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Asymptotic convergence results for a system of partial differential equations with hysteresis

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Abstract

A partial differential equation motivated by electromagnetic field equations in ferromagnetic media is considered with a relaxed rate dependent constitutive relation. It is shown that the solutions converge to the unique solution of the limit parabolic problem with a rate independent Preisach hysteresis constitutive operator as the relaxation parameter tends to zero.

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1 Introduction

The aim of this paper is to study the following system of partial differential equations

$$\begin{cases} \dfrac{\partial}{\partial t}(\alpha\, u + \beta\, w) - \triangle u = f \\ w = \overline{\mathcal{F}}\left(u - \gamma\, \dfrac{\partial w}{\partial t}\right) \end{cases} \qquad \text{in } \Omega \times (0,T), \tag{1.1}$$

where Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$, $\overline{\mathcal{F}}$ is a continuous rate independent invertible hysteresis operator, f is a given function, γ, α and β are given positive constants.

This system can be obtained by coupling the Maxwell equations, the Ohm law and a constitutive relation between the magnetic field and the magnetic induction, provided we neglect the displacement current. A detailed derivation will be given in Section 3 below. The meaning of the parameter γ is to take into account in the constitutive relation also a rate dependent component of the memory. A similar problem has been considered recently in [1] in the context of soil hydrology, with γ fixed and with a more general form of the elliptic part. The reason for introducing the parameter γ was to regularize the resulting P.D.E.s and obtain solvability of the new system.

Our aim here is to justify this regularization by proving that in the simpler case (1.1), the solutions of (1.1) converge as $\gamma \to 0$ to the (unique) strong solution (see [5]) of the system

$$\begin{cases} \dfrac{\partial}{\partial t}(\alpha\, u + \beta\, w) - \triangle u = f \\ w = \overline{\mathcal{F}}(u) \end{cases} \tag{1.2}$$

as an extension of the results contained in Chapter 4 of [4]. For γ positive, the second equation in (1.1) defines a constitutive operator $S : \mathbb{R} \times \mathcal{C}^0([0,T]) \to \mathcal{C}^1([0,T])$ which with each $u \in \mathcal{C}^0([0,T])$ and each initial condition $w^0 \in \mathbb{R}$ associates $w = S(w^0, u)$. Then (1.1) has the form

$$\frac{\partial}{\partial t}(\alpha\, u + \beta\, S(w^0, u)) - \triangle u = f. \tag{1.3}$$

The regularizing properties of S enable us to solve the problem by means of a simple application of the Banach contraction mapping principle. The passage to the limit as $\gamma \to 0$ is achieved in several steps, using in particular a lemma constructed ad hoc which allows us to pass to the limit in the nonlinear hysteresis term.

The outline of the paper is the following: after some remarks concerning Preisach operators (Section 2), we explain the physical interpretation of our model system in Section 3. Then we present in Section 4 the existence and uniqueness result while Section 5 is devoted to the asymptotic convergence of the solution as $\gamma \to 0$.

2 The Preisach operator

We describe the ferromagnetic behaviour using the Preisach model proposed in 1935 (see [16]). Mathematical aspects of this model were investigated by Krasnosel'skiĭ and Pokrovskiĭ (see [7], [8], and [9]). The model has been also studied in connection with partial differential equations by Visintin (see for example [17], [18]). The monograph of Mayergoyz ([15]) is mainly devoted to its modeling aspects.

Here we use the one-parametric representation of the Preisach operator which goes back to [10]. The starting point of our theory is the so called *play operator*. This operator constitutes the simplest example of continuous hysteresis operator in the space of continuous functions; it has been introduced in [9] but we can also find equivalent definitions in [2] and [18]; for its extension to less regular inputs, see also [12] and [13].

Let $r > 0$ be a given parameter. For a given input function $u \in \mathcal{C}^0([0,T])$ and initial condition $x^0 \in [-r, r]$, we define the output $\xi = \mathcal{P}_r(x^0, u) \in \mathcal{C}^0([0,T]) \cap BV(0,T)$ of the *play operator*

$$\mathcal{P}_r : [-r, r] \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T]) \cap BV(0,T)$$

as the solution of the variational inequality in Stieltjes integral form

$$\begin{cases} \displaystyle\int_0^T (u(t) - \xi(t) - y(t))\, d\xi(t) \geq 0 & \forall\, y \in \mathcal{C}^0([0,T]), \quad \displaystyle\max_{0\leq t\leq T} |y(t)| \leq r, \\ |u(t) - \xi(t)| \leq r & \forall\, t \in [0,T], \\ \xi(0) = u(0) - x^0. & \end{cases} \tag{2.1}$$

Let us consider now the whole family of play operators $\mathcal{P}_r$ parameterized by $r > 0$, which can be interpreted as a *memory variable*. Accordingly, we introduce the *hysteresis memory state space*

$$\Lambda := \{\lambda : \mathbb{R}_+ \to \mathbb{R} : \ |\lambda(r) - \lambda(s)| \leq |r - s| \ \ \forall\, r, s \in \mathbb{R}_+ : \lim_{r\to+\infty} \lambda(r) = 0\},$$

together with its subspaces

$$\Lambda_K = \{\lambda \in \Lambda : \lambda(r) = 0 \ \text{ for } \ r \geq K\}, \qquad \Lambda_\infty = \bigcup_{K>0} \Lambda_K. \tag{2.2}$$

