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Multiple disorder problems for Wiener and compound Poisson processes with exponential jumps

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Abstract

The multiple disorder problem consists of finding a sequence of stopping times which are as close as possible to the (unknown) times of 'disorder' when the distribution of an observed process changes its probability characteristics. We present a formulation and solution of the multiple disorder problem for a Wiener and a compound Poisson process with exponential jumps. The method of proof is based on reducing the initial optimal switching problems to the corresponding coupled optimal stopping problems and solving the equivalent coupled free-boundary problems by means of the smooth- and continuous-fit conditions.

1. Introduction

Assume that at time $t = 0$ we begin to observe a continuously updated process $X = (X_t)_{t \geq 0}$ which probability characteristics change at some unknown times $(\eta_n)_{n \in \mathbb{N}}$ when an unobservable (two-stated) continuous time Markov chain $\theta = (\theta_t)_{t \geq 0}$, called the *disorder process*, changes its state from one to another. Throughout the paper it is assumed that the process θ starts at 0 with probability $1 - \pi$, starts at 1 with probability π, and changes its state with intensity $\lambda > 0$. The multiple disorder problem (or the problem of quickest multiple disorder detection) is to decide by observing the process X at which time instants one should give alarms in order to indicate the occurrence of disorders $(\eta_n)_{n \in \mathbb{N}}$. In contrast to the problem of *single* disorder, in the *multiple* disorder problem one looks for an infinite sequence of alarm times which should be as close as possible to the times $(\eta_n)_{n \in \mathbb{N}}$ in the sense that the sum of probabilities of false alarms and the total average time between the occurrence of disorders and the alarms (when the latter are given correctly) should be minimal. The idea of consideration of multiple disorder problems in such formulation is due to A.N. Shiryaev. Note that the problem of quickest detection admits different formulations and appears in a number of applied sciences (see, e.g., [20] or [5]).

The problem of detecting a change in drift of a Wiener process was formulated and solved by Shiryaev [26]-[28] (see also [29] and [30; Chapter IV]). Some particular cases of the problem of detecting a change in the intensity of a Poisson process were considered in Gal'chuk and Rozovskii [13] and in Davis [6]. Peskir and Shiryaev [23] presented a complete solution of the disorder problem for a Poisson process in the Bayesian formulation. A complete solution to the problem for a compound Poisson process with exponential jumps in the Bayesian and variational formulations was derived in [14]. Recently, Dayanik and Sezer [7] obtained a solution to the disorder problem for a general compound Poisson process. A finite horizon version of the Wiener disorder problem was studied in [15]. In the present paper we formulate and solve the multiple disorder problem for observed Wiener and compound Poisson processes having exponentially distributed jumps. This problem can be reduced to an equivalent optimal switching problem.

Optimal switching problems are extensions of optimal stopping problems and optimal stopping games where one is looking for an infinite sequence optimal stopping times. A general approach for studying such problems was developed in Bensoussan and Friedman [2]-[3] and Friedman [11] (see also Friedman [12; Chapter XVI]). This investigation was continued in Brekke and Øksendal [4], Duckworth and Zervos [9], Hamadène and Jeanblanc [17] for the continuous-time case, and in Yushkevich [31] and Yushkevich and Gordienko [32] for the discrete-time case. A direct method for solving optimal switching problems for diffusion processes is described in Dayanik and Egami [8].

The paper is organized as follows. In Section 2, we give a formulation of the multiple disorder problem for a Wiener and a compound Poisson process with exponential jumps, and reduce it to the corresponding optimal switching problem. Then, using the strong Markov property of the a posteriori probability process, we construct an equivalent coupled optimal stopping problem and formulate the corresponding coupled free-boundary problem. In Section 3, we derive solutions to the coupled free-boundary problems for the both cases of Wiener and compound Poisson processes with exponential jumps, separately. In Section 4, we formulate and prove the main assertion of the paper showing that the specified solutions of the coupled free-boundary problems turn out to be solutions of the initial coupled optimal stopping problems. The main results of the paper are formulated in Theorem 4.1. The optimal switching procedure is displayed more explicitly in Remark 4.3.

2. Formulation of the problem

In order to simplify the further exposition, in this section we formulate the multiple disorder problem for the observed sum of a Wiener and a compound Poisson process having exponentially distributed jumps (see [30; Chapter IV, Sections 3-4] and [23] for the single disorder case).

2.1. For a precise formulation of the problem, it is convenient to assume that all our considerations take place on a probability space $(\Omega, \mathcal{F}, P_\pi)$ for $\pi \in [0,1]$. Let $\theta = (\theta_t)_{t \ge 0}$ be a continuous time Markov chain with two states 0 and 1, initial distribution $[1-\pi, \pi]$, transition-probability matrix $[e^{-\lambda t}, 1-e^{-\lambda t}; 1-e^{-\lambda t}, e^{-\lambda t}]$ for $t \ge 0$, and intensity-matrix $[-\lambda, \lambda; \lambda, -\lambda]$ with $\lambda > 0$. The process θ defined above is called a *'telegraphic signal'* (see [21; Chapter IX, Section 4]). It is assumed that the process θ is unobservable, so that, the switching times $\eta_n = \inf\{t \ge \eta_{n-1} \,|\, \theta_t \neq \theta_{\eta_{n-1}}\}$, when the process θ switches from 0 to 1 and from 1 to 0, are unknown (i.e., they cannot be observed directly).

It is further assumed that we observe a process $X = (X_t)_{t \geq 0}$ defined by:

$$X_t = \int_0^t \theta_{s-} \, dX_s^1 + \int_0^t (1 - \theta_{s-}) \, dX_s^0 \tag{2.1}$$

where $X_t^i = i\mu t + \sigma W_t + \sum_{j=1}^{N_t^i} Y_j^i$ for all $t \geq 0$. Here $W = (W_t)_{t \geq 0}$ is a standard Wiener process, $N^i = (N_t^i)_{t \geq 0}$ are Poisson processes with intensities $1/\lambda_i$, and $(Y_j^i)_{j \in \mathbb{N}}$ are sequences of independent random variables exponentially distributed with parameters $\lambda_i > 0$ for $i = 0, 1$, respectively. It is supposed that W, N^i, $(Y_j^i)_{j \in \mathbb{N}}$ and θ are independent for $i = 0, 1$.

