Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Attractors for the semiflow associated with a class of doubly nonlinear parabolic equations

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submitted: 15th December 2006

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No. 1194
Berlin 2006

2000 *Mathematics Subject Classification.* 35K55, 35B40, 35B41.

Key words and phrases. doubly nonlinear equation, singular potential, semiflow, global attractor, energy method, ω-limit.

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Abstract

A doubly nonlinear parabolic equation of the form $\alpha(u_t) - \Delta u + W'(u) = f$, complemented with initial and either Dirichlet or Neumann homogeneous boundary conditions, is addressed. The two nonlinearities are given by the maximal monotone function α and by the derivative W' of a smooth but possibly nonconvex potential W; f is a known source. After defining a proper notion of solution and recalling a related existence result, we show that from any initial datum emanates *at least* one solution which gains further regularity for $t > 0$. Such *regularizing solutions* constitute a semiflow $\mathcal{S}$ for which uniqueness is satisfied for strictly positive times and we can study long time behavior properties. In particular, we can prove existence of both global and exponential attractors and investigate the structure of ω-limits of single trajectories.

1 Introduction

In this paper we are interested in the following doubly non linear parabolic equation

$$\alpha(u_t) - \Delta u + W'(u) = f, \quad \text{for a.e. } (x,t) \in \Omega \times (0, +\infty), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $1 \le N \le 3$, is a bounded domain with smooth boundary $\partial\Omega$. Here α is a differentiable and strongly monotone (i.e., $\alpha' \ge \sigma > 0$) function in $\mathbb{R}$, W' is the derivative of a λ-convex (i.e., $W'' \ge -\lambda$, $\lambda \ge 0$) configuration potential, and f is a source. The equation is complemented with the initial conditions and with homogeneous boundary conditions of either Dirichlet or Neumann type. Equations like (1.1), apart from their own mathematical interest, can arise in large variety of applications, as the modelization of phase change phenomena [9, 11, 25, 33, 34], gas flow through porous media [23] and damaging of materials [10, 24, 37].

Existence of (at least) one solution to initial-boundary value problems for a class of doubly nonlinear equations including (1.1) was proved in the paper [15] (see also [3, 7, 44] for preceding related results). The questions of regularity, uniqueness, continuous dependence on data and long time behavior of solutions, however, were not dealt with in [15] and remained widely open for a long time. Moreover, the results of [15] require the restrictive assumption that α is bounded in the sense of operators (i.e. it maps bounded sets into bounded sets), which is not always fulfilled in physical applications (see the papers quoted above referring to specific models). On account of these considerations, in the former paper [43], written in collaboration with U. Stefanelli, we introduced a new concept of solution (*stronger* than that in [15], see Def. 3.1 below) and showed existence of this kind of solution with essentially no restriction on α. This permitted to prove also uniqueness, at least in some special cases, as well as existence of nonempty ω-limits. A further contribution in this field has been recently given in [19], where a doubly nonlinear equation strictly related to (1.1), but of *degenerate* type, is addressed from the viewpoint of both well-posedness and long time behavior.

One of the main issues of this paper is a regularization property, holding for $t > 0$, of the solutions to the IBV problem for (1.1). Due to the strong parabolicity of the system ($\alpha' \geq \sigma > 0$) such a fact is to be expected; however, the proof requires a somehow tricky machinery due to the presence of very general nonlinearities. The key point, resembling in some way the approach given also in [19], consists in an Alikakos-Moser [1] iteration scheme, operated here on the (formal) time derivative of (1.1), coupled with the use (infinitely many times) of the *uniform* Gronwall lemma (see, e.g., [49]). In this way we demonstrate that, if the source f is essentially bounded, then there exist solutions $u(t)$ (called "regularizing solutions" in the sequel, see Def. 3.4) which, for $t > 0$, are in $L^\infty(\Omega)$ together with their Laplacian and with $W'(u(t))$. Moreover, for $t > 0$ uniqueness holds, whereas from any initial datum can start more than one trajectory, unless the datum is more regular itself.

The regularization property serves also as a starting point to improve the results of [43] regarding the long time behavior. Actually, in case the potential W is *real analytic* we can show, using the *Simon-Łojasiewicz* method (cf. [31, 32, 48], see also [14, 26]), that ω-limits of all single trajectories contain only one point. This can be done without the severely restrictive assumptions on the growth of α at ∞ which were considered in [43]. We remark that the Simon-Łojasiewicz method is a deep and powerful tool that in recent year has been applied to characterize ω-limit sets of solutions to several different types of nonlinear evolution equations (see, e.g., [13, 14, 22, 28, 29] among the many related works).

From the viewpoint of long time behavior, however, our main result regards the existence and regularity properties of attractors. We have to stress that a contribution to this question was already provided in [45], where a (rather weak) notion of global attractor was introduced for a class of equations including (1.1). However, due to the very general and abstract setting adopted there (very similar to that of [15]), the attractor constructed in [45] seems not very flexible from the point of view of regularity (more precisely, it appears difficult to characterize it beyond its mere existence property). Moreover, the result in [45] holds only under the boundedness assumption on α considered in [15] and consequently is not suitable for our specific situation.

Here, also thanks to the much more specific form of equation (1.1), we can prove the existence of a global attractor in the natural phase space (i.e. under the precise conditions ensuring existence). The key point is the use of the so-called *energy method* by J. Ball (cf. [6], see also [39] and the references therein), which permits to prove this result without reinforcing the conditions on the source f (namely, we do not need to ask summability of its space derivatives) and despite the apparent lack of a dissipative estimate in the natural phase space (see Remark 5.1 below). We point out that, due to the (possible) non-uniqueness at $t = 0$, the semiflow $\mathcal{S}$ associated to (1.1) for which we can prove existence of the global attractor has to be carefully defined (in particular, "nonregularizing" solutions have to be excluded, see Remark 3.11). This is in agreement with other works where equations with (at least partial) lack of uniqueness are addressed (see, e.g., [5, 6, 35, 42, 45, 46]).

Our final issue is concerned with exponential attractors, whose existence is proved by using as a technical tool the so-called method of *short trajectories* (or *ℓ-trajectories*) due to Málek and Pražák [35]. Actually, this device permits to get in a simple way the contractive estimates required to have the exponential attraction property. We stress that this approach is quite similar to that used in [38], where the equation (strictly related to (1.1) or, more precisely, to its time derivative)

$$\alpha(u)_t - \Delta u + W'(u) = f, \quad \text{for a.e. } (x,t) \in \Omega \times (0, +\infty), \tag{1.2}$$

is addressed (although under partly different assumptions on the nonlinearities).

We conclude with the plan of the paper. In the next Section some preliminary material is recalled. Next, our results are presented in a rigorous way in Section 3, where in particular the required notions of solution are introduced. The subsequent Section 4 contains the proof of the regularization property and Sections 5 and 6 are devoted to global and exponential attractors, respectively. Finally, an abstract existence Theorem for global attractors, partially generalizing [5, Thm. 3.1], is reported in the Appendix.

2 Preliminaries

In this section we introduce some notations and recall some preliminary notions which are needed to state our problem in a precise way. First of all, we set $H := L^2(\Omega)$ and denote by $(\cdot,\cdot)$ the scalar product in H and by $\|\cdot\|$ the related norm. The symbol $\|\cdot\|_X$ will indicate the norm in the generic Banach space X. Moreover, focusing on the Dirichlet case, we set $V := H^1_0(\Omega)$, $V' := H^{-1}(\Omega)$ and identify H and H' so that we obtain the Hilbert triplet $V \subset H \subset V'$, where inclusions are continuous and compact. The notation $\langle\cdot,\cdot\rangle$ will stand for the duality between V and V'. We also let $B : V \to V'$ denote the distributional Laplace operator, namely

$$B : V \to V', \quad \langle Bu, v\rangle = (\nabla u, \nabla v) \ \ \forall\, u, v \in V. \tag{2.1}$$

Remark 2.1. Here and in the sequel we assumed Dirichlet conditions just for simplicity. Indeed, the (homogeneous) Neumann case works as well with the following simple change: we have to set $V := H^1(\Omega)$, $V' := H^1(\Omega)'$ and, in place of (2.1),

$$B : V \to V', \quad \langle Bu, v\rangle = (u, v) + (\nabla u, \nabla v) \ \ \forall\, u, v \in V. \tag{2.2}$$

All the results and proofs in the sequel then still work with no further change.

In order to correctly describe the asymptotic behavior of solutions, we need to introduce the space of $L^p_{\rm loc}$-*translation bounded* functions. As X is a Banach space and $p \in [1, +\infty)$ we set

$$\mathcal{T}^p(T, \infty; X) := \left\{ v \in L^p_{\rm loc}(T, \infty; X) : \sup_{t \ge T} \int_t^{t+1} \|v(s)\|_X^p \,{\rm d}s < \infty \right\}, \tag{2.3}$$

which is a Banach space with respect to the natural (graph) norm

$$\|v\|^p_{\mathcal{T}^p(T,\infty;X)} := \sup_{t \ge T} \int_t^{t+1} \|v(s)\|_X^p. \tag{2.4}$$

Next, we recall the *uniform* Gronwall Lemma (see, e.g., [49, Lemma III.1.1]), which will be repeatedly used in the sequel:

Lemma 2.2. *Let $y, a, b \in L^1_{\rm loc}(0, +\infty)$ three non negative functions such that $y' \in L^1_{\rm loc}(0, +\infty)$ and, for some $T \ge 0$,*

$$y'(t) \le a(t)y(t) + b(t) \qquad \text{for a.e. } t \ge T, \tag{2.5}$$

and let k_1, k_2, k_3 three nonnegative constants such that

$$\|a\|_{\mathcal{T}^1(T,\infty;\mathbb{R})} \le k_1, \qquad \|b\|_{\mathcal{T}^1(T,\infty;\mathbb{R})} \le k_2, \qquad \|y\|_{\mathcal{T}^1(T,\infty;\mathbb{R})} \le k_3. \tag{2.6}$$

Then, we have that

$$y(t + \tau) \le \Big(k_2 + k_3/\tau\Big)e^{k_1} \qquad \text{for all } t \ge T. \ \blacksquare \tag{2.7}$$

Now, let us recall some basic facts about absorbing sets and attractors. Assuming that $\mathcal{X}$ is a complete metric space, we shall (conventionally) call a *semiflow* on $\mathcal{X}$ a family $\mathcal{S}$ of maps from $[0, \infty)$ to $\mathcal{X}$, called *trajectories*, satisfying properties *(S1)*-*(S5)* listed below. We stress that this definition, which partly follows the approach in [5, 6] (see also [42]), is not standard at all. Actually, in Ball's terminology, $\mathcal{S}$ could be noted like a "strongly-weakly continuous generalized semiflow with unique continuation". We say here "semiflow" just for brevity.