For $\lambda \in \Lambda$, $u \in \mathcal{C}^0([0,T])$ and $r > 0$ we set

$$\wp_r[\lambda, u] := \mathcal{P}_r(x_r^0, u) \qquad \wp_0[\lambda, u] := u,$$

where x_r^0 is given by the formula

$$x_r^0 := \min\{r, \max\{-r, u(0) - \lambda(r)\}\}. \tag{2.3}$$

It turns out that

$$\wp_r : \Lambda \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$$

is Lipschitz continuous in the sense that, for every $u, v \in \mathcal{C}^0([0,T])$, $\lambda, \mu \in \Lambda$ and $r > 0$ we have

$$||\wp_r[\lambda, u] - \wp_r[\mu, v]||_{\mathcal{C}^0([0,T])} \leq \max\{|\lambda(r) - \mu(r)|, ||u - v||_{\mathcal{C}^0([0,T])}\}. \tag{2.4}$$

Moreover, if $\lambda \in \Lambda_R$ and $||u||_{\mathcal{C}^0([0,T])} \leq R$, then $\wp_r[\lambda, u](t) = 0$ for all $r \geq R$ and $t \in [0,T]$. For more details, see Sections II.3, II.4 of [11].

Now we introduce the *Preisach plane* as follows

$$\mathscr{P} := \{(r, v) \in \mathbb{R}^2 : r > 0\}$$

and consider a function $\varphi \in L^1_{\text{loc}}(\mathscr{P})$ such that there exists $\beta_1 \in L^1_{\text{loc}}(0, \infty)$ with

$$0 \leq \varphi(r, v) \leq \beta_1(r) \qquad \text{for a.e. } (r, v) \in \mathscr{P}.$$

We set

$$g(r, v) := \int_0^v \varphi(r, z)\, dz \qquad \text{for } (r, v) \in \mathscr{P}$$

and for $R > 0$, we put $b_1(R) := \displaystyle\int_0^R \beta_1(r)\, dr.$

Then the *Preisach operator*

$$\mathcal{W} : \Lambda_\infty \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$$

generated by the function g is defined by the formula

$$\mathcal{W}[\lambda, u](t) := \int_0^\infty g(r, \wp_r[\lambda, u](t))\, dr, \tag{2.5}$$

for any given $\lambda \in \Lambda_\infty$, $u \in \mathcal{C}^0([0,T])$ and $t \in [0,T]$. The equivalence of this definition and the classical one in [15], [18], e.g., is proved in [10].

Using the Lipschitz continuity (2.4) of the operator $\wp_r$, it is easy to prove that also $\mathcal{W}$ is locally Lipschitz continuous, in the sense that, for any given $R > 0$, for every $\lambda, \mu \in \Lambda_R$ and $u, v \in \mathcal{C}^0([0,T])$ with $||u||_{\mathcal{C}^0([0,T])}, ||v||_{\mathcal{C}^0([0,T])} \leq R$, we have

$$||\mathcal{W}[\lambda, u] - \mathcal{W}[\mu, v]||_{\mathcal{C}^0([0,T])} \leq \int_0^R |\lambda(r) - \mu(r)|\, \beta_1(r)\, dr + b_1(R)\, ||u - v||_{\mathcal{C}^0([0,T])}.$$

In view of (2.3), the initial value mapping $u(0) \mapsto \mathcal{W}[\lambda, u](0)$ can be represented by a locally Lipschitz continuous function

$$W_\lambda : \mathbb{R} \to \mathbb{R} : \qquad W_\lambda(u(0)) := \mathcal{W}[\lambda, u](0) = \int_0^\infty g(r, x_r^0)\, dr. \tag{2.6}$$

The first result on the inverse Preisach operator was proved in [3]. We make use of the following formulation proved in [11], Section II.3.

Theorem 2.1. *Let $\lambda \in \Lambda_\infty$ and $b > 0$ be given. Then the operator $b\,I + \mathcal{W}[\lambda, \cdot] : \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$ is invertible and its inverse is Lipschitz continuous.*

Finally we have the following local monotonicity result for the Preisach operator $\mathcal{W}$.

Theorem 2.2. *Consider $b \geq 0$, $R > 0$, $\lambda \in \Lambda_R$ and $u \in W^{1,1}(0,T)$ be given such that $||u||_{\mathcal{C}^0([0,T])} \leq R$. Put $w := b\,u + \mathcal{W}[\lambda, u]$. Then*

$$b \left(\frac{\partial u}{\partial t}(t)\right)^2 \leq \frac{\partial w}{\partial t}(t)\, \frac{\partial u}{\partial t}(t) \leq (b + b_1(R)) \left(\frac{\partial u}{\partial t}(t)\right)^2 .$$

As we are dealing with partial differential equations, we should consider both the input and the initial memory configuration λ that additionally depend on x. If for instance $\lambda(x, \cdot)$ belongs to Λ_∞ and $u(x, \cdot)$ belongs to $\mathcal{C}^0([0,T])$ for (almost) every x, then we define

$$\overline{\mathcal{W}}[\lambda, u](x,t) := \mathcal{W}[\lambda(x,\cdot), u(x,\cdot)](t) := \int_0^\infty g(r, \wp_r[\lambda(x,\cdot), u(x,\cdot)](t))\, dr. \tag{2.7}$$

