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A solution of Braess' approximation problem on Powers of the Distance Function

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Abstract

The polynomial approximation behaviour of the class of functions

$$F_s : \mathbb{R}^2 \backslash \{(x_0, y_0)\} \to \mathbb{R}, \quad F_s(x, y) = \big((x - x_0)^2 + (y - y_0)^2\big)^{-s}, \quad s \in (0, \infty),$$

is studied in [Bra01]. There it is claimed that the obtained results can be embedded in a more general setting. This conjecture will be confirmed and complemented by a different approach than in [Bra01]. The key is to connect the approximation rate of F_s with its holomorphic continuability for which the classical Bernstein approximation theorem is linked with the convexity of best approximants.
Approximation results of this kind also play a vital role in the numerical treatment of elliptic differential equations [Sau].

1 Introduction

We consider the following class of continuous functions

$$F_s : \mathbb{R}^2 \backslash \{(x_0, y_0)\} \to \mathbb{R}, \quad s \in (0, \infty),$$

$$F_s(x, y) = \Big((x - x_0)^2 + (y - y_0)^2 \Big)^{-s}.$$

The polynomial approximation behaviour for that type of functions is of special interest in the numerical treatment of elliptic differential equations when fundamental solutions are to be approximated, see [Sau].

In [Bra01] the polynomial approximation error of the functions F_s, $s \in (0, \infty)$, is examined for the closed unit disk $\overline{B}_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, where the singular point (x_0, y_0) of F_s lies in the complement of $\overline{B}_2$, i. e. $\rho := \sqrt{x_0^2 + y_0^2} > 1$.

To this end let us define the deviation of the set of real–valued polynomials P_n, $n \in \mathbb{N}$, to the function F_s, $s \in (0, \infty)$, by the standard approximation error

$$E_n(K, F_s) := \inf\{||F_s - P_n||_K,\ P_n : \mathbb{R}^2 \to \mathbb{R},\ P_n \text{ a polynomial of degree} \leq n\},$$

where $|| \cdot ||_K$ denotes the supremum norm on a compact set $K \subset \mathbb{R}^2$.

The results of [Bra01] can be summarized as follows:

For every function F_s, $s \in (0, \infty)$, the n–th approximation error satisfies the exponential decay

$$E_n(\overline{B}_2, F_s) \leq \frac{M}{R^n}, \tag{1}$$

where R is any real number of the interval $(1, \rho)$ and $M > 0$ is a constant independent of n. Consequently,

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F_s)} \leq \frac{1}{\rho} \tag{2}$$

for every $s \in (0, \infty)$ and $\rho \in (1, \infty)$.

In addition, if $\rho \in (3, \infty)$ and $s \in (0, \infty)$ or $\rho \in (1, \infty)$ and $s \in (0, 1]$ then R in inequality (1) can't be replaced by any number greater than ρ. Hence the relation

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F_s)} = \frac{1}{\rho} \tag{3}$$

holds for $\rho \in (3, \infty)$ and $s \in (0, \infty)$ or $\rho \in (1, \infty)$ and $s \in (0, 1]$.

Estimate (2) is verified by means of Newman's trick and Cauchy's estimates whereas the winding number theorem and the de la Valée–Poussin theorem are applied to establish relation (3). The restrictions for ρ and s in (3) are caused by the method of the proof and don't seem natural. Therefore Braess conjectures that (3) is true for any $\rho \in (1, \infty)$ and any $s \in (0, \infty)$.

The aim of this note is to establish relation (3) for all $\rho \in (1, \infty)$ and $s \in (0, \infty)$. Thus

we obtain a characterization of the asymptotic approximation behaviour for the functions F_s in terms of their singularities.
According to (1) we only have to focus on the lower estimate

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F_s)} \geq \frac{1}{\rho} \tag{4}$$

for $\rho \in (1,\infty)$ and $s \in (0,\infty)$.
The nub for this bound is to study the behaviour of the functions F_s, $s \in (0,\infty)$, outside the unit disk. This stands in contrast to [Bra01], where all the estimates are deduced from the special structure of the functions F_s on the closed unit disk.

2 Sharp asymptotic approximation results

A famous result which links the polynomial approximation rate of a function in $\mathbb{R}$ to its holomorphic continuability is Bernstein's classical approximation theorem, see Theorem 2.1. It is also an important tool for the verification of the lower bound (4).

Theorem 2.1 ([Ber52], 1912)
Let $f : [-1,1] \to \mathbb{R}$ be continuous and let $\rho > 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{E_n([-1,1], f)} \leq \frac{1}{\rho}$$

if and only if f has a holomorphic extension to the set

$$\{z \in \mathbb{C} : \ |h(z)| < \rho\},$$

where $h : \mathbb{C} \to \mathbb{C}\backslash\{z \in \mathbb{C} : |z| < 1\}$ is defined by $h(z) = z + \sqrt{z^2-1}$. The branch of the square root is chosen such that $h(x) > 1$ for $x > 1$.

A quick proof of Theorem 2.1 can be found in [DL93, p. 229–231].
Note, the function $h(z) = z + \sqrt{z^2-1}$ in Theorem 2.1 is the inverse of the Joukowski function with domain $\mathbb{C}$ and range $\mathbb{C}\backslash\mathbb{D}$.

Beside Bernstein's theorem the proof of inequality (4) requires the following property of best approximants.

Lemma 2.1
Let X be the Banach space of all real–valued continuous functions defined on a compact subset $K \subset \mathbb{R}^2$ and let X_n, $n \in \mathbb{N}$, be the subspace of all real–valued polynomials P_n, $P_n : K \to \mathbb{R}$, of degree $\leq n$.
If K is symmetric with respect to the y–axis[1], then an even function F in y has a best approximant $\hat{P}_n$ which is even in y.

