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Scattering Theory for Open Quantum Systems

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Dedicated to Pavel Exner on the ocassion of his 60^{th} birthday.

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Abstract

Quantum systems which interact with their environment are often modeled by maximal dissipative operators or so-called Pseudo-Hamiltonians. In this paper the scattering theory for such open systems is considered. First it is assumed that a single maximal dissipative operator A_D in a Hilbert space $\mathfrak{H}$ is used to describe an open quantum system. In this case the minimal self-adjoint dilation $\widetilde{K}$ of A_D can be regarded as the Hamiltonian of a closed system which contains the open system $\{A_D, \mathfrak{H}\}$, but since $\widetilde{K}$ is necessarily not semibounded from below, this model is difficult to interpret from a physical point of view. In the second part of the paper an open quantum system is modeled with a family $\{A(\mu)\}$ of maximal dissipative operators depending on energy μ, and it is shown that the open system can be embedded into a closed system where the Hamiltonian is semibounded. Surprisingly it turns out that the corresponding scattering matrix can be completely recovered from scattering matrices of single Pseudo-Hamiltonians as in the first part of the paper. The general results are applied to a class of Sturm-Liouville operators arising in dissipative and quantum transmitting Schrödinger-Poisson systems.

Contents

1 Introduction

Quantum systems which interact with their environment appear naturally in various physical problems and have been intensively studied in the last decades, see e.g. the monographes [18, 21, 35]. Such an open quantum system is often modeled with the help of a maximal dissipative operator, i.e., a closed linear operator A_D in some Hilbert space $\mathfrak{H}$ which satisfies

$$\Im\mathrm{m}\,(A_D f, f) \leq 0, \qquad f \in \mathrm{dom}\,(A_D),$$

and does not admit a proper extension in $\mathfrak{H}$ with this property. The dynamics in the open quantum system are described by the contraction semigroup e^{-itA_D}, $t \geq 0$. In the physical literature the maximal dissipative operator A_D is usually called a pseudo-Hamiltonian. It is well known that A_D admits a self-adjoint dilation $\widetilde{K}$ in a Hilbert space $\mathfrak{K}$ which contains $\mathfrak{H}$ as a closed subspace, that is, $\widetilde{K}$ is a self-adjoint operator in $\mathfrak{K}$ and

$$P_{\mathfrak{H}}\bigl(\widetilde{K} - \lambda\bigr)^{-1} \restriction_{\mathfrak{H}} = (A_D - \lambda)^{-1}$$

holds for all $\lambda \in \mathbb{C}_+ := \{z \in \mathbb{C} : \Im\mathrm{m}\,(z) > 0\}$, cf. [36]. Since the operator $\widetilde{K}$ is self-adjoint it can be regarded as the Hamiltonian or so-called quasi-Hamiltonian of a closed quantum system which contains the open quantum system $\{A_D, \mathfrak{H}\}$ as a subsystem.

In this paper we first assume that an open quantum system is described by a single pseudo-Hamiltonian A_D in $\mathfrak{H}$ and that A_D is an extension of a closed densely defined symmetric operator A in $\mathfrak{H}$ with finite equal deficiency indices. Then the self-adjoint dilation $\widetilde{K}$ can be realized as a self-adjoint extension of the symmetric operator $A \oplus G$ in $\mathfrak{K} = \mathfrak{H} \oplus L^2(\mathbb{R}, \mathcal{H}_D)$, where $\mathcal{H}_D$ is finite-dimensional and G is the symmetric operator in $L^2(\mathbb{R}, \mathcal{H}_D)$ given by

$$Gg := -i\frac{d}{dx}\,g, \quad \mathrm{dom}\,(G) = \bigl\{g \in W_2^1(\mathbb{R}, \mathcal{H}_D) : g(0) = 0\bigr\},$$

see Section 3.1. If A_0 is a self-adjoint extension of A in $\mathfrak{H}$ and G_0 denotes the usual self-adjoint momentum operator in $L^2(\mathbb{R}, \mathcal{H}_D)$,

$$G_0 g := -i\frac{d}{dx}\,g, \quad \mathrm{dom}\,(G) = W_2^1(\mathbb{R}, \mathcal{H}_D),$$

then the dilation $\widetilde{K}$ can be regarded as a singular perturbation (or more precisely a finite rank perturbation in resolvent sense) of the "unperturbed operator" $K_0 := A_0 \oplus G_0$, cf. [7, 42]. From a physical point of view K_0 describes a situation where both subsystems $\{A_0, \mathfrak{H}\}$ and $\{G_0, L^2(\mathbb{R}, \mathcal{H}_D)\}$ do not interact while $\widetilde{K}$ takes into account an interaction of the subsystems. Since the spectrum $\sigma(G_0)$ of the momentum operator is the whole real axis, standard perturbation results yield $\sigma(\widetilde{K}) = \sigma(K_0) = \mathbb{R}$ and, in particular, K_0 and $\widetilde{K}$ are necessarily not semibounded from below. For this reason K_0 and $\widetilde{K}$ are often called quasi-Hamiltonians rather than Hamiltonians.

The pair $\{\widetilde{K}, K_0\}$ is a complete scattering system in $\mathfrak{K} = \mathfrak{H} \oplus L^2(\mathbb{R}, \mathcal{H}_D)$, that is, the wave operators

$$W_\pm(\widetilde{K}, K_0) := \operatorname*{s-lim}_{t \to \pm\infty} e^{it\widetilde{K}} e^{-itK_0} P^{ac}(K_0)$$

exist and are complete, cf. [8, 13, 61, 62]. Here $P^{ac}(K_0)$ denotes the orthogonal projection in $\mathfrak{K}$ onto the absolutely continuous subspace $\mathfrak{K}^{ac}(K_0)$ of K_0. The scattering operator

$$S(\widetilde{K}, K_0) := W_+(\widetilde{K}, K_0)^* W_-(\widetilde{K}, K_0)$$

of the scattering system $\{\widetilde{K}, K_0\}$ regarded as an operator in $\mathfrak{K}^{ac}(K_0)$ is unitary, commutes with the absolutely continuous part K_0^{ac} of K_0 and is unitarily equivalent to a multiplication operator induced by a (matrix-valued) function $\{\widetilde{S}(\lambda)\}_{\lambda\in\mathbb{R}}$ in a spectral representation $L^2(\mathbb{R}, d\lambda, \mathcal{K}_\lambda)$ of $K_0^{ac} = A_0^{ac} \oplus G_0$, cf. [13]. The family $\{\widetilde{S}(\lambda)\}$ is called the scattering matrix of the scattering system $\{\widetilde{K}, K_0\}$ and is one of the most important quantities in the analysis of scattering processes.

In our setting the scattering matrix $\{\widetilde{S}(\lambda)\}$ decomposes into a 2×2 block matrix function in $L^2(\mathbb{R}, d\lambda, \mathcal{K}_\lambda)$ and it is one of our main goals in Section 3 to show that the left upper corner in this decomposition coincides with the scattering matrix $\{S_D(\lambda)\}$ of the dissipative scattering system $\{A_D, A_0\}$, cf. [55, 57, 58]. The right lower corner of $\{\widetilde{S}(\lambda)\}$ can be interpreted as the Lax-Phillips scattering matrix $\{S^{LP}(\lambda)\}$ corresponding to the Lax-Phillips scattering system $\{\widetilde{K}, \mathcal{D}_-, \mathcal{D}_+\}$. Here $\mathcal{D}_\pm := L^2(\mathbb{R}_\pm, \mathcal{H}_D)$ are so-called incoming and outgoing subspaces for the dilation $\widetilde{K}$, we refer to [13, 49] for details on Lax-Phillips scattering theory. The scattering matrices $\{\widetilde{S}(\lambda)\}$, $\{S_D(\lambda)\}$ and $\{S^{LP}(\lambda)\}$ are all explicitely expressed in terms of an "abstract" Titchmarsh-Weyl function $M(\cdot)$ and a dissipative matrix D which corresponds to the maximal dissipative operator A_D in $\mathfrak{H}$ and plays the role of an "abstract" boundary condition. With the help of this representation of $\{S^{LP}(\lambda)\}$ we easily recover the famous relation

$$S^{LP}(\lambda) = W_{A_D}(\lambda - i0)^*$$

found by Adamyan and Arov in [3, 4, 5, 6] between the Lax-Phillips scattering matrix and the characteristic function $W_{A_D}(\cdot)$ of the maximal dissipative operator A_D, cf. Corollary 3.11. We point out that $M(\cdot)$ and D are completely determined by the operators $A \subset A_0$ and A_D from the inner system. This is interesting also from the viewpoint of inverse problems, namely, the scattering matrix $\{\widetilde{S}(\lambda)\}$ of $\{\widetilde{K}, K_0\}$, in particular, the Lax-Phillips scattering matrix $\{S^{LP}(\lambda)\}$ can be recovered having to disposal only the dissipative scattering system $\{A_D, A_0\}$, see Theorem 3.6 and Remark 3.7.

We emphasize that this simple and somehow straightforward embedding method of an open quantum system into a closed quantum system by choosing a self-adjoint dilation $\widetilde{K}$ of the pseudo-Hamiltonian A_D is very convenient for mathematical scattering theory, but difficult to legitimate from a physical point of view, since the quasi-Hamiltonians $\widetilde{K}$ and K_0 are necessarily not semibounded from below.

In the second part of the paper we investigate open quantum systems which are described by an appropriate chosen family of maximal dissipative operators $\{A(\mu)\}$, $\mu \in \mathbb{C}_+$, instead of a single pseudo-Hamiltonian A_D. Similarly to the first part of the paper we assume that the maximal dissipative operators $A(\mu)$ are extensions of a fixed symmetric operator A in $\mathfrak{H}$ with equal finite deficiency indices. Under suitable (rather weak) assumptions on the family $\{A(\mu)\}$ there exists a symmetric operator T in a Hilbert space $\mathfrak{G}$ and a self-adjoint extension $\widetilde{L}$ of $L = A \oplus T$ in $\mathfrak{L} = \mathfrak{H} \oplus \mathfrak{G}$ such that

$$P_{\mathfrak{H}}\bigl(\widetilde{L} - \mu\bigr)^{-1} \upharpoonright_{\mathfrak{H}} = \bigl(A(\mu) - \mu\bigr)^{-1}, \qquad \mu \in \mathbb{C}_+, \tag{1.1}$$

holds, see Section 4.2. For example, in one-dimensional models for carrier transport in semiconductors the operators $A(\mu)$ are regular Sturm-Liouville differential operators in $L^2((a,b))$ with μ-dependent dissipative boundary conditions and the "linearization" $\widetilde{L}$ is a singular Sturm-Liouville operator in $L^2(\mathbb{R})$, cf. [10, 34, 37, 46] and Section 4.4. We remark that one can regard and interpret relation (1.1) also from an opposite point of view. Namely, if a self-adjoint operator $\widetilde{L}$ in a Hilbert space $\mathfrak{L}$ is given, then the compression of the resolvent of $\widetilde{L}$ onto any closed subspace $\mathfrak{H}$ of $\mathfrak{L}$ defines a family of maximal dissipative operators $\{A(\mu)\}$ via (1.1), so that each closed quantum system $\{\widetilde{L},\mathfrak{L}\}$ naturally contains open quantum subsystems $\{\{A(\mu)\},\mathfrak{H}\}$ of the type we investigate here. Nevertheless, since from a purely mathematical point of view both approaches are equivalent we will not explicitely discuss this second interpretation.

If A_0 and T_0 are self-adjoint extension of A and T in $\mathfrak{H}$ and $\mathfrak{G}$, respectively, then again $\widetilde{L}$ can be regarded as a singular perturbation of the self-adjoint operator $L_0 := A_0 \oplus T_0$ in $\mathfrak{L}$. As above L_0 describes a situation where the subsystems $\{A_0,\mathfrak{H}\}$ and $\{T_0,\mathfrak{G}\}$ do not interact while $\widetilde{L}$ takes into account a certain interaction. We note that if A and T have finite deficiency indices, then the operator $\widetilde{L}$ is semibounded from below if and only if A and T are semibounded from below. Well-known results imply that the pair $\{\widetilde{L},L_0\}$ is a complete scattering system in the closed quantum system and again the scattering matrix $\{\widetilde{S}(\lambda)\}$ decomposes into a 2×2 block matrix function which can be calculated in terms of abstract Titchmarsh-Weyl functions.

On the other hand it can be shown that the family $\{A(\mu)\}$, $\mu\in\mathbb{C}_+$, admits a continuation to $\mathbb{R}$, that is, the limit $A(\mu+i0)$ exists for a.e. $\mu\in\mathbb{R}$ in the strong resolvent sense and defines a maximal dissipative operator. The family $A(\mu+i0)$, $\mu\in\mathbb{R}$, can be regarded as a family of energy dependent pseudo-Hamiltonians in $\mathfrak{H}$ and, in particular, each pseudo-Hamiltonian $A(\mu+i0)$ gives rise to a quasi-Hamiltonian $\widetilde{K}_\mu$ in $\mathfrak{H}\oplus L^2(\mathbb{R},\mathcal{H}_\mu)$, a complete scattering system $\{\widetilde{K}_\mu, A_0\oplus -i\frac{d}{dx}\}$ and a corresponding scattering matrix $\{\widetilde{S}_\mu(\lambda)\}$ as illustrated in the first part of the introduction.

One of our main observations in Section 4 is that the scattering matrix $\{\widetilde{S}(\lambda)\}$ of the scattering system $\{\widetilde{L},L_0\}$ in $\mathfrak{H}\oplus\mathfrak{G}$ is related to the scattering matrices $\{\widetilde{S}_\mu(\lambda)\}$ of the systems $\{\widetilde{K}_\mu, A_0\oplus -i\frac{d}{dx}\}$, $\mu\in\mathbb{R}$, in $\mathfrak{H}\oplus L^2(\mathbb{R},\mathcal{H}_\mu)$ via

$$\widetilde{S}(\mu) = \widetilde{S}_\mu(\mu) \quad \text{for a.e. } \mu\in\mathbb{R}. \tag{1.2}$$

In other words, the scattering matrix $\{\widetilde{S}(\lambda)\}$ of the scattering system $\{\widetilde{L},L_0\}$ can be completely recovered from scattering matrices of scattering systems for single quasi-Hamiltonians. Furthermore, under certain continuity properties of the abstract Titchmarsh Weyl functions this implies $\widetilde{S}(\lambda)\approx\widetilde{S}_\mu(\lambda)$ for all λ in a sufficiently small neighborhood of the fixed energy $\mu\in\mathbb{R}$, which legitimizes the concept of single quasi-Hamiltonians for small energy ranges.

Similarly to the case of a single pseudo-Hamiltonian the diagonal entries of $\{\widetilde{S}(\mu)\}$ or $\{\widetilde{S}_\mu(\mu)\}$ can be interpreted as scattering matrices corresponding to energy dependent dissipative scattering systems and energy-dependent Lax-Phillips scattering systems. Moreover, if $\{S^{LP}_\mu(\lambda)\}$ is the scattering matrix of the Lax-Phillips scattering system $\{\widetilde{K}_\mu, L^2(\mathbb{R}_\pm,\mathcal{H}_\mu)\}$ and $W_{A(\mu)}(\cdot)$ denote the characteristic functions of the maximal dissipative operators $A(\mu)$ then an energy-dependent modification

$$S^{LP}_\mu(\mu) = W_{A(\mu)}(\mu - i0)^*$$

of the classical Adamyan-Arov result holds for a.e. $\mu \in \mathbb{R}$, cf. Section 4.3.

The paper is organized as follows. In Section 2 we give a brief introduction into extension and spectral theory of symmetric and self-adjoint operators with the help of boundary triplets and associated Weyl functions. These concepts will play an important role throughout the paper. Furthermore, we recall a recent result on the representation of the scattering matrix of a scattering system consisting of two self-adjoint extensions of a symmetric operator from [14]. Section 3 is devoted to open quantum systems described by a single pseudo-Hamiltonian A_D in $\mathfrak{H}$. In Theorem 3.2 a minimal self-adjoint dilation $\widetilde{K}$ in $\mathfrak{H} \oplus L^2(\mathbb{R}, \mathcal{H}_D)$ of the maximal dissipative operator A_D is explicitely constructed. Section 3.2 and Section 3.3 deal with the scattering matrix of $\{\widetilde{K}, K_0\}$ and the interpretation of the diagonal entries as scattering matrices of the dissipative scattering system $\{A_D, A_0\}$ and the Lax-Phillips scattering system $\{\widetilde{K}, L^2(\mathbb{R}_\pm, \mathcal{H}_D)\}$. In Section 3.4 we give an example of a pseudo-Hamiltonian which arises in the theory of dissipative Schrödinger-Poisson systems, cf. [11, 12, 43]. In Section 4 the family $\{A(\mu)\}$ of maximal dissipative operators in $\mathfrak{H}$ is introduced and, following ideas of [25], we construct a self-adjoint operator $\widetilde{L}$ in a Hilbert space $\mathfrak{L}$, $\mathfrak{H} \subset \mathfrak{L}$, such that (1.1) holds. After some preparatory work the relation (1.2) between the scattering matrices of $\{\widetilde{L}, L_0\}$ and the scattering systems consisting of quasi-Hamiltonians is verified in Section 4.3. Finally, in Section 4.4 we consider a so-called quantum transmitting Schrödinger-Poisson system as an example for an open quantum system which consists of a family of energy-dependent pseudo-Hamiltonians, cf. [10, 16, 19, 34, 37, 46].

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Notations. Throughout this paper $(\mathfrak{H}, (\cdot,\cdot))$ and $(\mathfrak{G}, (\cdot,\cdot))$ denote separable Hilbert spaces. The linear space of bounded linear operators defined on $\mathfrak{H}$ with values in $\mathfrak{G}$ will be denoted by $[\mathfrak{H}, \mathfrak{G}]$. If $\mathfrak{H} = \mathfrak{G}$ we simply write $[\mathfrak{H}]$. The set of closed operators in $\mathfrak{H}$ is denoted by $\mathcal{C}(\mathfrak{H})$. The resolvent set $\rho(S)$ of a closed linear operator $S \in \mathcal{C}(\mathfrak{H})$ is the set of all $\lambda \in \mathbb{C}$ such that $(S-\lambda)^{-1} \in [\mathfrak{H}]$, the spectrum $\sigma(S)$ of S is the complement of $\rho(S)$ in $\mathbb{C}$. $\sigma_p(S)$, $\sigma_c(S)$, $\sigma_{ac}(S)$ and $\sigma_r(S)$ stand for the point, continuous, absolutely continuous and residual spectrum of S, respectively. The domain, kernel and range of a linear operator are denoted by $\mathrm{dom}\,(\cdot)$, $\ker(\cdot)$ and $\mathrm{ran}\,(\cdot)$, respectively.

2 Self-adjoint extensions and scattering systems

In this section we briefly review the notion of abstract boundary triplets and associated Weyl functions in the extension theory of symmetric operators, see e.g. [27, 28, 30, 39]. For scattering systems consisting of a pair of self-adjoint extensions of a symmetric operator with finite deficiency indices we recall a result on the representation of the scattering matrix in terms of a Weyl function proved in [14].

2.1 Boundary triplets and closed extensions

Let A be a densely defined closed symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal deficiency indices $n_\pm(A) = \dim\ker(A^* \mp i) \le \infty$. We use the concept of boundary triplets for the description of the closed extensions $A_\Theta \subseteq A^*$ of A in $\mathfrak{H}$.

Definition 2.1 *A triplet* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is called a* boundary triplet *for the adjoint operator* A^* *if* $\mathcal{H}$ *is a Hilbert space and* $\Gamma_0, \Gamma_1 : \operatorname{dom}(A^*) \to \mathcal{H}$ *are linear mappings such that the "abstract Green identity"*

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g),$$

holds for all $f, g \in \operatorname{dom}(A^*)$ *and the map* $\Gamma := (\Gamma_0, \Gamma_1)^\top : \operatorname{dom}(A^*) \to \mathcal{H}\times\mathcal{H}$ *is surjective.*

We refer to [28] and [30] for a detailed study of boundary triplets and recall only some important facts. First of all a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* exists since the deficiency indices $n_\pm(A)$ of A are assumed to be equal. Then $n_\pm(A) = \dim\mathcal{H}$ and $A = A^* \upharpoonright \ker(\Gamma_0) \cap \ker(\Gamma_1)$ holds. We note that a boundary triplet for A^* is not unique.

In order to describe the closed extensions $A_\Theta \subseteq A^*$ of A with the help of a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* we have to consider the set $\widetilde{\mathcal{C}}(\mathcal{H})$ of closed linear relations in $\mathcal{H}$, that is, the set of closed linear subspaces of $\mathcal{H}\times\mathcal{H}$. We usually use a column vector notation for the elements in a linear relation Θ. A closed linear operator in $\mathcal{H}$ is identified with its graph, so that the set $\mathcal{C}(\mathcal{H})$ of closed linear operators in $\mathcal{H}$ is viewed as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$, in particular, a linear relation Θ is an operator if and only if the multivalued part $\operatorname{mul}\Theta = \left\{f' : \binom{0}{f'} \in \Theta\right\}$ is trivial. For the usual definitions of the linear operations with linear relations, the inverse, the resolvent set and the spectrum we refer to [32]. Recall that the adjoint relation $\Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ of a linear relation Θ in $\mathcal{H}$ is defined as

$$\Theta^* = \left\{ \begin{pmatrix} k \\ k' \end{pmatrix} : (h', k) = (h, k') \text{ for all } \begin{pmatrix} h \\ h' \end{pmatrix} \in \Theta \right\}$$

and Θ is said to be *symmetric* (*self-adjoint*) if $\Theta \subset \Theta^*$ (resp. $\Theta = \Theta^*$). Notice that this definition extends the definition of the adjoint operator. For a self-adjoint relation $\Theta = \Theta^*$ in $\mathcal{H}$ the multivalued part $\operatorname{mul}\Theta$ is the orthogonal complement of $\operatorname{dom}\Theta$ in $\mathcal{H}$. Setting $\mathcal{H}_{\rm op} := \overline{\operatorname{dom}\Theta}$ and $\mathcal{H}_\infty = \operatorname{mul}\Theta$ one verifies that Θ can be written as the direct orthogonal sum of a self-adjoint operator $\Theta_{\rm op}$ in the Hilbert space $\mathcal{H}_{\rm op}$ and the "pure" relation $\Theta_\infty = \left\{\binom{0}{f'} : f' \in \operatorname{mul}\Theta\right\}$ in the Hilbert space $\mathcal{H}_\infty$.

A linear relation Θ in $\mathcal{H}$ is called *dissipative* if $\Im\mathrm{m}\,(h', h) \le 0$ holds for all $(h, h')^\top \in \Theta$ and Θ is called *maximal dissipative* if it is dissipative and does not admit proper dissipative extensions in $\mathcal{H}$; then Θ is necessarily closed, $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$. We remark that a linear relation Θ is maximal dissipative if and only if Θ is dissipative and some $\lambda \in \mathbb{C}_+$ (and hence every $\lambda \in \mathbb{C}_+$) belongs to $\rho(\Theta)$.

