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Stationary solutions to an energy model for semiconductor devices where the equations are defined on different domains

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Abstract

We discuss a stationary energy model from semiconductor modelling. We accept the more realistic assumption that the continuity equations for electrons and holes have to be considered only in a subdomain Ω_0 of the domain of definition Ω of the energy balance equation and of the Poisson equation. Here Ω_0 corresponds to the region of semiconducting material, $\Omega \setminus \Omega_0$ represents passive layers. Metals serving as contacts are modelled by Dirichlet boundary conditions.

We prove a local existence and uniqueness result for the two-dimensional stationary energy model. For this purpose we derive a $W^{1,p}$-regularity result for solutions of systems of elliptic equations with different regions of definition and use the Implicit Function Theorem.

1 Stationary energy models for semiconductor devices

Semiconductor devices are heterostructures consisting of various materials (different semiconducting materials, passive layers and metals as contacts, for example). A typical situation is shown in Fig. 1. Metals used as contacts are substituted by Dirichlet boundary conditions on a part Γ_D of the boundary of the semiconducting material. In the domain Ω involving the passive layer (Ω_1) and semiconducting materials (Ω_0) we have to formulate a Poisson equation for the electrostatic potential and an energy balance equation with boundary conditions on $\Gamma := \partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_{N0}} \cup \overline{\Gamma_{N1}}$, where the subscripts D and N indicate the parts with Dirichlet and Neumann boundary conditions, respectively. Continuity equations for electrons and holes have to be taken into account only in the part Ω_0, and here we must formulate boundary conditions on $\Gamma_0 := \partial\Omega_0 = \overline{\Gamma_D} \cup \overline{\Gamma_{N01}} \cup \overline{\Gamma_{N0}}$. Especially on Γ_{N01}, which corresponds to the interface between semiconducting material and passive layers, no-flux conditions have to be formulated. In this paper we restrict our considerations to the case that the Dirichlet parts of Γ and Γ_0 coincide.

Let T and φ denote the lattice temperature and the electrostatic potential. Then the state equations for electrons and holes are given by the following expressions

$$n = N(\cdot,T)F\Big(\frac{\zeta_n + \varphi - E_n(\cdot,T)}{T}\Big),\ p = P(\cdot,T)F\Big(\frac{\zeta_p - \varphi + E_p(\cdot,T)}{T}\Big) \quad \text{in } \Omega_0,$$

where n and p are the electron and hole densities, N and P are the effective densities of state, ζ_n and ζ_p are the electrochemical potentials, E_n and E_p are the energy band edges, respectively. The function F arises from a distribution function (e.g. $F(y) = \mathrm{e}^y$ in the case of Boltzmann statistics or $F(y) = \mathcal{F}_{1/2}(y)$ in the case of Fermi-Dirac statistics). The electrostatic potential φ fulfils the Poisson equation

$$-\nabla\cdot(\varepsilon\nabla\varphi) = \begin{cases} f - n + p & \text{in } \Omega_0 \\ f & \text{in } \Omega_1 \end{cases}. \tag{1.1}$$

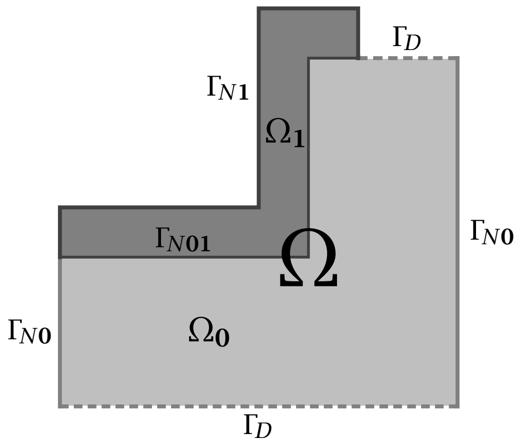

Figure 1: Schematic picture of a modelled semiconductor device

Here ε is the dielectric permittivity and f is a given doping profile. Mixed boundary conditions on Γ have to be prescribed. Next, we assume that the particle flux densities j_n, j_p and the total energy flux density j_e have the form (see e.g. Albinus, Gajewski, Hünlich [1])

$$\begin{aligned} j_n =& -(\sigma_n(x,n,p,T) + \sigma_{np}(x,n,p,T))(\nabla\zeta_n + P_n\nabla T) \\ & - \sigma_{np}(x,n,p,T)(\nabla\zeta_p + P_p\nabla T), \\ j_p =& -\sigma_{np}(x,n,p,T)(\nabla\zeta_n + P_n\nabla T) \\ & - (\sigma_p(x,n,p,T) + \sigma_{np}(x,n,p,T))(\nabla\zeta_p + P_p\nabla T), \\ j_e =& \begin{cases} -\kappa(x,n,p,T)\nabla T + \sum_{i=n,p}(\zeta_i + P_i T) j_i\,, & x \in \Omega_0 \\ -\widetilde{\kappa}(x,T)\nabla T\,, & x \in \Omega_1 \end{cases}, \end{aligned} \tag{1.2}$$

with conductivities κ, $\widetilde{\kappa}$, σ_n, $\sigma_p > 0$, $\sigma_{np} \geq 0$ and transported entropies P_n, P_p. The particle fluxes j_n, j_p only occur in the domain Ω_0 of the semiconducting material. A stationary energy model besides the Poisson equation (1.1) should contain two continuity equations for the densities n and p and a balance of the total energy

$$\nabla\cdot j_n = -R, \quad \nabla\cdot j_p = -R \quad \text{on } \Omega_0, \quad \nabla\cdot j_e = 0 \quad \text{on } \Omega, \tag{1.3}$$

where the net recombination rate R has the form

$$R = r(x,n,p,T)(\mathrm{e}^{(\zeta_n+\zeta_p)/T} - 1) \quad \text{in } \Omega_0.$$

Suitable boundary conditions for ζ_n, ζ_p resp. j_n, j_p on Γ_0 should to be added. The energy balance equation $\nabla\cdot j_e = 0$ with the corresponding flux term from (1.2) should be valid in the whole domain Ω and boundary conditions must be formulated on Γ.

In (1.2) on Ω_0 we used the fluxes (j_n, j_p, j_e) and the generalized forces $(\nabla\zeta_n, \nabla\zeta_p, \nabla T)$. In this setting Onsager relations are not valid. But this can be achieved by choosing other generalized forces, namely $(\nabla[\zeta_n/T], \nabla[\zeta_p/T], \nabla[-1/T])$. Then

$$\begin{pmatrix} j_n \\ j_p \\ j_e \end{pmatrix} = -\begin{pmatrix} (\sigma_n+\sigma_{np})T & \sigma_{np}T & \rho_n \\ \sigma_{np}T & (\sigma_p+\sigma_{np})T & \rho_p \\ \rho_n & \rho_p & \kappa T^2+\rho_e \end{pmatrix} \begin{pmatrix} \nabla[\zeta_n/T] \\ \nabla[\zeta_p/T] \\ \nabla[-1/T] \end{pmatrix} \quad \text{on } \Omega_0, \quad (1.4)$$

where

$$\begin{pmatrix} \rho_n \\ \rho_p \end{pmatrix} = \begin{pmatrix} (\sigma_n+\sigma_{np})T & \sigma_{np}T \\ \sigma_{np}T & (\sigma_p+\sigma_{np})T \end{pmatrix} \begin{pmatrix} \zeta_n+P_nT \\ \zeta_p+P_pT \end{pmatrix}, \ \rho_e = \rho_n(\zeta_n+P_nT)+\rho_p(\zeta_p+P_pT).$$

Now the matrix in (1.4) is symmetric and positive definite for non-degenerated states.

Based on the foregoing arguments we use the variables

$$z = (z_1, z_2, z_3, z_4) = \Big(\frac{\zeta_n}{T|_{\Omega_0}}, \frac{\zeta_p}{T|_{\Omega_0}}, -\frac{1}{T}, \varphi\Big),$$

where z_3 and z_4 live on Ω and z_1 and z_2 are defined on Ω_0 only. With suitable functions H_n, H_p we formulate the state equations on Ω_0 in these new variables

$$\begin{aligned} n(x) &= N(x,T)F\Big(\frac{\zeta_n+\varphi-E_n}{T}\Big) = H_n(x,z), \\ p(x) &= P(x,T)\,F\Big(\frac{\zeta_p-\varphi+E_p}{T}\Big) = H_p(x,z). \end{aligned}$$

Also the rate of generation-recombination of electrons and holes R can be expressed in the new variables

$$R = r(x,n,p,T)(\mathrm{e}^{(\zeta_n+\zeta_p)/T}-1) = r(x,H_n(z),H_p(z),-1/z_3)(\mathrm{e}^{z_1+z_2}-1) = R(x,z).$$

In summary, a stationary energy model for semiconductor devices can be written with suitable coefficient functions $a_{ik}(x,z)$, $a_{ik}\colon \Omega_0\times\mathbb{R}^2\times(-\infty,0)\times\mathbb{R}\to\mathbb{R}$, $i,k=1,\dots,3$, $\widetilde{a}_{33}(x,z_3)$, $\widetilde{a}_{33}\colon \Omega_1\times(-\infty,0)\to\mathbb{R}_+$ and $\varepsilon(x)$, $\varepsilon\colon\Omega\to\mathbb{R}_+$ as

$$-\nabla\cdot\begin{pmatrix} a_{11}(z) & a_{12}(z) & a_{13}(z) & 0 \\ a_{21}(z) & a_{22}(z) & a_{23}(z) & 0 \\ a_{31}(z) & a_{32}(z) & a_{33}(z) & 0 \\ 0 & 0 & 0 & \varepsilon \end{pmatrix}\begin{pmatrix} \nabla z_1 \\ \nabla z_2 \\ \nabla z_3 \\ \nabla z_4 \end{pmatrix} = \begin{pmatrix} -R(z) \\ -R(z) \\ 0 \\ f-H_n(z)+H_p(z) \end{pmatrix} \quad \text{on } \Omega_0 \quad (1.5)$$

and

$$-\nabla\cdot\begin{pmatrix} \widetilde{a}_{33}(z_3) & 0 \\ 0 & \varepsilon \end{pmatrix}\begin{pmatrix} \nabla z_3 \\ \nabla z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix} \quad \text{on } \Omega_1. \quad (1.6)$$

Here we have omitted the additional argument x of the coefficient functions. We formulate the boundary conditions in terms of z and the generalized forces ∇z

$$\begin{aligned} z_i = z_i^D,\ i = 1, \dots, 4, \quad &\text{on } \Gamma_D, \\ \nu \cdot \sum_{k=1,2,3} a_{ik}(x,z)\nabla z_k = g_i^{N0},\ i = 1,2,3,\ \nu \cdot (\varepsilon \nabla z_4) = g_4^{N0} \quad &\text{on } \Gamma_{N0}, \\ \nu \cdot \widetilde{a}_{33}(z_3) = g_3^{N1},\ \nu \cdot (\varepsilon \nabla z_4) = g_4^{N1} \quad &\text{on } \Gamma_{N1}, \\ \nu \cdot \sum_{k=1,2,3} a_{ik}(x,z)\nabla z_k = 0,\ i = 1,2, \quad &\text{on } \Gamma_{N01}. \end{aligned} \tag{1.7}$$

Remark 1.1 Let us mention that for the energy model introduced above the 3×3-matrix $(a_{ik}(x,z))$ for $x \in \Omega_0$ is symmetric and possesses the property that for each compact subset $K \subset \mathbb{R}^2 \times (-\infty, 0) \times \mathbb{R}$ there exists a constant $a_K > 0$ such that

$$\sum_{i,k=1,2,3} a_{ik}(x,z)\zeta_i\zeta_k \ge a_K \,\|t\|^2_{\mathbb{R}^3}, \quad x \in \Omega_0, \quad z \in K, \quad \zeta \in \mathbb{R}^3. \tag{1.8}$$

If no electron-hole scattering is involved in the model (this means $\sigma_{np} \equiv 0$), then the relations $a_{12}(x,z) = a_{21}(x,z) = 0$ are fulfilled.

