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Numerical approaches to rate-independent processes and applications in inelasticity

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Abstract

A general abstract approximation scheme for rate-independent processes in the energetic formulation is proposed and its convergence is proved under various rather mild data qualifications. The abstract theory is illustrated on several examples: plasticity with isotropic hardening, damage, debonding, magnetostriction, and two models of martensitic transformation in shape-memory alloys.

1 Introduction

Rate independent processes occur (after certain, and usually necessary, simplifications) in various physical (mainly mechanical but not only) systems exhibiting hysteretic response during isothermal evolution processes. Mathematical analysis of such processes, based on the notion of energetic solutions introduced in [40, 42], has been intensively scrutinized and develop in particular in [31, 35, 36, 37, 38, 41, 43, 53]. However, except for some particular attempts [7, 18, 32, 54], there has been no numerical analysis developed for such processes so far.

This paper fills the gap of a universally-applicable numerical scheme in the context of rate-independent processes and its analysis. After introducing the energetic formulation in Sect. 2, a fairly general conceptual numerical discretization is proposed and its convergence is analyzed in Sect. 3. Then, in Sect. 4, the generality is reduced to problems based on Banach spaces and with dissipation distances governed by degree-1 homogeneous potentials, which in turn allows for various specific constructions directly applicable in concrete situations. This is demonstrated in Sect. 5 on various examples from continuum mechanics of deformable bodies, namely plasticity with hardening, two models of martensitic transformation, damage, debonding, and magnetostriction.

In particular, it accompanies a large variety of existing models by conceptual finite-element discretizations supported by rigorous analysis as far as convergence concerns, and in some cases offers new results or improves known results as far as mere existence of solutions concerns.

2 An abstract setting: energetic solution

We consider a state space $\mathcal{Q}$ (independent of time) as a topological space. Typically, it is subset of a locally convex space. We will distinguish between a "non-dissipative" component $u \in \mathcal{U}$ and a "dissipative" component $z \in \mathcal{Z}$ of the state $q = (u, z) \in \mathcal{Q} := \mathcal{U} \times \mathcal{Z}$.

For a fixed time horizon $T > 0$, we consider a Gibbs-type *stored energy* $\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$. The further ingredient is a (time-independent but not necessarily symmetric) *dissipation distance* $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R} \cup \{+\infty\}$ which will later determine the dissipated energy and which is assumed to satisfy

$$\forall z_1, z_2, z_3 \in \mathcal{Z} : \qquad \mathcal{D}(z_1, z_1) = 0 \quad \& \quad \mathcal{D}(z_1, z_3) \leq \mathcal{D}(z_1, z_2) + \mathcal{D}(z_3, z_3). \tag{2.1}$$

Let us agree to write occasionally $\mathcal{D}(q_1, q_2)$ with the meaning $\mathcal{D}(z_1, z_2)$ for $q_1 = (u_1, z_1)$ and $q_2 = (u_2, z_2)$.

In case of Q having a linear structure, $\mathcal{D}(z_1, z_2) := R(z_2 - z_1)$ (as in Sect. 4 below) and convexity of both $\mathcal{E}$ and R, we want to address an evolution of $q = q(t)$ governed by the *doubly nonlinear* inclusion

$$\partial R\Big(\frac{\partial q}{\partial t}\Big) + \partial_q \mathcal{E}(t, q) \ni 0 \tag{2.2}$$

where "∂" denotes the subdifferential. Under some additional qualification, it is equivalent (see [36, 41]) to the *energetic formulation* based on Definition 2.1 below which, however, works under much weaker data qualification where (2.2) loses any sense. In fact, this definition is based on a global-minimization hypothesis competing with the maximum-dissipation principle (or rather Levitas' realizability principle [33]). In mathematical terms, it means *stability*

$$\forall \tilde{q} \in \mathcal{Q} : \qquad \mathcal{E}\big(t, q(t)\big) \leq \mathcal{E}(t, \tilde{q}) + \mathcal{D}\big(q(t), \tilde{q}\big), \tag{2.3}$$

and *energy equality*

$$\mathcal{E}(t, q(t)) + \mathrm{Var}_{\mathcal{D}}(q; s, t) = \mathcal{E}(s, q(s)) + \int_s^t \mathcal{P}(r, q(r))\, \mathrm{d}r, \tag{2.4}$$

where

$$\mathcal{P}(t, q) := \frac{\partial}{\partial t} \mathcal{E}(t, q) \quad \text{ and } \tag{2.5}$$

$$\mathrm{Var}_{\mathcal{D}}(q; s, t) := \sup \sum_{i=1}^{j} \mathcal{D}\big(q(t_{i-1}), q(t_i)\big) \tag{2.6}$$

with the supremum taken over all $j \in \mathbb{N}$ and over all partitions of $[s, t]$ in the form $s = t_0 < t_1 < \ldots < t_{j-1} < t_j = t$. The particular terms in (2.4) represent the stored energy at time t, the energy dissipated by changes of the internal variable during the time interval $[s, t]$, the stored energy at the initial time s, and the work done by external loadings during the time interval $[s, t]$; $\mathcal{P}$ is then the power.

Definition 2.1 *The process $q : [0, T] \to \mathcal{Q}$ is called an* ***energetic solution*** *to the initial-value problem given by the triple $(\mathcal{E}, \mathcal{D}, q_0)$ if*

(i) *it is* ***stable*** *in the sense that (2.3) holds for all* $t \in [0,T]$,

(ii) *the* ***energy balance*** *(2.4) holds for any* $0 \le s < t \le T$, *in particular* $t \mapsto \mathcal{P}(t, q(t))$ *is in* $L^1(0,T)$, *and*

(iii) *the* ***initial condition*** $q(0) = q_0$ *holds.*

For the analysis of the rate-independent problems, it is convenient to introduce the *sets of stable states* $\mathcal{S}(t)$ for any $t \in [0,T]$ by putting

$$\mathcal{S}(t) := \big\{ q \in \mathcal{Q};\ \ \mathcal{E}(t,q) < +\infty\ \ \&\ \ \forall \tilde{q} \in \mathcal{Q}:\ \ \mathcal{E}(t,q) \le \mathcal{E}(t,\tilde{q}) + \mathcal{D}(q,\tilde{q}) \big\}. \tag{2.7}$$

This allows us to recast the stability condition (i) in Definition 2.1 in the form $q(t) \in \mathcal{S}(t)$ for all $t \in [0,T]$. Yet, more importantly, we may address closedness properties of $\mathcal{S}(t)$.

In Sect. 4, we will specialize this setting by introducing an additional linear structure, i.e. $\mathcal{Q}$ will be (a subset of) a Banach space equipped with the weak or the norm topology. This will allow us to make the abstract properties more specific.

3 An abstract approximation

For an abstract approximation, we consider three positive parameters τ, h, and ε. Here $\tau > 0$ represents the fineness of a time discretization by a partition (not necessarily equidistant) of the time interval $[0,T]$. The parameter $h > 0$ denotes a spatial discretization of the state space $\mathcal{Q}$ by a subset $\mathcal{Q}_h$ again having the structure $\mathcal{Q}_h := \mathcal{U}_h \times \mathcal{Z}_h$. Moreover, $\varepsilon > 0$ is used for a possible approximation of the functionals $\mathcal{E}$ and $\mathcal{D}$ to be implemented more easily when restricted on $\mathcal{Q}_h$ (see also Remark 3.10 below) or just to guarantee the convergence in some more complicated cases. Typically, a penalization of some constraints may be involved by this way, cf. Sect.5. These last approximations lead to $\mathcal{E}_\varepsilon : [0,T] \times \bigcup_{h>0} \mathcal{Q}_h \to \mathbb{R} \cup \{+\infty\}$ and $\mathcal{D}_\varepsilon : \bigcup_{h>0} (\mathcal{Z}_h \times \mathcal{Z}_h) \to \mathbb{R} \cup \{+\infty\}$.

Using the indicator function $\delta_{\mathcal{Q}_h} : \mathcal{Q} \to \{0, +\infty\}$, i.e. $\delta_{\mathcal{Q}_h} = 0$ on $\mathcal{Q}_h$ and $\delta_{\mathcal{Q}_h} = +\infty$ on $\mathcal{Q} \setminus \mathcal{Q}_h$, it will occasionally be convenient to introduce the restriction to $\mathcal{Q}_h$ also by replacing $\mathcal{E}_\varepsilon$ and $\mathcal{D}_\varepsilon$ respectively by

$$\mathcal{E}_{\varepsilon,h} = \mathcal{E}_\varepsilon + \delta_{\mathcal{Q}_h} \quad \text{ and } \quad \mathcal{D}_{\varepsilon,h} : (q,\tilde{q}) \mapsto \mathcal{D}_\varepsilon(q,\tilde{q}) + \delta_{\mathcal{Q}_h}(q) + \delta_{\mathcal{Q}_h}(\tilde{q}). \tag{3.1}$$

3.1 Basic assumptions

We first collect a few basic assumptions. We assume (2.1) also for each $\mathcal{D}_\varepsilon$, i.e. for all $\varepsilon > 0$:

$$\forall z_1, z_2, z_3 \in \mathcal{Z}:\ \ \mathcal{D}_\varepsilon(z_1,z_1) = 0\ \ \&\ \ \mathcal{D}_\varepsilon(z_1,z_3) \le \mathcal{D}_\varepsilon(z_1,z_2) + \mathcal{D}_\varepsilon(z_3,z_3). \tag{3.2}$$

For proving existence results we will need the following lower semi-continuity and compactness results:

$$\forall\,\varepsilon,h>0:\quad \mathcal{D}_\varepsilon:\mathcal{Q}_h{\times}\mathcal{Q}_h\to\mathbb{R}_\infty \text{ are lower semi-continuous,} \tag{3.3}$$

$$\begin{aligned}&\forall\,\varepsilon,h>0\ \ \forall\,t{\in}[0,T]\ \ \forall\,a{\in}\mathbb{R}:\\ &\text{the sublevels } \{\,q{\in}\mathcal{Q}_h\,;\ \mathcal{E}_\varepsilon(t,q){\le}a\,\} \text{ are sequentially compact in } \mathcal{Q}.\end{aligned} \tag{3.4}$$

To pass to the limit will need a *uniform inf-compactness* of the collection $(\mathcal{E}_{\varepsilon,h}(t,\cdot))_{\varepsilon,h>0,\,t\in[0,T]}$:

$$\begin{aligned}&\forall\,a{\in}\mathbb{R}\ \ \forall\,\varepsilon,h>0,\ \ \theta{\in}[0,T],\ \ q_{h,\varepsilon}{\in}\mathcal{Q}_h:\quad \mathcal{E}_\varepsilon(\theta,q_{h,\varepsilon})\le a\\ &\Longrightarrow\quad \exists\,q{\in}\mathcal{Q}\ \ \exists \text{ subsequence } \{q_{h_n,\varepsilon_n}\}_{n\in\mathbb{N}}:\quad q=\lim_{n\to\infty}q_{h_n,\varepsilon_n}.\end{aligned} \tag{3.5}$$

Next we need a "Γ-liminf estimate" for the family $(\mathcal{D}_\varepsilon)_{\varepsilon>0}$ on $(\mathcal{Q}_h{\times}\mathcal{Q}_h)_{h>0}$ in the limit $\varepsilon,h\to0$:

$$\left.\begin{array}{l} z{\in}\mathcal{Z},\ \ z_{h,\varepsilon}{\in}\mathcal{Z}_h,\ \ z=\lim\limits_{(h,\varepsilon)\to(0,0)}z_{h,\varepsilon}\\ \tilde z{\in}\mathcal{Z},\ \ \tilde z_{h,\varepsilon}{\in}\mathcal{Z}_h,\ \ \tilde z=\lim\limits_{(h,\varepsilon)\to(0,0)}\tilde z_{h,\varepsilon}\end{array}\right\}\Rightarrow\ \ \mathcal{D}(z,\tilde z)\le\liminf_{(h,\varepsilon)\to(0,0)}\mathcal{D}_\varepsilon(z_{h,\varepsilon},\tilde z_{h,\varepsilon}). \tag{3.6}$$

The limit functional $\mathcal{D}$ has to satisfy a positivity condition:

$$\left.\begin{array}{l}\forall\,z{\in}\mathcal{Z}\ \forall\,\mathcal{K}\subset\mathcal{Z} \text{ sequentially compact } \forall\,z_n{\in}\mathcal{K}:\\ \lim\limits_{n\to\infty}\min\{\mathcal{D}(z_n,z),\mathcal{D}(z,z_n)\}=0\end{array}\right\}\Rightarrow\quad z=\lim_{n\to\infty}z_n. \tag{3.7}$$

Like for $\mathcal{D}_\varepsilon$ we also need a "Γ-liminf estimate" for the family $(\mathcal{E}_\varepsilon(t,\cdot))_{\varepsilon>0,\,t\in[0,T]}$ on $(\mathcal{Q}_h)_{h>0}$:

$$\forall\,q{\in}\mathcal{Q}\ \forall\,q_{h,\varepsilon}{\in}\mathcal{Q}_h \text{ with } q=\lim_{(h,\varepsilon)\to(0,0)}q_{h,\varepsilon}:\ \mathcal{E}(t,q)\le\liminf_{(h,\varepsilon,\theta)\to(0,0,t)}\mathcal{E}_\varepsilon(\theta,q_{h,\varepsilon}). \tag{3.8}$$

Note that (3.6) and (3.8) are only "lower" Γ-liminf estimates for $(\mathcal{D}_{\varepsilon,h})_{\varepsilon,h>0}$ and $(\mathcal{E}_{\varepsilon,h}(t,\cdot))_{\varepsilon,h>0,\,t\in[0,T]}$. The corresponding upper estimates are consequences of the central condition (3.16) which postulates the existence of joint recovery sequences.

So far all conditions above relate to static concepts. The next three conditions relate to the time dependence, which involves the power of external forces $\mathcal{P}_\varepsilon(t,q)=\frac{\partial}{\partial t}\mathcal{E}_\varepsilon(t,q)$. The first assumption provides a uniform energetic control of the power $\mathcal{P}_\varepsilon$, viz.,

$$\exists\,c_0,c_1{\in}\mathbb{R}\ \ \forall\,\varepsilon,h>0\ \ \forall q{\in}\mathcal{Q}_h:\ \ \Big(\exists t_0{\in}[0,T]:\ \ \mathcal{E}_\varepsilon(t_0,q)<+\infty\Big)\quad\Longrightarrow$$

$$\mathcal{E}_\varepsilon(\cdot,q)\in C^1([0,T])\quad\text{and} \tag{3.9a}$$

$$\forall\,t{\in}[0,T]:\quad\big|\mathcal{P}_\varepsilon(t,q)\big|\le c_1\big(\mathcal{E}_\varepsilon(t,q){+}c_0\big). \tag{3.9b}$$

Using a Gronwall estimate we immediately obtain the growth restrictions

$$\forall\,s,t\in[0,T]:\ \mathcal{E}_\varepsilon(s,q)+c_0\le\mathrm{e}^{c_1|t-s|}\big(\mathcal{E}_\varepsilon(t,q){+}c_0\big). \tag{3.10}$$

The second assumption is a conditioned (with respect to sublevels of $\mathcal{E}$) equi- (with respect to q) uniform (with respect to t) continuity of $\mathcal{P}(\cdot, q)$:

$$\begin{aligned}&\forall\, a\in\mathbb{R} \;\; \forall \sigma>0 \;\; \exists\, \delta>0 \;\; \forall s,t\in[0,T] \;\; \forall q\in\mathcal{Q}:\\ &\text{if } \mathcal{E}(0,q)\leq a \text{ and } |t-s|<\delta, \text{ then } \big|\mathcal{P}(s,q)-\mathcal{P}(t,q)\big|<\sigma.\end{aligned} \tag{3.11}$$

The third assumption on $\mathcal{P}_{\varepsilon,h}$ concerns the convergence of $\mathcal{P}_{\varepsilon,h}$ for $\varepsilon, h\to 0$. It is a "continuous convergence" but conditioned by the fact that the considered arguments are in the associated sets of stable states

$$\begin{aligned}\mathcal{S}_{\varepsilon,h}(t) := \big\{q\in\mathcal{Q}_h;\quad &\mathcal{E}_\varepsilon(t,q)<+\infty \quad \& \\ &\forall\tilde q\in\mathcal{Q}_h: \quad \mathcal{E}_\varepsilon(t,q)\leq\mathcal{E}_\varepsilon(t,\tilde q)+\mathcal{D}_\varepsilon(q,\tilde q)\big\},\end{aligned} \tag{3.12}$$

and that the energies are bounded:

$$\begin{aligned}&\text{If } (\varepsilon_n,h_n,t_n)\to(0,0,t), \quad q_n\in\mathcal{S}_{\varepsilon_n,h_n}(t_n), \;\; q_n\to q, \text{ and}\\ &\sup_{n\in\mathbb{N}}\mathcal{E}_{\varepsilon_n,h_n}(t_n,q_n)<+\infty, \text{ then } \lim_{n\to\infty}\mathcal{P}_{\varepsilon_n}(t_n,q_n)=\mathcal{P}(t,q).\end{aligned} \tag{3.13}$$

Recall that $\mathcal{D}_\varepsilon$ and $\mathcal{D}$ only depend on the z-component of $q=(u,z)$ and we have agreed to write occasionally, as e.g. in (3.12), $\mathcal{D}_\varepsilon(q,\tilde q)$ in the meaning of $\mathcal{D}_\varepsilon(z,\tilde z)$.

An essential ingredient for the convergence analysis is the abstract version of Helly's selection principle, which has been proved in the Appendix of [39] generalizing [35, Theorem 3.2].

Lemma 3.1 ***(Abstract Helly's selection principle [39].)*** *Under the conditions* (2.1), (3.6) *and* (3.7), *for every sequence* $z_n:[0,T]\to\mathcal{Z}$, $n\in\mathbb{N}$ *satisfying*

$$\exists C>0 \;\; \forall n\in\mathbb{N}: \quad \mathrm{Var}_{\mathcal{D}_{\varepsilon_n,h_n}}(z_n;0,T)\leq C, \tag{3.14a}$$

$$\exists \mathcal{K}\subset\mathcal{Z} \text{ sequentially compact} \quad \forall n\in\mathbb{N} \;\; \forall t\in[0,T]: \quad z_n(t)\in\mathcal{K}, \tag{3.14b}$$

there exists a subsequence $(z_{n_j})_{j\in\mathbb{N}}$, *a nondecreasing function* $\mathfrak{D}:[0,T]\to\mathbb{R}$, *and a limit process* $z:[0,T]\to\mathcal{Z}$ *such that we have*

$$\forall\, t\in[0,T]: \;\; z(t)=\lim_{j\to\infty}z_{n_j}(t), \quad \mathfrak{D}(t)=\lim_{j\to\infty}\mathrm{Var}_{\mathcal{D}_{\varepsilon_{n_j},h_{n_j}}}(z_{n_j};0,t), \text{ and} \tag{3.15a}$$

$$\forall\, s,t\in[0,T] \text{ with } s\leq t: \qquad \mathrm{Var}_{\mathcal{D}}(z;s,t)\leq\mathfrak{D}(t)-\mathfrak{D}(s). \tag{3.15b}$$

Remark 3.2 (*Weakening (3.13) on Banach spaces.*) In the applications presented in this paper we will not make use of the full strength of the "conditioned" continuous convergence. However, we refer to [14], where a setting is considered where $\mathcal{Q}$ is a Banach space equipped with its weak topology. It is shown that the assumptions in (3.13) first imply the energy convergence $\mathcal{E}_{\varepsilon_n,h_n}(t_n,q_n)\to\mathcal{E}(t,q)$. This, together with the weak convergence $q_n\rightharpoonup q$, can then be used to improve the convergence into the strong convergence. Hence, in that case the conditioning implies that only strongly convergent sequences have to be considered for the continuous convergence in (3.13).

3.2 Stability of sets of stable states

All the assumptions of the previous subsection are either on the family $(\mathcal{D}_{\varepsilon,h})_{\varepsilon,h>0}$ or on the family $(\mathcal{E}_{\varepsilon,h})_{\varepsilon,h>0}$. The final condition links the behavior of these two families and thus provide the upper Γ-limit estimates which are needed to complement the lower Γ-limit estimate for $\mathcal{D}$ in (3.6) and for $\mathcal{E}$ in (3.8). Sometimes, in particular when some holonomic-type constraints are involved in $\mathcal{E}$, it occurs that a *convergence criterion* of the type $h \leq H(\varepsilon)$, for some $H : \mathbb{R}^+ \to \mathbb{R}^+$ monotone and satisfying $H(\varepsilon) \to 0$ for $\varepsilon \to 0$, is needed.

The following central condition states the existence of a "joint recovery sequence" under suitable qualifications:

$$\begin{aligned}
&\forall\, q, \tilde{q} \in \mathcal{Q} \quad \forall t_n \in [0,T] \text{ with } t_n \to t \ \forall\, \varepsilon_n, h_n \to 0+ \ \text{ with } h_n \leq H(\varepsilon_n) \\
&\forall q_n \in \mathcal{S}_{\varepsilon_n,h_n}(t_n) \ \text{ with } \ q_n \to q \ \text{ and } \ \sup\nolimits_{n\in\mathbb{N}} \mathcal{E}_{\varepsilon_n,h_n}(t_n,q_n) < +\infty \\
&\exists\, \tilde{q}_n \in \mathcal{Q}_{h_n} \ \text{ with } \ \tilde{q}_n \to \tilde{q} : \\
&\limsup_{n\to\infty} \Big(\mathcal{E}_{\varepsilon_n,h_n}(t_n,\tilde{q}_n) + \mathcal{D}_{\varepsilon_n,h_n}(q_n,\tilde{q}_n) - \mathcal{E}_{\varepsilon_n,h_n}(t_n,q_n) \Big) \\
&\qquad\qquad\qquad\qquad \leq \mathcal{E}(t,\tilde{q}) + \mathcal{D}(q,\tilde{q}) - \mathcal{E}(t,q). \qquad (3.16)
\end{aligned}$$

The following assertion says, in other words, that the graph of the set-valued mapping $\mathcal{S} : [0,T] \rightrightarrows \mathcal{Q}$ contains Kuratowski's limes superior of the graphs of $\mathcal{S}_{\varepsilon,h} : [0,T] \rightrightarrows \mathcal{Q}_h$ at least if restricted to states with bounded energy as in (3.5) and if $h \leq H(\varepsilon)$ is taken into account. This upper semicontinuity result establishes a certain stability of sets of stable states that is crucial for the convergence analysis.