A description of all closed (symmetric, self-adjoint, (maximal) dissipative) extensions of A is given in the next proposition.

Proposition 2.2 *Let A be a densely defined closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping*

$$\Theta \mapsto A_\Theta := A^* \upharpoonright \Gamma^{(-1)}\Theta = A^* \upharpoonright \left\{f \in \operatorname{dom}(A^*) : \ (\Gamma_0 f, \Gamma_1 f)^\top \in \Theta\right\} \tag{2.1}$$

establishes a bijective correspondence between the set $\widetilde{\mathcal{C}}(\mathcal{H})$ *and the set of closed extensions* $A_\Theta \subseteq A^*$ *of* A. *Furthermore*

$$(A_\Theta)^* = A_{\Theta^*}$$

holds for any $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$. *The extension* A_Θ *in* (2.1) *is symmetric (self-adjoint, dissipative, maximal dissipative) if and only if* Θ *is symmetric (self-adjoint, dissipative, maximal dissipative).*

It follows immediately from this proposition that if $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^*, then the extensions

$$A_0 := A^* \upharpoonright \ker(\Gamma_0) \quad \text{and} \quad A_1 := A^* \upharpoonright \ker(\Gamma_1)$$

are self-adjoint. In the sequel usually the extension A_0 corresponding to the boundary mapping Γ_0 is regarded as a "fixed" self-adjoint extension. We note that the closed extension A_Θ in (2.1) is disjoint with A_0, that is $\mathrm{dom}\,(A_\Theta) \cap \mathrm{dom}\,(A_0) = \mathrm{dom}\,(A)$, if and only if $\Theta \in \mathcal{C}(\mathcal{H})$. In this case (2.1) takes the form

$$A_\Theta = A^* \upharpoonright \ker\big(\Gamma_1 - \Theta\Gamma_0\big). \tag{2.2}$$

For simplicity we will often restrict ourselves to simple symmetric operators. Recall that a symmetric operator is said to be *simple* if there is no nontrivial subspace which reduces it to a self-adjoint operator. By [47] each symmetric operator A in $\mathfrak{H}$ can be written as the direct orthogonal sum $\widehat{A} \oplus A_s$ of a simple symmetric operator $\widehat{A}$ in the Hilbert space

$$\widehat{\mathfrak{H}} = \mathrm{clospan}\big\{\ker(A^* - \lambda) : \lambda \in \mathbb{C}\backslash\mathbb{R}\big\}$$

and a self-adjoint operator A_s in $\mathfrak{H} \ominus \widehat{\mathfrak{H}}$. Here $\mathrm{clospan}\{\cdot\}$ denotes the closed linear span. Obviously A is simple if and only if $\widehat{\mathfrak{H}}$ coincides with $\mathfrak{H}$. Notice that if $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for the adjoint A^* of a non-simple symmetric operator $A = \widehat{A} \oplus A_s$, then $\widehat{\Pi} = \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$, where

$$\widehat{\Gamma}_0 := \Gamma_0 \upharpoonright \mathrm{dom}\,\big((\widehat{A})^*\big) \quad \text{and} \quad \widehat{\Gamma}_1 := \Gamma_1 \upharpoonright \mathrm{dom}\,\big((\widehat{A})^*\big),$$

is a boundary triplet for the simple part $(\widehat{A})^* \in \mathcal{C}(\widehat{\mathfrak{H}})$ such that the extension $A_\Theta = \Gamma^{(-1)}\Theta$, $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$, in $\mathfrak{H}$ is given by $\widehat{A}_\Theta \oplus A_s$, $\widehat{A}_\Theta := \widehat{\Gamma}^{(-1)}\Theta \in \mathcal{C}(\widehat{\mathfrak{H}})$, and the Weyl functions and γ-fields of $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and $\widehat{\Pi} = \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ coincide.

We say that a maximal dissipative operator is *completely non-self-adjoint* if there is no nontrivial reducing subspace in which it is self-adjoint. Notice that each maximal dissipative operator decomposes orthogonally into a self-adjoint part and a completely non-self-adjoint part, see e.g. [36].

2.2 Weyl functions, γ-fields and resolvents of extensions

Let, as in Section 2.1, A be a densely defined closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices. If $\lambda \in \mathbb{C}$ is a point of regular type of A, i.e. $(A - \lambda)^{-1}$ is bounded, we denote the *defect subspace* of A by $\mathcal{N}_\lambda = \ker(A^* - \lambda)$. The following definition can be found in [27, 28, 30].

Definition 2.3 *Let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* A^*. *The operator valued functions* $\gamma(\cdot):\ \rho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ *and* $M(\cdot):\ \rho(A_0) \to [\mathcal{H}]$ *defined by*

$$\gamma(\lambda) := \bigl(\Gamma_0 \upharpoonright \mathcal{N}_\lambda\bigr)^{-1} \qquad \textit{and} \qquad M(\lambda) := \Gamma_1 \gamma(\lambda), \quad \lambda \in \rho(A_0), \tag{2.3}$$

are called the γ-field *and the* Weyl function, *respectively, corresponding to the boundary triplet* Π.

It follows from the identity $\mathrm{dom}\,(A^*) = \ker(\Gamma_0) \dot{+} \mathcal{N}_\lambda$, $\lambda \in \rho(A_0)$, where as above $A_0 = A^* \upharpoonright \ker(\Gamma_0)$, that the γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ in (2.3) are well defined. Moreover both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\rho(A_0)$ and the relations

$$\gamma(\lambda) = \bigl(I + (\lambda - \mu)(A_0 - \lambda)^{-1}\bigr)\gamma(\mu), \qquad \lambda, \mu \in \rho(A_0),$$

and

$$M(\lambda) - M(\mu)^* = (\lambda - \overline{\mu})\gamma(\mu)^*\gamma(\lambda), \qquad \lambda, \mu \in \rho(A_0), \tag{2.4}$$

are valid (see [28]). The identity (2.4) yields that $M(\cdot)$ is a *Nevanlinna function*, that is, $M(\cdot)$ is a ($[\mathcal{H}]$-valued) holomorphic function on $\mathbb{C}\backslash\mathbb{R}$ and

$$M(\lambda) = M(\overline{\lambda})^* \qquad \text{and} \qquad \frac{\Im\mathrm{m}\,(M(\lambda))}{\Im\mathrm{m}\,(\lambda)} \geq 0 \tag{2.5}$$

hold for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$. The union of $\mathbb{C}\backslash\mathbb{R}$ and the set of all points $\lambda \in \mathbb{R}$ such that M can be analytically continued to λ and the continuations from $\mathbb{C}_+$ and $\mathbb{C}_-$ coincide is denoted by $\mathfrak{h}(M)$. Besides (2.5) it follows also from (2.4) that the Weyl function $M(\cdot)$ satisfies $0 \in \rho(\Im\mathrm{m}\,(M(\lambda)))$ for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$; Nevanlinna functions with this additional property are sometimes called uniformly strict, cf. [26]. Conversely, each $[\mathcal{H}]$-valued Nevanlinna function τ with the additional property $0 \in \rho(\Im\mathrm{m}\,(\tau(\lambda)))$ for some (and hence for all) $\lambda \in \mathbb{C}\backslash\mathbb{R}$ can be realized as a Weyl function corresponding to some boundary triplet, we refer to [28, 48, 50] for further details.

Let again $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* with corresponding γ-field $\gamma(\cdot)$ and Weyl function $M(\cdot)$. The spectrum and the resolvent set of the closed (not necessarily self-adjoint) extensions of A can be described with the help of the function $M(\cdot)$. More precisely, if $A_\Theta \subseteq A^*$ is the extension corresponding to $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ via (2.1), then a point $\lambda \in \rho(A_0)$ belongs to $\rho(A_\Theta)$ ($\sigma_i(A_\Theta)$, $i = p, c, r$) if and only if $0 \in \rho(\Theta - M(\lambda))$ (resp. $0 \in \sigma_i(\Theta - M(\lambda))$, $i = p, c, r$). Moreover, for $\lambda \in \rho(A_0) \cap \rho(A_\Theta)$ the well-known resolvent formula

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\bigl(\Theta - M(\lambda)\bigr)^{-1}\gamma(\overline{\lambda})^* \tag{2.6}$$

holds, cf. [27, 28, 30]. Formula (2.6) is a generalization of the known Krein formula for canonical resolvents. We emphasize that it is valid for any closed extension $A_\Theta \subseteq A^*$ of A with a nonempty resolvent set.

2.3 Self-adjoint extensions and scattering

Let A be a densely defined closed symmetric operator in the separable Hilbert space $\mathfrak{H}$ and assume that the deficiency indices of A coincide and are finite, i.e., $n_+(A) = n_-(A) < \infty$. Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $A_0 := A^* \upharpoonright \ker(\Gamma_0)$, be a boundary triplet for A^* and let A_Θ be a self-adjoint extension of A which corresponds to a self-adjoint $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$. Since here $\dim \mathcal{H}$ is finite by (2.6)

$$(A_\Theta - \lambda)^{-1} - (A_0 - \lambda)^{-1}, \qquad \lambda \in \rho(A_\Theta) \cap \rho(A_0),$$

is a finite rank operator and therefore the pair $\{A_\Theta, A_0\}$ performs a so-called *complete scattering system*, that is, the *wave operators*

$$W_\pm(A_\Theta, A_0) := \text{s-}\lim_{t\to\pm\infty} e^{itA_\Theta} e^{-itA_0} P^{ac}(A_0),$$

exist and their ranges coincide with the absolutely continuous subspace $\mathfrak{H}^{ac}(A_\Theta)$ of A_Θ, cf. [13, 45, 61, 62]. $P^{ac}(A_0)$ denotes the orthogonal projection onto the absolutely continuous subspace $\mathfrak{H}^{ac}(A_0)$ of A_0. The *scattering operator* $S(A_\Theta, A_0)$ of the *scattering system* $\{A_\Theta, A_0\}$ is then defined by

$$S(A_\Theta, A_0) := W_+(A_\Theta, A_0)^* W_-(A_\Theta, A_0).$$

If we regard the scattering operator as an operator in $\mathfrak{H}^{ac}(A_0)$, then $S(A_\Theta, A_0)$ is unitary, commutes with the absolutely continuous part

$$A_0^{ac} := A_0 \restriction \operatorname{dom}(A_0) \cap \mathfrak{H}^{ac}(A_0)$$

of A_0 and it follows that $S(A_\Theta, A_0)$ is unitarily equivalent to a multiplication operator induced by a family $\{S_\Theta(\lambda)\}$ of unitary operators in a spectral representation of A_0^{ac}, see e.g. [13, Proposition 9.57]. This family is called the *scattering matrix* of the scattering system $\{A_\Theta, A_0\}$ and is one of the most important quantities in the analysis of scattering processes.

We note that if the symmetric operator A is not simple, then the Hilbert space $\mathfrak{H}$ can be decomposed as $\mathfrak{H} = \widehat{\mathfrak{H}} \oplus (\widehat{\mathfrak{H}})^\perp$ (cf. the end of Section 2.1) such that the scattering operator is given by the orthogonal sum $S(\widehat{A}_\Theta, \widehat{A}_0) \oplus I$, where $A_\Theta = \widehat{A}_\Theta \oplus A_s$ and $A_0 = \widehat{A}_0 \oplus A_s$, and hence it is sufficient to consider simple symmetric operators A in the following.

Since the deficiency indices of A are finite the Weyl function $M(\cdot)$ corresponding to the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a matrix-valued Nevanlinna function. By Fatous theorem (see [33, 38]) then the limit

$$M(\lambda + i0) := \lim_{\epsilon\to+0} M(\lambda + i\epsilon) \tag{2.7}$$

from the upper half-plane exists for a.e. $\lambda \in \mathbb{R}$. We denote the set of real points where the limit in (2.7) exits by Σ^M and we agree to use a similar notation for arbitrary scalar and matrix-valued Nevanlinna functions. Furthermore we will make use of the notation

$$\mathcal{H}_{M(\lambda)} := \operatorname{ran}\big(\Im\mathrm{m}\,(M(\lambda))\big), \qquad \lambda \in \Sigma^M, \tag{2.8}$$

and we will in general regard $\mathcal{H}_{M(\lambda)}$ as a subspace of $\mathcal{H}$. The orthogonal projection and restriction onto $\mathcal{H}_{M(\lambda)}$ will be denoted by $P_{M(\lambda)}$ and $\restriction_{\mathcal{H}_{M(\lambda)}}$, respectively. Notice that for $\lambda \in \rho(A_0) \cap \mathbb{R}$ the Hilbert space $\mathcal{H}_{M(\lambda)}$ is trivial by (2.4). Again we agree to use a notation analogous to (2.8) for arbitrary Nevanlinna functions. The family $\{P_{M(\lambda)}\}_{\lambda\in\Sigma^M}$ of orthogonal projections in $\mathcal{H}$ onto $\mathcal{H}_{M(\lambda)}$, $\lambda \in \Sigma^M$, is measurable and defines an orthogonal projection in the Hilbert space $L^2(\mathbb{R}, d\lambda, \mathcal{H})$; sometimes we write $L^2(\mathbb{R}, \mathcal{H})$ instead of $L^2(\mathbb{R}, d\lambda, \mathcal{H})$. The range of this projection is denoted by $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)})$.

Besides the Weyl function $M(\cdot)$ we will also make use of the function

$$\lambda \mapsto N_\Theta(\lambda) := \big(\Theta - M(\lambda)\big)^{-1}, \qquad \lambda \in \mathbb{C}\backslash\mathbb{R}, \tag{2.9}$$

where $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ is the self-adjoint relation corresponding to the extension A_Θ via (2.1). Since $\lambda \in \rho(A_0) \cap \rho(A_\Theta)$ if and only if $0 \in \rho(\Theta - M(\lambda))$ the function $N_\Theta(\cdot)$ is well defined. It is not difficult to see that $N_\Theta(\cdot)$ is an $[\mathcal{H}]$-valued Nevanlinna function and hence $N_\Theta(\lambda + i0) = \lim_{\epsilon\to 0} N_\Theta(\lambda + i\epsilon)$ exists for almost every $\lambda \in \mathbb{R}$, we denote this set by Σ^{N_Θ}. We claim that

$$N_\Theta(\lambda + i0) = \big(\Theta - M(\lambda + i0)\big)^{-1}, \qquad \lambda \in \Sigma^M \cap \Sigma^{N_\Theta}, \tag{2.10}$$

holds. In fact, if Θ is a self-adjoint matrix then (2.10) follows immediately from $N_\Theta(\lambda)(\Theta - M(\lambda)) = (\Theta - M(\lambda))N_\Theta(\lambda) = I_{\mathcal{H}}$, $\lambda \in \mathbb{C}_+$. If $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ has a nontrivial multivalued part we decompose Θ as $\Theta = \Theta_{\rm op} \oplus \Theta_\infty$, where $\Theta_{\rm op}$ is a self-adjoint matrix in $\mathcal{H}_{\rm op} = \operatorname{dom}\Theta_{\rm op}$ and Θ_∞ is a pure relation in $\mathcal{H}_\infty = \mathcal{H} \ominus \mathcal{H}_{\rm op}$, cf. Section 2.1, and denote the orthogonal projection and restriction in $\mathcal{H}$ onto $\mathcal{H}_{\rm op}$ by $P_{\rm op}$ and $\restriction_{\mathcal{H}_{\rm op}}$, respectively. Then we have

$$\lambda \mapsto N_\Theta(\lambda) = \big(\Theta_{\rm op} - P_{\rm op}M(\lambda)\restriction_{\mathcal{H}_{\rm op}}\big)^{-1}P_{\rm op}, \qquad \lambda \in \mathbb{C}\backslash\mathbb{R},$$

(see e.g. [48, page 137]) and from $N_\Theta(\lambda + i0) = (\Theta_{\rm op} - P_{\rm op}M(\lambda + i0)\restriction_{\mathcal{H}_{\rm op}})^{-1}P_{\rm op}$ for all $\lambda \in \Sigma^M \cap \Sigma^{N_\Theta}$ we conclude (2.10). Notice that the set $\mathbb{R}\backslash(\Sigma^M \cap \Sigma^{N_\Theta})$ has Lebesgue measure zero.

The following representation theorem of the scattering matrix $\{S_\Theta(\lambda)\}_{\lambda\in\mathbb{R}}$ of the scattering system $\{A_\Theta, A_0\}$ is essential in the following, cf. [14, Theorem 3.8]. Since the scattering matrix is only determined up to a set of Lebesgue measure zero we choose the representative of the equivalence class defined on $\Sigma^M \cap \Sigma^{N_\Theta}$.

Theorem 2.4 *Let A be a densely defined closed simple symmetric operator with finite deficiency indices in the separable Hilbert space $\mathfrak{H}$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* with corresponding Weyl function $M(\cdot)$ and define $\mathcal{H}_{M(\lambda)}$, $\lambda \in \Sigma^M$, as in (2.8). Furthermore, let $A_0 = A^* \restriction \ker(\Gamma_0)$ and let $A_\Theta = A^* \restriction \Gamma^{(-1)}\Theta$, $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$, be a self-adjoint extension of A. Then the following holds.*

(i) *A_0^{ac} is unitarily equivalent to the multiplication operator with the free variable in $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)})$.*

(ii) *In $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)})$ the scattering matrix $\{S_\Theta(\lambda)\}$ of the complete scattering system $\{A_\Theta, A_0\}$ is given by*

$$S_\Theta(\lambda) = I_{\mathcal{H}_{M(\lambda)}} + 2iP_{M(\lambda)}\sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(\Theta - M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))}\restriction_{\mathcal{H}_{M(\lambda)}}$$

for all $\lambda \in \Sigma^M \cap \Sigma^{N_\Theta}$, where $M(\lambda) := M(\lambda + i0)$.

In order to show the usefulness of Theorem 2.4 and to make the reader more familiar with the notion of boundary triplets and associated Weyl functions we calculate the scattering matrix of the scattering system $\{-\frac{d^2}{dx^2} + \delta, -\frac{d^2}{dx^2}\}$ in the following simple example.

Example 2.5 Let us consider the densely defined closed simple symmetric operator

$$(Af)(x) := -f''(x), \quad \mathrm{dom}\,(A) = \bigl\{f \in W_2^2(\mathbb{R}) : f(0) = 0\bigr\},$$

in $L^2(\mathbb{R})$, see e.g. [1]. Clearly A has deficiency indices $n_+(A) = n_-(A) = 1$ and it is well-known that the adjoint operator A^* is given by $(A^*f)(x) = -f''(x)$,

$$\mathrm{dom}\,(A^*) = \bigl\{f \in W_2^2(\mathbb{R}\backslash\{0\}) : f(0+) = f(0-),\, f'' \in L^2(\mathbb{R})\bigr\}.$$

It is not difficult to verify that $\Pi = \{\mathbb{C}, \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0 f := f'(0+) - f'(0-) \quad \text{and} \quad \Gamma_1 f := -f(0+), \quad f \in \mathrm{dom}\,(A^*),$$

is a boundary triplet for A^* and $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ coincides with the usual self-adjoint second order differential operator defined on $W_2^2(\mathbb{R})$. Moreover the defect space $\ker(A^* - \lambda)$, $\lambda \notin [0, \infty)$, is spanned by the function

$$x \mapsto e^{i\sqrt{\lambda}x}\chi_{\mathbb{R}_+}(x) + e^{-i\sqrt{\lambda}x}\chi_{\mathbb{R}_-}(x), \quad \lambda \notin [0, \infty),$$

where the square root is defined on $\mathbb{C}$ with a cut along $[0, \infty)$ and fixed by $\Im\mathrm{m}\,(\sqrt{\lambda}) > 0$ for $\lambda \notin [0, \infty)$ and by $\sqrt{\lambda} \geq 0$ for $\lambda \in [0, \infty)$. Therefore we find that the Weyl function $M(\cdot)$ corresponding to $\Pi = \{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is given by

$$M(\lambda) = \frac{\Gamma_1 f_\lambda}{\Gamma_0 f_\lambda} = \frac{i}{2\sqrt{\lambda}}, \qquad f_\lambda \in \ker(A^* - \lambda), \quad \lambda \notin [0, \infty).$$

Let $\alpha \in \mathbb{R}\backslash\{0\}$ and consider the self-adjoint extension $A_{-\alpha^{-1}}$ corresponding to the parameter $-\alpha^{-1}$, $A_{-\alpha^{-1}} = A^* \upharpoonright \ker(\Gamma_1 + \alpha^{-1}\Gamma_0)$, i.e.

$$\begin{aligned}&(A_{-\alpha^{-1}}f)(x) = -f''(x)\\ &\mathrm{dom}\,(A_{-\alpha^{-1}}) = \bigl\{f \in \mathrm{dom}\,(A^*) : \alpha f(0\pm) = f'(0+) - f'(0-)\bigr\}.\end{aligned}$$

This self-adjoint operator is often denoted by $-\frac{d^2}{dx^2} + \alpha\delta$, see [1]. It follows immediately from Theorem 2.4 that the scattering matrix $\{S(\lambda)\}$ of the scattering system $\{A_{-\alpha^{-1}}, A_0\}$ is given by

$$S(\lambda) = \frac{2\sqrt{\lambda} - i\alpha}{2\sqrt{\lambda} + i\alpha}, \qquad \lambda > 0.$$

We note that scattering systems of the form $\{-\frac{d^2}{dx^2} + \alpha\delta', -\frac{d^2}{dx^2}\}$, $\alpha \in \mathbb{R}$, can be investigated in a similar way as above. Other examples can be found in [14].

3 Dissipative and Lax-Phillips scattering systems

In this section we regard scattering systems $\{A_D, A_0\}$ consisting of a maximal dissipative and a self-adjoint extension of a symmetric operator A with finite deficiency indices. In the theory of open quantum system the maximal dissipative operator A_D is often called a pseudo-Hamiltonian. We shall explicitely construct a dilation (or so-called quasi-Hamiltonian) $\widetilde{K}$ of A_D and calculate the scattering matrix of the scattering system $\{\widetilde{K}, A_0 \oplus G_0\}$, where G_0 is a self-adjoint first order differential operator. The diagonal entries of the scattering matrix then turn out to be the scattering matrix of the dissipative scattering system $\{A_D, A_0\}$ and of a so-called Lax-Phillips scattering system, respectively.

We emphasize that this efficient and somehow straightforward method for the analysis of scattering processes for open quantum systems has the essential disadvantage that the quasi-Hamiltonians $\widetilde{K}$ and $A_0 \oplus G_0$ are necessarily not semibounded from below.