Based upon the continuous observation of X, our task is to find a (nondecreasing) sequence of stopping times with respect to the natural filtration $\mathcal{F}_t^X = \sigma\{X_s \mid 0 \leq s \leq t\}$ generated by X for $t \geq 0$ being *'as close as possible'* to the unknown switching times of the process θ. More precisely, the problem consists of computing the risk function:

$$R_*(\pi) = \min\{V_*(\pi), W_*(\pi)\} \tag{2.2}$$

for $\pi \in [0, 1]$, where

$$V_*(\pi) = \inf_{(\tau_n)} \sum_{n=1}^{\infty} \left(b P_\pi[\theta_{\tau_{2n-1}} = 0] + a P_\pi[\theta_{\tau_{2n}} = 1] + \sum_{i=0}^{1} E_\pi \left[\int_{\tau_{2n-2+i}}^{\tau_{2n-1+i}} I(\theta_t = 1 - i) \, dt \right] \right) \tag{2.3}$$

$$W_*(\pi) = \inf_{(\sigma_n)} \sum_{n=1}^{\infty} \left(a P_\pi[\theta_{\sigma_{2n-1}} = 1] + b P_\pi[\theta_{\sigma_{2n}} = 0] + \sum_{i=0}^{1} E_\pi \left[\int_{\sigma_{2n-2+i}}^{\sigma_{2n-1+i}} I(\theta_t = i) \, dt \right] \right) \tag{2.4}$$

and finding the corresponding (nondecreasing) sequences of optimal stopping times $(\tau_n^*)_{n \in \mathbb{N}}$ and $(\sigma_n^*)_{n \in \mathbb{N}}$ at which the infimums in (2.3) and (2.4) are attained. In order to avoid difficulties with notations, we set $\tau_0 = \sigma_0 = 0$. Note that in (2.3) it is assumed that the process θ initially switches from 0 to 1 first, while in (2.4) it is assumed that θ initially switches from 1 to 0 first. Here $P_\pi[\theta_{\tau_n} = i]$ is the probability of a *'false alarm'* and $E_\pi\big[\int_{\tau_{n-1}}^{\tau_n} I(\theta_t = 1 - i) dt\big]$ is the *'average delay'* in detecting the *'disorder'* correctly after giving the alarm τ_{n-1} when the process θ switches from the state i to the state $1 - i$ for $i = 0, 1$ and $n \in \mathbb{N}$, and $a > 0$ and $b > 0$ are given constants (costs of false alarms). It follows that if $V_*(\pi) < W_*(\pi)$ then $(\tau_n^*)_{n \in \mathbb{N}}$ is the optimal strategy in (2.2), while if $V_*(\pi) > W_*(\pi)$ then $(\sigma_n^*)_{n \in \mathbb{N}}$ is optimal in (2.2), and either solution is good if $V_*(\pi) = W_*(\pi)$.

2.2. Straightforward calculations based on the fact that $(\tau_n)_{n \in \mathbb{N}}$ is a nondecreasing sequence of stopping times with respect to the filtration $(\mathcal{F}_t^X)_{t \geq 0}$ show that in (2.3)-(2.4) we have:

$$E_\pi \left[\int_{\tau_{n-1}}^{\tau_n} I(\theta_t = i) \, dt \right] = E_\pi \left[\int_0^\infty I(\tau_{n-1} \leq t) I(\theta_t = i) I(t \leq \tau_n) \, dt \right] \tag{2.5}$$

$$= E_\pi \left[\int_0^\infty E_\pi \big[I(\tau_{n-1} \leq t) I(\theta_t = i) I(t \leq \tau_n) \, \big| \, \mathcal{F}_t^X \big] \, dt \right] = E_\pi \left[\int_{\tau_{n-1}}^{\tau_n} P_\pi[\theta_t = i \, | \, \mathcal{F}_t^X] \, dt \right]$$

for $i = 0, 1$. Then, by means of similar arguments to those presented in [30; pages 195-197], one can reduce the functions (2.3)-(2.4) to the form:

$$V_*(\pi) = \inf_{(\tau_n)} E_\pi \left[\sum_{n=1}^{\infty} \left(b(1-\pi_{\tau_{2n-1}}) + a\pi_{\tau_{2n}} + \int_{\tau_{2n-2}}^{\tau_{2n-1}} \pi_t \, dt + \int_{\tau_{2n-1}}^{\tau_{2n}} (1-\pi_t) \, dt \right) \right] \tag{2.6}$$

$$W_*(\pi) = \inf_{(\sigma_n)} E_\pi \left[\sum_{n=1}^{\infty} \left(a\pi_{\sigma_{2n-1}} + b(1-\pi_{\sigma_{2n}}) + \int_{\sigma_{2n-2}}^{\sigma_{2n-1}} (1-\pi_t) \, dt + \int_{\sigma_{2n-1}}^{\sigma_{2n}} \pi_t \, dt \right) \right] \tag{2.7}$$

where $\pi_t = P_\pi[\theta_t = 1 \,|\, \mathcal{F}_t^X]$ for $t \geq 0$ is the *a posteriori probability process* with $P_\pi[\pi_0 = \pi] = 1$, and we set $\tau_0 = \sigma_0 = 0$. Moreover, it is easily seen that the infimums in (2.6) and (2.7) are taken over all sequences of stopping times $(\tau_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ such that $E_\pi[\tau_n \vee \sigma_n] < \infty$ for all $n \in \mathbb{N}$.