Lemma 3.3 ***(Conditioned upper semi-continuity of the sets of stable states.)*** *Let (3.8) and (3.16) hold and $t_n, \varepsilon_n, h_n, q_n$ and $q = \lim_{n\to\infty} q_n$ be as in* (3.16). *Then $q \in \mathcal{S}(t)$.*

Proof. By (3.8), we have

$$\mathcal{E}(t,q) \leq \liminf_{n\to\infty} \mathcal{E}_{\varepsilon_n,h_n}(t_n,q_n) \leq \sup_{n\in\mathbb{N}} \mathcal{E}_{\varepsilon_n,h_n}(t_n,q_n) < +\infty, \qquad (3.17)$$

where the last inequality is assumed in (3.16). Next, for $\tilde{q} \in \mathcal{Q}$ arbitrary, choose $\tilde{q}_n \in \mathcal{Q}_{h_n}$ as in (3.16). By definition (3.12), $q_n \in \mathcal{S}_{\varepsilon_n,h_n}(t_n)$ says that $\mathcal{E}_{\varepsilon_n,h_n}(t_n,\tilde{q}_n) + \mathcal{D}_{\varepsilon_n,h_n}(q_n,\tilde{q}_n) - \mathcal{E}_{\varepsilon_n,h_n}(t_n,q_n) \geq 0$. Using now the limsup estimate in (3.16) we obtain

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty} \mathcal{E}_{\varepsilon_n,h_n}(t_n,\tilde{q}_n) + \mathcal{D}_{\varepsilon_n,h_n}(q_n,\tilde{q}_n) - \mathcal{E}_{\varepsilon_n,h_n}(t_n,q_n) \\
&\leq \mathcal{E}(t,\tilde{q}) + \mathcal{D}(q,\tilde{q}) - \mathcal{E}(t,q). \qquad (3.18)
\end{aligned}$$

Since $\tilde{q}$ was arbitrary, definition (2.7) gives $q \in \mathcal{S}(t)$. □

Remark 3.4 (*Weakening of* (3.16).) In this proof the condition $\tilde{q}_n \to \tilde{q}$ was *not* used. Thus, in principle assumption (3.16) could be weakened by dropping this additional request. However, in doing so, the limsup estimate in (3.16) degenerates in the sense that the two sides in this estimate no longer depend on each other. In fact, the best choice for making the left-hand side small is, by recalling stability, the choice $\tilde{q}_n = q_n$, which makes each member in the sequence identical 0. Since this is independent of $\tilde{q}$, the weakened condition (3.16) just means $0 \leq \mathcal{E}(t,\tilde{q})+\mathcal{D}(q,\tilde{q})-\mathcal{E}(t,q)$, which is the desired stability of q. As we will see in the applications in Section 5, the strengthened condition is useful, since properly chosen joint recovery sequences allow us to prove

$$0 \leq \big(\mathcal{E}_{\varepsilon_n,h_n}(t_n,\tilde{q}_n)+\mathcal{D}_{\varepsilon_n,h_n}(q_n,\tilde{q}_n)-\mathcal{E}_{\varepsilon_n,h_n}(t_n,q_n)\big) \to \mathcal{E}(t,\tilde{q})+\mathcal{D}(q,\tilde{q})-\mathcal{E}(t,q),$$

from which we then conclude stability. See [39] for more discussion of this point.

Example 3.5 Quite typical way how the qualification (3.16) can be ensured is the situation when $\mathcal{D}_{\varepsilon,h}$ converges continuously to $\mathcal{D}$ in the sense

$$\lim_{\substack{\varepsilon\to 0,\ h\to 0\\ q_{h,\varepsilon}\to q,\,\tilde{q}_{h,\varepsilon}\to\tilde{q}\\ q_{h,\varepsilon},\tilde{q}_{h,\varepsilon}\in\mathcal{Q}_h}} \mathcal{D}_\varepsilon(q_{h,\varepsilon},\tilde{q}_{h,\varepsilon}) = \mathcal{D}(q,\tilde{q}) \tag{3.19}$$

and, in addition,

$$\forall \tilde{q}\in\mathcal{Q}\quad \forall h,\varepsilon>0\quad \exists \tilde{q}_{h,\varepsilon}\in\mathcal{Q}_h:\quad \lim_{(h,\varepsilon)\to(0,0)} \tilde{q}_{h,\varepsilon}=\tilde{q}\quad \text{and}$$
$$\limsup_{\substack{h\leq H(\varepsilon)\\ (h,\varepsilon,\theta)\to(0,0,t)}} \mathcal{E}_\varepsilon(\theta,\tilde{q}_{h,\varepsilon}) \leq \mathcal{E}(t,\tilde{q}). \tag{3.20}$$

Then (3.16) holds: indeed, it suffices to sum (3.20) used for $\tilde{q}_n = \tilde{q}_{h_n,\varepsilon_n}$ with (3.19) used for $q_n = q_{h_n,\varepsilon_n}$ and $\tilde{q}_n = \tilde{q}_{h_n,\varepsilon_n}$ and subtract (3.8) used for $q_n = q_{h_n,\varepsilon_n}$, and eventually estimate the sum of limits superior from below by limits superior of the sum. Let us still remark that (3.20) together with (3.8) is just the conditioned Γ-*convergence* (sometimes also called epi-convergence) of the collection $(\mathcal{E}_{\varepsilon,h}(\theta,\cdot)+\delta_{\mathcal{Q}_h})_{h,\varepsilon>0,\,\theta\in[0,T]}$ to $\mathcal{E}$ if $(h,\varepsilon,\theta)\to(0,0,t)$ conditioned by $h\leq H(\varepsilon)$.

3.3 Approximate solutions

We consider now $\tau>0$, and a partition $0=t_\tau^0<t_\tau^1<...<t_\tau^{k_\tau}=T$ with

$$t_i^\tau - t_{i-1}^\tau \leq \tau \quad \text{for}\ \ i=1,...,k_\tau. \tag{3.21}$$

We do not assume this partition to be equidistant. Further, we consider an approximation $[q_0]_{h,\varepsilon}$ of the initial condition q_0 and the following recursive incremental formula: we put $q_{\tau,h,\varepsilon}^0 = [q_0]_{h,\varepsilon}$ a given initial condition, and, for $k=1,...,k_\tau$ we define $q_{\tau,h,\varepsilon}^k$, an approximation of a solution at time t_τ^k, to be any solution of the minimization problem

$$\left.\begin{array}{ll}\text{Minimize} & \mathcal{E}_{\varepsilon,h}(t_\tau^k,q)+\mathcal{D}_{\varepsilon,h}(z_{\tau,h,\varepsilon}^{k-1},z)\\ \text{subject to} & q=(u,z)\in\mathcal{Q}_h.\end{array}\right\} \tag{3.22}$$

We define the approximate solution $q_{\tau,h,\varepsilon} : [0,T] \to \mathcal{Q}$ as a piecewise constant approximation, namely

$$q_{\tau,h,\varepsilon}(t) := \begin{cases} q^k_{\tau,h,\varepsilon} & \text{for } t^{k-1}_\tau < t \le t^k_\tau, \quad k = 1, ..., k_\tau, \\ q^0_{\tau,h,\varepsilon} = [q_0]_{h,\varepsilon} & \text{for } t = 0. \end{cases} \tag{3.23}$$

We also need the "retarded" approximate solution $q^{\mathrm{R}}_{\tau,h,\varepsilon} : [0,T] \to \mathcal{Q}$ with

$$q^{\mathrm{R}}_{\tau,h,\varepsilon}(t) := \begin{cases} q^k_{\tau,h,\varepsilon} & \text{for } t^{k-1}_\tau \le t < t^k_\tau, \quad k = 1, ..., k_\tau, \\ q^{k_\tau}_{\tau,h,\varepsilon} & \text{for } t = T, \end{cases} \tag{3.24}$$

Proposition 3.6 *(**Discrete stability and energy inequalities.**) Let* (3.2)*, the lower semicontinuity* (3.3)–(3.4) *of the approximate stored and the dissipated energies, and smoothness of external forcing* (3.9a) *hold. Then (3.22) has a solution $q^k_{\tau,h,\varepsilon}$ for any $k = 1, ..., k_\tau$ and $q_{\tau,h,\varepsilon}$ is stable in the sense*

$$q_{\tau,h,\varepsilon}(t) \in \mathcal{S}_{\varepsilon,h}(t^k_\tau) \quad \text{for any} \quad t \in (t^{k-1}_\tau, t^k_\tau], \quad k = 0, ..., k_\tau, \tag{3.25}$$

and satisfies the discrete upper energy inequality

$$\mathcal{E}_{\varepsilon,h}(s, q_{\tau,h,\varepsilon}(s)) + \mathrm{Var}_{\mathcal{D}_{\varepsilon,h}}(q_{\tau,h,\varepsilon}; r, s) - \mathcal{E}_{\varepsilon,h}(r, q_{\tau,h,\varepsilon}(r)) \le \int_r^s \frac{\partial \mathcal{E}_{\varepsilon,h}}{\partial t}\big(t, q^{\mathrm{R}}_{\tau,h,\varepsilon}(t)\big)\,\mathrm{d}t \tag{3.26}$$

for $r = t^{k_1}_\tau$ and $s = t^{k_2}_\tau$ with any $k_1, k_2 \in \mathbb{N} \cup \{0\}$, $0 \le k_1 \le k_2 \le k_\tau$, as well as a similar discrete lower energy inequality

$$\mathcal{E}_{\varepsilon,h}(s, q_{\tau,h,\varepsilon}(s)) + \mathrm{Var}_{\mathcal{D}_{\varepsilon,h}}(q_{\tau,h,\varepsilon}; r, s) - \mathcal{E}_{\varepsilon,h}(r, q_{\tau,h,\varepsilon}(r)) \ge \int_r^s \frac{\partial \mathcal{E}_{\varepsilon,h}}{\partial t}\big(t, q_{\tau,h,\varepsilon}(t)\big)\,\mathrm{d}t \tag{3.27}$$

for $r = t^{k_1}_\tau$ and $s = t^{k_2}_\tau$ but now only with $k_1, k_2 \in \mathbb{N}$, $1 \le k_1 \le k_2 \le k_\tau$.

Proof. The existence of $q^k_{\tau,h,\varepsilon}$ solving (3.22) follows from (3.3) and (3.4) via a recursive argument for $k = 1, ..., k_\tau$. Hence $q_{\tau,h,\varepsilon}$ and $q^{\mathrm{R}}_{\tau,h,\varepsilon}$ exist, too.

The discrete stability condition (3.25) follows by using successively that $q^k_{\tau,h,\varepsilon}$ is a solution to (3.22) and the triangle inequality (2.1) for $\mathcal{D}_{\varepsilon,h}$:

$$\begin{aligned} \mathcal{E}_{\varepsilon,h}(t^k_\tau, q^k_{\tau,h,\varepsilon}) &\le \mathcal{E}_{\varepsilon,h}(t^k_\tau, \tilde{q}) + \mathcal{D}_{\varepsilon,h}(q^{k-1}_{\tau,h,\varepsilon}, \tilde{q}) - \mathcal{D}_{\varepsilon,h}(q^{k-1}_{\tau,h,\varepsilon}, q^k_{\tau,h,\varepsilon}) \\ &\le \mathcal{E}_{\varepsilon,h}(t^k_\tau, \tilde{q}) + \mathcal{D}_{\varepsilon,h}(q^k_{\tau,h,\varepsilon}, \tilde{q}) \end{aligned} \tag{3.28}$$

for any $k = 1, ..., k_\tau$.

As to (3.26), we again use that $q^k_{\tau,h,\varepsilon}$ solves (3.22) and, comparing it with $q^{k-1}_{\tau,h,\varepsilon}$, we get

$$\begin{aligned} &\mathcal{E}_{\varepsilon,h}(t^k_\tau, q^k_{\tau,h,\varepsilon}) - \mathcal{E}_{\varepsilon,h}\big(t^{k-1}_\tau, q^{k-1}_{\tau,h,\varepsilon}\big) + \mathcal{D}_{\varepsilon,h}\big(q^{k-1}_{\tau,h,\varepsilon}, q^k_{\tau,h,\varepsilon}\big) \\ &\qquad \le \mathcal{E}_{\varepsilon,h}\big(t^k_\tau, q^{k-1}_{\tau,h,\varepsilon}\big) - \mathcal{E}_{\varepsilon,h}\big(t^{k-1}_\tau, q^{k-1}_{\tau,h,\varepsilon}\big) = \int_{t^{k-1}_\tau}^{t^k_\tau} \frac{\partial \mathcal{E}_{\varepsilon,h}(t, q^{k-1}_{\tau,h,\varepsilon})}{\partial t}\,\mathrm{d}t. \end{aligned} \tag{3.29}$$

Now the estimate (3.26) follows after a summation for $k = k_1{+}1, ..., k_2$. As to the estimate (3.27), by the stability (3.28) written for $q^{k-1}_{\tau,h,\varepsilon}$ $\tilde{q} = q^k_{\tau,h,\varepsilon}$, we find

$$\begin{aligned}\mathcal{E}_{\varepsilon,h}\big(t^k_\tau, q^k_{\tau,h,\varepsilon}\big) - \mathcal{E}_{\varepsilon,h}\big(t^{k-1}_\tau, q^{k-1}_{\tau,h,\varepsilon}\big) + \mathcal{D}_{\varepsilon,h}\big(q^{k-1}_{\tau,h,\varepsilon}, q^k_{\tau,h,\varepsilon}\big)\\ \geq \mathcal{E}_{\varepsilon,h}\big(t^k_\tau, q^k_{\tau,h,\varepsilon}\big) - \mathcal{E}_{\varepsilon,h}\big(t^{k-1}_\tau, q^k_{\tau,h,\varepsilon}\big) = \int_{t^{k-1}_\tau}^{t^k_\tau} \frac{\partial \mathcal{E}_{\varepsilon,h}(t, q^k_{\tau,h,\varepsilon})}{\partial t}\,\mathrm{d}t.\end{aligned} \tag{3.30}$$

By a summation for $k = k_1{+}1, ..., k_2$, we obtain (3.27). $\square$

Remark 3.7 (*Approximation of initial conditions.*) Note that (3.30) does not work for $k = 1$ because we (intentionally) did not assume "numerical" stability of the approximate initial condition, i.e. $[q_0]_{h,\varepsilon} \in \mathcal{S}_{\varepsilon,h}(0)$ which would only very hardly be guaranteed in concrete numerical schemes. This is also why (3.27) does not hold with $r = 0$, unlike (3.26).

3.4 Convergence of the approximate solutions

Now we investigate the asymptotics for $\tau \to 0$, $h \to 0$, and $\varepsilon \to 0$. Like for space discretization, we do not assume the partition of the time interval $[0, T]$ to be nested, but we assume that both time and space discretization refines when $\tau \to 0$ and $h \to 0$, respectively. Namely (3.21) for the time discretization while, for the spatial discretization, this refinement requirement is implicitly contained in (3.16); later it will be assumed explicitly (4.2) to prove (3.16).

Theorem 3.8 *Let the assumptions* (2.1), (3.2)– (3.9), (3.13), (3.16) *and* (3.21) *hold. Assume that the initial condition q_0 is stable, i.e.*

$$q_0 \in \mathcal{S}(0), \tag{3.31}$$

and is approximated by $[q_0]_{h,\varepsilon} \in \mathcal{Q}_h$ in the sense

$$[q_0]_{h,\varepsilon} \to q_0 \quad \textit{and} \quad \mathcal{E}_\varepsilon(0, [q_0]_{h,\varepsilon}) \to \mathcal{E}(0, q_0). \tag{3.32}$$

Then, there exists a subsequence $\{(\tau_n, h_n, \varepsilon_n)\}_{n\in\mathbb{N}}$ with $(\tau_n, h_n, \varepsilon_n) \to (0, 0, 0)$ for $n \to \infty$ satisfying the convergence criterion $h_n \leq H(\varepsilon_n)$ from condition (3.16) *and a process $q : [0, T] \to \mathcal{Q}$ being an energetic solution according to Definition 2.1 such that the following holds:*

(i) *for all $t \in [0, T]$ we have $\mathcal{E}_{\varepsilon_n}(t, q_n(t)) \to \mathcal{E}(t, q(t))$,*

(ii) *for all $t \in [0, T]$ we have $\mathrm{Var}_{\mathcal{D}_{\varepsilon_n}}(q_n; 0, t) \to \mathrm{Var}_{\mathcal{D}}(q; 0, t)$,*

(iii) *for all $t \in [0, T]$ we have $z_n(t) \to z(t)$ in $\mathcal{Z}$,*

(iv) $\frac{\partial}{\partial t}\mathcal{E}_{\varepsilon_n}(\cdot, q_n(\cdot)) \to \frac{\partial}{\partial t}\mathcal{E}(\cdot, q(\cdot))$ *in $L^1(0, T)$,*

(v) *for all $t \in [0, T]$ there is a subsequence $\{n_l\}_{l\in\mathbb{N}}$ such that $\lim_{l\to\infty} u_{n_l}(t) = u(t)$ in $\mathcal{U}$, hence $\lim_{l\to\infty} q_{n_l}(t) = q(t)$ in $\mathcal{Q}$,*

where we wrote shortly $q_n = (u_n, z_n)$ for $q^{\mathrm{R}}_{\tau_n,h_n,\varepsilon_n} = (u^{\mathrm{R}}_{\tau_n,h_n,\varepsilon_n}, z^{\mathrm{R}}_{\tau_n,h_n,\varepsilon_n})$.

Proof. We follow the steps for the existence proof formulated in [14, 39]. However, we are more general than [14, 39] as we do not require $[q_0]_{h,\varepsilon}$ to be stable.

Let us abbreviate

$$\mathfrak{G}_{\tau,h,\varepsilon}(t) := \mathcal{E}_{\varepsilon,h}(t, q^{\mathrm{R}}_{\tau,h,\varepsilon}(t)), \qquad \mathfrak{D}_{\tau,h,\varepsilon}(t) := \mathrm{Var}_{\mathcal{D}_{\varepsilon,h}}(q^{\mathrm{R}}_{\tau,h,\varepsilon}; 0, t). \tag{3.33}$$

Step 1: A priori estimates. By (3.9) and (3.10), we can estimate the right-hand side of (3.29) as

$$\begin{aligned}\int_{t_\tau^{k-1}}^{t_\tau^k} \frac{\partial \mathcal{E}_{\varepsilon,h}(t, q^{k-1}_{\tau,h,\varepsilon})}{\partial t}\,\mathrm{d}t \;&\le\; \int_{t_\tau^{k-1}}^{t_\tau^k} c_1\big(\mathcal{E}_{\varepsilon,h}(t, q^{k-1}_{\tau,h,\varepsilon}) + c_0\big)\,\mathrm{d}t \\ &\le\; \int_{t_\tau^{k-1}}^{t_\tau^k} c_1 \mathrm{e}^{c_1(t - t_\tau^{k-1})}\big(\mathcal{E}_{\varepsilon,h}(t_\tau^{k-1}, q^{k-1}_{\tau,h,\varepsilon}) + c_0\big)\,\mathrm{d}t \\ &=\; \big(\mathrm{e}^{c_1(t_\tau^k - t_\tau^{k-1})} - 1\big)\big(\mathcal{E}_{\varepsilon,h}(t_\tau^{k-1}, q^{k-1}_{\tau,h,\varepsilon}) + c_0\big).\end{aligned} \tag{3.34}$$

Forgetting, for a moment, $\mathcal{D}_{\varepsilon,h}$ in (3.29) and linking it with (3.34) yields $\mathcal{E}_{\varepsilon,h}(t_\tau^k, q^k_{\tau,h,\varepsilon}) + c_0 \le \mathrm{e}^{c_1(t_\tau^k - t_\tau^{k-1})}(\mathcal{E}_{\varepsilon,h}(t_\tau^{k-1}, q^{k-1}_{\tau,h,\varepsilon}) + c_0)$ from which, by induction for $k = 1, 2, ..., k_\tau$ we get

$$\mathcal{E}_{\varepsilon,h}(t_\tau^k, q^k_{\tau,h,\varepsilon}) \le \mathrm{e}^{c_1 t_\tau^k}\big(\mathcal{E}_{\varepsilon,h}(0, q^0_{\tau,h,\varepsilon}) + c_0\big) - c_0. \tag{3.35}$$

By (3.32), we conclude that $\mathcal{E}_{\varepsilon,h}(t_\tau^k, q^k_{\tau,h,\varepsilon})$ is upper bounded independently of k, h, τ, and ε. By (3.9b) we can bound $\mathfrak{G}_{\tau,h,\varepsilon}(t)$ from below and, by (3.35) with (3.10) after some still some calculations from above:

$$-c_0 \le \mathfrak{G}_{\tau,h,\varepsilon}(t) \le a_* \mathrm{e}^{c_1 t} - c_0 \quad \text{ with } \; a_* := c_0 + \sup_{\tau,h,\varepsilon} \mathcal{E}_{\varepsilon,h}(0, q^0_{\tau,h,\varepsilon}), \tag{3.36}$$

where the "sup" is considered for (τ, h, ε) small enough. Note that $a_* < +\infty$ due to (3.32) with the assumption $\mathcal{E}(0, q_0) < +\infty$.

