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On modeling, analytical study and homogenization for smart materials

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Abstract

We discuss existence, uniqueness, regularity, and homogenization results for some nonlinear time-dependent material models. One of the methods for proving existence and uniqueness is the so-called energetic formulation, based on a global stability condition and on an energy balance. As for the two-scale homogenization we use the recently developed method of periodic unfolding and periodic folding. We also take advantage of the abstract Γ-convergence theory for rate-independent evolutionary problems.

1 Introduction

The models analyzed here concern three types of materials of high interest for applications: shape memory alloys (SMA), ferroelectric materials, and a class of rate-independent systems within the theory of elastoplasticity with hardening. All of them work in the framework of small deformations and quasistatic approximation for the elastic or electrostatic equilibria. The last two are rate-independent, while in the first (SMA) so are the hysteretic flow rule for the phase transformation and the linear constitutive elasticity, but not the heat equation. For both rate-independent models we will apply the energetic method as introduced in Ref. [21] (for a survey see Ref. [18]). In each case the energetic formulation will be explicitely described.

In Section 2 we consider a thermomechanical model of shape memory alloys. This model (see Ref. [5]) takes into account the non-isothermal character of the phase transformations, as well as the existence of the intrinsic dissipation. For the governing equations we prove existence, uniqueness and regularity in several functions spaces.

In Section 3 we discuss rate-independent engineering models for multi-dimensional behavior of ferroelectric materials. These models capture the non-linear and hysteretic behavior of such materials. We show that these models can be formulated in an energetic framework based on the elastic and the electric displacements as reversible variables, and on internal irreversible variables such as the remanent polarization. Quite general conditions on the constitutive laws guarantee the existence of a solution. Under more restrictive assumptions uniqueness of the solutions holds.

Section 4 is devoted to the homogenization for a class of rate-independent systems described by the *energetic formulation*. The associated nonlinear partial differential system has periodically oscillating coefficients, but has the form of a standard evolutionary variational inequality. Thus, the model applies to standard linearized elastoplasticity with hardening. Using the recently developed methods of *two-scale convergence*, *periodic unfolding* and *periodic folding*, we show that the homogenized problem can be represented as a two-scale limit, which is again an energetic formulation, but now involving the macroscopic variable in the physical domain as well as the microscopic variable in the periodicity cell.

2 Shape Memory Alloys

This section is devoted to the mathematical study of a thermomechanical model describing the macroscopic behavior of shape memory alloys. The analyzed model takes into account the non-isothermal character of the phase transition, as well as the existence of the intrinsic dissipation. The model is published in Ref. [5], but a description of it can also be found in Refs. [31, 33]. A variant which neglects the intrinsic dissipation was studied in Refs. [3, 4]. The newest model from Ref. [5] is founded on a free energy which is a convex function with respect to the strain and to the martensitic volume fraction and concave with respect to the temperature. In the circular cylindrical case, uniqueness of solutions in a large class of spaces, as well as their existence in the space of continuous functions were established in Refs. [31, 32]. Existence, uniqueness and regularity of solutions in various functions spaces were proved in Ref. [28].

We next give a brief description of the mathematical problem and of our main results on it. The first law of thermodynamics, the balance of momentum in its quasistatic form, the evolution equation for the internal variables (the volume fraction of martensite), together with the second principle of thermodynamics (the entropy inequality), lead to a partial differential equations system. In the circular cylindrical case the problem reduces to the following ordinary differential system:

$$
(\mathcal{T})\left\{\begin{array}{l}
(\mathcal{H}): \quad \dot{\theta}+\dfrac{1}{\tau}\theta=\Gamma|\dot{\beta}|+\dfrac{L}{C}\dot{\beta} \\
\sigma=E(\varepsilon-g\beta) \\
(\mathcal{E}): \quad 0\le\beta\le 1, \left\{\begin{array}{l}
\text{If } \ \beta=0, \ \text{ then } \ \sigma\le\sigma^{+} \ \text{ and} \\
\qquad\qquad\qquad \dot{\beta}<0\Rightarrow\sigma\le\sigma^{-} \\
\text{If } \ 0<\beta<1, \ \text{ then } \ \sigma^{-}\le\sigma\le\sigma^{+} \ \text{ and} \\
\qquad\qquad\qquad \left\{\begin{array}{l}\dot{\beta}<0\Rightarrow\sigma=\sigma^{-}\\ \dot{\beta}>0\Rightarrow\sigma=\sigma^{+}\end{array}\right. \\
\text{If } \ \beta=1, \ \text{ then } \ \sigma\ge\sigma^{-} \ \text{ and} \\
\qquad\qquad\qquad \dot{\beta}>0\Rightarrow\sigma\ge\sigma^{+}
\end{array}\right. \\
\beta(0)=0, \ \theta(0)=0, \ \varepsilon(0)=0, \ \sigma(0)=0
\end{array}\right.
$$

The unknown data are: the temperature θ at the surface of the body, the total fraction β of the martensite in the body, and the axial elongation ϵ of the sample in the Ox_3 direction. The stress σ is supposed to be given. All these are real functions only depending on the time variable $t \geq 0$. The constants $\tau, \Gamma, L, C, E, g, p, q, T_0, T_a, \Delta T$ are all positive, $T_0 > T_a$, $\Gamma < L/C$, and $\sigma^{\pm} := p(T_0 - T_a + \theta + \beta\Delta T) \pm q$.

