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Numerical analysis of Lavrentiev-regularized state constrained elliptic control problems

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Abstract. In the present work, we apply semi-discretization proposed by the first author in [13] to Lavrentiev-regularized state constrained elliptic control problems. We extend the results of [17] and prove weak convergence of the adjoint states and multipliers of the regularized problems to their counterparts of the original problem. Further, we prove error estimates for finite element discretizations of the regularized problem and investigate the overall error imposed by the finite element discretization of the regularized problem compared to the continuous solution of the original problem. Finally we present numerical results which confirm our analytical findings.

1. Introduction. In the present work, we apply semi-discretization proposed by the first author in [13] to Lavrentiev-regularized state-constrained elliptic control problems. Let $\Omega \subset \mathbb{R}^n (n = 2, 3)$ denote an open, bounded domain with $C^{0,1}$-boundary Γ. As model problem we consider for states $y \in Y := H^1(\Omega) \cap C(\bar{\Omega})$ and controls $u \in L^2(\Omega)$

$$\text{(P)} \quad \begin{cases} \text{minimize} & J(y,u) := \dfrac{1}{2} \displaystyle\int_\Omega |y - y_d|^2 \, dx + \dfrac{\alpha}{2} \int_\Omega u^2 \, dx \\ \text{subject to} & y = S\,u \text{ and } y(x) \le y_c(x) \text{ a.e. in } \Omega, \end{cases}$$

where $y_d \in L^2(\Omega)$, $y_c \in C(\bar{\Omega})$ denote given functions, and $S : L^2(\Omega) \to Y$ denotes the control-to-state mapping, i.e. the solution operator of the Neumann problem

$$-\Delta\, y + y = u \text{ in } \Omega \text{ and } \partial_n y = 0 \text{ on } \Gamma.$$

Associated to (P) is the Lavrentiev-regularized control problem

$$\text{(P)} \quad \begin{cases} \text{minimize} & J(y,u) := \dfrac{1}{2} \displaystyle\int_\Omega |y - y_d|^2 \, dx + \dfrac{\alpha}{2} \int_\Omega u^2 \, dx \\ \text{subject to} & y = S\,u \text{ and } \lambda u(x) + y(x) \le y_c(x) \text{ a.e. in } \Omega, \end{cases}$$

where $\lambda > 0$ denotes the regularization parameter. Since the constraints in (P) and (P_λ), respectively, define closed convex sets, both problems admit unique solutions (y^*, u^*) and $(\bar{y}_\lambda, \bar{u}_\lambda)$.

The numerical treatment of problem (P) causes difficulties through the presence of the pointwise state constraints, since the corresponding Lagrange multiplier in general only represents a regular Borel measure (see Casas [6] or Alibert and Raymond [1]). In [17], Rösch, Tröltzsch, and the second author propose to circumvent these difficulties through approximating problem (P) by the family of problems (P_λ) ($\lambda > 0$). Among other things, they prove convergence of $(\bar{y}_\lambda, \bar{u}_\lambda) \to (y^*, u^*)$ in $L^2(\Omega)$ for $\lambda \to 0$. Furthermore, they show that the Lagrange multiplier assciated to the mixed control–state constriant in (P_λ) is an L^2-function for every $\lambda > 0$. The development of numerical approaches to tackle problem (P) is ongoing [3, 16, 18]. An excellent overview can be found in [11, 12], where also further references are given.

Numerical analysis for problem (P) is presented by the first author and Deckelnick in [9]. Among other things, they prove convergence of finite element approximations to the control and to the state of order $1 - \varepsilon$ in two-dimensions, and of order $1/2 - \varepsilon$ in three dimensions, in L^2 and H^1, respectively. In [15], the second author obtains the same convergence order for piecewise constant approximations of the controls, and also extends these results to problems with additional box constraints on the control. A general framework for numerical analysis of problems with pointwise state together with general constraints on the control is presented by Deckelnick and the first author in [10]. Error analysis for a full finite element discretization of problem (P_λ) with additional box constraints on the control is carried out in a very recent paper by Cherednichenko and Rösch [7].

In the present paper, we extend the results of [17] for problem (P_λ) and in Theorem 2.5 prove weak convergence for λ tending to zero of the adjoint states p_λ in L^2 and of the multipliers μ_λ in $C(\bar{\Omega})^*$ to their counterparts of problem (P). In Theorem 3.6, we prove error estimates for semi-discrete approximations to problem (P_λ) for $\lambda > 0$ fixed. More precisely, we show

$$\lambda\,\|\bar{u}_\lambda - \bar{u}_{\lambda,h}\| + \|\bar{y}_\lambda - \bar{y}_{\lambda,h}\| \leq C\,\frac{1}{\lambda^2}\,\big(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\big)$$

with C denoting a positive constant independent of the finite element grid size h and of λ. The key idea of the proof consists in the fact that the substitution

$$v(x) = \lambda\,u(x) + y(x) \tag{1.1}$$

transforms (P_λ) into the purely control constrained optimal control problem

$$(\mathrm{PV}) \quad \begin{cases} \text{minimize} & \tilde{J}(y,v) := \dfrac{1}{2}\,\|y - y_d\|^2 + \dfrac{\alpha}{2\,\lambda^2}\,\|v - y\|^2 \\ \text{subject to} & -\Delta\,y + c_\lambda\,y = \dfrac{1}{\lambda}\,v \quad \text{in } \Omega \\ & \qquad\qquad \partial_n\,y = 0 \qquad \text{on } \Gamma \\ \text{and} & v(x) \leq y_c(x) \text{ a.e. in } \Omega. \end{cases}$$

Here, $c_\lambda := 1 + 1/\lambda$. Since (PV) is a purely control-constrained problem, it admits a unique Lagrange multiplier in $L^2(\Omega)$ associated to the inequality constraint. Moreover, the discretization techniques developed in [13] are directly applicable to (PV). Furthermore, in Theorem 3.8 we also relate the finite element solution $(\bar{y}_{\lambda,h}, \bar{u}_{\lambda,h})$ to (y^*, u^*). We prove

$$\|u^* - \bar{u}_{\lambda,h}\| \leq C\,\Big(\sqrt{\lambda} + \frac{1}{\lambda^3}\big(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\big)\Big)$$

and

$$\|y^* - \bar{y}_{\lambda,h}\| \leq C\,\Big(\sqrt{\lambda} + \frac{1}{\lambda^2}\big(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\big)\Big).$$

We conclude that for fixed regularization parameter λ, the semi-discrete controls admit quadratic convergence to the optimal control of the regularized problem. This clearly forms a significant improvement compared to the convergence of finite element approximations to problem (P) reported above. Nevertheless, arguing from the numerical point of view, the overall error introduced by the Lavrentiev regularization consists of two different contributions: one arising from the *regularization* and another one caused by the *discretization*. This fact has to be considered when comparing the Lavrentiev regularization with a discretization of the original problem (P). Indeed, our numerical investigations indicate that the overall errors in case with and without regularization have the same order of magnitude for sufficiently small regularization parameters. Nevertheless, the numerical performance, i.e. the ratio of computational complexity and accuracy, is sometimes significantly improved by the Lavrentiev regularization (see Section 4 below). However, it should be noted that, in general, one can not guarantee that the corresponding states of the Lavrentiev–regularized problem (P_λ) are feasible for problem (P).

The paper is organized as follows. In Section 2 we prove that, beside control and state, also the adjoint state and the Lagrange multipliers converge in some weaker sense to the solution of the original problem. Section 3 addresses the error analysis for the regularized problems. In Section 4, these theoretical findings are confirmed by a numerical example that demonstrates how the regularization affects the numerical performance of an active set method.

1.1. Notation. Throughout this article, we use the following notation. Given an open, bounded set $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, we denote by $(.\,,\,.)$ the natural inner product of in $L^2(\Omega)$. The corresponding norm is denoted by $\|.\|$. Moreover, for the dual pairing between $C(\bar{\Omega})$ and $C(\bar{\Omega})^*$, we write $\langle .\,,\,.\rangle$.