(S1 – existence) For all $u_0 \in \mathcal{X}$ there exists at least one $u \in \mathcal{S}$ such that $u(0) = u_0$;
(S2 – translation invariance) For all $u \in \mathcal{S}$ and $T \ge 0$, the map $v : [0, \infty) \to \mathcal{X}$ given by $v(t) := u(T + t)$ still belongs to $\mathcal{S}$;
(S3 – concatenation) For all $u, v \in \mathcal{S}$ such that for some $T > 0$ it is $u(T) = v(0)$, the map $z : [0, \infty) \to \mathcal{X}$ coinciding con u on $[0, T]$ and given by $z(t) = v(t - T)$ on (T, ∞) belongs to $\mathcal{S}$;
(S4 – unique continuation for $T > 0$) For all $u, v \in \mathcal{S}$ such that $u(T) = v(T)$ for some $T > 0$, it is $u(t) = v(t)$ for all $t \in [T, \infty)$;
(S5 – strong-weak semicontinuity) We assume that, beyond the *strong* topology induced by the metric, $\mathcal{X}$ is endowed with a *weaker* topology. Then, we firstly ask that all elements of $\mathcal{S}$ are *weakly continuous* from $[0, \infty)$ to $\mathcal{X}$. Next, that for all sequence $\{u_n\} \subset \mathcal{S}$ such that $u_n(0) =: u_{0,n}$ tends *strongly* (i.e. with respect to the metric) to some $u_0 \in \mathcal{X}$, there exist a subsequence (not relabelled) of $\{u_n\}$ and $u \in \mathcal{S}$ with $u(0) = u_0$ such that, for all $t > 0$, $u_n(t)$ tends *weakly* to $u(t)$.

Remark 2.3. Regarding *(S5)*, if $\mathcal{X}$ is a Banach space, a natural choice for the "weak topology" mentioned there is of course that induced by the weak (or, in some cases, the weak star) convergence. We will show in the sequel (see in particular the Appendix) that the lack of a more usual "strong-strong" continuity property does not prevent use of time regularization-compactness methods to get existence of the global attractor. This fact has been noted also in other recent papers [40, 51].

We assumed property *(S4)*, which is not completely standard, just to fit the case of our system for which uniqueness holds only from $t > 0$. If $\mathcal{S}$ is a semiflow, we define the *space of regularized values* of $\mathcal{S}$ as

$$\mathcal{X}_{\mathrm{reg}} := \Big\{ u(t) : u \in \mathcal{S}, t > 0 \Big\}. \tag{2.8}$$

Moreover, if $u \in \mathcal{S}$, we recall that the (strong) ω-limit of u is the set of all limit (w.r.t. the metric) points of subsequences of $u(t)$ as $t \nearrow \infty$. >From *(S2)* and *(S4)*, it is also apparent that it can be naturally associated to a semiflow $\mathcal{S}$ the family $\{S(t)\}$, $t \in [0, \infty)$, of operators from $\mathcal{X}_{\mathrm{reg}}$ to itself, with $S(t)$ mapping $x \in \mathcal{X}_{\mathrm{reg}}$ into $u(t)$, where $u \in \mathcal{S}$ is the (unique) trajectory such that $u(0) = x$. It is then clear that $\{S(t)\}$ satisfies the usual semigroup properties. Due to the lack of uniqueness, $S(t)$ cannot be extended to the whole $\mathcal{X}$. Nevertheless, we can introduce the family of *multivalued* mappings $\{T(t)\}$, $t \in [0, \infty)$, given by

$$T(t) : \mathcal{X} \to 2^{\mathcal{X}}, \qquad T(t)u := \Big\{ v(t) : v \in \mathcal{S}, v(0) = u \Big\} \tag{2.9}$$

and by *(S4)* it is then clear that the restriction of $T(t)$ to $\mathcal{X}_{\mathrm{reg}}$ coincides with $S(t)$.

Next, we recall that a compact subset $\mathcal{A}$ of the *phase space* $\mathcal{X}$ is the *global attractor* for the semiflow $\mathcal{S}$ if the following conditions are satisfied:

(A1) The set $\mathcal{A}$ is strictly invariant, i.e., $T(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$;

(A2) $\mathcal{A}$ attracts the images of all bounded subsets of $\mathcal{X}$ as $t \nearrow +\infty$, namely

$$\lim_{t \nearrow +\infty} \operatorname{dist}(T(t)B, \mathcal{A}) = 0, \quad \text{for all bounded } B \subset \mathcal{X}, \tag{2.10}$$

where dist is the *non-symmetric* Hausdorff distance between sets in $\mathcal{X}$ (see, e.g., [21, 49]).

We point out that the global attractor represents the first (although extremely important) step in the understanding of the long-time dynamics of a given evolutive process. However, it may also present some drawbacks. First of all, it may be reduced to a single point, thus failing in capturing all the transient behaviour of the system. Moreover, in general it is extremely difficult to estimate the rate of convergence in (2.10) and to express it in terms of the physical parameters of the system. In this regard, simple examples show that this rate of convergence may be arbitrarily slow. This fact makes the global attractor very sensitive to perturbations and to numerical approximation. The concept of *exponential attractor* has then been proposed (see, e.g., [16]) to possibly overcome this difficulty. We recall that a compact subset $\mathcal{M}$ of the phase space $\mathcal{X}$ is called an *exponential attractor* for the semiflow $\mathcal{S}$ if the following conditions are satisfied:

(E1) The set $\mathcal{M}$ is positively invariant, i.e., $T(t)\mathcal{M} \subset \mathcal{M}$ for all $t \geq 0$;

(E2) The fractal dimension (see, e.g., [36, 49]) of $\mathcal{M}$ in $\mathcal{X}$ is finite;

(E3) The set $\mathcal{M}$ attracts exponentially fast the images of the bounded sets B of the phase space $\mathcal{X}$. Namely, for every bounded $B \subset \mathcal{X}$ there exist $C, \beta > 0$ depending on B and such that

$$\operatorname{dist}(T(t)B, \mathcal{M}) \leq C e^{-\beta t}, \quad \forall t \geq 0. \tag{2.11}$$

Thanks to $(E3)$ it follows that, compared to the global attractor, an exponential attractor is much more robust to perturbation and to the important issue of numerical approximation (see, e.g., [16] and [20]). Moreover, when the exponential attractor $\mathcal{M}$ exists, it contains the global attractor $\mathcal{A}$. Thus, in this case also $\mathcal{A}$ has finite fractal dimension. We point out that, however, also the theory of exponential attractors presents some disadvantages, like the lack of uniqueness of $\mathcal{M}$, whose choice or construction may be in some sense artificial. However, we refer to [18] where it is proposed a construction of an exponential attractor which selects a proper one valued branch of the exponential attractors depending in an Hölder continuous way on the dynamical system under study. In recent years several different techniques have been provided to guarantee existence of exponential attractors. Beyond the original method [16] based on a direct verification of the *discrete squeezing property*, we quote the "decomposition technique" developed in [17] and, in particular, the so-called method of "ℓ-trajectories" (or "short trajectories"), introduced by Málek and Pražák in [35], which provides a simplified framework which can be adopted to verify the theoretical conditions of [16] leading to existence of $\mathcal{M}$. Since we shall use this method in the sequel, we recall here, for convenience of the reader, its highlights, partly adapting the presentation in [35] to our more specific framework.

Let $\mathcal{X}$ be a *Hilbert* space and, for given $\tau > 0$, let us set $\mathcal{X}_\tau := L^2(0, \tau, \mathcal{X})$. We assume that there exists a subset B_1 of $\mathcal{X}$ such that for any $u_0 \in B_1$ there exists at least one map $u \in C_w([0, \infty); \mathcal{X})$ such that $u(0) = u_0$. These maps u are called "solutions" in what follows, and we assume that they form a semiflow $\mathcal{S}$ on the set B_1 endowed with the strong and weak topologies inherited from $\mathcal{X}$. We then introduce the space of ℓ-trajectories (where $\ell > 0$) as

$$\mathcal{B}^1_\ell := \left\{ \chi : (0, \ell) \to \mathcal{X}, \ \ \chi \text{ is a solution on } (0, \ell) \right\}. \tag{2.12}$$

The space $\mathcal{B}^1_\ell$ inherits its topology from $\mathcal{X}_\ell$. Moreover, according to *(S4)*, any ℓ-trajectory has, among all solutions, unique continuation. We shall assume that

$$\mathcal{B}^1_\ell \ \text{ is relatively compact in } \ \mathcal{X}_\ell, \tag{2.13}$$

where the closure is taken with respect to the topology of $\mathcal{X}_\ell$. Then, the method of ℓ-trajectories basically consists in lifting the dynamical system from the phase space of initial conditions to the space $\mathcal{B}^1_\ell$ of ℓ-trajectories. In particular, by *(S4)* we can define a semigroup L_t on $\mathcal{B}^1_\ell$ by setting

$$\{L_t \chi\}(\tau) := u(t+\tau), \quad \tau \in [0,\ell], \tag{2.14}$$

where χ is an ℓ-trajectory and u is the *unique* solution such that $u|_{[0,\ell]} = \chi$. We then define Then, the assumptions that lead to the existence of the exponential attractor in the space of ℓ-trajectories endowed with the topology of $\mathcal{X}_\ell$ read as follows (see [35]):

(M1) There exists a space W_ℓ compactly embedded into $\mathcal{X}_\ell$ and $\tau > 0$ such that $L_\tau : \mathcal{X}_\ell \to W_\ell$ is Lipschitz continuous on $\mathcal{B}^1_\ell$;
(M2) For all $\tau > 0$ the family of operators $L_t : \mathcal{X}_\ell \to \mathcal{X}_\ell$ is uniformly (w.r.t. $t \in [0,\tau]$) Lipschitz continuous on $\mathcal{B}^1_\ell$;
(M3) For all $\tau > 0$ there exist $c > 0$ and $\beta \in (0,1]$ such that for all $\chi \in \mathcal{B}^1_\ell$ and $t_1, t_2 \in [0,\tau]$ it holds that $\|L_{t_1}\chi - L_{t_2}\chi\|_{\mathcal{X}_\ell} \le c|t_1 - t_2|^\beta$.

In [35, Theorem 2.5] it is proved that, under the assumptions above, there exists an exponential attractor $\mathcal{M}_\ell$ in $\mathscr{X}_\ell$ for the dynamical system L_t on $\mathcal{B}^1_\ell$. One of the striking features of this method is that, once we have constructed an exponential attractor in the space of ℓ-trajectories, we can recover the dynamics in the original phase space B_1 and obtain an exponential attractor $\mathcal{M}$ for the semiflow $\mathcal{S}$. To this end, we introduce the evaluation map $e : \mathcal{B}^1_\ell \to \mathcal{X}$ which assigns to a given ℓ-trajectory χ its end point. More precisely, we define

$$e : \mathcal{B}^1_\ell \to \mathcal{X}, \quad \text{given by } \; e(\chi) := \chi(\ell). \tag{2.15}$$

By requiring

(M4) The map e is Hölder continuous on $\mathcal{B}^1_\ell$,

we obtain the exponential attractor in the phase space as the image of $\mathcal{E}_\ell$ (see [35, Theorem 2.6]), namely we have that $\mathcal{M} := e(\mathcal{M}_\ell)$ is an exponential attractor for the semiflow $\mathcal{S}$ on the space B_1.

