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New results on the stability of quasi-static paths of a single particle system with Coulomb friction and persistent contact

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Abstract

In this paper we announce some new mathematical results on the stability of quasi-static paths of a single particle linearly elastic system with Coulomb friction and persistent normal contact with a flat obstacle.A quasi-static path is said to be stable at some value of the load parameter if, for some finite interval of the load parameter thereafter, the dynamic solutions behave continuously with respect to the size of the initial perturbations (as in Lyapunov stability) and to the smallness of the rate of application of the external forces, ε (as in singular perturbation problems). In this paper we prove sufficient conditions for stability of quasi-static paths of a single particle linearly elastic system with Coulomb friction and persistent normal contact with a flat obstacle. The present system has the additional difficulty of its non-smoothness: the friction law is a multivalued operator and the dynamic evolutions of this system may have discontinuous accelerations.

1   Introduction

The study of the stability of frictional contact systems has deserved an increasing attention (Shevitz [SP94] , Adly & Goeleven [AG04] , Van de Wouv & Leine [WL04] , Brogliato [Bro04] , Sinou et al. [STJ03] , Duffour & Woodhouse [DW04]) due to its relevance in many engineering applications (Ibrahim [Ibr94] , Kinkaid et al. [KRP03] , Sinou et al. [STJ04]) as well as in geophysics (Gu et al. [GRRT84] , Scholz [Sch98]).

The concept of stability that one has in mind in many mechanical situations is the concept of Lyapunov stability, which, in particular, can be used to study the *stability of the equilibrium configurations under constant applied loads*(dynamic trajectories with zero velocity and acceleration). In what concerns the non-smooth friction problems, a discussion on the attractiveness of equilibrium sets with the application of LaSalle's principle can be found in Van de Wouw and Leine [WL04] , while the works of Shevitz [SP94] and Brogliato [Bro04] develop non-smooth Lyapunov functions.

A related but different issue is the *stability of quasi-static paths* of mechanical systems *under slowly varying applied loads.* In general, the concept of Lyapunov stability cannot be applied to quasi-static paths because such paths are not, in general, true solutions of the original governing dynamic equations (Loret et al. [LSM00]). But the "stability of quasi-static paths" can be related to the theory of singular perturbations (see again [LSM00]): the physical time t can be recognised as a fast (dynamic) time scale and a loading parameter λ, whose rate of change with respect to time, $\varepsilon = d\lambda/dt$, is arbitrarily small, can be recognised as a slow (quasi-static) time scale. Changing the independent variable t into λ in the governing system of dynamic differential equations or inclusions, one is led to a system in which some of the highest order derivatives with respect to λ appear multiplied by the small parameter ε. In this manner, following the mathematical definition of stability of quasi-static paths proposed by Martins et al. [MSPC04] , [MMPRSC05] a quasi-static path is stable at some point if, in some subsequent finite interval of the load parameter, any dynamic trajectory does not deviate from the quasi-static one more than some desired amount, provided that the initial conditions for the dynamic evolution are sufficiently close to the quasi-static path, and the loading is applied sufficiently slowly.

After the study of some smooth cases and some problems that have a not very severe non-smoothness (the elastic-plastic problems with linear hardening) [MMPRSC05] , this paper applies the same definition to a class of linearly elastic problems with friction that has a more severe non-smoothness: discontinuous acceleration and friction forces.

The structure of the article is the following. In Section 2, the governing dynamic and quasi-static equations and conditions are presented, and the definition of stability of quasi-static paths is recalled. In Section 3, existence results for dynamic and quasi-static problems with persistent frictional contact are recalled and refined. Section 4 contains an auxiliary result on the regularity of the solution of the quasi-static problem, which is shown to have a derivative with bounded variation. This result is essential to estimate, in Section 5, a contribution that involves the product of the inertia term in the dynamic equation with the derivative of the quasi-static solution. The main result of this paper, the stability of the quasi-static path, is then proved in the final Section 5.