2. Weak convergence of the Lagrange multipliers. In the present section we prove convergence of the adjoint states and of the Lagrange multipliers of problem (P_λ) to their counterparts of problem

(P). For this purpose it is convenient to introduce the reduced objective functional by $f(u) = J(S\,u, u)$ and the Lagrange functional $\mathcal{L} : L^2(\Omega) \times C(\bar{\Omega})^* \to \mathbb{R}$ by

$$\mathcal{L}(u, \mu) := f(u) + \langle S\,u - y_c\,,\, \mu\rangle.$$

Lagrange multipliers associated to the state constraint in (P) then are defined as follows:

Definition 2.1. *Let u^* denote the solution of* (P). *Then, $\mu \in C(\bar{\Omega})^*$ is called Lagrange multiplier, if it satisfies the following conditions:*

$$\frac{\partial \mathcal{L}}{\partial u}(u^*, \mu) = f'(u^*) + S^*\mu = 0 \tag{2.1}$$

$$\langle S\,u^* - y_c\,,\, \mu\rangle = 0 \tag{2.2}$$

$$\langle y\,,\, \mu\rangle \geq 0 \quad \forall\, y \in C(\bar{\Omega})^+, \tag{2.3}$$

where $C(\bar{\Omega})^+$ is defined by $C(\bar{\Omega})^+ = \{y \in C(\bar{\Omega}) \,|\, y(x) \geq 0 \;\forall x \in \bar{\Omega}\}$.

By means of the generalized Karush-Kuhn-Tucker theory, it can be proven that, under a certain Slater condition, problem (P) admits a Lagrange multiplier in $C^*(\bar{\Omega})$ that satisfies the conditions in Definition 2.1 (see for instance Casas [6] or Alibert and Raymond [1]). This Slater condition in the present setting is equivalent to the existence of a $\hat{u} \in L^2(\Omega)$ with $(S\,\hat{u})(x) < y_c(x)$ for all $x \in \bar{\Omega}$. Due to the special structure of the state equation, this is trivially fulfilled in our case, since every constant k with $k < y_c(x)$ everywhere in $\bar{\Omega}$, satisfies $(S\,k)(x) \equiv k < y_c(x)$ for all $x \in \bar{\Omega}$. Next, define $G : L^2(\Omega) \to L^2(\Omega)$ by the operator that arises if one considers the control-to-state operator as an operator with range in $L^2(\Omega)$, and set $p^* = G^*(G\,u^* - y_d) + S^*\,\mu$ such that $p^* \in L^2(\Omega)$. Casas [6] and Alibert and Raymond [1] proved that p^* is the unique very weak solution of

$$\begin{aligned} -\Delta\, p^* + p^* &= y^* - y_d + \mu|_\Omega \quad &&\text{in } \Omega \\ \partial_n\, p^* &= \mu|_\Gamma &&\text{on } \Gamma, \end{aligned} \tag{2.4}$$

that belongs to $W^{1,s}(\Omega)$, $1 \leq s < n/(n-1)$. With the definition of p^*, (2.1) is equivalent to

$$p^* + \alpha\, u^* = 0. \tag{2.5}$$

Notice that, together with the state equation and the pointwise state constraint, (2.2), (2.3), (2.4), and (2.5) are equivalent to the following optimality system

$$\left.\begin{aligned} &-\Delta\, y^* + y^* = u^* \;\;\text{in } \Omega \qquad -\Delta\, p^* + p^* = y^* - y_d + \mu_\Omega \;\;\text{in } \Omega \\ &\partial_n\, y^* = 0 \;\;\text{on } \Gamma \qquad\qquad\quad \partial_n\, p^* = \mu_\Gamma \;\;\text{on } \Gamma \\ &\alpha\, u^* + p^* = 0 \\ &\int_{\bar{\Omega}} (y^* - y_c)\, d\mu = 0\,, \quad y^*(x) \leq y_c(x) \quad \forall\, x \in \bar{\Omega} \\ &\int_{\bar{\Omega}} y\, d\mu \geq 0 \quad \forall\, y \in C(\bar{\Omega})^+, \end{aligned}\right\} \tag{2.6}$$

where μ_Ω and μ_Γ denote the restrictions of μ on Ω and Γ, respectively (cf. also [6] and [1]).

Based on the first-order necessary conditions for the auxiliary problem (PV) that was introduced in the introduction, it is straightforward to derive the optimality system for (P_λ). The latter is given by

$$\left.\begin{aligned} &-\Delta\, \bar{y} + \bar{y} = \bar{u} \;\;\text{in } \Omega \qquad -\Delta\, p + p = \bar{y} - y_d + \mu \;\;\text{in } \Omega \\ &\partial_n\, \bar{y} = 0 \;\;\text{on } \Gamma \qquad\qquad \partial_n\, p = 0 \;\;\text{on } \Gamma \\ &\alpha\, \bar{u}(x) + p(x) + \lambda\, \mu(x) = 0 \quad \text{a.e. in } \Omega \\ &(\mu\,,\, \lambda\, \bar{u} + \bar{y} - y_c) = 0 \\ &\mu(x) \geq 0 \quad \text{a.e. in } \Omega \\ &\lambda\, \bar{u}(x) \,+\, \bar{y}(x) \leq y_c(x) \quad \text{a.e. in } \Omega, \end{aligned}\right\} \tag{2.7}$$

where $(\bar{y}, \bar{u})$ denotes the unique optimal solution to (P_λ). Now, let us consider a sequence of positive real numbers $\{\lambda_n\}$ tending to zero for $n \to \infty$. The associated regularized problems are denoted by (P_n) and their solutions will be referred to as $(\bar{y}_n, \bar{u}_n) \in Y \times L^2(\Omega)$ with an adjoint state $p_n \in Y$ and Lagrange multiplier $\mu_n \in L^2(\Omega)$. In [17] and [16], it is proven that the control and the state converge strongly to the solution of (P), i.e.

$$\bar{u}_n \to u^* \quad \text{in } L^2(\Omega), \qquad \bar{y}_n \to y^* \quad \text{in } Y. \tag{2.8}$$

In the following, we establish corresponding convergence results for μ_n and p_n. It is clear that one cannot expect a result similar to (2.8) for μ_n as the multiplier in the limit is only an element of $C^*(\bar{\Omega})$. We start with the following lemma.

LEMMA 2.2. *The sequence of Lagrange multipliers associated to the mixed constraint in* (P_n)*, denoted by* $\{\mu_n\}$*, is uniformly bounded in* $L^1(\Omega)$.

Proof. The variational formulation of the adjoint equation is given by

$$\int_\Omega \nabla p_n \cdot \nabla w \, dx + \int_\Omega p_n \, w \, dx = \int_\Omega (\bar{y}_n - y_d + \mu_n) w \, dx \quad \forall\, w \in H^1(\Omega).$$

If we insert $w \equiv 1$ as test function, then

$$\int_\Omega \mu_n \, dx = \int_\Omega (p_n - \bar{y}_n + y_d) \, dx = \int_\Omega (-\alpha\, \bar{u}_n - \lambda_n\, \mu_n - \bar{y}_n + y_d) \, dx$$

follows due to the gradient equation in (2.7). Together with the positivity of the Lagrange multiplier, this implies

$$\|\mu_n\|_{L^1(\Omega)} \le (1 + \lambda_n)\, \|\mu_n\|_{L^1(\Omega)} \le \alpha\, \|\bar{u}_n\| + \|\bar{y}_n\| + \|y_d\| \le C_\mu$$

with a constant C_μ independent of n since the optimality of $(\bar{y}_n, \bar{u}_n)$ implies their uniform boundedness in $L^2(\Omega)$. ∎

LEMMA 2.3. *The sequence of Lagrange multipliers* $\{\mu_n\}$ *converges weakly-∗ in* $C(\bar{\Omega})^*$ *to a weak-∗ limit* $\tilde{\mu} \in C(\bar{\Omega})^*$*, i.e.*

$$\int_\Omega \mu_n \, w \, dx \to \langle w \,,\, \tilde{\mu} \rangle \quad \forall\, w \in C(\bar{\Omega})$$

Proof. First, let us identify the function $\mu_n \in L^2(\Omega)$ with an element $\tilde{\mu}_n$ in $C(\bar{\Omega})^*$ by defining

$$\langle w \,,\, \tilde{\mu}_n \rangle = \int_\Omega w \, d\tilde{\mu}_n := \int_\Omega w \, \mu_n \, dx \quad \forall\, w \in C(\bar{\Omega}).$$

Using Lemma 2.2, we obtain

$$\|\tilde{\mu}_n\|_{C(\bar{\Omega})^*} = \sup_{\substack{g \in C(\bar{\Omega}) \\ g \neq 0}} \frac{|\langle g \,,\, \tilde{\mu}_n \rangle|}{\|g\|_{C(\bar{\Omega})}} = \sup_{\substack{g \in C(\bar{\Omega}) \\ g \neq 0}} \frac{\left| \int_\Omega g \, \mu_n \, dx \right|}{\|g\|_{C(\bar{\Omega})}} \le \|\mu_n\|_{L^1(\Omega)} \le C_\mu,$$

i.e. the uniform boundedness of $\{\tilde{\mu}_n\}$ in $C(\bar{\Omega})^*$. Hence, since the closed unit ball in $C(\bar{\Omega})^*$ is weakly-∗ compact, we are allowed to select a subsequence, converging weakly-∗ in $C(\bar{\Omega})^*$ to a weak limit denoted by $\tilde{\mu}$. Because everything what follows is also valid for any other weakly-∗ converging subsequence, a known argument yields the assertion. ∎

Based on the previous lemma, we are now in the position to discuss the convergence of $\{p_n\}$. We will see that it converges weakly in $L^2(\Omega)$ which is also important for the finite element error analysis in the subsequent section (see the proof of Lemma 3.4 below).