3 Physical interpretation of the model system (1.1)

Let a ferromagnetic material occupy a bounded region $\mathscr{D} \subset \mathbb{R}^3$; we set $\mathscr{D}_T := \mathscr{D} \times (0,T)$ for a fixed $T > 0$, and we assume that the body is surrounded by vacuum. We denote by $\vec{g}$ a prescribed electromotive force; then Ohm's law reads

$$\vec{J} = \sigma\,(\vec{E} + \vec{g}) \qquad \text{in } \mathscr{D},$$

where σ is the electric conductivity, $\vec{J}$ is the electric current density and $\vec{E}$ is the electric field; we also prescribe $\vec{J} = 0$ outside $\mathscr{D}$.
In $\mathscr{D}_T$, we consider the Ampère and the Faraday laws in the form

$$\begin{aligned} c\,\nabla \times \vec{H} &= 4\pi\,\vec{J} + \frac{\partial \vec{D}}{\partial t}\,, \\ c\,\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}\,, \end{aligned}$$

where c is the speed of light in vacuum, $\vec{H}$ is the magnetic field, $\vec{D}$ is the electric displacement and $\vec{B}$ is the magnetic induction.

In case of a ferromagnetic metal, σ is very large, hence we can assume

$$4\,\pi\,|\vec{J}| \gg \left|\frac{\partial \vec{D}}{\partial t}\right| \qquad \text{in } \mathscr{D},$$

provided that the field $\vec{g}$ does not vary too rapidly.

Then we neglect the displacement current $\dfrac{\partial \vec{D}}{\partial t}$ in Ampère's law; this is the so-called *eddy current approximation*. By coupling this reduced law with Faraday's and Ohm's laws, in Gauss units we get

$$4\,\pi\,\sigma\,\frac{\partial \vec{B}}{\partial t} + c^2\,\nabla\times\nabla\times\vec{H} = 4\,\pi\,c\,\sigma\,\nabla\times\vec{g} \qquad \text{in } \mathscr{D}_T. \tag{3.1}$$

For more details on these topics, we refer to a classical text of electromagnetism, for example [6].

We now reduce this system to a scalar one describing *planar waves*. More precisely, let Ω be a domain of $\mathbb{R}^2$. We assume (using the orthogonal Cartesian coordinates x, y, z) that $\vec{H}$ is parallel to the $z-$axis and only depends on the coordinates x, y, i.e.

$$\vec{H} = (0, 0, H(x, y)).$$

Then

$$\nabla\times\nabla\times\vec{H} = (0, 0, -\triangle_{x,y}H) \qquad \left(\triangle_{x,y} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{3.2}$$

We also assume that

$$\vec{B} = (0, 0, B(x, y)), \qquad \nabla\times\vec{g} = (0, 0, \tilde{f});$$

then equation (3.1) is reduced to a scalar equation

$$\frac{4\pi\sigma}{c^2}\,\frac{\partial B}{\partial t} - \triangle_{x,y}H = f := \frac{4\,\pi\,\sigma}{c}\,\tilde{f}. \tag{3.3}$$

The constitutive law between B and H will be chosen according to the "rheological" circuit model $(F - L)|P$ as in [18, p. 54-55], where a combination in series of a ferromagnetic element

$$F:\ B^F = H^F + 4\pi M = (I + \overline{\mathcal{W}})(H^F)\,,$$

where M is the magnetization and $\overline{\mathcal{W}}$ is a Preisach operator, and of an induction element

$$L:\ H^L = \gamma\frac{\partial B^L}{\partial t}\,,$$

is coupled in parallel with a linear paramagnetic element

$$P:\ B^P = \mu H^P\,.$$

The general rheological rules for parallel and series combinations yield

$$B^F = B^L =: B^{FL}, \quad H = H^F + H^L = H^P, \quad B = B^{FL} + B^P,$$

where B is the total induction and H is the total field. Denoting $V := B^{FL}$, we obtain

$$H = \gamma \frac{\partial V}{\partial t} + (I + \overline{\mathcal{W}})^{-1}(V), \quad B = V + \mu H. \tag{3.4}$$

We thus rewrite (3.3) as

$$\begin{cases} \dfrac{4\pi\sigma}{c^2} \dfrac{\partial}{\partial t}(V + \mu H) - \triangle_{x,y} H = f, \\ V = (I + \overline{\mathcal{W}}) \left(H - \gamma \dfrac{\partial V}{\partial t} \right), \end{cases} \tag{3.5}$$

which is precisely (1.1). The case, where the influence of the induction element L is negligible, corresponds to the limit as $\gamma \to 0$.

4 Existence and uniqueness

In the setting (1.1) or (1.2), the space dimension is not relevant. We therefore consider an open bounded set of Lipschitz class $\Omega \subset \mathbb{R}^N$, $N \geq 1$, set $Q := \Omega \times (0,T)$, and fix an initial memory configuration

$$\lambda \in L^2(\Omega; \Lambda_K) \qquad \text{for some } K > 0, \tag{4.1}$$

where Λ_K is introduced in (2.2).
Let $\mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ be the Fréchet space of strongly measurable functions $\Omega \to \mathcal{C}^0([0,T])$, i.e. the space of functions $v : \Omega \to \mathcal{C}^0([0,T])$ such that there exists a sequence v_n of simple functions with $v_n \to v$ in $\mathcal{C}^0([0,T])$ a.e. in Ω.
We fix a constant $b_{\mathcal{F}} > 0$ and introduce the operator $\overline{\mathcal{F}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ in the following way

$$\overline{\mathcal{F}}(u)(x,t) := \mathcal{F}(u(x,\cdot))(t) := b_{\mathcal{F}}\, u(x,t) + \mathcal{W}[\lambda(x,\cdot), u(x,\cdot)](t); \tag{4.2}$$

here, $\mathcal{W}$ is the scalar Preisach operator defined in (2.5). In agreement with (2.6), we have

$$\overline{\mathcal{F}}(u)(x,0) = b_{\mathcal{F}}\, u^0(x) + W_{\lambda(x,\cdot)}(u^0(x)). \tag{4.3}$$