Using (3.35) again for (3.34) but summed for $k = 1, ..., k_\tau$, we obtain

$$\begin{aligned}\int_0^T \frac{\partial \mathcal{E}_{\varepsilon,h}(t, q^{\mathrm{R}}_{\tau,h,\varepsilon})}{\partial t}\,\mathrm{d}t \;&=\; \sum_{k=1}^{k_\tau} \int_{t_\tau^{k-1}}^{t_\tau^k} \frac{\partial \mathcal{E}_{\varepsilon,h}(t, q^{k-1}_{\tau,h,\varepsilon})}{\partial t}\,\mathrm{d}t \\ &\le\; \big(\mathcal{E}_{\varepsilon,h}(0, q^0_{\tau,h,\varepsilon}) + c_0\big) \sum_{k=1}^{k_\tau} \big(\mathrm{e}^{c_1 t_\tau^k} - \mathrm{e}^{c_1 t_\tau^{k-1}}\big) \\ &=\; \big(\mathcal{E}_{\varepsilon,h}(0, q^0_{\tau,h,\varepsilon}) + c_0\big)\big(\mathrm{e}^{c_1 T} - 1\big).\end{aligned} \tag{3.37}$$

Coming back to (3.29) and combining it with the lower bound (3.9b) for $\mathcal{E}_{\varepsilon,h}(T, q^{k_\tau}_{\tau,h,\varepsilon})$ and with (3.37), we now can estimate the total variation of $\mathfrak{D}_{\tau,h,\varepsilon}$ as

$$\begin{aligned}\mathrm{Var}(\mathfrak{D}_{\tau,h,\varepsilon}; 0, T) &= \sum_{k=1}^{k_\tau} \mathcal{D}_{\varepsilon,h}\big(q^{k-1}_{\tau,h,\varepsilon}, q^k_{\tau,h,\varepsilon}\big) \\ &\le \mathcal{E}_{\varepsilon,h}(0, q^0_{\tau,h,\varepsilon}) + c_0 + \big(\mathcal{E}_{\varepsilon,h}(0, q^0_{\tau,h,\varepsilon}) + c_0\big)\big(\mathrm{e}^{c_1 T} - 1\big) \\ &= \big(\mathcal{E}_{\varepsilon,h}(0, q^0_{\tau,h,\varepsilon}) + c_0\big)\mathrm{e}^{c_1 T} \le a_* \mathrm{e}^{c_1 T}\end{aligned} \tag{3.38}$$

with a_* from (3.36). We can now estimate also the total variation of $\mathfrak{G}_{\tau,h,\varepsilon}$ simply by (3.9b) and (3.10) as

$$\begin{aligned}
\mathrm{Var}(\mathfrak{G}_{\tau,h,\varepsilon};0,T) &= \int_0^T \Big|\frac{\partial\mathcal{E}_{\varepsilon,h}(t,q^{\mathrm{R}}_{\tau,h,\varepsilon})}{\partial t}\Big|\,\mathrm{d}t + \sum_{k=1}^{k_\tau}\big|\mathcal{E}_{\varepsilon,h}(t^k_\tau,q^k_{\tau,h,\varepsilon}) - \mathcal{E}_{\varepsilon,h}(t^k_\tau,q^{k-1}_{\tau,h,\varepsilon})\big| \\
&\le \int_0^T c_1\big(\mathfrak{G}_{\tau,h,\varepsilon}(t)+c_0\big)\mathrm{d}t + \sum_{k=1}^{k_\tau}\big|\mathcal{E}_{\varepsilon,h}(t^k_\tau,q^k_{\tau,h,\varepsilon}) - \mathcal{E}_{\varepsilon,h}(t^{k-1}_\tau,q^{k-1}_{\tau,h,\varepsilon})\big| \\
&\quad + \sum_{k=1}^{k_\tau}\int_{t^{k-1}_\tau}^{t^k_\tau}\Big|\frac{\partial\mathcal{E}_{\varepsilon,h}(t,q^{k-1}_{\tau,h,\varepsilon})}{\partial t}\Big|\,\mathrm{d}t =: T_1+T_2+T_3.
\end{aligned} \tag{3.39}$$

The term T_1 is bounded since we have already proved $|\mathfrak{G}_{\tau,h,\varepsilon}(t)|$ a-priori bounded, and also $T_3 \le T_1$, see (3.37), so it remains to bound T_2. By (3.29) and (3.30), we can estimate

$$\begin{aligned}
T_2 \le \mathrm{Var}(\mathfrak{G}_{\tau,h,\varepsilon};0,T) + \max\Big(T_1, \int_0^T \Big|\frac{\partial\mathcal{E}_{\varepsilon,h}(t,q_{\tau,h,\varepsilon})}{\partial t}\Big|\,\mathrm{d}t\Big) \\
+ \max\Big(0, \mathcal{E}_{\varepsilon,h}(0,q^0_{\tau,h,\varepsilon}) - \mathcal{E}_{\varepsilon,h}(t^1_\tau,q^1_{\tau,h,\varepsilon})\Big)
\end{aligned} \tag{3.40}$$

and, again by (3.9b) and (3.10),

$$\begin{aligned}
&\int_0^T \Big|\frac{\partial\mathcal{E}_{\varepsilon,h}(t,q_{\tau,h,\varepsilon})}{\partial t}\Big|\,\mathrm{d}t \le \int_0^T c_1\big(\mathcal{E}_{\varepsilon,h}(t,q_{\tau,h,\varepsilon})+c_0\big)\,\mathrm{d}t \\
&= \sum_{k=1}^{k_\tau}\int_{t^{k-1}_\tau}^{t^k_\tau} c_1\big(\mathcal{E}_{\varepsilon,h}(t,q^k_{\tau,h,\varepsilon})+c_0\big)\,\mathrm{d}t \le \sum_{k=1}^{k_\tau}\int_{t^{k-1}_\tau}^{t^k_\tau} c_1\mathrm{e}^{c_1(t^k_\tau-t)}\big(\mathcal{E}_{\varepsilon,h}(t^k_\tau,q^k_{\tau,h,\varepsilon})+c_0\big)\,\mathrm{d}t \\
&= \sum_{k=1}^{k_\tau}\big(\mathrm{e}^{c_1(t^k_\tau-t^{k-1}_\tau)}-1\big)\big(\mathcal{E}_{\varepsilon,h}(t^k_\tau,q^k_{\tau,h,\varepsilon})+c_0\big) \le \sum_{k=1}^{k_\tau}\big(\mathrm{e}^{c_1(t^k_\tau-t^{k-1}_\tau)}-1\big)\mathrm{e}^{c_1t^{k-1}_\tau}a_*\mathrm{e}^\tau \\
&= \sum_{k=1}^{k_\tau}\big(\mathrm{e}^{c_1t^k_\tau}-\mathrm{e}^{c_1t^{k-1}_\tau}\big)a_*\mathrm{e}^\tau = \big(\mathrm{e}^{c_1T}-1\big)a_*\mathrm{e}^\tau,
\end{aligned} \tag{3.41}$$

where we also used, by (3.35)–(3.36), the estimate

$$\mathcal{E}_{\varepsilon,h}(t^k_\tau,q^k_{\tau,h,\varepsilon})+c_0 \le \mathrm{e}^{c_1t^k_\tau}a_* \le \mathrm{e}^{c_1t^{k-1}_\tau}a_*\mathrm{e}^\tau,$$

and eventually the last term in (3.40) can be estimated simply because, by (3.9b) and (3.32), $\mathcal{E}_{\varepsilon,h}(t^1_\tau,q^1_{\tau,h,\varepsilon}) \ge -c_0$ and $\mathcal{E}_{\varepsilon,h}(0,q^0_{\tau,h,\varepsilon}) \le a_*-c_0$, hence this last term is bounded from above by a_* from (3.36).

Step 2: Selection of subsequences. Since the scalar functions $\mathfrak{G}_{\tau,h,\varepsilon}$ and $\mathfrak{D}_{\tau,h,\varepsilon}$ from (3.33) are uniformly bounded in $\mathrm{BV}([0,T])$ by (3.38) and (3.39) together with the obvious bounds on $|\mathfrak{G}_{\tau,h,\varepsilon}(0)| = |\mathcal{E}_\varepsilon(0,[q_0]_{h,\varepsilon})| \le \max(|c_0|,a_*)$ with a_* from (3.36) and $|\mathfrak{D}_{\tau,h,\varepsilon}(0)| = 0$, we may apply Helly's selection principle both in the classical form and, relying on the assumptions (3.2), (3.6), (3.7), also in the form of Lemma 3.1 to find a subsequence $\{(\tau_n, h_n, \varepsilon_n)\}_{n\in\mathbb{N}}$ such that for all $t \in [0,T]$ we have the following convergence:

$$\mathfrak{G}_{\tau_n,h_n,\varepsilon_n}(t) \to \mathfrak{G}(t), \quad \mathfrak{D}_{\tau_n,h_n,\varepsilon_n}(t) \to \mathfrak{D}(t), \text{ and } z_{\tau_n,h_n,\varepsilon_n}(t) \to z(t) \text{ in } \mathcal{Z}, \tag{3.42}$$

for suitable limit functions $\mathfrak{D}$, $\mathfrak{G}$ and z satisfying also (3.15b). This shows that the convergence at the point (iii) holds. We further set

$$\mathfrak{P}_n(t) := \frac{\partial}{\partial t}\mathcal{E}_{\varepsilon_n}(t, q_n(t)) \tag{3.43}$$

to denote the power of the external forces. Choosing another subsequence (not relabeled), if necessary, we also obtain

$$\mathfrak{P}_n \overset{\mathrm{w}^*}{\to} \mathfrak{p} \text{ in } L^\infty([0,T]), \tag{3.44}$$

since closed balls in $L^\infty([0,T])$ are sequentially weakly* compact. For fixed t, let

$$\mathfrak{P}(t) := \limsup_{n\to\infty} \mathfrak{P}_n(t). \tag{3.45}$$

Using Fatou's lemma we conclude $\mathfrak{P} \in L^\infty(0,T)$ and $\mathfrak{p}(t) \le \mathfrak{P}(t)$ for a.a. $t \in [0,T]$. Further, let us set

$$\mathcal{A}(t) := \Big\{\tilde{u}\in\mathcal{U};\ \ \frac{\partial}{\partial t}\mathcal{E}(t,\tilde{u},z(t)) = \mathfrak{P}(t)\Big\}. \tag{3.46}$$

For any t fixed, $\mathcal{A}(t)$ is nonempty: Indeed, we can choose a subsequence $(n_j^t)_{j\in\mathbb{N}}$ (depending on t!) such that

$$\mathfrak{P}(t) = \lim_{j\to\infty} \mathfrak{P}_{n_j^t}(t) = \lim_{j\to\infty} \frac{\partial}{\partial t}\mathcal{E}_{\varepsilon_{n_j^t}}(t, q_{n_j^t}(t)), \tag{3.47}$$

cf. (3.45) and (3.43). Taking into account the energy bound $\mathfrak{G}_{\tau,h,\varepsilon}(t)$ obtained in Step 1 and the compactness assumption (3.5), we can even assume that also $q_{n_j^t}(t)$ converges to some $q(t)$. By (3.42), $q(t) = (u(t), z(t))$ with $z(t)$ just from (3.42). Let $t_j := \max\{\theta \in [0,t];\ \theta = t_k^{\tau_{n_j}},\ k = 0, ..., k_{\tau_{n_j}}\}$. Then $q_{n_j^t}(t) \in \mathcal{S}_{\varepsilon_{n_j^t},h_{n_j^t}}(t_j)$. Obviously, also $t_j \to t$. Hence we can use (3.13) to obtain

$$\lim_{j\to\infty} \frac{\partial}{\partial t}\mathcal{E}_{\varepsilon_{n_j^t},h_{n_j^t}}(t_j, q_{n_j^t}(t)) = \frac{\partial}{\partial t}\mathcal{E}(t, q(t)). \tag{3.48}$$

Comparing it with (3.47) we get

$$\frac{\partial}{\partial t}\mathcal{E}(t, q(t)) = \mathfrak{P}(t). \tag{3.49}$$

Thus $u(t)$ forming the pair $q(t) = (u(t), z(t))$ lies in $\mathcal{A}(t)$ from (3.46). Ranging t over $[0,T]$ thus yields a mapping $u : [0,T] \to \mathcal{U}$ with $u(t) \in \mathcal{A}(t)$ for all $t \in [0,T]$.

Step 3: Stability of the limit process. The stability of the limit process q is now ensured by (3.16) as a direct consequence of Lemma 3.3. For fixed $t \in (0,T]$ consider $q_{n_j^t}(t)$ and t_j converging for $j \to \infty$ to $q(t)$ and t in the position of q_n and t_n in the condition (3.16), respectively, and then Lemma 3.3 just yields $q(t) \in \mathcal{S}(t)$. For $t = 0$, stability of $q(0) = q_0$ holds by assumption.

Step 4: Upper energy estimate. By (3.26) with $r = 0$ we have $\mathfrak{G}_{\tau_n,h_n,\varepsilon_n}(t) + \mathfrak{D}_{\tau_n,h_n,\varepsilon_n}(t) - \mathfrak{G}_{\tau_n,h_n,\varepsilon_n}(0) \leq \int_0^t \mathfrak{P}_n(s)\,\mathrm{d}s$ for any $t = t_{\tau_n}^k$, $k = 0, ..., k_{\tau_n}$. For a general $t \in [0,T]$, this inequality is fulfilled with an accuracy $\mathcal{O}(\tau_n)$; this is because $|\mathfrak{G}_{\tau_n,h_n,\varepsilon_n}(t) - \mathfrak{G}_{\tau_n,h_n,\varepsilon_n}(t_{\tau_n}^{k-1})| \leq \tau_n \|\mathfrak{P}_n\|_{L^\infty(0,T)}$ for $t \in [t_{\tau_n}^{k-1}, t_{\tau_n}^k)$ and because also $|\int_0^t \mathfrak{P}_n(s)\,\mathrm{d}s - \int_0^{t_{\tau_n}^{k-1}} \mathfrak{P}_n(s)\,\mathrm{d}s| \leq \tau_n \|\mathfrak{P}_n\|_{L^\infty(0,T)}$ while there is no additional error in the piece-wise constant $\mathfrak{D}_{\tau_n,h_n,\varepsilon_n}$. By the convergence properties (3.42), (3.44) and (3.45) with Fatou's lemma we get

$$\mathfrak{G}(t) + \mathfrak{D}(t) - \mathfrak{G}(0) \leq \int_0^t \mathfrak{p}(s)\mathrm{d}s \leq \int_0^t \mathfrak{P}(s)\mathrm{d}s. \tag{3.50}$$

Using further (3.8), (3.42), and the notation from Step 2, we have

$$\mathcal{E}(t, q(t)) \leq \liminf_{j\to\infty} \mathcal{E}_{\varepsilon_{n_j^t}, h_{n_j^t}}(t_j, q_{n_j^t}(t)) = \lim_{j\to\infty} \mathfrak{G}_{n_j^t}(t) = \mathfrak{G}(t). \tag{3.51}$$

By (3.15b) with $s = 0$ and $\mathfrak{D}(s) = \mathfrak{D}(0) = 0$, we have $\mathrm{Var}_{\mathcal{D}}(q; 0, t) \leq \mathfrak{D}(t)$. Moreover, by (3.32) we have $\mathfrak{G}(0) = \mathcal{E}(0, q(0))$. Inserting this into (3.50) and using still (3.49), we obtain

$$\begin{aligned} \mathcal{E}(t, q(t)) + \mathrm{Var}_{\mathcal{D}}(q; 0, t) - \mathcal{E}(0, q(0)) &\leq \mathfrak{G}(t) + \mathfrak{D}(t) - \mathfrak{G}(0) \\ &\leq \int_0^t \mathfrak{P}(s)\,\mathrm{d}s = \int_0^t \frac{\partial}{\partial s}\mathcal{E}(s, q(s))\,\mathrm{d}s, \end{aligned} \tag{3.52}$$

which is the desired upper energy estimate.

Step 5: Lower energy estimate. The opposite estimate $\mathcal{E}(t, q(t)) + \mathrm{Var}_{\mathcal{D}}(q; 0, t) - \mathcal{E}(0, q(0)) \geq \int_0^t \frac{\partial}{\partial s}\mathcal{E}(s, q(s))\mathrm{d}s$ is a consequence of the stability which is already established in Step 3. We refer to [36, Prop. 5.7] or also [39, Prop. 2.4] for this technical proof where (3.49) with $\mathfrak{P} \in L^\infty(0,T)$ and (3.11) have been used. Thus, we have proved that $q : [0,T] \to \mathcal{Q}$ is a solution.

Step 6: Improved convergence. Having energy equality, we conclude that in (3.50) all the inequalities must be equalities. In particular, this implies

$$\mathfrak{p}(t) = \mathfrak{P}(t), \quad \mathfrak{G}(t) = \mathcal{E}(t, q(t)) \quad \text{and} \quad \mathrm{Var}(q; 0, t) = \mathfrak{D}(t). \tag{3.53}$$

Together with the convergence properties established in Step 2, we obtain the assertions (i)–(iii). Finally, employing [14, Prop. A.2] together with $\mathfrak{p} = \mathfrak{P}$ yields (iv). $\square$

Remark 3.9 (*Two-sided energy estimate (3.26)–(3.27).*) In fact, (3.26)–(3.27) was used only to prove the a-priori BV-bound for $\mathfrak{G}_{\tau,h,\varepsilon}$ in Step 1. This bound is not really needed, since we may postpone the definition of $\mathfrak{G}:[0,T]\to\mathbb{R}$ from Step 2 to Step 4 and set $\mathfrak{G}(t)=\limsup_{n\to\infty}\mathfrak{G}_{\tau_n,h_n,\varepsilon_n}(t)$. Then (3.50) and (3.51) remain true but the last equality in (3.51) which has to be replaced by "$\leq$". Finally, Step 5 implies $\mathfrak{G}(t)=\mathcal{E}(t,q(t))$ as before. However, the two-sided energy estimate (3.26)–(3.27) has its own relevance as it can be used to check implementation of numerical calculations. Namely, evaluating the terms in (3.26)–(3.27) at each time step and checking a-posteriori the estimate (3.26)–(3.27) may detect, e.g., a failure of the minimization procedure, which we have to apply to solve numerically the global optimization problem (3.22) at every current time step; see [31] for numerical results in a concrete example. Violation of (3.26) or (3.27) mean that $q^k_{\tau,h,\varepsilon}$ or $q^{k-1}_{\tau,h,\varepsilon}$ cannot be stable, respectively.

Remark 3.10 (*Numerical integration.*) Another approximation of $\mathcal{E}_\varepsilon$ and $\mathcal{D}_\varepsilon$ involving, e.g., numerical integration can quite easily be incorporated, too. For this, $\mathcal{E}_\varepsilon$ and $\mathcal{D}_\varepsilon$ in the conditions in Sect. 3.1 as well as (3.16) should additionally depend on h by still another way than only by adding $\delta_{\mathcal{Q}_h}$. As such a generalization would complicate, in particular, Section 4 and as it will not be used in Section 5, we have omitted it completely.

4 Linear structure

We consider now the case that $\mathcal{U}$ and $\mathcal{Z}$ are subsets of some reflexive separable Banach spaces U and Z, respectively. This enables more detailed considerations.

4.1 Setting the data and their approximation

The weak topology, if restricted on bounded convex sets, will play the role of the sequentially compact topology used in Sect. 3 for (3.3)–(3.8), (3.13), and (3.16); yet cf. Remark 3.2 for (3.13). Here we will denote it by "w-lim" or "$\stackrel{\mathrm{w}}{\to}$" to distinguish it from the norm topology which we will denote by "s-lim" or "$\stackrel{\mathrm{s}}{\to}$". In case of non-reflexive spaces having preduals, we could work with weak* topologies instead of the weak ones. For an abstract parameter $h>0$, we consider finite-dimensional subspaces $U_h\subset U$ and $Z_h\subset Z$. The concrete constructions of $Q_h:=U_h\times Z_h$ used in numerical analysis are created by (here an abstract) "(quasi-)interpolation" linear bounded operators $\Pi_{U,h}:U\to U$ and $\Pi_{Z,h}:Z\to Z$. We put $\Pi_h=\Pi_{U,h}\times\Pi_{Z,h}:Q\to Q$, and

$$\mathcal{U}_h:=\Pi_{U,h}\mathcal{U},\qquad \mathcal{Z}_h:=\Pi_{Z,h}\mathcal{Z},\qquad \mathcal{Q}_h:=\mathcal{U}_h\times\mathcal{Z}_h=\Pi_h\mathcal{Q}. \tag{4.1}$$

To guarantee the central condition (3.16), we assume the natural basic approximation property that Π_h converges pointwise to the identity, i.e.

$$\forall q\in Q:\qquad \operatorname*{s-lim}_{h\to0}\Pi_h q=q. \tag{4.2}$$

The quasi-interpolation operators need not be conformal with constraints involved implicitly in $\mathcal{U}$ and $\mathcal{Z}$ so that $\mathcal{Q}_h$ need not be a subset of $\mathcal{Q}$. As an analytical tool the Γ-convergence approach allow also for such situations (cf. [39]) but, in order to use the theory from Sections 2–3 in a quantitative numerical way, we will always restrict ourselves on "conformal" situations when

$$\Pi_{U,h}\mathcal{U} \subset \mathcal{U} \qquad \text{and} \qquad \Pi_{U,h}\mathcal{Z} \subset \mathcal{Z}; \tag{4.3}$$

i.e. $\mathcal{Q}_h = \Pi_h \mathcal{Q} \subset \mathcal{Q}$. Possible "nonconformities" can be handled via the penalization parameter ε.