Lemma 2.4. *The sequence of adjoint states associated to* (P_n)*, denoted by* $\{p_n\}$*, converges weakly in* $L^2(\Omega)$ *to the solution of*

$$\begin{aligned} -\Delta\, p + p &= y^* - y_d + \tilde{\mu}|_\Omega && \text{in } \Omega \\ \partial_n\, p &= \tilde{\mu}|_\Gamma && \text{on } \Gamma, \end{aligned} \tag{2.9}$$

which is denoted by $\tilde{p}$ *in all what follows.*

Proof. Using again the identification of $\mu_n \in L^2(\Omega)$ with $\tilde{\mu}_n \in C(\bar{\Omega})^*$, one obtains for a fixed, but arbitrary $w \in L^2(\Omega)$

$$\begin{aligned} (w\,,\, p_n) &= \big(w\,,\, G^*(\bar{y}_n - y_d + \mu_n)\big) \\ &= \big(w\,,\, G^*(\bar{y}_n - y_d)\big) + (w\,,\, S^*\,\tilde{\mu}_n) \\ &= (G\,w\,,\, \bar{y}_n - y_d) + \langle S\,w\,,\, \mu_n\rangle \;\to\; (G\,w\,,\, y^* - y_d) + \langle S\,w\,,\, \tilde{\mu}\rangle \\ &\qquad\qquad\qquad = \big(w\,,\, G^*(y^* - y_d) + S^*\,\tilde{\mu}\big) = (w\,,\, \tilde{p}), \end{aligned}$$

where we used Lemma 2.3 and $\bar{y}_n \to y^*$ in $L^2(\Omega)$. Since $w \in L^2(\Omega)$ was chosen arbitrarily, this is equivalent to $p_n \rightharpoonup \tilde{p}$. ☐

Next, it is shown that the weak-$*$ limit $\tilde{\mu}$ indeed represents a Lagrange multiplier for problem (P).

Theorem 2.5. *The sequence of Lagrange multipliers associated to the regularized pointwise state constraints in* (P_λ)*, denoted by* $\{\mu_n\}$*, converges weakly-$*$ in* $C(\bar{\Omega})^*$ *to* $\tilde{\mu}$ *if* $n \to \infty$*. Moreover, the weak-$*$ limit* $\tilde{\mu}$ *is a Lagrange multiplier for the unregularized problem* (P) *according to Definition 2.1.*

Proof. The weak-$*$ convergence is stated in Lemma 2.3. It remains to show that the weak-$*$ limit satisfies the conditions in Definition 2.1, i.e. (2.1)–(2.3). Using Lemma 2.3, the positivity of $\tilde{\mu}$ is straightforward to show: the positivity property of μ_n in (2.7) implies

$$\int_\Omega \mu_n\, w\, dx \geq 0 \quad \forall\, w \in C(\bar{\Omega})^+$$

with $C(\bar{\Omega})^+$ as defined in Definition 2.1. Hence for every fixed, but arbitrary $w \in C(\bar{\Omega})^+$, Lemma 2.3 yields

$$0 \leq \int_\Omega \mu_n\, w\, dx \to \langle w\,,\, \tilde{\mu}\rangle$$

and thus (2.3).

To verify (2.1), we multiply the gradient equation in (2.7) with a fixed but arbitrary function $w \in C(\bar{\Omega})$ and integrate over Ω:

$$\int_\Omega (\alpha\, \bar{u}_n + p_n)\, w\, dx + \lambda_n \int_\Omega \mu_n\, w\, dx = 0 \quad \forall\, w \in C(\bar{\Omega}). \tag{2.10}$$

In view of Lemma 2.3, we have $\int_\Omega \mu_n\, w\, dx \to \langle w\,,\, \tilde{\mu}\rangle$, and hence

$$\lambda_n \int_\Omega \mu_n\, w\, dx \to 0, \tag{2.11}$$

for every fixed, but arbitrary $w \in C(\bar{\Omega})$, because of $\lambda_n \to 0$ for $n \to \infty$. Due to $\bar{u}_n \to u^*$ in $L^2(\Omega)$ and $p_n \rightharpoonup \tilde{p}$ in $L^2(\Omega)$, (2.11) implies for (2.10), when passing to the limit,

$$0 = \int_\Omega (\alpha\, \bar{u}_n + p_n)\, w\, dx + \lambda_n \int_\Omega \mu_n\, w\, dx \to \int_\Omega (\alpha\, u^* + \tilde{p})\, w\, dx \quad \forall\, w \in C(\bar{\Omega}),$$

and hence, $\alpha\, u^* + \tilde{p} = 0$, where $\tilde{p}$ solves (2.9). However, as already stated in context of (2.5), this equation is equivalent to (2.1) in Definition 2.1, i.e. $f'(u^*) + S^*\tilde{\mu} = 0$.

It remains to prove the complementary slackness condition (2.2). The slackness conditions in (2.7) read

$$\int_\Omega \lambda_n\, \mu_n\, \bar u_n\, dx + \int_\Omega (\bar y_n - y_c)\mu_n\, dx = 0,$$

where the second addend converges to $\langle y^* - y_c\,,\, \tilde\mu\rangle$ thanks to Lemma 2.3 and $\bar y_n \to y^*$ in Y. Notice that one can of course not apply (2.11) to the first addend since $\{\bar u_n\}$ does clearly not converge in $C(\bar\Omega)$. However, the gradient equation in (2.7) implies

$$\int_\Omega \lambda_n\, \mu_n\, \bar u_n\, dx = -\int_\Omega \bar u_n\, (\alpha\, \bar u_n + p_n)\, dx \to 0,$$

due to $\bar u_n \to u^*$ in $L^2(\Omega)$ and $(\alpha\, \bar u_n + p_n) \rightharpoonup (\alpha\, u^* + \tilde p) = 0$ in $L^2(\Omega)$ as derived above. Therefore, we obtain

$$\langle y^* - y_c\,,\, \tilde\mu\rangle = 0,$$

which is equivalent to (2.2). □

Remark 2.6. *In view of Lemma 2.5, $\tilde p$ is clearly an adjoint state associated to the original problem.*

2.1. The homogeneous Dirichlet case. Similarly to (P), one can discuss an analogous optimal control problem with homogeneous Dirichlet boundary conditions, i.e.

$$(\mathrm{Q})\quad \begin{cases} \text{minimize} \quad J(y,u) := \dfrac{1}{2}\displaystyle\int_\Omega |y - y_d|^2\, dx + \dfrac{\alpha}{2}\int_\Omega u^2\, dx \\ \text{subject to} \quad -\Delta\, y = u \quad \text{in } \Omega \\ \qquad\qquad\qquad y = 0 \quad \text{on } \Gamma \\ \text{and} \quad y(x) \le y_c(x) \quad \text{a.e. in } \Omega. \end{cases}$$

As will be seen subsequently, the weak-$*$ convergence of the Lagrange multipliers associated to the pointwise state constraints in (Q) can be proven similarly to the theory above. The main difference is the unifrom $L^1(\Omega)$-boundedness of the multipliers which is established by Lemma 2.8 above. It is well known that the state equation in (Q) admits a unique solution y in the state space $Y := H_0^1(\Omega)\cap C(\bar\Omega)$ for every $u \in L^2(\Omega)$. Again, we denote the associated control-to-state operator with range in $C(\bar\Omega)$ by S and with range in $L^2(\Omega)$ by G. In view of the homogeneous Dirichlet boundary conditions, (Q) is naturally only reasonable if $y_c(x) \ge 0$ everywhere on Γ. To satisfy the Slater condition for (Q), we further have to require $y_c(x) > 0$ for all $x \in \Gamma$. The Slater condition then reads