2 Assumptions

Definition 2.1 Let $V = \mathbb{R}^2 \times (-\infty, 0) \times \mathbb{R}$. We say that a function $b\colon \Omega_0 \times V \to \mathbb{R}$ is of the class (D_0) if it fulfils the following properties:

$z \mapsto b(x,z)$ is continuously differentiable for almost all $x \in \Omega_0$,
$x \mapsto b(x,z)$ is measurable for all $z \in V$.

For every compact subset $K \subset V$ there exists an $c_K > 0$ such that $|b(x,z)| \le c_K$ and $\|\partial_z b(x,z)\| \le c_K$ for all $z \in K$ and almost all $x \in \Omega_0$.

For every compact subset $K \subset V$ and $\epsilon > 0$ there exists a $\delta > 0$ such that for all z, $\overline{z} \in K$ holds $|z - \overline{z}| < \delta \Rightarrow |b(x,z) - b(x,\overline{z})| < \epsilon$ and $|\partial_z b(x,z) - \partial_z b(x,\overline{z})| < \epsilon$ for almost all $x \in \Omega_0$.

We say, a function $b\colon \Omega_1 \times V_1 \to \mathbb{R}$ is of the class (D_1) if in the previous definition V is substituted by $V_1 = (-\infty, 0)$ and Ω_0 is replaced by Ω_1.

For the analytical investigations of (1.5), (1.6), (1.7) we formulate the following general **assumptions:**

(A1) Ω_i is a bounded Lipschitzian domain in $\mathbb{R}^2$, $\Gamma_i := \partial\Omega_i$, $i = 0, 1$,

$\Omega_0 \cap \Omega_1 = \emptyset$, $\Gamma_{N01} \subset \Gamma_0 \cap \Gamma_1$,

$\Omega =: \Omega_0 \cup \Omega_1 \cup \Gamma_{N01}$ is a bounded Lipschitzian domain in $\mathbb{R}^2$, $\Gamma := \partial\Omega$,

Γ_{N0}, Γ_{N01}, Γ_D are disjoint open subsets of Γ_0, $\operatorname{mes}\Gamma_D > 0$,

$\Gamma_{0N} := \Gamma_{N0} \cup \Gamma_{N01} \cup (\overline{\Gamma_{N0}} \cap \overline{\Gamma_{N01}})$ is open in Γ_0,

$\Gamma_0 = \Gamma_{0N} \cup \Gamma_D \cup (\overline{\Gamma_{0N}} \cap \overline{\Gamma_D})$, $\overline{\Gamma_{0N}} \cap \overline{\Gamma_D}$ consists of finitely many points,

Γ_{N0}, Γ_{N1}, Γ_D are disjoint open subsets of Γ,

$\Gamma_N := \Gamma_{N0} \cup \Gamma_{N1} \cup (\overline{\Gamma_{N0}} \cap \overline{\Gamma_{N1}})$ is open in Γ,

$\Gamma = \Gamma_N \cup \Gamma_D \cup (\overline{\Gamma_N} \cap \overline{\Gamma_D})$, $\overline{\Gamma_N} \cap \overline{\Gamma_D}$ consists of finitely many points.

(A2) The functions $a_{ik} = a_{ki} \colon \Omega_0 \times V \to \mathbb{R}$ are of the class (D_0), $i, k = 1, 2, 3$.

For every compact subset $K \subset V$ there exists an $a_K > 0$ such that

$\sum_{i,k=1}^{3} a_{ik}(x, z)\xi_i\xi_k \geq a_K \|\xi\|^2$ for all $z \in K$, all $\xi \in \mathbb{R}^3$ and f.a.a. $x \in \Omega_0$.

The function $\widetilde{a}_{33} \colon \Omega_1 \times V_1 \to \mathbb{R}_+$ is of the class (D_1).

For every $k > 1$ there exists an $\widetilde{a}_k > 0$ such that $\widetilde{a}_{33}(x, z) \geq \widetilde{a}_k$ for all $z \in [-k, -1/k]$ and f.a.a. $x \in \Omega_1$.

(A3) $\varepsilon \in L^\infty(\Omega)$, $0 < \varepsilon_0 \leq \varepsilon(x) \leq \varepsilon^0 < \infty$ a.e. in Ω.

(A4) The functions $H_i \colon \Omega_0 \times V \to \mathbb{R}_+$, $i = n, p$, are of the class (D_0), let

$h_0 = H_n - H_p \colon \Omega_0 \times V \to \mathbb{R}$, $h_0(x, z_1, z_2, z_3, \cdot)$ is monotonic increasing for all $(z_1, \ldots, z_4) \in \mathbb{R}^2 \times (-\infty, 0) \times \mathbb{R}$ and f.a.a. $x \in \Omega_0$.

$|h_0(x, z_1, \ldots, z_4)| \leq c_k \mathrm{e}^{c|z_4|}$ for all $(z_1, z_2, z_3) \in [-k, k]^2 \times [-k, -1/k]$, $z_4 \in \mathbb{R}$ and f.a.a. $x \in \Omega_0$.

(A5) $R(x, z) = \widetilde{r}(x, z)\big(\mathrm{e}^{z_1+z_2} - 1\big)$, where $\widetilde{r} \colon \Omega_0 \times V \to \mathbb{R}_+$ is of the class (D_0).

The data z_i^D, g_i^{N0}, g_i^{N1} and f in the system (1.5), (1.6), (1.7) are assumed to have at least the following properties. There exists a $p > 2$, functions z_1^D, $z_2^D \in W^{1,p}(\Omega_0)$ and functions z_3^D, $z_4^D \in W^{1,p}(\Omega)$, such that $z_j^D|_{\Gamma_D} = z_j^D$, $j = 1, \ldots, 4$, and $z_3^D < 0$ in Ω. Moreover we suppose that $g_i^{N0} \in L^\infty(\Gamma_{N0})$, $i = 1, \ldots, 4$, $g_i^{N1} \in L^\infty(\Gamma_{N1})$, $i = 3, 4$, and $f \in L^\infty(\Omega)$.

3 Weak formulation

In abbreviation we set

$$G_0 = \Omega_0 \cup \Gamma_{0N}, \quad G = \Omega \cup \Gamma_N.$$

Due to (A1), G_0 and G are regular in the sense of Gröger [9]. In our analytical investigations we introduce the following names for the needed function spaces. Let $s \in [1, \infty)$,

$1/s + 1/s' = 1$, then we define the spaces

$$\begin{aligned} X_s &= (W_0^{1,s}(G_0))^2 \times (W_0^{1,s}(G))^2, \\ X_s^* &= (W^{-1,s'}(G_0))^2 \times (W^{-1,s'}(G))^2 \\ W_s &= (W^{1,s}(\Omega_0))^2 \times (W^{1,s}(\Omega))^2, \\ Y_{\Omega_0}^s &= (L^s(\Omega_0))^3, \quad Y_\Omega^s = (L^s(\Omega))^3, \\ \mathcal{L}^s &= (Y_{\Omega_0}^s)^2 \times (Y_\Omega^s)^2 \end{aligned}$$

with the norms

$$\|w\|_{W_s}^s = \|w_1\|_{W^{1,s}(\Omega_0)}^s + \|w_2\|_{W^{1,s}(\Omega_0)}^s + \|w_3\|_{W^{1,s}(\Omega)}^s + \|w_4\|_{W^{1,s}(\Omega)}^s, \quad w \in W_s,$$

$$\|y\|_{\mathcal{L}^s}^s = \|y_1\|_{Y_{\Omega_0}^s}^s + \|y_2\|_{Y_{\Omega_0}^s}^s + \|y_3\|_{Y_\Omega^s}^s + \|y_4\|_{Y_\Omega^s}^s, \quad y \in \mathcal{L}^s,$$

$$\|w\|_{W^{1,s}(\Omega)}^s = \int_\Omega \Big(w^2 + w_{x_1}^2 + w_{x_2}^2\Big)^{s/2} \mathrm{d}x, \quad w \in W^{1,s}(\Omega),$$

$$\|y\|_{Y_\Omega^s}^s = \int_\Omega \Big((y^1)^2 + (y^2)^2 + (y^3)^2\Big)^{s/2} \mathrm{d}x, \quad y = (y^1, y^2, y^3) \in Y_\Omega^s$$

and similar for the function spaces working on Ω_0. Note, that $W^{1,s}(\Omega)$ and Y_Ω^s are equipped with the norms used by Gröger [9].

We define the vectors

$$z^D = (z_1^D, \dots, z_4^D), \quad g = (g_1^{N0}, \dots, g_4^{N0}, g_3^{N1}, g_4^{N1}), \quad w = (z^D, g, f),$$

and we are looking for solutions of (1.5), (1.6), (1.7) in the form

$$z = Z + z^D,$$

where z^D corresponds to a function fulfilling the Dirichlet boundary conditions and Z represents the homogeneous part of the solution. Moreover, we use the notation $\mathcal{H}$ for the space of data, namely

$$\mathcal{H} = W_p \times L^\infty(\Gamma_{N0})^4 \times L^\infty(\Gamma_{N1})^2 \times L^\infty(\Omega).$$

Definition 3.1 Let $q \in (2, p]$ and $\tau > 1$. We define subsets $M_{q,\tau} \subset X_q \times W_p$ as follows,

$$\begin{aligned} M_{q,\tau} = \Big\{ &(Z, z^D) \in X_q \times W_p \colon |Z_i + z_i^D| < \tau,\ i = 1, 2, \text{ on } \Omega_0, \\ &-\tau < Z_3 + z_3^D < -\frac{1}{\tau}, \quad |Z_4 + z_4^D| < \tau \text{ on } \Omega \Big\}. \end{aligned} \tag{3.9}$$

Because of the continuous embedding of $W^{1,p}$, $W^{1,q}$ in the space of continuous functions the set $M_{q,\tau}$ is open in $X_q \times W_p$. Clearly, if $q_2 > q_1$ then $M_{q_2,\tau} \subset M_{q_1,\tau}$. Moreover, we have $M_{q,\tau_1} \subset M_{q,\tau_2}$ for $\tau_1 < \tau_2$.