For X another Banach space, it is often useful to consider a mapping $\Xi : U \times Z \to X$ to describe possible equality constraints of the form $\Xi(u,z) = 0$ that may implicitly be involved in the definition of $\mathcal{E}$. Moreover, we assume the forcing by $f : [0,T] \to U^* \times Z^*$ to be given explicitly in $\mathcal{E}$, which covers many applications (except, e.g., "hard-device" loading of mechanical systems through Dirichlet boundary conditions). Then, for $E : U \times Z \to \mathbb{R}$ we consider

$$\mathcal{E}(t,u,z) := \begin{cases} E(u,z) - \langle f(t),(u,z)\rangle & \text{if } u \in \mathcal{U},\ z \in \mathcal{Z},\ \Xi(u,z) = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.4}$$

The approximate energy deals with possible incompatibility of the finite-dimensional discretization with the equality constraints by a penalization of them (cf. [49]):

$$\mathcal{E}_\varepsilon(t,u,z) := \begin{cases} E(u,z) - \langle f(t),(u,z)\rangle + \dfrac{1}{\varepsilon}\big\|\Xi(u,z)\big\|_X^\alpha & \text{if } u \in \mathcal{U},\ z \in \mathcal{Z}, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.5}$$

To satisfy (3.4), we assume a super-linear growth of E to dominate the linear behavior of $\langle f(t), \cdot\rangle$:

$$\lim_{\substack{q \in \mathcal{Q} \\ \|q\| \to \infty}} \frac{E(q)}{\|q\|} = +\infty. \tag{4.6}$$

Obviously, (4.4) and (4.5) yield simply $\frac{\partial}{\partial t}\mathcal{E}(t,q) = \frac{\partial}{\partial t}\mathcal{E}_\varepsilon(t,q) = \langle \frac{\partial}{\partial t} f(t), q\rangle$ and (3.9a) requires

$$f \in C^1([0,T]; Q^*). \tag{4.7}$$

The coercivity (4.6) with (4.7) ensure also (3.9b), (3.11) and (3.13).

A quite canonical way to induce the dissipation distances in simpler cases is through a *degree-1 homogeneous dissipation potentials*. For this, we consider $K \subset Z$ a closed convex cone with the vertex at 0, $R : Z \to \mathbb{R}$ a continuous convex degree-1 homogeneous functional, i.e. $R(az) = aR(z)$ for any $z \in Z$ and $a \geq 0$. Then we consider the special case of $\mathcal{D}$ defined by

$$\mathcal{D}(z_1,z_2) := \begin{cases} R(z_2 - z_1) & \text{if } z_2 - z_1 \in K, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.8}$$

Note that $\mathcal{D}(z_1, z_1) = 0$ and the triangle inequality (2.1) holds. As R is convex and continuous and K convex closed, $\mathcal{D} : Z \times Z \to \mathbb{R} \cup \{+\infty\}$ is weakly lower semicontinuous.

If $K \neq Z$, then it might be numerically suitable to avoid the unilateral constraints involved by exact penalization by choosing the approximate potential $\mathcal{D}_\varepsilon$ in the form

$$\mathcal{D}_\varepsilon(z_1, z_2) := R_\varepsilon(z_2 - z_1) \quad \text{where} \quad R_\varepsilon(z) := R(z) + \inf_{\hat{z} \in K} \frac{\|z - \hat{z}\|}{\varepsilon}. \tag{4.9}$$

As K is a cone, R_ε is again a homogeneous degree-1 functional for any $\varepsilon > 0$ and (3.2) thus holds. As R is convex and continuous and K is convex, R_ε is convex and continuous, and the weak lower-semicontinuity (3.3) of R_ε holds, too. Note that always $R_\varepsilon \leq R + \delta_K$. Unfortunately, smoothening of $R + \delta_K$ e.g. by Yosida's approximation, which would be sometimes numerically desirable, does not fulfill (3.2) and expectedly nontrivial modifications of the theory in Sect. 3 would then be needed.

The stability (3.31) of the initial condition q_0 is, in general, difficult to verify and explicit constructions can be done in very special cases only. Anyhow, there is one universal way how to design a "gentle start", namely taking $q_0 = (u_0, z_0)$ minimizing $\mathcal{E}(0, \cdot)$, i.e. here a solution to the problem

$$\left.\begin{array}{ll} \text{minimize} & E(u, z) - \langle f(0), (u, z) \rangle, \\ \text{subject to} & \Xi(u, z) = 0, \quad u \in \mathcal{U}, \quad z \in \mathcal{Z}. \end{array}\right\} \tag{4.10}$$

Such a "gentle start" is, in fact, practically the only option applied in engineering simulations.

The other assumptions from Sect. 3 deserve a more detailed proof.

Proposition 4.1 *(**Verification of (3.5)–(3.8).**) Let E be weakly lower semicontinuous, $\Xi : Q \to X$ be weakly continuous, and let K be convex and closed, R be convex and also positive on $K \setminus \{0\}$, i.e.*

$$\forall z \in K : \quad z \neq 0 \quad \Rightarrow \quad R(z) > 0. \tag{4.11}$$

Then (3.5)–(3.8) with "$\to$" referring to the weak topology hold.

Proof. In view of (4.4), the condition $\mathcal{E}_\varepsilon(\theta, q_{h,\varepsilon}) \leq a < +\infty$ in (3.5) implies $E(q_{h,\varepsilon}) \leq C + \langle f(\theta), q_{h,\varepsilon} \rangle$, and by (4.6) a sequence of $\{q_{h,\varepsilon}\}_{h,\varepsilon>0}$ must be bounded hence it has a subsequence which converges weakly (recall that we assume reflexivity of Q), which proves (3.5).

As to (3.6), for $z_2 - z_1 \in K$ we have

$$\begin{aligned} \liminf_{(h,\varepsilon)\to(0,0)} \mathcal{D}_\varepsilon(z_{h,\varepsilon}, \tilde{z}_{h,\varepsilon}) &= \liminf_{(h,\varepsilon)\to(0,0)} R(\tilde{z}_{h,\varepsilon} - z_{h,\varepsilon}) + \inf_{\hat{z} \in K} \frac{\|\tilde{z}_{h,\varepsilon} - z_{h,\varepsilon} - \hat{z}\|}{\varepsilon} \\ &\geq \liminf_{(h,\varepsilon)\to(0,0)} R(\tilde{z}_{h,\varepsilon} - z_{h,\varepsilon}) \geq R(\tilde{z} - z) = \mathcal{D}(z, \tilde{z}) \end{aligned} \tag{4.12}$$

because R is weakly lower semicontinuous. If $z_2-z_1 \notin K$, then $\inf_{\hat{z}\in K} \|\tilde{z}-z-\hat{z}\| > 0$ because K is closed. Using also (4.11), we then have

$$\liminf_{(h,\varepsilon)\to(0,0)} \mathcal{D}_\varepsilon(z_{h,\varepsilon}, \tilde{z}_{h,\varepsilon}) \geq \lim_{(h,\varepsilon)\to(0,0)} \inf_{\hat{z}\in K} \frac{\|z_{h,\varepsilon} - \tilde{z}_{h,\varepsilon} - \hat{z}\|}{\varepsilon} = +\infty = \mathcal{D}(z, \tilde{z}). \tag{4.13}$$

To prove (3.7), take $z \in Z$ and a sequentially weakly compact $\mathcal{K}$ in Z and a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathcal{K}$ with $\lim_{n\to\infty} \min(\mathcal{D}(z_n, z), \mathcal{D}(z, z_n)) = 0$. For a subsequence we have $z_{n_j} \xrightarrow{\text{w}} \tilde{z}$, and the mentioned weak lower semicontinuity of $\mathcal{D}$ implies either $\mathcal{D}(z, \tilde{z}) = 0$ or $\mathcal{D}(\tilde{z}, z) = 0$. Thus we can conclude $\tilde{z} = z$ and the whole sequence must weakly converge, which proves (3.7).

As to (3.8), let us distinguish whether $\Xi(q) = 0$ or $\Xi(q) \neq 0$. The former case ensures the last equality in the following estimate:

$$\begin{aligned}\liminf_{(h,\varepsilon,\theta)\to(0,0,t)} \mathcal{E}_\varepsilon(\theta, q_{h,\varepsilon}) &= \liminf_{(h,\varepsilon)\to(0,0)} E(q_{h,\varepsilon}) - \langle f(\theta), q_{h,\varepsilon}\rangle + \frac{1}{\varepsilon}\|\Xi(q_{h,\varepsilon})\|_X^\alpha \\ &\geq \liminf_{(h,\varepsilon,\theta)\to(0,0,t)} E(q_{h,\varepsilon}) - \langle f(\theta), q_{h,\varepsilon}\rangle \geq E(q) - \langle f(t), q\rangle \;=\; \mathcal{E}(t, q),\end{aligned} \tag{4.14}$$

where the last inequality is by the weak lower semicontinuity of E. This proves that (3.8) holds with respect to the weak topology if $\Xi(q) = 0$. In the case $\Xi(q) \neq 0$, $q_{h,\varepsilon} \xrightarrow{\text{w}} q$ and the weak continuity of Ξ ensures $\liminf \|\Xi(q_{h,\varepsilon})\|_X \geq \|\text{w-}\lim \Xi(q_{h,\varepsilon})\|_X = \|\Xi(q)\|_X > 0$. Then, because of the coercivity (4.6) of E, we have

$$\liminf_{\substack{\theta\to t\\(h,\varepsilon)\to(0,0)}} \mathcal{E}_\varepsilon(\theta, q_{h,\varepsilon}) \geq \inf_{\substack{\tilde{q}\in Q\\ \theta\in[0,T]}} \big[E-f(\theta)\big](\tilde{q}) + \lim_{(h,\varepsilon)\to(0,0)} \frac{1}{\varepsilon}\big\|\Xi(q_{h,\varepsilon})\big\|_X^\alpha = +\infty = \mathcal{E}(t, q).$$

$\square$

In view of the above considerations, we have guaranteed the assumptions needed in Theorem 3.8 except (3.16) and (3.32). This conditions are still to be verified in particular cases, some of them scrutinized in Sections 4.2–4.4.

Remark 4.2 (*BV-estimates.*) Assuming coercivity of $R+\delta_K$ on some Banach space $Z_1 \supset Z$, i.e.

$$\lim_{z\in K,\ \|z\|_{Z_1}\to\infty} R(z) = +\infty, \tag{4.15}$$

together with the degree-1 homogeneity will make (4.11) more specific, namely $[R+\delta_K](z) \geq c\|z\|_{Z_1}$ with some $c > 0$, hence by (4.8) also $\mathcal{D}(q_1, q_2) \geq c\|z_1 - z_2\|_{Z_1}$, and by the definition of "Var" in (2.6) then also

$$\mathrm{Var}_{\mathcal{D}}(q; 0, T) \geq c\mathrm{Var}_{\|\cdot\|_{Z_1}}(z; 0, T). \tag{4.16}$$

In view of the definition (2.6) applied now with the norm $\|\cdot\|_{Z_1}$, the last expression is just the standard total variation and the estimate (3.38) yields boundedness of $z_{\tau,h,\varepsilon}$ and thus also the limit z in the bounded-variation space $\mathrm{BV}(0, T; Z_1)$.

4.2 The case $K = Z$

Let us consider an additional norm $|\cdot|$, which may induce a weaker topology than the canonical norm making Q a Banach space.

Proposition 4.3 ***(Verification of (3.16) and (3.32) for*** *$K{=}Z$.) Let (4.6) and (4.7) hold, and let $\alpha \geq 1$, let $E : Q \to \mathbb{R}$ in (4.4) be weakly lower semicontinuous and norm continuous, both $\Xi : Q \to X$ and $R : Z \to \mathbb{R}$ be weakly continuous, and $K = Z$ (hence $R_\varepsilon \equiv R$), and Ξ be also Lipschitz continuous with respect to $|\cdot|$, i.e.*

$$\exists \ell_\Xi \in \mathbb{R} \ \ \forall q_1, q_2 \in \mathcal{Q}: \qquad \big\|\Xi(q_1) - \Xi(q_2)\big\|_X \leq \ell_\Xi \big|q_1 - q_2\big| \tag{4.17}$$

and let the operator Π_h satisfies the convergence-rate estimate

$$\exists \gamma > 0,\ C \in \mathbb{R} \ \ \forall q \in \mathcal{Q}: \qquad \big|q - \Pi_h q\big| \leq C h^\gamma \big\|q\big\|. \tag{4.18}$$

Then (3.16) and (3.32) with $q_0 \in \mathcal{S}(0)$ are satisfied, the last two conditions relying on the convergence criterion

$$H(\varepsilon) = o\big(\varepsilon^{\frac{1}{\alpha\gamma}}\big) \quad \text{and with} \quad \tilde{q}_{h,\varepsilon} \overset{\mathrm{s}}{\to} \tilde{q}. \tag{4.19}$$

Proof. Let us prove (3.16). For any $\tilde{q} \in Q$, with $\Xi(\tilde{q}) = 0$, by (4.17) and (4.18), we have

$$\big\|\Xi(\Pi_h\tilde{q})\big\|_X = \big\|\Xi(\Pi_h\tilde{q}) - \Xi(\tilde{q})\big\|_X \leq \ell_\Xi\big|\tilde{q} - \Pi_h\tilde{q}\big| \leq C\ell_\Xi h^\gamma\big\|\tilde{q}\big\|. \tag{4.20}$$

For $(h,\varepsilon) \to (0,0)$ with $h \leq H(\varepsilon)$ with H from (4.19) we therefore have

$$\frac{1}{\varepsilon}\big\|\Xi(\Pi_h\tilde{q})\big\|_X^\alpha \leq C^\alpha \ell_\Xi^\alpha \frac{h^{\alpha\gamma}}{\varepsilon}\big\|\tilde{q}\big\|^\alpha \to 0. \tag{4.21}$$

We put $\tilde{q}_{h,\varepsilon} := \Pi_h\tilde{q}$ for (3.16); note that, in fact, we do not need any explicit dependence on ε except that we assume $h \leq H(\varepsilon)$. As E is strongly continuous and, by (4.2), $\tilde{q}_{h,\varepsilon} \overset{\mathrm{s}}{\to} \tilde{q}$, and as R is weakly continuous and $q_{h,\varepsilon} \overset{\mathrm{w}}{\to} q$ is assumed in (3.16), it holds

$$\begin{aligned}\lim_{\substack{h \leq H(\varepsilon)\\(\varepsilon,h)\to(0,0)}} \mathcal{E}_\varepsilon(\theta, \tilde{q}_{h,\varepsilon}) + \mathcal{D}(q_{h,\varepsilon}, \tilde{q}_{h,\varepsilon}) &= \lim_{\substack{h \leq H(\varepsilon)\\(\varepsilon,h)\to(0,0)}} E(\tilde{q}_{h,\varepsilon}) - \langle f(\theta), \tilde{q}_{h,\varepsilon}\rangle + R(\tilde{q}_{h,\varepsilon} - q_{h,\varepsilon})\\ &\quad + \frac{1}{\varepsilon}\big\|\Xi(\Pi_h\tilde{q})\big\|_X^\alpha = E(\tilde{q}) - \langle f(t), \tilde{q}\rangle + R(\tilde{q} - q) = \mathcal{E}(t, \tilde{q}) + \mathcal{D}(q, \tilde{q})\end{aligned}$$

whenever $\Xi(\tilde{q}) = 0$. Combining this with (3.8), we obtain (3.16) for $\Xi(\tilde{q}) = 0$. If $\Xi(\tilde{q}) \neq 0$, then due to the definition (4.4) the right-hand side in (3.16) is $+\infty$ and (3.16) is fulfilled trivially.

The stability of q_0 considered in Theorem 3.8 implies $\mathcal{E}(0, q_0) < +\infty$, and then the assumption (3.32) is fulfilled if one chooses

$$\big[q_0\big]_{h,\varepsilon} := \Pi_h q_0 \tag{4.22}$$

in (3.32). Indeed, $[q_0]_{h,\varepsilon} \overset{\mathrm{s}}{\to} q_0$ for $h \to 0$ just by (4.2) and then also $\mathcal{E}_\varepsilon(0, [q_0]_{h,\varepsilon}) = E(\Pi_h q_0) + \frac{1}{\varepsilon}\|\Xi(\Pi_h q_0)\|_X^\alpha - \langle f(0), \Pi_h q_0\rangle \to E(q_0) - \langle f(0), q_0\rangle = \mathcal{E}(0, q_0)$ because E is assumed norm continuous and because, since the finite energy of q_0 implies $\Xi(q_0) = 0$, we can employ the estimate (4.21) for $h \leq H(\varepsilon)$. $\square$

4.3 The case $K \subsetneqq Z$

Certain applications to unidirectional processes (like damage, delamination, debonding, or hardening in plasticity or in ferromagnets) require modelling with $K \subsetneqq Z$. This needs further finer investigations for which we consider some topology σ on $U \times Z$ which is finer than the weak one and coarser than the norm one; see the particular examples in Sect. 5.

Proposition 4.4 *(**Unconditional convergence for** $K \subsetneqq Z$.) Let $E : Q \to \mathbb{R}$ be weakly lower semicontinuous and σ-continuous. Assume both $R : Z \to \mathbb{R}$ and $\Xi : Q \to X$ be weakly continuous, and let (4.7), and that the following attainability condition, expressing certain consistency of the discretization with the constraints given by Ξ and K, hold:*

$$\begin{aligned}\forall q, \tilde{q} \in \mathcal{Q}, \quad \Xi(q) = 0, \quad \tilde{q}-q \in K, \quad \Xi(\tilde{q}) = 0, \ \forall q_h \in \mathcal{Q}_h, \ q_h \xrightarrow{\mathrm{w}} q \\ \exists \tilde{q}_h \in \mathcal{Q}_h : \quad \tilde{q}_h \xrightarrow{\sigma} \tilde{q}, \ \big\|\Xi(\tilde{q}_h)\big\|_X \le \big\|\Xi(q_h)\big\|_X, \ \tilde{q}_h - q_h \in K.\end{aligned} \tag{4.23}$$

Then (3.16) with $\mathcal{D}_{\varepsilon,h}$ from (4.9) is satisfied, now with $H \equiv 1$, i.e. "unconditionally". Moreover, the qualification (3.32) of the stable initial condition q_0 holds if

$$\exists q_{0h} \in \mathcal{Q}_h : \quad \Xi(q_{0h}) = 0 \ \ \& \ \ q_{0h} \xrightarrow{\sigma} q_0. \tag{4.24}$$

Proof. The a-priori bound $\mathcal{E}_{\varepsilon,h}(\theta, q_{h,\varepsilon}) \le C$ assumed in (3.16) means

$$\frac{1}{\varepsilon}\big\|\Xi(q_{h,\varepsilon})\big\|_X^\alpha \le C - E(q_{h,\varepsilon}) + \big\langle f(\theta), q_{h,\varepsilon}\big\rangle \le C + \sup_{\substack{q \in Q \\ \theta \in [0,T]}} \big[f(\theta) - E\big](q) < +\infty \tag{4.25}$$

due to (4.6) so that $\|\Xi(q_{h,\varepsilon})\|_X = \mathcal{O}(\varepsilon^{1/\alpha})$. In the limit therefore $\Xi(q) = 0$ because Ξ is assumed weakly continuous. Thus we take $q_{h,\varepsilon}$ from (3.16) for q_h in (4.23). As (3.16) is trivially satisfied if $\Xi(\tilde{q}) \ne 0$ because the right-hand side in (3.16) is $+\infty$, we can consider only $\Xi(\tilde{q}) = 0$. Then we can take $\tilde{q}_h$ from (4.23) for $\tilde{q}_{h,\varepsilon}$ in (3.16). Note that $\tilde{q}_{h,\varepsilon} - q_{h,\varepsilon} \in K$ in (4.23) ensures $\mathcal{D}_{\varepsilon,h}(q_{h,\varepsilon}, \tilde{q}_{h,\varepsilon}) = R(q_{h,\varepsilon} - \tilde{q}_{h,\varepsilon})$ due to the definition (4.9) and by the assumed weak continuity of R and closedness and convexity of K, we have

$$\lim_{(h,\varepsilon)\to(0,0)} \mathcal{D}_{\varepsilon,h}(q_{h,\varepsilon}, \tilde{q}_{h,\varepsilon}) = \lim_{(h,\varepsilon)\to(0,0)} R(\tilde{q}_{h,\varepsilon} - q_{h,\varepsilon}) = R(\tilde{q} - q) = \mathcal{D}(q, \tilde{q}). \tag{4.26}$$

Then, using the σ-continuity and weak lower semicontinuity of E the continuity of f (see (4.7)), and $\|\Xi(\tilde{q}_{h,\varepsilon})\|_X \le \|\Xi(q_{h,\varepsilon})\|_X$ (see (4.23)), we obtain

$$\begin{aligned}&\limsup_{(h,\varepsilon,\theta)\to(0,0,t)} \Big(\mathcal{E}_{\varepsilon,h}(\theta, \tilde{q}_{h,\varepsilon}) + \mathcal{D}_{\varepsilon,h}(q_{h,\varepsilon}, \tilde{q}_{h,\varepsilon}) - \mathcal{E}_{\varepsilon,h}(\theta, q_{h,\varepsilon})\Big) = \limsup_{(h,\varepsilon,\theta)\to(0,0,t)} \Big(E(\tilde{q}_{h,\varepsilon}) \\ &\quad - \langle f(\theta), \tilde{q}_{h,\varepsilon} - q_{h,\varepsilon}\rangle + \frac{1}{\varepsilon}\|\Xi(\tilde{q}_{h,\varepsilon})\|_X^\alpha + \mathcal{D}(q_{h,\varepsilon}, \tilde{q}_{h,\varepsilon}) - E(q_{h,\varepsilon}) - \frac{1}{\varepsilon}\|\Xi(\tilde{q}_{h,\varepsilon})\|_X^\alpha\Big) \\ &\quad \le \limsup_{(h,\varepsilon,\theta)\to(0,0,t)} \Big(E(\tilde{q}_{h,\varepsilon}) - \langle f(\theta), \tilde{q}_{h,\varepsilon} - q_{h,\varepsilon}\rangle + \mathcal{D}(q_{h,\varepsilon}, \tilde{q}_{h,\varepsilon}) - E(q_{h,\varepsilon})\Big) \\ &\quad = \lim_{(h,\varepsilon,\theta)\to(0,0,t)} \Big(E(\tilde{q}_{h,\varepsilon}) - \langle f(\theta), \tilde{q}_{h,\varepsilon} - q_{h,\varepsilon}\rangle + \mathcal{D}(q_{h,\varepsilon}, \tilde{q}_{h,\varepsilon})\Big) - \liminf_{(h,\varepsilon)\to(0,0)} E(q_{h,\varepsilon}) \\ &\quad \le E(\tilde{q}) - \langle f(t), \tilde{q}-q\rangle + \mathcal{D}(q, \tilde{q}) - E(q) = \mathcal{E}(t, \tilde{q}) + \mathcal{D}(q, \tilde{q}) - \mathcal{E}(t, q).\end{aligned} \tag{4.27}$$

Eventually, we are to prove (3.32) provided (4.24) and provided $q_0 \in \mathcal{S}(0)$; the last inclusion implies $\mathcal{E}(0,q_0) < +\infty$ which here further implies $\Xi(q_0) = 0$). Then, with $[q_0]_{h,\varepsilon} := q_{0h}$ in (4.24), it holds

$$\mathcal{E}_{\varepsilon,h}(0,[q_0]_{h,\varepsilon}) = E(q_{0h}) - \langle f(0), q_{0h}\rangle \to E(q_0) - \langle f(0), q_0\rangle = \mathcal{E}(0,q_0) \tag{4.28}$$

as required in (3.32) because $\mathcal{E}$ is assumed σ-continuous. Note that the last equality in (4.28) relies on $\Xi(q_0) = 0$ for which σ-continuity of Ξ is needed; in fact, we assumed even weak continuity of Ξ. □

4.4 The case $K \subsetneqq Z$ and "semiquadratic" E.

Some applications exhibits the "main" part of the stored energy E quadratic in terms of the dissipating variable z in the sense

$$E(u,z) := \frac{1}{2}\langle Bz, z\rangle + E_0(u,z)\,, \qquad B : Z \to Z^* \text{ linear and bounded,}$$
$$E_0 : U\times Z \to \mathbb{R} \text{ (s}\times\text{w)-continuous.} \tag{4.29}$$

In smooth cases, this corresponds to problems governed by "semilinear" mappings $E'(q) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & B\end{smallmatrix}\right) + E_0'(q)$. Such problems are well fitted for unconditional convergence under some particular circumstances.