3.1 Self-adjoint dilations of maximal dissipative operators

Let in the following A be a densely defined closed simple symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal finite deficiency indices $n_\pm(A) = n < \infty$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $A_0 = A^* \restriction \ker(\Gamma_0)$, be a boundary triplet for A^* and let $D \in [\mathcal{H}]$ be a dissipative $n \times n$-matrix. Then the closed extension

$$A_D = A^* \restriction \ker(\Gamma_1 - D\Gamma_0)$$

of A corresponding to $\Theta = D$ via (2.1)-(2.2) is maximal dissipative and $\mathbb{C}_+$ belongs to $\rho(A_D)$. Notice that here we restrict ourselves to maximal dissipative extensions A_D corresponding to dissipative matrices D instead of maximal dissipative relations in the finite dimensional space $\mathcal{H}$. This is no essential restriction, see Remark 3.3 at the end of this subsection. For $\lambda \in \rho(A_D) \cap \rho(A_0)$ the resolvent of the extension A_D is given by

$$(A_D - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(D - M(\lambda)\big)^{-1}\gamma(\overline{\lambda})^*, \tag{3.1}$$

cf. (2.6). Write the dissipative matrix $D \in [\mathcal{H}]$ as

$$D = \Re\mathrm{e}\,(D) + i\Im\mathrm{m}\,(D),$$

decompose $\mathcal{H}$ as the direct orthogonal sum of the finite dimensional subspaces $\ker(\Im\mathrm{m}\,(D))$ and $\mathcal{H}_D := \mathrm{ran}\,(\Im\mathrm{m}\,(D))$,

$$\mathcal{H} = \ker(\Im\mathrm{m}\,(D)) \oplus \mathcal{H}_D, \tag{3.2}$$

and denote by P_D and $\restriction_{\mathcal{H}_D}$ the orthogonal projection and restriction in $\mathcal{H}$ onto $\mathcal{H}_D$. Since $\Im\mathrm{m}\,(D) \le 0$ the self-adjoint matrix $-P_D\Im\mathrm{m}\,(D)\restriction_{\mathcal{H}_D} \in [\mathcal{H}_D]$ is strictly positive and the next lemma shows how $-iP_D\Im\mathrm{m}\,(D)\restriction_{\mathcal{H}_D}$ (and $iP_D\Im\mathrm{m}\,(D)\restriction_{\mathcal{H}_D}$) can be realized as a Weyl function of a differential operator.

Lemma 3.1 *Let G be the symmetric first order differential operator in the Hilbert space $L^2(\mathbb{R}, \mathcal{H}_D)$ defined by*

$$(Gg)(x) = -ig'(x), \qquad \mathrm{dom}\,(G) = \big\{g \in W_2^1(\mathbb{R}, \mathcal{H}_D) \,:\, g(0) = 0\big\}.$$

Then G is simple, $n_\pm(G) = \dim \mathcal{H}_D$ and the adjoint operator $G^*g = -ig'$ is defined on $\mathrm{dom}\,(G^*) = W_2^1(\mathbb{R}_-, \mathcal{H}_D) \oplus W_2^1(\mathbb{R}_+, \mathcal{H}_D)$. Moreover, the triplet $\Pi_G = \{\mathcal{H}_D, \Upsilon_0, \Upsilon_1\}$, where

$$\begin{aligned}
\Upsilon_0 g &:= \frac{1}{\sqrt{2}}\big(-P_D\Im\mathrm{m}\,(D)\restriction_{\mathcal{H}_D}\big)^{-\frac{1}{2}}\big(g(0+) - g(0-)\big),\\
\Upsilon_1 g &:= \frac{i}{\sqrt{2}}\big(-P_D\Im\mathrm{m}\,(D)\restriction_{\mathcal{H}_D}\big)^{\frac{1}{2}}\big(g(0+) + g(0-)\big),
\end{aligned}$$

$g \in \mathrm{dom}\,(G^*)$, is a boundary triplet for G^* and $G_0 := G^* \upharpoonright \ker(\Upsilon_0)$ is the usual self-adjoint first order differential operator in $L^2(\mathbb{R},\mathcal{H}_D)$ with domain $\mathrm{dom}\,(G_0) = W_2^1(\mathbb{R},\mathcal{H}_D)$ and $\sigma(G_0) = \mathbb{R}$. The Weyl function $\tau(\cdot)$ corresponding to the boundary triplet $\Pi_G = \{\mathcal{H}_D,\Upsilon_0,\Upsilon_1\}$ is given by

$$\tau(\lambda) = \begin{cases} -iP_D \Im\mathrm{m}\,(D)\upharpoonright_{\mathcal{H}_D}, & \lambda \in \mathbb{C}_+, \\ iP_D \Im\mathrm{m}\,(D)\upharpoonright_{\mathcal{H}_D}, & \lambda \in \mathbb{C}_-. \end{cases} \tag{3.3}$$

Proof. Besides the assertion that $\Pi_G = \{\mathcal{H}_D,\Upsilon_0,\Upsilon_1\}$ is a boundary triplet for G^* with Weyl function $\tau(\cdot)$ given by (3.3) the statements of the lemma are well-known. We note only that the simplicity of G follows from [2, VIII.104] and the fact that G can be written as a finite direct orthogonal sum of first order differential operators on $\mathbb{R}_-$ and $\mathbb{R}_+$.

A straightforward calculation shows that the identity

$$\begin{aligned}(G^*g,k) - (g,G^*k) &= i(g(0+),k(0+)) - i(g(0-),k(0-)) \\ &= (\Upsilon_1 g,\Upsilon_0 k) - (\Upsilon_0 g,\Upsilon_1 k)\end{aligned}$$

holds for all $g,k \in \mathrm{dom}\,(G^*)$. Moreover the mapping $(\Gamma_0,\Gamma_1)^\top$ is surjective. Indeed, for an element $(h,h')^\top \in \mathcal{H}_D \times \mathcal{H}_D$ we choose $g \in \mathrm{dom}\, G^*$ such that

$$g(0+) = \frac{1}{\sqrt{2}}\Big\{\big(-P_D\Im\mathrm{m}\,(D)\upharpoonright_{\mathcal{H}_D}\big)^{\frac12} h - i\big(-P_D\Im\mathrm{m}\,(D)\upharpoonright_{\mathcal{H}_D}\big)^{-\frac12} h'\Big\}$$

and

$$g(0-) = \frac{1}{\sqrt{2}}\Big\{-\big(-P_D\Im\mathrm{m}\,(D)\upharpoonright_{\mathcal{H}_D}\big)^{\frac12} h - i\big(-P_D\Im\mathrm{m}\,(D)\upharpoonright_{\mathcal{H}_D}\big)^{-\frac12} h'\Big\}$$

holds. Then a simple calculation shows $\Upsilon_0 g = h$, $\Upsilon_1 g = h'$ and therefore $\Pi_G = \{\mathcal{H}_D,\Upsilon_0,\Upsilon_1\}$ is a boundary triplet for G^*. It is not difficult to check that the defect subspace $\mathcal{N}_\lambda = \ker(G^* - \lambda)$ is

$$\mathcal{N}_\lambda = \begin{cases} \mathrm{sp}\,\big\{x \mapsto e^{i\lambda x}\chi_{\mathbb{R}_+}(x)\xi \,:\, \xi \in \mathcal{H}_D\big\}, & \lambda \in \mathbb{C}_+, \\ \mathrm{sp}\,\big\{x \mapsto e^{i\lambda x}\chi_{\mathbb{R}_-}(x)\xi \,:\, \xi \in \mathcal{H}_D\big\}, & \lambda \in \mathbb{C}_-, \end{cases}$$

and hence we conclude that the Weyl function of $\Pi_G = \{\mathcal{H}_D,\Upsilon_0,\Upsilon_1\}$ is given by (3.3). □

Let A_D be the maximal dissipative extension of A in $\mathfrak{H}$ from above and let G be the first order differential operator from Lemma 3.1. Clearly $K := A \oplus G$ is a densely defined closed simple symmetric operator in the separable Hilbert space

$$\mathfrak{K} := \mathfrak{H} \oplus L^2(\mathbb{R},\mathcal{H}_D)$$

with equal finite deficiency indices $n_\pm(K) = n_\pm(A) + n_\pm(G) < \infty$ and the adjoint is $K^* = A^* \oplus G^*$. The elements in $\mathrm{dom}\,(K^*) = \mathrm{dom}\,(A^*) \oplus \mathrm{dom}\,(G^*)$ will be written in the form $f \oplus g$, $f \in \mathrm{dom}\,(A^*)$, $g \in \mathrm{dom}\,(G^*)$. In the next theorem we construct a self-adjoint extension $\widetilde{K}$ of K in $\mathfrak{K}$ which is a minimal self-adjoint dilation of the dissipative operator A_D in $\mathfrak{H}$. The construction is based on the idea of the coupling method from [25]. It is worth to mention that in the case of a (scalar) Sturm-Liouville operator with real potential and dissipative boundary condition our construction coincides with the one proposed by B.S. Pavlov [60], cf. Example 3.5 below.

Theorem 3.2 *Let A, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and A_D be as in the beginning of this section, let G and $\Pi_G = \{\mathcal{H}_D, \Upsilon_0, \Upsilon_1\}$ be as in Lemma 3.1 and $K = A \oplus G$. Then*

$$\widetilde{K} = K^* \upharpoonright \left\{ f \oplus g \in \mathrm{dom}\,(K^*) : \begin{array}{r} P_D\Gamma_0 f - \Upsilon_0 g = 0, \\ (1 - P_D)(\Gamma_1 - \Re\mathrm{e}\,(D)\Gamma_0) f = 0, \\ P_D(\Gamma_1 - \Re\mathrm{e}\,(D)\Gamma_0) f + \Upsilon_1 g = 0 \end{array} \right\} \tag{3.4}$$

is a minimal self-adjoint dilation of the maximal dissipative operator A_D, that is, for all $\lambda \in \mathbb{C}_+$

$$P_{\mathfrak{H}} \big(\widetilde{K} - \lambda\big)^{-1} \upharpoonright_{\mathfrak{H}} = (A_D - \lambda)^{-1}$$

holds and the minimality condition $\mathfrak{K} = \mathrm{clospan}\{(\widetilde{K} - \lambda)^{-1}\mathfrak{H} : \lambda \in \mathbb{C}\backslash\mathbb{R}\}$ is satisfied. Moreover $\sigma(\widetilde{K}) = \mathbb{R}$.

Proof. Let $\gamma(\cdot), \nu(\cdot)$ and $M(\cdot), \tau(\cdot)$ be the γ-fields and Weyl functions of the boundary triplets $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and $\Pi_G = \{\mathcal{H}_D, \Upsilon_0, \Upsilon_1\}$, respectively. Then it is straightforward to check that $\widetilde{\Pi} = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$, where

$$\widetilde{\mathcal{H}} := \mathcal{H} \oplus \mathcal{H}_D, \quad \widetilde{\Gamma}_0 := \begin{pmatrix} \Gamma_0 \\ \Upsilon_0 \end{pmatrix} \quad \text{and} \quad \widetilde{\Gamma}_1 := \begin{pmatrix} \Gamma_1 - \Re\mathrm{e}\,(D)\Gamma_0 \\ \Upsilon_1 \end{pmatrix}, \tag{3.5}$$

is a boundary triplet for $K^* = A^* \oplus G^*$ and the corresponding Weyl function $\widetilde{M}(\cdot)$ and γ-field $\widetilde{\gamma}(\cdot)$ are given by

$$\widetilde{M}(\lambda) = \begin{pmatrix} M(\lambda) - \Re\mathrm{e}\,(D) & 0 \\ 0 & \tau(\lambda) \end{pmatrix}, \qquad \lambda \in \mathbb{C}\backslash\mathbb{R}, \tag{3.6}$$

and

$$\widetilde{\gamma}(\lambda) = \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & \nu(\lambda) \end{pmatrix}, \qquad \lambda \in \mathbb{C}\backslash\mathbb{R}, \tag{3.7}$$

respectively. Notice also that $K_0 := K^* \upharpoonright \ker(\widetilde{\Gamma}_0) = A_0 \oplus G_0$ holds.

With respect to the decomposition $\widetilde{\mathcal{H}} = \ker(\Im\mathrm{m}\,(D)) \oplus \mathcal{H}_D \oplus \mathcal{H}_D$ of $\widetilde{\mathcal{H}}$ (cf. (3.2)) we define the linear relation $\widetilde{\Theta}$ by

$$\widetilde{\Theta} := \left\{ \begin{pmatrix} (u, v, v)^\top \\ (0, -w, w)^\top \end{pmatrix} : u \in \ker(\Im\mathrm{m}\,(D),\ v, w \in \mathcal{H}_D \right\} \in \widetilde{\mathcal{C}}(\widetilde{\mathcal{H}}). \tag{3.8}$$

We leave it to the reader to check that $\widetilde{\Theta}$ is self-adjoint. Hence by Proposition 2.2 the operator $K_{\widetilde{\Theta}} = K^* \upharpoonright \widetilde{\Gamma}^{(-1)}\widetilde{\Theta}$ is a self-adjoint extension of the symmetric operator $K = A \oplus G$ in $\mathfrak{K} = \mathfrak{H} \oplus L^2(\mathbb{R}, \mathcal{H}_D)$ and one verifies without difficulty that this extension coincides with $\widetilde{K}$ from (3.4), $\widetilde{K} = K_{\widetilde{\Theta}}$.

In order to calculate $(\widetilde{K} - \lambda)^{-1}$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, we use the block matrix decomposition

$$M(\lambda) - \Re\mathrm{e}\,(D) = \begin{pmatrix} M_{11}^D(\lambda) & M_{12}^D(\lambda) \\ M_{21}^D(\lambda) & M_{22}^D(\lambda) \end{pmatrix} \in \big[\ker(\Im\mathrm{m}\,(D)) \oplus \mathcal{H}_D\big] \tag{3.9}$$

of $M(\lambda) - \Re\mathrm{e}\,(D) \in [\mathcal{H}]$. Then the definition of $\widetilde{\Theta}$ in (3.8) and (3.6) imply

$$\big(\widetilde{\Theta} - \widetilde{M}(\lambda)\big)^{-1} = \left\{ \left(\begin{pmatrix} -M_{11}^D(\lambda)u - M_{12}^D(\lambda)v \\ -w - M_{21}^D(\lambda)u - M_{22}^D(\lambda)v \\ w - \tau(\lambda)v \end{pmatrix} \atop (u, v, v)^\top \right) : \begin{array}{c} u \in \ker(\Im\mathrm{m}\,(D) \\ v, w \in \mathcal{H}_D \end{array} \right\}$$

and since every $\lambda \in \mathbb{C}\backslash\mathbb{R}$ belongs to $\rho(\widetilde{K}) \cap \rho(K_0)$, $K_0 = A_0 \oplus G_0$, it follows that $(\widetilde{\Theta} - \widetilde{M}(\lambda))^{-1}$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, is the graph of a bounded everywhere defined operator. In order to calculate $(\widetilde{\Theta} - \widetilde{M}(\lambda))^{-1}$ in a more explicit form we set

$$\begin{array}{rcl} x & := & -M_{11}^D(\lambda)u - M_{12}^D(\lambda)v, \\ y & := & -w - M_{21}^D(\lambda)u - M_{22}^D(\lambda)v, \\ z & := & w - \tau(\lambda)v. \end{array} \tag{3.10}$$

This yields

$$\begin{pmatrix} x \\ y+z \end{pmatrix} = -\begin{pmatrix} M_{11}^D(\lambda) & M_{12}^D(\lambda) \\ M_{21}^D(\lambda) & M_{22}^D(\lambda) + \tau(\lambda) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

and by (3.3) and (3.9) we have

$$-\begin{pmatrix} M_{11}^D(\lambda) & M_{12}^D(\lambda) \\ M_{21}^D(\lambda) & M_{22}^D(\lambda) + \tau(\lambda) \end{pmatrix} = \begin{cases} D - M(\lambda), & \lambda \in \mathbb{C}_+, \\ D^* - M(\lambda), & \lambda \in \mathbb{C}_- \end{cases}. \tag{3.11}$$

Hence for $\lambda \in \mathbb{C}_+$ we find

$$\begin{pmatrix} u \\ v \end{pmatrix} = \big(D - M(\lambda)\big)^{-1} \begin{pmatrix} x \\ y+z \end{pmatrix},$$

which implies

$$\begin{pmatrix} u \\ v \end{pmatrix} = \big(D - M(\lambda)\big)^{-1} \begin{pmatrix} x \\ y \end{pmatrix} + \big(D - M(\lambda)\big)^{-1} \upharpoonright_{\mathcal{H}_D} z \tag{3.12}$$

and

$$v = P_D\big(D - M(\lambda)\big)^{-1} \begin{pmatrix} x \\ y \end{pmatrix} + P_D\big(D - M(\lambda)\big)^{-1} \upharpoonright_{\mathcal{H}_D} z. \tag{3.13}$$

Therefore by inserting (3.10), (3.12) and (3.13) into the above expression for $(\widetilde{\Theta} - \widetilde{M}(\lambda))^{-1}$ we obtain

$$\big(\widetilde{\Theta} - \widetilde{M}(\lambda)\big)^{-1} = \begin{pmatrix} (D - M(\lambda))^{-1} & (D - M(\lambda))^{-1} \upharpoonright_{\mathcal{H}_D} \\ P_D(D - M(\lambda))^{-1} & P_D(D - M(\lambda))^{-1} \upharpoonright_{\mathcal{H}_D} \end{pmatrix} \tag{3.14}$$

for all $\lambda \in \mathbb{C}_+$ and by (2.6) the resolvent of the self-adjoint extension $\widetilde{K}$ admits the representation

$$\big(\widetilde{K} - \lambda\big)^{-1} = (K_0 - \lambda)^{-1} + \widetilde{\gamma}(\lambda)\big(\widetilde{\Theta} - \widetilde{M}(\lambda)\big)^{-1}\widetilde{\gamma}(\overline{\lambda})^*, \tag{3.15}$$

$\lambda \in \mathbb{C}\backslash\mathbb{R}$. It follows from $K_0 = A_0 \oplus G_0$, (3.7) and (3.14) that for $\lambda \in \mathbb{C}_+$ the compressed resolvent of $\widetilde{K}$ onto $\mathfrak{H}$ is given by

$$P_{\mathfrak{H}}\big(\widetilde{K} - \lambda\big)^{-1} \upharpoonright \mathfrak{H} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(D - M(\lambda)\big)^{-1}\gamma(\overline{\lambda})^*,$$

where $P_{\mathfrak{H}}$ denotes the orthogonal projection in $\mathfrak{K}$ onto $\mathfrak{H}$. Taking into account (3.1) we get

$$P_{\mathfrak{H}}\big(\widetilde{K} - \lambda\big)^{-1} \upharpoonright \mathfrak{H} = (A_D - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_+,$$

and hence $\widetilde{K}$ is a self-adjoint dilation of A_D. Since $\sigma(G_0) = \mathbb{R}$ it follows from well-known perturbation results and (3.15) that $\sigma(\widetilde{K}) = \mathbb{R}$ holds.

It remains to show that $\widetilde{K}$ satisfies the minimality condition

$$\mathfrak{K} = \mathfrak{H} \oplus L^2(\mathbb{R}, \mathcal{H}_D) = \operatorname{clospan}\big\{(\widetilde{K} - \lambda)^{-1}\mathfrak{H} : \lambda \in \mathbb{C}\backslash\mathbb{R}\big\}. \tag{3.16}$$

First of all $\text{s-lim}_{t\to+\infty}(-it)(\widetilde{K} - it)^{-1} = I_{\mathfrak{K}}$ implies that $\mathfrak{H}$ is a subset of the right hand side of (3.16). The orthogonal projection in $\mathfrak{K}$ onto $L^2(\mathbb{R}, \mathcal{H}_D)$ is denoted by P_{L^2}. Then we conclude from (3.7), (3.14) and (3.15) that for $\lambda \in \mathbb{C}_+$

$$P_{L^2}\big(\widetilde{K} - \lambda\big)^{-1} \restriction_{\mathfrak{H}} = \nu(\lambda) P_D \big(D - M(\lambda)\big)^{-1} \gamma(\overline{\lambda})^* \tag{3.17}$$

holds and this gives

$$\operatorname{ran}\big(P_{L^2}\big(\widetilde{K} - \lambda\big)^{-1} \restriction_{\mathfrak{H}}\big) = \ker(G^* - \lambda), \quad \lambda \in \mathbb{C}_+.$$

From (3.11) it follows that similar to the matrix representation (3.14) the left lower corner of $(\widetilde{\Theta} - \widetilde{M}(\lambda))^{-1}$ is given by $P_D(D^* - M(\lambda))^{-1}$ for $\lambda \in \mathbb{C}_-$. Hence, the analogon of (3.17) for $\lambda \in \mathbb{C}_-$ implies that

$$\operatorname{ran}\big(P_{L^2}\big(\widetilde{K} - \lambda\big)^{-1} \restriction_{\mathfrak{H}}\big) = \ker(G^* - \lambda)$$

is true for $\lambda \in \mathbb{C}_-$. Since by Lemma 3.1 the symmetric operator G is simple it follows that

$$L^2(\mathbb{R}, \mathcal{H}_D) = \operatorname{clospan}\big\{\ker(G^* - \lambda) : \lambda \in \mathbb{C}\backslash\mathbb{R}\big\}$$

holds, cf. Section 2.1, and therefore the minimality condition (3.16) holds. □

Remark 3.3 We note that also in the case where the parameter D is not a dissipative matrix but a maximal dissipative relation in $\mathcal{H}$ a minimal self-adjoint dilation of A_D can be constructed in a similar way as in Theorem 3.2.

Indeed, let A and $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be as in the beginning of this section and let $\widetilde{D} \in \widetilde{\mathcal{C}}(\mathcal{H})$ be a maximal dissipative relation in $\mathcal{H}$. Then $\widetilde{D}$ can be written as the direct orthogonal sum of a dissipative matrix $\widetilde{D}_{\text{op}}$ in $\mathcal{H}_{\text{op}} := \mathcal{H} \ominus \operatorname{mul}\widetilde{D}$ and an undetermined part or "pure relation" $\widetilde{D}_\infty := \{\binom{0}{y} : \ y \in \operatorname{mul}\widetilde{D}\}$. It follows that

$$B := A^* \restriction \Gamma^{(-1)}\big\{\tbinom{0}{y} : y \in \operatorname{mul}\widetilde{D}\big\} = A^* \restriction \Gamma^{(-1)}\widetilde{D}_\infty$$

is a closed symmetric extension of A and $\{\mathcal{H}_{\text{op}}, \Gamma_0 \restriction_{\operatorname{dom}(B^*)}, P_{\text{op}}\Gamma_1 \restriction_{\operatorname{dom}(B^*)}\}$ is a boundary triplet for

$$B^* = A^* \restriction \big\{f \in \operatorname{dom}(A^*) : (1 - P_{\text{op}})\Gamma_0 f = 0\big\}$$

with $A^* \restriction \ker(\Gamma_0) = B^* \restriction \ker(\Gamma_0 \restriction_{\operatorname{dom}(B^*)})$. In terms of this boundary triplet the maximal dissipative extension $A_{\widetilde{D}} = \Gamma^{(-1)}\widetilde{D}$ coincides with the extension

$$B_{\widetilde{D}_{\text{op}}} = B^* \restriction \ker\big(P_{\text{op}}\Gamma_1 \restriction_{\operatorname{dom}(B^*)} - \widetilde{D}_{\text{op}}\Gamma_0 \restriction_{\operatorname{dom}(B^*)}\big)$$

corresponding to the operator part $\widetilde{D}_{\text{op}} \in [\mathcal{H}_{\text{op}}]$ of $\widetilde{D}$.