Some comments are necessary in order to understand the mathematical problem raised by $(\mathcal{T})$:
1. The known data is an arbitrarily given continuous function $\sigma : J \to \mathbf{R}$ (J is an interval with $\min J = 0$) such that $\sigma(0) = 0$. The system $(\mathcal{T})$ is initially considered for unknown functions $\beta, \theta, \varepsilon : J \to \mathbf{R}$ having lateral derivatives everywhere on J, since they should satisfy $(\mathcal{H}), (\mathcal{E})$ with respect to these. If β is strictly increasing on some open subinterval $J_0 \subset J$, then[1] $\{t \in J_0 \mid \dot{\beta}_{\mathrm{f}}(t) > 0\}$ is dense in J_0, and so $\sigma = \sigma^+ = p(T_0 - T_a + \theta + \beta\Delta T) + q$ on J_0, by $(\mathcal{E})$. Consequently σ should have lateral derivatives on J_0. This poses a serious compatibility problem for our system if the given σ does not have lateral derivatives (e.g. if σ is continuous but nowhere differentiable).
2. If σ is such that $\dot{\beta}_{\mathrm{b}}(t_0) > 0$ and $\beta(t_0) = 1$, then β cannot be differentiable at t_0, since $\beta \leq 1$. This may happen even if σ is analytic on J, and so β can be less regular than σ. This is the reason to insist on lateral differentiability.
3. There exist strictly increasing continuous functions $u : J \to \mathbf{R}$, such that $\int_0^t \dot{u}(s)\mathrm{d}s = 0 \neq u(t) - u(0)$ for every $t > 0$. Since the usual derivative sometimes fails to characterize continuous and almost everywhere differentiable functions, its presence in $(\mathcal{T})$ may not guarantee the uniqueness of solutions.
4. Since for arbitrarily given σ a pronounced non-differentiability of solutions may occur, it would be natural to study $(\mathcal{T})$ in the space $C(J)$ of all real continuous functions on J, with the derivative in the sense of distributions. This is related to serious difficulties: what is the meaning of $|\dot{\beta}|$ in $(\mathcal{H})$ and of $\dot{\beta}(t)$ in $(\mathcal{E})$, if $\dot{\beta}$ is a distribution but not a function?
In order to remove the derivatives of β from $(\mathcal{E})$, we introduced in Ref. [31] a new notion. A point $t \in J_0$ (J_0 an interval) is said to be an *increment point* for $u \in C(J_0)$, if and only if for every neighborhood V of t, we have $t_1 < t_2$ and $u(t_1) < u(t_2)$ for some $t_1, t_2 \in V \cap J_0$. Let $M^+(u)$ denote the set of all increment points of u and set $M^-(u) := M^+(-u)$. If $X(J)$ is any of the spaces $AC_{\mathrm{loc}}(J), Lip_{\mathrm{loc}}(J), D^A_{\mathrm{f}}(J), D^A_{\mathrm{b}}(J), D^A_{\mathrm{l}}(J), D^{\aleph_0}(J),\ A_{\mathrm{f}}(J), A_{\mathrm{b}}(J), A_{\mathrm{l}}(J)$, endowed with its natural derivative (see the list below for details), then an equivalent form of $(\mathcal{E})$ for $\beta, \theta \in X(J)$ is

$$(\mathcal{E})_{X(J)} \begin{cases} \beta(t) > 0 \Rightarrow \sigma(t) \geq \sigma^-(t) \\ \beta(t) < 1 \Rightarrow \sigma(t) \leq \sigma^+(t) \\ t \in M^+(\beta) \Rightarrow \sigma(t) = \sigma^+(t) \\ t \in M^-(\beta) \Rightarrow \sigma(t) = \sigma^-(t). \end{cases}$$

If $\beta, \theta \in C(J)$ satisfy $(\mathcal{E})_{C(J)}$, then β must be locally monotone (see Ref. [31], Cor.4.2, p.455). If we write $(\mathcal{H})$ on every interval J_0 of monotonicity for β, we can then consider the following equation in distributions on $\overset{o}{J}_0$:

$$\dot{\theta} + \frac{1}{\tau}\theta = \left(\Gamma_0 + \frac{L}{C}\right)\dot{\beta} \quad \text{in } \mathcal{D}'(\overset{o}{J}_0), \tag{1}$$

[1] $\dot{u}_{\mathrm{f}}(t)$ and $\dot{u}_{\mathrm{b}}(t)$ denote the forward and the backward derivatives of u at t.

where $\Gamma_0 := \begin{cases} \Gamma, & \text{if } \beta \text{ is increasing on } J_0, \\ -\Gamma, & \text{else.} \end{cases}$

The system $(\mathcal{T})$ may be considered for any of the functions spaces and derivatives listed below (see Ref. [31] for the definition of an abstract derivation structure $X(J)$ and for the corresponding system $(\mathcal{T})_{X(J)}$).

List of functions spaces and associated derivatives

1) $C(J)$, with the derivative in the sense of distributions in $\mathcal{D}'(\overset{o}{J})$. We have the natural inclusions $C(J) \subset C(\overset{o}{J}) \subset \mathcal{D}'(\overset{o}{J})$. Let us recall that $u \in C(J)$ is increasing if and only if $u' \in \mathcal{D}'(\overset{o}{J})$ is positive.
2) $BV_{\text{loc}}(J) := \{u \in C(J) \,|\, u \text{ has locally bounded variation}\}$, with the derivative in the sense of distributions.
3) $AC_{\text{loc}}(J) := \{u \in C(J) \,|\, u \text{ is locally absolutely continuous}\}$, with the derivative almost everywhere.
4) $Lip_{\text{loc}}(J) := \{u \in C(J) \,|\, u \text{ is locally Lipschitz}\}$, with the derivative almost everywhere.
5) For every fixed at most countable subset A of J, consider the spaces:
a) $D_{\text{f}}^A(J) := \{u \in C(J) \,|\, u \text{ is differentiable to the right on } J \setminus A\}$ (respectively $D_{\text{b}}^A(J)$), with the forward (respectively backward) derivative on $J \setminus A$.
b) $D_{\text{l}}^A(J) = D_{\text{f}}^A(J) \cap D_{\text{b}}^A(J)$, with both forward and backward derivatives.
6) $D^{\aleph_0}(J) := \{u \in C(J) \,|$ the set of non-differentiability points of u is at most countable$\}$, with the usual derivative where this one exists.
7) a) $A_{\text{f}}(J) := \{u \in C(J) \,|\, u \text{ is forward-analytic}\}$, with the forward derivative. A function $u \in C(J)$ is said to be forward-analytic at $t \in J \setminus \{\sup J\}$, iff u is analytic on some $[t, s) \subset J$ $(s > t)$. We call u a *forward-analytic function*, iff u is forward-analytic at every $t \in J \setminus \{\sup J\}$.
b) $A_{\text{b}}(J) := \{u \in C(J) \,|\, u \text{ is backward-analytic}\}$, with the backward derivative (definitions are similar to those for $A_{\text{f}}(J)$).
c) $A_{\text{l}}(J) := A_{\text{f}}(J) \cap A_{\text{b}}(J)$, with both forward and backward derivatives.

Our problem is the following: for a fixed $X(J)$ in the above list and for a given $\sigma \in X(J)$ with $\sigma(0) = 0$, we wish to investigate the existence of solutions $\beta, \theta, \varepsilon \in X(J)$ of the system $(\mathcal{T})_{X(J)}$. The constitutive equation $\sigma = E(\varepsilon - g\beta)$ and the condition $\varepsilon(0) = 0$ from $(\mathcal{T})_{X(J)}$ can be ignored, since for $\beta, \theta \in X(J)$ satisfying all other conditions, we get a solution of $(\mathcal{T})_{X(J)}$ with $\varepsilon = \frac{\sigma}{E} + g\beta \in X(J)$. Therefore, every solution of $(\mathcal{T})_{X(J)}$ is given by a pair (β, θ) of functions from $X(J)$.

In Ref. [31] the following result is proved.