As to (3.32), we can guarantee it again through (4.24) now with σ the strong topology to have the quadratic term in (4.29) continuous. The verification of (3.16) is now more sophisticated:

Proposition 4.5 *(**"Semiquadratic" case: unconditional convergence.**) Let (4.7) and (4.29) hold, R be continuous, let further Ξ be independent of u, affine and continuous, i.e. in the form $\Xi(u,z) = \Xi_0(z) + \xi$ with $\xi \in X$, and $\Xi_0 \in \mathcal{L}(Z,X)$ compatible with the discretization operator $\Pi_{Z,h}$ in the sense that $\Pi_{Z,h}(\mathrm{Ker}\,\Xi_0) \subset \mathrm{Ker}\,\Xi_0$. Let also $\mathcal{Z} + K \subset \mathcal{Z}$, and the cone K be compatible with $\Pi_{Z,h}$ in the sense that $\Pi_{Z,h}K \subset K$. Then (3.16) with $H \equiv 1$ holds.*

Proof. We will prove (3.16) by using Proposition 4.4 and for this we will verify (4.23) with σ being the strong$\times$weak topology on $U \times Z$. The recovery element $\tilde{q}_h$ in (4.23) can be chosen simply as

$$\tilde{u}_h := \Pi_{U,h}\tilde{u}, \tag{4.30a}$$
$$\tilde{z}_h := z_h + \Pi_{Z,h}(\tilde{z} - z). \tag{4.30b}$$

It holds $\tilde{q}_h \in \mathcal{Q}_h$; indeed, $\tilde{u}_h \in \mathcal{U}_h$ just by the definitions (4.1) and (4.30a) while $\tilde{z}_h \in \mathcal{Z}_h$ because $\tilde{z} - z \in K$, assumed in (4.23), implies $\tilde{z}_h - z_h = \Pi_{Z,h}(\tilde{z} - z) \in \Pi_{Z,h}K$ and further $\mathcal{Z} + K \subset \mathcal{Z}$ implies $\mathcal{Z}_h = \Pi_{Z,h}\mathcal{Z} \supset \Pi_{Z,h}(\mathcal{Z} + K) = \mathcal{Z}_h + \Pi_{Z,h}K$ and eventually $z_h \in \mathcal{Z}_h$ is assumed in (4.23), hence $\tilde{z}_h \in \mathcal{Z}_h$ indeed follows.

Also, the inequality $\|\Xi(q_h)\|_X \le \|\Xi(\tilde q_h)\|_X$ in (4.23) follows from

$$\Xi(\tilde q_h) = \Xi_0\tilde z_h + \xi = \Xi_0(z_h + \Pi_{Z,h}(\tilde z - z)) + \xi = \Xi(q_h) \tag{4.31}$$

because $\Xi_0(\Pi_{Z,h}(z-\tilde z)) = 0$ holds. Indeed, $\Xi(\tilde q) = 0$ is also explicitly assumed in (4.23) while $\Xi(q) = 0$ follows from $q_h \overset{\text{w}}{\to} q$ assumed in (4.23) by the continuity of Ξ, and therefore $\Xi_0(z-\tilde z) = \Xi(q) - \Xi(\tilde q) = 0$, hence $z - \tilde z \in \operatorname{Ker}\Pi_0$, and by the assumed compatibility $\Pi_{Z,h}(\operatorname{Ker}\Xi_0) \subset \operatorname{Ker}\Xi_0$ also $\Pi_{Z,h}(z-\tilde z) \in \operatorname{Ker}\Xi_0$, hence eventually $\Xi_0(\Pi_{Z,h}(z-\tilde z)) = 0$. Then also, by using also (4.2), it holds

$$\begin{aligned}(\text{s}\times\text{w})\text{-}\lim_{h\to0}\ \tilde q_h &= \Big(\ \text{s-}\lim_{h\to0}\ \tilde u_h\ ,\ \text{w-}\lim_{h\to0}\ \tilde z_h\Big)\\ &= \Big(\ \text{s-}\lim_{h\to0}\ \Pi_{U,h}\tilde u\ ,\ \text{w-}\lim_{h\to0}\ z_h + \text{s-}\lim_{h\to0}\ \Pi_{Z,h}(\tilde z{-}z)\Big) = \big(\tilde u, z + (\tilde z{-}z)\big) = \tilde q.\end{aligned}$$

Although for $\sigma=$s$\times$w the energy E itself need not be σ-continuous like in Proposition 4.4, in the case (4.29) it is however possible to pass to the limit in the difference $\mathcal{E}(\theta,\tilde q_h) - \mathcal{E}(\theta,q_h)$ by using (4.31) and the binomial formula:

$$\begin{aligned}\mathcal{E}_\varepsilon(\theta,\tilde q_h) - \mathcal{E}_\varepsilon(\theta,q_h) &= \mathcal{E}(\theta,\tilde q_h) + \tfrac{1}{\varepsilon}\|\Xi(\tilde q_h)\|_X^\alpha - \mathcal{E}(\theta,q_h) - \tfrac{1}{\varepsilon}\|\Xi(q_h)\|_X^\alpha\\ &= \mathcal{E}(\theta,\tilde q_h) - \mathcal{E}(\theta,q_h)\\ &= \tfrac12\langle B\tilde z_h,\tilde z_h\rangle - \tfrac12\langle Bz_h,z_h\rangle + E_0(\tilde q_h) - E_0(q_h) - \langle f(\theta),\tilde q_h{-}q_h\rangle\\ &= \tfrac12\langle B(\tilde z_h - z_h),\tilde z_h + z_h\rangle\ +\ E_0(\tilde q_h) - E_0(q_h) - \langle f(\theta),\tilde q_h{-}q_h\rangle\\ &\to \tfrac12\langle B(\tilde z - z),\tilde z + z\rangle + E_0(\tilde q) - E_0(q) - \langle f(t),\tilde q - q\rangle\\ &= \tfrac12\langle B\tilde z,\tilde z\rangle - \tfrac12\langle Bz,z\rangle + E_0(\tilde q) - E_0(q) - \langle f(t),\tilde q - q\rangle\\ &= \mathcal{E}(t,\tilde q) - \mathcal{E}(t,q).\end{aligned} \tag{4.32}$$

For the limit passage it was important that $\tilde z_h{-}z_h = \Pi_{Z,h}(\tilde z{-}z) \overset{\text{s}}{\to} \tilde z{-}z$ because of (4.2) so that

$$\langle B(\tilde z_h{-}z_h),\tilde z_h{+}z_h\rangle \to \langle B(\tilde z - z),\tilde z + z\rangle \tag{4.33}$$

because $\tilde z_h{+}z_h \overset{\text{w}}{\to} z{+}\tilde z$. We have $\tilde z_h - z_h = \Pi_{Z,h}(\tilde z - z) \in \Pi_{Z,h}K \subset K$. Then, in view of the definition in (4.8) and the strong continuity of R, we have

$$\begin{aligned}\lim_{(\varepsilon,h)\to(0,0)} \mathcal{D}_\varepsilon(q_h,\tilde q_h) &= \lim_{(\varepsilon,h)\to(0,0)} R_\varepsilon(\tilde z_h{-}z_h) = \lim_{h\to0} R(\tilde z_h{-}z_h)\\ &= \lim_{h\to0} R\big(\Pi_{Z,h}(\tilde z{-}z)\big) = R\big(\lim_{h\to0}\Pi_{Z,h}(\tilde z{-}z)\big) = R(\tilde z{-}z) = \mathcal{D}(q,\tilde q).\end{aligned} \tag{4.34}$$

By (4.32) and (4.34), we can pass directly to the limit in (4.27). Thus (3.16) with $H\equiv1$ is proved in this case, too. $\square$

Alternatively to the setting (4.29), we can consider a variant with a fully quadratic "main" part of E:

Proposition 4.6 *(Semiquadratic case II: unconditional convergence.)* *Let*

$$E(q) := \frac{1}{2}\langle Bq, q\rangle + E_0(q)\,, \quad B : Q \to Q^* \ \textit{linear and bounded,}$$
$$E_0 : Q \to \mathbb{R} \ \textit{w-continuous.} \tag{4.35}$$

hold, R be continuous and $\mathcal{U} = U$ and and Ξ be affine and continuous, i.e. in the form $\Xi(q) = \Xi_0 q + \xi$ with $\xi \in X$ and $\Xi_0 \in \mathcal{L}(Q, X)$ such that $\Pi_h(\mathrm{Ker}\,\Xi_0) \subset \mathrm{Ker}\,\Xi_0$. Let again $\mathcal{Z} + K \subset \mathcal{Z}$, $\Pi_{Z,h}K \subset K$, and f satisfy (4.7). Then (3.16) with $H \equiv 1$ holds.

Proof. Instead of (4.30), we take

$$\tilde{q}_h := q_h + \Pi_h(\tilde{q} - q). \tag{4.36}$$

Then it suffices to modify the proof of Proposition 4.5 quite straightforwardly, e.g. to consider q's instead of z's in (4.31) and (4.32). □

Remark 4.7 (*No penalization.*) In case of the unconditional convergence, one can consider a numerical scheme with $\varepsilon = 0$, i.e. with the original $\mathcal{E}$ and $\mathcal{D}$ instead of $\mathcal{E}_{\varepsilon,h}$ and $\mathcal{D}_{\varepsilon,h}$. The corresponding incremental problem might then involve unilateral constraint; cf. also Remark 5.3.

5 Particular examples in continuum mechanics

The doubly-nonlinear inclusion (2.2) is a framework for description of so-called *generalized standard materials* with internal parameters as introduced by Halphen and Nguen [21] in those cases where convexity of stored and dissipated energies can be expected and inertial effects can be neglected. Here we have in mind various *inelastic rate-independent processes* in such materials having possibly a nonconvex stored energy. The following examples illustrate how the general theory applies in particular situations, cf. Table 1 for a survey. As a by-product of the presented numerical theory, we obtain analytical existence/convergence results which have not yet been derived in literature. For the sake of explanatory lucidity, we confine ourselves to rather conventional models from continuum mechanics although some less conventional models (e.g. those involving a microstructure described by so-called Young measures, see [32, 52, 53, 54]) allow for such numerical analysis, too. In Sect. 5.7 we present a combination of mechanical and ferromagnetical effects, i.e. magnetostriction with hysteretical effects, but the combination with ferroelectrical effects, i.e. piezoelectricity with hysteresis (see [43]), or even purely non-mechanical rate-independent models developed in ferromagnetics (e.g. [52, 53, 58, 59]) and ferroelectrics (e.g. [25, 48, 56]) could be treated similarly. We neglect any temperature dependence or, in other words, if there is a possible dependence of data on temperature, we consider sufficiently slow processes so that the released heat due to dissipative processes can efficiently be transferred away to allow for considering isothermal processes.

process / section	unidirectional (i.e. $K \subsetneqq Z$)	constraints (i.e. $X \neq \{0\}$)	quadratic energy E	proposition used
plasticity with hardening at small strains / 5.2	$+$	$-$	$+$	4.5 or 4.6
phase transformation: mixture approach / 5.3	$-$	$-$	$-$	4.4 (σ=s)
phase transformation: non-mixture approach/5.4	$-$	$+$	$-$	4.4 (σ=s)
damage / 5.5	$+$	$-$	$\pm$	4.5
debonding / 5.6	$+$	$-$	$-$	4.4 (σ=s$\times$w*)
magnetostriction / 5.7	$-$	$+$	$-$	4.3

Table 1. Organization and features of the examples presented in Section 5.

5.1 Sketch of continuum mechanics of deformable bodies

We assume a specimen occupying in its *reference configuration* a bounded domain $\Omega \subset \mathbb{R}^3$. As usual, $y : \Omega \to \mathbb{R}^3$ denotes the *deformation* and $u : \Omega \to \mathbb{R}^3$ the *displacement*, related to each other by $y(x) = x+u(x)$, $x \in \Omega$. Hence the *deformation gradient* equals $F = \nabla y = \mathbb{I} + \nabla u$ with $\mathbb{I} \in \mathbb{R}^{3\times3}$ being the identity matrix and ∇ is the gradient operator. For simplicity, we will treat only the soft-device loading realized through traction (Neumann or Robin-type) boundary conditions. The state of the material and possibly also of boundary conditions is assumed to depend on (a set of) certain parameters z that may evolve in time in a rate-independent manner. Then naturally U and Z used before will be the spaces of u's and of z's, respectively.

The specific energy stored in the inter-atomic links in the homogeneous (possibly anisotropic) continuum $\hat\varphi = \hat\varphi(F, z)$ is phenomenologically described as a function of the deformation gradient F and the mentioned variable $z \in \mathbb{R}^m$. Mostly the vector $z \in Z_0 \subset \mathbb{R}^m$ in not directly accessible for a macroscopical loading (for an exception see Sect. 5.7) and will thus play the role of *internal parameters.* The *frame-indifference*, i.e. $\hat\varphi(F, z) = \hat\varphi(RF, z)$ for any $R \in \mathrm{SO}(3)$ = the group of orientation-preserving rotations, requires that $\hat\varphi(\cdot, z)$ in fact depends only on the (right) *Cauchy-Green stretch* tensor

$$F^\top F = (\mathbb{I} + \nabla u)^\top (\mathbb{I} + \nabla u) = \mathbb{I} + (\nabla u)^\top + \nabla u + (\nabla u)^\top \nabla u. \tag{5.1}$$

An important property of $\hat\varphi(\cdot, z)$ is *quasiconvexity*, which means $\varphi(A, z) \leq \inf_{u\in W_0^{1,p}(\Omega;\mathbb{R}^3)} \int_\Omega \varphi(A+\nabla u, z)\,\mathrm{d}x$ for any $A \in \mathbb{R}^{3\times3}$. The following assertion modifies the celebrated result by Acerbi and Fusco [1]:

Lemma 5.1 *Let $\varphi : \mathbb{R}^{3\times3} \times \mathbb{R}^m \to \mathbb{R}$ be continuous, $\varphi(\cdot, z)$ quasiconvex, $p, p_1 \in (1, +\infty)$ and, for some $c_2 \geq c_1 > 0$,*

$$\forall A\in\mathbb{R}^{3\times3}\ \forall z\in Z_0 : \quad c_1\big(|A|^p+|z|^{p_1}-1\big) \leq \varphi(A, z) \leq c_2\big(1+|A|^p+|z|^{p_1}\big). \tag{5.2}$$

Then the functional $(u,z)\mapsto\int_\Omega\varphi(\nabla u,z)\,\mathrm{d}x$ *is (w×s)-lower semicontinuous on* $W^{1,p}(\Omega;\mathbb{R}^3)\times\{z{\in}L^{p_1}(\Omega;\mathbb{R}^m);\ z(\cdot){\in}Z_0$ *a.e. on* $\Omega\}$.

Sketch of the proof. By coercivity, we do not need to distinguish between sequential and topological lower semicontinuity.

Let us take a sequence $\{(u_n,z_n)\}_{n\in\mathbb{N}}$ (w×s)-converging to (u,z). Then $(\nabla u_n,z_n)$ (w×s)-converges to $(\nabla u,z)$ in $L^p(\Omega;\mathbb{R}^{3\times3})\times L^{p_1}(\Omega;\mathbb{R}^m)$. Also, selecting a suitable subsequence, it generates (a set) of $L^p{\times}L^{p_1}$-Young measures of the form $\nu{\otimes}\mu_z$ where $\mu_z=\{\delta_{z(x)}\}_{x\in\Omega}$ with $\delta_{z(x)}$ denoting here the Dirac distribution on $\mathbb{R}^m$ supported at $z(x)$; cf. [44, Corollary 3.4]. This means, in terms of a mentioned subsequence, that

$$\lim_{n\to\infty}\int_\Omega v(\nabla u_n,z_n)\mathrm{d}x=\int_\Omega\int_{\mathbb{R}^{3\times3}\times\mathbb{R}^m}v(A,r)\big[\nu_x\otimes\delta_{z(x)}\big](\mathrm{d}A{\times}\mathrm{d}r)\mathrm{d}x$$
$$=\int_\Omega\int_{\mathbb{R}^{3\times3}}v(A,z(x))\,\nu_x(\mathrm{d}A)\mathrm{d}x \tag{5.3}$$

for any v continuous of a growth less than p in the A-variable, while for φ continuous satisfying (5.2) we have only

$$\liminf_{n\to\infty}\int_\Omega\varphi(\nabla u_n,z_n)\,\mathrm{d}x\ge\int_\Omega\int_{\mathbb{R}^{3\times3}}\varphi(A,z(x))\,\nu_x(\mathrm{d}A)\mathrm{d}x; \tag{5.4}$$

cf. [45, Theorem 3.2].

As ν_x is a gradient L^p-Young measure with $\int_{\mathbb{R}^{3\times3}}A\nu_x(\mathrm{d}A)=\nabla u(x)$ for a.a. $x\in\Omega$, and as $\varphi(\cdot,z(x))$ is quasiconvex, for a.a. $x\in\Omega$ it holds

$$\int_{\mathbb{R}^{3\times3}}\varphi(A,z(x))\nu_x(\mathrm{d}A)\ge\varphi\Big(\int_{\mathbb{R}^{3\times3}}A\nu_x(\mathrm{d}A),z(x)\Big)=\varphi(\nabla u(x),z(x)). \tag{5.5}$$

see [30, 45]. Combining (5.4) and (5.5) yields $\liminf_{n\to\infty}\int_\Omega\varphi(\nabla u_n,z_n)\,\mathrm{d}x\ge\int_\Omega\varphi(\nabla u(x),z(x))\,\mathrm{d}x$. As the Young measure is not involved in the last estimate at all, this estimate holds, in fact, for the whole original sequence. $\square$

An example of a frame-indifferent quasiconvex (in fact even polyconvex, i.e. convex in terms of F and its determinant and cofactors) energy $\hat\varphi(F,z):=\tilde\varphi(F)$ satisfying (5.2) is the Ogden-type material

$$\varphi(F,z)=\alpha_1\mathrm{tr}\big(F^\top F-\mathbb{I}\big)^{p/2}+\alpha_2\big|\mathrm{tr}\big(\mathrm{cof}(F^\top F){-}\mathbb{I}\big)\big|^{p_0}+\phi_0\big(\det(F)\big); \tag{5.6}$$

here $\alpha_1,\alpha_2>0$, $p\ge3$, $p_0\le p/2$, ϕ_0 is a convex function of at most $p/3$ growth, and finally $\mathrm{tr}(\cdot)$ in (5.6) denotes the trace of a matrix.

As $F=\mathbb{I}{+}\nabla u$, we can express the specific stored energy in terms of the displacement gradient as

$$\varphi=\varphi(\nabla u,z)=\hat\varphi\big(\mathbb{I}{+}\nabla u,z\big). \tag{5.7}$$

The *Piola-Kirchhoff stress* $\sigma : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ is given by $\sigma = \varphi'_{\nabla u}(\nabla u, z) = \hat{\varphi}'_F(\mathbb{I}+\nabla u, z)$ with $\varphi'_{\nabla u}$ and $\hat{\varphi}'_F$ denoting the tensor-valued partial gradients.