Assumption 2.7. *There exists a $\hat u \in L^2(\Omega)$ such that*

$$(S\,\hat u)(x) < y_c(x) \quad \textit{for all } x \in \bar\Omega.$$

Notice that this condition need not be automatically fulfilled as in case of (P). However, if for instance $y_c(x) > 0$ everywhere in $\bar\Omega$, then the Slater condition is satisfied with $\hat u \equiv 0$. Based on Assumption 2.7, one can verify that the optimal control u^* satisfies the following optimality system (cf. for instance Casas [5]):

$$\left.\begin{aligned} &-\Delta\, y^* = u^* \quad \text{in } \Omega \qquad -\Delta\, p^* = y^* - y_d + \mu \quad \text{in } \Omega \\ &\quad y^* = 0 \quad \text{on } \Gamma \qquad\qquad p^* = 0 \quad \text{on } \Gamma \\ &\qquad\qquad \alpha\, u^* + p^* = 0 \\ &\int_{\bar\Omega} (y^* - y_c)\, d\mu = 0\,, \quad y^*(x) \le y_c(x) \quad \forall\, x \in \bar\Omega \\ &\qquad \int_{\bar\Omega} y\, d\mu \ge 0 \quad \forall\, y \in C(\bar\Omega)^+, \end{aligned}\right\} \tag{2.12}$$

where the Lagrange multiplier μ is again an element of $C(\bar{\Omega})^*$. In [5], it is shown that the adjoint equation admits a solution $p^* \in W^{1,s}$, $1 \leq s < n/(n-1)$. Notice that the adjoint equation exhibits homogeneous Dirichlet boundary conditions, i.e. the multiplier does not generate a measure on Γ. This is due to the fact that the singular part of μ is concentrated on the boundary of the active set which was proven by Bergounioux and Kunisch in [4]. Hence, thanks to the Slater condition which ensures that the state constraint is inactive on the boundary, we have $\mu_\Gamma = 0$ (see also [5]).

As above, we introduce the regularized counterpart of (Q) by

$$(\mathrm{Q}_\lambda)\quad \begin{cases} \text{minimize} & J(y,u) := \dfrac{1}{2}\,\|y-y_d\|^2 + \dfrac{\alpha}{2}\,\|u\|^2 \\ \text{subject to} & -\Delta\, y = u \quad \text{in } \Omega \\ & \quad\; y = 0 \quad \text{on } \Gamma \\ \qquad\text{and} & \lambda\, u(x) + y(x) \leq y_c(x) \quad \text{a.e. in } \Omega. \end{cases}$$

By the same arguments as in case of (P_λ), this problem exhibits a regular Lagrange multiplier in $L^2(\Omega)$. Similarly to (2.7), the optimality system, satisfied by the unique optimal solution $(\bar{y}, \bar{u})$, is given by

$$\left.\begin{array}{c} -\Delta\,\bar{y} = \bar{u} \quad \text{in } \Omega \qquad -\Delta\, p = \bar{y} - y_d + \mu \quad \text{in } \Omega \\ \bar{y} = 0 \quad \text{on } \Gamma \qquad\qquad p = 0 \qquad\qquad \text{on } \Gamma \\ \alpha\,\bar{u}(x) + p(x) + \lambda\,\mu(x) = 0 \quad \text{a.e. in } \Omega \\ (\mu\,,\,\lambda\,\bar{u} + \bar{y} - y_c) = 0 \\ \mu(x) \geq 0 \quad \text{a.e. in } \Omega \\ \lambda\,\bar{u}(x) \,+\, \bar{y}(x) \leq y_c(x) \quad \text{a.e. in } \Omega. \end{array}\right\} \tag{2.13}$$

As in the section above, we consider a sequence of regularization parameters tending to zero, i.e. $\{\lambda_n\}$ with $\lambda_n \to 0$ for $n \to \infty$. The associated regularized control problems as well as their solutions and the corresponding adjoint states and Lagrange multipliers are again referred to by the subscript n. It is easy to see that the analysis in [16] that yields the strong convergence of $\bar{u}_n$ to u^* in $L^2(\Omega)$ and $\bar{y}_n$ to y^* in Y, respectively, can be adapted to the case with homogeneous Dirichlet boundary conditions. To be more precise, the theory in [16] is mainly based on the fact that $G : L^2(\Omega) \to L^2(\Omega)$ is compact and self adjoint, which is clearly also fulfilled in case of (Q). For the adjoint state and the Lagrange multiplier, we derive a result analogous to Lemma 2.5 and Remark 2.6. We again start with the boundedness of the multipliers that follows from the Slater condition in assumption 2.7.

LEMMA 2.8. *Under Assumption 2.7, the sequence of Lagrange multipliers $\{\mu_n\}$ is uniformly bounded in $L^1(\Omega)$.*

Proof. Together with the maximum principle for the state equation, Assumption 2.7 yields the existence of a function $u_0 \in L^2(\Omega)$ with $u_0(x) \leq 0$ a.e. in Ω and $(S\,u_0)(x) < y_c(x)$ for all $x \in \bar{\Omega}$. Thus, there is a $\tau > 0$ such that, for all $\lambda \geq 0$,

$$\lambda\, u_0(x) + (S\,u_0)(x) \leq y_c(x) - \tau \quad \text{a.e. in } \Omega, \tag{2.14}$$

i.e. u_0 is a Slater point for the regularized problem (Q_λ), $\lambda \geq 0$. Next, let us define an auxiliary sequence $\{\hat{u}_n\}$ by

$$\hat{u}_n = u_0 - \bar{u}_n.$$

Together with (2.14), this definition immediately implies

$$-\big(\lambda_n \hat{u}_n(x) + (S\,\hat{u}_n)(x)\big) \geq \tau + \lambda_n \bar{u}_n(x) + (S\,\bar{u}_n)(x) - b(x) \quad \text{a.e. in } \Omega. \tag{2.15}$$

The gradient equation in (2.13) is equivalent to

$$\int_\Omega (\alpha\,\bar{u}_n + G^*(G\bar{u}_n - y_d + \mu_n) + \lambda_n \mu_n) u\, dx = 0 \quad \text{for all } u \in L^2(\Omega).$$

If we now choose $u = \hat{u}$, we obtain

$$\int_\Omega -(\lambda_n\, \hat{u}_n + G\, \hat{u}_n)\mu_n\, dx = \int_\Omega (\alpha\, \bar{u}_n + G^*(G\bar{u}_n - y_d))\hat{u}\, dx.$$

Together with the complementary slackness condition, i.e.

$$\int_\Omega (\lambda_n \bar{u}_n + G\, \bar{u}_n - b)\mu_n\, dx = 0$$

and (2.15), this gives in turn

$$\int_\Omega \tau\, \mu_n\, dx \le \Big((\alpha + \|G\|^2)\|\bar{u}_n\| + \|G\|\,\|y_d\|\Big)\,\big(\|u_0\| + \|\bar{u}_n\|\big).$$

Due to the uniform boundedness of $\{\bar{u}_n\}$ in $L^2(\Omega)$ that follows from the optimality of $\bar{u}_n$, this and the positivity property of μ_n imply the assertion. ∎

For the rest of the proof, we can proceed as in case of the homogeneous Neumann boundary conditions, since the underlying analysis does not depend on the concrete structure of the state equation. In this way, one obtains the following result:

THEOREM 2.9. *Suppose that Assumption 2.7 holds true and let $\{\mu_n\}$ denote the sequence of Lagrange multipliers associated to the regularized pointwise state constraints in* (Q_λ)*, while $\{p_n\}$ is the sequence of adjoint states. Then*

$$\mu_n \overset{*}{\rightharpoonup} \tilde{\mu} \quad \text{in } C(\bar{\Omega})^* \qquad \text{and} \qquad p_n \rightharpoonup \tilde{p} \quad \text{in } L^2(\Omega)$$

hold true, where $\tilde{\mu} \in C(\bar{\Omega})^*$ is a Lagrange multiplier for (Q) *in the sense of Definition 2.1 and $\tilde{p} \in W^{1,s}(\Omega)$, $1 \le s < n/(n-1)$, solves the adjoint equation in* (2.12) *with $\tilde{\mu}$ on the right-hand side.*

Now, we turn to the impact of the Lavrentiev regularization on the numerical treatment of state-constrained optimal control problems. To be more precise, we discuss the semi-discretization of the regularized problem in the spirit of Hinze [13]. The analysis is carried out for problem (P), i.e. the problem with homogeneous Neumann boundary conditions. Nevertheless, it is easy to verify that the same arguments apply in case of (Q) such that the error estimates in Theorem 3.6 and Remark 3.7 also hold for homogeneous Dirichlet boundary conditions.