[1]We call a set $K \subset \mathbb{R}^2$ symmetric with respect to the y–axis, if $(x,y) \in K$ implies $(x,-y) \in K$.

The proof follows immediately from the convexity of the (non–empty) set of best approximants, cf. [DL93].

Now we have all ingredients to establish the lower bound (4). Hence we achieve a complete characterization of the asymptotic behaviour of the approximation error for the functions F_s, $s \in (0, \infty)$.

Theorem 2.2
Let the function $F_s : \overline{B}_2 \to \mathbb{R}$ *be given by*

$$F_s(x, y) = \frac{1}{\big((x - x_0)^2 + (y - y_0)^2\big)^s}\,, \tag{5}$$

where $s \in (0, \infty)$ *and* $(x_0, y_0) \in \mathbb{R}^2$ *such that* $\rho = \sqrt{x_0^2 + y_0^2} > 1$*. Then*

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F_s)} = \frac{1}{\rho}\,. \tag{6}$$

Proof of Theorem 2.2: We justify the inequality

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F_s)} \geq \frac{1}{\rho}$$

for $\rho \in (1, \infty)$ and $s \in (0, \infty)$. Then the assertion follows in conjunction with estimate (2).

After rotating and translating coordinates we may assume that F_s takes the form

$$F_s(x, y) = \frac{1}{\big((x - \rho)^2 + y^2\big)^s}.$$

Since F_s is an even function in y there exists a best polynomial approximant $\hat{P}_n$ of degree $\leq n$ to F_s on $\overline{B}_2$ which is also even in y. Thus we can write

$$\hat{P}_n(x, y) = \sum_{\substack{j,\,k=0\\ 0\leq j+2k\leq n}}^{n} a_{jk} x^j y^{2k}, \qquad a_{jk} \in \mathbb{R}.$$

In view of the fact that $y^2 = 1 - x^2$ for a any point $(x, y) \in \partial B_2$, $\partial B_2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, we obtain

$$\begin{aligned} E_n(\overline{B}_2, F_s) &= ||F_s - \hat{P}_n||_{\overline{B}_2} \geq ||F_s - \hat{P}_n||_{\partial B_2} \\ &= \max_{x\in[-1,1]} \left| \frac{1}{(\rho^2 - 2x\rho + 1)^s} - \sum_{\substack{j,\,k=0\\ 0\leq j+2k\leq n}}^{n} a_{jk}\ x^j (1 - x^2)^k \right| \\ &= \max_{x\in[-1,1]} |f_s(x) - p_n(x)| \geq E_n([-1, 1], f_s)\,, \end{aligned}$$

where $f_s(x) = 1/(\rho^2 - 2x\rho + 1)^s$ and $p_n(x) = \sum_{\substack{j,\,k=0 \\ 0 \le j+2k \le n}}^{n} a_{jk}\, x^j (1-x^2)^k$.

Consequently,

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F_s)} \ge \limsup_{n\to\infty} \sqrt[n]{E_n([-1,1], f_s)}. \tag{7}$$

We next apply Theorem 2.1 to the function f_s. Note that

$$\hat{f}_s(z) = 1/(\rho^2 - 2z\rho + 1)^s$$

is a holomorphic extension of f_s to the set $L_\rho = \{z \in \mathbb{C} : \ |h(z)| < \rho\}$, where h is defined as in Theorem 2.1. Clearly, $\hat{f}_s$ has a non-removable singularity at the point $\hat{z} = 1/2(\rho + 1/\rho)$. Therefore the function $\hat{f}_s$ cannot be continued analytically to any neighborhood of the point $\hat{z}$. In other words, $\hat{f}_s$ has no holomorphic extension to any domain containing $\overline{L}_\rho$. Thus Theorem 2.1 implies

$$\limsup_{n\to\infty} \sqrt[n]{E_n([-1,1], f_s)} \ge \frac{1}{\rho}\,.$$

The latter, combined with equation (7), gives finally

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_2, F_s)} \ge \frac{1}{\rho}.$$

■

Theorem 2.2 can be also generalized to higher dimensions.

Theorem 2.3
Let $\overline{B}_d = \{x \in \mathbb{R}^d : ||x|| = \left(\sum_{k=1}^d x_k^2\right)^{1/2} \le 1\}$, $d \in \mathbb{N}\backslash\{1\}$, *and let the function* $F_s : \overline{B}_d \to \mathbb{R}$ *be given by*

$$F_s(x) = \frac{1}{||x - x_0||^{2s}} = \frac{1}{\left(\sum_{k=1}^d (x_k - x_{0,k})^2\right)^s}\,, \tag{8}$$

where $s \in (0, \infty)$ *and* $x_0 \in \mathbb{R}^d$ *such that* $\rho = ||x_0|| > 1$. *Then*

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_d, F_s)} = \frac{1}{\rho}\,. \tag{9}$$

Proof of Theorem 2.3: To establish the lower bound

$$\limsup_{n\to\infty} \sqrt[n]{E_n(\overline{B}_d, F_s)} \ge \frac{1}{\rho}$$

we only have to substitute y^2 by $\sum_{k=2}^d x_k^2$ and $\overline{B}_2$ by $\overline{B}_d$ in the proof of Theorem 2.2. A simple argument for the upper bound can be found in [Bra].

$\blacksquare$

Let us conclude by remarking that Theorem 2.3 extends easily to arbitrary closed balls in $\mathbb{R}^d$, $d \in \mathbb{N} \setminus \{1\}$.

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