We define the operator $F_{q,\tau}\colon M_{q,\tau} \times L^\infty(\Gamma_{N0})^4 \times L^\infty(\Gamma_{N1})^2 \times L^\infty(\Omega) \to X^*_{q'}$ by

$$\begin{aligned}
\langle F_{q,\tau}(Z,w),\psi\rangle_{X_{q'}} &= \int_{\Omega_0} \Big\{ \sum_{i,k=1}^{3} a_{ik}(\cdot,z)\nabla z_k\cdot\nabla\psi_i + \varepsilon\nabla z_4\cdot\nabla\psi_4\Big\}\mathrm{d}x \\
&+ \int_{\Omega_0}\Big\{R(\cdot,z)(\psi_1+\psi_2) + h_0(\cdot,z)\psi_4\Big\}\mathrm{d}x - \int_\Omega f\psi_4\,\mathrm{d}x \\
&+ \int_{\Omega_1}\Big\{\widetilde{a}_{33}(\cdot,z_3)\nabla z_3\cdot\nabla\psi_3 + \varepsilon\nabla z_4\cdot\nabla\psi_4\Big\}\mathrm{d}x \\
&- \int_{\Gamma_{N0}}\sum_{i=1}^{4} g_i^{N0}\psi_i\,\mathrm{d}\Gamma - \int_{\Gamma_{N1}}\sum_{i=3}^{4} g_i^{N1}\psi_i\,\mathrm{d}\Gamma, \quad \psi\in X_{q'}.
\end{aligned}$$

Here $q' = q/(q-1)$ denotes the dual exponent of q. Using this notation a weak formulation of the system (1.5), (1.6), (1.7) is

Problem (P):

$$\text{Find } (q,\tau,Z,w) \text{ such that } q\in(2,p],\ \tau>1,\ (Z,w)\in X_q\times\mathcal{H},$$

$$F_{q,\tau}(Z,w) = 0, \quad (Z,z^D)\in M_{q,\tau}.$$

Obviously, if (q,τ,Z,w) is a solution to (P) then $(\widetilde{q},\widetilde{\tau},Z,w)$ with $\widetilde{q}\in(2,q]$ and $\widetilde{\tau}\ge\tau$ is a solution to (P), too.

4 Analytical results

Lemma 4.1 *We assume* (A1) – (A5). *For all parameters $\tau > 1$, all exponents $q \in (2,p]$ the operator $F_{q,\tau}\colon M_{q,\tau}\times L^\infty(\Gamma_{N0})^4\times L^\infty(\Gamma_{N1})^2\times L^\infty(\Omega)\to X^*_{q'}$ is continuously differentiable.*

Proof. Let $q\in(2,p]$ and $\tau>1$ be fixed. We split up the operator $F_{q,\tau}$ into a sum $F_{q,\tau} = \sum_{i,k=1}^3 A^{ik} + \widetilde{A}^{33} + A^{44} + A^l - B$, where A^{ij}, $\widetilde{A}^{33}$, A^{44}, $A^l\colon M_{q,\tau}\to X^*_{q'}$, $B\colon L^\infty(\Gamma_{N0})^4\times L^\infty(\Gamma_{N1})^2\times L^\infty(\Omega)\to X^*_{q'}$,

$$\begin{aligned}
\langle A^{ik}(Z,z^D),\psi\rangle_{X_{q'}} &= \int_{\Omega_0} a_{ik}(\cdot,z)\nabla(Z_k+z_k^D)\cdot\nabla\psi_i\,\mathrm{d}x, \quad i,k=1,2,3,\\
\langle \widetilde{A}^{33}(Z,z^D),\psi\rangle_{X_{q'}} &= \int_{\Omega_1}\widetilde{a}_{33}(\cdot,z_3)\nabla(Z_3+z_3^D)\cdot\nabla\psi_3\,\mathrm{d}x,\\
\langle A^{44}(Z,z^D),\psi\rangle_{X_{q'}} &= \int_\Omega \varepsilon\nabla(Z_4+z_4^D)\cdot\nabla\psi_4\,\mathrm{d}x,\\
\langle A^{l}(Z,z^D),\psi\rangle_{X_{q'}} &= \int_{\Omega_0}\Big\{R(\cdot,z)(\psi_1+\psi_2)+h_0(\cdot,z)\psi_4\Big\}\,\mathrm{d}x,\\
\langle B(g,f),\psi\rangle_{X_{q'}} &= \int_\Omega f\psi_4\,\mathrm{d}x + \int_{\Gamma_{N0}}\sum_{i=1}^4 g_i^{N0}\psi_i\,\mathrm{d}\Gamma + \int_{\Gamma_{N1}}\sum_{i=3}^4 g_i^{N1}\psi_i\,\mathrm{d}\Gamma,\ \psi\in X_{q'},
\end{aligned}$$

where $z = Z + z^D$. Since $q > 2$ the continuous differentiability of the operator A^l is a direct consequence of the fact that $\widetilde{r}$ and h_0 are of the class (D_0), see (A4), (A5). The assertion for the operators A^{44} and B is verified by standard arguments. Now we do, as a representative of a non standard situation, the proof for a summand A^{ik}. First we show continuity. Let $(Z, z^D) \in M_{q,\tau}$ and let $(\overline{Z}, \overline{z}^D) \to 0$ in $X_q \times W_p$, then

$$
\begin{aligned}
&|\langle A^{ik}(Z+\overline{Z}, z^D+\overline{z}^D) - A^{ik}(Z, z^D), \psi\rangle_{X_{q'}}| \\
&\le \int_{\Omega_0} |a_{ik}(\cdot, z+\overline{z}) - a_{ik}(\cdot, z)||\nabla(Z_k + \overline{Z}_k + z_k^D + \overline{z}_k^D)||\nabla\psi_i| \,\mathrm{d}x \\
&+ \int_{\Omega_0} |a_{ik}(\cdot, z)||\nabla(\overline{Z}_k + \overline{z}_k^D)||\nabla\psi_i| \,\mathrm{d}x \\
&\le c_p \|a_{ik}(z+\overline{z}) - a_{ik}(z)\|_{L^\infty(\Omega_0)} (\|Z+\overline{Z}\|_{X_q} + \|z^D + \overline{z}^D\|_{W_p}) \|\psi\|_{X_{q'}} \\
&+ c_p \|a_{ik}(z)\|_{L^\infty(\Omega_0)} \big(\|\overline{Z}\|_{X_q} + \|\overline{z}^D\|_{W_p}\big) \|\psi\|_{X_{q'}}.
\end{aligned}
$$

Since the functions a_{ik} belong to the class (D_0), see (A2), the continuity follows. Next, let $(Z, z^D) \in M_{q,\tau}$ be arbitrarily fixed. We prove that the operator $\overline{A}^{ik}(Z, z^D) \in \mathcal{L}(X_q, X_{q'}^*)$,

$$
\begin{aligned}
\langle \overline{A}^{ik}(Z, z^D)\,\overline{Z}, \psi\rangle_{X_{q'}} &= \int_{\Omega_0} \partial_z a_{ik}(\cdot, z) \cdot \overline{Z} \nabla(Z_k + z_k^D) \cdot \nabla\psi_i \,\mathrm{d}x \\
&+ \int_{\Omega_0} a_{ik}(\cdot, z) \nabla\overline{Z}_k \cdot \nabla\psi_i \,\mathrm{d}x, \quad \psi \in X_{q'},
\end{aligned}
$$

is the Fréchet derivative of $A^{ik}(Z, z^D)$ with respect to Z: Let $\overline{Z} \to 0$ in X_q.

$$
\begin{aligned}
&|\langle A^{ik}(Z+\overline{Z}, z^D) - A^{ik}(Z, z^D) - \overline{A}^{ik}(Z, z^D)\overline{Z}, \psi\rangle_{X_{q'}}| \\
&\le \Big| \int_{\Omega_0} \Big(a_{ik}(\cdot, z+\overline{Z}) \nabla(z_k + \overline{Z}_k) - a_{ik}(\cdot, z)\nabla z_k\Big) \cdot \nabla\psi_i \,\mathrm{d}x \\
&\qquad - \int_{\Omega_0} \Big(\partial_z a_{ik}(\cdot, z) \cdot \overline{Z} \nabla z_k + a_{ik}(\cdot, z)\nabla\overline{Z}_k\Big) \cdot \nabla\psi_i \,\mathrm{d}x \Big| \\
&\le \int_{\Omega_0} \big|a_{ik}(\cdot, z+\overline{Z}) - a_{ik}(\cdot, z) - \partial_z a_{ik}(\cdot, z) \cdot \overline{Z}\big| |\nabla z_k||\nabla\psi_i| \,\mathrm{d}x \\
&+ \int_{\Omega_0} \big|a_{ik}(\cdot, z+\overline{Z}) - a_{ik}(\cdot, z)\big| |\nabla\overline{Z}_k||\nabla\psi_i| \,\mathrm{d}x \\
&\le c_p \|a_{ik}(z+\overline{Z}) - a_{ik}(z) - \partial_z a_{ik}(z) \cdot \overline{Z}\|_{L^\infty(\Omega_0)} \Big(\|Z\|_{X_q} + \|z^D\|_{W_p}\Big) \|\psi\|_{X_{q'}} \\
&+ \|a_{ik}(z+\overline{Z}) - a_{ik}(z)\|_{L^\infty(\Omega_0)} \|\overline{Z}\|_{X_q} \|\psi\|_{X_{q'}}.
\end{aligned}
$$

Exploiting, that a_{ik} are of the class (D_0) and $\overline{Z} \to 0$ the last two lines converge to zero and differentiability with respect to Z is shown. The continuity of this Fréchet derivative is guaranteed since the functions a_{ik} are of the class (D_0). Similarly one can prove the continuous differentiability of A^{ik} with respect to z^D. Substituting Ω_0 by Ω_1 we obtain the assertion for the operator $\widetilde{A}^{33}$ as a special case of the above if we take into account that $\widetilde{a}_{33}$ belongs to the class (D_1). Thus the sum $F_{q,\tau}$ of all the considered summands is continuously differentiable. $\square$

Especially we have

$$\begin{aligned}\langle \partial_Z F_{q,\tau}(Z,w)\overline{Z},\psi\rangle_{X_{q'}} &= \int_{\Omega_0}\sum_{i,k=1}^{3}\Big\{a_{ik}(\cdot,z)\nabla\overline{Z}_k + \partial_z a_{ik}(\cdot,z)\cdot\overline{Z}\nabla z_k\Big\}\cdot\nabla\psi_i\,\mathrm{d}x\\
&+\int_{\Omega_0}\Big\{\partial_z R(\cdot,z)\cdot\overline{Z}\,(\psi_1+\psi_2)+\partial_z h_0(\cdot,z)\cdot\overline{Z}\,\psi_4\Big\}\mathrm{d}x\\
&+\int_{\Omega_1}\Big\{\widetilde{a}_{33}(\cdot,z_3)\nabla\overline{Z}_3+\frac{\partial\widetilde{a}_{33}}{\partial z_3}(\cdot,z_3)\,\overline{Z}_3\,\nabla z_3\Big\}\cdot\nabla\psi_3\,\mathrm{d}x\\
&+\int_{\Omega}\varepsilon\,\nabla\overline{Z}_4\cdot\nabla\psi_4\,\mathrm{d}x,\qquad \psi\in X_{q'}.\end{aligned}\tag{4.10}$$