Remark 2.4. In general, the semiflow $\mathcal{S}$ is originally defined on a space $\mathcal{X}$ "larger" than the bounded set B_1 (usually, but not in our case, on the whole $\mathcal{X}$), and B_1 is chosen "a posteriori" as a bounded, absorbing and positively invariant set for the "original" $\mathcal{S}$. One of the advantages of this approach is then that property (2.13) requires in general very little smoothing effect (and is usually straighforward to be checked in concrete situations). We also note that, once we have the exponential attractor $\mathcal{M}$ on B_1, as B_1 is absorbing, $\mathcal{M}$ turns out to be an exponential attractor on the whole space $\mathcal{X}$.

3 Main results

We begin by specifying our basic assumptions on data. First of all, we ask

$$\alpha \in C^1(\mathbb{R};\mathbb{R}), \quad \alpha(0) = 0, \quad \alpha'(r) \ge \sigma > 0 \;\; \text{for all } \; r \in \mathbb{R}. \tag{hpα}$$

Next, given $\lambda \geq 0$ and an *open* (either bounded or unbounded) interval $I \subset \mathbb{R}$ with $0 \in I$, we assume that the potential W fulfills

$$W \in C^{1,1}_{\mathrm{loc}}(I;\mathbb{R}), \quad W'(0) = 0, \quad W'' \geq -\lambda \ \text{ a.e. in } \ I, \tag{hpW1}$$

$$\lim_{r \to \partial I} W'(r) \operatorname{sign} r = +\infty. \tag{hpW2}$$

Property (hpW1) is called λ*-convexity* in what follows (see [2] for the definition). Since W is defined up to an additive constant, it is also not restrictive to suppose that

$$\exists\, \eta > 0 : \quad W(r) \geq \frac{\eta r^2}{2} \ \text{ for all } \ r \in I. \tag{3.1}$$

We then introduce the basic *phase space* for our analysis:

$$\mathcal{X}_2 := \Big\{ u \in H : Bu,\, W'(u) \in H \Big\}, \tag{3.2}$$

which is endowed with the metric

$$d_2^2(u,v) := \|u-v\|^2 + \|Bu - Bv\|^2 + \|(W'+\lambda)(u) - (W'+\lambda)(v)\|^2. \tag{3.3}$$

Proceeding as in cite [41, Lemma 3.8] (compare also with [45, Sec. 3]), it is easy to show that $\mathcal{X}_2$ is a *complete* metric space. It is also clear that $\mathcal{X}_2 \subset V \cap H^2(\Omega)$ (continuously); however, if $I \neq \mathbb{R}$, in general the inclusion is strict.

We can now list our hypotheses on the initial and source data:

$$u_0 \in \mathcal{X}_2, \tag{hpu_0}$$

$$f \in L^\infty(\Omega). \tag{hp$f$}$$

Then, standardly identifying α and W' as operators from H to itself, we introduce the

Definition 3.1. *We call an* $\mathcal{X}_2$-solution *to the Problem* (P) *given by*

$$\alpha(u_t) + Bu + W'(u) = f, \quad \text{in } \ H, \quad \text{a.e. in } \ (0,\infty), \tag{3.4}$$

$$u|_{t=0} = u_0, \quad \text{in } \ H \tag{3.5}$$

one function $u : [0,\infty) \to H$ *satisfying* (3.4), (3.5)*, and, for some* $C > 0$,

$$u,\, u_t,\, \alpha(u_t),\, Bu,\, W'(u) \in L^\infty(0,\infty;H), \tag{3.6}$$

$$d_2^2(u(t),0) = \|u(t)\|^2 + \|Bu(t)\|^2 + \|(W'+\lambda)(u(t))\|^2 \leq C^2 \quad \text{for all } \ t \in [0,\infty). \tag{3.7}$$

We note that (3.4)–(3.5) give a rigorous formulation of the IBV problem for (1.1). With condition (3.7) we ask the solution to stay in the phase space $\mathcal{X}_2$ for any (and not just a.e.) value of the time variable. We can now recall the statement of the existence result proved in [43, Thm. 2.5]:

Theorem 3.2. *Assume* (hpα), (hpW1)–(hpW2), *and* (hpu_0)–(hp$f$). *More precisely, suppose that for some* $\kappa > 0$ *it is*

$$d_2^2(u_0, 0) = \|u_0\|^2 + \|Bu_0\|^2 + \|(W' + \lambda)(u_0)\|^2 \leq \kappa^2. \tag{3.8}$$

Then, Problem (P) *admits at least one* $\mathcal{X}_2$*-solution, which additionally satisfies*

$$\|u_t\|^2_{L^2(0,t;V)} \leq C^2. \tag{3.9}$$

Moreover, the constants C *in* (3.7), (3.9) *depend only on* Ω, α, W, f, *and (linearly) on* κ *in* (3.8). *In particular, they do not depend on the time* t.

We remark that (3.9), which was not stated in [43, Thm. 2.5] since the coercivity hypotheses on α considered there were weaker, follows easily from the proof in [43, Sec. 3] thanks to the last assumption in (hpα). Let us now see that *some* solutions to Problem (P) gain more spatial regularity for $t > 0$. With this aim, we introduce the new space

$$\mathcal{X}_\infty := \Big\{ u \in L^\infty(\Omega) : Bu,\ W'(u) \in L^\infty(\Omega) \Big\}, \tag{3.10}$$

which is naturally endowed with the (complete) metric

$$d_\infty^2(u, v) := \|u-v\|^2_{L^\infty(\Omega)} + \|Bu-Bv\|^2_{L^\infty(\Omega)} + \|(W'+\lambda)(u)-(W'+\lambda)(v)\|^2_{L^\infty(\Omega)}. \tag{3.11}$$

We also introduce weaker notions of convergence (and, in fact, weaker topologies) on the spaces $\mathcal{X}_2$, $\mathcal{X}_\infty$. Namely, we say that a sequence $\{u_n\}$ tends to u weakly in $\mathcal{X}_2$ (in $\mathcal{X}_\infty$) if $u_n \to u$, $Bu_n \to Bu$, and $(W' + \lambda)(u_n) \to (W' + \lambda)(u)$ weakly in H (weakly *star* in $L^\infty(\Omega)$, respectively). When we construct below the semiflow $\mathcal{S}$ on $\mathcal{X}_2$, property *(S5)* will be implicitly intended with respect to this weak structure.

To proceed, we need to introduce a couple of functionals defined on the space $\mathcal{X}_2$, the first of which has the meaning of *energy*:

$$\mathcal{E}(u) := \int_\Omega \left[\frac{|\nabla u|^2}{2} + W(u) - fu \right], \tag{3.12}$$

$$\mathcal{F}(u) := \frac{1}{2}\|Bu + W'(u)\|^2 - (f, Bu + W'(u)). \tag{3.13}$$

It is clear that, since (3.1) and (hp$f$) hold, both functionals are finite and bounded from below on $\mathcal{X}_2$. Moreover, mimicking the procedure given in [43, Sec. 3], i.e., *formally* testing (3.4) by $\lambda u_t + (Bu + W'(u))_t$, and using in particular (hpW1), one can expect that solutions u to Problem (P) satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\lambda\mathcal{E} + \mathcal{F}\Big)(u(t)) \leq 0 \quad \text{for a.e. } t \geq 0. \tag{3.14}$$

Setting then $\mathcal{G} := \lambda\mathcal{E} + \mathcal{F}$ and noting that there exist $\eta_1, \eta_3 > 0$ and $\eta_2 \geq 0$ such that

$$\eta_1 d_2^2(u, 0) - \eta_2 \leq \mathcal{G}(u) \leq \eta_3\Big(d_2^2(u, 0) + 1\Big) \quad \forall\, u \in \mathcal{X}_2, \tag{3.15}$$

relation (3.14) takes the form of a decay (or Liapounov) condition for the distance d_2.

However, the formal procedure used to get (3.14) seems very difficult to be justified if we just know that u is an $\mathcal{X}_2$-solution. Actually, (3.4) is settled in H and (3.6) does not imply that the test function $\lambda u_t + (Bu + W'(u))_t$ takes values in H.

To overcome this difficulty, we recall that the existence Theorem 3.2 was shown in [43] via approximation and compactness methods. We sketch here, and partly refine, just the highlights of this procedure. Let us substitute α and W in (3.4) with regularized functions α_n and W_n still satisfying (hpα), (hpW1) and such that

$$\alpha_n, (W_n' + 2\lambda\,\mathrm{Id}) \text{ are Lipschitz continuous with their inverses,} \tag{3.16}$$

$$\alpha_n, (W_n' + 2\lambda\,\mathrm{Id}) \to \alpha, (W' + 2\lambda\,\mathrm{Id}) \text{ in the sense of graphs [4],} \tag{3.17}$$

the latter convergences intended as $n \nearrow \infty$. Then, noting as (P$_n$) the problem still given by (3.4) (with the regularized functions) and (3.5) (note that the initial datum is *not* regularized), it is not difficult to show the

Proposition 3.3. *For every $n > 0$, Problem* (P$_n$) *has one and only one solution u_n such that*

$$u_{n,tt} \in L^2(0, \infty, H), \qquad u_n, u_{n,t} \in L^2(0, \infty, H^2(\Omega)). \tag{3.18}$$

Moreover, u_n satisfies estimates (3.6), (3.7) *with C independent of n. Finally, for any subsequence of $n \nearrow \infty$, there exists a subsubsequence (still noted here as u_n) such that u_n suitably (i.e., in the sense specified by* (3.6) *and* (3.7)*) tends to u, where u is an $\mathcal{X}_2$-solution to Problem* (P).

We point out that the proof of the above Proposition could be performed just by refining the estimates and the passage to the limit in [43, Sec. 3]. We omit, for brevity, the technical details of the argument and rather focus our attentions on the more subtle consequences of working with solutions u_n of (P$_n$). Of course, the functions u_n do satisfy (3.14) (where, of course, W_n replaces W in $\mathcal{G}$). However, the convergence $u_n \to u$ specified by estimate (3.7) is too weak to let (3.14) pass to the limit with n. Moreover, due to nonuniqueness for the problem (P), there might exist some $\mathcal{X}_2$-solutions which are not, or at least are not known to be, limit of (sub)sequences of solutions to (P$_n$). Actually, we shall note in the sequel as *limiting* (respectively, *nonlimiting*) the solutions to (P) which are (respectively, are not) limits of (sub)sequences of solutions to (P$_n$). For all these reasons, we have to introduce a new concept of solution, where a (much weaker than (3.14)) form of Liapounov property (cf. (3.20) below) for $\mathcal{G}$ is *postulated*. From the proofs, it will be clear that all limiting solutions satisfy (3.20), but there might also exist nonlimiting solutions satisfying it.