Remark 3.4 In the special case $\ker(\Im\mathrm{m}\, D) = \{0\}$ the relations (3.4) take the form

$$\Gamma_0 f - \Upsilon_0 g = 0 \quad \text{and} \quad (\Gamma_1 - \Re\mathrm{e}\,(D)\Gamma_0)f + \Upsilon_1 g = 0,$$

so that $\widetilde{K}$ is a coupling of the self-adjoint operators A_0 and G_0 corresponding to the coupling of the boundary triplets $\Pi_A = \{\mathcal{H}, \Gamma_0, \Gamma_1 - \Re\mathrm{e}\,(D)\Gamma_0\}$ and $\Pi_G = \{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$ in the sense of [25]. In the case $\ker(\Im\mathrm{m}\, D) \neq \{0\}$ another construction of $\widetilde{K}$ is based on the concept of boundary relations (see [26]).

A minimal self-adjoint dilation $\widetilde{K}$ for a scalar Sturm-Liouville operator with a complex (dissipative) boundary condition has originally been constructed by B.S. Pavlov in [60]. For the scalar case ($n = 1$) the operator in (3.20) in the following example coincides with the one in [60].

Example 3.5 Let $Q_+ \in L^1_{\mathrm{loc}}(\mathbb{R}_+, [\mathbb{C}^n])$ be a matrix valued function such that $Q_+(\cdot) = Q_+(\cdot)^*$, and let A be the usual minimal operator in $\mathfrak{H} = L^2(\mathbb{R}_+, \mathbb{C}^n)$ associated to the Sturm-Liouville differential expression $-\frac{d^2}{dx^2} + Q_+$,

$$A = -\frac{d^2}{dx^2} + Q_+, \quad \mathrm{dom}\,(A) = \bigl\{ f \in \mathcal{D}_{\max,+} : f(0) = f'(0) = 0 \bigr\},$$

where $\mathcal{D}_{\max,+}$ is the maximal domain defined by

$$\mathcal{D}_{\max,+} = \bigl\{ f \in L^2(\mathbb{R}_+, \mathbb{C}^n) : f, f' \in AC(\mathbb{R}_+, \mathbb{C}^n), -f'' + Q_+ f \in L^2(\mathbb{R}_+, \mathbb{C}^n) \bigr\}.$$

It is well known that the adjoint operator A^* is given by $A^* = -\frac{d^2}{dx^2} + Q_+$, $\mathrm{dom}\,(A^*) = \mathcal{D}_{\max,+}$.

In the following we assume that the limit point case prevails at $+\infty$, so that the deficiency indices $n_\pm(A)$ of A are both equal to n. In this case a boundary triplet $\Pi = \{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ for A^* is

$$\Gamma_0 f := f(0), \quad \Gamma_1 f := f'(0), \quad f \in \mathrm{dom}\,(A^*) = \mathcal{D}_{\max,+}. \tag{3.18}$$

For any dissipative matrix $D \in [\mathbb{C}^n]$ we consider the (maximal) dissipative extension A_D of A determined by

$$A_D = A^* \restriction \ker(\Gamma_1 - D\Gamma_0), \qquad \Im\mathrm{m}\, D \leq 0. \tag{3.19}$$

(a) First suppose $0 \in \rho(\Im\mathrm{m}\, D)$. Then $\mathcal{H}_D = \mathbb{C}^n$ and by Theorem 3.2 and Remark 3.4 the (minimal) self-adjoint dilation $\widetilde{K}$ of the operator A_D is a self-adjoint operator in $\mathfrak{K} = L^2(\mathbb{R}_+, \mathbb{C}^n) \oplus L^2(\mathbb{R}, \mathbb{C}^n)$ defined by

$$\begin{aligned} &\widetilde{K}(f \oplus g) = \bigl(-f'' + Q_+ f\bigr) \oplus -ig', \\ &\mathrm{dom}\,(\widetilde{K}) = \left\{ \begin{array}{c} f \in \mathcal{D}_{\max,+},\ g \in W^1_2(\mathbb{R}_-, \mathbb{C}^n) \oplus W^1_2(\mathbb{R}_+, \mathbb{C}^n) \\ f'(0) - Df(0) = -i(-2\Im\mathrm{m}\, D)^{1/2} g(0-), \\ f'(0) - D^* f(0) = -i(-2\Im\mathrm{m}\, D)^{1/2} g(0+) \end{array} \right\}. \end{aligned} \tag{3.20}$$

(b) Let now $\ker(\Im\mathrm{m}\, D) \neq \{0\}$, so that $\mathcal{H}_D = \mathrm{ran}\,(\Im\mathrm{m}\, D) = \mathbb{C}^k \neq \mathbb{C}^n$. According to Theorem 3.2 the (minimal) self-adjoint dilation $\widetilde{K}$ of the operator A_D in $\mathfrak{K} = L^2(\mathbb{R}_+, \mathbb{C}^n) \oplus$

$L^2(\mathbb{R},\mathbb{C}^k)$ is defined by

$$\widetilde{K}(f\oplus g) = \bigl(-f'' + Q_+ f\bigr)\oplus -ig',$$
$$\mathrm{dom}\,(\widetilde{K}) = \left\{\begin{matrix} f\in\mathcal{D}_{\max,+},\ g\in W_2^1(\mathbb{R}_-,\mathbb{C}^k)\oplus W_2^1(\mathbb{R}_+,\mathbb{C}^k) \\ P_D[f'(0) - Df(0)] = -i(-2P_D\Im\mathrm{m}\,(D)\restriction_{\mathcal{H}_D})^{1/2}g(0-), \\ P_D[f'(0) - D^* f(0)] = -i(-2P_D\Im\mathrm{m}\,(D)\restriction_{\mathcal{H}_D})^{1/2}g(0+), \\ f'(0) - \mathrm{Re}\,(D)f(0)\in\mathcal{H}_D \end{matrix}\right\}.$$

3.2 Dilations and dissipative scattering systems

Let, as in the previous section, A be a densely defined closed simple symmetric operator in $\mathfrak{H}$ with equal finite deficiency indices and let $\Pi = \{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^*, $A_0 = A^*\restriction\ker\Gamma_0$, with corresponding Weyl function $M(\cdot)$. Let $D\in[\mathcal{H}]$ be a dissipative matrix and let $A_D = A^*\restriction\ker(\Gamma_1 - D\Gamma_0)$ be the corresponding maximal dissipative extension in $\mathfrak{H}$. Since $\mathbb{C}_+\ni\lambda\mapsto M(\lambda) - D$ is a Nevanlinna function the limits

$$M(\lambda + i0) - D = \lim_{\epsilon\to+0} M(\lambda + i0) - D$$

and

$$N_D(\lambda + i0) = \lim_{\epsilon\to+0} N_D(\lambda + i\epsilon) = \lim_{\epsilon\to+0}\bigl(D - M(\lambda + i\epsilon)\bigr)^{-1}$$

exist for a.e. $\lambda\in\mathbb{R}$. We denote these sets of real points λ by Σ^M and Σ^{N_D}. Then we have

$$N_D(\lambda + i0) = \bigl(D - M(\lambda + i0)\bigr)^{-1},\qquad \lambda\in\Sigma^M\cap\Sigma^{N_D}, \tag{3.21}$$

cf. Section 2.3. Let G be the symmetric first order differential operator in $L^2(\mathbb{R},\mathcal{H}_D)$ and let $\Pi_G = \{\mathcal{H}_D,\Upsilon_0,\Upsilon_1\}$ be the boundary triplet from Lemma 3.1. Then $G_0 = G^*\restriction\ker(\Upsilon_0)$ is the usual self-adjoint differentiation operator in $L^2(\mathbb{R},\mathcal{H}_D)$ and $K_0 = A_0\oplus G_0$ is self-adjoint in $\mathfrak{K} = \mathfrak{H}\oplus L^2(\mathbb{R},\mathcal{H}_D)$. In the next theorem we consider the complete scattering system $\{\widetilde{K},K_0\}$, where $\widetilde{K}$ is the minimal self-adjoint dilation of A_D in $\mathfrak{K}$ from Theorem 3.2.

Theorem 3.6 *Let A, $\Pi = \{\mathcal{H},\Gamma_0,\Gamma_1\}$, $M(\cdot)$ and A_D be as above and define $\mathcal{H}_{M(\lambda)}$, $\lambda\in\Sigma^M$, as in* (2.8). *Let $K_0 = A_0\oplus G_0$ and let $\widetilde{K}$ be the minimal self-adjoint dilation of A_D from Theorem* 3.2. *Then the following holds.*

(i) $K_0^{ac} = A_0^{ac}\oplus G_0$ *is unitarily equivalent to the multiplication operator with the free variable in $L^2(\mathbb{R},d\lambda,\mathcal{H}_{M(\lambda)}\oplus\mathcal{H}_D)$.*

(ii) *In $L^2(\mathbb{R},d\lambda,\mathcal{H}_{M(\lambda)}\oplus\mathcal{H}_D)$ the scattering matrix $\{\widetilde{S}(\lambda)\}$ of the complete scattering system $\{\widetilde{K},K_0\}$ is given by*

$$\widetilde{S}(\lambda) = \begin{pmatrix} I_{\mathcal{H}_{M(\lambda)}} & 0 \\ 0 & I_{\mathcal{H}_D}\end{pmatrix} + 2i\begin{pmatrix}\widetilde{T}_{11}(\lambda) & \widetilde{T}_{12}(\lambda) \\ \widetilde{T}_{21}(\lambda) & \widetilde{T}_{22}(\lambda)\end{pmatrix}\in[\mathcal{H}_{M(\lambda)}\oplus\mathcal{H}_D],$$

for all $\lambda \in \Sigma^M \cap \Sigma^{N_D}$, *where*

$$\begin{aligned}
\widetilde{T}_{11}(\lambda) &= P_{M(\lambda)}\sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(D - M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))}\upharpoonright_{\mathcal{H}_{M(\lambda)}},\\
\widetilde{T}_{12}(\lambda) &= P_{M(\lambda)}\sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(D - M(\lambda)\big)^{-1}\sqrt{-\Im\mathrm{m}\,(D)}\upharpoonright_{\mathcal{H}_D},\\
\widetilde{T}_{21}(\lambda) &= P_D\sqrt{-\Im\mathrm{m}\,(D)}\big(D - M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))}\upharpoonright_{\mathcal{H}_{M(\lambda)}},\\
\widetilde{T}_{22}(\lambda) &= P_D\sqrt{-\Im\mathrm{m}\,(D)}\big(D - M(\lambda)\big)^{-1}\sqrt{-\Im\mathrm{m}\,(D)}\upharpoonright_{\mathcal{H}_D}
\end{aligned}$$

and $M(\lambda) = M(\lambda + i0)$.

Proof. Let $K = A \oplus G$ and let $\widetilde{\Pi} = \{\mathcal{H}\oplus\mathcal{H}_D, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ be the boundary triplet for K^* from (3.5). Notice that since A and G are densely defined closed simple symmetric operators also K is a densely defined closed simple symmetric operator. Recall that for $\lambda \in \mathbb{C}_+$ the Weyl function of $\widetilde{\Pi} = \{\mathcal{H}\oplus\mathcal{H}_D, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is given by

$$\widetilde{M}(\lambda) = \begin{pmatrix} M(\lambda) - \Re\mathrm{e}\,(D) & 0 \\ 0 & -iP_D\Im\mathrm{m}\,(D)\upharpoonright_{\mathcal{H}_D}\end{pmatrix}. \tag{3.22}$$

Then Theorem 2.4 implies that

$$L^2\big(\mathbb{R}, d\lambda, \mathcal{H}_{\widetilde{M}(\lambda)}\big), \quad \mathcal{H}_{\widetilde{M}(\lambda)} = \mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_D, \quad \lambda \in \Sigma^M,$$

performs a spectral representation of the absolutely continuous part

$$\begin{aligned}
K_0^{ac} &= K_0 \upharpoonright \mathrm{dom}\,(K_0) \cap \mathfrak{K}^{ac}(K_0)\\
&= A_0 \oplus G_0 \upharpoonright \big(\mathrm{dom}\,(A_0) \cap \mathfrak{H}^{ac}(A_0)\big) \oplus L^2(\mathbb{R}, \mathcal{H}_D) = A_0^{ac} \oplus G_0
\end{aligned}$$

of K_0 such that the scattering matrix $\{\widetilde{S}(\lambda)\}$ of the scattering system $\{\widetilde{K}, K_0\}$ is given by

$$\begin{aligned}
\widetilde{S}(\lambda) &= I_{\mathcal{H}_{\widetilde{M}(\lambda)}}\\
&\quad + 2iP_{\widetilde{M}(\lambda)}\sqrt{\Im\mathrm{m}\,(\widetilde{M}(\lambda))}\big(\widetilde{\Theta} - \widetilde{M}(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(\widetilde{M}(\lambda))}\upharpoonright_{\mathcal{H}_{\widetilde{M}(\lambda)}}
\end{aligned} \tag{3.23}$$

for all $\lambda \in \Sigma^{\widetilde{M}} \cap \Sigma^{N_{\widetilde{\Theta}}}$, where $P_{\widetilde{M}(\lambda)}$ and $\upharpoonright_{\mathcal{H}_{\widetilde{M}(\lambda)}}$ are the projection and restriction in $\widetilde{\mathcal{H}} = \mathcal{H}\oplus\mathcal{H}_D$ onto $\mathcal{H}_{\widetilde{M}(\lambda)}$. Here $\widetilde{\Theta}$ is the self-adjoint relation from (3.8) and the function $N_{\widetilde{\Theta}}$ is defined analogously to (2.9) and

$$N_{\widetilde{\Theta}}(\lambda + i0) = \big(\widetilde{\Theta} - \widetilde{M}(\lambda + i0)\big)^{-1}$$

holds for all $\lambda \in \Sigma^{\widetilde{M}} \cap \Sigma^{N_{\widetilde{\Theta}}}$, cf. (2.10).

By (3.22) we have

$$\sqrt{\Im\mathrm{m}\,(\widetilde{M}(\lambda + i0))} = \begin{pmatrix} \sqrt{\Im\mathrm{m}\,(M(\lambda + i0))} & 0 \\ 0 & P_D\sqrt{-\Im\mathrm{m}\,(D)}\upharpoonright_{\mathcal{H}_D}\end{pmatrix}$$

for all $\lambda \in \Sigma^{\widetilde{M}} = \Sigma^M$ and (3.14) yields

$$\big(\widetilde{\Theta} - \widetilde{M}(\lambda + i0)\big)^{-1} = \begin{pmatrix} (D - M(\lambda + i0))^{-1} & (D - M(\lambda + i0))^{-1}\upharpoonright_{\mathcal{H}_D} \\ P_D(D - M(\lambda + i0))^{-1} & P_D(D - M(\lambda + i0))^{-1}\upharpoonright_{\mathcal{H}_D}\end{pmatrix}$$

for $\lambda \in \Sigma^M \cap \Sigma^{N_{\widetilde{\Theta}}}$. It follows that the sets $\Sigma^M \cap \Sigma^{N_{\widetilde{\Theta}}}$ and $\Sigma^M \cap \Sigma^{N_D}$, see (3.21), coincide and by inserting the above expressions into (3.23) we conclude that for each $\lambda \in \Sigma^M \cap \Sigma^{N_D}$ the scattering matrix $\{\widetilde{S}(\lambda)\}$ is a two-by-two block operator matrix with respect to the decomposition

$$\mathcal{H}_{\widetilde{M}(\lambda)} = \mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_D, \qquad \lambda \in \Sigma^M \cap \Sigma^{N_D},$$

with the entries from assertion (ii). □

Remark 3.7 It is worth to note that the scattering matrix $\{\widetilde{S}(\lambda)\}$ of the scattering system $\{\widetilde{K}, K_0\}$ in Theorem 3.6 depends only on the dissipative matrix D and the Weyl function $M(\cdot)$ of the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^*. In other words, the scattering matrix $\{\widetilde{S}(\lambda)\}$ is completely determined by objects corresponding to the operators A, A_0 and A_D in $\mathfrak{H}$.

Let A_D and A_0 be as in the beginning of this section. In the following we will focus on the so-called *dissipative scattering system* $\{A_D, A_0\}$ and we refer the reader to [22, 23, 51, 52, 53, 54, 55, 56, 57] for a detailed investigation of such scattering systems. We recall only that the wave operators $W_\pm(A_D, A_0)$ of the dissipative scattering system $\{A_D, A_0\}$ are defined by

$$W_+(A_D, A_0) = \text{s-}\lim_{t\to+\infty} e^{itA_D^*} e^{-itA_0} P^{ac}(A_0)$$

and

$$W_-(A_D, A_0) = \text{s-}\lim_{t\to+\infty} e^{-itA_D} e^{itA_0} P^{ac}(A_0),$$

where $e^{-itA_D} := \text{s-lim}_{n\to\infty}(1 + \frac{it}{n} A_D)^{-n}$, see e.g. [45, §IX]. The scattering operator

$$S_D := W_+(A_D, A_0)^* W_-(A_D, A_0)$$

of the dissipative scattering system $\{A_D, A_0\}$ will be regarded as an operator in $\mathfrak{H}^{ac}(A_0)$. Then S_D is a contraction which in general is not unitary. Since S_D and A_0^{ac} commute it follows that S_D is unitarily equivalent to a multiplication operator induced by a family $\{S_D(\lambda)\}$ of contractive operators in a spectral representation of A_0^{ac}.

With the help of Theorem 3.6 we obtain a representation of the scattering matrix of the dissipative scattering system $\{A_D, A_0\}$ in terms of the Weyl function $M(\cdot)$ of $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ in the following corollary, cf. Theorem 2.4.

Corollary 3.8 *Let A, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $A_0 = A^* \restriction \ker(\Gamma_0)$, $M(\cdot)$ and A_D be as above and define $\mathcal{H}_{M(\lambda)}$, $\lambda \in \Sigma^M$, as in* (2.8). *Then the following holds.*

(i) *A_0^{ac} is unitarily equivalent to the multiplication operator with the free variable in $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)})$.*

(ii) *In $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)})$ the scattering matrix $\{S_D(\lambda)\}$ of the dissipative scattering system $\{A_D, A_0\}$ is given by*

$$S_D(\lambda) = I_{\mathcal{H}_{M(\lambda)}} + 2iP_{M(\lambda)}\sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(D - M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))}\restriction_{\mathcal{H}_{M(\lambda)}}$$

for all $\lambda \in \Sigma^M \cap \Sigma^{N_D}$, where $M(\lambda) = M(\lambda + i0)$.

Proof. Let $\widetilde{K}$ be the minimal self-adjoint dilation of A_D from Theorem 3.2. Since for $t \geq 0$ we have

$$P_{\mathfrak{H}} e^{-it\widetilde{K}} \upharpoonright \mathfrak{H} = \text{s-}\lim_{n\to\infty} P_{\mathfrak{H}} \big(1 + \tfrac{it}{n}\widetilde{K}\big)^{-n} \upharpoonright_{\mathfrak{H}} = \text{s-}\lim_{n\to\infty} \big(1 + \tfrac{it}{n} A_D\big)^{-n} = e^{-itA_D}$$

it follows that the wave operators $W_+(A_D, A_0)$ and $W_-(A_D, A_0)$ coincide with

$$\begin{aligned} P_{\mathfrak{H}} W_+(\widetilde{K}, K_0) \upharpoonright_{\mathfrak{H}} &= \text{s-}\lim_{t\to+\infty} P_{\mathfrak{H}} e^{it\widetilde{K}} e^{-itK_0} P^{ac}(K_0) \upharpoonright_{\mathfrak{H}} \\ &= \text{s-}\lim_{t\to+\infty} P_{\mathfrak{H}} e^{it\widetilde{K}} \upharpoonright_{\mathfrak{H}} e^{-itA_0} P^{ac}(A_0) \end{aligned}$$

and

$$\begin{aligned} P_{\mathfrak{H}} W_-(\widetilde{K}, K_0) \upharpoonright_{\mathfrak{H}} &= \text{s-}\lim_{t\to-\infty} P_{\mathfrak{H}} e^{it\widetilde{K}} e^{-itK_0} P^{ac}(K_0) \upharpoonright_{\mathfrak{H}} \\ &= \text{s-}\lim_{t\to+\infty} P_{\mathfrak{H}} e^{-it\widetilde{K}} \upharpoonright_{\mathfrak{H}} e^{itA_0} P^{ac}(A_0), \end{aligned}$$

respectively. This implies that the scattering operator S_D coincides with the compression $P_{\mathfrak{H}^{ac}(A_0)} S(\widetilde{K}, K_0) \upharpoonright_{\mathfrak{H}^{ac}(A_0)}$ of the scattering operator $S(\widetilde{K}, K_0)$ onto $\mathfrak{H}^{ac}(A_0)$. Therefore the scattering matrix $S_D(\lambda)$ of the dissipative scattering system is given by the upper left corner

$$\big\{I_{\mathcal{H}_{M(\lambda)}} + 2i\widetilde{T}_{11}(\lambda)\big\}, \quad \lambda \in \Sigma^M \cap \Sigma^{N_D},$$

of the scattering matrix $\{\widetilde{S}(\lambda)\}$ of the scattering system $\{\widetilde{K}, K_0\}$, see Theorem 3.6. $\square$

3.3 Lax-Phillips scattering systems

Let again A, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $\{A_D, A_0\}$ and G, G_0, $\Pi_G = \{\mathcal{H}_D, \Upsilon_0, \Upsilon_1\}$ be as in the previous subsections. In Corollary 3.8 we have shown that the scattering matrix of the dissipative scattering system $\{A_D, A_0\}$ is the left upper corner in the block operator matrix representation of the scattering matrix $\{\widetilde{S}(\lambda)\}$ of the scattering system $\{\widetilde{K}, K_0\}$, where $\widetilde{K}$ is a minimal self-adjoint dilation of A_D in $\mathfrak{K} = \mathfrak{H} \oplus L^2(\mathbb{R}, \mathcal{H}_D)$ and $K_0 = A_0 \oplus G_0$, cf. Theorem 3.6.

In the following we are going to interpret the right lower corner of $\{\widetilde{S}(\lambda)\}$ as the scattering matrix corresponding to a Lax-Phillips scattering system, see e.g. [13, 49] for further details. To this end we decompose the space $L^2(\mathbb{R}, \mathcal{H}_D)$ into the orthogonal sum of the subspaces

$$\mathcal{D}_- := L^2(\mathbb{R}_-, \mathcal{H}_D) \quad \text{and} \quad \mathcal{D}_+ := L^2(\mathbb{R}_+, \mathcal{H}_D). \tag{3.24}$$

Then clearly $\mathfrak{K} = \mathfrak{H} \oplus \mathcal{D}_- \oplus \mathcal{D}_+$ and we agree to denote the elements in $\mathfrak{K}$ in the form $f \oplus g_- \oplus g_+$, $f \in \mathfrak{H}$, $g_\pm \in \mathcal{D}_\pm$ and $g = g_- \oplus g_+ \in L^2(\mathbb{R}, \mathcal{H}_D)$. By J_+ and J_- we denote the operators

$$J_+ : L^2(\mathbb{R}, \mathcal{H}_D) \to \mathfrak{K}, \quad g \mapsto 0 \oplus 0 \oplus g_+,$$

and

$$J_- : L^2(\mathbb{R}, \mathcal{H}_D) \to \mathfrak{K}, \quad g \mapsto 0 \oplus g_- \oplus 0,$$

respectively. Notice that $J_+ + J_-$ is the embedding of $L^2(\mathbb{R}, \mathcal{H}_D)$ into $\mathfrak{K}$. In the next lemma we show that $\mathcal{D}_+$ and $\mathcal{D}_-$ are so-called *outgoing* and *incoming subspaces* for the self-adjoint dilation $\widetilde{K}$ in $\mathfrak{K}$.