We define a set of data, which is compatible with thermodynamic equilibrium,

$$\begin{aligned}Q := \Big\{w=(z^D,g,f)\in\mathcal{H}\colon\ & z_i^D=\text{const},\ g_i^{N0}=0,\ i=1,2,3,\\
& g_3^{N1}=0,\ z_1^D+z_2^D=0,\ z_3^D<0\Big\}.\end{aligned}$$

Theorem 4.1 (Existence and uniqueness of thermodynamic equilibria). *We make the assumptions* (A1) – (A5). *Let* $w^*=(z^{D*},g^*,f^*)\in Q$ *be given.*

i) *Then there exist a* $q_0\in(2,p]$, *a constant* $\tau>1$ *and a function* $Z_4^*\in W_0^{1,q_0}(G)$ *such that* $(Z^*,z^{D*})=((0,0,0,Z_4^*),z^{D*})\in M_{q_0,\tau}$ *and* $F_{q_0,\tau}(Z^*,w^*)=0$. *In other words,* (q_0,τ,Z^*,w^*) *is a solution to* (P).

ii) $z^*=Z^*+z^{D*}$ *is a thermodynamic equilibrium of* (1.5), (1.6), (1.7).

iii) *If* $(\widetilde{q},\widetilde{\tau},\widetilde{Z},w^*)$ *is a solution to* (P), *then* $\widetilde{Z}=Z^*$ *in* $X_{\widehat{q}}$ *with* $\widehat{q}=\min\{q_0,\widetilde{q}\}$ *holds.*

Proof. 1. For $w^*=(z^{D*},g^*,f^*)\in Q$ we define the function $h_1\colon\Omega_0\times\mathbb{R}\to\mathbb{R}$ by

$$h_1(x,\phi)=h_0(x,(0,0,0,\phi)+z^{D*}(x))$$

and consider the operator $\mathcal{E}\colon H_0^1(G)\to H^{-1}(G)$,

$$\begin{aligned}\langle\mathcal{E}(\phi),\overline{\phi}\rangle_{H_0^1(G)} &= \int_\Omega\Big\{\varepsilon\,\nabla(\phi+z_4^{D*})\cdot\nabla\overline{\phi}\ -f^*\overline{\phi}\Big\}\,\mathrm{d}x+\int_{\Omega_0}h_1(\cdot,\phi)\,\overline{\phi}\,\mathrm{d}x\\
&-\int_{\Gamma_{N0}}g_4^{N0*}\overline{\phi}\,\mathrm{d}\Gamma-\int_{\Gamma_{N1}}g_4^{N1*}\overline{\phi}\,\mathrm{d}\Gamma\qquad\forall\overline{\phi}\in H_0^1(G).\end{aligned}$$

For ϕ_1, $\phi_2\in H_0^1(G)$ we have

$$\begin{aligned}&\langle\mathcal{E}(\phi_1)-\mathcal{E}(\phi_2),\phi_1-\phi_2\rangle_{H_0^1(G)}\\
&=\int_\Omega\varepsilon\,|\nabla(\phi_1-\phi_2)|^2\,\mathrm{d}x+\int_{\Omega_0}(h_1(\cdot,\phi_1)-h_1(\cdot,\phi_2))(\phi_1-\phi_2)\,\mathrm{d}x,\end{aligned}$$

and the properties (A1), (A3), (A4) of Γ_D, ε and h_0 supply the strong monotonicity of the operator $\mathcal{E}$. Next we prove the hemicontinuity of $\mathcal{E}$. We have to show that the mapping

$t \mapsto \langle \mathcal{E}(\phi + t\widehat{\phi}), \overline{\phi} \rangle_{H_0^1(G)}$ for arbitrarily given ϕ, $\widehat{\phi}$, $\overline{\phi} \in H_0^1(G)$ is continuous on $[0,1]$. Let $t_0 \in [0,1]$, $t_n \to t_0$, $t_n \in [0,1]$. Then

$$\begin{aligned} &\langle \mathcal{E}(\phi + t_n\widehat{\phi}) - \mathcal{E}(\phi + t_0\widehat{\phi}), \overline{\phi} \rangle_{H_0^1(G)} \\ &\le c|t_n - t_0| \|\widehat{\phi}\|_{H^1} \|\overline{\phi}\|_{H^1} + \Big| \int_{\Omega_0} \Big[h_1(\cdot, \phi + t_n\widehat{\phi}) - h_1(\cdot, \phi + t_0\widehat{\phi}) \Big] \, \overline{\phi} \, \mathrm{d}x \Big|. \end{aligned} \tag{4.11}$$

According to (A4) we have $h_1(x, \phi + t_n\widehat{\phi}) \to h_1(x, \phi + t_0\widehat{\phi})$ and

$$|h_1(x, \phi + t_n\widehat{\phi})| \le \widetilde{c} \, \mathrm{e}^{\widetilde{c}\,(|\phi| + |\widehat{\phi}|)} \quad \text{for almost all } x \in \Omega_0.$$

Now we use the embedding result of Trudinger [12] for two dimensional Lipschitzian domains which tells us that

$$\|\mathrm{e}^{|v|}\|_{L^2(\Omega_0)} \le d(\|v\|_{H^1(\Omega_0)}) \quad \forall v \in H^1(\Omega_0),$$

where $d\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, monotone increasing function with $\lim_{y\to\infty} d(y) = \infty$. Since $\overline{\phi} \in L^2(\Omega_0)$ we get an integrable upper bound for the integrand in the last term in (4.11) and Lebesgue's Dominated Convergence Theorem leads to the hemicontinuity of $\mathcal{E}$. Since $\mathcal{E}$ is strongly monotone and hemicontinuous, there exists a unique solution $\phi \in H_0^1(G)$ of $\mathcal{E}(\phi) = 0$. Especially we have $\|\phi\|_{H^1(\Omega)} \le \widehat{c}$, where $\widehat{c}$ depends only on the data w^*.

2. Next we prove that this solution possesses more regularity. We define

$$\begin{aligned} \langle \mathcal{E}_0(\phi), \overline{\phi} \rangle_{H_0^1(G)} &= \int_\Omega \Big\{ \varepsilon \, \nabla\phi \cdot \nabla\overline{\phi} + \phi \, \overline{\phi} \Big\} \mathrm{d}x, \\ \langle \mathcal{T}, \overline{\phi} \rangle_{H_0^1(G)} &= \int_\Omega \Big\{ -\varepsilon \, \nabla z_4^{D*} \cdot \nabla\overline{\phi} + \big(f^* + \phi\big)\overline{\phi} \Big\} \, \mathrm{d}x - \int_{\Omega_0} h_1(\cdot, \phi)\overline{\phi} \, \mathrm{d}x \\ &\quad + \int_{\Gamma_{N0}} g_4^{N0*} \, \overline{\phi} \, \mathrm{d}\Gamma + \int_{\Gamma_{N1}} g_4^{N1*} \, \overline{\phi} \, \mathrm{d}\Gamma \quad \forall \overline{\phi} \in H_0^1(G). \end{aligned}$$

Since $z_4^{D*} \in W^{1,p}(\Omega)$ is a fixed element there is a $\overline{c} > 0$ such that $|z_4^{D*}| \le \overline{c}$. From the properties (A4) of h_0 we find $|h_1(x, \phi)| \le c(z^{D*}) \, \mathrm{e}^{c\,|z_4^{D*} + \phi|} \le \widetilde{c}(z^{D*}) \, \mathrm{e}^{c\,\overline{c}\,|\phi|}$ f.a.a. $x \in \Omega_0$. Thus the embedding result of Trudinger mentioned in the first step of this proof yields

$$\|h_1(\cdot, \phi)\|_{L^2(\Omega_0)} \le \widetilde{c}(z^{D*}) \, d(\|\phi\|_{H^1(\Omega_0)}) \le \widehat{c}.$$

Furthermore, using that $(z^{D*}, g^*, f^*) \in \mathcal{H}$ is fixed it results that $\mathcal{T} \in W^{-1,p}(G)$. Thus taking benefit from Grögers regularity result [9] applied to the equation $\mathcal{E}_0(\phi) = \mathcal{T}$ we obtain a $q_0 \in (2, p]$ such that $\phi \in W^{1,q_0}(G)$ and $\|\phi\|_{W^{1,q_0}} \le c_{q_0} \|\mathcal{T}\|_{W^{-1,p}(G)}$. Note that our assumption concerning the domain Ω and its boundary ensure that G is regular in the sense of Gröger [9].

3. The continuous embedding $W^{1,q_0}(\Omega) \hookrightarrow C(\overline{\Omega})$ ensures that $\|\phi\|_{C(\overline{\Omega})} \le c(q_0, w^*)$. Setting $Z_i^* = 0$, $i = 1, 2, 3$, $Z_4^* = \phi$ and using that $w^* \in Q$ we find a constant $\tau > 1$ such that that $(Z^*, z^{D*}) \in M_{q_0,\tau}$ and $F_{q_0,\tau}(Z^*, w^*) = 0$. In other words, (q_0, τ, Z^*, w^*) is a solution to Problem (P). Moreover, $z^* = Z^* + z^{D*}$ is a thermodynamic equilibrium of (1.5), (1.6), (1.7).

4. Uniqueness: Let $(\widetilde{q}, \widetilde{\tau}, \widetilde{Z}, w^*)$ be a solution to Problem (P) and let $\widetilde{z} = \widetilde{Z} + z^{D*}$. Then we have $(Z^*, z^{D*}) \in M_{q_0,\tau}$, $(\widetilde{Z}, z^{D*}) \in M_{\widetilde{q},\widetilde{\tau}}$ and $F_{q_0,\tau}(Z^*, w^*) = F_{\widetilde{q},\widetilde{\tau}}(\widetilde{Z}, w^*) = 0$. Let $\widehat{q} = \min\{q_0, \widetilde{q}\}$, $\widehat{\tau} = \max\{\tau, \widetilde{\tau}\}$. Then we have that (Z^*, z^{D*}), $(\widetilde{Z}, z^{D*}) \in M_{\widehat{q},\widehat{\tau}}$ and $F_{\widehat{q},\widehat{\tau}}(Z^*, w^*) = F_{\widehat{q},\widehat{\tau}}(\widetilde{Z}, w^*) = 0$. We test the last equation with $(\widetilde{Z}_1, \widetilde{Z}_2, \widetilde{Z}_3, 0)$. Since w^*, $w^* + (Z^*, 0, 0) \in Q$ we obtain

$$
\begin{aligned}
0 &= \langle F_{\widehat{q},\widehat{\tau}}(\widetilde{Z}, w^*) - F_{\widehat{q},\widehat{\tau}}(Z^*, w^*), (\widetilde{Z}_1, \widetilde{Z}_2, \widetilde{Z}_3, 0)\rangle_{X_{q'}} \\
&= \int_{\Omega_0} \sum_{i,k=1}^{3} a_{ik}(\cdot, \widetilde{z}) \nabla \widetilde{Z}_k \cdot \nabla \widetilde{Z}_i \,\mathrm{d}x + \int_{\Omega_1} \widetilde{a}_{33}(\cdot, \widetilde{z}_3) |\nabla \widetilde{Z}_3|^2 \,\mathrm{d}x \\
&\quad + \int_{\Omega_0} \widetilde{r}(\cdot, \widetilde{z}) \Big(\mathrm{e}^{\widetilde{Z}_1 + \widetilde{Z}_2} - 1\Big)(\widetilde{Z}_1 + \widetilde{Z}_2) \,\mathrm{d}x.
\end{aligned}
$$