Definition 3.4. *A* regularizing solution *to Problem* (P) *is an $\mathcal{X}_2$-solution which, additionally, fulfills the* regularization property

$$u_t,\ \alpha(u_t),\ Bu,\ W'(u) \in L^\infty(\Omega \times (T, \infty)) \quad \forall T > 0 \tag{3.19}$$

and the Liapounov condition

$$\mathcal{G}(u(t)) \le \mathcal{G}(u(0)) \quad \textit{for all } t \ge 0. \tag{3.20}$$

Then, we have the following result, which will be proved in the next Section 4:

Theorem 3.5 (Regularizing solutions)**.** *Let* (hpα), (hpW1)–(hpW2) *and* (hpu_0)–(hp$f$) *with* (3.8) *hold. Then, Problem* (P) *admits at least one regularizing solution. Moreover, there exist constants $\mathfrak{c}_1, \mathfrak{c}_2 > 0$ and a continuous and monotone function $\phi : [0,\infty) \to [0,\infty)$, all independent both of the initial data and of time, and explicitly computable in terms of Ω, α, W, f, such that, for* every *regularizing solution and all $T > 0$, it is*

$$\|u_t(t)\|^2_{L^\infty(\Omega)} \le \mathfrak{c}_1 \frac{1 + \mathcal{G}(u_0)}{T^{\mathfrak{c}_2}} \quad \forall\, t \ge T, \tag{3.21}$$

$$d^2_\infty(u(t), 0) \le \phi\left(\mathfrak{c}_1 \frac{1 + \mathcal{G}(u_0)}{T^{\mathfrak{c}_2}}\right) \quad \forall\, t \ge T. \tag{3.22}$$

In particular, thanks to the second inequality in (3.15) and to (3.8), the bounds (3.21), (3.22) depend only on the "radius" κ of the initial datum with respect to d_2.

Theorem 3.5 is the starting point for all the subsequent investigations. As a first consequence, using the last of (3.21) and (hpW2), from straightforward arguments there follows the

Corollary 3.6 (Separation)**.** *Let* (hpα), (hpW1)–(hpW2) *and* (hpu_0)–(hp$f$) *hold, and let u be a regularizing solution. Then, for any $T > 0$ there exist $\underline{r} < 0, \overline{r} > 0$, with $\inf I < \underline{r} < 0 < \overline{r} < \sup I$, such that*

$$\underline{r} \le u(x,t) \le \overline{r} \quad \forall\, x \in \Omega,\ t \ge T. \tag{3.23}$$

Remark 3.7. The *separation property* (3.23) stated in the Corollary improves the analogous property shown in [43, Prop. 2.10] and holding for less regular solutions (i.e., $\mathcal{X}_2$-solutions in our notation) under additional assumptions on W.

The local Lipschitz continuity of W' (following from (hpW1)) and the simple argument used in [43, Proof of Thm. 2.11] permit then to obtain immediately the

Corollary 3.8 (Uniqueness)**.** *Assume* (hpα), (hpW1)–(hpW2) *and* (hpu_0)–(hp$f$). *Let also u, v be a pair of $\mathcal{X}_2$-solutions satisfying, for some $T, c \ge 0$,*

$$d_\infty(u(t), 0) + d_\infty(v(t), 0) \le c \quad \forall\, t \ge T, \tag{3.24}$$

with c independent of t. Then, $u \equiv v$ on $[T, \infty)$.

The proof of the next result will be detailed in Section 4.

Corollary 3.9. *Under assumptions* (hpα), (hpW1)–(hpW2) *and* (hpu_0)–(hp$f$), *the set $\mathcal{S}$ of regularizing solutions to Problem* (P) *is a semiflow, whose space of regularized values is contained into $\mathcal{X}_\infty$.*

Remark 3.10. Comparing our assumptions on α, W with those taken in [43], we point out that here (cf. (hpW2)), if $I \neq \mathbb{R}$, we are not able to consider potentials *bounded* in $\overline{I}$ (like, e.g., the "double obstacle" $W(r) \sim I_{[-1,1]}(r) - \lambda r^2/2$, $I_{[-1,1]}$ being the *indicator function* of $[-1,1]$). More precisely, this restriction is not required in the proof of Theorem 3.5, where only (hpW1) is used, but in the subsequent Corollaries 3.6 and 3.8. Concerning α, differently from [43], we cannot consider here the case in which α is a maximal monotone function with some multivalued branch, and in particular we are not able to deal with the situation where the domain of α is *strictly* included in $\mathbb{R}$ (as it happens, e.g., in the application to *irreversible* phase transitions considered in [25, 33, 34]). Indeed, in case $\operatorname{dom}\alpha \neq \mathbb{R}$, one can still deduce (3.21), but not (3.22), which is crucial for the long time analysis.

Remark 3.11. The non-uniqueness of solutions to (P) can be precised as follows. Given an initial datum $u_0 \in \mathcal{X}_2$, from it more than one solution can emanate. In particular, there are one, or more, regularizing solutions, starting from u_0, at least one of which is limiting, and all these regularizing solutions are taken as elements of the semiflow $\mathcal{S}$. Other solutions can also exist which are not elements of $\mathcal{S}$. In particular, (nonlimiting) smooth solutions enjoying (3.21) but not (3.20) are excluded from $\mathcal{S}$.

Let us now come to the long time behavior.

Theorem 3.12 (Global attractor)**.** *Assume* (hpα), (hpW1)–(hpW2) *and* (hpu_0)–(hp$f$). *Then, the semiflow $\mathcal{S}$ associated with Problem* (P) *admits the global attractor $\mathcal{A}$, which is compact in $\mathcal{X}_2$ and "sequentially weakly compact" in $\mathcal{X}_\infty$ (i.e., sequences in $\mathcal{A}$ admit subsequences "weakly" converging in $\mathcal{X}_\infty$).*

Theorem 3.13 (Exponential attractors)**.** *Suppose that* (hpα), (hpW1)–(hpW2) *and* (hpu_0)–(hp$f$) *hold. Then, the semiflow $\mathcal{S}$ associated with Problem* (P) *admits an exponential attractor $\mathcal{M}$. More precisely, $\mathcal{M}$ is a compact subset of V which attract exponentially fast with respect to the V-norm any d_2-bounded subsets of $\mathcal{X}_2$.*

Remark 3.14. We showed existence of $\mathcal{M}$ by working in V rather than in $\mathcal{X}_2$ since, due to the nonlinear character of W, it seems difficult to prove a contracting estimate in the metric d_2. Instead, refining our procedure it should be possible to choose, at least, $\mathcal{X} = V \cap H^2(\Omega)$. Nevertheless, in this case, the argument (especially the proof of *(M1)*) would have become very technical.

As recalled in Section 2, the existence of $\mathcal{M}$ entails that the global attractor $\mathcal{A}$ is contained in $\mathcal{M}$ and has finite fractal dimension in V.

As a final issue, by virtue of the L^∞-bound on u_t, we are able to sharpen the results in [43] concerning ω-limits of the elements of $\mathcal{S}$. Actually, since $\alpha(0) = 0$, it is clear (cf. [43, Thm. 2.13]) that the stationary states u_∞ of (3.4) are solutions of

$$Bu_\infty + W'(u_\infty) = f \quad \text{in } H. \tag{3.25}$$

It is well known that, since W needs not be convex, (3.25) may well admit infinitely many solutions [27], all of which, due to (hpW1), (hpW2) and standard elliptic regularity results, belong to $\mathcal{X}_\infty$. Thus, given $u \in \mathcal{S}$, the question of the convergence of all the trajectory $u(t)$ to one of these solutions may be non trivial. As in [43], we are able to show this property by making use of the so-called Łojasiewicz-Simon inequality [31, 32, 48], at least provided that

$$W|_{I_0} \quad \text{is real analytic,} \tag{3.26}$$

where $I_0 \subset I$ is an open interval containing 0 and such that $W'(r)r > 0$ for all $r \in I \setminus I_0$. Clearly, I_0 exists thanks to (hpW2); moreover, by maximum principle arguments, any solution to (3.25) takes values in a compact subset of I_0. Then, we have the following

Theorem 3.15 (Convergence to the stationary states)**.** *Let us assume hypotheses* (hpα), (hpW1)–(hpW2), (hpu_0)–(hp$f$) *and* (3.26). *Then, letting u be a regularizing solution, the ω-limit of u consists of a* unique *function u_∞ solving* (3.25). *Furthermore, as $t \nearrow +\infty$,*

$$u(t) \to u_\infty \qquad \textit{strongly in } V \cap C(\overline{\Omega}), \tag{3.27}$$

i.e., we have convergence for the whole trajectory $u(t)$.

The difference between this result and [43, Thm. 2.18] lies in the fact that, thanks to (3.19), we need not assume any growth condition on α. Roughly speaking, the L^∞-bound on u_t combined with the regularity and the coercivity of α (see (hpα)) reduces the nonlinearity α to an almost "linear" contribution and makes the analysis of the convergence of the trajectory simpler. In fact, Theorem 3.15 can be proved by simply adapting the proof given in [14]. We leave the details to the reader.

Remark 3.16 (The asymptotically autonomous case)**.** For the sake of studying ω-limits, we could also consider time dependent sources, by assuming, instead of (hp$f$),

$$f \in L^2(0,+\infty; L^\infty(\Omega)), \quad f_t \in L^1(0,+\infty; L^\infty(\Omega)). \tag{3.28}$$

Indeed, it could be shown that Theorem 3.5 and Corollaries 3.6, 3.8, and 3.9 still hold in this setting. Moreover, assuming also that there exist $c, \xi > 0$ such that

$$t^{1+\xi} \int_t^\infty \|f(s)\|^2 \, \mathrm{d}s \le c \quad \text{for all } \ t \ge 0, \tag{3.29}$$

Theorem 3.15 could be extended as well (see also [14, 26] for this kind of assumptions).

4 Regularization in time

Proof of Theorem 3.5. We shall use an Alikakos-Moser [1] iteration argument for which some a priori estimates are needed. In particular, we shall work on the (formal) time derivative of (3.4), namely given by

$$\alpha'(u_t)u_{tt} + Bu_t + W''(u)u_t = 0. \tag{4.1}$$

Of course, (4.1) needs not make sense if u is just an $\mathcal{X}_2$-solution. However, we can write it for Problem (P_n), derive the estimates at the level n, and then let them pass to the limit $n \nearrow \infty$ using the semicontinuity properties of norms w.r.t. weak convergences. This approach has the drawback that, at a first stage, the estimates will hold only for the "limiting solutions". They will be properly extended to all regularizing solutions in the second part of the proof.