If the displacement gradient ∇u is small, one can neglect the quadratic term (5.1) so that the Green-Lagrange strain tensor E from (5.6) turns into a so-called small-strain tensor $e(u) := \frac{1}{2}\nabla u + \frac{1}{2}(\nabla u)^\top$, i.e.

$$e_{ij}(u) = \frac{1}{2}\frac{\partial u_i}{\partial x_j} + \frac{1}{2}\frac{\partial u_j}{\partial x_i}, \qquad i,j = 1,...,3. \tag{5.8}$$

For all examples below, we assume $\Omega \subset \mathbb{R}^3$ to be a *polyhedral domain*. The discretization is made by a nested family of regular triangulations of Ω with the *mesh parameter* $h > 0$ and $\Pi_{U,h}$ and $\Pi_{Z,h}$ will always be considered as quasi-interpolation operators related with standard *conformal finite elements* of polynomial type, namely P0 (i.e. element-wise constant functions) or P1 (i.e. element-wise affine continuous functions). To be more explicit, we can consider a mollifier $u \mapsto \tilde{u}_h$ with $\tilde{u}_h(x) = \int_\Omega k_h(x,\xi)u(\xi)\mathrm{d}\xi$ using a continuous kernel $k_h : \Omega\times\Omega \to \mathbb{R}^+$ supported on an h-neighbourhood of the diagonal in $\Omega\times\Omega$ and $\int_\Omega k_h(x,\xi)\,\mathrm{d}\xi = 1$ for all $x \in \Omega$. Then define $u_h = \Pi_{U,h}u$ as a Lagrange piecewise affine interpolation of $\tilde{u}_h$ using the nodal points in case of P1-elements, or piecewise constant interpolation using barycenters of the simplexes of the particular triangulation in case of P0-elements. Moreover, we will assume the nested triangulations conformal with the specific disjoint partition of Γ where possibly different boundary conditions are prescribed. As to the initial condition q_0, we will always assume its stability (3.31), e.g. ensured through a "gentle start" (4.10) and thus not discussed in particular cases.

5.2 Plasticity with hardening at small strains

The first example on which we want to demonstrate our theory is a fully rate-independent plasticity with isotropic hardening. The vector of the internal parameters $z := (\pi, \eta) \in L^2(\Omega; \mathbb{R}^{3\times3}_{\mathrm{sym},0}) \times L^2(\Omega) =: Z$ is therefore now composed from the *plastic strain* π and a *hardening* variable η; here we used the notation

$$\mathbb{R}^{3\times3}_{\mathrm{sym},0} := \big\{A \in \mathbb{R}^{3\times3};\ \ A^\top = A,\ \ \mathrm{tr}(A) = 0\big\}. \tag{5.9}$$

For simplicity, we consider homogeneous Dirichlet boundary conditions on a part Γ_0 of the boundary $\partial\Omega$ with nonvanishing surface measure, so that

$$\mathcal{U} \ := \ U = \big\{u \in W^{1,2}(\Omega;\mathbb{R}^3);\ \ u = 0 \ \text{ a.e. on } \Gamma_0\big\}, \tag{5.10}$$

$$\mathcal{Z} \ := \ \big(L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym},0}) \times L^2(\Omega)\big) \cap K \tag{5.11}$$

where K is the cone of admissible evolution directions, see (5.14) below. The coincidence that the z-component of states can be restricted equally in the stored energy and dissipation energy is important for (5.17) below. We postulate the stored energy as

$$E(u,z) \equiv E(u,\pi,\eta) := \frac{1}{2}\int_\Omega (e(u)-\pi)^\top \mathbb{C}(e(u)-\pi) + b\eta^2 \,\mathrm{d}x \tag{5.12}$$

where $\mathbb{C} = [\mathbb{C}_{ijkl}] \in \mathbb{R}^{3\times3\times3\times3}$ is a positive-definite 4th-order tensor of elastic moduli and $b > 0$ a hardening parameter. There are no constraints of the type $\Xi(u, \pi, \eta) = 0$ so we consider $\mathcal{E}_\varepsilon \equiv \mathcal{E}$. In view of Remark 3.10, it also means that no numerical-integration error is expected. Considering the loading by a time-varying force g acting on $\Gamma_1 := \partial\Omega \setminus \Gamma_0$, we postulate f as

$$\langle f(t), (u, z)\rangle := \int_{\Gamma_1} g(t, x)\cdot u(x)\,\mathrm{d}S. \tag{5.13}$$

The hardening is a unidirectional process and is, in standardly accepted models, reflected by the cone of admissible evolution directions in the form

$$K := \{z = (\pi, \eta);\ \ \eta \geq \delta_P^*(\pi) \ \text{ a.e. on } \Omega\}. \tag{5.14}$$

Here $P \subset \mathbb{R}^{3\times3}$ is a convex closed neighbourhood of the origin, δ_P is its indicator function, and δ_P^* the conjugate functional to δ_P with respect to the duality pairing $\sigma : e = \sum_{i,j=1}^3 \sigma_{ij} e_{ij}$. Note that the physical dimension of this pairing is Pa=J/m^3. Hence, δ_P^* is convex, *homogeneous degree-1* and positive except at the origin, and thus K is indeed a cone. The interior of P is called *elasticity domain* while its boundary is called the *yield surface*. More precisely, it corresponds to the initial elasticity domain if $\eta = 1$ is considered as an initial condition while the actual elasticity domain may be inflated during the loading process just by the isotropical hardening. The continuous part of the degree-1 homogeneous dissipation potential is

$$R(z) := \int_\Omega \delta_P^*(\pi)\,\mathrm{d}x \tag{5.15}$$

so that the overall dissipation distance is, in view of (4.8),

$$\mathcal{D}(z_1, z_2) \equiv \mathcal{D}(\pi_1, \eta_1, \pi_2, \eta_2) := \begin{cases} \int_\Omega \delta_P^*(\pi_2{-}\pi_1)\,\mathrm{d}x & \text{if } \eta_2{-}\eta_1 \geq \delta_P^*(\pi_2{-}\pi_1) \text{ on } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

This leads naturally to $Z_1 := L^1(\Omega; \mathbb{R}^{3\times3}_{\mathrm{sym},0}) \times L^1(\Omega)$ in Remark 4.2. Beside the mentioned initial condition $\eta(0, \cdot) = 1$, we must prescribe $\pi(0, \cdot) = \pi_0 \in L^2(\Omega; \mathbb{R}^{3\times3}_{\mathrm{sym},0})$. The required stability (3.31) of q_0, achieved e.g. through the "gentle start" (4.10) as suggested in Sect. 5.1, yields $z_0 = (\pi_0, \eta_0) \in K$, i.e. here $\delta_K^*(\pi_0) \leq 1$. The mentioned initial condition $\eta_0 = 1$ is, in general, guaranteed by this way only if $f(0)$ is small enough. Moreover, it is well-known (cf. [22, 36]) that this problem has a unique energetic solution $(u, z) \in W^{1,\infty}([0, T]; U \times Z)$.

We assume a polyhedral domain Ω with also Γ_0 and Γ_1 having a polyhedral shape, and assume Ω triangulated by a nested family of regular triangulations with the mesh parameter $h > 0$ conformal with the partition $\Gamma = \Gamma_0 \cup \Gamma_1$, and $\Pi_{U,h}$ and $\Pi_{Z,h}$ quasi-interpolation operators related with conformal P1-elements and P0-elements, respectively. It is also important that the P0-elements are conformal with the cone K from (5.14) used also for $\mathcal{Z}$ in (5.11) in the sense $\Pi_{Z,h}K \subset K$, as needed for Propositions 4.5 and 4.6. As there is no Ξ in this problem, we have $\mathcal{E}_\varepsilon = \mathcal{E}$ but R_ε

from (4.9) is to be considered (unless one thinks about $R+\delta_K$ in place of R_e as suggested in Remark 4.7), and also (4.24) with σ the norm topology works simply for $[q_0]_{h,\varepsilon} := \Pi_h q_0$.

Corollary 5.2 *Let the data Ω, Γ_0, Γ_1, P, and q_0 be qualified as above, and $g \in C^1([0,T]; L^{4/3}(\Gamma_1;\mathbb{R}^3))$ and $[q_0]_{h,\varepsilon}$ be taken as above. Then the approximate solutions $q_{\varepsilon,\tau,h} = (u_{\varepsilon,\tau,h}, \pi_{\varepsilon,\tau,h}, \eta_{\varepsilon,\tau,h})$ with*

$$u_{\varepsilon,\tau,h} \in L^\infty(0,T; W^{1,2}(\Omega;\mathbb{R}^3)), \tag{5.16a}$$
$$\pi_{\varepsilon,\tau,h} \in L^\infty(0,T; L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym},0})) \cap \mathrm{BV}([0,T]; L^1(\Omega;\mathbb{R}^{3\times3})), \tag{5.16b}$$
$$\eta_{\varepsilon,\tau,h} \in L^\infty(0,T; L^2(\Omega)) \;\cap\; \mathrm{BV}([0,T]; L^1(\Omega)), \tag{5.16c}$$

based on the P0-elements for π and η and the P1-elements for u converge for $(\varepsilon,\tau,h) \to (0,0,0)$ (even as the whole sequence in the sense of Theorem 3.8 with Remark 4.2) to the energetic solution of the problem given by E, R, K, f and q_0 above.

Proof. The coercivity (4.6) is ensured due to the Poincaré inequality through the Dirichlet boundary conditions, ensuring

$$E(u,\pi,\eta) \geq c\big(\|u\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)} + \|\pi\|^2_{L^2(\Omega;\mathbb{R}^{3\times3})} + \|\eta\|^2_{L^2(\Omega)}\big) \tag{5.17}$$

provided also $\delta^*_P(\pi) \leq \eta$; note that such coercivity does not hold for general $(\pi,\eta) \in Z$, which is why for the definition (5.11) of $\mathcal{Z}$ the restriction to K had to be used.

As P is assumed bounded, δ^*_P is Lipschitz continuous, and hence R is continuous.

Moreover, stability of q_0 as well as (4.24) have already been discussed above.

Using the coercivity of E already proved, we can verify (3.9b) with $\mathcal{E}_\varepsilon = \mathcal{E}$ by $\frac{\partial\mathcal{E}}{\partial t}(t,q) = -\langle\frac{\partial f}{\partial t}, q\rangle = -\int_{\Gamma_1}\frac{\partial g}{\partial t}(t,x)\cdot u(x)\,\mathrm{d}S$ and the estimate

$$\begin{aligned}\Big|\frac{\partial\mathcal{E}}{\partial t}(t,q)\Big| \leq N\Big\|\frac{\partial g}{\partial t}\Big\|_{L^{4/3}(\Gamma_1;\mathbb{R}^3)}\big\|u\big\|_{W^{1,2}(\Omega;\mathbb{R}^3)} &\leq NG_1\|q\| \leq \frac{N^2G_1^2}{2c} + \frac{c}{2}\big\|q\big\|^2 \\ &\leq \frac{N^2G_1^2}{c} + \mathcal{E}(t,q) = c_1\big(\mathcal{E}(t,q)+c_0\big)\end{aligned}$$

with $c_1 = 1$ and $c_0 = N^2G_1^2/c$. Here c is from (5.17) and N is the norm of the trace operator $u \mapsto u|_{\Gamma_1}$ in $\mathrm{Lin}(W^{1,2}(\Omega), L^4(\Gamma_1))$ and $G_1 = \|g\|_{C^1([0,T];L^{4/3}(\Gamma_1))}$. Here we used the estimate $\mathcal{E}(t,q) = E(q) - \langle f(t), q\rangle = E(q) - \int_{\Gamma_1}\frac{\partial g}{\partial t}(t,x)\cdot u(x)\,\mathrm{d}S \geq c\|q\|^2 - NG_1\|q\| \geq \frac{c}{2}\|q\|^2 - \frac{1}{2c}N^2G_1^2$.

Then we use the assertions from Sect. 3 through either Propositions 4.5 or 4.6. In the former case, the setting (4.29) takes now

$$B(\pi,\eta) := \big(\mathbb{C}\pi\,,\, b\eta\big), \qquad E_0(u,\pi,\eta) := \int_\Omega \frac{e(u)^\top\mathbb{C}e(u)}{2} - e(u)^\top\mathbb{C}\pi\,\mathrm{d}x,$$

while for the latter case the setting (4.35) works simply with $B = E'$ and $E_0 = 0$, i.e.

$$B(u,\pi,\eta) := \big(\mathrm{div}\big(\mathbb{C}(e(u)-\pi)\big)\,,\, \mathbb{C}(\pi - e(u))\,,\, b\eta\big)\,, \qquad E_0 := 0,$$

with the "div" term considered in the weak sense, of course. Note that $\mathcal{Z} + K \subset \mathcal{Z}$, holds, too. Eventually, due to the uniqueness result [22, 36] or [51, Sect.11.1.3], we conclude that the whole sequence converges. □

Remark 5.3 (*Implementation without regularization by LQ-programme.*) In anisotropic media like single-crystals, the domain P is considered to be polyhedral, cf. e.g. [12], hence δ_P^* has a polyhedral epigraph and the incremental problem (3.22) without any regularization (cf. Remark 4.7) represents a minimization problem of a sum of a quadratic and a polyhedral-graph functional which can be, after a computationally cheap enhancement, solved by efficient linear-quadratic solvers; cf. [52, Lemma 4] for this enhancement.

Remark 5.4 The P0/P1-discretization of this plasticity problem has been already used by Alberty and Carstensen [2] and thus Corollary 5.2 recovers some results from [2]. Note that our convergence result does not use higher-order regularity of the solutions $(u,z) \in W^{1,\infty}(0,T;U{\times}Z)$. Hence we cannot expect convergence rates as in [2] and thus our results are closer to [23] where the above convergence result was established already by a more elaborate method.

5.3 Phase transformation: a mixture approach

In engineering, modelling of inelastic response of the materials undergoing *martensitic transformation* is of high interest. Here we want to demonstrate our theory on a simplified mixture-like model for martensitic transformation.

Taking Γ_0 as in Section 5.2 and $Z_0 := \{s \in \mathbb{R}^m;\ s_l \geq 0\ \&\ \sum_{l=1}^m s_l = 1\}$ the Gibbs simplex, we put

$$\mathcal{U} := U = \big\{u \in W^{1,p}(\Omega;\mathbb{R}^3);\quad u = 0 \ \text{ a.e. on } \Gamma_0\big\}, \tag{5.18}$$

$$\mathcal{Z} := \big\{z \in Z := W^{\alpha,2}(\Omega;\mathbb{R}^m);\ \ z(x) \in Z_0 \ \text{ for a.a. } x \in \Omega\big\} \tag{5.19}$$

with $\alpha > 0$ denoting (possibly a fractional) order of derivatives of the vector of the *internal parameters* z which now represents *volume fractions* referring to m phases (or phase variants). For simplicity, we consider the loading again through g as in Sect. 5.2, i.e. f is again defined by (5.13). We postulate the stored energy as

$$E(u,z) := \int_\Omega \varphi(\nabla u, z)\,\mathrm{d}x + \frac{\kappa}{2}|z|_\alpha^2 \tag{5.20}$$

with $\kappa, \alpha > 0$ and $|\cdot|_\alpha$ denoting the usual seminorm in the Sobolev (or, for α noninteger, Sobolev-Slobodetskiĭ) space, i.e.

$$|z|_\alpha^2 = \begin{cases} \displaystyle\int_\Omega |\nabla^\alpha z|^2 \,\mathrm{d}x & \text{for } \alpha \in \mathbb{N}, \\ \displaystyle\frac{1}{4}\int_\Omega\int_\Omega \frac{|\nabla^{[\alpha]}z(x) - \nabla^{[\alpha]}z(\xi)|^2}{|x-\xi|^{3+2(\alpha-[\alpha])}} \,\mathrm{d}x\mathrm{d}\xi & \text{for } \alpha > 0 \text{ noninteger} \end{cases} \tag{5.21}$$

with $[\alpha]$ the integer part of α. In principle, more physically justified kernels with a support localized around the diagonal $\{x = \xi\}$ with the same singular behaviour as $|x-\xi|^{-3-2(\alpha-[\alpha])}$ for $|x-\xi| \to 0$ could equally be used in (5.21).

The degree-1 homogeneous dissipation potential is now postulated as

$$R(z) := \int_\Omega \delta_M^*(z)\,\mathrm{d}x \tag{5.22}$$

where δ_M^* is determined, in analogy with δ_P^* from Sect. 5.2, by a convex compact neighbourhood $M \subset \mathbb{R}^m$ of the origin which prescribes activation energies for martensite/austenite phase-transformation or for re-orientation of particular martensitic variants. In particular, the martensitic transformation is a reversible process, so that $K = Z$. Also, there is nor Ξ neither $K \neq Z$ and thus both $\mathcal{E}_\varepsilon \equiv \mathcal{E}$ and $\mathcal{D}_\varepsilon \equiv \mathcal{D}$ and the ε-regularization is irrelevant here.

For the discretization, we consider naturally P1-elements for u and either P0-elements for z (if $\alpha < 1/2$) or P1-element also for z if ($\alpha < 3/2$). Again, taking $[q_0]_{h,\varepsilon} := \Pi_h q_0$ guarantees (4.24) with σ being the norm topology.

Corollary 5.5 *Let the data Ω, Γ_0, and Γ_1 be qualified as in Sect. 5.2, let φ be qualified as in Lemma 5.1 (note that p_1 is irrelevant as Z_0 is bounded here), and further let*

$$g \in C^1([0,T]; L^{p^\#/(p^\#-1)}(\Gamma_1;\mathbb{R}^3)), \qquad \text{where } p^\# \begin{cases} = \frac{2p}{3-p} & \text{for } p < 3, \\ < +\infty & \text{for } p = 3, \\ = +\infty & \text{for } p > 3, \end{cases} \tag{5.23}$$

and $q_0 \in \mathcal{S}(0)$ be approximated by $[q_0]_{h,\varepsilon} := \Pi_h q_0$. Then the approximate solutions $q_{\tau,h} = (u_{\tau,h}, z_{\tau,h})$ with

$$u_{\tau,h} \in L^\infty(0,T; W^{1,p}(\Omega;\mathbb{R}^3)), \tag{5.24a}$$
$$z_{\tau,h} \in L^\infty(0,T; W^{\alpha,2}(\Omega;\mathbb{R}^m)) \cap \mathrm{BV}([0,T]; L^1(\Omega;\mathbb{R}^m)), \tag{5.24b}$$

based on the P1-elements for u and the P0- or P1-elements for z converge for $(\tau, h) \to (0,0)$ (in terms of subsequences in the sense of Theorem 3.8 with Remark 4.2) to energetic solutions of the problem given by E, R, f and q_0 above.

Proof. Coercivity on $\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$ follows from the assumed coercivity (5.2) of $\varphi(\cdot, z)$ by Poincaré inequality combined with the Dirichlet condition on Γ_0 and by the regularizing κ-term in (5.20) combined with the constraint $z(x) \in Z_0$ involved in $\mathcal{Z}$ in (5.19).

The lower-semicontinuity of the first term in (5.20) needed for (3.8) follows by Lemma 5.1 with $p_1 < +\infty$ arbitrary since Z_0 is now bounded.

The continuity of $R : L^1(\Omega) \to \mathbb{R}$ follows from (in fact is equivalent to) the assumed boundedness of $M \subset \mathbb{R}^m$.

The assumption in Proposition 4.4 are satisfied simply if σ : equals the strong topology on $W^{1,p}(\Omega; \mathbb{R}^3) \times W^{\alpha,2}(\Omega; \mathbb{R}^m)$. Here the convexity of the Gibbs simplex Z_0 involved in $\mathcal{Z}$ is used, which makes both P0- and P1-elements compatible with $\mathcal{Z}$ in the sense $\Pi_{Z,h}\mathcal{Z} \subset \mathcal{Z}$, cf. (4.3), which makes our results from Section 4.3 working. $\square$

Example 5.6 At small strains, a popular model takes a "mixture" of quadratic energies in the form

$$\varphi(\nabla u, z) := \sum_{\ell=1}^{m} z_\ell \frac{(e(u){-}e_\ell)^\top \mathbb{C}_\ell (e(u){-}e_\ell)}{2} + \psi(z) \quad \text{where} \quad e_\ell := \frac{U_\ell^\top {+} U_\ell}{2},$$

with the distortion matrices U_ℓ of particular pure phases (or phase variants). The setting here is related with the situation of martensitic transformation in a single-crystal and z's are *volume fractions* of the so-called austenite and of particular variants of martensite, e.g. $m = 4$ or 7 for tetragonal or orthorhombic martensite, respectively. The function ψ reflects the difference between chemical energies of austenite and martensite and also between pure phases and "mixtures". As $\varphi(\cdot, z)$ is now convex, it qualifies for Lemma 5.1 with Z_0 bounded. The philosophy of mixtures of austenite/martensite phases in so-called *shape-memory alloys* has been proposed by Frémond [15]; in rate-independent variant also presented in [16]. For its analysis and numerical implementation see [10, 11, 17, 24]. Gradients of mesoscopical volume fractions (i.e.. (5.20) with $\alpha = 1$ has already been used in Frémond's model [16, p.364] or [17, Formula (7.20)]. Another way for obtaining physically relevant mixture energies is the quasiconvexification under volume constrains, also called cross-quasiconvexification, see [42].

Example 5.7 If the elastic-moduli tensors $C_\ell = \mathbb{C}$ are equal for all phases, the specific energy in Example 5.6) transforms to

$$\varphi(\nabla u, z) = \sum_{\ell=1}^{m} \frac{(e(u){-}e_{\rm tr}(z))^\top \mathbb{C} (e(u){-}e_{\rm tr}(z))}{2} + \tilde{\psi}(z) \quad \text{with} \quad e_{\rm tr}(z) = \sum_{\ell=1}^{m} z_\ell e_\ell,$$

where $e_{\rm tr}(z)$ the is so-called *transformation strain*. Note that, although (5.20) has got now a quadratic form except the lower-order term $\tilde{\psi}(z)$, we cannot use Proposition 4.5 or 4.6 because of the constraint $z(x) \in Z_0$. Hence, the quadratic structure of the regularizing term $\kappa|z|_\alpha^2$ cannot be exploited and a non-quadratic regularizing term could equally be considered through this section. For such a model we refer e.g. to [5, 6, 8, 19, 20, 28, 60].

5.4 Phase transformation: non-mixture approach

The mixture approach in Sect. 5.3 is rather designed for phenomenological models of polycrystals but is too coarse for the description of complicated microstructures occurring in shape-memory-alloy single-crystals. An attempt to build a microscopical model has been done in [34] (see also [35]) by restricting z to be valued in vertices of the Gibbs' simplex, i.e. only pure phase(variant)s are allowed; then $\alpha < 1/2$ should be taken in (5.20) or, as considered in [34, 35], a BV-like term $\kappa|\nabla z|$. In this model, z "switches" φ.