Exploiting the assumption (A5) for $\widetilde{r}$ and the fact that $(\mathrm{e}^y - 1)y \geq 0$ we find

$$
\int_{\Omega_0} \sum_{i,k=1}^{3} a_{ik}(\cdot, \widetilde{z}) \nabla \widetilde{Z}_k \cdot \nabla \widetilde{Z}_i \,\mathrm{d}x + \int_{\Omega_1} \widetilde{a}_{33}(\cdot, \widetilde{z}_3) |\nabla \widetilde{Z}_3|^2 \,\mathrm{d}x \leq 0.
$$

According to (A2) we have $\widetilde{a}_{33}(x, \widetilde{z}) \geq c(\widetilde{z}) > 0$ and the matrix $(a_{ik}(x, \widetilde{z}))_{i,k=1,2,3}$ is strongly elliptic. Therefore we obtain $\nabla \widetilde{Z}_i = 0$, $i = 1, 2, 3$. And $|\Gamma_D| > 0$ supplies that $\widetilde{Z}_i = 0$, $i = 1, 2$, on Ω_0 and $\widetilde{Z}_3 = 0$ on Ω.

Finally, the test of $F_{\widehat{q},\widehat{\tau}}(\widetilde{Z}, w^*) - F_{\widehat{q},\widehat{\tau}}(Z^*, w^*) = 0$ with $(0, 0, 0, \widetilde{Z}_4 - Z_4^*)$ leads to $\widetilde{Z}_4 = Z_4^*$, since the operator $\mathcal{E}$ is strongly monotone. In summary we find $\widetilde{Z} = Z^*$, which gives the last assertion. □

Lemma 4.2 (Fredholm property of the linearization). *We assume* (A1) – (A5). *Let* $w^* = (z^{D*}, g^*, f^*) \in Q$ *be given. Let* (q_0, τ, Z^*, w^*) *be the equilibrium solution to Problem* (P) (*see Theorem 4.1*) *and let* $z^* = Z^* + z^{D*}$. *Then there exists a* $q_1 \in (2, q_0]$ *such that the operator* $\partial_Z F_{q_1,\tau}(Z^*, w^*)$ *is a Fredholm operator of index zero.*

Proof. Let $q \in (2, q_0]$ and $\overline{Z} \in X_q$. The linearization is given in (4.10) and has now to be calculated at (Z^*, w^*). Since $\nabla z_i^* = 0$, $i = 1, 2, 3$, $\mathrm{e}^{z_1^* + z_2^*} = 1$ and

$$
\partial_z R(\cdot, z^*) \cdot \overline{Z} = \partial_z \widetilde{r}(\cdot, z^*) \cdot \overline{Z} \left(\mathrm{e}^{z_1^* + z_2^*} - 1\right) + \widetilde{r}(\cdot, z^*) \mathrm{e}^{z_1^* + z_2^*} (\overline{Z}_1 + \overline{Z}_2),
$$

we obtain according to (4.10) that

$$
\begin{aligned}
\langle \partial_Z F_{q,\tau}(Z^*, w^*) \overline{Z}, \psi \rangle_{X_{q'}} &= \int_{\Omega_0} \Big(\sum_{i,k=1}^{3} a_{ik}(\cdot, z^*) \nabla \overline{Z}_k \cdot \nabla \psi_i + \varepsilon \nabla \overline{Z}_4 \cdot \nabla \psi_4 \Big) \mathrm{d}x \\
&\quad + \int_{\Omega_0} \Big(\widetilde{r}(\cdot, z^*)(\overline{Z}_1 + \overline{Z}_2)(\psi_1 + \psi_2) + \partial_z h_0(\cdot, z^*) \cdot \overline{Z}\, \psi_4 \Big) \mathrm{d}x \\
&\quad + \int_{\Omega_1} \Big(\widetilde{a}_{33}(\cdot, z_3^*) \nabla \overline{Z}_3 \cdot \nabla \psi_3 + \varepsilon \nabla \overline{Z}_4 \cdot \nabla \psi_4 \Big) \mathrm{d}x.
\end{aligned}
\tag{4.12}
$$

Now we follow ideas in the proof of Theorem 4.1 of Recke [11]. For $q \in (2, q_0]$ we write $\partial_Z F_{q,\tau}(Z^*, w^*)$ as sum $\partial_Z F_{q,\tau}(Z^*, w^*) = L_q + K_q$ with operators L_q, $K_q\colon X_q \to X^*_{q'}$, where

$$\begin{aligned}
\langle L_q \overline{Z}, \psi\rangle_{X_{q'}} &= \int_{\Omega_0} \Big(\sum_{i,k=1}^{3} a_{ik}(\cdot, z^*) \nabla \overline{Z}_k \cdot \nabla \psi_i + \varepsilon \nabla \overline{Z}_4 \cdot \nabla \psi_4 + \sum_{i=1}^{4} \overline{Z}_i \, \psi_i \Big) \mathrm{d}x \\
&\quad + \int_{\Omega_1} \Big(\widetilde{a}_{33}(\cdot, z_3^*) \nabla \overline{Z}_3 \cdot \nabla \psi_3 + \varepsilon \nabla \overline{Z}_4 \cdot \nabla \psi_4 + \sum_{i=3}^{4} \overline{Z}_i \, \psi_i \Big) \mathrm{d}x, \\
\langle K_q \overline{Z}, \psi\rangle_{X_{q'}} &= \int_{\Omega_0} \Big\{ \widetilde{r}(\cdot, z^*) (\overline{Z}_1 + \overline{Z}_2)(\psi_1 + \psi_2) + \partial_z h_0(\cdot, z^*) \cdot \overline{Z} \, \psi_4 - \sum_{i=1}^{4} \overline{Z}_i \, \psi_i \Big\} \, \mathrm{d}x \\
&\quad - \int_{\Omega_1} \sum_{i=3}^{4} \overline{Z}_i \, \psi_i \, \mathrm{d}x \, , \quad \psi \in X_{q'}.
\end{aligned}$$

The operator K_q is compact because of the compact embedding of $W^{1,q}(\Omega)$ into $L^\infty(\Omega)$. The operator L_q is injective. Next, we apply Theorem 5.1 from Section 5. We set $d_{ik} = a_{ik}(\cdot, z^*)$, $i, k = 1, 2, 3$, $d_{4k} = d_{k4} = 0$, $k = 1, 2, 3$, $d_{44} = \varepsilon$, $d_i = 1$, $i = 1, \ldots, 4$, in Ω_0 and $\widetilde{d}_{33} = \widetilde{a}_{33}(\cdot, z_3^*)$, $\widetilde{d}_{43} = \widetilde{d}_{34} = 0$, $\widetilde{d}_{44} = \varepsilon$, $\widetilde{d}_i = 1$, $i = 3, 4$, in Ω_1. Due to our assumptions (A2), (A3) the properties (B) at the beginning of Section 5 are fulfilled. Since L_q is the restriction of $\overline{A}$ (see (5.18)) to X_q, Theorem 5.1 guarantees the existence of an exponent $q_1 \in (2, q_0]$ such that the operator L_{q_1} is surjective. Then by Banach's open mapping theorem and Nikolsky's criterion for Fredholm operators the assertion of the lemma follows. □

Lemma 4.3 (Injectivity of the linearization). *We assume* (A1) – (A5). *Let the vector of data $w^* = (z^{D*}, g^*, f^*) \in Q$ be given, and let (q_0, τ, Z^*, w^*) be the equilibrium solution to Problem* (P), *and $z^* = Z^* + z^{D*}$ (see Theorem 4.1). Then the Fréchet derivative $\partial_Z F_{q_1,\tau}(Z^*, w^*)\colon X_{q_1} \to X^*_{q_1'}$ is injective, where q_1 is chosen as in Lemma 4.2.*

Proof. It is sufficient to prove the injectivity of the operator on X_2. The derivative has the form (4.12). Let $\partial_Z F_{q_1,\tau}(Z^*, w^*)\overline{Z} = 0$, $\overline{Z} \in X_2$.

We test this equation with $\psi = (\overline{Z}_1, \overline{Z}_2, \overline{Z}_3, 0)$ and take into account the strong ellipticity condition for $(a_{ik}(x, z^*))_{i,k=1,2,3}$, the fact that $|\Gamma_D| > 0$ and the property that $\widetilde{r}(z^*) \geq 0$ and get that $\overline{Z}_i = 0$, $i = 1, 2, 3$. Next, we use the test function $\psi = (0, 0, 0, \overline{Z}_4)$ and obtain

$$\int_{\Omega} \Big\{ \varepsilon \, |\nabla \overline{Z}_4|^2 + \frac{\partial}{\partial z_4} h_0(\cdot, z^*) \, \overline{Z}_4^2 \Big\} \mathrm{d}x = 0.$$

Since h_0 is continuously differentiable and monotonic increasing in the argument z_4 (see (A4)) we have $\frac{\partial}{\partial z_4} h_0(x, z^*) \geq 0$ a.e. on Ω which together with $\varepsilon \geq \varepsilon_0$ a.e. on Ω and $|\Gamma_D| > 0$ leads to $\overline{Z}_4 = 0$. Thus the injectivity of $\partial_Z F_{q_1,\tau}(Z^*, w^*)\colon X_{q_1} \to X^*_{q_1'}$ follows, too. □

Theorem 4.2 (Local existence and uniqueness of steady states). *We assume* (A1) – (A5). *Let $w^* = (z^{D*}, g^*, f^*) \in Q$ be given, and let (q_0, τ, Z^*, w^*) be the equilibrium solution to Problem* (P), *and $z^* = Z^* + z^{D*}$ (see Theorem 4.1).*

Then there exists a $q_1 \in (2, q_0]$ *such that the following assertion holds: There exist neighbourhoods* $U \subset X_{q_1}$ *of* Z^* *and* $W \subset \mathcal{H}$ *of* $w^* = (z^{D*}, g^*, f^*)$ *and a* C^1*-map* $\Phi\colon W \to U$ *such that* $Z = \Phi(w)$ *iff*

$$F_{q_1,\tau}(Z, w) = 0, \quad (Z, z^D) \in M_{q_1,\tau}, \quad Z \in U, \quad w = (z^D, g, f) \in W.$$

Proof. According to Lemma 4.2 and Lemma 4.3 there exists a q_1 such that the operator $\partial_Z F_{q_1,\tau}(Z^*, w^*)\colon X_{q_1} \to X^*_{q_1'}$ is an injective Fredholm Operator of index zero. Therefore the assertion of the theorem is a direct consequence of the Implicit Function Theorem. □

Finally, let us draw a conclusion from Theorem 4.2. First, we define the sets

$$\begin{aligned} Q_1 = \Big\{ & w = (z^D, g, f) \in \mathcal{H}\colon g_i^{N0} = 0,\ i = 1,2,3,\ g_3^{N1} = 0, \\ & \int_{\Gamma_D} (z_1^D + z_2^D)\, \mathrm{d}\Gamma = 0,\ z_3^D < 0 \Big\}, \\ Q_2 = \Big\{ & w = (z^D, g, f) \in \mathcal{H}\colon z_3^D < 0 \Big\}. \end{aligned}$$

Obviously $Q \subset Q_1 \subset Q_2$ holds, but Q_1 and Q_2 contain also elements which are not compatible with thermodynamic equilibria.