2.3. It can be shown (see [21; Chapters IX, XVIII and XIX]) that the a posteriori probability process $(\pi_t)_{t\geq 0}$ solves the stochastic differential equation:

$$\begin{aligned} d\pi_t = {} & \lambda(1-2\pi_t)\, dt + \frac{\mu}{\sigma}\pi_t(1-\pi_t)\, d\overline{W}_t \\ & + \int_0^\infty \frac{\pi_{t-}(1-\pi_{t-})(e^{-\lambda_1 x} - e^{-\lambda_0 x})}{\pi_{t-}e^{-\lambda_1 x} + (1-\pi_{t-})e^{-\lambda_0 x}} \Big(\mu^X(dt, dx) \\ & - (\pi_{t-}e^{-\lambda_1 x} + (1-\pi_{t-})e^{-\lambda_0 x})\, dt dx \Big) (\pi_0 = \pi) \end{aligned} \tag{2.8}$$

where the innovation process $\overline{W} = (\overline{W}_t)_{t\geq 0}$ defined by:

$$\overline{W}_t = \frac{1}{\sigma}\Big(X_t^c - \mu \int_0^t \pi_s \, ds \Big) \tag{2.9}$$

is a standard Wiener process (see also [21; Chapter IX]). Here $X^c = (X_t^c)_{t\geq 0}$ denotes the continuous part and $\mu^X(dt, dx)$ is the measure of jumps of the process X (see [19; Chapters I and II]). It can be verified that $(\pi_t)_{t\geq 0}$ is a time-homogeneous (strong) Markov process under P_π with respect to its natural filtration. As the latter clearly coincides with $(\mathcal{F}_t^X)_{t\geq 0}$, it is also clear that the infimums in (2.6) and (2.7) can equivalently be taken over all stopping times of $(\pi_t)_{t\geq 0}$. This shows that the process $(\pi_t)_{t\geq 0}$ plays the role of a *sufficient statistic* in the problems (2.6) and (2.7).

2.4. Using the strong Markov property of the process $(\pi_t)_{t\geq 0}$, we can reduce the system (2.6)-(2.7) to the following *coupled optimal stopping problem*:

$$V_*(\pi) = \inf_\tau E_\pi \left[b(1-\pi_\tau) + \int_0^\tau \pi_t \, dt + W_*(\pi_\tau) \right] \tag{2.10}$$

$$W_*(\pi) = \inf_\sigma E_\pi \left[a\pi_\sigma + \int_0^\sigma (1-\pi_t) \, dt + V_*(\pi_\sigma) \right] \tag{2.11}$$

where the infimums in (2.10) and (2.11) are taken over all stopping times τ and σ such that $E_\pi[\tau \vee \sigma] < \infty$, respectively. By using the arguments in [30; pages 197-198] and [23] it can be verified that the function $V_*(\pi)$ from (2.10) is concave and decreasing, while the function $W_*(\pi)$ from (2.11) is concave and increasing on $[0,1]$. Then it follows that the optimal stopping times in (2.10) and in (2.11) have the form:

$$\tau_* = \inf\{t \geq 0 \mid \pi_t \geq B_*\} \tag{2.12}$$
$$\sigma_* = \inf\{t \geq 0 \mid \pi_t \leq A_*\} \tag{2.13}$$

where B_* is the smallest number π from $[0,1]$ such that $V_*(\pi) = b(1-\pi)$, and A_* is the largest number π from $[0,1]$ such that $W_*(\pi) = a\pi$. Hence, we may conclude that the sequence of stopping times $(\tau_n^*)_{n\in\mathbb{N}}$ given by:

$$\tau_{2n-1}^* = \inf\{t \geq \tau_{2n-2}^* \mid \pi_t \geq B_*\} \tag{2.14}$$
$$\tau_{2n}^* = \inf\{t \geq \tau_{2n-1}^* \mid \pi_t \leq A_*\} \tag{2.15}$$

is optimal in (2.6) and thus in (2.3), while the sequence of stopping times $(\sigma_n^*)_{n\in\mathbb{N}}$ given by:

$$\sigma_{2n-1}^* = \inf\{t \geq \sigma_{2n-2}^* \mid \pi_t \leq A_*\} \tag{2.16}$$
$$\sigma_{2n}^* = \inf\{t \geq \sigma_{2n-1}^* \mid \pi_t \geq B_*\} \tag{2.17}$$

is optimal in (2.7) and thus in (2.4). In order to avoid difficulties in notations, here we set $\tau_0^* = \sigma_0^* = 0$.

It is also seen that there exist a unique point $0 < \pi_* < 1$ such that $V_*(\pi_*) = W_*(\pi_*)$. Therefore, for a given number π from the interval $[0,1]$ it follows that if $\pi_* < \pi \leq 1$ then the sequence (2.14)-(2.15) is optimal in the problem (2.2), while if $0 \leq \pi < \pi_*$ then the sequence (2.16)-(2.17) is optimal in (2.2), and either solution is good if $\pi = \pi_*$.

2.5. Standard arguments imply that the infinitesimal operator $\mathbb{L}$ of the process $(\pi_t)_{t\geq 0}$ acts on a function $F \in C^2([0,1])$ according to the rule:

$$\begin{aligned}(\mathbb{L}F)(\pi) = &\left(\lambda(1-2\pi) - \frac{\lambda_0 - \lambda_1}{\lambda_0\lambda_1}\pi(1-\pi)\right) F'(\pi) + \frac{\mu^2}{2\sigma^2}\pi^2(1-\pi)^2 F''(\pi) \\ &+ \int_0^\infty \left[F\left(\frac{\pi e^{-\lambda_1 x}}{\pi e^{-\lambda_1 x} + (1-\pi)e^{-\lambda_0 x}}\right) - F(\pi)\right] \left(\pi e^{-\lambda_1 x} + (1-\pi)e^{-\lambda_0 x}\right) dx\end{aligned} \tag{2.18}$$