Lemma 3.9 *Let $\widetilde{K}$ be the self-adjoint operator from Theorem 3.2, let $\mathcal{D}_\pm$ be as in (3.24) and $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ be as above. Then*

$$e^{-it\widetilde{K}} \subseteq \mathcal{D}_\pm, \quad t \in \mathbb{R}_\pm, \quad \text{and} \quad \bigcap_{t\in\mathbb{R}} e^{-it\widetilde{K}}\mathcal{D}_\pm = \{0\},$$

and, if in addition $\sigma(A_0)$ is singular, then

$$\overline{\bigcup_{t\in\mathbb{R}} e^{-it\widetilde{K}}\mathcal{D}_+} = \overline{\bigcup_{t\in\mathbb{R}} e^{-it\widetilde{K}}\mathcal{D}_-} = \mathfrak{K}^{ac}(\widetilde{K}). \tag{3.25}$$

Proof. Let us first show that

$$e^{-it\widetilde{K}} \upharpoonright \mathcal{D}_\pm = J_\pm e^{-itG_0} \upharpoonright \mathcal{D}_\pm, \qquad t \in \mathbb{R}_\pm, \tag{3.26}$$

holds. In fact, since e^{-itG_0} is the right shift group we have

$$e^{-itG_0}(\mathrm{dom}\,(G) \cap \mathcal{D}_\pm) \subseteq \mathrm{dom}\,(G) \cap \mathcal{D}_\pm, \quad t \in \mathbb{R}_\pm,$$

where $\mathrm{dom}\,(G) \cap \mathcal{D}_\pm = \{W^{1,2}(\mathbb{R}, \mathcal{H}_D) : f(x) = 0,\, x \in \mathbb{R}_\pm\}$. Let us fix some $t \in \mathbb{R}_\pm$ and denote the symmetric operator $A \oplus G$ by K. Since

$$J_\pm\big(\mathrm{dom}\,(G) \cap \mathcal{D}_\pm\big) \subset \mathrm{dom}\,(K) \subset \mathrm{dom}\,(\widetilde{K})$$

the function

$$f_{t,\pm}(s) := e^{i(s-t)\widetilde{K}} J_\pm e^{-isG_0} \upharpoonright_{\mathcal{D}_\pm} f_\pm, \quad s \in \mathbb{R}_\pm, \quad f_\pm \in \mathrm{dom}\,(G) \cap \mathcal{D}_\pm,$$

is differentiable and

$$\frac{d}{ds} f_{t,\pm}(s) = ie^{i(s-t)\widetilde{K}}\big(\widetilde{K} - 0_{\mathfrak{H}} \oplus G_0\big) J_\pm e^{-isG_0} \upharpoonright_{\mathcal{D}_\pm} f_\pm = 0, \quad t \in \mathbb{R}_\pm,$$

holds. Hence we have $f_{t,\pm}(0) = f_{t,\pm}(t)$ and together with the observation that the set $\mathrm{dom}\,(G) \cap \mathcal{D}_\pm$ is dense in $\mathcal{D}_\pm$ this immediately implies (3.26). Then we obtain $e^{-it\widetilde{K}}\mathcal{D}_\pm \subseteq \mathcal{D}_\pm$, $t \in \mathbb{R}_\pm$ and

$$\bigcap_{t\in\mathbb{R}} e^{-it\widetilde{K}}\mathcal{D}_\pm \subseteq \bigcap_{t\in\mathbb{R}_\pm} e^{-it\widetilde{K}}\mathcal{D}_\pm = \bigcap_{t\in\mathbb{R}_\pm} J_\pm e^{-itG_0}\mathcal{D}_\pm = \{0\}.$$

Let us show (3.25). Since A has finite deficiency indices the wave operators $W_\pm(\widetilde{K}, A_0 \oplus G_0)$ exist and are complete, i.e., $\mathrm{ran}\,(W_\pm(\widetilde{K}, A_0 \oplus G_0)) = \mathfrak{K}^{ac}(\widetilde{K})$ holds. Since A_0 is singular we have

$$W_\pm(\widetilde{K}, A_0 \oplus G_0) = \text{s-}\lim_{t\to\pm\infty} e^{it\widetilde{K}}(J_+ + J_-)e^{-itG_0} \upharpoonright_{L^2}$$

and it follows from (3.26) that $W_\pm(\widetilde{K}, A_0 \oplus G_0)f_\pm = f_\pm$ for $f_\pm \in \mathcal{D}_\pm$, so that in particular $\mathcal{D}_\pm$ and $e^{-itG_0}\mathcal{D}_\pm \in \mathfrak{K}^{ac}(\widetilde{K})$ for $t \in \mathbb{R}_\pm$. Assume now that $g \in L^2(\mathbb{R}, \mathcal{H}_D)$ vanishes identically on some open interval $(-\infty, \alpha)$. Then for $r > 0$ sufficiently large $e^{-irG_0}g \in \mathcal{D}_+$ and by (3.26) for $t > r$

$$e^{it\widetilde{K}}(J_+ + J_-)e^{-i(t-r)G_0}e^{-irG_0}g = e^{ir\widetilde{K}}J_+e^{-irG_0}g.$$

Since the elements $g \in L^2(\mathbb{R}, \mathcal{H}_D)$ which vanish on intervals $(-\infty, \alpha)$ form a dense set in $L^2(\mathbb{R}, \mathcal{H}_D)$ and the wave operator $W_+(\widetilde{K}, A_0 \oplus G_0)$ is complete we conclude that

$$\bigcup_{r \in \mathbb{R}_+} e^{ir\widetilde{K}} \mathcal{D}_+ \tag{3.27}$$

is a dense set in $\mathfrak{K}^{ac}(\widetilde{K})$. A similar argument shows that the set (3.27) with $\mathbb{R}_+$ and $\mathcal{D}_+$ replaced by $\mathbb{R}_-$ and $\mathcal{D}_-$, respectively, is also dense in $\mathfrak{K}^{ac}(\widetilde{K})$. This implies (3.25). □

According to Lemma 3.9 the system $\{\widetilde{K}, \mathcal{D}_-, \mathcal{D}_+\}$ is a Lax-Phillips scattering system and in particular the *Lax-Phillips wave operators*

$$\Omega_\pm := \text{s-}\lim_{t \to \pm\infty} e^{it\widetilde{K}} J_\pm e^{-itG_0} : L^2(\mathbb{R}, \mathcal{H}_D) \to \mathfrak{K}$$

exist, cf. [13]. We note that $\text{s-lim}_{t \to \pm\infty} J_\mp e^{-itG_0} = 0$ and therefore the restrictions of the wave operators $W_\pm(\widetilde{K}, K_0)$ of the scattering system $\{\widetilde{K}, K_0\}$, $K_0 = A_0 \oplus G_0$, onto $L^2(\mathbb{R}, \mathcal{H}_D)$,

$$W_\pm(\widetilde{K}, K_0) \restriction_{L^2} = \text{s-}\lim_{t \to \pm\infty} e^{it\widetilde{K}} (J_+ + J_-) e^{-itG_0},$$

coincide with the Lax-Phillips wave operators $\Omega_\pm$. Hence the *Lax-Phillips scattering operator* $S^{LP} := \Omega_+^* \Omega_-$ admits the representation

$$S^{LP} = P_{L^2} S(\widetilde{K}, K_0) \restriction_{L^2}$$

where $S(\widetilde{K}, K_0) = W_+(\widetilde{K}, K_0)^* W_-(\widetilde{K}, K_0)$ is the scattering operator of the scattering system $\{\widetilde{K}, K_0\}$. The Lax-Phillips scattering operator S^{LP} is a contraction in $L^2(\mathbb{R}, \mathcal{H}_D)$ and commutes with the self-adjoint differential operator G_0. Hence S^{LP} is unitarily equivalent to a multiplication operator induced by a family $\{S^{LP}(\lambda)\}$ of contractive operators in $L^2(\mathbb{R}, \mathcal{H}_D)$, this family is called the *Lax-Phillips scattering matrix.*

The above considerations together with Theorem 3.6 immediately imply the following corollary on the representation of the Lax-Phillips scattering matrix.

Corollary 3.10 *Let $\{\widetilde{K}, \mathcal{D}_-, \mathcal{D}_+\}$ be the Lax-Phillips scattering system considered in Lemma 3.9 and let A, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, A_D, $M(\cdot)$ and G_0 be as in the previous subsections. Then $G_0 = G_0^{ac}$ is unitarily equivalent to the multiplication operator with the free variable in $L^2(\mathbb{R}, \mathcal{H}_D) = L^2(\mathbb{R}, d\lambda, \mathcal{H}_D)$ and the Lax-Phillips scattering matrix $\{S^{LP}(\lambda)\}$ admits the representation*

$$S^{LP}(\lambda) = I_{\mathcal{H}_D} + 2iP_D\sqrt{\Im\mathrm{m}\,(-D)}\big(D - M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(-D)} \restriction_{\mathcal{H}_D} \tag{3.28}$$

for $\lambda \in \Sigma^M \cap \Sigma^{N_D}$, where $M(\lambda) = M(\lambda + i0)$.

Let again A_D be the maximal dissipative extension of A corresponding to the maximal dissipative matrix $D \in [\mathcal{H}]$ and let $\mathcal{H}_D = \mathrm{ran}\,(\Im\mathrm{m}\,(D))$. By [29] the characteristic function W_{A_D} of the completely non-self-adjoint part of A_D is given by

$$\begin{aligned} &W_{A_D} : \mathbb{C}_- \to [\mathcal{H}_D] \\ &\mu \mapsto I_{\mathcal{H}_D} - 2iP_D\sqrt{-\Im\mathrm{m}\,(D)}\big(D^* - M(\mu)\big)^{-1}\sqrt{-\Im\mathrm{m}\,(D)} \restriction_{\mathcal{H}_D} . \end{aligned} \tag{3.29}$$

Comparing (3.28) and (3.29) we obtain the famous relation between the Lax-Phillips scattering matrix and the characteristic function found by Adamyan and Arov in [3, 4, 5, 6].

Corollary 3.11 *Let the assumption be as in Corollary* 3.10*. Then the Lax-Phillips scattering matrix* $\{S^{LP}(\lambda)\}$ *and the characteristic function* W_{A_D} *of the maximal dissipative operator* A_D *are related by*

$$S^{LP}(\lambda) = W_{A_D}(\lambda - i0)^*, \qquad \lambda \in \Sigma^M \cap \Sigma^{N_D}.$$

Next we consider the special case that the spectrum $\sigma(A_0)$ of the self-adjoint extension $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ is purely singular, $\mathfrak{H}^{ac}(A_0) = \{0\}$. As usual let $M(\cdot)$ be the Weyl function corresponding to $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$. Then we have $\mathcal{H}_{M(\lambda)} = \operatorname{ran}(\Im m\,(M(\lambda + i0))) = \{0\}$ for a.e. $\lambda \in \Sigma^M$, cf. [17], and if even $\sigma(A_0) = \sigma_p(A_0)$ then $\mathcal{H}_{M(\lambda)} = \{0\}$ for all $\lambda \in \Sigma^M$. Therefore Theorem 3.6 and Corollaries 3.10 and 3.11 imply the following statement.

Corollary 3.12 *Let the assumption be as in Corollary* 3.10*, let* $K_0 = A_0 \oplus G_0$ *and assume in addition that* $\sigma(A_0)$ *is purely singular. Then the scattering matrix* $\{\widetilde{S}(\lambda)\}$ *of the complete scattering system* $\{\widetilde{K}, K_0\}$ *coincides with the Lax-Phillips scattering matrix* $\{S^{LP}(\lambda)\}$ *of the Lax-Phillips scattering system* $\{\widetilde{K}, \mathcal{D}_-, \mathcal{D}_+\}$*, that is,*

$$\widetilde{S}(\lambda) = S^{LP}(\lambda) = W_{A_D}(\lambda - i0)^* \tag{3.30}$$

for a.e. $\lambda \in \mathbb{R}$*. If even* $\sigma(A_0) = \sigma_p(A_0)$*, then* (3.30) *holds for all* $\lambda \in \Sigma^M \cap \Sigma^{N_D}$*.*

3.4 A dissipative Schrödinger-Poisson system

In this subsection we consider an open quantum system consisting of a self-adjoint and a maximal dissipative extension of a symmetric regular Sturm-Liouville differential operator. Such maximal dissipative operators or pseudo-Hamiltonians are used in the description of carrier transport in semi-conductors, see e.g. [9, 11, 34, 37, 43, 44, 46].

Assume that $-\infty < x_l < x_r < \infty$ and let $V \in L^\infty((x_l, x_r))$ be a real valued function. Moreover let $m \in L^\infty((x_l, x_r))$ be a real function such that $m > 0$ and $m^{-1} \in L^\infty((x_l, x_r))$. It is well-known that

$$\begin{aligned}
(Af)(x) &:= -\frac{1}{2}\frac{d}{dx}\frac{1}{m(x)}\frac{d}{dx}f(x) + V(x)f(x),\\
\operatorname{dom}(A) &:= \left\{ f \in L^2((x_l, x_r)) : \begin{array}{l} f, \frac{1}{m}f' \in W_2^1((x_l, x_r)) \\ f(x_l) = f(x_r) = 0 \\ \left(\frac{1}{m}f'\right)(x_l) = \left(\frac{1}{m}f'\right)(x_r) = 0 \end{array} \right\},
\end{aligned}$$

is a densely defined closed simple symmetric operator in the Hilbert space $\mathfrak{H} := L^2((x_l, x_r))$. The deficiency indices of A are $n_+(A) = n_-(A) = 2$ and the adjoint operator A^* is given by

$$\begin{aligned}
(A^*f)(x) &= -\frac{1}{2}\frac{d}{dx}\frac{1}{m(x)}\frac{d}{dx}f(x) + V(x)f(x),\\
\operatorname{dom}(A^*) &= \left\{ f \in \mathfrak{H} : f, \frac{1}{m}f' \in W_2^1((x_l, x_r)) \right\}.
\end{aligned}$$

It is straightforward to verify that $\Pi = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0 f := \begin{pmatrix} f(x_l) \\ f(x_r) \end{pmatrix} \quad \text{and} \quad \Gamma_1 f := \begin{pmatrix} \left(\frac{1}{2m} f'\right)(x_l) \\ -\left(\frac{1}{2m} f'\right)(x_r) \end{pmatrix}, \tag{3.31}$$

$f \in \mathrm{dom}\,(A^*)$, is a boundary triplet for A^*. Notice that the self-adjoint extension $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ corresponds to Dirichlet boundary conditions, that is,

$$\mathrm{dom}\,(A_0) = \left\{ f \in \mathfrak{H} : f, \frac{1}{m} f' \in W_2^1((x_l, x_r)), f(x_l) = f(x_r) = 0 \right\}.$$

It is well known that A_0 is semibounded from below and that $\sigma(A_0)$ consists of eigenvalues accumulating to $+\infty$. As usual we denote the Weyl function corresponding to $\Pi = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ by $M(\cdot)$. Here $M(\cdot)$ is a two-by-two matrix-valued function which has poles at the eigenvalues of A_0 and in particular we have

$$\mathcal{H}_{M(\lambda)} = \mathrm{ran}\left(\Im\mathrm{m}\,(M(\lambda))\right) = \{0\} \quad \text{for all} \quad \lambda \in \Sigma^M. \tag{3.32}$$

If $\varphi_\lambda, \psi_\lambda \in L^2((x_l, x_r))$ are fundamental solutions of $-\frac{1}{2}\left(\frac{1}{m} f'\right)' + Vf = \lambda f$ satisfying the boundary conditions

$$\varphi_\lambda(x_l) = 1, \quad \left(\tfrac{1}{m}\varphi_\lambda'\right)(x_l) = 0, \qquad \psi_\lambda(x_l) = 0, \quad \left(\tfrac{1}{m}\psi_\lambda'\right)(x_l) = 1, \tag{3.33}$$

then M can be written as

$$M(\lambda) = \frac{1}{2\psi_\lambda(x_r)} \begin{pmatrix} -\varphi_\lambda(x_r) & 1 \\ 1 & -\left(\frac{1}{m}\psi_\lambda'\right)(x_r) \end{pmatrix}, \qquad \lambda \in \rho(A_0). \tag{3.34}$$

We are interested in maximal dissipative extensions

$$A_D = A^* \upharpoonright \ker(\Gamma_1 - D\Gamma_0)$$

of A where $D \in [\mathbb{C}^2]$ has the special form

$$D = \begin{pmatrix} -\kappa_l & 0 \\ 0 & -\kappa_r \end{pmatrix}, \quad \Im\mathrm{m}\,(\kappa_l) \geq 0, \quad \Im\mathrm{m}\,(\kappa_r) \geq 0. \tag{3.35}$$

Of course, if both κ_l and κ_r are real constants then $\mathcal{H}_D = \mathrm{ran}\,(\Im\mathrm{m}\,(D)) = \{0\}$ and A_D is self-adjoint. In this case A_D can be identified with the self-adjoint dilation $\widetilde{K}$ acting in $\mathfrak{H} \oplus L^2(\mathbb{R}, \{0\}) \widetilde{=} \mathfrak{H}$, cf. Theorem 3.2.

Let us first consider the situation where both κ_l and κ_r have positive imaginary parts. Then $\mathcal{H}_D = \mathbb{C}^2$ and the self-adjoint dilation $\widetilde{K}$ from Theorem 3.2 is given by

$$\widetilde{K}(f \oplus g_- \oplus g_+) = \left(-\tfrac{1}{2}\left(\tfrac{1}{m} f'\right)' + Vf\right) \oplus -ig_-' \oplus -ig_+',$$
$$\mathrm{dom}\,\widetilde{K} = \left\{ \begin{matrix} f, \frac{1}{m} f' \in W_2^1((x_l, x_r)), \\ g_\pm \in W_2^1(\mathbb{R}_\pm, \mathbb{C}^2) \end{matrix} : \begin{matrix} \Gamma_0 f - \Upsilon_0 g = 0, \\ (\Gamma_1 - \Re\mathrm{e}\,(D)\Gamma_0) f + \Upsilon_1 g = 0 \end{matrix} \right\}.$$

Here $\Pi_G = \{\mathbb{C}^2, \Upsilon_0, \Upsilon_1\}$ is the boundary triplet for first order differential operator $G \subset G^*$ in $L^2(\mathbb{R}, \mathbb{C}^2)$ from Lemma 3.1 and we have decomposed the elements $f \oplus g$ in $\mathfrak{H} \oplus L^2(\mathbb{R}, \mathbb{C}^2)$ as agreed in the beginning of Section 3.3. Let us set

$$g_-(0-) = \begin{pmatrix} g_l(0-) \\ g_r(0-) \end{pmatrix} \quad \text{and} \quad g_+(0+) = \begin{pmatrix} g_l(0+) \\ g_r(0+) \end{pmatrix}.$$

Then a straightforward calculation using the definitions of $\Pi = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ and $\Pi_G = \{\mathbb{C}^2, \Upsilon_0, \Upsilon_1\}$ in (3.31) and Lemma 3.1, respectively, shows that an element $f \oplus g_- \oplus g_+$ belongs to $\mathrm{dom}\,(\widetilde{K})$ if and only if

$$\begin{aligned}
\big(\tfrac{1}{2m} f'\big)(x_l) + \kappa_l f(x_l) &= -i\sqrt{2\Im\mathrm{m}\,(\kappa_l)}\, g_l(0-)\\
\big(\tfrac{1}{2m} f'\big)(x_l) + \overline{\kappa}_l f(x_l) &= -i\sqrt{2\Im\mathrm{m}\,(\kappa_l)}\, g_l(0+)\\
\big(\tfrac{1}{2m} f'\big)(x_r) - \kappa_r f(x_r) &= i\sqrt{2\Im\mathrm{m}\,(\kappa_r)}\, g_r(0-)\\
\big(\tfrac{1}{2m} f'\big)(x_r) - \overline{\kappa}_r f(x_r) &= i\sqrt{2\Im\mathrm{m}\,(\kappa_r)}\, g_r(0+)
\end{aligned}$$

holds. We note that this dilation $\widetilde{K}$ is isomorph in the sense of [36, Section I.4] to those used in [11, 12, 43, 44].

Theorem 3.6 and the fact that $\sigma(A_0)$ is singular (cf. (3.32)) imply that the scattering matrix $\{\widetilde{S}(\lambda)\}$ of the scattering system $\{\widetilde{K}, K_0\}$, $K_0 = A_0 \oplus G_0$, coincides with

$$S^{LP}(\lambda) = I_{\mathbb{C}^2} + 2i\sqrt{-\Im\mathrm{m}\,(D)}\big(D - M(\lambda)\big)^{-1}\sqrt{-\Im\mathrm{m}\,(D)} \in [\mathbb{C}^2]$$

for all $\lambda \notin \sigma_p(A_0) \cap \mathbb{R}$, where $M(\lambda) = M(\lambda + i0)$ (cf. Corollary 3.12). By (3.35) here $\sqrt{-\Im\mathrm{m}\,(D)}$ is a diagonal matrix with entries $\sqrt{\Im\mathrm{m}\,(\kappa_l)}$ and $\sqrt{\Im\mathrm{m}\,(\kappa_r)}$. We leave it to the reader to compute $S^{LP}(\lambda)$ explicitely in terms of the fundamental solutions φ_λ and ψ_λ in (3.33). According to Corollary 3.11 the continuation of the characteristic function W_{A_D} of the completely non-self-adjoint pseudo-Hamiltonian A_D from $\mathbb{C}_-$ to $\mathbb{R}\backslash\{\sigma_p(A_0)\}$ coincides with $S^{LP}(\lambda)^*$,

$$W_{A_D}(\lambda - i0) = I_{\mathbb{C}^2} - 2i\sqrt{-\Im\mathrm{m}\,(D)}\big(D^* - M(\overline{\lambda})\big)^{-1}\sqrt{-\Im\mathrm{m}\,(D)} = S^{LP}(\lambda)^*.$$

Next we consider briefly the case where one of the entries of D in (3.35) is real. Assume e.g. $\kappa_l \in \mathbb{R}$. In this case $\mathcal{H}_D = \mathbb{C} \widehat{=} \{0\} \oplus \mathbb{C}$, P_D is the orthogonal projection onto the second component in $\mathbb{C}^2$ and G is a first order differential operator in $L^2(\mathbb{R}, \mathbb{C})$. The self-adjoint dilation $\widetilde{K}$ is

$$\widetilde{K}(f \oplus g_- \oplus g_+) = \big(-\tfrac{1}{2}\big(\tfrac{1}{m} f'\big)' + Vf\big) \oplus -ig_-' \oplus -ig_+',$$

$$\mathrm{dom}\,\widetilde{K} = \left\{ \begin{matrix} f, \frac{1}{m} f' \in W_2^1((x_l, x_r)),\\ g_\pm \in W_2^1(\mathbb{R}_\pm, \mathbb{C}^2) \end{matrix} : \begin{matrix} P_D\Gamma_0 f - \Upsilon_0 g = 0,\\ (1 - P_D)(\Gamma_1 - \Re\mathrm{e}\,(D)\Gamma_0) f = 0,\\ P_D(\Gamma_1 - \Re\mathrm{e}\,(D)\Gamma_0) f + \Upsilon_1 g = 0 \end{matrix} \right\},$$

and explicitely this means that an element $f \oplus g_- \oplus g_+$ belongs to $\mathrm{dom}\,(\widetilde{K})$ if and only if

$$\begin{aligned}
\big(\tfrac{1}{2m} f\big)'(x_r) - \overline{\kappa}_r f(x_r) &= i\sqrt{2\Im\mathrm{m}\,(\kappa_r)}\, g_+(0+)\\
\big(\tfrac{1}{2m} f\big)'(x_r) - \kappa_r f(x_r) &= i\sqrt{2\Im\mathrm{m}\,(\kappa_r)}\, g_-(0-)\\
\big(\tfrac{1}{2m} f\big)'(x_l) + \kappa_l f(x_l) &= 0
\end{aligned}$$

holds. The scattering matrix of $\{\widetilde{K}, K_0\}$ is given by

$$S^{LP}(\lambda) = I_{\mathcal{H}_D} + 2i\Im\mathrm{m}\,(\kappa_r) P_D\big(D - M(\lambda)\big)^{-1} \upharpoonright_{\mathcal{H}_D}, \quad \lambda \in \Sigma^M,$$

which is now a scalar function, and is related to the characteristic function of the maximal dissipative operator A_D by $S^{LP}(\lambda) = W_{A_D}(\lambda - i0)^*$.

