n)$ open for this case.

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