for all $\pi \in [0,1]$. In order to find the unknown value functions $V_*(\pi)$ and $W_*(\pi)$ from (2.10) and (2.11) as well as the unknown boundaries A_* and B_* from (2.12) and (2.13), using the general theory of optimal stopping problems for continuous time Markov processes (see, e.g., [16] and [30; Chapter III, Section 8]), we can formulate the following *coupled free-boundary problem*:

$$(\mathbb{L}W)(\pi) = -(1-\pi) \quad \text{for} \quad A < \pi < 1, \quad (\mathbb{L}V)(\pi) = -\pi \quad \text{for} \quad 0 < \pi < B \tag{2.19}$$

$$W(A+) = aA + V(A+), \quad V(B-) = b(1-B) + W(B-) \tag{2.20}$$

$$W(\pi) = a\pi + V(\pi) \quad \text{for} \quad 0 \le \pi < A, \quad V(\pi) = b(1-\pi) + W(\pi) \quad \text{for} \quad B < \pi \le 1 \tag{2.21}$$

$$W(\pi) < a\pi + V(\pi) \quad \text{for} \quad A < \pi < 1, \quad V(\pi) < b(1-\pi) + W(\pi) \quad \text{for} \quad 0 < \pi < B \tag{2.22}$$

with $0 < A_* < B_* < 1$, where the conditions (2.20), which are satisfied by virtue of the concavity arguments above, play the role of *instantaneous-stopping conditions*. Note that by the superharmonic characterization of the value function (see [10] or [30]) it follows that $V_*(\pi)$ from (2.10) and $W_*(\pi)$ from (2.11) are the largest functions satisfying (2.19)-(2.22). Moreover, we assume that the *smooth-fit conditions*:

$$(\text{if } \mu \neq 0 \text{ or } \lambda_0 > \lambda_1) \; W'(A+) = a + V'(A+), \quad V'(B-) = -b + W'(B-) \; (\text{if } \mu \neq 0 \text{ or } \lambda_0 < \lambda_1) \tag{2.23}$$

are satisfied. The latter can be explained by the fact that in these cases the process $(\pi_t)_{t \ge 0}$ can pass through the corresponding boundaries A_* and B_* continuously. Such property was earlier observed in [22]-[23] by solving some other optimal stopping problems for jump processes (see also [1] for necessary and sufficient conditions for the occurrence of smooth fit and references to the related literature, and [24] for an extensive overview).

In order to find the optimal boundaries A_* and B_*, let us introduce the *reference (difference) function* $U(\pi) = V(\pi) - W(\pi)$ for all $\pi \in [0, 1]$. Then from (2.19)-(2.22) and (2.23) it follows that $U(\pi)$ solves the system:

$$(\mathbb{L}U)(\pi) = 1 - 2\pi \quad \text{for} \quad A < \pi < B \tag{2.24}$$

$$U(A+) = -aA, \quad U(B-) = b(1-B) \tag{2.25}$$

$$U(\pi) = -a\pi \quad \text{for} \quad 0 \le \pi < A, \quad U(\pi) = b(1-\pi) \quad \text{for} \quad B < \pi \le 1 \tag{2.26}$$

$$U(\pi) > -a\pi \quad \text{for} \quad A < \pi \le 1, \quad U(\pi) < b(1-\pi) \quad \text{for} \quad 0 \le \pi < B \tag{2.27}$$

and the following conditions hold:

$$(\text{if } \mu \neq 0 \text{ or } \lambda_0 > \lambda_1) \quad U'(A+) = -a, \quad U'(B-) = -b \quad (\text{if } \mu \neq 0 \text{ or } \lambda_0 < \lambda_1). \tag{2.28}$$

3. Solutions of the coupled free-boundary problem

In this section we solve the systems (2.24)-(2.27)+(2.28) and (2.19)-(2.22)+(2.23) for the both cases $\mu \neq 0$ with $\lambda_0 = \lambda_1$ and $\mu = 0$ with $\lambda_0 \neq \lambda_1$, separately.

3.1. By means of straightforward calculations it can be checked that in case $\mu \neq 0$ and $\lambda_0 = \lambda_1$ the solution of the system (2.24)-(2.26)+(2.28) takes the form:

$$U(\pi; A, B) = \frac{b-a}{F_0(A) - F_0(B)} \int_A^\pi F_0(x)\, dx + \frac{\pi - A}{\lambda} - aA \tag{3.1}$$

for all $A_* < \pi < B_*$ and the boundaries A_* and B_* such that $0 < A_* < B_* < 1$ are uniquely determined by the following coupled system of equations:

$$bF_0(A) - aF_0(B) = \frac{1}{\lambda}\Big(F_0(B) - F_0(A)\Big) \tag{3.2}$$

$$(b-a)\int_A^B F_0(x)\, dx = \left(aA + b(1-B) - \frac{1}{\lambda}(B-A)\right)\Big(F_0(A) - F_0(B)\Big) \tag{3.3}$$

with the function $F_0(x)$ defined by:

$$F_0(x) = \exp\left(\frac{2\lambda\sigma^2}{\mu^2 x(1-x)}\right) \tag{3.4}$$

for all $0 < x < 1$ (see Figure 1 below).

