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Moderate deviations for random walk in random scenery

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Abstract: We investigate random walks in independent, identically distributed random sceneries under the assumption that the scenery variables satisfy Cramér's condition. We prove moderate deviation principles in dimensions $d \geq 2$, covering all those regimes where rate and speed do not depend on the actual distribution of the scenery. In the case $d \geq 4$ we even obtain precise asymptotics for the annealed probability of a moderate deviation, extending a classical central limit theorem of Kesten and Spitzer. In $d \geq 3$, an important ingredient in the proofs are new concentration inequalities for self-intersection local times of random walks, which are of independent interest, whilst in $d = 2$ we use a recent moderate deviation result for self-intersection local times, which is due to Bass, Chen and Rosen.

1. Introduction

In the world of stochastic processes in random environments, random walks in random scenery represent a class of processes with fairly weak interaction. Nevertheless, they have deservedly received a lot of attention since their introduction by Kesten and Spitzer [KS79] and, independently, by Borodin [Bo79a, Bo79b]. A major reason for this interest is that in $d \leq 2$ the simple random walk in random scenery exhibits *super-diffusive* behaviour. However, in dimensions $d \geq 3$, when the underlying random walk visits most sites only once, the behaviour of the random walk in random scenery is diffusive. Here finer features, like large deviation behaviour, have to be studied in order to get an understanding of the interaction of walk and scenery.

To define random walk in random scenery, suppose $\{S_n \colon n \geq 0\}$ is an underlying random walk on $\mathbb{Z}^d$ started at the origin, and $\{\xi(z) : z \in \mathbb{Z}^d\}$ are independent, identically distributed real-valued random variables, which are independent of the random walk and which are called the scenery. *Random walk in random scenery* is the process $\{X_n \colon n \geq 0\}$ given by

$$X_n := \sum_{1 \leq k \leq n} \xi(S_k) = \sum_{z \in \mathbb{Z}^d} \ell_n(z)\, \xi(z) \qquad \text{for } n \geq 0,$$

where $\ell_n(z) := \sum_{1 \leq k \leq n} 1\{S_k = z\}$ are the local times of the random walk at the site z.

Throughout this paper we make the following additional *assumptions* on the model. The underlying walk is a symmetric and aperiodic walk in dimensions $d \geq 2$, such that the covariance matrix Γ of S_1 is finite and nondegenerate. Moreover, the random variable $\xi(0)$ is centred, i.e. $\mathbb{E}\xi(0) = 0$, with variance $\sigma^2 > 0$, and satisfies $\mathbb{E}|\xi(0)|^3 < \infty$ and *Cramér's condition*,

$$\mathbb{E}\big\{ e^{\theta \xi(0)} \big\} < \infty \qquad \text{for some } \theta > 0. \tag{1}$$

The early papers by Kesten, Spitzer and Borodin establish *central limit theorems* for the random walk in random scenery. Indeed, it is (implicitly) shown in [KS79] that, for $d \geq 3$,

$$\frac{X_n}{\sqrt{n}} \overset{n\uparrow\infty}{\Longrightarrow} \mathcal{N}\big(0, \sigma(2G(0) - 1)\big), \tag{2}$$

where G is the Green's function of the underlying random walk. Bolthausen in [Bo89] extended this to the planar case by showing that

$$\frac{X_n}{\sqrt{n \log n}} \overset{n\uparrow\infty}{\Longrightarrow} \mathcal{N}(0, \pi^{-1}).$$

Hence, *moderate and large deviation problems* for the random walk in random scenery deal with the asymptotic behaviour of $\mathbb{P}\{X_n \geq b_n\}$ for $b_n \gg \sqrt{n}$, i.e. $\lim b_n/\sqrt{n} = \infty$, if $d \geq 3$, and $b_n \gg \sqrt{n \log n}$ if $d = 2$. Let us remark for completeness that Kesten and Spitzer have also established a limit theorem in distribution for $X_n/n^{3/4}$ with non-Gaussian limits for $d = 1$, a case we do not consider in this paper as large and moderate deviations are more or less fully understood in this case.[1]