Proposition 2.1 *Let $X(J)$ be an abstract derivation structure. For $\beta, \theta \in C(J)$, the following statements are equivalent:*

(a) *(β, θ) is a solution of $(\mathcal{T})_{X(J)}$.*

(b) *(β, θ) is a solution of $(\mathcal{T})_{C(J)}$ and $\beta, \theta \in X(J)$.*

Now let $\sigma \in X(J)$ be fixed, such that $\sigma(0) = 0$. Since every solution of $(\mathcal{T})_{X(J)}$ also satisfies $(\mathcal{T})_{C(J)}$, we deduce that $(\mathcal{T})_{X(J)}$ is compatible if and only if for the unique solution (β, θ) of $(\mathcal{T})_{C(J)}$ (see Ref. [32], Th.3.1, p.543) we have $\beta, \theta \in X(J)$. Hence, for our problem, regularity of solutions (that is $\beta, \theta \in X(J)$ whenever $\sigma \in X(J)$) is equivalent to their existence.

Let $X(J)$ be any of the spaces

$$\begin{aligned}&BV_{\text{loc}}(J), AC_{\text{loc}}(J), Lip_{\text{loc}}(J), D^A_{\text{f}}(J), D^A_{\text{b}}(J), D^A_{\text{l}}(J), D^{\aleph_0}(J),\\ &A_{\text{f}}(J), A_{\text{b}}(J), A_{\text{l}}(J),\end{aligned} \tag{2}$$

endowed with its natural derivative from the above list.

Our main result is the following (for the proof, see Ref. [28]):

Theorem 2.2 *For any given $\sigma \in X(J)$, the system $(\mathcal{T})_{X(J)}$ has a unique solution.*

3 Ferroelectric Materials

Here we give a general description of a class of time-dependent models for ferroelectric materials. Our class of models is inspired by the engineering models from Refs. [13, 14, 15, 25, 29]. However, we will rephrase the theories there in such a way that it can be formulated in terms of two energetic functionals, namely the stored energy $\mathcal{E}$ and the pseudo-potential $\mathcal{R}$ for the dissipation. Thus, we will be able to take advantage of the recently developed energetic approach to rate-independent models, (see Refs. [11, 20, 17, 21] and the survey [18]).
The basic quantities in the theory are the elastic displacement field $u : \Omega \to \mathbb{R}^d$ and the electric displacement field $D : \mathbb{R}^d \to \mathbb{R}^d$. Here, the electric displacement is also defined outside the body, as interior polarization of a ferroelectric material generates an electric field E and displacement D in all of $\mathbb{R}^d$ via the static Maxwell equation in $\mathbb{R}^d$. Commonly, the polarization P is used for modeling, and is defined via

$$D = \epsilon_0 E + P,$$

where ϵ_0 is the dielectric constant (or permetivity) in the medium surrounding the body Ω. Our formulation stays with D, since it leads to a simple and consistent thermomechanical model.

In addition we use internal variables $Q : \Omega \to \mathbb{R}^{d_Q}$ which, for instance, may be taken as a remanent strain $\boldsymbol{\varepsilon}_{\text{rem}}$ or a remanent polarization P_{rem}.

The stored-energy functional has the form

$$\begin{aligned}\mathcal{E}(t, u, D, Q) \;\; &= \;\; \int_\Omega \Big(W(x, \boldsymbol{\varepsilon}(u), D, Q) + \alpha(x, \nabla Q) \Big)\, \mathrm{d}x + \int_{\mathbb{R}^d \setminus \Omega} \frac{1}{2\epsilon_0} |D|^2 \,\mathrm{d}x \\ &\quad - \langle \ell(t), (u, D) \rangle,\end{aligned}$$

where W is the Helmholtz free energy and $\boldsymbol{\varepsilon}(u)$ is the infinitesimal strain tensor given by

$$\boldsymbol{\varepsilon}(u) = \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}}) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} := \{\, \boldsymbol{\varepsilon} \in \mathbb{R}^{d\times d} \mid \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathsf{T}} \,\}.$$

The nonlocal term $\alpha(x, \nabla Q)$ in $\mathcal{E}$ usually takes the form $\frac{k}{2}|\nabla Q|^2$ with $k > 0$. This term penalizes rapid changes of the internal variable by introducing a length scale which determines the minimal width of the interfaces between domains of different polarization.
The external loading $\ell(t)$ depends on the process time t and is usually given by

$$\begin{aligned}\langle \ell(t), (u, D)\rangle &= \int_{\mathbb{R}^d} E_{\mathrm{ext}}(t, x)\cdot D(x)\,\mathrm{d}x + \int_{\Omega} f_{\mathrm{vol}}(t, x)\cdot u(x)\,\mathrm{d}x \\ &\quad + \int_{\Gamma_{\mathrm{Neu}}} f_{\mathrm{surf}}(t, x)\cdot u(x)\,\mathrm{d}a(x),\end{aligned}$$

where E_{ext}, f_{vol} and f_{surf} are applied, external fields.

For the dissipation potential $\mathcal{R}$ we take the very simple form

$$\mathcal{R}(\dot{Q}) = \int_{\Omega} R(x, \dot{Q}(x))\,\mathrm{d}x,$$

where $R(x, \cdot) : \mathbb{R}^{d_Q} \to [0, \infty)$ is convex and positively homogeneous of degree 1. Note that the dissipation potential only acts on the rate $\dot{Q} = \frac{\partial}{\partial t}Q$ of the internal variable. The classical way to describe dissipation in ferroelectrics is a switching function of the form

$$\Phi(x, X_Q) \le 0, \quad \text{with } X_Q = \frac{\partial}{\partial Q}W - \mathrm{div}(\mathrm{D}\alpha(\nabla Q)). \tag{3}$$

This is equivalent to our dissipation potential R by the relation

$$R(x, \dot{Q}) = \max\{\, \dot{Q}\cdot X_Q \mid \Phi(x, X_Q) \le 0 \,\}.$$

To formulate the rate-independent evolution law we use the thermomechanically conjugated forces

$$\sigma = \frac{\partial}{\partial \boldsymbol{\varepsilon}}W \in \mathbb{R}^{d\times d}, \quad E = \begin{cases} \frac{\partial}{\partial D}W & \text{on } \Omega, \\ \frac{1}{\epsilon_0}D & \text{on } \mathbb{R}^d \setminus \Omega \end{cases}, \quad X_Q \in \mathbb{R}^{d_Q}, \tag{4}$$

where σ is the stress tensor and E the electric field. The elastic equilibrium equation and the Maxwell equations read

$$\begin{array}{ll} -\operatorname{div}\sigma + f_{\mathrm{vol}}(t, \cdot) = 0 & \text{in } \Omega, \\ \operatorname{div} D = 0 \text{ and } \operatorname{curl}(E - E_{\mathrm{ext}}(t, \cdot)) = 0 & \text{in } \mathbb{R}^d, \end{array} \tag{5}$$

where $\operatorname{curl} E$ is defined as $\nabla E - (\nabla E)^{\mathsf{T}}$ for general dimensions.