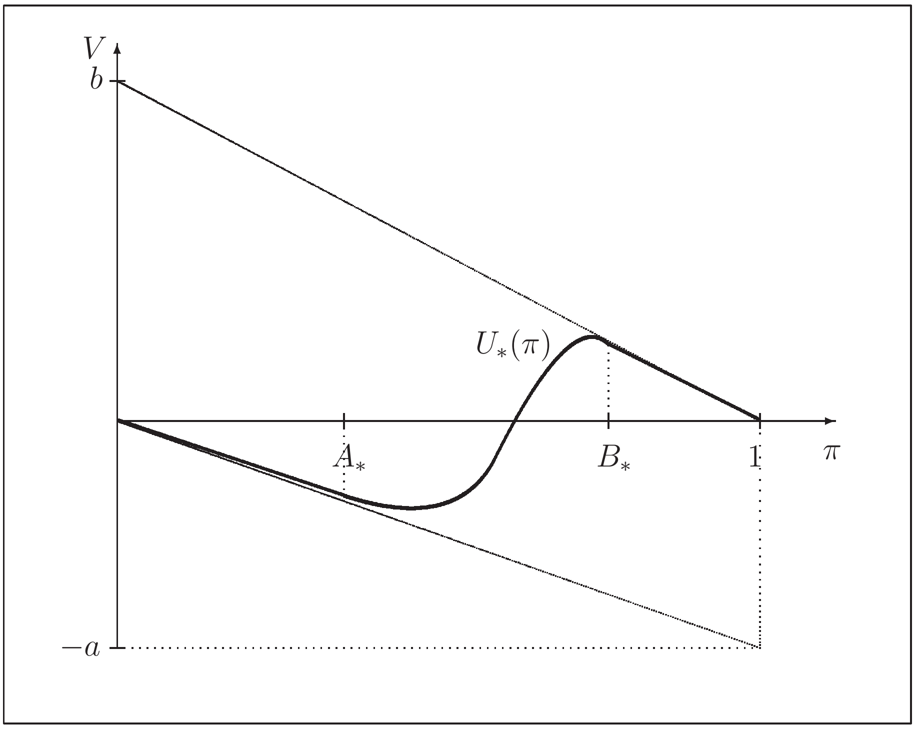

Figure 1: A computer drawing of the reference (difference) function $\pi \mapsto U_*(\pi)$ on $[0, 1]$.

Therefore, solving equations (2.19) and using conditions (2.20) for A and B fixed (as well as taking into account the fact that the value functions should be bounded), we obtain the expressions:

$$V(\pi; B) = b(1-B) + \frac{2\sigma^2}{\mu^2} \int_\pi^B \int_0^x \frac{F_0(x)}{F_0(y)} \frac{dy}{y(1-y)^2}\, dx \tag{3.5}$$

$$W(\pi; A) = aA + \frac{2\sigma^2}{\mu^2} \int_A^\pi \int_x^1 \frac{F_0(x)}{F_0(y)} \frac{dy}{y^2(1-y)}\, dx \tag{3.6}$$

where the function $F_0(x)$ is defined in (3.4).

3.2. Let us now assume that $\mu = 0$ and $\lambda_0 \neq \lambda_1$. In this case, by making straightforward calculations it is shown that when $\lambda_0 > \lambda_1$ the solution of the system (2.24)-(2.26)+(2.28) takes the form:

$$U(\pi; A, B) = b(1-B) - \int_\pi^B \frac{\gamma\lambda_1 H_1(x, B)(1-x)x^\gamma}{[\lambda_1 + (\lambda_0 - \lambda_1)x](1-x)^\gamma}\, dx \tag{3.7}$$

with

$$H_1(x, B) = \frac{1}{D(x)} \left(C_1(x, B) - \int_x^B \frac{C_1(y, B) G_1(y, B)}{D(y) G_1(x, B)}\, dy \right) \tag{3.8}$$

$$C_1(x, B) = \frac{bB(1-B)^\gamma}{\gamma(\gamma-1)B^\gamma} - \frac{\lambda_0(1-2x)(1-x)^\gamma}{\gamma(1-x)x^\gamma} \tag{3.9}$$

$$D(x) = \frac{x[\lambda'\gamma(\gamma-1)(1-2x) - x(1-x)]}{(1-x)(x+\gamma-1)} \tag{3.10}$$

$$G_1(x, B) = \exp\left(-\int_x^B \frac{dz}{D(z)} \right) \tag{3.11}$$

and $\gamma = \lambda_0/(\lambda_0 - \lambda_1) > 1$, $\lambda' = \lambda_0(\lambda_0 - \lambda_1) > 0$ as well as the boundaries A_* and B_* such that $0 < A_* < B_* < 1$ are uniquely determined by the following coupled system of equations:

$$\frac{\gamma\lambda_1 H_1(A, B)(1-A)A^\gamma}{[\lambda_1 + (\lambda_0 - \lambda_1)A](1-A)^\gamma} = -a \tag{3.12}$$

$$\int_A^B \frac{\gamma\lambda_1 H_1(x, B)(1-x)x^\gamma}{[\lambda_1 + (\lambda_0 - \lambda_1)x](1-x)^\gamma}\, dx = aA + b(1-B) \tag{3.13}$$

and when $\lambda_0 < \lambda_1$ the function $U(\pi; A, B)$ is given by:

$$U(\pi; A, B) = -aA + \int_A^\pi \frac{\gamma\lambda_1 H_2(x, A)(1-x)x^\gamma}{[\lambda_1 + (\lambda_0 - \lambda_1)x](1-x)^\gamma}\, dx \tag{3.14}$$

with

$$H_2(x, A) = \frac{1}{D(x)} \left(C_2(x, A) + \int_A^x \frac{C_2(y, A) G_2(y, A)}{D(y) G_2(x, A)}\, dy \right) \tag{3.15}$$

$$C_2(x, A) = -\frac{aA(1-A)^\gamma}{\gamma(\gamma-1)A^\gamma} - \frac{\lambda_0(1-2x)(1-x)^\gamma}{\gamma(1-x)x^\gamma} \tag{3.16}$$

$$G_2(x, A) = \exp\left(\int_A^x \frac{dz}{D(z)} \right) \tag{3.17}$$

and $\gamma = \lambda_0/(\lambda_0 - \lambda_1) < 0$, $\lambda' = \lambda_0(\lambda_0 - \lambda_1) < 0$ as well as the boundaries A_* and B_* such that $0 < A_* < B_* < 1$ are uniquely determined by the following coupled system of equations:

$$\frac{\gamma\lambda_1 H_2(B,A)(1-B)B^\gamma}{[\lambda_1 + (\lambda_0 - \lambda_1)B](1-B)^\gamma} = -b \tag{3.18}$$

$$\int_A^B \frac{\gamma\lambda_1 H_2(x,A)(1-x)x^\gamma}{[\lambda_1 + (\lambda_0 - \lambda_1)x](1-x)^\gamma}\, dx = aA + b(1-B). \tag{3.19}$$