2 Governing equations and definition of stability of the quasi-static path

We consider a linear elastic system with two degrees of freedom: a single particle system. Its configuration is determined by the displacement $u \in \mathbb{R}^2$ of the particle. In the following we write u_t and u_n for the *tangential* and *normal* displacement components, respectively. This is motivated by the assumption that the particle cannot penetrate a rigid obstacle and this restriction is modelled by the inequality $u_n \geq 0$. The evolution of the system is described in terms of the load parameter λ, which is linked via the small load rate parameter $\varepsilon > 0$ to the physical time t: $\lambda = \varepsilon t$. The elastic behaviour is modelled by the 2×2 positive definite stiffness matrix K, while the applied and the reaction forces acting on the particle are represented by the vector functions with values in $\mathbb{R}^2$, $f(\lambda)$ and $r(\lambda)$, respectively:

$$K = \begin{pmatrix} k_{tt} & k_{tn} \\ k_{nt} & k_{nn} \end{pmatrix}, \quad f(\lambda) = \begin{pmatrix} f_t(\lambda) \\ f_n(\lambda) \end{pmatrix}, \quad r(\lambda) = \begin{pmatrix} r_t(\lambda) \\ r_n(\lambda) \end{pmatrix}. \tag{2.1}$$

The derivative $\mathrm{d}(\ )/\mathrm{d}\lambda$ is denoted by $(\ )'$. Furthermore, $\mu \geq 0$ denotes the coefficient of friction, and in the whole article we assume that for some given time $\Lambda > 0$ we have

$$f \in \mathrm{C}^1([0, \Lambda], \mathbb{R}^2). \tag{2.2}$$

The **equation of motion** in the dynamic case is

$$\varepsilon^2 u''(\lambda) + Ku(\lambda) - f(\lambda) = r(\lambda), \tag{3-a}$$

where, without loss of generality, we assume a unit mass. The equation of motion in the quasi-static case reads

$$Ku(\lambda) - f(\lambda) = r(\lambda). \tag{3-b}$$

The **unilateral contact conditions** satisfied by the solutions are given by

$$u_n \geq 0, \quad u_n r_n = 0, \quad r_n \geq 0. \tag{4}$$

Introducing the set-valued sign function

$$\mathrm{sign}(s) := \begin{cases} -1 & \text{for } s < 0 \\ [-1, 1] & \text{for } s = 0 \\ +1 & \text{for } s > 0 \end{cases}$$

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we can formulate the **Coulomb friction law** as follows

$$-r_t(\lambda) \in \mu r_n(\lambda)\text{sign}\left(u'_t(\lambda)\right). \tag{5}$$

In the whole article we assume that we are in situations of **persistent contact**, so that $u_n \equiv 0$. Then, in the dynamic case, the equations (3-a), (4) and (5) lead to the **dynamic problem with persistent contact:**

For given $u_0, v_0 \in \mathbb{R}$ find $u_t \in W^{2,\infty}([0, \Lambda], \mathbb{R})$ satisfying the initial conditions

$$u_t(0) = u_0, \quad \varepsilon u'_t(0) = v_0, \tag{6}$$

and such that, for all $\lambda \in [0, \Lambda]$,

$$\varepsilon^2 u''_t(\lambda) + k_{tt}u_t(\lambda) - f_t(\lambda) \in -\mu(k_{nt}u_t(\lambda) - f_n(\lambda))\text{sign}\left(u'_t(\lambda)\right), \tag{7}$$
$$k_{nt}u_t(\lambda) - f_n(\lambda) \geq 0. \tag{8}$$

To distinguish the dynamic solution from the quasi-static one, the latter is denoted by $\bar{u}_t$. By taking $\varepsilon = 0$ in (7) we formally get the corresponding **quasi-static problem with persistent contact**, which reads:

For given $\bar{u}_0 \in \mathbb{R}$, find $\bar{u}_t \in W^{1,\infty}([0, \Lambda], \mathbb{R})$ satisfying

$$\bar{u}_t(0) = \bar{u}_0, \tag{9}$$

and such that, for all $\lambda \in [0, \Lambda]$,

$$k_{tt}\bar{u}_t(\lambda) - f_t(\lambda) \in -\mu\left(k_{nt}\bar{u}_t(\lambda) - f_n(\lambda)\right)\text{sign}\left(\bar{u}'_t(0)\right), \tag{10}$$
$$k_{nt}\bar{u}_t(\lambda) - f_n(\lambda) \geq 0. \tag{11}$$

Note that for $\lambda = 0$ this immediately implies some restrictions on the initial condition $\bar{u}_0$. We can now introduce the

Definition 2.1 (stability of a quasi-static path) *Let $\bar{u}_t$ be a quasi-static path, i.e. a solution of the quasi-static problem (9)-(11). We call the quasi-static path $\bar{u}_t$ **stable** at $\lambda = 0$, if there exists some positive interval of loading parameter values $0 < \Delta\lambda \leq \Lambda$, such that for every $\delta > 0$, we can find constants $C_{ini}(\delta) > 0$ and $C_\varepsilon(\delta) > 0$, such that for each parameter ε and initial conditions u_0 and v_0 at $\lambda = 0$ with*

$$|v_0| + |u_0 - \bar{u}_0| < C_{ini}(\delta) \quad and \quad \varepsilon < C_\varepsilon(\delta) \tag{12}$$

the dynamic solution u_t of (6)-(8) remains near the quasi-static path in the following sense

$$|\varepsilon u'_t(\lambda)| + |u_t(\lambda) - \bar{u}_t(\lambda)| < \delta, \tag{13}$$

for all $\lambda \in [0, \Delta\lambda]$.

3   Existence of solutions

In the article of Martins et al. [MMMP05] it is shown in a quite more general situation, that for initial conditions with positive normal reaction (i.e. $k_{nt}u_0 - f_n(0) > 0$) there exists some $\lambda_* \in (0, \Lambda]$ for which a solution of (6)-(8) exists in the interval $[0, \lambda_*]$.
In order to guarantee existence of solution up to an arbitrary given load parameter $\Lambda > 0$, we need a stronger assumption on f that holds on the whole interval $[0, \Lambda]$.

Lemma 3.1 (Existence of a dynamic solution) *There exists a constant $C > 0$ that depends on all data except the external normal force f_n, such that, for each normal force f_n satisfying*

$$-f_n(\lambda) > C, \ \ for \ all \ \lambda \in [0, \Lambda], \tag{1}$$

a solution of (6)-(8) exists in $[0, \Lambda]$.

Remark 3.2 *By classical results from the theory of differential inclusions we know that for general $f \in C^2([0, \Lambda], \mathbb{R}^2)$ there exists a solution $u_t \in W^{2,\infty}([0, \Lambda], \mathbb{R})$ of the inclusion (7) with the initial conditions (6). See for example Aubin [AC84] , Page 98, Theorem 3. The rest of the Proof consists of using energy estimates to show that under assumption (1) on the external normal force f_n, this solution u_t automatically satisfies (8) for all $\lambda \in [0, \Lambda]$. The full proof can be found in [MMRS06] .*

Lemma 3.3 (Existence of a quasi-static solution) *Assume that $\mu > 0$ and the quasi-static initial condition $\bar{u}_0 \in \mathbb{R}$ satisfies*

$$|k_{tt}\bar{u}_0 - f_t(0)| \leq \mu \left(k_{nt}\bar{u}_0 - f_n(0) \right), \tag{2}$$

and let

$$k_{tt} - \mu|k_{nt}| > 0 \tag{3}$$

hold. Then there exists a constant $C > 0$ that depends on all data except the external normal force f_n, such that, for each normal force f_n satisfying

$$-f_n(\lambda) > C, \ \ for \ all \ \lambda \in [0, \Lambda], \tag{4}$$

there exists a quasi-static solution $\bar{u}_t \in W^{1,\infty}([0, \Lambda], \mathbb{R}^2)$ of the problem (9)-(11). Furthermore the solution satisfies

$$|\bar{u}_t'| \leq \frac{(\mu + 1)\|f'\|_{L^\infty([0,\Lambda])}}{k_{tt} - \mu|k_{nt}|}. \tag{5}$$