Now Theorem 2.1 yields that $\mathcal{F}$ is invertible and its inverse is a Lipschitz continuous operator in $\mathcal{C}^0([0,T])$. Let us set $\mathcal{G} = \mathcal{F}^{-1}$ and let $L_{\mathcal{G}}$ be the Lipschitz constant of the operator $\mathcal{G}$.
At this point we introduce the operator

$$\overline{\mathcal{G}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \qquad \overline{\mathcal{G}} := \overline{\mathcal{F}}^{-1}. \tag{4.4}$$

It turns out that

$$\overline{\mathcal{G}}(w)(x,t) = \mathcal{G}(w(x,\cdot))(t) \qquad \forall\, w \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])). \tag{4.5}$$

It follows from Theorem 2.1 that $\overline{\mathcal{G}}$ is Lipschitz continuous in the following sense

$$||\overline{\mathcal{G}}(u_1)(x,\cdot) - \overline{\mathcal{G}}(u_2)(x,\cdot)||_{\mathcal{C}^0([0,T])} \leq L_{\mathcal{G}}\, ||u_1(x,\cdot) - u_2(x,\cdot)||_{\mathcal{C}^0([0,T])}$$
$$\text{for any } u_1, u_2 \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])), \text{ a.e. in } \Omega.$$

Moreover Theorem 2.2 entails that there exist two constants $c_{\mathcal{F}}$ and $C_{\mathcal{F}}$ such that

$$c_{\mathcal{F}} \left(\frac{\partial u}{\partial t}\right)^2 \leq \frac{\partial \overline{\mathcal{F}}(u)}{\partial t}\frac{\partial u}{\partial t} \leq C_{\mathcal{F}} \left(\frac{\partial u}{\partial t}\right)^2 \qquad \text{a.e. in } Q\,. \tag{4.6}$$

On the other hand, (4.6) entails

$$c_{\mathcal{G}} \left(\frac{\partial w}{\partial t}\right)^2 \leq \frac{\partial \overline{\mathcal{G}}(w)}{\partial t}\frac{\partial w}{\partial t} \leq C_{\mathcal{G}} \left(\frac{\partial w}{\partial t}\right)^2 \quad \text{a.e. in } Q, \text{ with } C_{\mathcal{G}} = \frac{1}{c_{\mathcal{F}}},\ c_{\mathcal{G}} = \frac{1}{C_{\mathcal{F}}}. \tag{4.7}$$

Consider now system (1.1) with homogeneous Dirichlet boundary conditions and set $V := H_0^1(\Omega)$. We first state the existence and uniqueness result.

Theorem 4.1. (Existence and uniqueness)
Let α, β, γ be given positive constants. Suppose that the following assumptions on the data

$$f \in L^2(Q),\ \ u^0 \in V,\ \ w^0 \in L^2(\Omega)$$

hold. Then (1.1) *with homogeneous Dirichlet boundary conditions and initial conditions*

$$u(x,0) = u^0(x), \qquad w(x,0) = w^0(x), \tag{4.8}$$

admits a unique solution

$$u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;W^{2,2}(\Omega)),\ w \in L^2(\Omega; \mathcal{C}^1([0,T])).$$

Proof. The proof is divided into two steps.
• STEP 1: THE SOLUTION OPERATOR S. We neglect for the moment the dependence on the space parameter x within the constitutive relation

$$\gamma \frac{\partial w}{\partial t} + \overline{\mathcal{G}}(w) = u. \tag{4.9}$$

This means that we deal here with the following problem: for a given $u \in \mathcal{C}^0([0,T])$, find $w \in \mathcal{C}^1([0,T])$ such that

$$\begin{cases} \gamma \dfrac{dw}{dt} + \mathcal{G}(w) &= u \qquad \text{in } [0,T], \\ \qquad\qquad w(0) &= w^0. \end{cases} \tag{4.10}$$

Clearly, due to the Lipschitz continuity of $\mathcal{G}$, problem (4.10) admits a unique solution $w \in \mathcal{C}^1([0,T])$ for every $u \in \mathcal{C}^0([0,T])$. In this manner, we can define the solution operator

$$S : \mathcal{C}^0([0,T]) \to \mathcal{C}^1([0,T]) : \ u \mapsto w.$$

Let us show now that S is Lipschitz continuous in the sense that we prove that there exists a constant L_S such that

$$||S(u_1)-S(u_2)||_{\mathcal{C}^1([0,t])} \leq L_S\,||u_1-u_2||_{\mathcal{C}^0([0,t])}, \quad \forall\, u_1,u_2 \in \mathcal{C}^0([0,t]), \quad \forall\, t \in [0,T]. \quad (4.11)$$