Therefore, solving equations (2.19) and using conditions (2.20) for A and B fixed (as well as taking into account the fact that the value functions should be bounded), we obtain the expressions:

$$V(\pi; B) = b(1-B) - \int_\pi^B \frac{\gamma\lambda_1 F_1(x,B)(1-x)x^\gamma}{[\lambda_1 + (\lambda_0 - \lambda_1)x](1-x)^\gamma}\, dx \tag{3.20}$$

$$W(\pi; A) = aA + \int_A^\pi \frac{\gamma\lambda_1 F_2(x,A)(1-x)x^\gamma}{[\lambda_1 + (\lambda_0 - \lambda_1)x](1-x)^\gamma}\, dx \tag{3.21}$$

where when $\lambda_0 > \lambda_1$ we have:

$$F_1(x,B) = \frac{1}{D(x)}\left(C_3(x,B) - \int_x^B \frac{C_3(y,B)G_1(y,B)}{D(y)G_1(x,B)}\, dy\right) \tag{3.22}$$

$$F_2(x,A) = \frac{1}{D(x)}\left(C_4(x,A) + \int_A^x \frac{C_4(y,A)G_2(y,A)}{D(y)G_2(x,A)}\, dy\right) \tag{3.23}$$

$$C_3(x,B) = \frac{bB(1-B)^\gamma}{\gamma(\gamma-1)B^\gamma} - \frac{\lambda_0 x(1-x)^\gamma}{\gamma(1-x)x^\gamma} \tag{3.24}$$

$$C_4(x,A) = -\frac{aA(1-A)^\gamma}{\gamma(\gamma-1)A^\gamma} - \frac{\lambda_0 x(1-x)^\gamma}{\gamma(1-x)x^\gamma} \tag{3.25}$$

while when $\lambda_0 < \lambda_1$ we have:

$$F_1(x,B) = \frac{1}{D(x)}\left(C_5(x,B) - \int_x^B \frac{C_5(y,B)G_1(y,B)}{D(y)G_1(x,B)}\, dy\right) \tag{3.26}$$

$$F_2(x,A) = \frac{1}{D(x)}\left(C_6(x,A) + \int_A^x \frac{C_6(y,A)G_2(y,A)}{D(y)G_2(x,A)}\, dy\right) \tag{3.27}$$

$$C_5(x,B) = \frac{bB(1-B)^\gamma}{\gamma(\gamma-1)B^\gamma} - \frac{\lambda_0 (1-x)(1-x)^\gamma}{\gamma(1-x)x^\gamma} \tag{3.28}$$

$$C_6(x,A) = -\frac{aA(1-A)^\gamma}{\gamma(\gamma-1)A^\gamma} - \frac{\lambda_0 (1-x)(1-x)^\gamma}{\gamma(1-x)x^\gamma}. \tag{3.29}$$

4. Main result and proof

Taking into account the facts proved above, we are now ready to formulate and prove the main assertion of the paper.

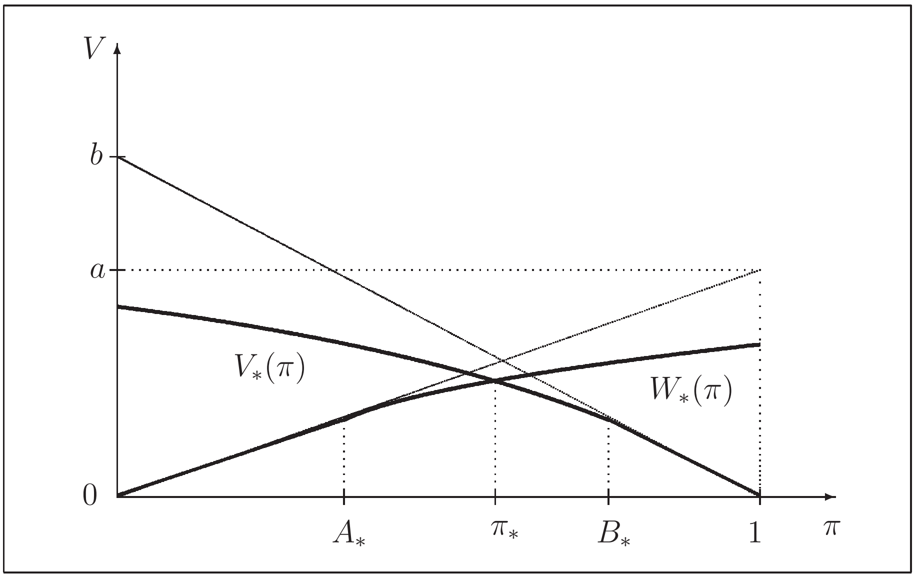

Figure 2: A computer drawing of the value functions $\pi \mapsto V_*(\pi)$ and $\pi \mapsto W_*(\pi)$ for $\pi \in [0,1]$.