Before proceeding, we introduce some further notation. For simplicity, we shall omit the index n of the approximation in all what follows. The symbol c will stand for a positive constant, possibly varying even inside one single line, which is allowed to depend on the data Ω, α, W, f, but neither on the initial values, nor on time. The constant(s) c will be also independent of the exponents p_j of the iteration process (see below) and, of course, of n. Some c's whose precise value is needed will be distinguished by noting them as c_i, $i \geq 0$. Let us now set, for $p \in [2, \infty)$,

$$a_p(s) := \int_0^s \alpha'(r)|r|^{p-2}r \,\mathrm{d}r \tag{4.2}$$

and notice that (recall that $\alpha(0) = 0$)

$$\frac{\sigma}{p}|s|^p \leq a_p(s) \leq \alpha(s)|s|^{p-2}s \quad \forall\, s \in \mathbb{R}. \tag{4.3}$$

Moreover, it is clear that (at least formally, as noted above)

$$\frac{\mathrm{d}}{\mathrm{d}t}a_p(u_t) = \alpha'(u_t)|u_t|^{p-2}u_t u_{tt}. \tag{4.4}$$

Then, testing (4.1) by u_t, recalling the second of (hpW1) and adding $\lambda\|u_t\|^2$ on both hands sides, and integrating over $(0, t)$, we get

$$2\|a_2(u_t(t))\|_{L^1(\Omega)} + 2\|u_t\|^2_{L^2(0,t;V)} \leq 2\|a_2(u_t(0))\|_{L^1(\Omega)} + c\|u_t\|^2_{L^2(0,t;H)}. \tag{4.5}$$

To control the latter term in the right hand side above, we can use (3.9). The other one, by (4.3) with $p = 2$ and Young's inequality, becomes

$$2\|a_2(u_t(0))\|_{L^1(\Omega))} \leq \|\alpha(u_t(0))\|^2 + \|u_t(0)\|^2 \leq c(1+\kappa)^2, \tag{4.6}$$

where the latter inequality is a consequence of a comparison in (3.4) (written for (P_n)) and of assumption (hpu_0) (κ is as in (3.8)). Actually, α^{-1} is Lipschitz continuous due to (hpα). In conclusion, from (4.5) we obtain

$$2\|a_2(u_t)\|_{L^\infty(0,\infty;L^1(\Omega))} + 2\|u_t\|^2_{L^2(0,\infty;V)} \leq c_0(1+\kappa)^2. \tag{4.7}$$

We can now describe the two estimates which are at the core of the iteration process.

First estimate. Let $j \geq 1$, $p_j > 1$, and let us test (4.1) by $|u_t|^{p_j-2}u_t$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega a_{p_j}(u_t) + \Big(Bu_t, |u_t|^{p_j-2}u_t\Big) \leq \lambda \|u_t\|_{p_j}^{p_j} \tag{4.8}$$

(we agree, here and in the sequel, to note by $\|\cdot\|_p$ the norm in $L^p(\Omega)$ for $p \in [1,\infty]$). By definition of B and Poincaré's inequality (everything works with minor changes also in the Neumann case),

$$\Big(Bu_t, |u_t|^{p_j-2}u_t\Big) \geq \frac{4(p_j-1)}{p_j^2}\int_\Omega \left|\nabla\Big(|u_t|^{\frac{p_j-2}{2}}u_t\Big)\right|^2 \geq \frac{c_1}{p_j}\|u_t\|_{3p_j}^{p_j}, \tag{4.9}$$

for some $c_1 > 0$. Assuming then that there exist $T_j, \ell_j > 0$ such that

$$p_j\|a_{p_j}(u_t)\|_{\mathcal{T}^1(T_j,\infty;L^1(\Omega))} \leq \ell_j, \qquad p_j\|u_t\|^{p_j}_{\mathcal{T}^{p_j}(T_j,\infty;L^{p_j}(\Omega))} \leq \ell_j \tag{4.10}$$

and multiplying (4.8) by p_j, from Lemma 2.2 we get, for $\tau_j \in (0,1]$,

$$p_j\|a_{p_j}(u_t(t+\tau_j))\|_{L^1(\Omega)} \leq \ell_j\bigg(\lambda + \frac{1}{\tau_j}\bigg) \qquad \forall\, t \geq T_j, \tag{4.11}$$

whence, recalling (4.3), we also have

$$\|u_t(t+\tau_j)\|_{p_j}^{p_j} \leq \frac{\ell_j}{\sigma}\bigg(\lambda + \frac{1}{\tau_j}\bigg) \qquad \forall\, t \geq T_j. \tag{4.12}$$

Moreover, integrating p_j times (4.8) over $(t, t+1)$ for $t \geq T_j + \tau_j$, and taking (4.9), (4.11) into account, it is not difficult to infer

$$\int_t^{t+1} \|u_t(s)\|_{3p_j}^{p_j}\,\mathrm{d}s \leq \frac{\ell_j}{c_1}\bigg(2\lambda + \frac{1}{\tau_j}\bigg) \qquad \forall\, t \geq T_j + \tau_j. \tag{4.13}$$

Interpolation argument. By elementary interpolation of L^p spaces, we have

$$\|u_t(t)\|_{7p_j/3} \leq \|u_t(t)\|_{p_j}^{1/7}\|u_t(t)\|_{3p_j}^{6/7} \qquad \forall\, t \geq T_j + \tau_j. \tag{4.14}$$

Hence, still for $t \geq T_j + \tau_j$,

$$\int_t^{t+1} \|u_t(s)\|_{7p_j/3}^{7p_j/6}\,\mathrm{d}s \leq \|u_t\|^{p_j/6}_{L^\infty(t,t+1,L^{p_j}(\Omega))}\int_t^{t+1}\|u_t(s)\|_{3p_j}^{p_j}\,\mathrm{d}s. \tag{4.15}$$

Thus, from (4.12) and (4.13),

$$\|u_t\|^{7p_j/6}_{\mathcal{T}^{7p_j/6}(T_j+\tau_j,\infty;L^{7p_j/3}(\Omega))} \leq \bigg(\frac{\ell_j}{\sigma}\bigg)^{1/6}\bigg(\lambda + \frac{1}{\tau_j}\bigg)^{1/6}\frac{\ell_j}{c_1}\bigg(2\lambda + \frac{1}{\tau_j}\bigg). \tag{4.16}$$

In conclusion, there exists c_2 depending only on c_1, σ, λ and such that

$$\|u_t\|^{p_j}_{\mathcal{T}^{7p_j/6}(T_j+\tau_j,\infty;L^{7p_j/3}(\Omega))} \leq c_2\ell_j\bigg(1 + \frac{1}{\tau_j}\bigg). \tag{4.17}$$

Second estimate. We now test (3.4) by $|u_t|^{q-2}u_t$, with $q > 1$ to be chosen later. Owing to the bound (3.7) and using (hpα), it is clear that

$$\int_\Omega \alpha(u_t)|u_t|^{q-2}u_t \le \| - Bu - W'(u) + f\|_2 \|u_t\|_{2q-2}^{q-1} \le c(1+\kappa)\|u_t\|_{2q-2}^{q-1}. \tag{4.18}$$

Consequently,

$$\sigma\|u_t\|_q^q \le c(1+\kappa)\|u_t\|_{2q-2}^{q-1}. \tag{4.19}$$

The above relations (4.18)–(4.19) hold pointwise in t. Then, integrating (4.18) over $(t, t+1)$ for t greater than a suitable S and using the latter inequality in (4.3), we get, for some c_3 depending only on C, σ,

$$q\|a_q(u_t)\|_{\mathcal{T}^1(S,\infty;L^1(\Omega))} + q\|u_t\|_{\mathcal{T}^q(S,\infty;L^q(\Omega))}^q \le c_3 q(1+\kappa)\int_t^{t+1} \|u_t(s)\|_{2q-2}^{q-1}\, \mathrm{d}s. \tag{4.20}$$

Bootstrap. At this point, if we take in the previous argument

$$S = T_{j+1} := T_j + \tau_j, \qquad q = p_{j+1} := \frac{7p_j}{6} + 1, \tag{4.21}$$

relation (4.20) is readily rewritten as

$$\begin{aligned} &p_{j+1}\|a_{p_{j+1}}(u_t)\|_{\mathcal{T}^1(T_{j+1},\infty;L^1(\Omega))} + p_{j+1}\|u_t\|_{\mathcal{T}^{p_{j+1}}(T_{j+1},\infty;L^{p_{j+1}}(\Omega))}^{p_{j+1}} \\ &\qquad \le c_3 p_{j+1}(1+\kappa)\int_t^{t+1} \|u_t(s)\|_{2p_{j+1}-2}^{p_{j+1}-1}\, \mathrm{d}s. \end{aligned} \tag{4.22}$$

Hence, recalling (4.17), the left hand side above is majorized by

$$c_3 p_{j+1}(1+\kappa)c_2^{7/6}\ell_j^{7/6}\left(1+\frac{1}{\tau_j}\right)^{7/6} \le c_4 \ell_j^{7/6} p_j \left(1+\frac{1}{\tau_j}\right)^{7/6}(1+\kappa). \tag{4.23}$$

Thus, we can define

$$\ell_{j+1} := c_4 \ell_j^{7/6} p_j \left(1+\frac{1}{\tau_j}\right)^{7/6}(1+\kappa), \tag{4.24}$$

so that (4.23) implies (4.10) at the step $j+1$. More precisely, since by (4.7) we can take

$$T_1 := 0, \qquad p_1 := 2, \qquad \ell_1 := c_0(1+\kappa)^2, \tag{4.25}$$

assuming that $\epsilon \in (0,1)$ is given, we also choose

$$\tau_j := \frac{\epsilon}{j^2}, \quad \text{so that } T_{j+1} = T_j + \tau_j \le c\epsilon \quad \forall\, j \ge 1 \tag{4.26}$$

and for $c > 0$ independent of j. At this point, let us set, for notational simplicity,

$$b := 7/6, \qquad B_j := \sum_{i=0}^{j} b^i \le 6b^{j+1}. \tag{4.27}$$

Then, it is not difficult to get from (4.24) (cf. also (4.25))

$$\ell_{j+1} \le c_4^{B_{j-1}} c_0^{b^j} (1+\kappa)^{p_{j+1}} \prod_{i=1}^{j} p_i^{b^{j-i}} \prod_{i=1}^{j} \left(1 + \frac{i^2}{\epsilon}\right)^{b^{j-i+1}}, \tag{4.28}$$

whence, noting that

$$c_5 b^j \le p_j \le c_6 b^{2j} \quad \forall\, j \ge 1 \tag{4.29}$$

and for some $c_5, c_6 > 0$ independent of j, and passing to the logarithm, it is not difficult to show that

$$\left(\prod_{i=1}^{j} p_i^{b^{j-i}} \right)^{1/p_j} \le c, \tag{4.30}$$

$$\left(\prod_{i=1}^{j} \left(1 + \frac{i^2}{\epsilon}\right)^{b^{j-i+1}} \right)^{1/p_j} \le \frac{c}{\epsilon^{c_7}}. \tag{4.31}$$