Corollary 4.1 *We assume* (A1) – (A5).
i) *Let* $w = (z^D, g, f) \in Q_1$ *be given. Then there are constants* $q \in (2, p]$, $\tau > 1$, $\epsilon > 0$ *such that the following assertions hold: If*

$$\|\nabla z_i^D\|_{L^p(\Omega_0)} < \epsilon, \quad i = 1,2, \quad \|\nabla z_3^D\|_{L^p(\Omega)} < \epsilon \tag{4.13}$$

then there exists a $Z \in X_q$ *such that* (q, τ, Z, w) *is a solution to* (P)*. This solution lies in a neighbourhood of an equilibrium solution* (q, τ, Z^*, w^*) *to* (P)*, and in this neighbourhood there are no solutions* $(q, \tau, \widetilde{Z}, w)$ *with* $\widetilde{Z} \neq Z$.

ii) *Let* $w = (z^D, g, f) \in Q_2$ *be given. Then there are constants* $q \in (2, p]$, $\tau > 1$, $\epsilon > 0$ *such that the following assertions hold: If*

$$\begin{aligned} & \|\nabla z_i^D\|_{L^p(\Omega_0)} < \epsilon, \quad i = 1,2, \quad \|\nabla z_3^D\|_{L^p(\Omega)} < \epsilon, \\ & \|z_1^D + z_2^D\|_{L^1(\Gamma_D)} < \epsilon, \\ & \|g_i^{N0}\|_{L^\infty(\Gamma_{N0})} \le \epsilon, \quad i = 1,2,3, \quad \|g_3^{N1}\|_{L^\infty(\Gamma_{N1})} \le \epsilon, \end{aligned} \tag{4.14}$$

then there exists a $Z \in X_q$ *such that* (q, τ, Z, w) *is a solution to* (P)*. This solution lies in a neighbourhood of an equilibrium solution* (q, τ, Z^*, w^*) *to* (P)*, and in this neighbourhood there are no solutions* $(q, \tau, \widetilde{Z}, w)$ *with* $\widetilde{Z} \neq Z$.

Proof. 1. Let $w = (z^D, g, f) \in Q_1$ be given. We define

$$z_i^{D*} = \frac{1}{|\Gamma_D|} \int_{\Gamma_D} z_i^D \, \mathrm{d}\Gamma, \ i = 1,2,3, \quad z_4^{D*} = z_4^D, \quad w^* = (z^{D*}, g, f)$$

and find that $w^* \in Q$. Let (q_0, τ, Z^*, w^*) be the corresponding equilibrium solution to (P). Because of Theorem 4.2 there exist constants $q \in (2, q_0]$, $\epsilon' > 0$ such that the equation $F_{q,\tau}(Z, w) = 0$ has a locally unique solution $Z \in X_q$ if

$$\|w - w^*\|_{\mathcal{H}} = \sum_{i=1}^{2} \|z_i^D - z_i^{D*}\|_{W^{1,p}(\Omega_0)} + \|z_3^D - z_3^{D*}\|_{W^{1,p}(\Omega)} < \epsilon'. \tag{4.15}$$

Since for $i = 1, 2, 3$ the mean values of $z_i^D - z_i^{D*}$ on Γ_D vanish we can apply the Friedrich inequality to obtain

$$\begin{aligned} \|z_i^D - z_i^{D*}\|_{W^{1,p}(\Omega_0)} &\le c\, \|\nabla z_i^D\|_{L^p(\Omega_0)}, \quad i = 1, 2, \\ \|z_3^D - z_3^{D*}\|_{W^{1,p}(\Omega)} &\le c\, \|\nabla z_3^D\|_{L^p(\Omega)}. \end{aligned}$$

Choosing ϵ in (4.13) sufficiently small the inequality (4.15) can be fulfilled.

2. We decompose $\mathbb{R}^2 = \mathcal{S} \oplus \mathcal{S}^\perp$, where $\mathcal{S} = \operatorname{span}\{(1,1)\}$, $\mathcal{S}^\perp = \operatorname{span}\{(1,-1)\}$. The corresponding projection operators are denoted by $\Pi_{\mathcal{S}} \colon \mathbb{R}^2 \to \mathcal{S}$ and $\Pi_{\mathcal{S}^\perp} \colon \mathbb{R}^2 \to \mathcal{S}^\perp$. Obviously, there is a constant $c > 0$ such that

$$\|\lambda - \Pi_{\mathcal{S}^\perp}\lambda\|_{\mathbb{R}^2} = \|\Pi_{\mathcal{S}}\lambda\|_{\mathbb{R}^2} \le c|\lambda_1 + \lambda_2| \quad \forall \lambda \in \mathbb{R}^2. \tag{4.16}$$

Let $w = (z^D, g, f) \in Q_2$ be given. We define

$$\begin{aligned} \overline{z}_i^D &= \frac{1}{|\Gamma_D|} \int_{\Gamma_D} z_i^D \, \mathrm{d}\Gamma, \ i = 1, 2, 3, \\ z^{D*} &= (z_1^{D*}, z_2^{D*}, z_3^{D*}, z_4^{D*}) = \Big(\Pi_{\mathcal{S}^\perp}(\overline{z}_1^D, \overline{z}_2^D), \overline{z}_3^D, z_4^D\Big), \\ w^* &= (z^{D*}, (0, 0, 0, g_4^{N0}, 0, g_4^{N1}), f) \end{aligned}$$

and find again that $w^* \in Q$. Let (q_0, τ, Z^*, w^*) be the corresponding equilibrium solution to (P). Because of Theorem 4.2 there are constants $q \in (2, q_0]$, $\epsilon' > 0$ such that the equation $F_{q,\tau}(Z, w) = 0$ has a locally unique solution $Z \in X_q$ if

$$\begin{aligned} \|w - w^*\|_{\mathcal{H}} = \sum_{i=1}^{2} &\Big\{ \|z_i^D - z_i^{D*}\|_{W^{1,p}(\Omega_0)} + \|g_i\|_{L^\infty(\Gamma_{N0})} \Big\} \\ &+ \|z_3^D - z_3^{D*}\|_{W^{1,p}(\Omega)} + \|g_3^{N0}\|_{L^\infty(\Gamma_{N0})} + \|g_3^{N1}\|_{L^\infty(\Gamma_{N1})} < \epsilon'. \end{aligned} \tag{4.17}$$

From the Friedrich inequality and inequality (4.16) it follows that

$$\begin{aligned} \|w - w^*\|_{\mathcal{H}} \le c \Big(&\sum_{i=1}^{2} \Big\{ \|\nabla z_i^D\|_{L^p(\Omega_0)} + \|g_i^{N0}\|_{L^\infty(\Gamma_{N0})} \Big\} + \|z_1^D + z_2^D\|_{L^1(\Gamma_D)} \\ &+ \|\nabla z_3^D\|_{L^p(\Omega)} + \|g_3^{N0}\|_{L^\infty(\Gamma_{N0})} + \|g_3^{N1}\|_{L^\infty(\Gamma_{N1})} \Big), \end{aligned}$$

and ϵ in (4.14) can be chosen such that (4.17) is fulfilled. $\square$

The assertions of Corollary 4.1 can be interpreted as follows. Let the source terms for the Poisson equation (i.e. f, z_4^D, g_4^{N0}, g_4^{N1}) be given. Then the stationary energy model has a solution, if the driving forces for the fluxes induced by the boundary data (i.e. the gradients $\nabla z_1^D, \nabla z_2^D, \nabla z_3^D$), the driving forces for the generation-recombination of electrons and holes on the boundary (i.e. the affinities $z_1^D + z_2^D$ on Γ_D) and the prescribed fluxes on the boundary (i.e. g_1^{N0}, g_2^{N0}, g_3^{N0} and g_3^{N1}) are small enough. This solution is locally unique.

Remark 4.1 If all equations are defined on the same domain Ω and mixed boundary conditions are formulated on Γ_D and Γ_N analogous results concerning the stationary energy model (1.5), (1.6), (1.7) are obtained by setting formally $\Omega_0 = \Omega$, $\Omega_1 = \emptyset$, $\Gamma_{N01} = \Gamma_{N1} = \emptyset$, $\Gamma_{N0} = \Gamma_N$ and $G_0 = G$.

Remark 4.2 Theorem 4.2 gives a local existence and uniqueness result for the stationary energy model (1.5), (1.6), (1.7) for semiconductor devices in two space dimensions. Moreover, the different domains of definition of the relevant model equations are taken into account. For the case that Ω and Ω_0 coincide we have investigated an energy model containing incompletely ionized impurities in [8] and a multi species version of the above energy model in [7].

Remark 4.3 Gröger, Recke [10] study quasilinear second order elliptic systems given on the same domain, where for the divers equations the partition of the mixed boundary conditions into Dirichlet and Neumann parts differs. There is shown that such boundary value problems with triangular main part generate Fredholm maps between appropriate Sobolev-Campanato spaces and that the Implicit Function Theorem can be applied to this situation.

Remark 4.4 If in the energy model (1.1), (1.3) the temperature is considered as a constant positive parameter and the balance equation for the density of the total energy is omitted, then the remaining equations form a drift-diffusion model (van Roosbroeck equations). For the case that Ω and Ω_0 coincide there is a lot of papers dealing with this model (e.g. Chen, Jüngel [2] (here electron hole scattering is involved), [3, 4, 5] and papers cited there). But Gajewski, Gröger [6] considered the Poisson equation in a larger domain Ω containing the domain of definition $\Omega_0 \subset \Omega$ of the continuity equations, too.