Theorem 4.1. *Let the process $X = (X_t)_{t \ge 0}$ be given by (2.1) with $\mu \neq 0$ or $\lambda_0 \neq \lambda_1$. Then the value functions (2.3) and (2.4) take the expressions:*

$$V_*(\pi) = \begin{cases} V(\pi; B_*), & \text{if } 0 \le \pi < B_* \\ b(1-\pi), & \text{if } B_* \le \pi \le 1 \end{cases} \tag{4.1}$$

and

$$W_*(\pi) = \begin{cases} W(\pi; A_*), & \text{if } A_* < \pi \le 1 \\ a\pi, & \text{if } 0 \le \pi \le A_* \end{cases} \tag{4.2}$$

and the optimal stopping times $(\tau_n^*)_{n \in \mathbb{N}}$ and $(\sigma_n^*)_{n \in \mathbb{N}}$ have the structure (2.14)-(2.15) and (2.16)-(2.17), where the functions $V(\pi; B)$ and $W(\pi; A)$ and the boundaries A_* and B_* are specified as follows [see Figure 2 above]:

(i) if $\mu \neq 0$ and $\lambda_0 = \lambda_1$, then $V(\pi; B)$ and $W(\pi; A)$ are given by (3.5) and (3.6), as well as the optimal boundaries A_* and B_* satisfy the inequalities $0 < A_* < B_* < 1$ and are uniquely determined by the coupled system of equations (3.2)-(3.3);

(ii) if $\mu = 0$ and $\lambda_0 > \lambda_1$, then $V(\pi; B)$ and $W(\pi; A)$ are given by (3.20) and (3.21), as well as the optimal boundaries A_* and B_* satisfy the inequalities $0 < A_* < B_* < 1$ and are uniquely determined by the coupled system of equations (3.12)-(3.13);

(iii) if $\mu = 0$ and $\lambda_0 < \lambda_1$, then $V(\pi; B)$ and $W(\pi; A)$ are given by (3.20) and (3.21), as well as the optimal boundaries A_* and B_* satisfy the inequalities $0 < A_* < B_* < 1$ and are uniquely determined by the coupled system of equations (3.18)-(3.19).

Proof. In order to verify the related assertions, it remains to show that the functions (4.1) and (4.2) coincide with the value functions (2.10) and (2.11), respectively, and the stopping times τ_* and σ_* from (2.12) and (2.13) with the boundaries A_* and B_* specified above are optimal. For this, let us denote by $V(\pi)$ and $W(\pi)$ the right-hand sides of the expressions (4.1) and (4.2), respectively. In these cases, by means of straightforward calculations and the assumptions above it follows that the functions $V(\pi)$ and $W(\pi)$ solve the system (2.19)-(2.22), and conditions (2.23) are satisfied under the corresponding relationships on the parameters of the model. Note that from the formulas of the previous section it is seen that the both functions $V(\pi)$ and $W(\pi)$ are concave on $[0,1]$. The latter can be shown directly by analyzing the expressions (3.5)-(3.6) and (3.20)-(3.21). Then, applying Itô-Tanaka-Meyer formula (see, e.g., [18; Chapter V, Theorem 5.52] or [25; Chapter IV, Theorem 51]) to $V(\pi_t)$ and $W(\pi_t)$, we obtain:

$$V(\pi_t) = V(\pi) + \int_0^t (\mathbb{L}V)(\pi_s)\, I(\pi_s \neq B_*)\, ds + M_t \tag{4.3}$$

$$W(\pi_t) = W(\pi) + \int_0^t (\mathbb{L}W)(\pi_s)\, I(\pi_s \neq A_*)\, ds + N_t \tag{4.4}$$

where the processes $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ defined by:

$$\begin{aligned} M_t = &\int_0^t V'(\pi_s)\, \frac{\mu}{\sigma} \pi_s(1-\pi_s)\, d\overline{W}_s \\ &+ \int_0^t \int_0^\infty \left[V\left(\frac{\pi_{s-} e^{-\lambda_1 x}}{\pi_{s-} e^{-\lambda_1 x} + (1-\pi_{s-}) e^{-\lambda_0 x}} \right) - V(\pi_{s-}) \right] \Big(\mu^X(ds,dx) - \overline{\nu}(ds,dx) \Big) \end{aligned} \tag{4.5}$$

$$\begin{aligned} N_t = &\int_0^t W'(\pi_s)\, \frac{\mu}{\sigma} \pi_s(1-\pi_s)\, d\overline{W}_s \\ &+ \int_0^t \int_0^\infty \left[W\left(\frac{\pi_{s-} e^{-\lambda_1 x}}{\pi_{s-} e^{-\lambda_1 x} + (1-\pi_{s-}) e^{-\lambda_0 x}} \right) - W(\pi_{s-}) \right] \Big(\mu^X(ds,dx) - \overline{\nu}(ds,dx) \Big) \end{aligned} \tag{4.6}$$

are local martingales under the measure P_π with respect to $(\mathcal{F}_t^X)_{t\geq 0}$ and we set $\overline{\nu}(dt,dx) = (\pi_{t-} e^{-\lambda_1 x} + (1-\pi_{t-}) e^{-\lambda_0 x}) dt dx$.

By the construction of $V(\pi)$ and $W(\pi)$ from the previous sections and by using the straightforward calculations it can be checked that $(\mathbb{L}V)(\pi) \geq -\pi$ for all $B < \pi < 1$ and $(\mathbb{L}W)(\pi) \geq -(1-\pi)$ for all $0 < \pi < A$. Moreover, by means of standard arguments it can be shown that the function $V(\pi; B_*)$ is decreasing, while the function $W(\pi; A_*)$ is increasing on the intervals $(0, B_*)$ and $(A_*, 1)$, respectively, since for their derivatives we have $-b < V'(\pi; B_*) < 0$ and $0 < W'(\pi; A_*) < a$. Then the properties (2.22) also hold, that together with (2.20)-(2.21) yields $V(\pi) \leq b(1-\pi) + W(\pi)$ and $W(\pi) \leq a\pi + V(\pi)$ for all $\pi \in [0,1]$. Observe that by using (2.8) it is shown that the time spent by the process $(\pi_t)_{t\geq 0}$ at the points A_* and B_* is of Lebesgue measure zero. Hence, from the expressions (4.3)-(4.4) and the structure of stopping times in (2.12)-(2.13), by using the fact that $A_* \leq (a\lambda + 1)/(2a\lambda + 1)$,

$B_* \geq b\lambda/(2b\lambda+1)$ and $0 < A_* < B_* < 1$ it follows that the inequalities:

$$b(1-\pi_\tau) + \int_0^\tau \pi_s\, ds + W(\pi_\tau) \geq V(\pi_\tau) + \int_0^\tau \pi_s\, ds \geq V(\pi) + M_\tau \tag{4.7}$$

$$a\pi_\sigma + \int_0^\sigma (1-\pi_s)\, ds + V(\pi_\sigma) \geq W(\pi_\sigma) + \int_0^\sigma (1-\pi_s)\, ds \geq W(\pi) + N_\sigma \tag{4.8}$$

hold for any stopping times τ and σ of the process $(\pi_t)_{t\geq 0}$.