Large deviation problems for random walks in random scenery in dimensions $d \geq 2$ have only recently attracted attention, see [GP02, GHK06, GKS05, As06, AC05, AC06], and also [CP01, AC03, Ca04] where Brownian motions are used in place of random walks. The fascination of this subject stems from the rich behaviour that comes to light when large deviations are investigated. The intricate interplay of the walk with the scenery leads to a large number of different regimes depending on

- the dimension d of the underlying lattice $\mathbb{Z}^d$,
- the upper tail behaviour of the scenery variable,
- the size of the deviation studied,

to name just the most important ones. For example, Asselah and Castell [AC06], restricting attention to dimensions $d \geq 5$ and scenery variables with superexponential decay of upper tails, have identified *five* regimes with different large deviation speeds. Heuristically, in each regime the walk and the scenery 'cooperate' in a different way to obtain the deviating behaviour. Up to now only one of these regimes has been fully treated, including the discussion of explicit rate functions. This is the *very large deviation* regime discussed (together with a number of boundary cases) by Gantert, König and Shi in [GKS05]. In this regime it is assumed that

$$\log \mathbb{P}\{\xi(0) > x\} \sim -D\, x^q \qquad \text{as } x \uparrow \infty,$$

for some $D > 0$ and $q > d/2$. Then, for any $n \ll b_n \ll n^{\frac{1+q}{q}}$, as $n \uparrow \infty$,

$$\log \mathbb{P}\big\{X_n > b_n\big\} \sim K\, n^{-\frac{2q-d}{d+2}}\, b_n^{\frac{2q}{d+2}}, \tag{3}$$

where $K = K(D, q, d) > 0$ is a constant given explicitly in terms of a variational problem. The underlying strategy is that the random walk *contracts* to grow at a speed of

$$n^{\frac{1+q}{d+2}} / b_n^{\frac{q}{d+2}} \ll n^{\frac{1}{2}},$$

and the scenery adopts values of size b_n/n on the range of the walk. The right hand side in (3) represents the combined cost of these two deviations.

In the present paper we study *moderate deviation principles*, providing a full analysis including explicit rate functions and, in dimensions $d \geq 4$, even exact asymptotics of moderate deviation probabilities. We consider as moderate deviations the regimes extending from the central limit scaling up to the point where either the deviation speed or the rate function start to depend on the actual distribution of the scenery, or in other words where tail conditions stronger than Cramér's condition would have an impact on the speed or rate of the deviations.

Heuristically, our results, which will be described in detail in the next section, show that in $d \geq 3$ throughout the moderate deviation regime the deviation is achieved by a moderate deviation of the scenery without any contribution from the walk. The rates therefore agree with those obtained for *fixed* walk in a random scenery by Guillotin-Plantard in [GP02]. Crucial ingredients of our proofs are concentration inequalities for self-intersection local times of random walks, see Proposition 11. Our exact asymptotic results for the moderate deviation probabilities build on classical ideas of Cramér.

[1] This information was communicated to us by F. Castell.

In $d = 2$, by contrast, the moderate deviation regime splits in two parts. If $\sqrt{n \log n} \ll b_n \ll \sqrt{n} \log n$ then, again, we only have a contribution from the scenery and the walk exhibits typical behaviour. However, if $\sqrt{n} \log n \ll b_n \ll n/\log n$ the random walk *contracts*, though in a much more delicate way than in the very large deviation regime: The self-intersection local times of the walk, which normally are of order $n \log n$ are now increased to be of order $\sqrt{n} b_n$. At the same time, on the (contracted) range of the walk, the scenery values perform a moderate deviation and take values of size b_n/n. Our results in the case $d = 2$ rely on moderate deviation principles for renormalised self-intersection local times of planar random walks recently obtained by Bass, Chen and Rosen [BCR06].