4 Energy dependent scattering systems

In this section we consider families $\{A_{-\tau(\lambda)}, A_0\}$ of scattering systems, where $\tau(\cdot)$ is a matrix Nevanlinna function and $\{A_{-\tau(\lambda)}\}$ is a family of maximal dissipative extensions of a symmetric operator A with finite deficiency indices. Such scattering systems arise naturally in the description of open quantum systems, see e.g. Section 4.4 where a simple model of a so-called quantum transmitting Schrödinger-Poisson system is described. Following ideas in [25] (see also [15, 20, 31, 40, 41]) the family $\{A_{-\tau(\lambda)}\}$ is "linearized" in an abstract way, that is, we construct a self-adjoint extension $\widetilde{L}$ of A which acts in a larger Hilbert space $\mathfrak{H} \oplus \mathfrak{G}$ and satisfies

$$P_{\mathfrak{H}}\big(\widetilde{L} - \lambda\big)^{-1} \restriction_{\mathfrak{H}} = \big(A_{-\tau(\lambda)} - \lambda\big)^{-1},$$

so that, roughly speaking, the open quantum system is embedded into a closed system. The corresponding Hamiltonian $\widetilde{L}$ is semibounded if and only if A_0 is semibounded and $\tau(\cdot)$ is holomorphic on some interval $(-\infty, \eta)$. The essential observation here is that the scattering matrix of $\{\widetilde{L}, L_0\}$, where L_0 is the direct orthogonal sum of A_0 and a self-adjoint operator connected with $\tau(\cdot)$, pointwise coincides with the scattering matrix of a scattering system $\{\widetilde{K}, K_0\}$ as investigated in the previous section. From a physical point of view this in particular justifies the use of quasi-Hamiltonians $\widetilde{K}$ for the analysis of scattering processes in suitable small energy ranges.

4.1 The Štraus family and its characteristic functions

Let A be a densely defined closed simple symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal finite deficiency indices $n_\pm(A) = n < \infty$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Assume that $\tau(\cdot)$ is an $[\mathcal{H}]$-valued Nevanlinna function and consider the family $\{A_{-\tau(\lambda)}\}$,

$$A_{-\tau(\lambda)} := A^* \restriction \ker\big(\Gamma_1 + \tau(\lambda)\Gamma_0\big), \qquad \lambda \in \mathbb{C}_+,$$

of closed extension of A. Sometimes it is convenient to consider $A_{-\tau(\lambda)}$ for all $\lambda \in \mathfrak{h}(\tau)$, that is, for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and all real points λ where τ is holomorphic, cf. Section 2.2. Since $\Im\mathrm{m}\,\tau(\lambda) \geq 0$ for $\lambda \in \mathbb{C}_+$ it follows that each $A_{-\tau(\lambda)}$, $\lambda \in \mathbb{C}_+$, is a maximal dissipative extension of A in $\mathfrak{H}$. The family $\{A_{-\tau(\lambda)}\}_{\lambda\in\mathbb{C}_+}$ is called the *Štraus family of A associated with τ* (cf. [59] and e.g. [24, Section 3.3]) and for brevity we shall often call $\{A_{-\tau(\lambda)}\}$ simply *Štraus family*.

Since $\mathcal{H}$ is finite dimensional Fatous theorem (see [33, 38]) implies that the limit $\tau(\lambda+i0) = \lim_{\epsilon\to+0}\tau(\lambda + i\epsilon)$ from the upper half-plane exists for a.e. $\lambda \in \mathbb{R}$. As in Section 2.3 we denote set of real points λ where this limit exists by Σ^τ. If there is no danger of confusion we will usually write $\tau(\lambda)$ instead of $\tau(\lambda+i0)$ for $\lambda \in \Sigma^\tau$. Obviously, the Lebesgue measure of $\mathbb{R} \setminus \Sigma^\tau$ is zero. Hence the Štraus family $\{A_{-\tau(\lambda)}\}_{\lambda\in\mathbb{C}_+}$ admits a continuation to $\mathbb{C}_+ \cup \Sigma^\tau$ which is also denoted by $\{A_{-\tau(\lambda)}\}$, $\lambda \in \mathbb{C}_+ \cup \Sigma^\tau$. We remark that in the case $\Im\mathrm{m}\,(\tau(\lambda)) = 0$ for some $\lambda \in \mathbb{C}_+ \cup \Sigma^\tau$ the maximal dissipative operator $A_{-\tau(\lambda)}$ is self-adjoint.

Let $M(\cdot)$ be the Weyl function of the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$. Then $M(\cdot)$ is an $[\mathcal{H}]$-valued Nevanlinna function and $\Im\mathrm{m}\,(M(\lambda))$ is strictly positive for $\lambda \in \mathbb{C}_+$. Therefore

$$N_{-\tau(\lambda)}(\lambda) := -\big(\tau(\lambda) + M(\lambda)\big)^{-1}, \quad \lambda \in \mathbb{C}_+,$$

is a well-defined Nevanlinna function, see also (2.9). The set of all real λ where the limit

$$N_{-\tau(\lambda+i0)}(\lambda+i0) = \lim_{\epsilon\to+0} -\big(\tau(\lambda+i\epsilon)+M(\lambda+i\epsilon)\big)^{-1}$$

exists will for brevity be denoted by Σ^N. Furthermore, for fixed $\lambda \in \Sigma^\tau$ we define an $[\mathcal{H}]$-valued Nevanlinna function $Q_{-\tau(\lambda)}(\cdot)$ by

$$Q_{-\tau(\lambda)}(\mu) := -\big(\tau(\lambda)+M(\mu)\big)^{-1}, \quad \mu \in \mathbb{C}_+, \tag{4.1}$$

and denote by Σ^{Q_λ} the set of all real points μ where the limit

$$Q_{-\tau(\lambda)}(\mu+i0) = \lim_{\epsilon\to+0} Q_{-\tau(\lambda)}(\mu+i\epsilon) \tag{4.2}$$

exists. Notice that the complements $\mathbb{R}\setminus\Sigma^N$ and $\mathbb{R}\setminus\Sigma^{Q_\lambda}$ are of Lebesgue measure zero. The next lemma will be used in Section 4.3.

Lemma 4.1 *Let A, $\Pi = \{\mathcal{H},\Gamma_0,\Gamma_1\}$, $M(\cdot)$ and $\tau(\cdot)$ be as above. Then the following assertions* (i)-(iii) *are true.*

(i) *If $\lambda\in\Sigma^\tau$ and $\mu\in\Sigma^M\cap\Sigma^{Q_\lambda}$, then the operator $\tau(\lambda)+M(\mu)$ is invertible and*

$$\big(\tau(\lambda)+M(\mu)\big)^{-1} = \lim_{\epsilon\to+0}\big(\tau(\lambda)+M(\mu+i\epsilon)\big)^{-1}. \tag{4.3}$$

(ii) *If $\lambda\in\Sigma^\tau\cap\Sigma^M\cap\Sigma^N$, then the operator $\tau(\lambda)+M(\lambda)$ is invertible and*

$$\big(\tau(\lambda)+M(\lambda)\big)^{-1} = \lim_{\epsilon\to+0}\big(\tau(\lambda+i\epsilon)+M(\lambda+i\epsilon)\big)^{-1}. \tag{4.4}$$

(iii) *If $\lambda\in\Sigma^\tau\cap\Sigma^M\cap\Sigma^N$, then $\lambda\in\Sigma^{Q_\lambda}$ and*

$$\big(\tau(\lambda)+M(\lambda)\big)^{-1} = \lim_{\epsilon\to+0}\big(\tau(\lambda)+M(\lambda+i\epsilon)\big)^{-1}. \tag{4.5}$$

Proof. (i) If $\lambda\in\Sigma^\tau$, $\mu\in\Sigma^M$, then $\lim_{\epsilon\to+0}(\tau(\lambda)+M(\mu+i\epsilon)) = \tau(\lambda)+M(\mu)$. Since

$$\big(\tau(\lambda)+M(\mu+i\epsilon)\big)Q_{-\tau(\lambda)}(\mu+i\epsilon) = Q_{-\tau(\lambda)}(\mu+i\epsilon)\big(\tau(\lambda)+M(\mu+i\epsilon)\big) = -I_{\mathcal{H}}$$

for all $\epsilon>0$, we get

$$-I_{\mathcal{H}} = \big(\tau(\lambda)+M(\mu)\big)Q_{-\tau(\lambda)}(\mu) = Q_{-\tau(\lambda)}(\mu)\big(\tau(\lambda)+M(\mu)\big)$$

for $\lambda\in\Sigma^\tau$ and $\mu\in\Sigma^M\cap\Sigma^{Q_\lambda}$ which proves (4.3).

(ii) For $\lambda\in\Sigma^\tau\cap\Sigma^M$ clearly

$$\lim_{\epsilon\to+0}\big(\tau(\lambda+i\epsilon)+M(\lambda+i\epsilon)\big) = \tau(\lambda)+M(\lambda)$$

exists. Since $(\tau(\lambda)+M(\lambda))N_{-\tau(\lambda)}(\lambda) = N_{-\tau(\lambda)}(\lambda)(\tau(\lambda)+M(\lambda)) = -I_{\mathcal{H}}$ for all $\lambda\in\mathbb{C}_+$ we have

$$-I_{\mathcal{H}} = \big(\tau(\lambda)+M(\lambda)\big)N_{-\tau(\lambda)}(\lambda) = N_{-\tau(\lambda)}(\lambda)\big(\tau(\lambda)+M(\lambda)\big)$$

for $\lambda \in \Sigma^\tau \cap \Sigma^M \cap \Sigma^N$ which verifies (4.4).

(iii) Let $\lambda \in \Sigma^\tau \cap \Sigma^M \cap \Sigma^N$. Let us show that $\lambda \in \Sigma^{Q_\lambda}$, i.e., we have to show that $\lim_{\epsilon\to+0}(\tau(\lambda)+M(\lambda+i\epsilon))^{-1}$ exists. Since $\tau(\lambda)+M(\lambda)$ is boundedly invertible and $\tau(\lambda)+M(\lambda+i\epsilon)$, $\epsilon>0$, converges in the operator norm to $\tau(\lambda)+M(\lambda)$ the family $\{(\tau(\lambda)+M(\lambda+i\epsilon))^{-1}\}_{\epsilon>0}$ is uniformly bounded. Using

$$\begin{aligned}&\big(\tau(\lambda)+M(\lambda+i\epsilon)\big)^{-1}-\big(\tau(\lambda)+M(\lambda)\big)^{-1}\\&\quad=-\big(\tau(\lambda)+M(\lambda+i\epsilon)\big)^{-1}\big(M(\lambda+i\epsilon)-M(\lambda)\big)\big(\tau(\lambda)+M(\lambda)\big)^{-1},\quad \epsilon>0,\end{aligned}$$

one obtains the existence of $\lim_{\epsilon\to+0}(\tau(\lambda)+M(\lambda+i\epsilon))^{-1}$ and (4.5). □

Let A, $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ and $M(\cdot)$ be as in the beginning of this section and let as above $\tau(\cdot)$ be a matrix Nevanlinna function with values in $[\mathcal{H}]$. For each maximal dissipative operator from the Štraus family $\{A_{-\tau(\lambda)}\}_{\lambda\in\mathbb{C}_+}$ the characteristic function $W_{A_{-\tau(\lambda)}}$ is given by

$$\begin{aligned}&W_{A_{-\tau(\lambda)}}:\mathbb{C}_-\longrightarrow[\mathcal{H}_{\tau(\lambda)}]\\&\quad\mu\mapsto I_{\mathcal{H}_{\tau(\lambda)}}+2iP_{\tau(\lambda)}\sqrt{\Im\mathrm{m}\,(\tau(\lambda))}\big(\tau(\lambda)^*+M(\mu)\big)^{-1}\sqrt{\Im\mathrm{m}\,(\tau(\lambda))}\restriction_{\mathcal{H}_{\tau(\lambda)}},\end{aligned}\tag{4.6}$$

(see [29] and (3.29)), where we have used $\mathcal{H}_{\tau(\lambda)}=\mathrm{ran}\,(\Im\mathrm{m}\,(\tau(\lambda)))$, $\lambda\in\Sigma^\tau$, and denoted the projection and restriction onto $\mathcal{H}_{\tau(\lambda)}$ by $P_{\tau(\lambda)}$ and $\restriction_{\mathcal{H}_{\tau(\lambda)}}$, respectively.

If we regard the Štraus family $\{A_{-\tau(\lambda)}\}$ on the larger set $\mathbb{C}_+\cup\Sigma^\tau$, then for $\lambda\in\Sigma^\tau$ the characteristic function $W_{A_{-\tau(\lambda)}}(\cdot)$ is defined as in (4.6). Notice that in the case $\Im\mathrm{m}\,(\tau(\lambda))=0$ for $\lambda\in\Sigma^\tau$ the characteristic function of the self-adjoint extension $A_{-\tau(\lambda)}$ of A is the identity operator on the trivial space $\mathcal{H}_{\tau(\lambda)}=\{0\}$. Since the characteristic functions $W_{A_{-\tau(\lambda)}}(\cdot)$, $\lambda\in\mathbb{C}_+\cup\Sigma^\tau$, are contractive $[\mathcal{H}_{\tau(\lambda)}]$-valued functions in the lower half-plane, the limits

$$W_{A_{-\tau(\lambda)}}(\mu-i0)=\lim_{\epsilon\to+0}W_{A_{-\tau(\lambda)}}(\mu-i\epsilon)$$

exist for a.e. $\mu\in\mathbb{R}$, cf. [36]. The next proposition is a simple consequence of Lemma 4.1.

Proposition 4.2 *Let A, $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ and $M(\cdot)$ be as above and let $\tau(\cdot)$ be an $[\mathcal{H}]$-valued Nevanlinna function. Let $\{A_{-\tau(\lambda)}\}_{\lambda\in\mathbb{C}_+\cup\Sigma^\tau}$ be the Štraus family of maximal dissipative extensions of A and let $W_{A_{-\tau(\lambda)}}(\cdot)$ be the corresponding characteristic functions. Then the following holds.*

(i) *If $\lambda\in\Sigma^\tau$ and $\mu\in\Sigma^M\cap\Sigma^{Q_\lambda}$, then the limit $W_{A_{-\tau(\lambda)}}(\mu-i0)$ exists and*

$$\begin{aligned}&W_{A_{-\tau(\lambda)}}(\mu-i0)=\\&\qquad I_{\mathcal{H}_{\tau(\lambda)}}+2iP_{\tau(\lambda)}\sqrt{\Im\mathrm{m}\,(\tau(\lambda))}(\tau(\lambda)^*+M(\mu)^*)^{-1}\sqrt{\Im\mathrm{m}\,(\tau(\lambda))}\restriction_{\mathcal{H}_{\tau(\lambda)}}.\end{aligned}$$

(ii) *If $\lambda\in\Sigma^\tau\cap\Sigma^M\cap\Sigma^N$, then the limit $W_{A_{-\tau(\lambda)}}(\lambda-i0)$ exists and*

$$\begin{aligned}&W_{A_{-\tau(\lambda)}}(\lambda-i0)=\\&\qquad I_{\mathcal{H}_{\tau(\lambda)}}+2iP_{\tau(\lambda)}\sqrt{\Im\mathrm{m}\,(\tau(\lambda))}(\tau(\lambda)^*+M(\lambda)^*)^{-1}\sqrt{\Im\mathrm{m}\,(\tau(\lambda))}\restriction_{\mathcal{H}_{\tau(\lambda)}}.\end{aligned}$$

4.2 Coupling of symmetric operators and coupled scattering systems

Let, as in the previous subsection A be a densely defined closed simple symmetric operator in $\mathfrak{H}$ with equal finite deficiency indices and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* with corresponding Weyl function $M(\cdot)$. Let $\tau(\cdot)$ be an $[\mathcal{H}]$-valued Nevanlinna function and assume in addition that τ can be realized as the Weyl function corresponding to a densely defined closed simple symmetric operator T in some separable Hilbert space $\mathfrak{G}$ and a suitable boundary triplet $\Pi_T = \{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$ for T^*. It is worth to note that the Nevanlinna function $\tau(\cdot)$ has this property if and only if $\Im\mathrm{m}\,(\tau(\lambda))$ is invertible for some (and hence for all) $\lambda \in \mathbb{C}_+$ and

$$\lim_{y\to\infty} \frac{1}{y}\big(\tau(iy)h, h\big) = 0 \quad \text{and} \quad \lim_{y\to\infty} y\,\Im\mathrm{m}\,\big(\tau(iy)h, h\big) = \infty \tag{4.7}$$

hold for all $h \in \mathcal{H}$, $h \neq 0$, (see e.g. [48, Corollary 2.5 and Corollary 2.6] and [28, 50]).

In the following the function $-\tau(\cdot)$ and the Štraus family

$$A_{-\tau(\lambda)} = A^* \upharpoonright \ker\big(\Gamma_1 + \tau(\lambda)\Gamma_0\big) \tag{4.8}$$

are in a certain sense the counterparts of the dissipative matrix $D \in [\mathcal{H}]$ and the corresponding maximal dissipative extension A_D from Section 3.1. Similarly to Theorem 3.2 we construct an "energy dependent dilation" in Theorem 4.3 below, that is, we find a self-adjoint operator $\widetilde{L}$ such that

$$P_{\mathfrak{H}}\big(\widetilde{L} - \lambda\big)^{-1} \upharpoonright_{\mathfrak{H}} = \big(A_{-\tau(\lambda)} - \lambda\big)^{-1}$$

holds.

First we fix a separable Hilbert space $\mathfrak{G}$, a densely defined closed simple symmetric operator $T \in \mathcal{C}(\mathfrak{G})$ and a boundary triplet $\Pi_T = \{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$ for T^* such that $\tau(\cdot)$ is the corresponding Weyl function. We note that T and $\mathfrak{G}$ are unique up to unitary equivalence and the resolvent set $\rho(T_0)$ of the self-adjoint operator $T_0 := T^* \upharpoonright \ker(\Upsilon_0)$ coincides with the set $\mathfrak{h}(\tau)$ of points of holomorphy of τ, cf. Section 2.2. Since the deficiency indices of T are $n_+(T) = n_-(T) = n$ it follows that

$$L := A \oplus T, \qquad \mathrm{dom}\, L = \mathrm{dom}\, A \oplus \mathrm{dom}\, T,$$

is a densely defined closed simple symmetric operator in the separable Hilbert space $\mathfrak{L} := \mathfrak{H} \oplus \mathfrak{G}$ with deficiency indices $n_\pm(L) = n_\pm(A) + n_\pm(T) = 2n$.

The following theorem has originally been proved in [25, § 5]. For the sake of completeness we present another proof that differs from the original one, cf. [15].

Theorem 4.3 *Let A, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $M(\cdot)$ and $\tau(\cdot)$ be as above, let T be a densely defined closed simple symmetric operator in $\mathfrak{G}$ and $\Pi_T = \{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$ be a boundary triplet for T^* with Weyl function $\tau(\cdot)$. Then*

$$\widetilde{L} = L^* \upharpoonright \left\{ f \oplus g \in \mathrm{dom}\,(L^*) : \begin{array}{l} \Gamma_0 f - \Upsilon_0 g = 0 \\ \Gamma_1 f + \Upsilon_1 g = 0 \end{array} \right\}, \tag{4.9}$$

is a self-adjoint operator in $\mathfrak{L}$ such that

$$P_{\mathfrak{H}}\big(\widetilde{L}-\lambda\big)^{-1}\restriction_{\mathfrak{H}} = \big(A_{-\tau(\lambda)}-\lambda\big)^{-1}$$

holds for all $\lambda \in \rho(A_0)\cap\mathfrak{h}(\tau)\cap\mathfrak{h}(-(M+\tau)^{-1})$ and the minimality condition

$$\mathfrak{L} = \operatorname{clospan}\big\{\big(\widetilde{L}-\lambda\big)^{-1}\mathfrak{H} : \lambda\in\mathbb{C}\backslash\mathbb{R}\big\}$$

is satisfied. Moreover, $\widetilde{L}$ is semibounded from below if and only if A_0 is semibounded from below and $(-\infty,\eta)\subset\mathfrak{h}(\tau)$ for some $\eta\in\mathbb{R}$.