Let $(\tau_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ be arbitrary localizing sequences of stopping times for the processes $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$, respectively. Then, using (4.7)-(4.8) and taking the expectations with respect to P_π, by means of the optional sampling theorem (see, e.g., [19; Chapter I, Theorem 1.39]), we get:

$$\begin{aligned} &E_\pi\left[b(1-\pi_{\tau\wedge\tau_n}) + \int_0^{\tau\wedge\tau_n} \pi_s\, ds + W(\pi_{\tau\wedge\tau_n})\right] \\ &\geq E_\pi\left[V(\pi_{\tau\wedge\tau_n}) + \int_0^{\tau\wedge\tau_n} \pi_s\, ds\right] \geq V(\pi) + E_\pi\big[M_{\tau\wedge\tau_n}\big] = V(\pi) \end{aligned} \tag{4.9}$$

$$\begin{aligned} &E_\pi\left[a\pi_{\sigma\wedge\sigma_n} + \int_0^{\sigma\wedge\sigma_n} (1-\pi_s)\, ds + V(\pi_{\sigma\wedge\sigma_n})\right] \\ &\geq E_\pi\left[W(\pi_{\sigma\wedge\sigma_n}) + \int_0^{\sigma\wedge\sigma_n} (1-\pi_s)\, ds\right] \geq W(\pi) + E_\pi\big[N_{\sigma\wedge\sigma_n}\big] = W(\pi) \end{aligned} \tag{4.10}$$

for all $\pi \in [0,1]$. Hence, letting n go to infinity and using Fatou's lemma, for any stopping times τ and σ such that $E_\pi[\tau \vee \sigma] < \infty$ we obtain that the inequalities:

$$E_\pi\left[b(1-\pi_\tau) + \int_0^\tau \pi_s\, ds + W(\pi_\tau)\right] \geq V(\pi) \tag{4.11}$$

$$E_\pi\left[a\pi_\sigma + \int_0^\sigma (1-\pi_s)\, ds + V(\pi_\sigma)\right] \geq W(\pi) \tag{4.12}$$

are satisfied for all $\pi \in [0,1]$.

By virtue of the fact that the functions $V(\pi)$ and $W(\pi)$ satisfy the system (2.19)-(2.22) with the boundaries A_* and B_*, by the structure of the stopping times τ_* in (2.12) and σ_* in (2.13) as well as by the expressions (4.3) and (4.4) it follows that the equalities:

$$V(\pi_{\tau_*\wedge\tau_n}) + \int_0^{\tau_*\wedge\tau_n} \pi_s\, ds = V(\pi) + M_{\tau_*\wedge\tau_n} \tag{4.13}$$

$$W(\pi_{\sigma_*\wedge\sigma_n}) + \int_0^{\sigma_*\wedge\sigma_n} (1-\pi_s)\, ds = W(\pi) + N_{\sigma_*\wedge\sigma_n} \tag{4.14}$$

hold for all $\pi \in [0,1]$. Note that, by means of standard arguments and using the structure of the process (2.8) and of the stopping times (2.12) and (2.13), we have $E_\pi[\tau_* \vee \sigma_*] < \infty$ for all $\pi \in [0,1]$. Hence, letting n go to infinity in (4.13)-(4.14) and using conditions (2.21)-(2.22), by means of the Lebesgue bounded convergence theorem, we obtain the equalities:

$$E_\pi \left[b(1-\pi_{\tau_*}) + \int_0^{\tau_*} \pi_s \, ds + W(\pi_{\tau_*}) \right] = V(\pi) \tag{4.15}$$

$$E_\pi \left[a\pi_{\sigma_*} + \int_0^{\sigma_*} (1-\pi_s) \, ds + V(\pi_{\sigma_*}) \right] = W(\pi) \tag{4.16}$$

for all $\pi \in [0,1]$, that together with (4.11)-(4.12) directly imply the desired assertion. $\square$

Remark 4.2. By means of straightforward calculations from the previous section it can be verified that in case $\mu = 0$ with $\lambda_0 > \lambda_1$ we have $V'(B_*-; B_*) > -b + W'(B_*-; A_*)$, while in case $\mu = 0$ with $\lambda_0 < \lambda_1$ we have $W'(A_*+; A_*) < a + V'(A_*+; B_*)$. According to the arguments in [22]-[23] such effects can be explained by the fact that in those cases the process $(\pi_t)_{t \ge 0}$ can pass through the corresponding boundaries B_* or A_* only by jumping. According to the results in [1] we may conclude that this property appears because of finite intensity of jumps and exponential distribution of jump sizes of the compound Poisson process J.

Remark 4.3. The results formulated above show that the following sequential procedure is optimal. Being based on the observations $X = (X_t)_{t \ge 0}$ we construct the sufficient statistic process $(\pi_t)_{t \ge 0}$ and stop the observations as soon as the latter process comes into the region $[0, A_*]$ or $[B_*, 1]$ and then conclude that the continuous Markov chain $\theta = (\theta_t)_{t \ge 0}$ has switched from 1 to 0 or from 0 to 1, respectively. Starting from one of those regions $[0, A_*]$ or $[B_*, 1]$, we stop the observations as soon as the process $(\pi_t)_{t \ge 0}$ comes to the opposite region and then conclude that θ has switsched from 0 to 1 or from 1 to 0, respectively. Then we continue the procedure from the beginning.

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