2. Main results

Recall that we assume that the random variable $\xi(0)$ satisfies Cramér's condition (1) and $\sigma^2 > 0$ denotes its variance. For $d \geq 3$ we define the Green's function of the random walk by

$$G(x) := \sum_{k=0}^{\infty} \mathbb{P}\{S_k = x\} \qquad \text{for } x \in \mathbb{Z}^d.$$

Theorem 1 (Refined moderate deviations in dimensions $d \geq 4$).
There exists a regularly varying sequence (a_n) of index $\frac{2}{3}$, such that, if $d \geq 4$ and $n^{\frac{1}{2}} \ll b_n \ll a_n$, then

$$\mathbb{P}\big\{X_n \geq b_n\big\} \sim 1 - \Phi\bigg(\frac{b_n}{\sqrt{\sigma^2\, n\, (2G(0) - 1)}}\bigg) \qquad \text{as } n \uparrow \infty,$$

where Φ denotes the standard normal distribution function.

Remark 2. This result extends the central limit theorem (2) to the moderate deviation regime. Note that asymptotics of this degree of precision are very rarely encountered in stochastic processes beyond the independent case. In this theorem we are restricted to dimensions $d \geq 4$ as our proof requires an analysis of *triple* self-intersections of random walks, for which $d = 3$ is the critical dimension.

In dimension $d = 3$ we can no longer provide *precise* asymptotics, but we can still prove a full moderate deviation principle with the same speed and rate function as in $d \geq 4$.

Theorem 3 (Moderate deviations in dimensions $d \geq 3$).
If $d \geq 3$ and $n^{\frac{1}{2}} \ll b_n \ll n^{\frac{2}{3}}$, then, as $n \uparrow \infty$,

$$\log \mathbb{P}\big\{X_n \geq b_n\big\} \sim -\frac{b_n^2}{n}\, \frac{1}{2\sigma^2\, (2G(0) - 1)}\,.$$

Remark 4. In this regime the deviation is entirely due to the moderate deviation behaviour of the scenery, whereas the random walk does not contribute and behaves in a typical way. Asselah and Castell [AC06] show that the regime in this result is maximal possible under Cramér's condition, more precisely, higher regularity features of the scenery distribution decide whether this behaviour persists when b_n grows faster than $n^{2/3}$.

Remark 5. For the sequence $b_n = n^{\beta}$ with $1/2 < \beta \leq 2/3$, the deviation speed $n^{2\beta - 1}$, but not the rate function, in this result was identified by Asselah and Castell [AC06] in $d \geq 5$ and by Asselah [As06] in $d = 3$, under the additional assumptions that the law of $\xi(0)$ has a symmetric density which is decreasing on the positive half-axis.

Turning to $d = 2$, we define $\varkappa$ to be the optimal constant in the *Gagliardo-Nirenberg* inequality,

$$\varkappa := \inf \left\{ c \,:\, \|f\|_4 \le c \, \|\nabla f\|_2^{\frac{1}{2}} \, \|f\|_2^{\frac{1}{2}} \text{ for all } f \in C_c^1(\mathbb{R}^2) \right\}.$$

This constant features prominently in large deviation results for intersection local times of Brownian motion and random walk intersection local times, see [Ch04] for further discussion of the Gagliardo-Nirenberg inequality and the associated constant $\varkappa$.

Theorem 6 (Moderate deviations in dimension $d = 2$)**.**

(a) *If* $n^{\frac{1}{2}} \sqrt{\log n} \ll b_n \ll n^{\frac{1}{2}} \log n$, *then, as* $n \uparrow \infty$,

$$\log \mathbb{P}\{X_n \ge b_n\} \sim -\frac{b_n^2}{n \log n} \, \frac{\pi (\det \Gamma)^{1/2}}{2\sigma^2}.$$

(b) *If* $n^{\frac{1}{2}} \log n \ll b_n \ll n / \log n$, *then, as* $n \uparrow \infty$,

$$\log \mathbb{P}\{X_n \ge b_n\} \sim -\frac{b_n}{\sqrt{n}} \, \frac{(\det \Gamma)^{1/4}}{\varkappa^2 \sigma}.$$