The evolution of Q follows the force balance law:

$$0 \in \partial R(x, \dot{Q}) + X_Q,$$

where $\partial R(x,\cdot)$ is the subdifferential of the convex function $R(x,\cdot)$.

We now want to rewrite these relations, as equations in function spaces. For this purpose we introduce a suitable state space $\mathcal{Y} = \mathcal{F} \times \mathcal{Q}$ as follows. The space $\mathcal{F}$ contains the functions u and D, and takes the form

$$\mathcal{F} = \mathcal{H} \times \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d), \text{ where } \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d) := \{\, \psi \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{R}^d) \mid \operatorname{div}\psi = 0 \,\}$$

and $\mathcal{H}$ is a closed affine subspace of $\mathrm{H}^1(\Omega;\mathbb{R}^d)$. The space $\mathcal{Q}$ contains the internal state functions Q and is taken to be $\mathrm{W}^{1,q_Q}(\Omega;\mathbb{R}^{d_Q})$ for a suitable $q_Q > 1$.

Using the well-known fact (cf. Ref. [30], Th.1.4) that the total space $\mathrm{L}^2(\mathbb{R}^d;\mathbb{R}^d)$ decomposes in two orthogonal closed subspaces $\mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d)$ and

$$\mathrm{L}^2_{\mathrm{curl}}(\mathbb{R}^d) = \{\, \psi \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{R}^d) \mid \operatorname{curl}\psi = 0 \,\},$$

we obtain the following result.

Proposition 3.1 *Let $\mathrm{D}_D\mathcal{E}(t,u,D,Q)[\widehat{D}]$ denote the Gâteaux derivative of $\mathcal{E}$ in the direction $\widehat{D}$. Then*

$$\big(\forall\, \widehat{D} \in \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d) :\ \mathrm{D}_D\mathcal{E}(t,u,D,Q)[\widehat{D}] = 0\big) \iff \operatorname{curl}(E - E_{\mathrm{ext}}(t,\cdot)) = 0 \quad \textit{in } \mathbb{R}^d.$$

Thus, we implement the Maxwell equations by the condition $\mathrm{D}_D\mathcal{E}(t,u,D,Q) = 0$ in a suitable function space. Similarly, the elastic equilibrium is obtained by $\mathrm{D}_u\mathcal{E}(t,u,D,Q) = 0$. The full problem may be written as

$$\begin{gathered} \mathrm{D}_u\mathcal{E}(t,u(t),D(t),Q(t)) = 0, \quad \mathrm{D}_D\mathcal{E}(t,u(t),D(t),Q(t)) = 0, \\ 0 \in \partial\mathcal{R}(\dot{Q}(t)) + \mathrm{D}_Q\mathcal{E}(t,u(t),D(t),Q(t)), \end{gathered} \tag{6}$$

where the last above equation corresponds to the dissipative force balance.

In fact, our theory is not based on the force balance (6). Instead, following Refs. [18, 21], we use a weaker formulation only based on energies. This energetic formulation avoids derivatives of $\mathcal{E}$ and of the solution (u,D,Q). Under suitable smoothness and convexity assumptions the energetic formulation is equivalent to (6). We call (u,D,Q) an ***energetic solution*** of the problem associated with $\mathcal{E}$ and $\mathcal{R}$, if for all $t \in [0,T]$ the *stability condition* (S) and the *energy balance* (E) hold:

$$\begin{aligned} &\text{(S)} \quad \mathcal{E}(t,u(t),D(t),Q(t)) \le \mathcal{E}(t,\widehat{u},\widehat{D},\widehat{Q}) + \mathcal{R}(\widehat{Q}-Q(t)) \text{ for all } \widehat{u},\widehat{D},\widehat{Q}; \\ &\text{(E)} \quad \mathcal{E}(t,u(t),D(t),Q(t)) + \textstyle\int_0^t \mathcal{R}(\dot{Q}(s))\,\mathrm{d}s \\ &\qquad = \mathcal{E}(0,u(0),D(0),Q(0)) - \textstyle\int_0^t \langle \dot{\ell}(s),(u(s),D(s))\rangle\,\mathrm{d}s. \end{aligned} \tag{7}$$

In Refs. [22, 23] we showed that (S) & (E) has solutions for suitable initial data, if the constitutive functions W, α, and R satisfy reasonable continuity and convexity assumptions. Under stronger conditions we also proved uniqueness of solutions.

We now provide conditions on the constitutive functions W, α and R, in order to get the existence result.

The first assumption concerns the domain and the Dirichlet boundary:

$$\begin{aligned}&\Omega \subset \mathbb{R}^d \text{ is a connected bounded open set with Lipschitz boundary } \Gamma,\\ &\text{and } \Gamma_{\mathrm{Dir}} \text{ a measurable subset of } \Gamma, \text{ such that } \textstyle\int_{\Gamma_{\mathrm{Dir}}} 1 \, \mathrm{d}a > 0.\end{aligned} \tag{B0}$$

The function $R : \Omega \times \mathbb{R}^{d_Q} \to [0, \infty)$ satisfies

$$R \in \mathrm{C}^0(\overline{\Omega} \times \mathbb{R}^{d_Q}) \text{ and } \exists\, c_R, C_R > 0\ \forall V \in \mathbb{R}^{d_Q} :\ c_R|V| \leq R(x, V) \leq C_R|V|. \tag{B1}$$

$$\forall\, x \in \Omega :\ R(x, \cdot) : \mathbb{R}^{d_Q} \to [0, \infty) \text{ is 1-homogeneous and convex.} \tag{B2}$$

The functions W and α have to fulfill the following three conditions:

$$W : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^d \times \mathbb{R}^{d_Q} \to [0, \infty) \text{ is a Caratheodory function,} \tag{B3}$$

which means that the function $W(\cdot, \boldsymbol{\varepsilon}, D, Q)$ is measurable on Ω for each $(\boldsymbol{\varepsilon}, D, Q)$, and that the mapping $W(x, \cdot, \cdot, \cdot)$ is continuous on $\mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^d \times \mathbb{R}^{d_Q}$ for a.e. $x \in \Omega$.