Remark 3.4 *The proof follows directly from a result in Mielke& Schmid [MS07] , where existence of a quasi-static solution even without the limitation of persistent contact was proven. Persistent contact is shown under the assumption $-f_n > C$ analogous to the prove of Lemma 3.1. Klarbring [Kla90] has shown that if (3) does not hold, one cannot expect in general the existence of a continuous solution to (9)-(11).*

In the following we assume that f_n always satisfies a condition of the form (1) and we focus on the inclusions (7) and (10).

4 Variation of the derivative of the quasi-static path

A short calculation shows that the inclusion (10) is equivalent to the following sweeping process formulation

$$-\bar{u}_t'(\lambda) \in \mathcal{N}_{\mathcal{C}(\lambda)}(\bar{u}_t(\lambda) \subset \mathbb{R}, \tag{1}$$

where the set $\mathcal{C}(\lambda)$ is defined by

$$\mathcal{C}(\lambda) := \left[\frac{f_t(\lambda) + \mu f_n(\lambda)}{k_{tt} + \mu k_{nt}}, \frac{f_t(\lambda) - \mu f_n(\lambda)}{k_{tt} - \mu k_{nt}} \right]$$

and the corresponding normal cone in u is denoted by $\mathcal{N}_{\mathcal{C}}(u)$.

In the following we denote by $\Pi : 0 = \tau_0 < \tau_1 < \cdots < \tau_{N_\Pi} = \Lambda$, with $N_\Pi \in \mathbb{N}$, a partition of the interval $[0, \Lambda]$, and we denote by $\Pi[0, \Lambda]$ the set of all partitions of $[0, \Lambda]$.

The variation of a function of one variable, $f : [0, \Lambda] \to \mathbb{R}^n$, is defined as

$$\text{var}(f; 0, \Lambda) := \sup_{\Pi \in \Pi[0,\Lambda]} \sum_{j=1}^{N_\Pi} \|f(\tau_j) - f(\tau_{j-1})\|.$$

We say that f is of bounded variation if $\text{var}(f; 0, \Lambda) < \infty$ holds.

Lemma 4.1 (bounded variation of the derivative) *Assume that the moving set $\mathcal{C}(\lambda) \subset \mathbb{R}$ is defined by $\mathcal{C}(\lambda) := [g(\lambda), h(\lambda)]$ with functions $g, h \in \text{C}^1([0, \Lambda], \mathbb{R})$ satisfying $\text{var}(g'; 0, \Lambda) < \infty$ and $\text{var}(h'; 0, \Lambda) < \infty$, and also $h(\lambda) - g(\lambda) > 0$ for all $\lambda \in [0, \Lambda]$. Then there exists a solution of*

$$-u'(\lambda) \in \mathcal{N}_{\mathcal{C}(\lambda)}(u(\lambda). \tag{2}$$

Further u is differentiable from the right for all $\lambda \in [0, \Lambda)$, i.e.

$$u'(\lambda) = \lim_{h \searrow 0} \frac{u(\lambda + h) - u(\lambda)}{h}.$$

Additionally the right derivative u' is a right continuous function with bounded variation, $\text{var}(u'; 0, \Lambda) \leq \text{var}(g'; 0, \Lambda) + |g'(0)| + \text{var}(h'; 0, \Lambda) + |h'(0)|.$

Remark 4.2 *The proof of this Lemma adapts to the present context of a particle with non-prescribed normal force, some arguments used by Marques [Mon94] and Martins et al. [MMR06] for cases with prescribed normal force. The full proof can be found again in [MMRS06] .*