Collecting the above estimates, we infer

$$\ell_{j+1}^{1/p_{j+1}} \le \frac{c(1+\kappa)}{\epsilon^{c_7}}. \tag{4.32}$$

Thus, (4.12) (written at the step $j+1$) gives, for all $j \in \mathbb{N}$,

$$\|u_t(t)\|_{p_j} \le \frac{c(1+\kappa)}{\epsilon^{c_8}} \qquad \forall\, t \ge T_{j+1}. \tag{4.33}$$

>From (4.17) we also have

$$\|u_t\|_{\mathcal{T}^{p_{j+1}-1}(T_{j+1},\infty;L^{2(p_{j+1}-1)}(\Omega))} \le \frac{c(1+\kappa)}{\epsilon^{c_8}}. \tag{4.34}$$

Finally, taking the limit of (4.33) as $j \nearrow \infty$ we obtain

$$\|u_t(t)\|_\infty \le \frac{c_9(1+\kappa)}{\epsilon^{c_8}} \quad \forall\, t \ge c\epsilon, \tag{4.35}$$

where the last c is the same as in (4.26). Hence, by arbitrariness of ϵ, $u_t(t)$ is essentially bounded for a.e. $t > 0$. More precisely, squaring (4.35), recalling (3.8), and owing also to the first inequality in (3.15), (3.21) follows at once. Recalling (hpα), and using in particular that α is defined on the whole real line, we also obtain

$$\|\alpha(u_t)\|_\infty \le \phi\left(\mathfrak{c}_1 \frac{1 + \mathcal{G}(u_0)}{T^{\mathfrak{c}_2}} \right) \quad \forall\, t \ge T, \tag{4.36}$$

where ϕ depends only on α. Then, rewriting (3.4) as

$$Bu + W'(u) + \lambda u = f + \lambda u - \alpha(u_t), \tag{4.37}$$

and viewing it as a time dependent family of elliptic problems with monotone nonlinearity and uniformly bounded forcing term, it is not difficult to obtain also (3.22) as a consequence of standard maximum principle arguments. More precisely, one can test (4.37) by $|W'(u)+\lambda u|^{p-2}(W'(u)+\lambda u)$ for $p\in[2,\infty)$ and then let $p\nearrow\infty$.

To conclude the proof of Theorem 3.5, we recall that the procedure above has to be intended in the framework of Problem (P_n). Then, the bounds (3.21), (3.22), as well as the Liapounov condition (3.20), pass easily to the limit $n\nearrow\infty$ thanks to lower semicontinuity of norms with respect to weak and weak star convergences. More precisely, to obtain (3.20) the following property (of straightforward proof) is used:

Lemma 4.1. *The functional $\mathcal{G}$ is weakly sequentially lower semicontinuous in $\mathcal{X}_2$, namely, we have*

$$\mathcal{G}(u)\leq\liminf_{n\nearrow\infty}\mathcal{G}(u_n) \tag{4.38}$$

if $\{u_n\}\subset\mathcal{X}_2$ tends to some limit u weakly *in $\mathcal{X}_2$. The same property holds also for $\mathcal{F}$.*

The proof of Theorem 3.5 is however not yet complete since, up to now, we have just showed that any *limiting solution* is a *regularizing solution* and fulfills (3.21), (3.22) and (3.20). To conclude, we have to prove that *any* regularizing solution u (i.e. also a *nonlimiting* one) satisfies (3.21) and (3.22) (while (3.20) is now postulated in Definition 3.4). Here, the key point is to notice that, by (3.19) and Cor. 3.8, taken any $s>0$, from the "datum" $u(s)$ at most one solution emanates. Thus, any regularizing u is also "limiting" as it is restricted to $[s,\infty)$. This means that, referring for instance to (3.21), we have at least

$$\|u_t(t)\|^2_{L^\infty(\Omega)}\leq\mathfrak{c}_1\frac{1+\mathcal{G}(u(s))}{(T-s)^{\mathfrak{c}_2}}\quad\forall\,t\geq T>s>0. \tag{4.39}$$

Then, (3.21) follows easily by first using (3.20) (with s in place of t) and then taking the limit for $s\searrow 0$. The bound (3.22) is proved exactly in the same way and concludes the proof of Theorem 3.5. ∎

Remark 4.2. Notice that, for *any* regularizing solution, there holds the property (slightly stronger than (3.20))

$$\mathcal{G}(u(t))\leq\mathcal{G}(u(s))\quad\text{for all }t\geq s\geq 0. \tag{4.40}$$

Indeed, if $s=0$, then (4.40) reduces to (3.20). Otherwise, u coincides on $[s,\infty)$ with a limiting solution. Thus, (4.40) can be shown by noting as before that u is limiting on $[s,\infty)$, considering (P_n) w.r.t. the "initial" datum $u(s)$, and finally letting $n\nearrow\infty$.

Proof of Corollary 3.9. Property *(S1)* is evident and *(S4)* follows from Cor. 3.8. Next, *(S2)* and *(S3)* are immediate once one notes that v (in *(S2)*) and z (in *(S3)*) fulfill (3.20) thanks to Remark 4.2. Finally, let us prove *(S5)*. Although we could use here the regularization properties (3.21), (3.22), we rather give a proof which essentially relies only on (3.7), since we think it is interesting to notice that the strong-weak semicontinuity properties require no smoothing effect.

Thus, to show the first of *(S5)*, we start by observing that, due to (3.6), any $u \in \mathcal{S}$ stays in $C_w([0,\infty);H^2(\Omega))$, so that we just have to prove that, as $s,t \in [0,\infty)$ and s tends to t, $(W'+\lambda)(u(s))$ goes to $(W'+\lambda)(u(t))$ weakly in H. To see this, we first notice (cf. also [41, Sec. 6]) that there exists $c \geq 0$ such that $\|(W'+\lambda)(u(s))\| \leq c$ for *all* (not just a.e.) $s \in [0,\infty)$. Then, it is clear that, as $s \to t$, any subsequence of $(W'+\lambda)(u(s))$ admits a subsequence weakly convergent in H, whose limit is identified as $(W'+\lambda)(u(t))$ thanks to the convergence $u(s) \to u(t)$, which is strong in H, the monotonicity of $W' + \lambda\,\mathrm{Id}$, and [8, Lemma 1.3, p. 42]. This proves weak continuity of single trajectories. If we use (3.21), (3.22), we actually get more, namely $W'(u(\cdot))$ is strongly continuous with values in $C(\overline{\Omega})$ at least for strictly positive times.

To conclude, let us show the second property in *(S5)*. Letting then $u_n, u_{0,n}$ as in *(S5)*, as $u_{0,n}$ tends to u_0 in $\mathcal{X}_2$, it is in particular bounded in $\mathcal{X}_2$. This entails that (3.7), (3.21), (3.22) hold uniformly in n. By compactness arguments (similar to those in [43, Subsec. 3.3]) and using [47, Cor. 4], we then obtain that (a not relabelled subsequence of) u_n satisfies, for all $T > 0$,

$$u_n \to u \quad \text{strongly in } C^0([0,T];V), \tag{4.41}$$

$$(W'+\lambda)(u_n) \to (W'+\lambda)(u) \quad \text{weakly in } L^2(0,T;H), \tag{4.42}$$

where u is an $\mathcal{X}_2$-solution to Problem (P) with initial datum u_0, and it satisfies (3.7), (3.21) and (3.22). In particular, given any $t > 0$, by (4.41) $u_n(t)$ goes to $u(t)$ strongly in V. Then, by uniform boundedness, this convergence is also weak in $H^2(\Omega)$. As before, the monotonicity of $W' + \lambda\,\mathrm{Id}$ and the bound $\|(W'+\lambda)(u_n(t))\| \leq c$, which is uniform both in n and in t, permit to show that $(W'+\lambda)(u_n(t)) \to (W'+\lambda)(u(t))$ weakly in H (no further extraction of subsequence is required here, since the limit is already identified). To conclude, we have to see that u is a regularizing solution (i.e. it also fulfills (3.20)). To prove this, it suffices to write (3.20) for u_n and take the liminf as $n \nearrow \infty$. Indeed, the left hand side can be treated by Lemma 4.1, while the right hand side passes directly to the limit since $u_{0,n} \to u_0$ *strongly* in $\mathcal{X}_2$ and it is easy to check that $\mathcal{G}$ is continuous with respect to d_2. ∎

5 Long time behavior

Proof of Theorem 3.12. We shall show the following facts:

(L1) The semiflow $\mathcal{S}$ possesses a Liapounov function;

(L2) The set of stationary points of $\mathcal{S}$ is bounded in $\mathcal{X}_2$;

(L3) The semiflow $\mathcal{S}$ is *asymptotically compact*, namely for any sequence $\{u_n^0\}_{n\in\mathbb{N}}$ bounded in $\mathcal{X}_2$ and any positive sequence $\{t_n\}_{n\in\mathbb{N}}$, $t_n \nearrow \infty$, any sequence of the form $\{u_n(t_n)\}$, where $u_n \in \mathcal{S}$ and $u_n(0) = u_n^0$, is precompact in $\mathcal{X}_2$.

By the theory of global attractors (see, e.g., [30, Theorem 3.2] or [5, Thm. 5.1]), *(L1)–(L3)* would imply the existence of a global attractor compact in $\mathcal{X}_2$. However, here neither the "standard" theory in [30], nor the "generalized" theory in [5], can be directly applied since we have no uniqueness and just strong-weak semicontinuity. Nevertheless, we shall show in the Appendix that the validity of [5, Thm. 5.1] can be extended also to this case.

Remark 5.1. The use of this method permits to bypass the direct proof of existence of an $\mathcal{X}_2$-bounded absorbing set, which seems difficult to get here due to the possibly fast growth of α at ∞. Of course, *a posteriori* the dissipativity property will be satisfied just as a consequence of the existence of the global attractor.

To proceed, we first notice that, by the *energy estimate* (obtained testing (3.4) by u_t), $\mathcal{E}$ is a Liapounov functional. Note that the regularity of *any* $\mathcal{X}_2$-solution is sufficient to justify this estimate (and this is the reason why we do not use here the functional $\mathcal{G}$, which also enjoys a Liapounov property, at least for *regularizing* solutions, by Remark 4.2). Thus, *(L1)* holds. Second, *(L2)* is an easy consequence of well-known elliptic regularity results (we even have boundedness in $\mathcal{X}_\infty$). Thus, it just remains to show *(L3)*, whose proof will be split in a number of steps.