3. Error analysis for the regularized problem. In the following, we discuss a semi-discretization of problem (P_λ) according to the approach proposed in [13]. To that end, let us introduce a family of regular triangulations $\{\mathcal{T}_h\}_{h>0}$ of Ω, i.e. $\bar{\Omega} = \bigcup_{T\in\mathcal{T}_h} \bar{T}$. With each element $T \in \mathcal{T}_h$, we associate two parameters $\rho(T)$ and $R(T)$, where $\rho(T)$ denotes the diameter of the set T and $R(T)$ is the diameter of the largest ball contained in T. The mesh size of $\mathcal{T}_h$ is defined by $h = \max_{T\in\mathcal{T}_h} \rho(T)$. For the upcoming error analysis, we have to require some additional conditions on $\mathcal{T}_h$ and the domain.

ASSUMPTION 3.1. *The domain Ω is a open bounded and convex subset of $\mathbb{R}^n$, $n = 2, 3$ and its boundary Γ is a polygon ($n = 2$) or a polyhedron ($n = 3$). Moreover, there exist two positive constants ρ and R such that*

$$\frac{\rho(T)}{R(T)} \le R\,, \quad \frac{h}{\rho(T)} \le \rho$$

hold for all $T \in \mathcal{T}_h$ and all $h > 0$. Furthermore, the regularization parameter is bounded from above by by a constant $\lambda_{\max} < \infty$.

Notice that the last assumption on the values for λ is not really restrictive, since our aim is to approximate the original state-constrained problem (P). For domains satisfying Assumption 3.1, one finds the following result (cf. for instance Dauge [8]):

LEMMA 3.2. *Suppose that Ω fulfills the condition in Assumption 3.1 and let f be a given function in $L^2(\Omega)$, while w solves*

$$\begin{aligned} -\Delta\, w + w &= f \quad \textit{in } \Omega \\ \partial_n\, w &= 0 \quad \textit{on } \Gamma. \end{aligned} \tag{3.1}$$

Then, $w \in H^2(\Omega)$ and the estimate

$$\|w\|_{H^2(\Omega)} \le c\,\|f\|$$

holds true with a constant c independent of f and h.

The overall error analysis is based on a consideration of the transformed problem (PV) with the auxiliary control v. In a standard way, one deduces the optimality conditions for (PV) that read

$$(\nabla\bar y\,,\,\nabla w) + c_\lambda\,(\bar y\,,\,w) = (\bar v\,,\,w) \quad \forall\, w \in H^1(\Omega) \tag{3.2}$$

$$(\nabla p\,,\,\nabla w) + c_\lambda\,(p\,,\,w) = \big(\bar y - y_d + \frac{\alpha}{\lambda^2}\,(\bar y - \bar v)\,,\,w\big) \quad \forall\, w \in H^1(\Omega) \tag{3.3}$$

$$\big(\bar v - \bar y + \frac{\lambda}{\alpha}\,p\,,\,v - \bar v\big) \ge 0 \quad \forall\, v \in V_{ad} \tag{3.4}$$

with $V_{ad} := \{v \in L^2(\Omega)\,|\,v(x) \le y_c(x) \text{ a.e. in } \Omega\}$. As in the section before, the optimal solution of the regularized problem is indicated by a bar. Following [13], the semi-discretized version of (PV) is given by

$$(\mathrm{PV}_{h,s}) \quad \left\{ \begin{array}{ll} \text{minimize} & \tilde J(y_h, v) := \dfrac{1}{2}\,\|y_h - y_d\|^2 + \dfrac{\alpha}{2\,\lambda^2}\,\|v - y_h\|^2 \\ \text{subject to} & (\nabla y_h\,,\,\nabla w_h) + c_\lambda\,(y_h\,,\,w_h) = \dfrac{1}{\lambda}\,(v\,,\,w_h) \\ & \qquad\qquad\qquad\qquad \forall\, w_h \in W_h \subset H^1(\Omega) \\ \text{and} & v(x) \le y_c(x) \text{ a.e. in } \Omega, \end{array} \right.$$

i.e. we do not discretize the control v. Here, W_h denotes the space of linear finite elements, i.e. $W_h = \{w \in C(\bar\Omega)\mid w|_T \in \mathcal{P}_1\ \forall\, T \in \mathcal{T}_h\}$, where $\mathcal{T}_h$ is a regular triangulation of Ω. For the optimality system of $(\mathrm{PV}_{h,s})$, we find

$$(\nabla\bar y_h\,,\,\nabla w_h) + c_\lambda\,(\bar y_h\,,\,w_h) = (\bar v_h\,,\,w_h) \quad \forall\, w_h \in W_h \tag{3.5}$$

$$(\nabla p_h\,,\,\nabla w_h) + c_\lambda\,(p_h\,,\,w_h) = \big(\bar y_h - y_d + \frac{\alpha}{\lambda^2}\,(\bar y_h - \bar v_h)\,,\,w_h\big) \quad \forall\, w_h \in W_h \tag{3.6}$$

$$\big(\bar v_h - \bar y_h + \frac{\lambda}{\alpha}\,p_h\,,\,v - \bar v_h\big) \ge 0 \quad \forall\, v \in V_{ad}. \tag{3.7}$$

Notice that $\bar v_h \notin W_h$. Due to the semi-discrete approach, the solution $\bar v$ of (PV) is feasible for $(\mathrm{PV}_{h,s})$ and therefore, we are allowed to insert $\bar v$ in the variational inequality (3.7). On the other hand, we insert $\bar v_h$ in (3.4). Adding both inequalities then yields

$$\big(\bar v - \bar v_h - (\bar y - \bar y_h) + \frac{\lambda}{\alpha}\,(p - p_h)\,,\,\bar v_h - \bar v\big) \ge 0,$$

which in turn gives

$$\begin{aligned} \|\bar v - \bar v_h\|^2 &\le \Big(\frac{\lambda}{\alpha}\,(p - p_h) - (\bar y - \bar y_h)\,,\,\bar v_h - \bar v\Big) \\ &\le \big(y_h(\bar v) - \bar y\,,\,\bar v_h - \bar v\big) + \frac{\lambda}{\alpha}\,\big(p - p_h(\bar v)\,,\,\bar v_h - \bar v\big) \\ &\quad + \underbrace{\frac{\lambda}{\alpha}\,\big(p_h(\bar v) - p_{hh}(\bar v)\,,\,\bar v_h - \bar v\big)}_{=:\,A} \\ &\quad + \underbrace{\big(\bar y_h - y_h(\bar v)\,,\,\bar v_h - \bar v\big)}_{=:\,B} + \underbrace{\frac{\lambda}{\alpha}\,\big(p_{hh}(\bar v) - p_h\,,\,\bar v_h - \bar v\big)}_{=:\,C}. \end{aligned} \tag{3.8}$$

Here, the notation $y(v)$ with an arbitrary $v \in L^2(\Omega)$ corresponds to the solution of

$$(\nabla y\,,\,\nabla w) + c_\lambda\,(y\,,\,w) = \frac{1}{\lambda}\,(v\,,\,w) \quad \forall\, w \in H^1(\Omega), \tag{3.9}$$

while $y_h(v)$ solves

$$(\nabla y_h\,,\,\nabla w_h) + c_\lambda\,(y_h\,,\,w_h) = \frac{1}{\lambda}\,(v\,,\,w_h) \quad \forall\, w_h \in W_h. \tag{3.10}$$

Moreover, $p_h(v)$ is defined as solution of

$$(\nabla p_h\,,\,\nabla w_h) + c_\lambda\,(p_h\,,\,w_h) = \big(y(v) - y_d + \frac{\alpha}{\lambda^2}\,(y(v) - v)\,,\,w_h\big) \quad \forall\, w_h \in W_h \tag{3.11}$$

and similarly, $p_{hh}(v)$ denotes the solution to

$$\begin{aligned}(\nabla p_{hh}\,,\,\nabla w_h) + {}& c_\lambda\,(p_{hh}\,,\,w_h) = \\ & \big(y_h(v) - y_d + \frac{\alpha}{\lambda^2}\,(y_h(v) - v)\,,\,w_h\big) \quad \forall\, w_h \in W_h\end{aligned} \tag{3.12}$$

Notice that, with these notations at hand, we have $\bar y = y(\bar v)$, $\bar y_h = y_h(\bar v_h)$, $p = p(\bar v)$, and $p_h = p_{hh}(\bar v_h)$. Now, let us consider a slightly more general equation given by

$$(\nabla z\,,\,\nabla w) + c_\lambda\,(z\,,\,w) = (g\,,\,w) \quad \forall\, w \in H^1(\Omega) \tag{3.13}$$

with some $g \in L^2(\Omega)$. Similarly to above, we introduce the discrete version of (3.13) by

$$(\nabla z_h\,,\,\nabla w_h) + c_\lambda\,(z_h\,,\,w_h) = (g\,,\,w_h) \quad \forall\, w \in W_h$$

and denote the associated solution by $z_h(g)$.