Let us consider $u_1, u_2 \in \mathcal{C}^0([0,T])$ and let $w_1, w_2 \in \mathcal{C}^1([0,T])$ be such that $w_i = S(u_i)$, $i = 1,2$. The initial data are fixed, that is, $w_1(0) = w_2(0) = w^0$. For any $t \in [0,T]$ we have

$$\begin{aligned}\left|\frac{dw_1}{dt}(t) - \frac{dw_2}{dt}(t)\right| &\leq \frac{1}{\gamma}\,|u_1(t)-u_2(t)| + \frac{L_{\mathcal{G}}}{\gamma}\max_{0\leq\tau\leq t}|w_1(\tau)-w_2(\tau)| \\ &\leq \frac{1}{\gamma}\,|u_1(t)-u_2(t)| + \frac{L_{\mathcal{G}}}{\gamma}\int_0^t \left|\frac{dw_1}{dt} - \frac{dw_2}{dt}\right|(\tau)\,d\tau.\end{aligned}$$

Hence, by Gronwall's argument,

$$\int_0^t \left|\frac{dw_1}{dt} - \frac{dw_2}{dt}\right|(\tau)\,d\tau \leq \frac{1}{\gamma}\int_0^t e^{\frac{L_{\mathcal{G}}}{\gamma}(t-\tau)}\,|u_1(\tau)-u_2(\tau)|\,d\tau,$$

which yields

$$\left|\frac{dw_1}{dt}(t) - \frac{dw_2}{dt}(t)\right| \leq \frac{1}{\gamma}\,e^{\frac{L_{\mathcal{G}}}{\gamma}T}\,||u_1-u_2||_{\mathcal{C}^0([0,t])}$$

for every $t \in [0,T]$. Hence (4.11) holds with $L_S = \left(\frac{1}{\gamma} + \frac{1}{L_{\mathcal{G}}}\right)\,e^{\frac{L_{\mathcal{G}}}{\gamma}T}$.

We easily extend this estimate to the space dependent problem

$$\begin{cases} \gamma\,\dfrac{\partial w}{\partial t} + \overline{\mathcal{G}}(w) = u \\ w(\cdot,0) = w^0(\cdot) \end{cases} \qquad \text{a.e. in } Q, \tag{4.12}$$

with given functions $u \in L^2(\Omega;\mathcal{C}^0([0,T]))$, $w^0 \in L^2(\Omega)$. It immediately follows from (4.11) that the solution mapping

$$\bar{S} : L^2(\Omega;\mathcal{C}^0([0,T])) \to L^2(\Omega;\mathcal{C}^1([0,T])) : \qquad u \mapsto w \tag{4.13}$$

associated with (4.12) is well defined and Lipschitz continuous, with Lipschitz constant L_S.

STEP 2: FIXED POINT. Our model problem can be rewritten now as

$$\frac{\partial}{\partial t}(\alpha\,u + \beta\,\bar{S}(u)) - \triangle u = f \tag{4.14}$$

with $u(\cdot,0) = u^0(\cdot)$ and homogeneous Dirichlet boundary conditions. The unique solution will be found by the Banach contraction mapping principle.

Let us fix $z \in H^1(0,T;L^2(\Omega))$; then $z \in L^2(\Omega;\mathcal{C}^0([0,T]))$ and therefore $\bar{S}(z)$ is well-defined and belongs to $L^2(\Omega;\mathcal{C}^1([0,T]))$. Instead of (4.14), we consider the equation

$$\frac{\partial}{\partial t}(\alpha\,u + \beta\,\bar{S}(z)) - \triangle u = f \tag{4.15}$$

which is nothing but the linear heat equation. As $f \in L^2(Q)$, this means that (4.15) admits a unique solution $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)) \cap L^\infty(0,T;V)$.

We now introduce the set

$$\tilde{B} = \{z \in H^1(0,T;L^2(\Omega)) : \, z(\cdot,0) = u^0(\cdot)\}$$

and the operator

$$\tilde{J} : \tilde{B} \to \tilde{B} : \qquad z \mapsto u,$$

which with every $z \in \tilde{B}$ associates the solution $u \in \tilde{B}$ of (4.15). In order to prove that $\tilde{J}$ is a contraction, consider now two elements $z_1, z_2 \in \tilde{B}$, and set $u_1 := \tilde{J}(z_1)$, $u_2 := \tilde{J}(z_2)$. Then we have

$$\frac{\partial}{\partial t}(\alpha\,(u_1 - u_2) + \beta\,(\bar{S}(z_1) - \bar{S}(z_2))) - \triangle(u_1 - u_2) = 0.$$

We test this equation by $\dfrac{\partial}{\partial t}(u_1 - u_2)$ and obtain

$$\begin{aligned}
&\alpha \int_\Omega \left|\frac{\partial}{\partial t}(u_1 - u_2)\right|^2 (x,t)\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla(u_1 - u_2)|^2(x,t)\,dx \\
&\leq \frac{\alpha}{2}\int_\Omega \left|\frac{\partial}{\partial t}(u_1 - u_2)\right|^2 (x,t)\,dx + \frac{L_S^2\,\beta^2}{2\,\alpha}\int_\Omega \max_{0\leq\tau\leq t}|z_1 - z_2|^2(x,\tau)\,dx,
\end{aligned}$$

where L_S is the Lispchitz constant of the operator $\bar{S}$. This implies that

$$\begin{aligned}
&\int_\Omega \left|\frac{\partial}{\partial t}(u_1 - u_2)\right|^2 (x,t)\,dx + \frac{1}{\alpha}\frac{d}{dt}\int_\Omega |\nabla(u_1 - u_2)|^2(x,t)\,dx \\
&\leq \frac{L_S^2\,\beta^2\,t}{\alpha^2}\int_0^t \int_\Omega \left|\frac{\partial}{\partial t}(z_1 - z_2)\right|^2 (x,\tau)\,dx\,dt.
\end{aligned} \tag{4.16}$$