5 A surjectivity result for a system of second order linear elliptic equations defined on different domains

First, we collect some results concerning equivalent norms on cross products of spaces, its duals and on properties of the duality map on cross products (see Lemmata 5.1, 5.2, 5.3). These results enable us to adapt results of Gröger [9] to systems of elliptic equations with different domains of definition. Second, we state a surjectivity property of operators related to strongly coupled linear elliptic equations with homogeneous mixed boundary conditions and different domains of definition. We use the notation, spaces and norms of

Section 3. We consider the operator $\overline{A}\colon X_2 \to X_2^*$ defined by

$$\begin{aligned}\langle \overline{A}z, \overline{z}\rangle_{X_2} = &\int_{\Omega_0} \Big\{ \sum_{i,k=1}^{4} d_{ik}\nabla z_k \cdot \nabla \overline{z}_i + \sum_{i=1}^{4} d_i z_i \overline{z}_i \Big\}\,\mathrm{d}x \\ &+ \int_{\Omega_1} \Big\{ \sum_{i,k=3}^{4} \widetilde{d}_{ik}\nabla z_k \cdot \nabla \overline{z}_i + \sum_{i=3}^{4} \widetilde{d}_i z_i \overline{z}_i \Big\}\,\mathrm{d}x, \quad z,\, \overline{z} \in X_2.\end{aligned} \tag{5.18}$$

Concerning the coefficient functions we suppose

(B) $d_{ik},\, d_i \in L^\infty(\Omega_0),\ i,k = 1,\dots,4,\ \widetilde{d}_{ik},\, \widetilde{d}_i \in L^\infty(\Omega_1),\ i,k = 3,4.$

There exist $M,\, m > 0$ such that

$$\sum_{i,k=1}^{4} d_{ik}t_k t_i \ge m|t|^2_{\mathbb{R}^4}, \quad \sum_{i=1}^{4}\Big|\sum_{k=1}^{4} d_{ik}t_k\Big|^2 \le M^2|t|^2_{\mathbb{R}^4}\ \forall t \in \mathbb{R}^4,\ \text{a.e. on } \Omega_0,$$

$$\sum_{i,k=3}^{4} \widetilde{d}_{ik}t_k t_i \ge m|t|^2_{\mathbb{R}^2}, \quad \sum_{i=3}^{4}\Big|\sum_{k=3}^{4} \widetilde{d}_{ik}t_k\Big|^2 \le M^2|t|^2_{\mathbb{R}^2}\ \forall t \in \mathbb{R}^2,\ \text{a.e. on } \Omega_1,$$

$$m \le d_i \le M,\ i = 1,\dots,4,\ \text{a.e. on } \Omega_0,\ \ m \le \widetilde{d}_i \le M,\ i = 3,4,\ \text{a.e. on } \Omega_1.$$

For $s \ge 2$ the operator $\overline{A}$ maps X_s continuously into $X_{s'}^*$. In Theorem 5.1 we show that for $s \ge 2$ and sufficiently near to 2 the operator $\overline{A}$ from X_s to $X_{s'}^*$ is onto, too. We start with some preliminary results.

Lemma 5.1 *Let X be a Banach space with norm $\|\cdot\|$, and let $\|\cdot\|_0$ be an equivalent norm,*

$$c_1\|u\| \le \|u\|_0 \le c_2\|u\| \quad \forall u \in X.$$

Let $\|\cdot\|_*$ and $\|\cdot\|_{0*}$ denote the canonical norms in the dual space X^*. Then

$$\tfrac{1}{c_2}\|h\|_* \le \|h\|_{0*} \le \tfrac{1}{c_1}\|h\|_* \quad \forall h \in X^*.$$

Lemma 5.2 *For all $k \in \mathbb{N}$, all $a = (a_1,\dots,a_k) \in \mathbb{R}^k_+$ and all $p \in (1,\infty)$ there holds the estimate*

$$c_1(p)\Big(\sum_{i=1}^{k} a_i^2\Big)^{\frac{p}{2}} \le \sum_{i=1}^{k} a_i^p \le c_2(p)\Big(\sum_{i=1}^{k} a_i^2\Big)^{\frac{p}{2}},$$

where

$$\begin{aligned} p < 2:&\quad c_1(p) = 1, &&c_2(p) = k^{1-\frac{p}{2}}\\ p = 2:&\quad c_1(p) = 1, &&c_2(p) = 1\\ p > 2:&\quad c_1(p) = k^{1-\frac{p}{2}}, &&c_2(p) = 1.\end{aligned}$$

Note that for every fixed $k \in \mathbb{N}$ the functions c_1 and c_2 are continuous functions of p and that $c_1(p),\, c_2(p) \to 1$ if $p \to 2$.

Lemma 5.3 *Let $p \geq 2$, p' the dual exponent and let $B_{i,p}$, $B^*_{i,p'}$, $i = 1, \ldots, m$, be a scale of Banach spaces. On the cross products $B_p = B_{1,p} \times \cdots \times B_{m,p}$ and $B^*_{p'} = B^*_{1,p'} \times \cdots \times B^*_{m,p'}$ we use the norms*

$$\|u\|^p_{B_p} = \sum_{i=1}^m \|u_i\|^p_{B_{i,p}} \text{ analogously } B_{p'}, \qquad \|h\|^{p'}_{B^*_p} = \sum_{i=1}^m \|h_i\|^{p'}_{B^*_{i,p}} \text{ analogously } B^*_{p'}.$$

Suppose that $D_i \colon B^*_{i,p'} \to B_{i,p}$ are linear continuous maps with $\|D_i\| = M_{i,p}$. Then $D \colon B^*_{p'} \to B_p$ with $Dh = (D_1 h_1, \ldots, D_m h_m)$ is a linear continuous map with

$$\|D\| \leq \max_{i=1,\ldots,m} \Big\{ M_{i,p} \Big\}.$$

Proof. Because of

$$\|D\| = \sup_{h \in B^*_{p'},\, \|h\|_{B^*_{p'}} \leq 1} \|Dh\|_{B_p} = \sup_{h \in B^*_{p'},\, \|h\|_{B^*_{p'}} \leq 1} \; \sup_{\psi \in B^*_p,\, \|\psi\|_{B^*_p} \leq 1} \langle \psi, Dh \rangle_{B_p}$$

and the estimate

$$\begin{aligned} \langle \psi, Dh \rangle_{B_p} &= \sum_{i=1}^m \langle \psi_i, D_i h_i \rangle_{B_{i,p}} \leq \sum_{i=1}^m M_{i,p} \|\psi_i\|_{B^*_{i,p}} \|h_i\|_{B^*_{i,p'}} \\ &\leq \max_{i=1,\ldots,m} M_{i,p} \sum_{i=1}^m \|\psi_i\|_{B^*_{i,p}} \|h_i\|_{B^*_{i,p'}} \\ &\leq \max_{i=1,\ldots,m} M_{i,p} \Big(\sum_{i=1}^m \|\psi_i\|^{p'}_{B^*_{i,p}} \Big)^{\frac{1}{p'}} \Big(\sum_{i=1}^m \|h_i\|^p_{B^*_{i,p'}} \Big)^{\frac{1}{p}} \\ &\leq \max_{i=1,\ldots,m} M_{i,p} \|\psi\|_{B^*_p} \|h\|_{B^*_{p'}} \end{aligned}$$

we obtain $\|D\| \leq \max_{i=1,\ldots,m} M_{i,p}$. □

Remember the definition of function spaces at the beginning of Section 3. We define the operator $L_G \colon W^{1,2}_0(G) \to L^2(\Omega)^3 = Y^2_\Omega$ by $L_G y = (y, \nabla y)$, $y \in W^{1,2}_0(G)$. Then $L^*_G L_G = J_G$, where J_G denotes the duality map of the space $W^{1,2}_0(G)$,

$$\langle J_G\, y, \phi \rangle_{W^{1,2}_0(G)} = \int_\Omega (y\phi + \nabla y \cdot \nabla \phi)\,\mathrm{d}x, \quad y, \phi \in W^{1,2}_0(G).$$

If $s > 2$ then L_G maps $W^{1,s}_0(G)$ continuously into Y^s_Ω, and L^*_G maps Y^s_Ω continuously into $W^{1,s'}_0(G)^*$. Moreover, for $s > 2$ we obtain that J_G maps $W^{1,s}_0(G)$ into $W^{1,s'}_0(G)^*$ and that J_G is continuous as a map from $W^{1,s}_0(G)$ into $W^{1,s'}_0(G)^*$. We use M^G_s as abbreviation for

$$M^G_s = \sup \Big\{ \|y\|_{W^{1,s}_0(G)} \colon y \in W^{1,s}_0(G),\ \|J_G\, y\|_{W^{1,s'}_0(G)^*} \leq 1 \Big\}.$$

Since by assumption (A1) the set $G \subset \mathbb{R}^2$ is regular in the sense of Gröger [9] there exists $r_G > 2$ such that J_G maps $W^{1,r_G}_0(G)$ onto $W^{1,r'_G}_0(G)^*$. Moreover, $M^G_2 = 1$, and for $s \in [2, r_G]$ the mappings are onto, too, and we have

$$M^G_s \leq {M^G_{r_G}}^\theta, \quad \text{where } \theta \text{ is given by the relation } \quad \frac{1}{s} = \frac{1-\theta}{2} + \frac{\theta}{r_G} \tag{5.19}$$

(see Sect. 3 and Lemma 1 in [9]).

Analogously we define for the regular set G_0 operators L_{G_0}, J_{G_0} and obtain a corresponding exponent r_{G_0} and quantities $M_s^{G_0}$. Now we define the four component operators L, J working componentwise

$$\begin{aligned}
&L : X_2 \to \mathcal{L}^2, && Lz = \big(L_{G_0}z_1, L_{G_0}z_2, L_G z_3, L_G z_4\big), && z \in X_2,\\
&L^* : \mathcal{L}^2 \to X_2^*, && L^*u = \big(L^*_{G_0}u_1, L^*_{G_0}u_2, L^*_G u_3, L^*_G u_4\big), && u \in \mathcal{L}^2,\\
&J : X_2 \to X_2^*, && Jz = \big(J_{G_0}z_1, J_{G_0}z_2, J_G z_3, J_G z_4\big), && z \in X_2.
\end{aligned}$$

Note that for $s > 2$ the (restricted) operators $L\colon X_s \to \mathcal{L}^s$ as well as $L^*\colon \mathcal{L}^s \to X^*_{s'}$ are linear continuous maps with norm less or equal to one. Moreover, it results that J maps X_s into $X^*_{s'}$ and that J is continuous as a map from X_s into $X^*_{s'}$, too. We will use M_s as abbreviation for

$$M_s = \sup\Big\{\|z\|_{X_s} : z \in X_s,\ \|Jz\|_{X^*_{s'}} \le 1\Big\}.$$

Let $\widehat{r} := \min\{r_G,\, r_{G_0}\}$. Then

$$\begin{aligned}
M^{G_0}_{\widehat{r}} &= \Big(M^{G_0}_{r_{G_0}}\Big)^{\widehat{\theta}}, \quad \text{where } \frac{1}{\widehat{r}} = \frac{\widehat{\theta}}{r_{G_0}} + \frac{1-\widehat{\theta}}{2},\\
M^{G}_{\widehat{r}} &= \Big(M^{G}_{r_G}\Big)^{\widetilde{\theta}}, \quad \text{where } \frac{1}{\widehat{r}} = \frac{\widetilde{\theta}}{r_G} + \frac{1-\widetilde{\theta}}{2}.
\end{aligned} \tag{5.20}$$

Lemma 5.4 *We assume* (A1). *For $\widehat{r}$ the map J is from $X_{\widehat{r}}$ onto $X^*_{\widehat{r}'}$. Moreover, for all $s \in [2, \widehat{r}]$ the estimate*

$$M_s \le \max\Big\{\Big(M^{G_0}_{\widehat{r}}\Big)^{\theta}, \big(M^{G}_{\widehat{r}}\big)^{\theta}\Big\} \quad \textit{with } \frac{1}{s} = \frac{\theta}{\widehat{r}} + \frac{1-\theta}{2}$$

is fulfilled.