$$\begin{aligned}&\exists\, c_{\mathcal{E}}, C_{\mathcal{E}} > 0,\ q > 1\ \forall\, (x, \boldsymbol{\varepsilon}, D, Q, V) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^d \times \mathbb{R}^{d_Q} \times \mathbb{R}^{d_Q} :\\ &W(x, \boldsymbol{\varepsilon}, D, Q) + \alpha(V) \geq c_{\mathcal{E}}(|\boldsymbol{\varepsilon}|^2 + |D|^2 + |Q|^q + |V|^q) - C_{\mathcal{E}}.\end{aligned} \tag{B4}$$

$$\begin{aligned}&\alpha : \mathbb{R}^{d_Q\times d} \to \mathbb{R} \text{ is convex and}\\ &\forall\, (x, Q) \in \Omega \times \mathbb{R}^{d_Q} :\ W(x, \cdot, \cdot, Q) : \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^d \to \mathbb{R} \text{ is convex.}\end{aligned} \tag{B5}$$

For the external loading $\ell(t)$ we assume

$$\ell \in \mathrm{C}^1\big([0, T], (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^d))^* \times \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d)^*\big). \tag{B6}$$

Let us consider the following functions spaces:

$$\mathcal{F} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega, \mathbb{R}^d)_{\mathrm{weak}} \times \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d)_{\mathrm{weak}}, \quad \mathcal{Z} = \mathrm{L}^1(\Omega, \mathbb{R}^{d_Q})_{\mathrm{strong}}.$$

Here the subscripts "weak" and "strong" indicate the use of the weak or strong (norm) topology in the corresponding Banach spaces. The functional $\mathcal{E}$ is defined as above on $[0, T] \times \mathcal{F} \times \mathcal{Z}$, where $\mathcal{E}(t, u, D, Q)$ takes the value $+\infty$ if $Q \notin \mathrm{W}^{1,q}(\Omega; \mathbb{R}^{d_Q})$ or if the integrand is not in $\mathrm{L}^1(\Omega)$.

We can now state our existence theorem.

Theorem 3.2 (Existence theorem)
If the assumptions (B0)–(B6) *hold, then for each stable initial condition* $(u_0, D_0, Q_0) \in \mathcal{F} \times \mathcal{Z}$*, the energetic problem* (S) & (E) *has a solution* $(u, D, Q) : [0, T] \to \mathcal{F} \times \mathcal{Z}$*, such that* $(u(0), D(0), Q(0)) = (u_0, D_0, Q_0)$*, and*

$$(u, D, Q) \in \mathrm{L}^\infty([0, T]; \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^d) \times \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d) \times \mathrm{W}^{1,q}(\Omega; \mathbb{R}^{d_Q})).$$

4 Homogenization for rate-independent systems

Our aim is to provide homogenization results for evolutionary variational inequalities of the type:

$$\langle \mathcal{A}q - \ell(t), v - \dot{q} \rangle + \mathcal{R}(v) - \mathcal{R}(\dot{q}) \geq 0 \quad \text{for every } v \in \mathcal{Q}. \tag{8}$$

Here $\mathcal{Q}$ is a Hilbert space with dual $\mathcal{Q}^*$, the continuous linear operator $\mathcal{A} : \mathcal{Q} \to \mathcal{Q}^*$ is symmetric and positive definite, the forcing ℓ lies in $\mathrm{C}^1([0,T], \mathcal{Q}^*)$, and the dissipation functional $\mathcal{R} : \mathcal{Q} \to [0, \infty)$ is convex, lower semi-continuous and positively homogeneous of degree 1, i.e., $\mathcal{R}(\gamma q) = \gamma \mathcal{R}(q)$ for all $\gamma \geq 0$ and $q \in \mathcal{Q}$. The last property of $\mathcal{R}$ leads to rate independence.

The problem (8) has many different equivalent formulations. For our purposes the so-called *energetic formulation* for rate-independent hysteresis problem is especially appropriate, cf. Refs. [18, 20]. This formulation is solely based on the energy-storage functional $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R}$ defined via

$$\mathcal{E}(t,q) = \frac{1}{2}\langle \mathcal{A}q, q \rangle - \langle \ell(t), q \rangle,$$

and on the dissipation functional $\mathcal{R}$. Thus, homogenization of an evolutionary problem can be reduced to some extend to homogenization of functionals. We formulate our rate-independent evolutions systems and we provide existence and uniqueness theorems for the initial and expected two-scale homogenized problems. We present some Γ-convergence results and finally our main homogenization theorem.

The notion of two-scale convergence has been introduced by Nguetseng (see Ref. [27]) in 1989 and developed by Allaire in 1992 (see Ref. [2]). Ref. [16] provides an overview of the main homogenization problems studied by this technique. The periodic unfolding method recently introduced (2002) by Cioranescu, Damlamian and Griso in Ref. [8], reduces the two-scale convergence to a weak convergence in an appropriate space. This concept is now applied in a variety of quite different applications in continuum mechanics, see e.g., Refs. [1, 10, 26, 34, 35]. To the best of our knowledge there is no theory for nonsmooth evolutionary systems like the variational inequalities here.

Throughout, the domain Ω will be a bounded open subset of $\mathbb{R}^d$. For the semi-open unit cell $Y = [0,1)^d$, we have $\cup_{\lambda \in \mathbb{Z}^d}(\lambda + Y) = \mathbb{R}^d$ and $(\lambda + Y) \cap (\mu + Y) = \emptyset$ for $\lambda, \mu \in \mathbb{Z}^d$ with $\lambda \neq \mu$. From now on we will assume that $p \in (1, \infty)$.

Let us recall the definition of the *classical* two-scale convergence.

Definition 4.1 *Let $(v_\varepsilon)_\varepsilon$ be a sequence in $\mathrm{L}^p(\Omega)$. One says that $(v_\varepsilon)_\varepsilon$ two-scale converges to $V = V(x,y)$ in $\mathrm{L}^p(\Omega \times Y)$ (we write $v_\varepsilon \overset{\mathrm{ts}}{\rightharpoonup} V$), if for any function $\psi = \psi(x,y)$ in $\mathrm{C}^\infty_{\mathrm{c}}(\Omega; \mathrm{C}^\infty_{\mathrm{per}}(Y))$, one has*

$$\lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) \,\mathrm{d}x = \int_\Omega \int_Y V(x,y)\psi(x,y) \,\mathrm{d}y \,\mathrm{d}x. \tag{9}$$

The *periodic unfolding operator* $\mathcal{T}_\varepsilon$ was introduced in Ref. [8] and then used for homogenization of nonlinear integrals in Refs. [6, 7]. On the full space $\mathbb{R}^d$, it is defined by

$$\mathcal{T}_\varepsilon : \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d \times Y); \quad \mathcal{T}_\varepsilon v(x,y) = v\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right).$$

We next introduce the notions of *weak/strong two-scale convergence.*

Definition 4.2 *Let $V \in \mathrm{L}^p(\Omega \times Y)$. A bounded sequence $(v_\varepsilon)_\varepsilon$ in $\mathrm{L}^p(\Omega)$*

(w2): *weakly two-scale converges to V (we write $v_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} V$), if*

$$\mathcal{T}_\varepsilon v_\varepsilon \rightharpoonup V \ \textit{(weakly) in } \mathrm{L}^p(\mathbb{R}^d \times Y).$$

(s2): *strongly two-scale converges to V (we write $v_\varepsilon \overset{\mathrm{s2}}{\rightharpoonup} V$), if*

$$\mathcal{T}_\varepsilon v_\varepsilon \to V \ \textit{(strongly) in } \mathrm{L}^p(\mathbb{R}^d \times Y).$$

Clearly, the above weak two-scale convergence is stronger than the classical.