We set $\theta := \dfrac{L_S^2\,\beta^2}{\alpha^2}$ and we introduce the following equivalent norm on $H^1(0,T;L^2(\Omega))$

$$|||\eta||| = \left(||\eta(0)||^2_{L^2(\Omega)} + \int_0^T e^{-\theta\,t^2}\,\left|\left|\frac{\partial\eta}{\partial t}\right|\right|^2_{L^2(\Omega)}(t)\,dt\right)^{1/2} \qquad \forall\,\eta \in H^1(0,T;L^2(\Omega)).$$

If now we multiply (4.16) by $e^{-\theta\,t^2}$ and integrate over $t \in (0,T)$, we obtain that

$$|||u_1 - u_2||| \leq \frac{1}{2}\,|||z_1 - z_2|||$$

and thus $\tilde{J}$ is a contraction on the closed subset $\tilde{B}$ of $H^1(0,T;L^2(\Omega))$, which yields the existence and uniqueness of the solution $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)) \cap L^\infty(0,T;V)$. $\square$

5 Asymptotic convergence

In this section we investigate the behaviour of the solution of our model problem if the parameter γ goes to zero. We prove the following theorem.

Theorem 5.1. *Under the assumptions of Theorem* 4.1*, let* (u_γ, w_γ) *for* $\gamma > 0$ *be the unique solution of* (1.1) *with homogeneous Dirichlet boundary conditions, and initial conditions*

$$u_\gamma(x,0) = u^0(x), \qquad w_\gamma^0(x,0) = \overline{\mathcal{F}}(u_\gamma)(x,0) = b_{\mathcal{F}}\, u^0(x) + W_{\lambda(x,\cdot)}(u^0(x)) \tag{5.1}$$

according to (4.3)*. Then there exists*

$$u \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;W^{2,2}(\Omega))$$

such that

$$\begin{aligned}
&\frac{\partial u_\gamma}{\partial t} \to \frac{\partial u}{\partial t} && \text{weakly in } L^2(Q_T)\\
&\frac{\partial w_\gamma}{\partial t} \to \frac{\partial \overline{\mathcal{F}}(u)}{\partial t} && \text{weakly in } L^2(Q_T)\\
&\triangle u_\gamma \to \triangle u && \text{weakly in } L^2(Q_T)\\
&u_\gamma \to u && \text{strongly in } L^2(\Omega; \mathcal{C}^0([0,T]))\\
&w_\gamma \to \overline{\mathcal{F}}(u) && \text{strongly in } L^2(\Omega; \mathcal{C}^0([0,T]))
\end{aligned}$$

as $\gamma \to 0$*, and* u *is the unique solution of the equation*

$$\frac{\partial}{\partial t}(\alpha\, u + \beta\, \overline{\mathcal{F}}(u)) - \triangle u = f \qquad \text{in the } L^2(Q_T) \text{ sense,} \tag{5.2}$$

with initial condition $u(x,0) = u^0(x)$ *and homogeneous Dirichlet boundary condition.*

Proof. The regularity of u_γ and w_γ allows us to differentiate (4.12) in time and obtain

$$\gamma\,\frac{\partial^2 w_\gamma}{\partial t^2} + \frac{\partial \overline{\mathcal{G}}(w_\gamma)}{\partial t} = \frac{\partial u_\gamma}{\partial t} \qquad \text{a. e.} \tag{5.3}$$

In the series of estimates below, we denote by $C_1, C_2, \ldots$ any positive constant depending only on the data of the problem, but independent of γ.

We now test the first equation of (1.1) by $\dfrac{\partial u_\gamma}{\partial t}$ and (5.3) by $\beta\,\dfrac{\partial w_\gamma}{\partial t}$. This yields

$$\int_\Omega \left(\alpha \left(\frac{\partial u_\gamma}{\partial t}\right)^2 + \beta\, \frac{\partial u_\gamma}{\partial t}\, \frac{\partial w_\gamma}{\partial t} \right) dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_\gamma|^2\, dx = \int_\Omega \left(f\, \frac{\partial u_\gamma}{\partial t}\right) dx \tag{5.4}$$

and

$$\beta\,\frac{\gamma}{2}\frac{d}{dt}\int_\Omega \left(\frac{\partial w_\gamma}{\partial t}\right)^2 dx + \beta \int_\Omega \frac{\partial \overline{\mathcal{G}}(w_\gamma)}{\partial t}\,\frac{\partial w_\gamma}{\partial t}\, dx = \beta \int_\Omega \frac{\partial u_\gamma}{\partial t}\,\frac{\partial w_\gamma}{\partial t}\, dx. \tag{5.5}$$