A different philosophy with presumably similar effects, pioneered by Falk [13], considers the vectorial "order parameter" z related to the deformation gradient ∇u and particular shapes are then switched rather by ∇u. Spinodal regions are then allowed instead of mixtures. The specific stored energy φ now depends only on ∇u but need not be quasiconvex. For example, in [3, 4, 32, 50, 54], a *multiwell potential* $\hat{\varphi}$ (related with φ by (5.7)) arises by the combination of *St. Venant-Kirchhoff materials* considered for each particular phase:

$$\hat{f}(F) := \min_{\ell=1,..,m} \Big(\frac{1}{2}(U_\ell^{-\top} F^\top F U_\ell^{-1} - \mathbb{I})^\top \mathbb{C}_\ell (U_\ell^{-\top} F^\top F U_\ell^{-1} - \mathbb{I}) + c_\ell\Big), \qquad (5.25)$$

where U_ℓ are distortion matrices as in Example 5.6, $\mathbb{C}_\ell$ are elastic-moduli tensors, c_ℓ are some constants, and $U_\ell^{-\top} := (U_\ell^\top)^{-1}$. Now naturally $p = 4$.

We postulate the stored energy in terms of E and Ξ as

$$E(u,z) := \int_\Omega \varphi(\nabla u)\,\mathrm{d}x + \frac{\kappa}{2}|u|_\alpha^2, \qquad (5.26)$$
$$\Xi(u,z) := z - \mathcal{L}(\nabla u), \qquad (5.27)$$

with $\kappa > 0$, $\alpha > 1$ and $\mathcal{L} : \mathbb{R}^{3\times3} \to Z_0$ playing the role of a "phase indicator" with Z_0 being again the Gibbs simplex. The seminorm $|\cdot|_\alpha$ defined in (5.21) used for $1 < \alpha < 2$ with the Frobenius norm in the enumerator, now acting on (3×3)-matrices is frame-indifferent, as observed by Arndt in [3]. We consider the same loading as in Sects. 5.2 and 5.3, i.e. f from (5.13), but now we put

$$\mathcal{U} \;:=\; U = \big\{u \in W^{\alpha,2}(\Omega;\mathbb{R}^3);\;\; u = 0 \;\text{ a.e. on } \Gamma_0\big\}, \qquad (5.28)$$
$$\mathcal{Z} \;:=\; \big\{z \in Z := L^2(\Omega;\mathbb{R}^m);\;\; z(x) \in Z_0 \;\text{ for a.a. } x \in \Omega\big\}, \qquad (5.29)$$

and then naturally $X := Z$. The dissipation potential R is again from (5.22). There is no K involved, hence $\mathcal{D}_\varepsilon = \mathcal{D}$, but as Ξ from (5.27) occurs, the regularization $\mathcal{E}_\varepsilon$ is, in principle, to be considered.

Choosing $\alpha < 3/2$ allows for the usage of P1-elements for u and P0-elements for z. As now $\mathcal{Q} = Q$ and $K = Z$, so in particular their conformity (4.3) is automatic. The proof of the following assertion shows that they are conformal also with the constraints $\Xi(q) = 0$ so, in view of Remark 4.7, it would be possible to avoid the ε-regularization at all. When taking $[u_0]_{h,\varepsilon} = \Pi_{U,h} u_0$, we have $\nabla[u_0]_{h,\varepsilon}$ element-wise constant and so is $\mathcal{L}(\nabla[u_0]_{h,\varepsilon}) =: [z_0]_{h,\varepsilon}$, and (4.24) is satisfied.

Corollary 5.8 *Let $\varphi : \mathbb{R}^{3\times3} \to \mathbb{R}$ be continuous (not necessarily quasiconvex) satisfying (5.2) here with $m := 0$ (so no z-dependence), let g satisfy (5.23), $\mathcal{L} : \mathbb{R}^{3\times3} \to Z_0$ be continuous, and $\alpha \in (1, 3/2)$ and $p < 6/(5-2\alpha)$ in (5.2), and q_0 and $[q_0]_{h,\varepsilon}$ as specified above. Then the approximate solutions $q_{\varepsilon,\tau,h} = (u_{\varepsilon,\tau,h}, z_{\varepsilon,\tau,h})$ with*

$$u_{\varepsilon,\tau,h} \in L^\infty(0,T;W^{\alpha,2}(\Omega;\mathbb{R}^3)), \tag{5.30a}$$
$$z_{\varepsilon,\tau,h} \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^m)) \cap \mathrm{BV}([0,T];L^1(\Omega;\mathbb{R}^m)), \tag{5.30b}$$

based on the P1-elements for u and the P0-elements for z converge for $(\varepsilon,\tau,h) \to (0,0,0)$ (in terms of subsequences in the sense of Theorem 3.8 with Remark 4.2) to energetic solutions of the problem given by E, R, f and q_0 above.

Proof. Weak lower semicontinuity of E is due to convexity of the regularizing term $\kappa|u|_\alpha^2$ in (5.26) while φ is now treated by compactness of the embedding $W^{\alpha,2}(\Omega) \subset W^{1,p}(\Omega)$ (guaranteed if $p < 6/(5-2\alpha)$) as a lower-order term. The limit passage in the z-variable is trivial. This compactness also ensures the weak continuity of $\Xi : U \times Z \to X$.

As $K = Z$, condition (4.23) with σ being the strong topology holds, if we show, for given $\tilde{z} = \mathcal{L}(\nabla\tilde{u})$, the existence of $(\tilde{u}_h, \tilde{z}_h) \xrightarrow{\sigma} (\tilde{u}, \tilde{z})$ such that $\tilde{z}_h = \mathcal{L}(\nabla\tilde{u}_h)$. As far as $\tilde{u}_h$, this can be done by a density argument of smooth functions in $W^{\alpha,2}(\Omega;\mathbb{R}^3)$, and then the usual Lagrange interpolation. By the embedding $W^{\alpha,2}(\Omega) \subset W^{1,p}(\Omega)$, $\nabla\tilde{u}_h \xrightarrow{\mathrm{S}} \nabla\tilde{u}$ in $L^p(\Omega;\mathbb{R}^{3\times3})$ and $\tilde{z}_h = \mathcal{L}(\nabla\tilde{u}_h) \xrightarrow{\mathrm{S}} \mathcal{L}(\nabla\tilde{u}) = \tilde{z}$ by continuity of the Nemytskiĭ mapping induced by $\mathcal{L}$.

Then we use the results from Sect. 3 via Proposition 4.4 with σ being the strong topology on $W^{\alpha,2}(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R}^m)$. $\square$

Remark 5.9 The inequalities $\alpha < 3/2$ and $p < 6/(5-2\alpha)$ restrict us to $p < 3$, which unfortunately excludes (5.25). Hence we are tempted to take higher α which, however, brings the necessity to use higher-order elements (or to split the problem to a system). Considering P2-elements for u would allow for $\alpha < 5/2$ which, in turn, would allow for arbitrarily high p. Since $\mathcal{L}$ is inevitable nonlinear, it is no longer conformal with the constraint $\Xi(q) = 0$ no matter what (polynomial) elements are taken for z. This would drive us to a penalization technique based on Proposition 4.3. However, here it is simpler to modify our analysis to allow for expressing the model in terms of u only, cf. the following Remark 5.10.

Remark 5.10 In fact, a "viscous" rate-dependent variant of the above model was proposed in [50], for the rate-independent dissipation term cf. [50, Formula (33)]. The regularizing term $|\cdot|_\alpha$ used for $\alpha < 1/2$ and the P0/P1-discretization was suggested and implemented in [3] and computational experiments on NiMnGa single crystals reported in [4]. In [46], the model was analyzed and implemented in the 1-dimensional case with $\alpha = 2$. Pure analysis then followed also in [47]; in particular for $\alpha \geq 3$, [47, Prop.3] investigated an inviscid variant of this model accounting, contrary to our paper, also for inertial effects. In fact, the model was formulated only in terms of u in [3, 46, 47, 50] but then the dissipation distance took the form $\mathcal{D}(u_1,u_2) = \int_\Omega |\mathcal{L}(\nabla u_1) - \mathcal{L}(\nabla u_2)|\,\mathrm{d}x$, having lost the structure based on the degree-1 homogeneous potential R. Neglecting difficulties in numerical evaluation of such $\mathcal{D}$ if $\alpha = 2$ would be considered, by this way one gets rid of the necessity to penalize Ξ (which, in case $\alpha < 1/2$, is made possible due to Corollary 5.8 together with Remark 4.7 in our case too). Nevertheless, a fully rate-independent model, used in fact for calculations in [4], has not been subjected to any rigorous mathematical/numerical analysis, and therefore Corollary 5.8 brings indeed new results.

5.5 Damage at large strains

In engineering, other inelastic process in the materials of a high interest is *damage*. We consider a fully rate-independent isotropic and nonlocal damage, and again consider the body Ω fixed on a nonvanishing part Γ_0 and loaded by a surface force g on $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, so that $\mathcal{U} = U$ is again from (5.18). As we consider isotropic damage, the internal parameter $z \in \mathcal{Z}$ will be scalar valued with

$$\mathcal{Z} \;:=\; \big\{ z \in Z := W^{\alpha,2}(\Omega);\;\; z(x) \geq 0 \;\text{ for a.a. } x \in \Omega \big\}. \tag{5.31}$$

We postulate the stored energy again by the formula (5.20); $\kappa > 0$ in (5.20) is now a coefficient responsible for *nonlocal effects* in gradient-of-damage theories as, e.g., in [16], cf. [38] for a discussion and more references. Note that we admitted, rather formally, φ operating on the argument z nonrestricted from above to allow for a simple construction of the recovery sequence (4.30). The loading f is considered again by (5.13).

Like isotropic hardening in Sect. 5.2, the process of damaging is unidirectional in the sense that, if in progress, it can only increase but the material never can heal, which leads us to define the cone of admissible evolution directions as

$$K := \{ z \in W^{\alpha,2}(\Omega);\;\; z \geq 0 \;\text{ a.e. on } \Omega \} \equiv \mathcal{Z}. \tag{5.32}$$

The degree-1 homogeneous dissipation potential is considered as

$$R(z) := \int_\Omega c_1 z \,\mathrm{d}x, \tag{5.33}$$

where c_1 is a phenomenological specific energy (with physical dimension J/m^3=Pa) expressing the energy needed for a damage of a unit volume described by a unit jump of the damage parameter z. Considering the initial condition for $z_0 = 0$ and $\varphi(A,\cdot)$ decreasing for $z \in [0,1]$ and with $\varphi(A,z) = \varphi(A,1) + (z-1)^2$ for $z \in (1,+\infty)$, we effectively force the values of z to range only the interval $[0,1]$ and c_1 refers to the specific energy dissipated by damaging the original material (having the stored-energy $\varphi(\cdot,0)$) to the fully damaged material (having the stored-energy $\varphi(\cdot,1)$ assumed to be still coercive so we exclude the case when the material fully disintegrates).

As no equality constraints of the type $\Xi(q) = 0$ are involved, we have $\mathcal{E}_\varepsilon = \mathcal{E}$ but the ε-regularization $\mathcal{D}_\varepsilon$ from (4.9) is to be still considered unless one takes $R + \delta_K$ instead of R_ε, cf. Remark 4.7. For the discretization, as in Sect. 5.3, we consider P1-elements for u and either P0-elements for z (if $\alpha < 1/2$) or P1-element also for z if ($\alpha < 3/2$). Again, both P0- and P1-elements are conformal with the constraints in $\mathcal{Z} = K$ from (5.31)–(5.32) in the sense $\Pi_{Z,h}\mathcal{Z} \subset \mathcal{Z}$ and $\Pi_{Z,h}K \subset K$, as required in Proposition 4.5.

Corollary 5.11 *Let the data Ω, Γ_0, and Γ_1 be qualified as in Sect. 5.2, let φ be qualified as in Lemma 5.1 with $m := 1$ and $Z_0 := \{z \geq 0\}$ and $p_1 := 2$, let g satisfy (5.23), and let $q_0 \in \mathcal{S}(0)$ and $[q_0]_{h,\varepsilon} := \Pi_h q_0$. Then the approximate solutions $q_{\varepsilon,\tau,h} = (u_{\varepsilon,\tau,h}, z_{\varepsilon,\tau,h})$ with*

$$u_{\varepsilon,\tau,h} \in L^\infty(0,T; W^{1,p}(\Omega;\mathbb{R}^3)), \tag{5.34a}$$

$$z_{\varepsilon,\tau,h} \in L^\infty(0,T; W^{\alpha,2}(\Omega)) \cap \mathrm{BV}([0,T]; L^1(\Omega)), \tag{5.34b}$$

based on the P1-elements for u and the P0- or P1-elements for z converge for $(\varepsilon,\tau,h) \to (0,0,0)$ *(in terms of subsequences in the sense of Theorem 3.8 with Remark 4.2) to energetic solutions of the problem given by E, R, K, f and q_0 above.*

Proof. Coercivity on $\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$ follows from the assumed condition $|A|^p \leq \varphi(A,z)$ by the Poincaré inequality combined with the Dirichlet condition on Γ_0 and by the regularizing κ-term in (5.20) combined with the constraint $z(x) \geq 0$ involved in $\mathcal{Z}$ in (5.31). Then we use Proposition 4.5 with the decomposition (4.29) using $B = E_1'$ with $E_1(z) := \frac{\kappa}{2}|z|_\alpha^2$ and $E_0(u,z) = \int_\Omega \varphi(\nabla u, z)\,\mathrm{d}x$. Note also that $[q_0]_{h,\varepsilon} := \Pi_h q_0$ satisfies (4.24). $\square$

Remark 5.12 The partial damage at large strains has been analyzed in [38] but without any numerical approximation and nonquadratic regularizing term $\frac{\kappa}{p_1}|\nabla z|^{p_1}$ with $p_1 > 3$ had to be used, contrary to the quadratic term in (5.20) which is usual in engineering literature but never was mathematically analyzed so far. Hence Corollary 5.11 represents a new extension in this field.

Example 5.13 (*Engineering "(1−d)-model".*) Considering two materials having linear response described in small strains by elastic moduli tensors $\mathbb{C}_1$ and $\mathbb{C}_2$, the first one undergoing a damage in a linear way leads to the potential φ in the form

$$\varphi(\nabla u, z) = (1-z)^+ \frac{e(u)^\top \mathbb{C}_1 e(u)}{2} + \frac{e(u)^\top \mathbb{C}_2 e(u)}{2} + ((z-1)^+)^2$$

where $(\cdot)^+ = \max(0,\cdot)$. This potential satisfies all our assumptions with $p = p_1 = 2$ in (5.2) if $\mathbb{C}_1$ is positive semidefinite and $\mathbb{C}_2$ positive definite. Such a model is called in engineering literature a $1-d$ model (here rather $1-z$) and can be used for two-component materials as e.g. filled polymers or filled rubbers which do not undergo a full damage.

5.6 Debonding at large strains

Other inelastic processes may occur not in the materials themselves but on the boundary. Here we want to consider a possible *debonding* of an elastic support on a part Γ_2 of the boundary $\partial\Omega$. The internal parameter $z \in L^\infty(\Gamma_2)$ is therefore now a scalar debonding parameter assumed to range $[0,1]$ and expressing *volume fraction* of the adhesive which fixes elastically the body on Γ_2 if not debonded. It is natural also to consider a *unilateral Signorini contact* on Γ_2. Moreover, we again consider the body Ω fixed on a nonvanishing part Γ_0 of $\partial\Omega$ (disjoint with Γ_2) and loaded by a surface time-varying force g on $\Gamma_1 = \partial\Omega \setminus (\Gamma_0 \cup \Gamma_2)$, so that

$$\mathcal{U} := \big\{u \in W^{1,p}(\Omega;\mathbb{R}^3);\quad u = 0 \ \text{ a.e. on } \Gamma_0, \quad \nu \cdot u \geq 0 \ \text{ a.e. on } \Gamma_2\big\}, \tag{5.35}$$

$$\mathcal{Z} := \big\{z \in Z := L^\infty(\Gamma_2);\quad 0 \leq z \leq 1 \text{ a.e. on } \Gamma_2\big\} \tag{5.36}$$

with $\nu = \nu(x)$ a normal to Γ_2. We postulate the stored energy as

$$E(u,z) := \int_\Omega \varphi(\nabla u)\,\mathrm{d}x + \int_{\Gamma_2} (1-z)\psi(u)\,\mathrm{d}S, \tag{5.37}$$

where $\psi : \mathbb{R}^3 \to \mathbb{R}_+$ describes the elastic response of the adhesive fixing the body on Γ_2.

Considering naturally that debonding can only develop but never heal back leads us to pose the cone of admissible evolution directions as

$$K := \{z \in L^\infty(\Gamma_2);\quad z \geq 0 \quad \text{a.e. on } \Gamma_2\}. \tag{5.38}$$

Similarly like in (5.33), the degree-1 homogeneous dissipation potential is

$$R(z) := \int_{\Gamma_2} c_2 z\,\mathrm{d}S \tag{5.39}$$

with c_2 a phenomenological specific energy (with physical dimension $\mathrm{J/m^2}$) expressing the energy needed for a full debonding of a unit area of Γ_2.

Natural finite-element approximation is now P1-elements for u and P0-elements on the boundary for z. We assume that the disjoint partition $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ is polyhedral and that the nested triangulations are conformal with this partition. To simplify technical details, let us assume that Γ_2 is flat; this ensures $\Pi_{U,h}\mathcal{U} \subset \mathcal{U}$, cf. (4.3). Also the constraints in (5.36) are conformal with P0-elements in the sense $\Pi_{Z,h}\mathcal{Z} \subset \mathcal{Z}$. As there is no Ξ here, we have $\mathcal{E}_\varepsilon \equiv \mathcal{E}$ but $\mathcal{D}_\varepsilon \neq \mathcal{D}$ is still to be considered.

Corollary 5.14 *Let the disjoint partition $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ be polyhedral, Γ_2 flat, and the nested triangulations be conformal with this partition, φ be qualified as in Lemma 5.1 with $n := 0$ (i.e. with no z-dependence in φ), g satisfy (5.23), and $\psi : \mathbb{R}^3 \to \mathbb{R}$ be continuous satisfying the growth condition* $0 \leq \psi(u) \leq C(1+|u|^{p^\# - \epsilon})$

with $p^{\#}$ from (5.23) and $\epsilon > 0$, and $q_0 \in \mathcal{S}(0)$ is approximated by $[q_0]_{h,\varepsilon} := \Pi_h q_0$. Then the approximate solutions $q_{\varepsilon,\tau,h} = (u_{\varepsilon,\tau,h}, z_{\varepsilon,\tau,h})$ with

$$u_{\varepsilon,\tau,h} \in L^\infty(0,T; W^{1,p}(\Omega;\mathbb{R}^3)), \tag{5.40a}$$

$$z_{\varepsilon,\tau,h} \in L^\infty(0,T; L^\infty(\Gamma_2)) \cap \mathrm{BV}([0,T]; L^1(\Gamma_2)), \tag{5.40b}$$

based on the P1-elements for u and the P0-elements for z converge for $(\varepsilon,\tau,h) \to (0,0,0)$ (in terms of subsequences in the sense of Theorem 3.8 with Remark 4.2) to some energetic solutions of the problem given by E, R, K, f and q_0 above.

Proof. The coercivity of E follows as in Corollary 5.5; note that the term on Γ_2, being nonnegative, cannot destroy it. The weak lower-semicontinuity is again as in Corollary 5.5, the term on Γ_2 being even weakly continuous due to affinity in z-variable and due to the compactness of the trace operator $u \mapsto u|_{\Gamma_2} : W^{1,p}(\Omega;\mathbb{R}^3) \to L^{p^{\#}-\epsilon}(\Gamma_2;\mathbb{R}^3)$.

We will explicitly construct the recovery sequence $\{\tilde{q}_h\}_{h>0}$ for (4.23). As to $\tilde{u}_h$ we use the construction (4.30a); as Γ_2 is flat, ν is constant on Γ_2, and $\Pi_{U,h}\mathcal{U} = U_h \cap \mathcal{U}$, which ensures $\mathcal{U}_h \subset \mathcal{U}$ because $\mathcal{U}_h := \Pi_{U,h}\mathcal{U}$. As for $\Pi_{Z,h}$, we have in mind the standard Clément's quasi-interpolation by element-wise constant averages, hence e.g. functions valued in [0,1] are again mapped to (element-wise constant) functions valued in [0,1]. Then we put

$$\tilde{z}_h := 1 - (1 - z_h)\Pi_{Z,h}\Big(\frac{1-\tilde{z}}{1-z}\Big). \tag{5.41}$$

If $z(x) = 1$, then also $\tilde{z}(x) = 1$ because always $z \le \tilde{z} \le 1$ and the fraction in (5.41) can be defined arbitrarily in valued $[0,1]$. The product of element-wise constant functions $1 - z_h$ and $\Pi_{Z,h}(\frac{1-\tilde{z}}{1-z})$ is again element-wise constant, hence $z_h \in Z_h$. As $0 \le \Pi_{Z,h}(\frac{1-\tilde{z}}{1-z}) \le 1$, we have also $z_h \le \tilde{z}_h \le 1$, hence $\tilde{z}_h \in \mathcal{Z}_h$ and $\tilde{z}_h - z_h \in K$. As $\Pi_{Z,h}(\frac{1-\tilde{z}}{1-z}) \overset{\mathrm{s}}{\to} \frac{1-\tilde{z}}{1-z}$ in any $L^p(\Gamma_2)$, $p < +\infty$, and $z_h \overset{\mathrm{w}^*}{\to} z$, from (5.41) we have $\tilde{z}_h \overset{\mathrm{w}^*}{\to} 1 - (1-z)\frac{1-\tilde{z}}{1-z} = \tilde{z}$ in fact in $L^\infty(\Gamma_2)$ due to the a-priori bound of values in [0,1].