(c) *Finally, for every* $a > 0$,

$$\log \mathbb{P}\{X_n \ge a n^{\frac{1}{2}} \log n\} \sim -I(a) \log n,$$

where

$$I(a) := \begin{cases} \dfrac{\pi a^2 (\det \Gamma)^{1/2}}{2\sigma^2}, & \text{for } a \le \dfrac{\sigma}{\pi \varkappa^2 (\det \Gamma)^{1/4}}, \\[2ex] \dfrac{a \, (\det \Gamma)^{1/4}}{\sigma \varkappa^2} - \dfrac{1}{2\pi \varkappa^4}, & \text{for } a \ge \dfrac{\sigma}{\pi \varkappa^2 (\det \Gamma)^{1/4}}. \end{cases}$$

Remark 7. In regime (a) the deviation is due to the moderate deviation behaviour of the scenery only, but in regimes (b) and (c) there is an additional contraction of the walks to achieve the moderate deviation. There is only a very small gap between our moderate deviation regime and the large deviation regime studied in [GKS05]: Assuming that *all* exponential moments of $\xi(0)$ are finite and $b_n = an$, for some $a > 0$, they obtain a large deviation principle with speed $n^{1/2}$ and a rate function which is strongly dependent on the moment generating function of the scenery variable.

Remark 8. In the special case of simple random walk in *Gaussian* scenery, Theorem 6(a) is known from [GKS05].

The regime $n^{\frac{1}{2}} \sqrt{\log n} \ll b_n \ll n / \log n$, which we consider in Theorem 6, is *maximal* for a moderate deviation principle using only Cramér's condition. The following large deviation principle shows that for $b_n \gg n / \log n$ finer features of the scenery distribution (in this particular case the constant D) enter into the large deviation rate.

Proposition 9 (Special large deviations for $d = 2$). *Assume that, for some $D > 0$,*

$$\log \mathbb{P}\{\xi(0) > x\} \sim -D\,x \qquad \text{as } x \uparrow \infty, \tag{4}$$

and suppose that $(b_n \log n)/n \to \infty$ and $\log b_n / \log n \to \beta \in [1, 2)$. Then, as $n \uparrow \infty$,

$$\log \mathbb{P}\{X_n \geq b_n\} \sim -\Big(\frac{b_n}{\log n}\Big)^{1/2} \Big(\frac{8\,K_2 D}{2-\beta}\Big)^{1/2}, \tag{5}$$

provided the underlying random walk is such that the limit $K_2 := \lim_{n\to\infty} \frac{\mathbb{E}[\ell_n(0)]}{\log n} \in (0, \infty)$ exists.

Remark 10. Note that this result is the planar case of the regime

$$\log \mathbb{P}\{\xi(0) > x\} \sim -Dx^{\frac{d}{2}} \qquad \text{as } x \uparrow \infty,$$

which is described as 'delicate' in [GKS05, Remark 1.2]. The proof of Proposition 9 is based on large deviation results for the maximum of the local times obtained in [GHK06].

The remainder of the paper is structured as follows. Section 3 is devoted to statements about self-intersection local times of our random walk, which are of independent interest. The proofs of our three theorems and Proposition 9 follow in the subsequent four sections.

Throughout this paper we use the symbols P and E to denote probabilities, resp. expectations, with respect to the scenery variables only, and the symbols $\mathbb{P}$ and $\mathbb{E}$ to denote probabilities, resp. expectations, with respect to both the random walk and scenery.

We use the letters c, C to denote positive, finite constants, whose value can change at every occurrence, and which never depend on random quantities. For nonnegative functions f_n, g_n, possibly depending on the sampled walk or scenery, the Landau symbols $f_n = o(g_n)$ and $f_n = O(g_n)$ denote $\lim f_n/g_n = 0$, respectively $\limsup f_n/g_n < \infty$, *uniformly in the sampled walk or scenery.*

3. Concentration inequalities for self-intersection local times

Recall that $\{S_n : n \geq 0\}$ is a symmetric, aperiodic random walk on the lattice $\mathbb{Z}^d$, $d \geq 2$, with nondegenerate covariance matrix Γ. For integers $q > 1$ we define the *q-fold self-intersection local time* $\{\ell_n^{(q)} : n \geq 0\}$ of the random walk as

$$\ell_n^{(q)} := \sum_{z\in\mathbb{Z}^d} \ell_n^q(z) = \sum_{1\leq i_1,\cdots,i_q\leq n} 1\{S_{i_1} = \cdots = S_{i_q}\} \qquad \text{for } n \geq 0\,.$$