4.1 ε problem

Let us consider:

$\Omega \subset \mathbb{R}^d$, a connected bounded open set, with Lipschitz boundary Γ,

$Y = [0,1)^d \subset \mathbb{R}^d$, unit periodicity cell,

$u : \Omega \to \mathbb{R}^d$, displacement,

$z : \Omega \to \mathbb{R}^m$, internal variable.

For every $\varepsilon > 0$, define the *energy functional* $\mathcal{E}_\varepsilon$ and the *dissipation functional* $\mathcal{R}_\varepsilon$ by

$$\begin{aligned}
\mathcal{E}_\varepsilon(t,u,z) &= \int_\Omega \frac{1}{2}\Big\langle \mathbb{C}\left(\tfrac{x}{\varepsilon}\right)\left(\mathbf{e}(u) - \mathbb{B}\left(\tfrac{x}{\varepsilon}\right) z\right), \mathbf{e}(u) - \mathbb{B}\left(\tfrac{x}{\varepsilon}\right) z\Big\rangle \,\mathrm{d}x \\
&\quad + \int_\Omega \frac{1}{2}\left\langle \mathbb{H}\left(\tfrac{x}{\varepsilon}\right) z, z\right\rangle \mathrm{d}x - \int_\Omega u(x) f_{\mathrm{ext}}(t,x)\,\mathrm{d}x \\
\mathcal{R}_\varepsilon(\dot{z}) &= \int_\Omega \rho\left(\tfrac{x}{\varepsilon}, \dot{z}(x)\right) \mathrm{d}x,
\end{aligned}$$

where

$$\mathbf{e}(u) = \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}}) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} := \{\, \sigma \in \mathbb{R}^{d\times d} \mid \sigma = \sigma^{\mathsf{T}} \,\}.$$

The tensors $\mathbb{C}, \mathbb{H}, \mathbb{B}$ defined on $\mathbb{R}^d$ are Y-periodic, and take values in:

$$\mathbb{C}(y) \in \text{Sym4}^{\text{th}} \text{ order tensor}, \quad \mathbb{B}(y) \in \text{Lin}(\mathbb{R}^m, \mathbb{R}^{d\times d}_{\text{sym}}), \quad \mathbb{H}(y) \in \mathbb{R}^{m\times m}_{\text{sym}}.$$

We work under the hypotheses stated below.

Assumptions for $\mathbb{C}, \mathbb{H}, \mathbb{B}$: for all $y \in \mathbb{R}^d$ and $z \in \mathbb{R}^m$, we have

$$\frac{1}{C}|\mathbf{e}|^2 \leq \langle \mathbb{C}(y)\mathbf{e}, \mathbf{e}\rangle \leq C|\mathbf{e}|^2, \quad \frac{1}{C}|z|^2 \leq \langle \mathbb{H}z, z\rangle \leq C|z|^2, \quad \|\mathbb{B}(y)\| \leq C$$

(for some constant $C > 0$).

Assumptions for ρ:

$$(\mathrm{H}_\rho) \begin{cases} \rho : \mathbb{R}^d \times \mathbb{R}^m \to [0, \infty), \\ \rho(\cdot, v) \text{ Lebesgue measurable and } Y\text{-periodic, for every } v \in \mathbb{R}^m, \\ \rho(y, \cdot) \text{ 1-homogeneous and convex for a.e. } y \in \mathbb{R}^d, \\ \frac{1}{C}|v| \leq \rho(y, v) \text{ for a.e. } y \in \mathbb{R}^m \text{ and every } v \in \mathbb{R}^m, \\ |\rho(y, v) - \rho(y, v')| \leq C|v - v'| \text{ for a.e. } y \in \mathbb{R}^d \text{ and all } v, v' \in \mathbb{R}^m. \end{cases}$$

Let us consider the Hilbert space $\mathcal{Q} = \mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega)^m$.
We call $q_\varepsilon = (u_\varepsilon, z_\varepsilon) : [0, T] \to \mathcal{Q}$ an ***energetic solution*** of the problem associated with $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$, if for every $t \in [0, T]$ the *stability condition* (S^ε) and the *energy balance* (E^ε) hold:

$$\begin{aligned} (\mathrm{S}^\varepsilon):\ & \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) \leq \mathcal{E}_\varepsilon(t, u, z) + \mathcal{R}_\varepsilon(z - z_\varepsilon(t)) \quad \text{for every } (u, z) \in \mathcal{Q}, \\ (\mathrm{E}^\varepsilon):\ & \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \int_0^t \mathcal{R}_\varepsilon(\dot{z}_\varepsilon(s))\,\mathrm{d}s = \mathcal{E}_\varepsilon(0, u_\varepsilon(0), z_\varepsilon(0)) \\ & \qquad - \int_0^t \int_\Omega \dot{f}_{\text{ext}}(s, x) \cdot u(x)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

We now state our existence and uniqueness result for (S^ε) & (E^ε).

Proposition 4.3 *Let* $f_{\text{ext}} \in \mathrm{C}^{\text{Lip}}([0, T], (\mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d)^*)$. *Then for all* $\varepsilon > 0$ *and stable* $(u^0_\varepsilon, z^0_\varepsilon) \in \mathcal{Q}$, *there is a unique solution* $(u_\varepsilon, z_\varepsilon) \in \mathrm{C}^{\text{Lip}}([0, T], \mathcal{Q})$ *of* (S^ε) & (E^ε), *with* $(u_\varepsilon(0), z_\varepsilon(0)) = (u^0_\varepsilon, z^0_\varepsilon)$.