Lemma 5.2. *Given $0 < \tau < T < \infty$, there exists c depending on τ, T and on the initial datum such that any regularizing solution u satisfies the further bounds*

$$\|u_{tt}\|_{L^2(\tau,T;H)} + \|u_t\|_{L^\infty(\tau,T;V)} \leq c, \tag{5.1}$$

$$\|Bu_t\|_{L^2(\tau,T;H)} \leq c. \tag{5.2}$$

Proof. We can prove (5.1)–(5.2) by working on (P_n) and then letting $n \nearrow \infty$. As before, we omit the subscript n, for simplicity. Indeed, since we just consider strictly positive times, u can be thought as a limiting solution. In this regard, (5.1) is obtained by testing (4.1) by $(t-\tau)u_{tt}$ and using monotonicity of α together with (3.9) and (3.19). Next, (5.2) follows by making a comparison in (4.1) and using the continuity of W'', (3.19) and (3.23). The technical details of the procedure, as well as the standard argument for passing to the limit with n, are left to the reader. ∎

To proceed, we set, just to avoid some technicalities, $f \equiv 0$. We have the

Lemma 5.3. *Let $z \in \mathcal{S}$. Setting, for $s > 0$,*

$$H(z(s)) := -\Big(\alpha(z_t(s)), (Bz_t + W''(z)z_t)(s)\Big) - \frac{1}{2}\Big(\alpha(z_t(s)), (Bz + W'(z))(s)\Big), \tag{5.3}$$

for any $\tau, M > 0$ there holds

$$\mathcal{F}(z(\tau+M)) = e^{-M}\mathcal{F}(z(\tau)) + \int_\tau^{\tau+M} e^{s-\tau-M} H(z(s))\,\mathrm{d}s. \tag{5.4}$$

Proof. Since we work on $[\tau, \infty)$, we can use the further regularity properties (5.1)–(5.2), which allow us to test (3.4) by $(Bz_t + W''(z)z_t) + \frac{1}{2}(Bz + W'(z))$. Integrating over $(\tau, \tau+M)$, we readily get 5.4. ∎

Remark 5.4. Let us note that, using, e.g., [12, Lemme 3.3, p. 73], we get, more precisely, that the function $t \mapsto \mathcal{G}(z(t))$ is absolutely continuous on $[\tau, \infty)$ for all $\tau > 0$. This permits, in particular, to improve (in our specific case) the first condition in *(S5)*. Namely, the elements of our semiflow $\mathcal{S}$ belong to $C((0,\infty);\mathcal{X}_2)$ (compare also with [5, (C1)].

Let us now complete the proof of *(L3)*. We use here the "energy method" originally devised by Ball in [6] (see also [39] for an extension to nonautonomous systems). Take τ, M as before, and let v_n be the (unique) regularizing solution satisfying, for $t \in [0,\infty)$, $v_n(t) = u_n(t_n + t - M - \tau)$ (so that, in particular, $v_n(0) = u_n(t_n - M - \tau)$, $v_n(\tau) = u_n(t_n - M)$ and $v_n(\tau + M) = u_n(t_n)$). Since by (3.22) there exists $k > 0$ such that $d_\infty(v_n(t), 0) \le k$ for all $n \in \mathbb{N}$ and $t \in [0,\infty)$, by weak compactness we have that there exist $\chi_{-M}, \chi \in \mathcal{X}_2$ such that $v_n(\tau) \to \chi_{-M}$ and $v_n(\tau + M) \to \chi$ weakly in $\mathcal{X}_\infty$. Then, writing (5.4) for $z = v_n$, we get

$$\begin{aligned} \mathcal{F}(u_n(t_n)) - e^{-M}\mathcal{F}(u_n(t_n - M)) &= \mathcal{F}(v_n(\tau + M)) - e^{-M}\mathcal{F}(v_n(\tau)) \\ &= \int_\tau^{\tau+M} e^{s-\tau-M} H(v_n(s))\,\mathrm{d}s =: \mathcal{H}(v_n). \end{aligned} \tag{5.5}$$

Next, let us note that, at least up to a not relabelled subsequence, v_n properly tends to an $\mathcal{X}_2$-solution v. Thus, in particular, we have that $v(\tau) = \chi_{-M}$ and $v(\tau + M) = \chi$. Moreover, still by (3.22), $d_\infty(v(t), 0) \le k$ for all $t \in [0,\infty)$. Thus, setting $v_0 := \lim_{n\nearrow\infty} v_n(0)$, since by the existence property there must exist at least one $z \in \mathcal{S}$ such that $z(0) = v_0$, by Corollary 3.8 it must be $z \equiv v$ on $[0,\infty)$, which means that v is itself an element of $\mathcal{S}$ and, consequently, satisfies (5.4). Thus, noting that, by (5.1), (5.2) and weak compactness, $\mathcal{H}(v_n)$ tends to $\mathcal{H}(v)$, taking the lim sup in (5.5) one gets

$$\begin{aligned} \limsup_{n\nearrow\infty} \mathcal{F}(u_n(t_n)) &\le k e^{-M} + \limsup_{n\nearrow\infty} \mathcal{H}(v_n) \\ &= k e^{-M} + \mathcal{H}(v) \\ &= k e^{-M} + \mathcal{F}(v(\tau + M)) - \mathcal{F}(v(\tau)) e^{-M} \\ &\le k e^{-M} + \mathcal{F}(\chi). \end{aligned} \tag{5.6}$$

Since $u_n(t_n)$ tends to χ weakly in $\mathcal{X}_2$ and using once more Lemma 4.1, it is then easy to see that $\mathcal{F}(u_n(t_n))$ tends to $\mathcal{F}(\chi)$, which readily entails that $u_n(t_n) \to \chi$ *strongly* in $\mathcal{X}_2$, i.e. *(L3)*. ∎

Remark 5.5. We point out that the attractor $\mathcal{A}$ turns out to be more regular. More precisely, it is bounded and hence "weakly" compact in $\mathcal{X}_\infty$. Indeed, it is easy to realize that the set of stationary points of (P) mentioned in property *(L2)* is also bounded in $\mathcal{X}_\infty$. Moreover, (3.22) entails that $\mathcal{S}$ is (sequentially) "weakly" compact, i.e. *(L3)* holds, in $\mathcal{X}_\infty$. Thus, Ball's procedure sketched in the Appendix can be repeated with respect to the "weak" topology in $\mathcal{X}_\infty$. As a further consequence, it is now easy to see that $\mathcal{A}$ is also *strongly* compact in $W^{2,p}(\Omega)$ for all $p \in [1,\infty)$.

Remark 5.6. On account of the previous Remark, our procedure entails existence of an absorbing ser B_0 for $\mathcal{S}$ bounded in $\mathcal{X}_\infty$ (not just in $\mathcal{X}_2$).

6 Exponential attractors

In this section we prove Theorem 3.13 by means of the method of ℓ-trajectories. In order to apply the theory of [35] sketched in Section 2, we take $\mathcal{X} := V$ endowed with its standard norm. In comparison with the global attractor, which was constructed in the smaller space $\mathcal{X}_2$, we are thus working with weaker norm and topology.

We know from the previous Section that $\mathcal{S}$ admits an absorbing set B_0 bounded in $\mathcal{X}_\infty$. We let (uniqueness holds on B_0, thus we can use the "semigroup" $S(\cdot)$)

$$B_1 := \overline{\cup_{t\in[0,T_0]} S(t)B_0}, \tag{6.1}$$

where $T_0 > 0$ is such that $S(t)B_0 \subset B_0$ for all $t \geq T_0$ and the closure is taken w.r.t. the *weak* topology of $\mathcal{X}_\infty$. Due to the uniform character of estimate (3.22) (now the initial data are in B_0, so they are uniformly bounded in $\mathcal{X}_\infty$), B_1 is still absorbing and bounded in $\mathcal{X}_\infty$. Moreover, we claim that B_1 is positively invariant. To prove this fact, we let $\tau > 0$ and assume that $u_0 \in B_1$ is given by

$$u_0 = \lim_{n\nearrow\infty} S(t_n)u_{0,n}, \tag{6.2}$$

where $\{u_{0,n}\} \subset B_0$ and $\{t_n\} \subset [0, T_0]$. Then, using uniform boundedness, weak compactness arguments and the uniqueness property of solutions it is not difficult to realize that

$$S(t_n + \tau)u_{0,n} = S(\tau)\Big(S(t_n)u_{0,n}\Big) \to S(\tau)u_0 \tag{6.3}$$

weakly in $\mathcal{X}_\infty$ as $n \nearrow \infty$ (note that we cannot use directly *(S5)* since we do not know that $S(t_n)u_{0,n}$ converges *strongly* in $\mathcal{X}_2$). This readily entails that $S(\tau)u_0 \in B_1$, which is then positively invariant.

At this point, possibly making a positive and finite time shift, we consider elements of $\mathcal{S}$ starting from initial data in B_1. Following [35, Sec. 2] and Section 2 in this paper, we set $\mathcal{X}_\ell := L^2(0, \ell; \mathcal{X})$, where the choice of $\ell \in (0, \infty)$ is here arbitrary, and define $\mathcal{B}^1_\ell$ as the set of ℓ-trajectories whose initial datum lies in B_1. Using that B_1 is positively invariant and *weakly* closed in $\mathcal{X}_\infty$, it is not difficult to show that $\mathcal{B}^1_\ell$ is also closed with respect to the norm in $\mathcal{X}_\ell$.

We now show the validity of conditions *(M1)*, *(M2)* and *(M3)* reported in Section 2. To do this, we prove a number of a priori estimates involving the difference of two solutions. Namely, we take u_1, u_2 solving (P) and starting from $u_{0,1}, u_{0,2} \in B_1$, respectively, and set $u := u_1 - u_2$. Then, writing (3.4) for $u = u_1$ and for $u = u_2$, and taking the difference, we have

$$\alpha(u_{1,t}) - \alpha(u_{2,t}) + Bu + W'(u_1) - W'(u_2) = 0. \tag{6.4}$$

In the sequel, the varying constant $c > 0$ and the constants $c_1, c_2, \dots > 0$, whose numeration is restarted, will be allowed to depend on B_1 and on ℓ, additionally. Thus, let us test (6.4) by u_t. We get

$$\sigma\|u_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|u\|_V^2 \leq c\|u\|^2, \tag{6.5}$$

where we also used the Young inequality and that, thanks to (3.22), there exists $c > 0$ depending on B_1 such that $\|W''(u_1(r))\|_\infty + \|W''(u_2(r))\|_\infty \le c$ for all $r \in [0, \infty)$. Then, by Gronwall's Lemma,

$$\|u(y)\|_V^2 \le e^{c(y-s)}\|u(s)\|_V^2 \le e^{2c\ell}\|u(s)\|_V^2 =: c_1\|u(s)\|_V^2 \tag{6.6}$$

for all s, y such that $0 \le y - s \le 2\ell$. Then, taking $s \in [0, \ell]$, $t \in [s, 2\ell]$ and integrating (6.5) over $[s, t]$, we infer

$$\sigma \int_s^t \|u_t(r)\|^2 \,\mathrm{d}r + \|u(t)\|_V^2 \le c \int_s^t \|u(r)\|^2 + \|u(s)\|_V^2. \tag{6.7}$$