We also denote the *maximum* of the local times by

$$\ell_n^{(\infty)} := \max_{z\in\mathbb{Z}^d} \ell_n(z)\,.$$

The most important quantity is $\{\ell_n^{(2)} : n \geq 0\}$, which is simply called the self-intersection local time. Its asymptotic expectations are

$$\mathbb{E}\ell_n^{(2)} \sim \begin{cases} n\,(2G(0)-1) & \text{if } d \geq 3\,, \\ n \log n\, \frac{1}{\pi\sqrt{\det\Gamma}} & \text{if } d = 2\,. \end{cases} \tag{6}$$

In $d \geq 3$ this is easy, for $d = 2$ in the strongly aperiodic case this follows from the local central limit theorem in the form $\mathbb{P}\{S_n = 0\} = 1/(n\,2\pi\sqrt{\det\Gamma}) + o(1/n)$, see [Sp76, Proposition P7.9, p.75], and can be extended to the periodic case using Spitzer's trick, see [Sp76, proof of Proposition P26.1, p.310].

The main results of this section are the following concentration inequalities for double and triple self-intersection local times, which are of independent interest. They are therefore given in somewhat greater generality than needed for the proof of our main results.

Proposition 11 (Concentration inequalities). *Let $n \geq 2$. There exists a constant $c > 0$ such that,*

(a) *if $d > 4$, then for $x \geq n^{\frac{2}{3}} \log^2 n$,*

$$\mathbb{P}\big\{|\ell_n^{(2)} - \mathbb{E}\ell_n^{(2)}| \geq x\big\} \leq \exp\Big\{ - c\,\frac{x^{\frac{1}{2}}}{\log n}\Big\}\,;$$

(b) *if $d = 4$, then for $x \geq n^{\frac{2}{3}} \log^3 n$,*

$$\mathbb{P}\big\{|\ell_n^{(2)} - \mathbb{E}\ell_n^{(2)}| \geq x\big\} \leq \exp\Big\{ - c\,\frac{x^{\frac{1}{2}}}{\log^{3/2} n}\Big\}\,;$$

(c) *if $d = 3$, then for $x \geq n^{\frac{1}{2}} \log^{9/2} n$,*

$$\mathbb{P}\big\{|\ell_n^{(2)} - \mathbb{E}\ell_n^{(2)}| \geq x\big\} \leq \exp\Big\{ - c\,\frac{x^{\frac{2}{3}}}{n^{\frac{1}{3}}}\Big\}\,;$$

(d) *if $d > 4$, then for $x \geq n^{\frac{3}{5}} \log^2 n$,*

$$\mathbb{P}\big\{|\ell_n^{(3)} - \mathbb{E}\ell_n^{(3)}| \geq x\big\} \leq \exp\Big\{ - c\,\frac{x^{\frac{1}{3}}}{\log^{2/3} n}\Big\}\,;$$

(e) *if $d = 4$, then for $x \geq n^{\frac{3}{5}} \log^{7/2} n$,*

$$\mathbb{P}\big\{|\ell_n^{(3)} - \mathbb{E}\ell_n^{(3)}| \geq x\big\} \leq \exp\Big\{ - c\,\frac{x^{\frac{1}{3}}}{\log^{7/6} n}\Big\}\,.$$

Remark 12. All of these inequalities are, to the best of our knowledge, new. Similar concentration inequalities, but only for simple random walk and under considerably stronger assumptions on the relationship of x and n, have been found by Asselah and Castell in [AC06, Propositions 1.4 and 1.6] if $d \geq 5$, and by Asselah in [As06, Proposition 1.1] if $d = 3$. In particular, if $d \geq 5$, for the special case $x = yn$ they obtain an upper bound of $\exp\{-c\sqrt{n}\}$, which is an improvement of (a). The proofs in [As06, AC06] are based on a delicate and powerful analysis of the number of sites in $\mathbb{Z}^d$ visited a certain number of times, and are therefore of independent interest. In this paper we give a direct proof of Proposition 11, which entirely avoids the discussion of the number of visits to individual sites, and is therefore much easier than the method of Asselah and Castell.