Summing up (5.4), (5.5) and using (4.7), we obtain

$$\frac{\alpha}{2}\int_\Omega\left|\frac{\partial u_\gamma}{\partial t}\right|^2 dx+\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla u_\gamma|^2\,dx+c_{\mathcal{G}}\,\beta\int_\Omega\left|\frac{\partial w_\gamma}{\partial t}\right|^2 dx+\beta\,\frac{\gamma}{2}\frac{d}{dt}\int_\Omega\left|\frac{\partial w_\gamma}{\partial t}\right|^2 dx\le C_1.$$

Note that $\dfrac{\partial w_\gamma}{\partial t}(x,0)=0$ by the choice of $w_\gamma(x,0)$. This allows us to obtain the following estimates

$$\begin{cases} ||u_\gamma||_{H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;V)} \le C_2, & (5.6a)\\ ||w_\gamma||_{H^1(0,T;L^2(\Omega))} \le C_3, & (5.6b)\\ \sqrt{\gamma}\left|\left|\dfrac{\partial w_\gamma}{\partial t}\right|\right|_{L^\infty(0,T;L^2(\Omega))} \le C_4, & (5.6c)\end{cases}$$

and, by comparison, $||\triangle u_\gamma||_{L^2(Q)}\le C_5$. This entails that there exists a function u and a sequence $\gamma_n\to 0$ such that

$$u_{\gamma_n}\to u \text{ weakly star in } H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;V)\cap L^2(0,T;W^{2,2}(\Omega)).$$

On the other hand, by interpolation and after a suitable choice of representatives, we deduce that (see [14], Chapter 4)

$$H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;V)\subset L^2(\Omega;\mathcal{C}^0([0,T]))$$

with continuous and compact injection; this ensures that

$$u_{\gamma_n}\to u \text{ strongly in } L^2(\Omega;\mathcal{C}^0([0,T])),$$

in particular (passing to subsequences if necessary),

$$u_{\gamma_n}\to u \text{ uniformly in } [0,T], \text{ a.e. in } \Omega. \tag{5.7}$$

One might be tempted to use direcly equation (4.9), which yields

$$||u_\gamma-\overline{\mathcal{G}}(w_\gamma)||_{L^\infty(0,T;L^2(\Omega))}\le\gamma\left|\left|\frac{\partial w_\gamma}{\partial t}\right|\right|_{L^\infty(0,T;L^2(\Omega))}$$

and this, together with (5.6c), entails that

$$u_\gamma-\overline{\mathcal{G}}(w_\gamma)\to 0 \qquad \text{strongly in } L^\infty(0,T;L^2(\Omega)) \text{ as } \gamma\to 0.$$

However, this does not seem to be enough to conclude that $w_\gamma\to\overline{\mathcal{F}}(u)$, as neither $\overline{\mathcal{G}}$ nor $\overline{\mathcal{F}}$ are continuous in $L^\infty(0,T;L^2(\Omega))$, and a refined argument has to be used.

From now on, we keep the sequence $\gamma_n\to 0$ fixed as in (5.7). Our aim is now to show that there exists a function w such that

$$w_{\gamma_n}\to w \text{ uniformly in } [0,T], \text{ a.e. in } \Omega. \tag{5.8}$$

In fact, this will allow us to pass to the limit in the nonlinear hysteresis term. We show that (5.8) is obtained from (5.7) by using the following lemma:

Lemma 5.2. *Consider a sequence of functions* $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{C}^0([0,T])$ *such that*

$$||u_n - u||_{\mathcal{C}^0([0,T])} \to 0 \qquad \text{as } n \to \infty\,.$$

Let $0 < a_n \le \alpha_n(t) \le b_n$ *be measurable functions, with* $\lim\limits_{n\to\infty} b_n = 0$. *Finally let* $\{v_n\}_{n\in\mathbb{N}}$ *be solutions of the following Cauchy problem*

$$\begin{cases} \alpha_n(t)\,\dfrac{dv_n}{dt}(t) + v_n(t) = u_n(t), \\ v_n(0) = u_n(0). \end{cases}$$

Then

$$||v_n - u||_{\mathcal{C}^0([0,T])} \to 0 \qquad \text{as } n \to \infty\,.$$

Proof. Put $\beta_n(t) = \dfrac{1}{\alpha_n(t)}$. Then

$$v_n(t) = e^{-\int_0^t \beta_n(\tau)\,d\tau}\, u_n(0) + \int_0^t \beta_n(s)\, e^{-\int_s^t \beta_n(\tau)\,d\tau}\, u_n(s)\,ds$$

hence, for all $t \in [0,T]$, we get

$$v_n(t) - u_n(t) = e^{-\int_0^t \beta_n(\tau)\,d\tau}\,(u_n(0) - u_n(t)) + \int_0^t \beta_n(s)\, e^{-\int_s^t \beta_n(\tau)\,d\tau}\,(u_n(s) - u_n(t))\,ds.$$

Let now $\varepsilon > 0$ be given. Using the Ascoli-Arzelà theorem, we find $\delta > 0$ *independent of* n such that

$$|t_1 - t_2| < \delta \Rightarrow |u_n(t_1) - u_n(t_2)| < \varepsilon.$$

For $t \in [0,\delta]$ we have

$$|v_n(t) - u_n(t)| \le \varepsilon\, \left(e^{-\int_0^s \beta_n(\tau)\,d\tau} + \int_0^t \beta_n(s)\, e^{-\int_s^t \beta_n(\tau)\,d\tau} ds \right) = \varepsilon.$$