5 Stability of the quasi-static path

From Lemma (3.1) and (3.3) we know that there exist solutions $u_t, \bar{u}_t : [0, \Lambda] \to \mathbb{R}$ of the dynamic problem (6)-(8) and of the quasi-static problem (9)-(11), respectively. First we rewrite the inclusions (7) and (10) by using the functions $\rho, \bar{\rho} : [0, \Lambda] \to [-1, 1]$ as follows

$$\varepsilon^2 u_t''(\lambda) + k_{tt} u_t(\lambda) - f_t(\lambda) = \rho(\lambda) \mu(k_{tn} u_t(\lambda) - f_n(\lambda)),$$
$$-\rho(\lambda) \in \text{sign}(u_t'(\lambda)), \tag{1}$$

$$k_{tt} \bar{u}_t(\lambda) - f_t(\lambda) = \bar{\rho}(\lambda) \mu(k_{tn} \bar{u}_t(\lambda) - f_n(\lambda)),$$
$$-\bar{\rho}(\lambda) \in \text{sign}(\bar{u}_t'(\lambda)). \tag{2}$$

Note that due to our assumption on f_n we have $\bar{r}_n = (k_{nt} \bar{u}_t - f_n) \geq c > 0$ and then $\bar{\rho}(\lambda) = \frac{k_{tt} \bar{u}_t(\lambda) - f_t(\lambda)}{\mu(k_{tn} \bar{u}_t(\lambda) - f_n(\lambda))} \in \text{W}^{1,\infty}([0, \Lambda], [-1, 1])$. Consequently the right derivative of $\bar{\rho}(\lambda)$ that we will denote by $\bar{\rho}'(\lambda)$ exists for almost all $\lambda \in [0, \Lambda]$. By differentiating the first line in (2) we have

$$k_{tt} \bar{u}_t' - f_t' = \bar{\rho}' \mu(k_{tn} \bar{u}_t(\lambda) - f_n(\lambda)) + \bar{\rho} \mu(k_{nt} \bar{u}_t' - f_n').$$

To simplify the formula we use the fact that $\bar{\rho}' \neq 0$ implies $\bar{u}_t'(\lambda) = 0$ due to the right continuity of the right derivative $\bar{u}_t'(\lambda)$ and the inclusion in (2) .We deduce the following estimate

$$\frac{|f_t'| + \mu |f_n'|}{\mu(k_{nt} \bar{u}_t - f_n)} \geq |\bar{\rho}'|. \tag{3}$$

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Hence, due to $\bar{r}_n = (k_{nt}\bar{u}_t - f_n) \geq c > 0$, $|\bar{\rho}'|$ is uniformly bounded.
Subtracting the equations in (1) and (2) leads us to

$$\varepsilon^2 u_t'' + k_{tt}(u_t - \bar{u}_t) = \mu\left(\rho r_n - \bar{\rho}\bar{r}_n\right) = \mu(\rho - \bar{\rho})r_n + \mu\bar{\rho}(r_n - \bar{r}_n). \tag{4}$$

Before multiplying the above equation by $(u_t' - \bar{u}_t')$, we observe that the inclusion in (1) is equivalent to $-u_t'(y - \rho) \leq 0$, for all $y \in [-1,1]$, and (2) is equivalent to $-\bar{u}_t'(\bar{y} - \bar{\rho}) \leq 0$, for all $\bar{y} \in [-1,1]$. Choosing $y = \bar{\rho}$ and $\bar{y} = \rho$ we get the monotonicity condition

$$(u_t' - \bar{u}_t')(\rho - \bar{\rho}) \leq 0.$$

Multiplying then (4) by $(u_t' - \bar{u}_t')$, we are immediately led to the estimate

$$\varepsilon^2 u_t''(u_t' - \bar{u}_t') + k_{tt}(u_t - \bar{u}_t)(u_t' - \bar{u}_t') \leq \mu\bar{\rho}(r_n - \bar{r}_n)(u_t' - \bar{u}_t'). \tag{5}$$

This estimate is rewritten after some rearrangements as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\varepsilon^2}{2}(u_t')^2\right) + \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{(k_{tt} - \mu\bar{\rho}k_{nt})(u_t - \bar{u}_t)^2}{2} + \frac{\mu k_{nt}(u_t - \bar{u}_t)^2}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\bar{\rho} \leq \varepsilon^2 u_t''\bar{u}_t' \tag{6}$$