LEMMA 3.3. *Under Assumption 3.1, there exists a constant $C(\Omega)$ independent of λ such that*

$$\|z_h(g) - z(g)\|_{L^2(\Omega)} \le C(\Omega)\,\big(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\big)\,\|z(g)\|_{H^2(\Omega)}$$

holds true.

Proof. The proof follows standard arguments. Using the Galerkin orthogonality and standard interpolation error estimates, one obtains

$$\begin{aligned}\|z_h(g) - z(g)\|_{H^1(\Omega)} &\le \|z(g) - I_h z(g)\|_{H^1(\Omega)} + \frac{1}{\lambda}\,\|z(g) - I_h z(g)\| \\ &\le C(\Omega)\,\big(h + \frac{1}{\lambda}\,h^2\big)\,\|z(g)\|_{H^2(\Omega)}\end{aligned}$$

where I_h denotes the linear interpolation operator. Applying the well known argument according to Nitsche then gives the assertion. ∎

LEMMA 3.4. *Suppose that Assumption 3.1 is fulfilled. Then there exists a constant $C(\Omega)$ independent of λ such that the following estimate is valid*

$$\|y_h(\bar v) - \bar y\| \le C(\Omega)\,\big(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\big). \tag{3.14}$$

In addition

$$\lambda\,\|p_h(\bar v) - p\| \le C(\alpha,\Omega)\,\big(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\big) \tag{3.15}$$

holds true with a constant $C(\alpha,\Omega)$ independent of λ.

Proof. By construction, $\bar{y} = y(\bar{v})$ is also the solution of the state equation in (P_λ) with $\bar{u} = 1/\lambda\,(\bar{v} - \bar{y})$ on the right hand side, i.e. it solves (3.1) with $\bar{u}$ as inhomogeneity. Therefore, Lemma 3.2 together with (2.8) yields

$$\|\bar{y}\|_{H^2(\Omega)} \le c\,\|\bar{u}\| \le c,$$

where the optimality of $\bar{u}$ guarantees its uniform boundedness w.r.t. λ in $L^2(\Omega)$. Together with Lemma 3.3, this implies (3.14).

Moreover, again due to (1.1), i.e. $\bar{u} = 1/\lambda\,(\bar{v} - \bar{y})$, the adjoint state solves

$$\begin{aligned} -\Delta\,p + p &= \bar{y} - y_d - \frac{1}{\lambda}\,p + \frac{\alpha}{\lambda}\,\bar{u} && \text{in } \Omega\\ \partial_n\,p &= 0 && \text{on } \Gamma, \end{aligned}$$

and hence, again by Lemma 3.2,

$$\lambda\,\|p\|_{H^2(\Omega)} \le c\,\Big(\lambda\,\|\bar{y}\| + \lambda\,\|y_d\| + \alpha\,\|\bar{u}\| + \|p\|\Big)$$

follows with a constant c independent of λ. Thanks to their optimality, $\bar{u}$ and $\bar{y}$ are uniformly bounded in $L^2(\Omega)$ independent of λ. Moreover, consider again an arbitrary sequence $\{\lambda_n\}$ tending to zero for $n \to \infty$. Then, from Lemma 2.4, we know that the associated sequence of adjoint states converges weakly in $L^2(\Omega)$, giving in turn its uniform boundedness such that $\|p\| \le c$ independent of λ. Thus, we obtain $\lambda\,\|p\|_{H^2(\Omega)} \le c$ and consequently, Lemma 3.3 gives the assertion. ◻

Lemma 3.5. *The solution operator associated to the discrete equation* (3.10) *is Lipschitz continuous with Lipschitz constant 1, i.e.*

$$\|y_h(v_1) - y_h(v_2)\| \le \|v_1 - v_2\|$$

for all $v_1, v_2 \in L^2(\Omega)$.

Proof. Let us consider the difference of the equations associated to $y(v_1)$ and $y(v_2)$, respectively. If one inserts $y_h(v_1) - y_h(v_2)$ itself as test function into the arising equation, then

$$\begin{aligned} \|y_h(v_1) - y_h(v_2)\|^2 &\le \|y_h(v_1) - y_h(v_2)\|^2 + \lambda\,\|y_h(v_1) - y_h(v_2)\|^2_{H^1(\Omega)}\\ &= \big(v_1 - v_2\,,\, y_h(v_1) - y_h(v_2)\big) \end{aligned}$$

is obtained, which gives the assertion. ◻

Theorem 3.6. *Suppose that Assumption 3.1 is fulfilled. Then, there is a constant* $C(\alpha,\Omega)$ *independent of* λ *such that*

$$\|\bar{v} - \bar{v}_h\| \le C(\alpha,\Omega)\,\frac{1}{\lambda^2}\,\big(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\big) \tag{3.16}$$

is satisfied.

Proof. The result will be obtained by estimating the addends on the right hand side of (3.8). We start with (3.12) with $\bar{v}$ as inhomogeneity and substract the analogous equation for $\bar{v}_h$ on the right hand side such that

$$\begin{aligned} &\Big(\nabla\big[p_{hh}(\bar{v}) - p_{hh}(\bar{v}_h)\big]\,,\,\nabla w_h\Big) + c_\lambda\,\Big(p_{hh}(\bar{v}) - p_{hh}(\bar{v}_h)\,,\,w_h\Big) =\\ &\qquad\Big(y_h(\bar{v}) - y_h(\bar{v}_h) + \frac{\alpha}{\lambda^2}\,\big(y_h(\bar{v}) - y_h(\bar{v}_h) - \bar{v} + \bar{v}_h\big)\,,\,w_h\Big) \quad \forall\, w_h \in W_h \end{aligned} \tag{3.17}$$

is obtained. Notice that, by definition, $\bar{y}_h = y_h(\bar{v}_h)$ and $p_{hh}(\bar{v}_h) = p_h$. If we now consider (3.10) with $\bar{v}_h - \bar{v}$ on the right hand side, insert $p_{hh}(\bar{v}) - p_h$ as test function, and choose $\bar{y}_h - y_h(\bar{v})$ as test function in (3.17), then subtracting the arising equations from each other yields

$$\begin{aligned} &\frac{1}{\lambda}\,\Big(p_{hh}(\bar{v}) - p_h\,,\bar{v}_h - \bar{v}\Big)\\ &\qquad = \Big(y_h(\bar{v}) - \bar{y}_h + \frac{\alpha}{\lambda^2}\,\big(y_h(\bar{v}) - \bar{y}_h - \bar{v} + \bar{v}_h\big)\,,\,\bar{y}_h - y_h(\bar{v})\Big). \end{aligned} \tag{3.18}$$

Thus, we obtain for C as defined in (3.8)

$$C = -\Big(1 + \frac{\lambda^2}{\alpha}\Big)\, \|y_h(\bar v) - \bar y_h\|^2 + \big(\bar y_h - y_h(\bar v)\,,\, \bar v_h - \bar v\big).$$

For the sum $B + C$ in (3.8), we therefore find by using Young's inequality

$$\begin{aligned} B + C &= -\Big(1 + \frac{\lambda^2}{\alpha}\Big)\, \|y_h(\bar v) - \bar y_h\|^2 + 2\,\big(\bar y_h - y_h(\bar v)\,,\, \bar v_h - \bar v\big) \\ &\leq \Big(\frac{1}{\kappa} - 1 - \frac{\lambda^2}{\alpha}\Big)\, \|y_h(\bar v) - \bar y_h\|^2 + \kappa\, \|\bar v_h - \bar v\|^2 \end{aligned} \tag{3.19}$$

with some real number $\kappa > 0$ that will be specified subsequently. With $\bar y = y(\bar v)$, a discussion, analogous to (3.18), for the difference $p_h(\bar v) - p_{hh}(\bar v)$ implies for A as introduced in (3.8)

$$\begin{aligned} A &= \Big(1 + \frac{\lambda^2}{\alpha}\Big)\, \big(\bar y - y_h(\bar v)\,,\, y_h(\bar v_h) - y_h(\bar v)\big) \\ &\leq \Big(1 + \frac{\lambda^2}{\alpha}\Big)\, C(\Omega)\, \big(h^2 + \frac{1}{\lambda}\, h^3 + \frac{1}{\lambda^2}\, h^4\big)\, \|\bar v_h - \bar v\|, \end{aligned} \tag{3.20}$$

where we used Lemma 3.5 and (3.14) for the last inequality. Similarly, we apply (3.14) and (3.15), respectively, to the first two addends on the right hand side of (3.8). Together with (3.19), we finally end up with

$$\begin{aligned} (1 - \kappa)\, \|\bar v_h - \bar v\|^2 \leq \Big(2 + \frac{\lambda^2}{\alpha} + \frac{1}{\alpha}\Big)\, C(\Omega)\, \big(h^2 + \frac{1}{\lambda}\, h^3 + \frac{1}{\lambda^2}\, h^4\big)\, \|\bar v_h - \bar v\| \\ + \Big(\frac{1}{\kappa} - 1 - \frac{\lambda^2}{\alpha}\Big)\, \|y_h(\bar v) - \bar y_h\|^2. \end{aligned} \tag{3.21}$$