Proof. It is easy to see that $\widetilde{\Pi} = \{\mathcal{H}\oplus\mathcal{H},\widetilde{\Gamma}_0,\widetilde{\Gamma}_1\}$, where $\widetilde{\Gamma}_0 := (\Gamma_0,\Upsilon_0)^\top$ and $\widetilde{\Gamma}_1 := (\Gamma_1,\Upsilon_1)^\top$, is a boundary triplet for $L^* = A^*\oplus T^*$. If $\gamma(\cdot)$ and $\nu(\cdot)$ denote the γ-fields of $\Pi = \{\mathcal{H},\Gamma_0,\Gamma_1\}$ and $\Pi_T = \{\mathcal{H},\Upsilon_0,\Upsilon_1\}$, respectively, then the γ-field $\widetilde{\gamma}$ and Weyl function $\widetilde{M}$ of $\widetilde{\Pi} = \{\mathcal{H}\oplus\mathcal{H},\widetilde{\Gamma}_0,\widetilde{\Gamma}_1\}$ are given by

$$\lambda\mapsto\widetilde{\gamma}(\lambda) = \begin{pmatrix}\gamma(\lambda) & 0\\ 0 & \nu(\lambda)\end{pmatrix} \quad\text{and}\quad \lambda\mapsto\widetilde{M}(\lambda) = \begin{pmatrix}M(\lambda) & 0\\ 0 & \tau(\lambda)\end{pmatrix},$$

$\lambda\in\rho(A_0)\cap\rho(T_0)$, $A_0 = A^*\restriction\ker(\Gamma_0)$, $T_0 = T^*\restriction\ker(\Upsilon_0)$. A simple calculation shows that the relation

$$\Theta := \left\{\begin{pmatrix}(v,v)^\top\\ (w,-w)^\top\end{pmatrix} : v,w\in\mathcal{H}\right\}\in\widetilde{\mathcal{C}}(\mathcal{H}\oplus\mathcal{H}) \tag{4.10}$$

is self-adjoint in $\mathcal{H}\oplus\mathcal{H}$, hence the operator $L_\Theta = L^*\restriction\widetilde{\Gamma}^{(-1)}\Theta$ is a self-adjoint extension of L in $\mathfrak{L} = \mathfrak{H}\oplus\mathfrak{G}$ and L_Θ coincides with $\widetilde{L}$ in (4.9). Hence, with $L_0 = L^*\restriction\ker(\widetilde{\Gamma}_0) = A_0\oplus T_0$ we have

$$\big(\widetilde{L}-\lambda\big)^{-1} = (L_0-\lambda)^{-1} + \widetilde{\gamma}(\lambda)\big(\Theta-\widetilde{M}(\lambda)\big)^{-1}\widetilde{\gamma}(\overline{\lambda})^*, \tag{4.11}$$

for all $\lambda\in\rho(\widetilde{L})\cap\rho(L_0)$ by (2.6). Note that the difference of the resolvents of $\widetilde{L}$ and L_0 is a finite rank operator and therefore by well-known perturbation results $\widetilde{L}$ is semibounded if and only if L_0 is semibounded, that is, A_0 and T_0 are both semibounded. From $\rho(T_0) = \mathfrak{h}(\tau)$ we conclude that the last assertion of the theorem holds.

Similar considerations as in the proof of Theorem 3.2 show that

$$\big(\Theta-\widetilde{M}(\lambda)\big)^{-1} = -\begin{pmatrix}(M(\lambda)+\tau(\lambda))^{-1} & (M(\lambda)+\tau(\lambda))^{-1}\\ (M(\lambda)+\tau(\lambda))^{-1} & (M(\lambda)+\tau(\lambda))^{-1}\end{pmatrix} \tag{4.12}$$

holds for all $\lambda\in\rho(\widetilde{L})\cap\rho(L_0)$. Therefore the compressed resolvent of $\widetilde{L}$ has the form

$$P_{\mathfrak{H}}\big(L-\lambda\big)^{-1}\restriction\mathfrak{H} = (A_0-\lambda)^{-1} - \gamma(\lambda)\big(M(\lambda)+\tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^*$$

and coincides with $(A_{-\tau(\lambda)}-\lambda)^{-1}$ for all λ belonging to

$$\rho(L_0)\cap\rho(\widetilde{L}) = \rho(A_0)\cap\mathfrak{h}(\tau)\cap\mathfrak{h}\big(-(M+\tau)^{-1}\big),$$

see Section 2.2. The minimality condition follows from the fact that T is simple, $\operatorname{clospan}\{\ker(T^*-\lambda) : \lambda\in\mathbb{C}\backslash\mathbb{R}\}$, and (4.11) in a similar way as in the proof of Theorem 3.2 □

Example 4.4 Let A be the symmetric Sturm-Liouville operator from Example 3.5 and let $\Pi = \{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be the boundary triplet for A^* defined by (3.18). Besides the operator A we consider the minimal operator T in $\mathfrak{G} = L^2(\mathbb{R}_-, \mathbb{C}^n)$ associated to the Sturm-Liouville differential expression $-\frac{d^2}{dx^2} + Q_-$,

$$T = -\frac{d^2}{dx^2} + Q_-, \quad \mathrm{dom}\,(T) = \bigl\{g \in \mathcal{D}_{\max,-} : g(0) = g'(0) = 0\bigr\}.$$

Analogously to Example 3.5 it is assumed that $Q_- \in L^1_{\mathrm{loc}}(\mathbb{R}_-, [\mathbb{C}^n])$ satisfies $Q_-(\cdot) = Q_-(\cdot)^*$, that the limit point case prevails at $-\infty$ and the maximal domain $\mathcal{D}_{\max,-}$ is defined in the same way as $\mathcal{D}_{\max,+}$ in Example 3.5 with $\mathbb{R}_+$ and Q_+ replaced by $\mathbb{R}_-$ and Q_-, respectively.

It is easy to see that $\Pi_T = \{\mathbb{C}^n, \Upsilon_0, \Upsilon_1\}$, where

$$\Upsilon_0 g := g(0), \quad \Upsilon_1 g := -g'(0), \quad g \in \mathrm{dom}\,(T^*) = \mathcal{D}_{\max,-}, \tag{4.13}$$

is a boundary triplet for T^*. For $f \in \mathrm{dom}\,(A^*)$ and $g \in \mathrm{dom}\,(T^*)$ the conditions $\Gamma_0 f - \Upsilon_0 g = 0$ and $\Gamma_1 f + \Upsilon_1 g = 0$ in (4.9) stand for

$$f(0+) = g(0-) \quad \text{and} \quad f'(0+) = g'(0-),$$

so that the operator $\widetilde{L}$ in Theorem 4.3 is the self-adjoint Sturm-Liouville operator

$$\widetilde{L} = -\frac{d^2}{dx^2} + Q, \qquad Q(x) = \begin{cases} Q_+(x), & x \in \mathbb{R}_+, \\ Q_-(x), & x \in \mathbb{R}_-, \end{cases}$$

in $L^2(\mathbb{R}, \mathbb{C}^n)$.

Let A, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $M(\cdot)$ and T, $\Pi_T = \{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$, $\tau(\cdot)$ be as in the beginning of this subsection. We define the families $\{\mathcal{H}_{M(\lambda)}\}_{\lambda \in \Sigma^M}$ and $\{\mathcal{H}_{\tau(\lambda)}\}_{\lambda \in \Sigma^\tau}$ of Hilbert spaces $\mathcal{H}_{M(\lambda)}$ and $\mathcal{H}_{\tau(\lambda)}$ by

$$\mathcal{H}_{M(\lambda)} = \mathrm{ran}\bigl(\Im\mathrm{m}\,(M(\lambda + i0))\bigr) \quad \text{and} \quad \mathcal{H}_{\tau(\lambda)} = \mathrm{ran}\bigl(\Im\mathrm{m}\,(\tau(\lambda + i0))\bigr) \tag{4.14}$$

for all real points λ belonging to Σ^M and Σ^τ, respectively, cf. Section 2.3. As usual the projections and restrictions in $\mathcal{H}$ onto $\mathcal{H}_{M(\lambda)}$ and $\mathcal{H}_{\tau(\lambda)}$ are denoted by $P_{M(\lambda)}$, $\restriction_{\mathcal{H}_{M(\lambda)}}$ and $P_{\tau(\lambda)}$, $\restriction_{\mathcal{H}_{\tau(\lambda)}}$, respectively.

The next theorem is the counterpart of Theorem 3.6 in the present framework. We consider the complete scattering system $\{\widetilde{L}, L_0\}$ consisting of the self-adjoint operators $\widetilde{L}$ from Theorem 4.3 and

$$L_0 := A_0 \oplus T_0, \quad A_0 = A^* \restriction \ker(\Gamma_0), \quad T_0 = T^* \restriction \ker(\Upsilon_0),$$

and express the scattering matrix $\{\widetilde{S}(\lambda)\}$ in terms of the function $M(\cdot)$ and $\tau(\cdot)$.

Theorem 4.5 *Let A, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $M(\cdot)$ and T, $\Pi_T = \{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$, $\tau(\cdot)$ be as above. Define $\mathcal{H}_{M(\lambda)}$, $\mathcal{H}_{\tau(\lambda)}$ as in (4.14) and let $L_0 = A_0 \oplus T_0$ and $\widetilde{L}$ be as in Theorem 4.3. Then the following holds.*

(i) $L_0^{ac} = A_0^{ac} \oplus T_0^{ac}$ *is unitarily equivalent to the multiplication operator with the free variable in* $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_{\tau(\lambda)})$.

(ii) *In* $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_{\tau(\lambda)})$ *the scattering matrix* $\{\widetilde{S}(\lambda)\}$ *of the complete scattering system* $\{\widetilde{L}, L_0\}$ *is given by*

$$\widetilde{S}(\lambda) = I_{\mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_{\tau(\lambda)}} - 2i \begin{pmatrix} \widetilde{T}_{11}(\lambda) & \widetilde{T}_{12}(\lambda) \\ \widetilde{T}_{21}(\lambda) & \widetilde{T}_{22}(\lambda) \end{pmatrix} \in [\mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_{\tau(\lambda)}], \tag{4.15}$$

for all $\lambda \in \Sigma^M \cap \Sigma^\tau \cap \Sigma^N$, *where*

$$\begin{aligned}
\widetilde{T}_{11}(\lambda) &= P_{M(\lambda)} \sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(M(\lambda) + \tau(\lambda)\big)^{-1} \sqrt{\Im\mathrm{m}\,(M(\lambda))} \upharpoonright_{\mathcal{H}_{M(\lambda)}}, \\
\widetilde{T}_{12}(\lambda) &= P_{M(\lambda)} \sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(M(\lambda) + \tau(\lambda)\big)^{-1} \sqrt{\Im\mathrm{m}\,(\tau(\lambda))} \upharpoonright_{\mathcal{H}_{\tau(\lambda)}}, \\
\widetilde{T}_{21}(\lambda) &= P_{\tau(\lambda)} \sqrt{\Im\mathrm{m}\,(\tau(\lambda))}\big(M(\lambda) + \tau(\lambda)\big)^{-1} \sqrt{\Im\mathrm{m}\,(M(\lambda))} \upharpoonright_{\mathcal{H}_{M(\lambda)}}, \\
\widetilde{T}_{22}(\lambda) &= P_{\tau(\lambda)} \sqrt{\Im\mathrm{m}\,(\tau(\lambda))}\big(M(\lambda) + \tau(\lambda)\big)^{-1} \sqrt{\Im\mathrm{m}\,(\tau(\lambda))} \upharpoonright_{\mathcal{H}_{\tau(\lambda)}}
\end{aligned}$$

and $M(\lambda) = M(\lambda + i0)$, $\tau(\lambda) = \tau(\lambda + i0)$.

Proof. Let $L = A \oplus T$ and let $\widetilde{\Pi} = \{\mathcal{H} \oplus \mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ be the boundary triplet for L^* from the proof of Theorem 4.3. The corresponding Weyl function $\widetilde{M}$ is

$$\lambda \mapsto \widetilde{M}(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ 0 & \tau(\lambda) \end{pmatrix}, \qquad \lambda \in \rho(A_0) \cap \rho(T_0), \tag{4.16}$$

and since L is a densely defined closed simple symmetric operator in the separable Hilbert space $\mathfrak{L} = \mathfrak{H} \oplus \mathfrak{G}$ we can apply Theorem 2.4. First of all we immediately conclude from

$$\mathcal{H}_{\widetilde{M}(\lambda)} = \mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_{\tau(\lambda)}, \qquad \lambda \in \Sigma^{\widetilde{M}} = \Sigma^M \cap \Sigma^\tau,$$

that the absolutely continuous part $L_0^{ac} = A_0^{ac} \oplus T_0^{ac}$ of L_0 is unitarily equivalent to the multiplication operator with the free variable in the direct integral $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_{\tau(\lambda)})$. Moreover

$$\widetilde{S}(\lambda) = I_{\widetilde{\mathcal{H}}_\lambda} + 2i P_{\widetilde{M}(\lambda)} \sqrt{\Im\mathrm{m}\,(\widetilde{M}(\lambda))}\big(\Theta - \widetilde{M}(\lambda)\big)^{-1} \sqrt{\Im\mathrm{m}\,(\widetilde{M}(\lambda))} \upharpoonright_{\mathcal{H}_{\widetilde{M}(\lambda)}} \tag{4.17}$$

holds for $\lambda \in \Sigma^{\widetilde{M}} \cap \Sigma^{N_\Theta}$, where Θ is the self-adjoint relation from (4.10), the set Σ^{N_Θ} is defined as in Section 2.3 and $P_{\widetilde{M}(\lambda)}$ and $\upharpoonright_{\mathcal{H}_{\widetilde{M}(\lambda)}}$ denote the projection and restriction in $\mathcal{H} \oplus \mathcal{H}$ onto $\mathcal{H}_{\widetilde{M}(\lambda)}$, respectively.

For $\lambda \in \Sigma^{\widetilde{M}} \cap \Sigma^{N_\Theta}$ we have

$$\lim_{\epsilon \to +0} \big(\Theta - \widetilde{M}(\lambda + i\epsilon)\big)^{-1} = \big(\Theta - \widetilde{M}(\lambda + i0)\big)^{-1}$$

and

$$\big(\Theta - \widetilde{M}(\lambda)\big)^{-1} = -\begin{pmatrix} (M(\lambda) + \tau(\lambda))^{-1} & (M(\lambda) + \tau(\lambda))^{-1} \\ (M(\lambda) + \tau(\lambda))^{-1} & (M(\lambda) + \tau(\lambda))^{-1} \end{pmatrix},$$

cf. (4.12). This implies that the sets $\Sigma^{\widetilde{M}} \cap \Sigma^{N_\Theta}$ and $\Sigma^M \cap \Sigma^\tau \cap \Sigma^N$ coincide. Moreover, by inserting the above expression for $(\Theta - \widetilde{M}(\lambda))^{-1}$, $\lambda \in \Sigma^M \cap \Sigma^\tau \cap \Sigma^N$ into (4.17) and taking into account (4.16) we find that the scattering matrix $\{\widetilde{S}(\lambda)\}$ of the scattering system $\{\widetilde{L}, L_0\}$ has the form asserted in (ii). □

The following corollary, which is of similar type as Corollary 3.12, is a simple consequence of Theorem 4.5 and Proposition 4.2.

Corollary 4.6 *Let the assumptions be as in Theorem* 4.5*, let $W_{A_{-\tau(\lambda)}}(\cdot)$ be the characteristic function of the extension $A_{-\tau(\lambda)}$ in* (4.8) *and assume in addition that $\sigma(A_0)$ is purely singular. Then L_0^{ac} is unitarily equivalent to the multiplication operator with the free variable in $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{\tau(\lambda)})$ and the scattering matrix $\{\widetilde{S}(\lambda)\}$ of the complete scattering system $\{\widetilde{L}, L_0\}$ is given by*

$$\begin{aligned}\widetilde{S}(\lambda) &= W_{A_{-\tau(\lambda)}}(\lambda - i0)^* \\ &= I_{\mathcal{H}_{\tau(\lambda)}} - 2iP_{\tau(\lambda)}\sqrt{\Im\mathrm{m}\,(\tau(\lambda))}\big(M(\lambda) + \tau(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(\tau(\lambda))}\restriction_{\mathcal{H}_{\tau(\lambda)}}\end{aligned}$$

for a.e. $\lambda \in \mathbb{R}$. In the special case $\sigma(A_0) = \sigma_p(A_0)$ this relation holds for all $\lambda \in \Sigma^M \cap \Sigma^\tau \cap \Sigma^N$.

Corollary 4.7 *Let the assumptions be as in Corollary* 4.6 *and suppose that the defect of A is one, $n_\pm(A) = 1$. Then*

$$\widetilde{S}(\lambda) = W_{A_{-\tau(\lambda)}}(\lambda - i0)^* = \frac{M(\lambda) + \overline{\tau(\lambda)}}{M(\lambda) + \tau(\lambda)}$$

holds for a.e. $\lambda \in \mathbb{R}$ with $\mathrm{Im}\,\tau(\lambda + i0) \neq 0$.

4.3 Scattering matrices of energy dependent and fixed dissipative scattering systems

Let A, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $A_0 = A^* \restriction \ker(\Gamma_0)$ and $\tau(\cdot)$ be as in the previous subsections and let $\{A_{-\tau(\lambda)}\}$ be the Štraus family associated with τ from (4.8). In the following we first fix some $\mu \in \mathbb{C}_+ \cup \Sigma^\tau$ and consider the fixed dissipative scattering system $\{A_{-\tau(\mu)}, A_0\}$. Notice that if $\mu \in \Sigma^\tau$ it may happen that $A_{-\tau(\mu)}$ is self-adjoint. Let us denote by $\widetilde{K}_\mu$ the minimal self-adjoint dilation of the maximal dissipative extension $A_{-\tau(\mu)}$ in $\mathfrak{H} \oplus L^2(\mathbb{R}, d\lambda, \mathcal{H}_{\tau(\mu)})$ constructed in Theorem 3.2. Here the fixed Hilbert space $\mathcal{H}_{\tau(\mu)} = \mathrm{ran}\,(\Im\mathrm{m}\,(\tau(\mu)))$ coincides with $\mathcal{H}$ if $\mu \in \mathbb{C}_+$ or $\mathcal{H}_{\tau(\mu)}$ is a (possibly trivial) subspace of $\mathcal{H}$ if $\mu \in \Sigma^\tau$. Furthermore, if $K_0 = A_0 \oplus G_0$, where G_0 is the first order differential operator in $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{\tau(\mu)})$ from Lemma 3.1, then according to Theorem 3.6 the absolutely continuous part $K_0^{ac} = A_0^{ac} \oplus G_0$ of K_0 is unitarily equivalent to the multiplication operator with the free variable in the direct integral $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_{\tau(\mu)})$ and the scattering matrix $\{\widetilde{S}_\mu(\lambda)\}$ of the scattering system $\{\widetilde{K}_\mu, K_0\}$ is given by

$$\widetilde{S}_\mu(\lambda) = I_{\mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_{\tau(\mu)}} - 2i\begin{pmatrix}\widetilde{T}_{11,\mu}(\lambda) & \widetilde{T}_{12,\mu}(\lambda) \\ \widetilde{T}_{21,\mu}(\lambda) & \widetilde{T}_{22,\mu}(\lambda)\end{pmatrix} \in \big[\mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_{\tau(\mu)}\big], \tag{4.18}$$

for all $\lambda \in \Sigma^M \cap \Sigma^{Q_\mu}$, where

$$\begin{aligned}
\widetilde{T}_{11,\mu}(\lambda) &= P_{M(\lambda)}\sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(\tau(\mu)+M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))}\upharpoonright_{\mathcal{H}_{M(\lambda)}},\\
\widetilde{T}_{12,\mu}(\lambda) &= P_{M(\lambda)}\sqrt{\Im\mathrm{m}\,(M(\lambda))}\big(\tau(\mu)+M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(\tau(\mu))}\upharpoonright_{\mathcal{H}_{\tau(\mu)}},\\
\widetilde{T}_{21,\mu}(\lambda) &= P_{\tau(\mu)}\sqrt{\Im\mathrm{m}\,(\tau(\mu))}\big(\tau(\mu)+M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(M(\lambda))}\upharpoonright_{\mathcal{H}_{M(\lambda)}},\\
\widetilde{T}_{22,\mu}(\lambda) &= P_{\tau(\lambda)}\sqrt{\Im\mathrm{m}\,(\tau(\mu))}\big(\tau(\mu)+M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(\tau(\mu))}\upharpoonright_{\mathcal{H}_{\tau(\mu)}}
\end{aligned}$$

and $M(\lambda) = M(\lambda + i0)$. Here the set Σ^{Q_μ} and the corresponding function $\lambda \mapsto Q_{-\tau(\mu)}(\lambda)$ defined in (4.1)-(4.2) replace Σ^{N_D} and $\lambda \mapsto (D - M(\lambda))^{-1}$ in Theorem 3.6, respectively.

The following theorem is one of the main results of this paper. Roughly speaking it says that the scattering matrix of the scattering system $\{\widetilde{L}, L_0\}$ from Theorem 4.5 pointwise coincides with scattering matrices of scattering systems $\{\widetilde{K}_\mu, K_0\}$ of the above form.

Theorem 4.8 *Let A, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, $M(\cdot)$ and T, $\Pi_T = \{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$, $\tau(\cdot)$ be as in the beginning of Section 4.2 and let $L_0 = A_0 \oplus T_0$ and $\widetilde{L}$ be as in Theorem 4.3. For $\mu \in \Sigma^\tau$ denote the minimal self-adjoint dilation of $A_{-\tau(\mu)}$ in $\mathfrak{H} \oplus L^2(\mathbb{R}, \mathcal{H}_{\tau(\mu)})$ by $\widetilde{K}_\mu$ and let $K_0 = A_0 \oplus G_0$, where G_0 is the self-adjoint first order differential operator in $L^2(\mathbb{R}, \mathcal{H}_{\tau(\mu)})$.*

Then for each $\mu \in \Sigma^M \cap \Sigma^\tau \cap \Sigma^N$ the value of the scattering matrix $\{\widetilde{S}_\mu(\lambda)\}$ of the scattering system $\{\widetilde{K}_\mu, K_0\}$ at energy $\lambda = \mu$ coincides with the value of the scattering matrix $\{\widetilde{S}(\lambda)\}$ of the scattering system $\{\widetilde{L}, L_0\}$ at energy $\lambda = \mu$, that is,

$$\widetilde{S}(\mu) = \widetilde{S}_\mu(\mu) \qquad \textit{for all } \ \mu \in \Sigma^M \cap \Sigma^\tau \cap \Sigma^N. \tag{4.19}$$

Proof. According to Lemma 4.1 (iii) each real $\mu \in \Sigma^M \cap \Sigma^\tau \cap \Sigma^N$ belongs also to the set Σ^{Q_μ}. Therefore by comparing Theorem 4.5 with the scattering matrix $\{\widetilde{S}_\mu(\lambda)\}$ of $\{\widetilde{K}_\mu, K_0\}$ at energy $\lambda = \mu$ in (4.18) we conclude (4.19). □

Remark 4.9 We note that Theorem 4.8 in a certain sense justifies the use of self-adjoint dilations (or quasi-Hamiltonians) in the analysis of scattering processes for open quantum systems. Indeed, if we e.g. assume that the functions $M(\cdot)$, $\tau(\cdot)$ and $(M(\cdot) + \tau(\cdot))^{-1}$ are continuous on an interval $I \subset \mathbb{R}$ containing the point μ, then for $\lambda \in I$ the scattering matrix $\{\widetilde{S}_\mu(\lambda)\}$ of the scattering system $\{\widetilde{K}_\mu, K_0\}$ is a "good" approximation of the "real" scattering matrix $\{\widetilde{S}(\lambda)\}$, $\lambda \in I$, of the scattering system $\{\widetilde{L}, L_0\}$.