Let now $t > \delta$, and let

$$C = \sup\{|u_n(t_1) - u_n(t_2)|,\; t_1, t_2 \in [0,T], n \in \mathbb{N}\}.$$

Then

$$\begin{aligned} |v_n(t) - u_n(t)| &\le C e^{-\int_0^t \beta_n(\tau)\,d\tau} + \varepsilon \int_{t-\delta}^t \beta_n(s) e^{-\int_s^t \beta_n(\tau)\,d\tau}\,ds + C \int_0^{t-\delta} \beta_n(s) e^{-\int_s^t \beta_n(\tau)\,d\tau}\,ds \\ &= \varepsilon\, \left(1 - e^{-\int_{t-\delta}^t \beta_n(\tau)\,d\tau}\right) + C\, e^{-\int_{t-\delta}^t \beta_n(\tau)\,d\tau} \le \varepsilon + C\, e^{-\frac{\delta}{b_n}}, \end{aligned}$$

and thus Lemma 5.2 follows. □

Let $\Omega' \subset \Omega$ be a set of full measure ($\mathrm{meas}(\Omega \setminus \Omega') = 0$) such that, by virtue of (5.7), $u_{\gamma_n}(x,\cdot) \to u(x,\cdot)$ converge uniformly for all $x \in \Omega'$. Keeping now $x \in \Omega'$ fixed, set

$$u_\gamma(x,\cdot) =: \tilde{u}_\gamma(\cdot), \qquad w_\gamma(x,\cdot) =: \tilde{w}_\gamma(\cdot).$$

We recall from (4.2) that

$$\mathcal{F}(v(x,\cdot))(t) = \overline{\mathcal{F}}(v)(x,t) \qquad \forall\, v \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])).$$

Our idea is to apply Lemma 5.2 to the Cauchy problem

$$\begin{cases} \tilde{w}_\gamma = \mathcal{F}\left(\tilde{u}_\gamma - \gamma\, \dfrac{d\tilde{w}_\gamma}{dt}\right), \\ \tilde{w}_\gamma(0) = \mathcal{F}(\tilde{u}_\gamma)(0), \end{cases} \tag{5.9}$$

which we rewrite as

$$\begin{cases} \gamma\, \dfrac{d\tilde{w}_\gamma}{dt} + \tilde{v}_\gamma = \tilde{u}_\gamma, \\ \tilde{w}_\gamma = \mathcal{F}(\tilde{v}_\gamma), \\ \tilde{w}_\gamma(0) = \mathcal{F}(\tilde{u}_\gamma)(0). \end{cases} \tag{5.10}$$

We now set

$$\alpha_\gamma(t) = \begin{cases} \gamma\, \dfrac{d\mathcal{F}(\tilde{v}_\gamma)}{dt}(t) \Big/ \dfrac{d\tilde{v}_\gamma}{dt}(t) & \text{if } \dfrac{d\tilde{v}_\gamma}{dt} \neq 0 \\ \gamma\, c_{\mathcal{F}} & \text{if } \dfrac{d\tilde{v}_\gamma}{dt} = 0. \end{cases}$$

>From (4.6) we obtain that

$$0 < \gamma\, c_{\mathcal{F}} \leq \alpha_\gamma(t) \leq \gamma\, C_{\mathcal{F}}.$$

Hence, system (5.10) can be rewritten in the form

$$\begin{cases} \alpha_\gamma(t)\, \dfrac{d\tilde{v}_\gamma}{dt}(t) + \tilde{v}_\gamma(t) = \tilde{u}_\gamma(t), \\ \tilde{v}_\gamma(0) = \tilde{u}_\gamma(0). \end{cases}$$

We have that

$$\tilde{u}_{\gamma_n} \to \tilde{u} \;\text{ uniformly in }\; \mathcal{C}^0([0,T]) \;\text{ as }\; \gamma_n \to 0\,,$$

hence by Lemma 5.2,

$$\tilde{v}_{\gamma_n} \to \tilde{u} \;\text{ uniformly in }\; \mathcal{C}^0([0,T]) \;\text{ as }\; \gamma_n \to 0\,.$$

This in turn entails that

$$\tilde{w}_{\gamma_n} \to \mathcal{F}(\tilde{u}) \;\text{ uniformly in }\; \mathcal{C}^0([0,T]) \;\text{ as }\; \gamma_n \to 0\,.$$

Since $x \in \Omega'$ has been chosen arbitrarily, we obtain

$$w_{\gamma_n} \to \overline{\mathcal{F}}(u) \;\text{ uniformly in }\; \mathcal{C}^0([0,T])\,,\text{ a.e. in }\; \Omega \;\text{ as }\; \gamma_n \to 0\,.$$

By (5.6b),

$$\frac{\partial w_{\gamma_n}}{\partial t} \to \frac{\partial \overline{\mathcal{F}}(u)}{\partial t} \qquad \text{weakly in }\; L^2(Q_T)\,.$$

This enables us to pass to the limit in the equation

$$\frac{\partial}{\partial t}(\alpha u_\gamma + \beta w_\gamma) - \triangle u_\gamma = f.$$

We thus checked that u is a solution of (5.2) with the required boundary and initial condition. Since this solution is unique by the argument of [5], we conclude that u_γ converges to u independently of how γ tends to 0. This completes the proof of Theorem 5.1. □

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