To ensure the non-positivity of the coefficient in front of the last addend, we have to choose $\kappa \geq \alpha/(\alpha + \lambda^2)$. Hence, $1 - \kappa > 0$ follows. Clearly, the best choice of κ is the smallest possible value, i.e. $\kappa = \alpha/(\alpha + \lambda^2)$ such that (3.21) gives

$$\|\bar v_h - \bar v\| \leq \frac{\alpha + \lambda^2}{\lambda^2}\, \Big(2 + \frac{\lambda^2}{\alpha} + \frac{1}{\alpha}\Big)\, C(\Omega)\, \big(h^2 + \frac{1}{\lambda}\, h^3 + \frac{1}{\lambda^2}\, h^4\big).$$

For $\lambda \leq \lambda_{\max}$, according to Assumption 3.1, this implies (3.16). □

With Theorem 3.6 at hand, it is straightforward to derive an estimate for the error with respect to the state: using Lemma 3.5, (3.14), and (3.16), we find

$$\begin{aligned} \|\bar y - \bar y_h\| &\leq \|\bar y - y_h(\bar v)\| + \|\bar v - \bar v_h\| \\ &\leq \Big(C(\Omega) + \frac{1}{\lambda^2}\, C(\alpha, \Omega)\Big)\, \big(h^2 + \frac{1}{\lambda}\, h^3 + \frac{1}{\lambda^2}\, h^4\big). \end{aligned} \tag{3.22}$$

Clearly, for small values of λ, the factor $1/\lambda^2$ also dominates this error. Now, let us turn to the original control u. In view of the transformation formula (1.1), we define the discrete optimal control by $\bar u_h = (1/\lambda)(\bar v_h - \bar y_h)$ and hence $\bar u_h \notin W_h$. The optimality of $\bar v_h$ for $(\mathrm{PV}_{h,s})$ clearly implies that $\bar u_h$ solves

$$(\mathrm{P}_{\lambda,h,s}) \quad \begin{cases} \text{minimize} & J(y_h, u) := \dfrac{1}{2}\, \|y_h - y_d\|^2 + \dfrac{\alpha}{2}\, \|u\|^2 \\ \text{subject to} & (\nabla y_h\,,\, \nabla w_h) + (y_h\,,\, w_h) = (u\,,\, w_h) \quad \forall\, w_h \in W_h \\ \text{and} & \lambda\, u(x) + y_h(x) \leq y_c(x) \text{ a.e. in } \Omega, \end{cases}$$

i.e. the semi-discrete counterpart of (P_λ). Due to $\bar u - \bar u_h = (1/\lambda)[(\bar v - \bar v_h) + (\bar y_h - \bar y)]$, (3.16) and (3.22) immediately give the following result:

Corollary 3.7. *Under Assumption 3.1, we find for the error in the original control*

$$\|\bar u - \bar u_h\| \leq C(\alpha, \Omega)\, \frac{1}{\lambda^3}\, \big(h^2 + \frac{1}{\lambda}\, h^3 + \frac{1}{\lambda^2}\, h^4\big), \tag{3.23}$$

where $\bar{u}$ denotes the solution of (P_λ)*, while $\bar{u}_h$ is the solution of the associated semi-discrete problem.*

Thus, for a fixed λ, we obtain quadratic convergence of the control as in case of the purely control-constrained case discussed in [13]. On the other hand, Corollary 3.7 already indicates that the approximation behavior of the solution of $(\mathrm{P}_{\lambda,h})$ strongly depends on the value of λ. As we would like to approximate the purely state-constrained problem for $\lambda = 0$, we now investigate how to couple h and λ. For the overall approximation error, we find

$$\|u^* - \bar{u}_{\lambda,h}\| \leq \|u^* - \bar{u}_\lambda\| + \|\bar{u}_\lambda - \bar{u}_{\lambda,h}\|,$$

where, as before, u^* denotes the solution of the original purely state-constrained problem (P). Moreover, $\bar{u}_\lambda$ is the exact solution of (P_λ) for a given $\lambda > 0$ and $\bar{u}_{\lambda,h}$ denotes the associated discrete solution. Assuming the rather strict condition that the sequence $\{\bar{u}_\lambda\}_{\lambda\downarrow 0}$ is uniformly bounded in $L^\infty(\Omega)$, it is shown in [18] that

$$\|u^* - \bar{u}_\lambda\| \leq c\sqrt{\lambda} \tag{3.24}$$

holds true with a constant c independent of λ. Using this together with (3.23), the continuity of the control-to-state mapping, and (3.26) proves

Theorem 3.8. *Let Assumption 3.1 be fulfilled and assume that the sequence of optimal solutions to* (P_λ) *for $\lambda \downarrow 0$, denoted by $\{\bar{u}_\lambda\}$, is uniformly bounded in $L^\infty(\Omega)$. Then, with the notations introduced above there holds*

$$\|u^* - \bar{u}_{\lambda,h}\| \leq C(\alpha,\Omega)\left(\sqrt{\lambda} + \frac{1}{\lambda^3}\big(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\big)\right) =: C(\alpha,\Omega)\,F(\lambda,h), \tag{3.25}$$

and

$$\|y^* - \bar{y}_{\lambda,h}\| \leq C(\alpha,\Omega)\left(\sqrt{\lambda} + \frac{1}{\lambda^2}\big(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\big)\right) \tag{3.26}$$

with a generic positive constant $C(\alpha,\Omega)$ independent of λ and h.

From (3.25) and (3.26), we deduce two different theoretical predictions concering the qualitative impact of the Lavrentiev regularization on the numerical approximation of (P).

Remark 3.9. *We observe that, for the minimization of $\|u^* - \bar{u}_\lambda\|$ a small value of λ is favorable, while the discretization error is increased by a reduction of λ. Hence, the two different contributions to the overall error behave contrarily. Secondly, in case of the state, the discretization error is less important than in case of the control because of the coefficient $1/\lambda^2$ instead of $1/\lambda^3$. This indicates that the overall error with respet to the state might be dominated by the regularization error.*

4. Numerical investigation. Both theoretical predictions, stated by Remark 3.9, will be confirmed by the numerical example, presented below. Due to the opposite behavior of the two different error contributions, there should be an optimal value for λ. As a coarse indicator for the dependency of this optimum on h, we compute the minimizer of F in (3.25), denoted by $\lambda_0(h)$, as solution of $\partial F(\lambda,h)/\partial\lambda = 0$. The result for different values of h is displayed in Table 4.1, where δ is defined by

$$\delta := \frac{\log(\lambda_0(h_1)) - \log(\lambda_0(h_2))}{\log(h_1) - \log(h_2)}.$$

As one can see, the values for λ_0 approximately satisfy $\lambda_0 \sim h^{0.6}$. This relation is also used for the choice of λ in the following example. Moreover, we see from Table 4.1 that, in order to balance the different error contributions, one has to reduce the regularization parameter if h is decreased.

The test case, used for the following numerical investigation, is taken from [14]. It is constructed such that the Lagrange multipliers associated to the pure state constraints are continuous. The considered control problem coincides with (P) unless that there is an additional bound from below in the state constraint, i.e. $y_a(x) \leq y(x) \leq y_b(x)$ a.e. in Ω. It is easy to verify that this additional bound does not influence the

Table 4.1
Dependency of the Lavrentiev parameter on h.