Then, having (4.23) proved, we can verify (3.16) through Proposition 4.4 used with the topology $\sigma := \mathrm{s} \times \mathrm{w}^*$ on $W^{1,p}(\Omega;\mathbb{R}^3) \times L^\infty(\Gamma_2)$. $\square$

Remark 5.15 As we do not have any gradient-type regularization like in Sect. 5.5, we had to assume $\psi(u,\cdot)$ affine to allow for a passage via weak convergence. It however does not allow for any artificial definition of ψ like we did for ψ in Sect. 5.5 for $z > 1$, which is why here we had to include the constraint $z(x) \in [0,1]$ explicitly into $\mathcal{Z}$ in (5.36) but this, in turn, destroyed any quadratic structure in z and hence we had to rely on Proposition 4.4 supported by the rather sophisticated construction (5.41).

Remark 5.16 A debonding on a-priori prescribed surfaces inside the body, called then rather a *delamination*, could be treated similarly only by introducing a more extensive notation, cf. [31]. Let us emphasize that Corollary 5.14 adapted to such a problem substantially improves results from [31], where convergence has only been proved for a semidiscretization in time while the convergence of the full time-space discretization has only silently been assumed under an additional convergence criterion $h/\tau \to 0$.

5.7 Magnetostriction at small strains

In this section, we illustrate our theory on a *deformable ferromagnet* occupying a domain $\Omega \subset \mathbb{R}^3$ and undergoing quasistatic isothermal evolution at small strains. Again, the non-dissipative component $u : \Omega \to \mathbb{R}^3$ will be the *displacement* while the dissipating variable $z : \Omega \to \mathbb{R}^3$ will now be the *magnetization* vector; thus $m = 3$ here. The stored energy is then considered in the form

$$E(u, z) := \int_\Omega \Big(\varphi\big(\nabla u(x), z(x)\big) + \frac{\kappa}{2}\big|\nabla z\big|^2\Big)\,\mathrm{d}x + \frac{\mu_0}{2}\int_{\mathbb{R}^3}\big|\nabla\phi\big|^2\,\mathrm{d}x. \tag{5.42}$$

The particular terms in (5.42) represent the *mechanical stored energy* interacting with an *anisotropic magnetization energy*, the *exchange energy* (with $\kappa > 0$ a coefficient having a quantum-mechanical origin), and the energy of the *demagnetizing field* $\phi \in W^{1,2}(\Omega)$ (with $\mu_0 > 0$ the vacuum permeability) which is determined by the magnetization z by the (weak solution to the) following 2nd-order linear elliptic equation on the whole space $\mathbb{R}^3$:

$$\mathrm{div}(\mu_0\nabla\phi - \chi_\Omega z) = 0 \text{ on } \mathbb{R}^3, \tag{5.43}$$

where $\chi_\Omega : \mathbb{R}^3 \to \{0, 1\}$ denotes the characteristic function on Ω. The external forcing might be both mechanical and magnetical. Let us consider it again via a surface force g (as in Section 5.2) and by an external magnetic field h_{ext}:

$$\langle f(t), (u, z)\rangle := \int_{\Gamma_1} g(t, x)\cdot u(x)\,\mathrm{d}S + \int_\Omega h_{\mathrm{ext}}(t, x)\cdot z(x)\,\mathrm{d}x. \tag{5.44}$$

Contrary to the previous sections, z is not any internal parameter because it can be subjected directly to outer loading by h_{ext}. For notational simplicity, we consider again the Dirichlet condition on Γ_0 and then take $\mathcal{U} = U$ from (5.18) while Z is naturally to be taken as $W^{1,2}(\Omega; \mathbb{R}^3)$. The standard model involves also the so-called *Heisenberg constraint*

$$\big|z(x)\big| = m_{\mathrm{s}} \qquad \text{for a.a. } x \tag{5.45}$$

with $m_{\mathrm{s}} > 0$ a given saturation magnetization. In fact, due to (5.45) we can redefine $\varphi(A, z)$ for $|z| > m_{\mathrm{s}}$, if needed, suitably so that the coercivity (5.2) holds. For the dissipation potential R we consider, for example,

$$R(z) := \int_\Omega d_0\big|z\big| + d_1\big|e_3\cdot z\big|\,\mathrm{d}x \tag{5.46}$$

where $d_0 > 0$ and $d_1 \geq 0$ and $e_3 = (0,0,1)$. The d_0-term has been considered in [58] while for the d_1-term see [59] or also [52, 53]. The former term corresponds to a basic dissipation and ensures coercivity of R while the latter term describes dissipation during remagnetization in a uniaxial magnet with easy-magnetization axis in the direction e_3; then the anisotropic energy $\varphi(A,\cdot)$ is assumed to have minima along this axis and $d_0 + d_1$ is a so-called coercive force which determines the width of a parent hysteresis loop during cyclic magnetization processes. The magnetization process is fully reversible (because we do not consider any sort of unidirectional "hardening" like in [53]) and therefore we put $K = Z = W^{1,2}(\Omega;\mathbb{R}^3)$. The initial magnetization z_0 should satisfy the constraint (5.45) and, together with u_0, be stable with respect to the loading $h_{\rm ext}(0,\cdot)$ and $g(0,\cdot)$; we will not specify this rather technical condition.

We cannot simply involve the constraint (5.45) into $\mathcal{Z}$ because (4.3) cannot conventionally be achieved because no polynomial finite elements are compatible with the Heisenberg constraints (5.45). Hence we implement it by Ξ and then take simply $\mathcal{Z} := Z = W^{1,2}(\Omega;\mathbb{R}^3)$ and define Ξ as

$$\Xi : U \times Z \to X := L^2(\Omega) : (u,z) \mapsto \frac{|z|^2 - m_{\rm s}^2}{\sqrt{|z|^2 + m_{\rm s}^2}}. \tag{5.47}$$

Note that the nonlinearity $r \mapsto (|r|^2 - m_{\rm s}^2)/\sqrt{|r|^2 + m_{\rm s}^2}$ involved in (5.47) has a linear growth and is Lipschitz continuous, and so is $\Xi : L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^3) \to L^2(\Omega)$. Simultaneously, Ξ is weakly continuous on $U \times Z$ due to the compact embedding of $U \times Z$ into $L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^3)$.

Again we consider a polyhedral domain Ω and its nested regular triangulations, and in view of (5.42) take P1-elements both for u and z. Then, in principle, exact integration formulae can be exploited for (5.43) and for the last term in (5.42), too. So no discretization of φ would be needed, although practical calculations usually exploit some numerical approximation of this procedure (and hence of E itself, too). As we did not consider it in previous sections, we omit it here too. Because of the mentioned incompatibility of the P1-elements (and in fact with any polynomial finite-elements), with the constraint $\Xi(u,z) = 0$, i.e. $|z| = m_{\rm s}$, we must consider the penalization method. Using $\alpha = 2$ in (4.5), it yields

$$\mathcal{E}_\varepsilon(u,z) = \int_\Omega \bigg(\varphi\big(\nabla u(x), z(x)\big) + \frac{\kappa}{2}|\nabla z|^2 + \frac{\big(|z|^2 - m_{\rm s}^2\big)^2}{\varepsilon\big(|z|^2 + m_{\rm s}^2\big)}\bigg)\,{\rm d}x + \frac{\mu_0}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\,{\rm d}x. \tag{5.48}$$

The conformity (4.3) is here automatic because there are no other constraints involved, i.e. $\mathcal{Q} \equiv Q$ and $K \equiv Z$.

Corollary 5.17 *Let the data Ω, Γ_0, and Γ_1 be qualified as in Sect. 5.2, let φ be qualified as in Lemma 5.1 with $Z_0 := \mathbb{R}^m$, $m = 3$, $p_1 = 2$, let g satisfy (5.23), and let further $h_{\rm ext} \in C^1([0,T]; L^{6/5}(\Omega;\mathbb{R}^3))$, $q_0 \in \mathcal{S}(0)$ and $[q_0]_{h,\varepsilon} := \Pi_h q_0$. Then the* *approximate solutions* $q_{\varepsilon,\tau,h} = (u_{\varepsilon,\tau,h}, z_{\varepsilon,\tau,h})$ *with*

$$u_{\varepsilon,\tau,h} \in L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^3)), \tag{5.49a}$$

$$z_{\varepsilon,\tau,h} \in L^\infty(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \cap \mathrm{BV}([0,T];L^1(\Omega;\mathbb{R}^3)), \tag{5.49b}$$

based on the P1-elements and the penalization of the Heisenberg constraint (5.45) as in (5.48) converge for $(\varepsilon,\tau,h) \to (0,0,0)$ *(in terms of subsequences in the sense of Theorem 3.8 with Remark 4.2) to energetic solutions of the problem given by* E*,* R*,* Ξ*,* f *and* q_0 *above under the convergence criterion* $h^2/\varepsilon \to 0$*.*

For the convergence criterion $h \le H(\varepsilon)$ can take H, e.g., in the form

$$H(\varepsilon) = \varepsilon^a \quad \text{with any} \;\; 0 < a < 1/2. \tag{5.50}$$

Proof of Corollary 5.17. The weak lower semicontinuity in the sense (3.8) of the φ-term in (5.48) is by Lemma 5.1, while that of the terms $|\nabla z|^2$ and $|\nabla\phi|^2$ is due to the convexity and linearity of (5.43). The penalty term in (5.48) has the 2nd-order-polynomial growth and is therefore continuous because of the compact embedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$. The coercivity of $\mathcal{E}$ on $U \times Z$ follows from (5.2) through Poincaré's inequality.

For our P1-elements, the estimate (4.18) with $\gamma = 1$ is then known to hold with $|\cdot|$ and $\|\cdot\|$ being respectively the L^2- and the $W^{1,2}$-norms. The Lipschitz continuity (4.17) of Ξ from (5.47) holds for $X := L^2(\Omega)$, which just makes the penalty form in (5.48) with $\alpha = 2$. The choice $[q_0]_{h,\varepsilon} := \Pi_h q_0$ again satisfies (4.24). Our assertion then follows from Theorem 3.8 through Proposition 4.3 where (4.19) just says that $h = o(\sqrt{\varepsilon})$, as claimed. □

Remark 5.18 References for magnetostriction usually addresses large strains where more complications arise, cf. [9, 26, 27, 55, 57]. Mathematical analysis at large strains needs some additional regularization, e.g. like [55]. A conventional form of φ in (5.42) in term of small strains, as considered here, is $\varphi(\nabla u, z) = \varphi_0(z) + \frac{1}{2}(e(u) - e_z)^\top \mathbb{C}(e(u) - e_z)$ with e_z a preferred strain tensor corresponding to the magnetization z; for the concrete form of e_z we refer to [27, 57]. No numerical and even purely theoretical analysis of this rate-independent evolution problem seems to be reported in literature hence Corollary 5.17 represents a new result for this conceptual algorithm.

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References

[1] E. Acerbi, N. Fusco, Semicontinuity problems in the calculus of variations. *Arch. Ration. Mech. Anal.* **86** (1984) 125-145.

[2] Alberty, J., Carstensen, C.: Numerical analysis of time-dependent primal elastoplasticity with hardening. *SIAM J. Numer. Anal.* **37** (2000), 1271–1294.

[3] Arndt, M., Griebel, M., Roubíček, T.: Modelling and numerical simulation of martensitic transformation in shape memory alloys. *Continuum Mech. Thermodyn.* **15** (2003), 463-485.

[4] Arndt, M., Griebel, M., Novák, V., Roubíček, T., Šittner, P.: Martensitic transformation in NiMnGa single crystals: numerical simulations and experiments. *Int. J. Plasticity*, **22** (2006), 1943-1961.

[5] Auricchio, F., Mielke, A., Steffaneli, U.: A rate-independent model for the isothermal quasi-static evolution of shape-memory materials. In preparation.

[6] Auricchio, F., Petrini, L.: Improvements and algorithmical considerations on a recent three-dimensional model describing stress-induced solid phase transformations. *Int. J. Numer. Meth. in Engr.* **55** (2002), 1255–1284.

[7] B. Bourdin, G. Francfort, J.-J. Marigo: Numerical experiments in revisited brittle fracture. *J. Mech. Phys. Solids* **48** (2000), 797–826.

[8] Boyd, J.G., Lagoudas, D.C.: A thermodynamical constitutive model for shape memory materials. Part I. The monolithic shape memory alloys. *Int. J. Plasticity* **12** (1996), 805–842.

[9] Brown, W.F.: Magnetoelastic interactions. In: C.Truesdel, ed., Springer Tracts in Natural Philosophy **9**, Springer, 1966.

[10] Colli, P., Frémond, M. and Visintin, A.: Thermo-mechanical evolution of shape memory alloys. *Quarterly Appl. Math.* **48** (1990), 31–47.

[11] Colli, P., Sprekels, J.: Global existence for a three-dimensional model for the thermo-mechanical evolution of shape memory alloys. *Nonlinear Anal.* **18** (1992), 873–888.

[12] Conti, S., Ortiz, M.: Dislocation microstructures and efective behaviour of single crystals. *Archive Rat. Mech. Anal.* **176** (2005), 103–147.

[13] Falk, F.: Model free energy, mechanics and thermodynamics of shape memory alloys. *Acta Metallurgica* **28** (1980), 1773-1780.

[14] Francfort, G., Mielke, A.: An existence result for a rate-independent material model in the case of nonconvex energies. *J. reine u. angew. Math.* **595** (2006), 55–91.

[15] Frémond, M.: Matériaux à mémoire de forme. *C.R. Acad. Sci. Paris* **304**, Série II (1987), 239–244.

[16] Frémond, M.: *Non-Smooth Thermomechanics.* Springer, Berlin, 2002.

[17] Fried, E., Gurtin, M.E.: Dynamic solid-solid transitions with phase characterized by an order perameter. *Physica D* **72** (1994), 287–308.

[18] A. Giacomini, M. Ponsiglione: Discontinuous finite element approximation of quasistatic crack growth in nonlinear elasticity. *Math. Models Methods Appl. Sci.* **16** (2006), 77–118.

[19] Govindjee, S., Miehe, Ch.: A multi-variant martensitic phase transformation model: formulation and numerical implementation. *Comp. Meth. in Appl. Mech. Engr.* **191** (2001), 215–238.

[20] Govindjee, S., Mielke, A., Hall, G.J.: Free-energy of mixing for n-variant martensitic phase transformations using quasi-convex analysis. *J. Mech. Physics Solids* **50** (2002), 1897–1922.

[21] Halphen, B, Nguyen, Q.S.: Sur les matériaux standards généralisés. *J. Mécanique* **14** (1975), 39–63.

[22] Han, W., Reddy, B.D: *Plasticity. Mathematical theory and numerical analysis.* Springer, New York, 1999.

[23] Han, W., Reddy, B.D: Convergence of approximations to the primal problem in plasticity under conditions of minimal regularity. *Numer. Math.* **87** (2000), 283–315.

[24] Hoffmann, K.-H., Niezgódka, M., Zheng Songmu: Existence and uniqueness of global solutions to an extended model of the dynamical development in shape memory alloys. *Nonlinear Analysis, Th. Meth. Appl.* **15** (1990), 977–990.

[25] Huber, J.E., Fleck, N.A., Landis, C.M., McMeeking, R.M.: A constitutive model for ferroelectric polycrystals. *J. Mech. Phys. Solids* **49** (2001), 785–811.

[26] James, R.D., Kinderlehrer, D.: Theory of magnetostriction with applications to $\mathrm{Tb}_x\mathrm{Dy}_{1-x}\mathrm{Fe}_2$. *Phil. Mag.* **68** (1993), 237–274.

[27] James, R.D., Wuttig, M.: Magnetostriction of martensite. *Phil. Mag.* A **77** (1998), 1273.

[28] Jung, Y., Papadopoulos, P., Ritchie, R.O.: Constitutive modeling and numerical simulation of multivariant phase transformation in superelastic shape-memory alloys. *Int. J. Numer. Meth. Engineering* **60** (2004), 429–460.

[29] Kamlah, M.: Ferroelectric and ferroelastic piezoceramics-modelling of electromechanical hysteresis phenomenona. *Cont. Mech. Thermodyn.* **13** (2001), 219–268.

[30] Kinderlehrer, D., Pedregal, P.: Gradient Young measures generated by sequences in Sobolev spaces. *J. Geom. Anal.* **4** (1994), 59–90.

[31] Kočvara, M., Mielke, A., Roubíček, T.: A rate-independent approach to the delamination problem. *Mathematics and Mechanics of Solids* **11** (2006), 423-447.

[32] Kružík, M., Mielke, A., Roubíček, T.: Modelling of microstructure and its evolution in shape-memory-alloy single-crystals, in particular in CuAlNi. *Meccanica* **40** (2005), 389-418..

[33] Levitas, V.I.: The postulate of realizibility: formulation and applications to postbifurcational behavior and phase transitions in elastoplastic materials. *Int. J. Eng. Sci.* **33** (1995), 921–971.

[34] Mainik, A.: A rate-independent model for phase transformations in shape-memory alloys. PhD-thesis, Fachbereich Mathematik, Universität Stuttgart, 2004.

[35] Mainik, A., Mielke, A.: Existence results for energetic models for rate-independent systems. *Calc. Var.* **22** (2005), 73–99.

[36] Mielke, A.: Evolution of rate-independent systems. In: *Handbook of Differential Equations: Evolutionary Diff. Eqs.* (Eds. C.Dafermos, E.Feireisl), pp. 461–559, Elsevier, Amsterdam, 2005.

[37] Mielke, A., Roubíček, T.: Rate-independent model of inelastic behaviour of shape-memory alloys. *Multiscale Modeling Simul.* **1** (2003), 571-597.

[38] Mielke, A., Roubíček, T.: Rate-independent damage processes in nonlinear inelasticity. *M3AS* **16** (2006), 177-209.

[39] Mielke, A., Roubíček, T., Steffaneli, U.: Relaxation and Γ-limits for rate-independent evolution equations. (Preprint No.1156, WIAS, Berlin, 2006) *Calc. Var. P.D.E.*, submitted.

[40] Mielke, A., Theil, F.: A mathematical model for rate-independent phase transformations with hysteresis. In: *Models of continuum mechanics in analysis and engineering.* (Eds.: H.-D.Alber, R.Balean, R.Farwig), Shaker Ver., Aachen, 1999, pp.117-129.

[41] Mielke, A., Theil, F.: On rate-independent hysteresis models. *Nonlin. Diff. Eq. Appl.* **11** (2004), 151–189.

[42] Mielke, A., Theil, F., Levitas, V.I.: A variational formulation of rate-independent phase transformations using an extremum principle. *Archive Rat. Mech. Anal.* **162** (2002), 137–177.

[43] Mielke, A., Timofte, A.: An energetic material model for time-dependent ferroelectric behavior: existence and uniqueness. *Math. Methods in the Appl. Sci.* **29** (2006), 1393-1410.

[44] Müller, S.: Variational models for microstructure and phase transitions. (Lect.Notes No.2, Max-Planck-Institut für Math., Leipzig, 1998). In: *Calculus of variations and geometric evolution problems.* (Eds.: S.Hildebrandt et al.) Lect. Notes in Math. **1713** (1999), Springer, Berlin, pp.85–210.

[45] Pedregal, P.: *Parametrized Measures and Variational Principles.* Birkhäuser, Basel, 1997.

[46] Plecháč, P., Roubíček, T.: Visco-elasto-plastic model for martensitic phase transformation in shape-memory alloys. *Math. Methods in the Appl. Sci.* **25** (2002), 1281-1298.

[47] Rajagopal, K.R., Roubíček, T.: On the effect of dissipation in shape-memory alloys. *Nonlinear Anal., Real World Appl.* **4** (2003), 581-597.

[48] Romanowski, H., Schröder, J.: Modelling of the nonlinear ferroelectric hysteresis within a thermodynamically consistent framework. In: *Trends in Applications of Mathematics to Mechanics* (Eds. Y.Wang, K.Hutter), Shaker Ver., Aachen, 2005, pp.419–428.

[49] Roubíček, T.: A note on an interaction between penalization and discretization. In: Proc. IFIP-IIASA Conf. *Modelling and Inverse Problems of Control for Distributed Parameter Systems.* (A.Kurzhanski, I.Lasiecka, eds.) Lect. Notes in Control and Inf. Sci. **154**, Springer, 1991, pp.145-150.

[50] Roubíček, T.: Dissipative evolution of microstructure in shape memory alloys. In: *Lectures on Applied Mathematics.* (H-J.Bungartz, R.H.W.Hoppe, C.Zenger, eds.) Springer, Berlin, 2000, pp.45–63.

[51] Roubíček, T.: *Nonlinear Partial Differential Equations with Applications.* Birkhäuser, Basel, 2005.

[52] Roubíček, T., Kružík, M.: Microstructure evolution model in micromagnetics. *Zeit. für angew. Math. und Physik*, **55** (2004), 159-182.

[53] Roubíček, T., Kružík, M.: Mesoscopic model for ferromagnets with isotropic hardening. *Zeit. für angew. Math. und Physik* **56** (2005), 107-135.

[54] Roubíček, T., Kružík, M.: Mesoscopic model of microstructure evolution in shape memory alloys, its numerical analysis and computer implementation. 3rd GAMM Seminar on microstructures. (Ed.C.Miehe), *GAMM Mitteilungen* 2006 **29**, J.Wiley, in print.

[55] Rybka, P., Luskin, M.: Existence od energy minimizers for magnetostrictive materials. *SIAM J. Math. Anal.* **36** (2005), 2004-2019.

[56] Shu, Y., Bhattacharya, K.: Domain patterns and macroscopic behaviour of ferroelectric materials. *Phil. Mag. B* **81** (2001), 2021–2054.

[57] Tickle, R.: Ferromagnetic shape memory materials. PhD-thesis, Univ. of Minnesota, Minneapolis, 2000.

[58] Visintin, A., Modified Landau-Lifshitz equation for ferromagnetism. *Physica B* **233** (1997), 365–369.

[59] Visintin, A., Maxwell's equations with vector hysteresis. *Arch. Rat. Mech. Anal.* **175** (2005), 1–37.

[60] Vivet, A., Lexcellent, C.: Micromechanical modelling for tension-compression pseudoelastic behaviour of AuCd single crystals. *Euro Phys.J. A.P.***4**(1998), 125-132.