Proof. For $s \in [2, \widehat{r}]$ the onto-properties of J_{G_0} and J_G supply the onto-property of the four component map J. The proof of the inequality is based on Lemma 5.3 and uses (5.19), (5.20). Setting $m = 4$,

$$\begin{aligned}
&B_{i,s} = W^{1,s}_0(G_0), && D_i = \big(J_{G_0}\big)^{-1}\colon B^*_{i,s'} \to B_{i,s}, && M_{i,s} = \Big(M^{G_0}_{\widehat{r}}\Big)^{\theta}, && i = 1,2,\\
&B_{i,s} = W^{1,s}_0(G), && D_i = \big(J_{G}\big)^{-1}\colon B^*_{i,s'} \to B_{i,s}, && M_{i,s} = \Big(M^{G}_{\widehat{r}}\Big)^{\theta}, && i = 3,4,
\end{aligned}$$

where

$$\frac{1}{s} = \frac{\theta}{\widehat{r}} + \frac{1-\theta}{2}$$

we can apply Lemma 5.3. Then the assertion of the lemma follows. □

According to Lemma 5.4 it results $M_2 = 1$ and the quantity M_s depends continuously on s such that $M_s \to 1$ as $s \to 2$.

We introduce the function $b = (b_1, b_2, b_3, b_4)$, where $b_i \colon \Omega_0 \times (\mathbb{R}^3)^4 \to \mathbb{R}^3$, $i = 1, 2$, $b_i \colon \Omega \times (\mathbb{R}^3)^4 \to \mathbb{R}^3$, $i = 3, 4$, are defined by

$$b_i(x, \eta_1, \eta_2, \eta_3, \eta_4) = \Big(d_i(x)\eta_i^1, \sum_{k=1}^4 d_{ik}(x)\eta_k^2, \sum_{k=1}^4 d_{ik}(x)\eta_k^3\Big), \quad i = 1, 2,$$

$$b_i(x, \eta_1, \eta_2, \eta_3, \eta_4) = \begin{cases} \Big(d_i(x)\eta_i^1, \sum_{k=1}^4 d_{ik}(x)\eta_k^2, \sum_{k=1}^4 d_{ik}(x)\eta_k^3\Big) & \text{if } x \in \Omega_0 \\ \Big(\widetilde{d}_i(x)\eta_i^1, \sum_{k=3}^4 \widetilde{d}_{ik}(x)\eta_i^2, \sum_{k=3}^4 \widetilde{d}_{ik}(x)\eta_i^3\Big) & \text{if } x \in \Omega \setminus \Omega_0, \end{cases} \qquad i = 3, 4,$$

$$\eta = (\eta_1, \eta_2, \eta_3, \eta_4) \in (\mathbb{R}^3)^4, \ \eta_i = (\eta_i^1, \eta_i^2, \eta_i^3), \ i = 1, \ldots, 4.$$

Here d_{ik}, $\widetilde{d}_{ik}$, d_i and $\widetilde{d}_i$ are the coefficient functions from (5.18). $\mathbb{R}^3$ as well as $(\mathbb{R}^3)^4$ is considered with the usual Euclidean norm. Clearly, b is linear in the argument η and according to assumption (B) the estimates

$$\begin{aligned} &b(x, \eta) \cdot \eta \geq m\,|\eta|^2_{(\mathbb{R}^3)^4}, \\ &|b(x, \eta)|^2_{(\mathbb{R}^3)^4} \leq M^2 |\eta|^2_{(\mathbb{R}^3)^4} \quad \forall x \in \Omega_0, \end{aligned} \tag{5.21}$$

$$\begin{aligned} &b_3(x, \eta) \cdot \eta_3 + b_4(x, \eta) \cdot \eta_4 \geq m\,(|\eta_3|^2_{\mathbb{R}^3} + |\eta_4|^2_{\mathbb{R}^3}), \\ &|b_3(x, \eta)|^2_{\mathbb{R}^3} + |b_4(x, \eta)|^2_{\mathbb{R}^3} \leq M^2(|\eta_3|^2_{\mathbb{R}^3} + |\eta_4|^2_{\mathbb{R}^3}) \quad \forall x \in \Omega_1 \end{aligned} \tag{5.22}$$

are fulfilled. We set $\alpha = m/M^2$ and define on $\mathcal{L}^2$ the operator B_2 by

$$(B_2\, y)(x) = y(x) - \alpha\, b(x, y(x)), \quad y \in \mathcal{L}^2.$$

Now we restrict this operator B_2 to the space $\mathcal{L}^s$ with $s > 2$ and obtain a linear mapping $B_s = B_2|_{\mathcal{L}^s}$ from $\mathcal{L}^s$ into itself. In the next estimate we make use of an equivalent norm of $\mathcal{L}^s$:

$$\|y\|_{0,\mathcal{L}^s} = \left(\int_{\Omega_0} (|y(x)|^2_{(\mathbb{R}^3)^4})^{s/2}\, \mathrm{d}x + \int_{\Omega_1} \Big(|y_3(x)|^2_{\mathbb{R}^3} + |y_4(x)|^2_{\mathbb{R}^3}\Big)^{s/2} \mathrm{d}x \right)^{1/s}, \quad y \in \mathcal{L}^s.$$

Using (5.21), (5.22) the norm of the mapping B_s can be estimated as follows

$$\begin{aligned} \|B_s\, y\|^s_{0,\mathcal{L}^s} &= \int_{\Omega_0} (|(B_s\, y)(x)|^2_{(\mathbb{R}^3)^4})^{s/2}\, \mathrm{d}x + \int_{\Omega_1} \Big(|(B_s\, y)_3(x)|^2_{\mathbb{R}^3} + |(B_s\, y)_4(x)|^2_{\mathbb{R}^3}\Big)^{s/2} \mathrm{d}x \\ &= \int_{\Omega_0} \Big(|y|^2_{(\mathbb{R}^3)^4} + \alpha^2\, |b(\cdot, y(\cdot))|^2_{(\mathbb{R}^3)^4} - 2\alpha\, b(x, y(\cdot)) \cdot y\Big)^{s/2} \mathrm{d}x \\ &+ \int_{\Omega_1} \Big(\sum_{i=3,4} \Big(|y_i(x)|^2_{\mathbb{R}^3} + \alpha^2\, |b_i(\cdot, y(\cdot))|^2_{\mathbb{R}^3} - 2\alpha\, b_i(\cdot, y(\cdot)) \cdot y_i\Big)\Big)^{s/2} \mathrm{d}x \\ &\leq \Big(1 - \frac{m^2}{M^2}\Big)^{s/2} \Big(\int_{\Omega_0} (|y(x)|^2_{(\mathbb{R}^3)^4})^{s/2} \mathrm{d}x + \int_{\Omega_1} \Big(|y_3(x)|^2_{\mathbb{R}^3} + |y_4(x)|^2_{\mathbb{R}^3}\Big)^{s/2} \mathrm{d}x \Big) \\ &\leq \Big(1 - \frac{m^2}{M^2}\Big)^{s/2} \|y\|^s_{0,\mathcal{L}^s} \qquad \forall y \in \mathcal{L}^s. \end{aligned}$$

Lemma 5.2 ensures the estimate

$$4^{1/s-1/2}\|y\|_{0,\mathcal{L}^s} \le \|y\|_{\mathcal{L}^s} \le \|y\|_{0,\mathcal{L}^s} \quad \forall y \in \mathcal{L}^s,$$

which leads to

$$\begin{aligned} \|B_s\, y\|_{\mathcal{L}^s} \le \|B_s\, y\|_{0,\mathcal{L}^s} &\le \Big(1-\frac{m^2}{M^2}\Big)^{1/2}\|y\|_{0,\mathcal{L}^s} \\ &\le 4^{1/2-1/s}\Big(1-\frac{m^2}{M^2}\Big)^{1/2}\|y\|_{\mathcal{L}^s} \quad \forall y \in \mathcal{L}^s. \end{aligned} \tag{5.23}$$

Theorem 5.1 *We suppose* (A1) *and* (B). *Then, the operator* $\overline{A}$ *defined in* (5.18) *maps* X_s *onto the space* $X^*_{s'}$ *provided that* $s \in [2, q_0]$ *and*

$$4^{1/2-1/s}\, M_s\Big(1-\frac{m^2}{M^2}\Big)^{1/2} < 1.$$

In particular, there exists a $q_1 \in (2, q_0]$ *such that* $\overline{A}$ *maps* X_{q_1} *onto the space* $X^*_{q_1{}'}$.

Proof. Here we adapt the proof of Theorem 1 in Gröger [9] to strongly coupled systems of elliptic equations with different domains of definition. Note that for $z \in X_s$ we have $\frac{m}{M^2}J^{-1}\overline{A}z = z - J^{-1}L^*B_sLz$. For every fixed $h \in X^*_{s'}$, $s \in [2, q]$, we define the operator $Q_h : X_s \to X_s$ by

$$Q_h z := J^{-1}\Big(L^*B_sLz + \frac{m}{M^2}h\Big) = z - \frac{m}{M^2}J^{-1}(\overline{A}z - h), \quad z \in X_s.$$

Due to the properties of the operators B_s, L, L^* and J^{-1} (in particular see (5.23) and Lemma 5.4) we find

$$\begin{aligned} \|Q_h z - Q_h \overline{z}\|_{X_s} &\le M_s\|L^*\|_{\mathcal{L}(\mathcal{L}^s, X^*_{s'})}\|B_s\|_{\mathcal{L}(\mathcal{L}^s,\mathcal{L}^s)}\|L\|_{\mathcal{L}(X_s,\mathcal{L}^s)}\|z-\overline{z}\|_{X_s} \\ &\le 4^{1/2-1/s}\, M_s\Big(1-\frac{m^2}{M^2}\Big)^{1/2}\|z-\overline{z}\|_{X_s}. \end{aligned}$$

Note that $4^{1/2-1/s}\, M_s\Big(1-\frac{m^2}{M^2}\Big)^{1/2}$ continuously depends on s and

$$4^{1/2-1/s}\, M_s\Big(1-\frac{m^2}{M^2}\Big)^{1/2} \to \Big(1-\frac{m^2}{M^2}\Big)^{1/2} < 1 \text{ for } s \to 2.$$

Thus, there exists an exponent $s_0 \in (2, q_0]$ such that for all $s \in [2, s_0)$, we have

$$4^{1/2-1/s}\, M_s\Big(1-\frac{m^2}{M^2}\Big)^{1/2} < 1,$$

which guarantees that $Q_h : X_s \to X_s$ is strictly contractive. According to the definition of Q_h the fixed point $z \in X_s$ is a solution of $\overline{A}z = h$. Therefore $\overline{A}$ maps the space X_s onto $X^*_{s'}$. □

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