	$h = 0.04$	$h = 0.02$	$h = 0.01$	$h = 0.005$
$\lambda_0(h)$	0.5835	0.3843	0.2542	0.1688
δ	–	0.6025	0.5962	0.5909

theory presented above. We choose $\Omega = (0,1)^2$ as test domain and set $\alpha = 10^{-4}$. Moreover, the desired state y_d and the bounds y_a and y_b are defined by

$$y_a(x) = \begin{cases} g(x) \text{ , if } g(x) \leq -0.7 \\ -0.7 \text{ , if } g(x) > -0.7 \end{cases} , \; y_b(x) = \begin{cases} g(x) \text{ , if } g(x) \geq 0.7 \\ 0.7 \text{ , if } g(x) < 0.7 \end{cases}$$

$$y_d(x) = \begin{cases} \big((2\pi^2\alpha - 1)(2\pi^2 + 1) + 11\big) g(x) - 7 \text{ , if } g(x) \geq 0.7 \\ \big((2\pi^2\alpha - 1)(2\pi^2 + 1) + 11\big) g(x) + 7 \text{ , if } g(x) < 0.7 \end{cases}$$

with $x = (x_1, x_2)$ and $g(x) := \cos(\pi\, x_1)\cos(\pi\, x_2)$. It is straightforward to verify that the exact solution for this problem is given by

$$y^*(x) = g(x) \text{ , } u^*(x) = (2\,\pi^2 + 1)\, g(x) \text{ , } p^*(x) = -\alpha\,(2\,\pi^2 + 1)\, g(x)$$

$$\mu_a(x) = \begin{cases} -10\, g(x) - 7 \text{ , if } g(x) \leq -0.7 \\ 0 \qquad\qquad \text{, if } g(x) > -0.7 \end{cases} , \; \mu_b(x) = \begin{cases} 10\, g(x) - 7 \text{ , if } g(x) \geq 0.7 \\ 0 \qquad\quad \text{, if } g(x) < 0.7. \end{cases}$$

The aim of the following numerical investigation is to confirm the conclusions, stated in Remark 3.9, on the principle influence of the Lavrentiev regularization on the approximation behavior. As we will see in the following, these effects are also visible in case of a full discretization of the control problems instead of the semi-discrete approach, discussed above. The advantage of a full discretization, where the control is also discretized by linear ansatz functions, is that the corresponding implementation is significantly less costly compared to the semi-discrete approach. The fully discretized problems read

$$(\mathrm{P}_h) \quad \begin{cases} \displaystyle\min_{u_h \in W_h} & J(y_h, v) := \frac{1}{2}\,\|y_h - y_d\|^2 + \frac{\alpha}{2}\,\|u_h\|^2 \\ \text{subject to} & (\nabla y_h\,,\, \nabla w_h) + (y_h\,,\, w_h) = (u_h\,,\, w_h) \quad \forall\, w_h \in W_h \\ \text{and} & y_h(x) \leq y_c(x) \text{ a.e. in } \Omega \end{cases}$$

and

$$(\mathrm{P}_{\lambda,h}) \quad \begin{cases} \displaystyle\min_{u_h \in W_h} & J(y_h, v) := \frac{1}{2}\,\|y_h - y_d\|^2 + \frac{\alpha}{2}\,\|u_h\|^2 \\ \text{subject to} & (\nabla y_h\,,\, \nabla w_h) + (y_h\,,\, w_h) = (u_h\,,\, w_h) \quad \forall\, w_h \in W_h \\ \text{and} & \lambda\, u_h(x) + y_h(x) \leq y_c(x) \text{ a.e. in } \Omega. \end{cases}$$

In [9], it is shown that, in case of the purely state-constrained problem, the fully discretized and the semi-discrete approach coincide. The discrete problems (P_h) and $(\mathrm{P}_{\lambda,h})$ are numerically solved by a primal-dual active set strategy (cf. for instance [2] or [3]). Moreover, the overall systems of equations for the unknowns u_h, y_h, p_h, $\mu_{a,h}$, and $\mu_{b,h}$, arising in each active set step, are solved using a direct sparse LU decomposition. Figure 4.1 and 4.2 show the L^2-norms of the relative difference between the discrete and the exact solutions for difference values for h and λ, in particular $\lambda = 4 \cdot 10^{-4}\, h^{0.6}$. Figure 4.1 illustrates that the most accurate approximation of u^* is not achieved with $\lambda = 0$. This confirms the first statement of Remark 3.9, i.e. that the two different error components behave contrarily, which leads to an optimal value for λ depending on h. In fact, Figure 4.1 demonstrates that this optimal value is indeed given by $\lambda \sim h^{0.6}$. If we further decrease the value for λ, then the error is increased again and approaches the values for $\lambda = 0$, as exemplarily shown for $\lambda = 10^{-6}$. Furthermore, from Figure 4.2, we

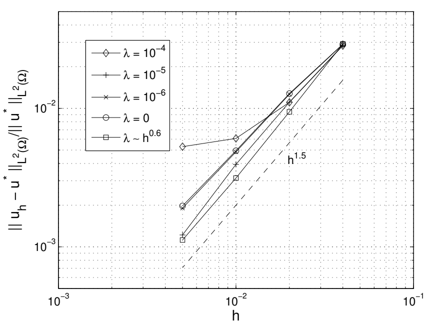

FIG. 4.1. *The relative error in the control for different values of h and λ.*

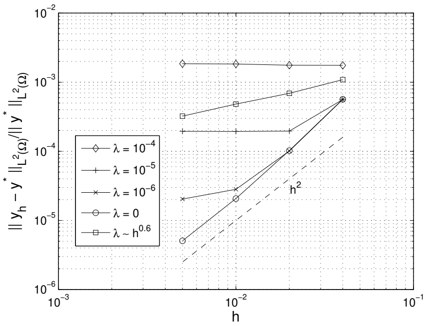

FIG. 4.2. *The relative error in the state for different values of h and λ.*

notice that the best approximation of the state is realized with $\lambda = 0$. In addition to that, for $\lambda = 10^{-4}$, a reduction of h does not influence the overall error. This indicates that the error with respect to the state is indeed dominated by the regularization error, as stated in the second part of Remark 3.9.

Assessing the overall impact of the regularization on the accuracy, one has to conclude that the difference between the relative errors in the control is fairly small (cf. Figure 4.1), while the error in the state is even increased by the regularization. Hence, this example indicates that the Lavrentiev regularization only yields in parts a slight improvement of the accuracy. A more significant reason for the regularization is the improvement of the performance that is depicted in Figures 4.3–4.6. In particular, Figures 4.3

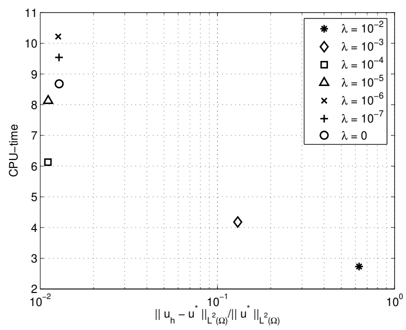

FIG. 4.3. *CPU-time in sec. and relative error of the control for different λ and $h = 0.02$.*

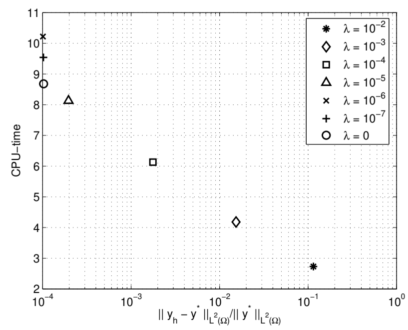

FIG. 4.4. *CPU-time in sec. and relative error of the state for different λ and $h = 0.02$.*

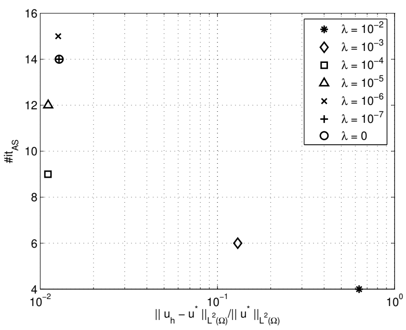

FIG. 4.5. *Number of active set iterations and relative error of the control for different λ and $h = 0.02$.*

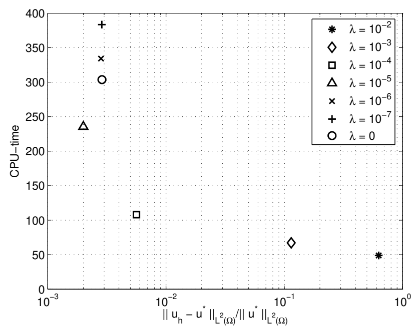

FIG. 4.6. *CPU-time in sec. and relative error of the state for different λ and $h = 0.00667$.*

and 4.6 show that the performance is indeed improved by the regularization, i.e. a smaller approximation error is achieved with less computational effort. Moreover, for certain values of λ, the performance with respect to the approximation of the state is comparable to the case $\lambda = 0$ as Figure 4.4 illustrates. In addition, Figure 4.5 demonstrates that the number of iteration required by the active set algorithm is also reduced by the regularization. In summary, we conclude that a regularization of optimal control problems with pointwise state constraints is reasonable, also from a numerical point of view. However, the results strongly depend on the coupling of reagularization parameter and mesh size, and the optimal value for λ is of course not known a priori.

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