Moreover, we have ε*-independent Lipschitz bounds for the solutions, that is, for some constant* $c_1 > 0$ *we have*

$$\|(u_\varepsilon(t), z_\varepsilon(t)) - (u_\varepsilon(s), z_\varepsilon(s))\|_{\mathrm{H}^1 \times \mathrm{L}^2} \leq c_1|t - s| \quad \text{for all } t, s \in [0, T], \varepsilon > 0. \tag{10}$$

4.2 Two-scale homogenized problem

We now formulate the problem $(\mathbf{S})$ & $(\mathbf{E})$, which will turn out to be the two-scale homogenized problem for (S^ε) & (E^ε).
Let $\mathbf{Q} = \mathbf{H} \times \mathbf{Z}$, where

$$\mathbf{H} = \mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\text{av}}(Y))^d, \quad \mathbf{Z} = \mathrm{L}^2(\Omega; \mathrm{L}^2(Y))^m = \mathrm{L}^2(\Omega \times Y)^m.$$

Here, $\mathrm{H}^1_{\mathrm{av}}(Y) = \{\, U \in \mathrm{H}^1_{\mathrm{per}}(Y) \mid \int_Y U(y)\,\mathrm{d}y = 0 \,\}$. For all $Q = (U, Z)$ in $\mathbf{Q}$, with $U = (U_0, U_1)$, let us define the two-scale functionals $\mathbf{E}$ and $\mathbf{R}$

$$\begin{aligned}
\mathbf{E}(t, U, Z) \;&=\; \int_\Omega\!\int_Y \frac{1}{2}\Big\langle \mathbb{C}(y)(\widehat{\mathbf{e}}(U) - \mathbb{B}(y)Z), \widehat{\mathbf{e}}(U) - \mathbb{B}(y)Z \Big\rangle \,\mathrm{d}y\,\mathrm{d}x \\
&\quad + \int_\Omega\!\int_Y \frac{1}{2}\langle \mathbb{H}(y)Z, Z\rangle \,\mathrm{d}y\,\mathrm{d}x - \int_\Omega\!\int_Y U_0(x) f_{\mathrm{ext}}(t, x)\,\mathrm{d}y\,\mathrm{d}x, \\
\mathbf{R}(\dot{Z}) \;&=\; \int_\Omega\!\int_Y \rho(y, \dot{Z}(x, y))\,\mathrm{d}y\,\mathrm{d}x,
\end{aligned}$$

where $\widehat{\mathbf{e}}(U) = \mathbf{e}_x(U_0) + \mathbf{e}_y(U_1)$, which means $\widehat{\mathbf{e}}(U)(x, y) = \mathbf{e}_x(U_0(\cdot))(x) + \mathbf{e}_y(U_1(x, \cdot))(y)$.

The energetic formulation for the two-scale homogenized problem $(\mathbf{S})$ & $(\mathbf{E})$ reads: for every $t \in [0, T]$, the stability condition $(\mathbf{S})$ and the energy balance $(\mathbf{E})$ hold, that is,

$$\begin{aligned}
&(\mathbf{S}):\;\; \mathbf{E}(t, U(t), Z(t)) \le \mathbf{E}(t, \widetilde{U}, \widetilde{Z}) + \mathbf{R}(\widetilde{Z} - Z(t)) \;\forall\, \widetilde{Q} = (\widetilde{U}, \widetilde{Z}) \in \mathbf{H} \times \mathbf{Z}, \\
&(\mathbf{E}):\;\; \mathbf{E}(t, U(t), Z(t)) + \int_0^t \mathbf{R}(\dot{Z}(s))\,\mathrm{d}s = \mathbf{E}(0, U(0), Z(0)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \int_0^t\!\int_\Omega \dot{f}_{\mathrm{ext}}(s, x) \cdot U_0(x)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

We next state our existence and uniqueness result for the problem $(\mathbf{S})$ & $(\mathbf{E})$.

Proposition 4.4 *Let $f_{\mathrm{ext}} \in \mathrm{C}^{\mathrm{Lip}}([0, T], (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$. Then for every stable $Q^0 = (U^0, Z^0) \in \mathbf{Q}$, the problem $(\mathbf{S})$& $(\mathbf{E})$ has a unique solution $Q = (U, Z) \in \mathrm{C}^{\mathrm{Lip}}([0, T], \mathbf{Q})$, with $Q(0) = Q^0$.*

The convergence of $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ to $\mathbf{E}$ and $\mathbf{R}$ can be viewed as a type of two-scale Mosco convergence, i.e., Γ-convergence in the weak and in the strong topology (see Ref. [19]). Here we rely on the unfolding operator and on the folding operator in order to construct suitable recovery sequences, also called *realizing sequences* in Ref. [12].

The crucial tool for proving the convergence of the solutions q_ε to the energetic solution Q associated with $\mathbf{E}$ and $\mathbf{R}$ is the abstract Γ-convergence theory developed in Ref. [19]. There, the simple theory relies on the fact that the dissipation functions converge continuously in the weak topology; yet this is not the case in our situation. However, we are able to use the quadratic nature of the energies allowing some cancelations in differences of energies. For instance, $\mathcal{E}_\varepsilon(t, q_\varepsilon) - \mathcal{E}_\varepsilon(t, \widetilde{q}_\varepsilon)$ converges to $\mathbf{E}(t, Q) - \mathbf{E}(t, \widetilde{Q})$, if q_ε has the "weak" two-scale limit Q and $q_\varepsilon - \widetilde{q}_\varepsilon$ has a strong(!) two-scale limit $Q - \widetilde{Q}$.

Our homogenization theorem is based on the notion of *two-scale cross-convergence.*

Definition 4.5 *Let $Q = (U, Z) \in \mathbf{Q}$, with $U = (U_0, U_1)$. A sequence $q_\varepsilon = (u_\varepsilon, z_\varepsilon)_\varepsilon$ in $\mathcal{Q}$ is called two-scale cross-convergent to (U, Z), if*

$$u_\varepsilon \stackrel{\mathrm{w2}}{\rightharpoonup} U_0, \quad \nabla u_\varepsilon \stackrel{\mathrm{w2}}{\rightharpoonup} \nabla U_0 + \nabla_y U_1, \quad z_\varepsilon \stackrel{\mathrm{w2}}{\rightharpoonup} Z.$$

We write this as $(u_\varepsilon, z_\varepsilon) \stackrel{\mathrm{cw2}}{\rightharpoonup} (U, Z)$.

We can now formulate our main result stating that $(\mathbf{S})\,\&\,(\mathbf{E})$ is the two-scale homogenized problem for $(\mathrm{S}^\varepsilon)\,\&\,(\mathrm{E}^\varepsilon)$ (see Ref. [24] for the proof).

Theorem 4.6 *Let* $q_\varepsilon = (u_\varepsilon, z_\varepsilon)$ *be the solution of* $(\mathrm{S}^\varepsilon)\,\&\,(\mathrm{E}^\varepsilon)$. *For the initial data, assume that:*

$$
\begin{aligned}
&q_\varepsilon^0 = (u_\varepsilon^0, z_\varepsilon^0) \textit{ is stable for every } \varepsilon > 0,\\
&q_\varepsilon^0 = (u_\varepsilon^0, z_\varepsilon^0) \stackrel{\mathrm{cw2}}{\rightharpoonup} Q^0 = (U^0, Z^0) \in \mathbf{Q},\\
&\mathcal{E}_\varepsilon(0, q_\varepsilon^0) \to \mathbf{E}(0, Q^0).
\end{aligned}
$$

Then $(q_\varepsilon)_\varepsilon$ *two-scale cross-converges to the unique solution* $Q = (U, Z)$ *of the two-scale homogenized problem* $(\mathbf{S})\,\&\,(\mathbf{E})$, *with* $Q(0) = Q^0$.