Remark 4.10 The statements of Theorem 4.5 and Theorem 4.8 are also interesting from the viewpoint of inverse problems. Namely, if $\tau(\cdot)$ is a matrix Nevanlinna function, satisfying $\ker(\Im\mathrm{m}\,(\tau(\lambda))) = 0$, $\lambda \in \mathbb{C}_+$, and the conditions (4.7), and if $\{A_{-\tau(\lambda)}, A_0\}$ is a family of energy dependent dissipative scattering systems as considered above, then in general the Hilbert space $\mathfrak{G}$ and the operators $T \subset T_0$ are not explicitely known, and hence also the scattering system $\{\widetilde{L}, L_0\}$ is not explicitely known. However, according to Theorem 4.5 the scattering matrix $\{\widetilde{S}(\lambda)\}$ can be expressed in terms of $\tau(\cdot)$ and the Weyl function $M(\cdot)$, and by Theorem 4.8 $\{\widetilde{S}(\lambda)\}$ can be obtained with the help of the scattering matrices $\{\widetilde{S}_\mu(\lambda)\}$ of the scattering systems $\{\widetilde{K}_\mu, K_0\}$.

The following corollary concerns the scattering matrices $\{S_{-\tau(\mu)}(\lambda)\}$ of the energy dependent dissipative scattering systems $\{A_{-\tau(\mu)}, A_0\}$, $\mu \in \Sigma^\tau$.

Corollary 4.11 *Let the assumptions be as in Theorem 4.8 and let $\mu \in \Sigma^M \cap \Sigma^\tau \cap \Sigma^N$. Then the scattering matrix $\{S_{-\tau(\mu)}(\lambda)\}$ of the dissipative scattering system $\{A_{-\tau(\mu)}, A_0\}$ at energy $\lambda = \mu$ coincides with the upper left corner of the scattering matrix $\{\widetilde{S}(\lambda)\}$ of the scattering system $\{\widetilde{L}, L_0\}$ at energy $\lambda = \mu$.*

Let again $\widetilde{K}_\mu$ be the minimal self-adjoint dilation of the maximal dissipative operator $A_{-\tau(\mu)}$ in $\mathfrak{H} \oplus L^2(\mathbb{R}, d\lambda, \mathcal{H}_{\tau(\mu)})$. In the next corollary we focus on the Lax-Phillips scattering matrices $\{S_\mu^{LP}(\lambda)\}$ of the Lax-Phillips scattering systems $\{\widetilde{K}_\mu, \mathcal{D}_{-,\mu}, \mathcal{D}_{+,\mu}\}$, where

$$\mathcal{D}_{-,\mu} := L^2\big(\mathbb{R}_-, \mathcal{H}_{\tau(\mu)}\big) \quad \text{and} \quad \mathcal{D}_{+,\mu} := L^2\big(\mathbb{R}_+, \mathcal{H}_{\tau(\mu)}\big)$$

are incoming and outgoing subspaces for $\widetilde{K}_\mu$, cf. Lemma 3.9. If $W_{A_{-\tau(\mu)}}(\cdot)$ is the characteristic function of $A_{-\tau(\mu)}$, cf. (4.6), then according to Corollaries 3.10 and 3.11 we have

$$\begin{aligned} S_\mu^{LP}(\lambda) &= W_{A_{-\tau(\mu)}}(\lambda - i0)^* \\ &= I_{\mathcal{H}_{\tau(\lambda)}} - 2i P_{\tau(\lambda)} \sqrt{\Im\mathrm{m}\,(\tau(\lambda))}\big(\tau(\mu) + M(\lambda)\big)^{-1} \sqrt{\Im\mathrm{m}\,(\tau(\lambda))} \upharpoonright_{\mathcal{H}_{\tau(\lambda)}} \end{aligned}$$

for all $\lambda \in \Sigma^M \cap \Sigma^{Q_\mu}$, cf. Proposition 4.2 and Corollary 4.6. Statements (ii) and (iii) of the following corollary can be regarded as generalizations of the classical Adamyan-Arov result, cf. [3, 4, 5, 6] and Corollary 3.11.

Corollary 4.12 *Let the assumptions be as in Theorem 4.8 and let $\mu \in \Sigma^M \cap \Sigma^\tau \cap \Sigma^N$.*

(i) *The scattering matrix $\{S_\mu^{LP}(\lambda)\}$ of the Lax Phillips scattering system $\{\widetilde{K}_\mu, \mathcal{D}_{-,\mu}, \mathcal{D}_{+,\mu}\}$ at energy $\lambda = \mu$ coincides with the lower right corner of the scattering matrix $\{\widetilde{S}(\lambda)\}$ of the scattering system $\{\widetilde{L}, L_0\}$ at $\lambda = \mu$.*

(ii) *The characteristic function $W_{A_{-\tau(\mu)}}(\cdot)$ of $A_{-\tau(\mu)}$ satisfies*

$$\begin{aligned} S_\mu^{LP}(\mu) &= W_{A_{-\tau(\mu)}}(\mu - i0)^* \\ &= I_{\mathcal{H}_{\tau(\mu)}} - 2i P_{\tau(\mu)} \sqrt{\Im\mathrm{m}\,(\tau(\mu))}\big(\tau(\mu) + M(\mu)\big)^{-1} \sqrt{\Im\mathrm{m}\,(\tau(\mu))} \upharpoonright_{\mathcal{H}_{\tau(\mu)}} . \end{aligned}$$

(iii) *If $\sigma(A_0)$ is purely singular, then*

$$\widetilde{S}(\mu) = S_\mu^{LP}(\mu) = W_{A_{-\tau(\mu)}}(\mu - i0)^*$$

holds for a.e. $\mu \in \mathbb{R}$. In the special case $\sigma(A_0) = \sigma_p(A_0)$ this is true for all $\mu \in \Sigma^M \cap \Sigma^\tau \cap \Sigma^N$.

4.4 A quantum transmitting Schrödinger-Poisson system

As an example we consider an open quantum system of similar type as in Section 3.4. Instead of a single pseudo-Hamiltonian A_D here the open quantum system is described by a family of energy dependent pseudo-Hamiltonians $\{A_{-\tau(\lambda)}\}$ which is sometimes called a quantum transmitting family.

Let, as in Section 3.4, $(x_l, x_r) \subset \mathbb{R}$ be a bounded interval and let A be the symmetric Sturm-Liouville operator in $\mathfrak{H} = L^2((x_l, x_r))$ given by

$$(Af)(x) = -\frac{1}{2}\frac{d}{dx}\frac{1}{m(x)}\frac{d}{dx}f(x) + V(x)f(x),$$

$$\mathrm{dom}\,(A) = \left\{ f \in \mathfrak{H} : \begin{array}{l} f, \frac{1}{m}f' \in W_2^1((x_l, x_r)) \\ f(x_l) = f(x_r) = 0 \\ \left(\frac{1}{m}f'\right)(x_l) = \left(\frac{1}{m}f'\right)(x_r) = 0 \end{array} \right\},$$

where $V, m, m^{-1} \in L^\infty((x_l, x_r))$ are real functions and $m > 0$. Let v_l, v_r be real constants, let $m_l, m_r > 0$ and define $\widetilde{V}, \widetilde{m} \in L^\infty(\mathbb{R})$ by

$$\widetilde{V}(x) := \begin{cases} v_l & x \in (-\infty, x_l] \\ V(x) & x \in (x_l, x_r) \\ v_r & x \in [x_r, \infty) \end{cases} \tag{4.20}$$

and

$$\widetilde{m}(x) := \begin{cases} m_l & x \in (-\infty, x_l] \\ m(x) & x \in (x_l, x_r) \\ m_r & x \in [x_r, \infty) \end{cases}, \tag{4.21}$$

respectively. We choose the boundary triplet $\Pi = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$,

$$\Gamma_0 f = \begin{pmatrix} f(x_l) \\ f(x_r) \end{pmatrix}, \quad \Gamma_1 f = \begin{pmatrix} \left(\frac{1}{2m}f'\right)(x_l) \\ -\left(\frac{1}{2m}f'\right)(x_r) \end{pmatrix}, \quad f \in \mathrm{dom}\,(A^*),$$

from (3.31) for A^*.

In the following we consider the Štraus family

$$A_{-\tau(\lambda)} = A^* \upharpoonright \ker\bigl(\Gamma_1 + \tau(\lambda)\Gamma_0\bigr), \qquad \lambda \in \mathbb{C}_+ \cup \Sigma^\tau,$$

associated with the 2×2-matrix Nevanlinna function

$$\lambda \mapsto \tau(\lambda) = \begin{pmatrix} i\sqrt{\frac{\lambda - v_l}{2m_l}} & 0 \\ 0 & i\sqrt{\frac{\lambda - v_r}{2m_r}} \end{pmatrix}; \tag{4.22}$$

here the square root is defined on $\mathbb{C}$ with a cut along $[0, \infty)$ and fixed by $\Im\mathrm{m}\,(\sqrt{\lambda}) > 0$ for $\lambda \notin [0, \infty)$ and by $\sqrt{\lambda} \geq 0$ for $\lambda \in [0, \infty)$, cf. Example 2.5, so that indeed $\Im\mathrm{m}\,(\tau(\lambda)) > 0$ for $\lambda \in \mathbb{C}_+$ and $\overline{\tau(\lambda)} = \tau(\overline{\lambda})$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Moreover it is not difficult to see that $\tau(\cdot)$ is holomorphic on $\mathbb{C}\backslash[\min\{v_l, v_r\}, \infty)$ and $\Sigma^\tau = \mathbb{R}$. The Štraus family $\{A_{-\tau(\lambda)}\}$, $\lambda \in \mathbb{C}_+ \cup \Sigma^\tau$, has the explicit form

$$\begin{aligned} \bigl(A_{-\tau(\lambda)}f\bigr)(x) &:= -\frac{1}{2}\frac{d}{dx}\frac{1}{m}\frac{d}{dx}f(x) + V(x)f(x), \\ \mathrm{dom}\,\bigl(A_{-\tau(\lambda)}\bigr) &= \left\{ f \in \mathfrak{H} : \begin{array}{l} f, \frac{1}{m}f' \in W_2^1((x_l, x_r)), \\ \left(\frac{1}{2m}f'\right)(x_l) = -i\sqrt{\frac{\lambda - v_l}{2m_l}}f(x_l), \\ \left(\frac{1}{2m}f'\right)(x_r) = i\sqrt{\frac{\lambda - v_r}{2m_r}}f(x_r) \end{array} \right\}. \end{aligned} \tag{4.23}$$

The operator $A_{-\tau(\lambda)}$ is self-adjoint if $\lambda \in (-\infty, \min\{v_l, v_r\}]$ and maximal dissipative if $\lambda \in (\min\{v_l, v_r\}, \infty)$. We note that the Štraus family in (4.23) plays an important role for the quantum transmitting Schrödinger-Poisson system in [10] where it was called the quantum transmitting family. For this open quantum system the boundary conditions in (4.23) are often called transparent boundary conditions.

We leave it to the reader to verify that the Nevanlinna function $\tau(\cdot)$ in (4.22) satisfies the conditions (4.7). Hence by [28, 48, 50] there exists a separable Hilbert space $\mathfrak{G}$, a densely defined closed simple symmetric operator T in $\mathfrak{G}$ and a boundary triplet $\Pi_T = \{\mathbb{C}^2, \Upsilon_0, \Upsilon_1\}$ for T^* such that $\tau(\cdot)$ is the corresponding Weyl function. Here $\mathfrak{G}$, T and $\Pi_T = \{\mathbb{C}^2, \Upsilon_0, \Upsilon_1\}$ can be explicitly described. Indeed, as Hilbert space $\mathfrak{G}$ we choose $L^2((-\infty, x_l) \cup (x_r, \infty))$ and frequently we identify this space with $L^2((-\infty, x_l)) \oplus L^2((x_r, \infty))$. An element $g \in \mathfrak{G}$ will be written in the form $g = g_l \oplus g_r$, where $g_l \in L^2((-\infty, x_l))$ and $g_r \in L^2((x_r, \infty))$. The operator T in $\mathfrak{G}$ is defined by

$$(Tg)(x) := \begin{pmatrix} -\frac{1}{2}\frac{d}{dx}\frac{1}{m_l}\frac{d}{dx}g_l(x) + v_l g_l(x) & 0 \\ 0 & -\frac{1}{2}\frac{d}{dx}\frac{1}{m_r}\frac{d}{dx}g_r(x) + v_r g_r(x) \end{pmatrix},$$

$$\mathrm{dom}\,(T) := \left\{ g = g_l \oplus g_r \in \mathfrak{G} : \begin{array}{c} g \in W_2^2((-\infty, x_l)) \oplus W_2^2((x_r, \infty)) \\ g_l(x_l) = g_r(x_r) = g_l'(x_l) = g_r'(x_l) = 0 \end{array} \right\},$$

and it is well-known that T is a densely defined closed simple symmetric operator in $\mathfrak{G}$ with deficiency indices $n_+(T) = n_-(T) = 2$. The adjoint operator T^* is given by

$$(T^*g)(x) = \begin{pmatrix} -\frac{1}{2}\frac{d}{dx}\frac{1}{m_l}\frac{d}{dx}g_l(x) + v_l g_l(x) & 0 \\ 0 & -\frac{1}{2}\frac{d}{dx}\frac{1}{m_r}\frac{d}{dx}g_r(x) + v_r g_r(x) \end{pmatrix},$$

$$\mathrm{dom}\,(T^*) = \big\{ g = g_l \oplus g_r \in \mathfrak{G} : W_2^2((-\infty, x_l)) \oplus W_2^2((x_r, \infty)) \big\}.$$

We leave it to the reader to check that $\Pi_T = \{\mathbb{C}^2, \Upsilon_0, \Upsilon_1\}$, where

$$\Upsilon_0 g := \begin{pmatrix} g_l(x_l) \\ g_r(x_r) \end{pmatrix} \quad \text{and} \quad \Upsilon_1 g := \begin{pmatrix} -\frac{1}{2m_l} g_l'(x_l) \\ \frac{1}{2m_r} g_r'(x_r) \end{pmatrix},$$

$g = g_l \oplus g_r \in \mathrm{dom}\,(T^*)$, is a boundary triplet for T^*. Notice that $T_0 = T^* \upharpoonright \ker(\Upsilon_0)$ is the restriction of T^* to the domain

$$\mathrm{dom}\,(T_0) = \big\{ g \in \mathrm{dom}\,(T^*) : g_l(x_l) = g_r(x_r) = 0 \big\},$$

that is, T_0 corresponds to Dirichlet boundary conditions. It is not difficult to see that $\sigma(T_0) = [\min\{v_l, v_r\}, \infty)$ and hence the Weyl function corresponding to $\Pi_T = \{\mathbb{C}^2, \Upsilon_0, \Upsilon_1\}$ is holomorphic on $\mathbb{C}\backslash[\min\{v_l, v_r\}, \infty)$.

Lemma 4.13 *Let $T \subset T^*$ and $\Pi_T = \{\mathbb{C}^2, \Upsilon_0, \Upsilon_1\}$ be as above. Then the corresponding Weyl function coincides with $\tau(\cdot)$ in (4.22).*

Proof. A straightforward calculation shows that

$$h_{l,\lambda}(x) := \frac{i}{\sqrt{2m_l(\lambda - v_l)}} \exp\left\{ -i\sqrt{2m_l(\lambda - v_l)}(x - x_l) \right\}$$

belongs to $L^2((-\infty, x_l))$ for $\lambda \in \mathbb{C}\backslash[v_l, \infty)$ and satisfies

$$-\frac{1}{2}\frac{d}{dx}\frac{1}{m_l}\frac{d}{dx}h_{l,\lambda}(x) + v_l h_{l,\lambda}(x) = \lambda h_{l,\lambda}(x).$$

Analogously the function

$$k_{r,\lambda}(x) := \frac{i}{\sqrt{2m_l(\lambda - v_r)}} \exp\left\{i\sqrt{2m_r(\lambda - v_r)}(x - x_r)\right\}$$

belongs to $L^2((x_r,\infty))$ for $\lambda \in \mathbb{C}\backslash[v_r,\infty)$ and satisfies

$$-\frac{1}{2}\frac{d}{dx}\frac{1}{m_r}\frac{d}{dx}k_{r,\lambda}(x) + v_r k_{r,\lambda}(x) = \lambda k_{r,\lambda}(x).$$

Therefore the functions

$$h_\lambda := h_{l,\lambda} \oplus 0 \quad \text{and} \quad k_\lambda := 0 \oplus k_{r,\lambda}$$

belong to $\mathfrak{G}$ and we have $\ker(T^* - \lambda) = \mathrm{sp}\{h_\lambda, k_\lambda\}$.

As the Weyl function $\widehat{\tau}(\cdot)$ corresponding to T and $\Pi_T = \{\mathbb{C}^2, \Upsilon_0, \Upsilon_1\}$ is defined by

$$\Upsilon_1 g_\lambda = \widehat{\tau}(\lambda)\Upsilon_0 g_\lambda \quad \text{for all} \quad g_\lambda \in \ker(T^* - \lambda),$$

$\lambda \in \mathbb{C}\backslash[\min\{v_l, v_r\},\infty)$, we conclude from

$$\Upsilon_1 h_\lambda = \frac{1}{2}\begin{pmatrix} -\frac{1}{m_l} \\ 0 \end{pmatrix} \quad \text{and} \quad \Upsilon_0 h_\lambda = \begin{pmatrix} \frac{i}{\sqrt{2m_l(\lambda - v_l)}} \\ 0 \end{pmatrix}$$

and

$$\Upsilon_1 k_\lambda = \frac{1}{2}\begin{pmatrix} 0 \\ -\frac{1}{m_r} \end{pmatrix} \quad \text{and} \quad \Upsilon_0 k_\lambda = \begin{pmatrix} 0 \\ \frac{i}{\sqrt{2m_r(\lambda - v_r)}} \end{pmatrix}$$

that $\widehat{\tau}$ has the form (4.22), $\widehat{\tau}(\cdot) = \tau(\cdot)$. □

Let A, $\Pi = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ and T, $\Pi_T = \{\mathbb{C}^2, \Upsilon_0, \Upsilon_1\}$ be as above. Then according to Theorem 4.3 the operator

$$\widetilde{L} := A^* \oplus T^* \upharpoonright \left\{ f \oplus g \in \mathrm{dom}\,(A^* \oplus T^*) : \begin{array}{l} \Gamma_0 f - \Upsilon_0 g = 0 \\ \Gamma_1 f + \Upsilon_1 g = 0 \end{array} \right\} \tag{4.24}$$

is a self-adjoint extension of $A \oplus T$ in $\mathfrak{H} \oplus \mathfrak{G}$. We can identify $\mathfrak{H} \oplus \mathfrak{G}$ with $L^2((-\infty, x_l)) \oplus L^2((x_l, x_r)) \oplus L^2((x_r,\infty))$ and $L^2(\mathbb{R})$. The elements $f \oplus g$ in $\mathfrak{H} \oplus \mathfrak{G}$, $f \in \mathfrak{H}$, $g = g_l \oplus g_r \in \mathfrak{G}$ will be written in the form $g_l \oplus f \oplus g_r$. The conditions $\Gamma_0 f = \Upsilon_0 g$ and $\Gamma_1 f = -\Upsilon_1 g$, $f \in \mathrm{dom}\,(A^*)$, $g \in \mathrm{dom}\,(T^*)$, have the form

$$\begin{pmatrix} f(x_l) \\ f(x_r) \end{pmatrix} = \begin{pmatrix} g_l(x_l) \\ g_r(x_r) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \left(\frac{1}{2m}f_l'\right)(x_l) \\ -\left(\frac{1}{2m}f_r'\right)(x_r) \end{pmatrix} = \begin{pmatrix} \frac{1}{2m_l}g'(x_l) \\ -\frac{1}{2m_r}g'(x_r) \end{pmatrix}.$$

Therefore an element $g_l \oplus f \oplus g_r$ in the domain of (4.24) has the properties

$$g_l(x_l) = f(x_l) \quad \text{and} \quad f(x_r) = g_r(x_r)$$

as well as

$$\frac{1}{m_l}g_l'(x_l) = \left(\frac{1}{m}f'\right)(x_l) \quad \text{and} \quad \left(\frac{1}{m}f'\right)(x_r) = \frac{1}{m_r}g_r'(x_r)$$

and the self-adjoint operator $\widetilde{L}$ in (4.24) becomes

$$\widetilde{L}(g_l \oplus f \oplus g_r) = \begin{pmatrix} -\frac{1}{2}\frac{d}{dx}\frac{1}{m_l}\frac{d}{dx}g_l + v_l g_l & 0 & 0 \\ 0 & -\frac{1}{2}\frac{d}{dx}\frac{1}{m}\frac{d}{dx}f + Vf & 0 \\ 0 & 0 & -\frac{1}{2}\frac{d}{dx}\frac{1}{m_r}\frac{d}{dx}g_r + v_r g_r \end{pmatrix}.$$

With the help of (4.20) and (4.21) we see that (4.24) can be regarded as the usual self-adjoint second order differential operator

$$\widetilde{L} = -\frac{1}{2}\frac{d}{dx}\frac{1}{\widetilde{m}}\frac{d}{dx} + \widetilde{V}$$

on the maximal domain in $L^2(\mathbb{R})$, that is, (4.24) coincides with the so-called Buslaev-Fomin operator from [10].

Denote by $M(\cdot)$ the Weyl function corresponding to A and the boundary triplet $\Pi = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$, cf. (3.33)-(3.34). Since $\sigma(A_0)$ consists of eigenvalues Corollary 4.6 implies that the scattering matrix $\{\widetilde{S}(\lambda)\}$ of the scattering system $\{\widetilde{L}, L_0\}$, $L_0 = A_0 \oplus T_0$, is given by

$$\widetilde{S}(\lambda) = I_{\mathcal{H}_{\tau(\lambda)}} - 2iP_{\tau(\lambda)}\sqrt{\Im\mathrm{m}\,(\tau(\lambda))}\big(M(\lambda) + \tau(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}\,(\tau(\lambda))} \restriction_{\mathcal{H}_{\tau(\lambda)}}$$

for all $\lambda \in \rho(A_0) \cap \Sigma^N$, where

$$\mathcal{H}_{\tau(\lambda)} = \mathrm{ran}\,(\Im\mathrm{m}\,(\tau(\lambda))) = \begin{cases} \{0\}, & \lambda \in (-\infty, \min\{v_l, v_r\}], \\ \mathbb{C}, & \lambda \in (\min\{v_l, v_r\}, \max\{v_l, v_r\}], \\ \mathbb{C}^2, & \lambda \in (\max\{v_l, v_r\}, \infty). \end{cases}$$

The scattering system $\{\widetilde{L}, L_0\}$ was already investigated in [9, 10]. There it was in particular shown that the scattering matrix $\{\widetilde{S}(\lambda)\}$ and the characteristic function $W_{A_{-\tau(\lambda)}}(\cdot)$ of the maximal dissipative extension $A_{-\tau(\lambda)}$ from (4.23) are connected via

$$\widetilde{S}(\lambda) = W_{A_{-\tau(\lambda)}}(\lambda - i0)^*,$$

which we here immediately obtain from Corollary 4.6.

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