Thus, using (6.6) integrated for $y \in [s, t]$ to estimate the first term in the right hand side above, we get, for $t = 2\ell$,

$$\sigma \int_s^{2\ell} \|u_t(r)\|^2 \,\mathrm{d}r + \|u(2\ell)\|_V^2 \le c_2\|u(s)\|_V^2, \tag{6.8}$$

whence, integrating for $s \in [0, \ell]$,

$$\sigma\ell\|u_t\|_{L^2(\ell,2\ell;H)}^2 + \ell\|u(2\ell)\|_V^2 \le c_2\|u\|_{L^2(0,\ell;V)}^2. \tag{6.9}$$

Now, let us notice that a direct comparison argument in (3.4) gives

$$\|u\|_{H^2(\Omega)}^2 \le c\Big(\|u\|^2 + \|Bu\|^2\Big) \le c_3\|u\|^2 + c_3\|u_t\|^2, \tag{6.10}$$

where the last inequality holds by the local Lipschitz continuity of α and W' and Theorem 3.5. Thus, evaluating the above formula in $y \in [\ell, 2\ell]$, and using (6.6),

$$\|u(y)\|_{H^2(\Omega)}^2 \le c_3c_1\|u(s)\|_V^2 + c_3\|u_t(y)\|^2. \tag{6.11}$$

Finally, integrating for $s \in [0, \ell]$ and $y \in [\ell, 2\ell]$ and recalling (6.9),

$$\|u\|_{L^2(\ell,2\ell;H^2(\Omega))}^2 \le c_4\|u\|_{L^2(0,\ell;V)}^2. \tag{6.12}$$

We are in the position to show properties *(M1)*, *(M2)* and *(M3)*. Setting

$$W_\ell := \Big\{v \in L^2(0, \ell; H^2(\Omega)) : \; v_t \in L^2(0, \ell; H)\Big\}, \tag{6.13}$$

from (6.12) and (6.9) we have, respectively,

$$\|L_\ell u_1 - L_\ell u_2\|_{L^2(0,\ell;H^2(\Omega))} \le c\|u_1 - u_2\|_{L^2(0,\ell;V)}, \tag{6.14}$$

$$\Big\|(L_\ell u_1 - L_\ell u_2)_t\Big\|_{L^2(0,\ell;H)} \le c\|u_1 - u_2\|_{L^2(0,\ell;V)}, \tag{6.15}$$

which imply property *(M1)* thanks to a straightforward application of the Aubin-Lions compactness Lemma.

Concerning property *(M2)*, this follows from (6.6) taking $y = s + t$, with t varying in $[0, \tau]$, $\tau > 0$, and integrating for $s \in [0, \ell]$ (the constant c_1 will actually take the value $e^{2c\tau}$, instead of $e^{2c\ell}$, with these choices).

Finally, property *(M3)* is a simple and direct consequence of the time-regularity (3.9) of the time derivatives of the solutions (cf. [35, Lemma 2.2]).

According now to [35, Theorem 2.5], our procedure entails existence of an exponential attractor $\mathcal{M}_\ell$ in the space of short trajectories. To show the existence of an exponential attractor also in the physical state space, we have to check the regularity *(M4)* for the evaluation map e, which follows easily from (6.6) by taking $y = \ell$ and integrating for $s \in [0, \ell]$. Thus, thanks also to Remark 2.4, the set $\mathcal{M} := e(\mathcal{M}_\ell)$ is an exponential attractor in $\mathcal{X} = V$ for the semiflow $\mathcal{S}$.

Remark 6.1. We stress once more that $\mathcal{M}$ is a compact set in $\mathcal{V}$, but it is able to attract exponentially fast only the sets which are bounded in $\mathcal{X}_2$ (actually for initial data lying in V also the existence theory requires additional conditions).

7 Appendix

We show here that the construction of global attractors for *generalized* semiflows (i.e., in our terminology, semiflows with "strong-strong" continuity properties but with no uniqueness at all) given in [5] can be extended to our situation. Actually, in comparison with J. Ball's proof, we have some simplification (mainly of technical character) due to the unique continuation *(S3)*. On the other hand, since our property *(S5)* is weaker than J. Ball's "strong-strong" continuity [5, *(H4)*], we have to suitably modify some points, which become now slightly more complicated. For the reader's convenience we report at least the highlights of all steps of J. Ball's argument. Concerning the proofs, we just point out the different points, instead. Basically, we will see that when in J. Ball's proofs [5, *(H4)*] is used, we can replace it by the combined use of *(S5)* and the *asymptotic compactness (L3)*. In agreement with our specific situation, the phase space will be indicated as $\mathcal{X}_2$ in what follows, but of course everything holds for a generic metric space additionally endowed with some "weak" topology.

Proposition 7.1 (Lemma 3.4 in [5])**.** *Let (S1)–(S5) and (L3) hold and let $B \subset \mathcal{X}_2$ a bounded set. Then, the ω-limit $\omega(B)$ is nonempty, compact,* fully *invariant and attracts B.*

Proof. It is obvious from *(L3)* that $\omega(B)$ is nonempty and easy to show directly that it is closed. We now prove that, for all $z \in \omega(B)$, there exists a *complete trajectory* ψ taking values in $\omega(B)$ and such that $\psi(0) = z$ (we recall that "complete trajectory" means that $\psi : \mathbb{R} \to \mathcal{X}_2$ is such that $\psi(\cdot + \tau) \in \mathcal{S}$ for all $\tau \in \mathbb{R}$). Let then $\{u_n\} \subset \mathcal{S}$ and $t_n \nearrow \infty$ such that $u_n(t_n) \to z$ and $\{u_n(0)\} \subset B$. By *(S2)*, the sequence $\{v_n\}$, defined by $v_n(\cdot) := u_n(t_n + \cdot)$, lies in $\mathcal{S}$ and satisfies $v_n(0) \to z$

strongly. Then, by *(S5)*, there exist a nonrelabelled subsequence of n and a solution $v \in \mathcal{S}$ such that, for all $t > 0$, $u_n(t_n + t) = v_n(t) \to v(t)$ *weakly* in $\mathcal{X}_2$. On the other hand, setting $w_n(\cdot) := u_n(t_n/2 + \cdot)$, it is $w_n \in \mathcal{S}$. Moreover, we notice that, with no modifications in the proof, it is still valid here [5, Prop. 3.1], which says that *(L3)* entails *eventual boundedness*, i.e., that for any bounded B there exists $\tau_B \geq 0$ such that $\cup_{t \geq \tau_B} T(t)B$ is still bounded. Thus, we have that $\{w_n(0)\}$ is bounded and consequently, thanks to *(L3)*, $u_n(t_n + t) = w_n(t_n/2 + t)$ converges strongly to its limit which is already identified as $v(t)$. Moreover, it is clear that $v(t) \in \omega(B)$ for all $t \geq 0$. This shows that from z originates a (semi)trajectory v taking values in $\omega(B)$. The same trick used above permits to adapt also J. Ball's proof that v extends to a complete trajectory ψ. Next, noting that on $\omega(B)$ uniqueness holds, the above property also entails the *complete invariance* of $\omega(B)$ (which did not necessarily hold in Ball's case). Finally, the proof that $\omega(B)$ is compact and attracts B is essentially the same as in [5]. ∎

Proposition 7.2 (Lemma 3.5 in [5]). *Let (S1)–(S5) and (L3) hold and let $\mathcal{S}$ be* pointwise dissipative, *namely let there exist B_0 bounded in $\mathcal{X}_2$ such that any $u \in \mathcal{S}$ eventually takes values in B_0. Then, there exists $\tau > 0$ such that, for any $\delta > 0$, the set*

$$B_1 := \bigcup_{t \geq \tau} T(t)(B(B_0, \delta)), \tag{7.1}$$

with $B(B_0, \delta)$ denoting the open δ-neighbourhood of B_0, is an absorbing set for $\mathcal{S}$.

Proof. Let $\delta > 0$. By contradiction, let us assume that some bounded B is not absorbed by B_1. Then, there exist $\{u_n\} \subset \mathcal{S}$ and $t_n \nearrow \infty$ with $\{u_n(0)\} \subset B$ such that, for all n, $u_n(t_n) \notin B_1$. By eventual boundedness, there exists $\tau > 0$ (note it does not depend on δ) such that $\gamma^\tau(B) = \cup_{t \geq \tau} T(t)B$ is bounded. Let us then set $v_n(\cdot) := u_n(t_n/2 + \cdot)$, so that $v_n(0) = u_n(t_n/2)$ and $v_n(t_n/2) = u_n(t_n)$. By *(L3)*, at least for a subsequence, $v_n(0) \to z$ strongly. This entails by *(S5)* that there exists $v \in \mathcal{S}$ such that $v_n(t) \to v(t)$ weakly for all $t \in [0, \infty)$. As before, since $v_n(t) = u_n(t_n/2 + t)$ and $\{u_n(0)\}$ is bounded, by *(L3)* the convergence $v_n(t) \to v(t)$ is actually strong. Moreover, it is easy to see (proceed exactly as in [5]) that $v_n(t) \notin B(B_0, \delta)$ for all $t \in [0, t_n/2 - \tau]$. Thus, passing to the (strong) limit, we have that $v(t) \notin B(B_0, \delta)$ for all $t \in [0, \infty)$. Since v is a trajectory, this contradicts the point dissipativity of $\mathcal{S}$ and gives the assert. ∎

Proposition 7.3 (Theorem 3.3 in [5]). *Let (S1)–(S5) and (L3) hold and let $\mathcal{S}$ be* pointwise dissipative. *Then, $\mathcal{S}$ admits the global attractor $\mathcal{A}$.*

Proof. It is as in [5], up to minor modifications. ∎

Proposition 7.4 (Theorem 5.1 in [5]). *Let (S1)–(S5) and (L1)–(L3) hold. Then, $\mathcal{S}$ is* pointwise dissipative *(hence, by the previous result, it admits the global attractor).*

Proof. Although it is similar to that in [5], we prefer to give some more detail. First, it is easy to prove that, noting as V the Liapounov functional and as $\mathcal{E}_0$ the set of rest (i.e., stationary) points of $\mathcal{S}$, given $u \in \mathcal{S}$, V is constant on $\omega(u)$ and $\omega(u)$ is contained in $\mathcal{E}_0$. To conclude, we show that, given an arbitrary $\delta > 0$, any $u \in \mathcal{S}$ eventually takes values in the (bounded) set $B_0 := B(\mathcal{E}_0, \delta)$. Actually, if by contradiction $u(t_n) \notin B_0$ for a diverging sequence $\{t_n\}$, defining $v_n(\cdot) := u(t_n/2 + \cdot)$ and being, as before, $\{v_n\} \subset \mathcal{S}$ and $\{v_n(0)\}$ bounded, by asymptotic compactness $u(t_n) = v_n(t_n/2)$ has a subsequence which converges to an element of $\mathcal{E}_0$. ∎

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