Next we integrate the above estimate and further use the estimate (3) on $\bar{\rho}'$ to obtain

$$\frac{\varepsilon^2}{2}(u_t'(\lambda))^2 + \frac{k_{tt} - \mu|k_{nt}|}{2}(u_t(\lambda) - \bar{u}_t(\lambda))^2 \leq C\int_0^\lambda (u_t(s) - \bar{u}_t(s))^2 \, \mathrm{d}s$$

$$+ \varepsilon^2 \int_0^\lambda u_t''(s)\bar{u}_t'(s) \, \mathrm{d}s \tag{7}$$

$$+ \frac{\varepsilon^2}{2}u_1^2 + \frac{k_{tt} + \mu|k_{nt}|}{2}(u_0 - \bar{u}_0)^2 \tag{8}$$

for some finite constant $C > 0$ defined by estimate (3). Next we apply Gronwall's Lemma to estimate $(u_t(\lambda) - \bar{u}_t(\lambda))^2$. In a first step we divide both sides by $\frac{k_{tt} - \mu|k_{nt}|}{2}$, which is positive due to the assumption (3). We omit the exact and lengthy result and we represent it in the following simplified form. There exist positive constants $C_1 < \infty$ and $C_2 < \infty$, depending on the data K, μ and f only, such that

$$\left(u_t(\lambda) - \bar{u}_t(\lambda)\right)^2 \leq C_1 G(\varepsilon, \lambda) \exp\left(C_2\lambda\right)$$

holds, with $G(\varepsilon, \lambda) := \varepsilon^2 \int_0^\lambda u_t''(s)\bar{u}_t'(s) \, \mathrm{d}s + \frac{\varepsilon^2}{2}u_1^2 + (u_0 - \bar{u}_0)^2$. We can use this to estimate the first integral on the right hand side of (8) by $\int_0^\lambda (u_t(s) - \bar{u}_t(s))^2 \, \mathrm{d}s \leq C_3 G(\varepsilon, \lambda)$ with C_3 depending on K, μ, f and Λ only.
Using this estimate there exists a constant $C_4(K, \mu, f, \Lambda)$ such that

$$\frac{\varepsilon^2}{2}(u_t'(\lambda))^2 + \frac{k_{tt} - \mu|k_{nt}|}{2}(u_t(\lambda) - \bar{u}_t(\lambda))^2$$

$$\leq C_4\left(\varepsilon^2 \int_0^\lambda u_t''(s)\bar{u}_t'(s) \, \mathrm{d}s + \frac{\varepsilon^2}{2}u_1^2 + (u_0 - \bar{u}_0)^2\right) \tag{9}$$

holds for all $\lambda \in [0, \Lambda]$. The remaining task is to estimate the integral $\varepsilon^2 \int_0^\lambda u_t''(s)\bar{u}_t' XS(s) \, \mathrm{d}s$. This uses the results proved in Lemma 4.1 and adapts the argument in [MMR06] to the present case of non-prescribed normal force. The full proof can be found in the article [MMRS06] .
We now summarise our results in

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Theorem 5.1 (Stability of the Quasi-Static Path) *Let the stiffness matrix K be positive definite and the coefficient of friction $\mu > 0$ be such that $k_{tt} > \mu|k_{nt}|$ holds. In addition, let the initial condition $\bar{u}_0$ of the quasi-static problem satisfy $|k_{tt}\bar{u}_0 - f_t(0)| \leq \mu(k_{nt}\bar{u}_0 - f_n(0))_+$, and let the external force f satisfy $f \in C^1\left([0,\Lambda],\mathbb{R}^2\right)$, $var(f';0,\Lambda) < \infty$ and $\inf\{-f_n(\lambda) : \lambda \in [0,\Lambda]\} \geq$ C, for some constant $C > 0$ that depends on all data except f_n (see (1)) . Then the dynamic (6)-(8) and the quasi-static (9)-(11) problems with persistent contact have solutions and the quasi-static path is stable at time 0 in the sense of definition 2.1.*

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