References

[1] Alber, H.D., Evolving microstructure and homogenization, *Contin. Mech. Thermodyn.* **12** (2000), 235–286.

[2] Allaire, A., Homogenization and two-scale convergence, *SIAM J. Math. Anal.* **23** (1992), 1482–1518.

[3] X. Balandraud, E. Ernst and E. Soós, Relaxation and creep phenomena in shape memory alloys, Part I: Hysteresis loop and pseudoelastical behavior, *Z. Angew. Math. Phys.* **51** (2000), 171–203.

[4] X. Balandraud, E. Ernst and E. Soós, Relaxation and creep phenomena in shape memory alloys, Part II: Stress relaxation and strain creep during phase transformation, *Z. Angew. Math. Phys.* **51** (2000), 419–448.

[5] X. Balandraud, E. Ernst and E. Soós, Monotone strain-stress models for Shape Memory Alloys hysteresis loop and pseudoelastic behavior, *Z. Angew. Math. Phys.* **56** (2005), 304–356.

[6] Cioranescu, D., Damlamian, A., De Arcangelis, R., Homogenization of nonlinear integrals via the periodic unfolding method, *C.R. Acad. Sci. Paris* **Ser. I 339** (2004), 77–82.

[7] D. Cioranescu, A. Damlamian, R. De Arcangelis, Homogenization of quasiconvex integrals via the periodic unfolding method, *SIAM J. Math. Anal.* **37** (2006), no. 5, 1435–1453.

[8] D. Cioranescu, A. Damlamian, G. Griso, Periodic unfolding and homogenization, *C. R. Acad. Sci. Paris,* **Ser. I 335** (2002), 99–104.

[9] D. Cioranescu, P. Donato, An introduction to Homogenization, *Oxford Lecture Series in Mathematics and its Applications*, **17**, Oxford University Press, (1999).

[10] C. Eck, P. Knaber, S. Korotov, A two-scale method for the computation of solid-liquid phase transitions with dendritic microstructure, *J. Comput. Phys.* 178 (2002), 58–80.

[11] G. Francfort and A. Mielke, Existence results for a class of rate-independent material models with non-convex elastic energies, *J. reine angew. Math.*, **595** (2006), 55–91.

[12] V. V. Jikov, S. M. Kozlov, O. A. Oleinik, Homogenization of differential operators and integral functions, Springer-Verlag, (1994).

[13] M. Kamlah, Ferroelectric and ferroelastic piezoceramics-modelling of electromechanical hysteresis phenomena, *Continuum Mech.Thermodyn.*, **13** (2001), 219–268.

[14] M. Kamlah and Q. Jiang, A model for PZT ceramics under uni-axial loading, *Forschungszentrum Karlsruhe*, FZKA 6211, (1998).

[15] M. Kamlah and Z. Wang, A thermodynamically and microscopically motivated constitutive model for piezoceramics, *Comput. Materials Science*, **28** (2003) 409–418.

[16] D. Lukkassen, G. Nguetseng, P. Wall, Two-scale convergence, *International Journal of Pure and Applied Mathematics* **2** (2002), 35–86.

[17] A. Mainik and A. Mielke, Existence results for energetic models for rate-independent systems, *Calc. Var. PDSs* **22** (2005), 73–99.

[18] Mielke, A., Evolution for rate-independent systems. In *Handbook of differential equations.* (ed. by Dafermos, C. and Feireisl, E.). Evolutionary Equations Vol. II, Elsevier B.V. 2005, 461–559.

[19] A. Mielke, T. Roubicek, U. Stefanelli, Gamma limits and relaxations for rate-independent evolution equations, WIAS Preprint 1156 (2006).

[20] A. Mielke and F. Theil, On rate-independent hysteresis models, *Nonl. Diff. Eqns. Appl.(NoDEA)* **11** (2004) 151–189.

[21] Mielke, A., Theil, F., Levitas, V., A variational formulation of rate-independent phase transformations using an extremum principle, *Arch. Rational Mech. Anal.* **162** (2002), 137–177.

[22] A. Mielke, A. M. Timofte, An energetic material model for time-dependent ferroelectric behavior: existence and uniqueness, *Math. Meth. Appl. Sciences*, **29** (2006), 1393–1410.

[23] A. Mielke and A. M. Timofte, Modeling and Analytical Study for Ferroelectric Materials, *Mechanics of Advanced Materials and Structures* **13** (2006), 457–462.

[24] A. Mielke and A. Timofte, Two-scale homogenization for evolutionary variational inequalities via the energetic formulation, submitted.

[25] R. McMeeking and C. Landis, A phenomenological multi-axial constitutive law for switching in polycrystalline ferroelectric ceramics, *Internat. J. Engrg. Sci* **40** (2002) 1553-1557.

[26] S. Nesenenko, Homogenization and regularity in viscoplasticity, PhD Thesis, Logos Verlag Berlin, (2006).

[27] Nguetseng, G., A general convergence result for a functional related to the theory of homogenization, *SIAM J. Math. Anal.* **20** (1989), 608–623.

[28] T. Ratiu, A. Timofte and V. Timofte, Existence, uniqueness and regularity of solutions for a thermomechanical model of shape memory alloys, *Math. Mech. Solids* (in press).

[29] H. Romanowski and J. Schröder, Coordinate invariant modelling of the ferroelectric hysteresis within a thermodynamically consistent framework. A mesoscopic approach. In *Trends in Applications of Mathematics to Mechanics*, Y. Wang and K. Hutter (eds), Shaker Verlag (2005), 419–428.

[30] R. Temam, Navier-Stokes equations, volume 2 of *Studies in Mathematics and its Applications*, North-Holland Publishing Co., Amsterdam, third edition, (1984).

[31] A. Timofte, V. Timofte, Uniqueness theorem for a thermomechanical model of shape memory alloys, *Math. Mech. Solids* **6** (2001), 447–466.

[32] A. Timofte, V. Timofte, Existence theorem for a thermomechanical model of shape memory alloys, *Math. Mech. Solids* **6** (2001), 541–545.

[33] A. Timofte, V. Timofte, Qualitative results concerning the behavior of shape memory alloys, *Rev. Roumaine Math. Pures Appl.* **47** (2002), 121–133.

[34] A. Visintin, Two-scale Stefan problem, *Nonlinear analysis and applications (Warsaw, 1994)* **7** (1996), 405–424.

[35] A. Visintin, Two-scale model of phase transitions, *Phys. D* **106** (1997), 66–80.