3.1 Proof of Proposition 11

We start with some useful estimates for the partial Green's functions,

$$G_n(x) := \sum_{k=0}^{n} \mathbb{P}\{S_k = x\}, \qquad \text{for } n \geq 2 \text{ and } x \in \mathbb{Z}^d.$$

Lemma 13. *For all $n \geq 2$,*

$$\sum_{z \in \mathbb{Z}^d} G_n^2(z) \leq \begin{cases} C\sqrt{n} & \text{if } d = 3, \\ C \log n & \text{if } d = 4, \\ C & \text{if } d > 4. \end{cases}$$

Proof. If $d = 3$ we have from [Sp76, Proposition P26.1, p.308] that $G(z) \leq C/(1 + |z|)$. Then

$$\sum_{z \in \mathbb{Z}^3} G_n^2(z) = \sum_{|z| \leq \sqrt{n}} G_n^2(z) + \sum_{|z| > \sqrt{n}} G_n^2(z) \leq \sum_{|z| \leq \sqrt{n}} G^2(z) + \Big(\sup_{|z| > \sqrt{n}} G(z)\Big) \sum_{|z| > \sqrt{n}} G_n(z)\,.$$

The estimate for $G(z)$ shows that the first sum on the right is bounded by $C\sqrt{n}$. We further have, from the definition of G_n and Chebyshev's inequality,

$$\Big(\sup_{|z|>\sqrt{n}} G(z)\Big) \sum_{|z|>\sqrt{n}} G_n(z) \le C\, n^{-1/2} \sum_{k=0}^{n} \mathbb{P}\{|S_k| > \sqrt{n}\} \le C\, n^{-1/2} \sum_{k=0}^{n} \frac{\mathbb{E}|S_k|^2}{n} \le C\sqrt{n}\,,$$

which completes the argument. In dimension $d \ge 4$ we use that, by [Uc98, (1.4)], we have

$$G(z) \le \sum_{x\in\mathbb{Z}^d} \frac{\pi(x)}{1+|x-z|^{d-2}} \qquad \text{for all } z \in \mathbb{Z}^d, \tag{7}$$

where $(\pi(x)\colon x \in \mathbb{Z}^d)$ is a summable family of nonnegative weights. If $d > 4$, by the triangle inequality,

$$\Big(\sum_{z\in\mathbb{Z}^d} G^2(z)\Big)^{1/2} \le \sum_{x\in\mathbb{Z}^d} \Big(\sum_{z\in\mathbb{Z}^d} \frac{\pi^2(x)}{(1+|x-z|^{d-2})^2}\Big)^{1/2} = \Big(\sum_{x\in\mathbb{Z}^d} \pi(x)\Big)\Big(\sum_{z\in\mathbb{Z}^d} \frac{1}{(1+|z|^{d-2})^2}\Big)^{1/2},$$

which is bounded by a constant. If $d = 4$ we use first that

$$\sum_{z\in\mathbb{Z}^4} G_n^2(z) = \sum_{|z|\le n} G_n^2(z) + \sum_{|z|>n} G_n^2(z) \le \sum_{|z|\le n} G^2(z) + \Big(\sup_{z\in\mathbb{Z}^4} G(z)\Big) \sum_{|z|>n} G_n(z)\,.$$

Clearly, G is bounded, see (7), and an argument analogous to the case $d = 3$ shows that the second sum on the right is bounded by a constant. Using the triangle inequality as in the case $d > 4$ we obtain for the first sum on the right

$$\Big(\sum_{|z|\le n} G^2(z)\Big)^{1/2} \le \sum_{x\in\mathbb{Z}^4} \pi(x) \Big(\sum_{|z+x|\le n} \frac{1}{(1+|z|^2)^2}\Big)^{1/2}.$$

It suffices to show that the content of the round bracket on the right is bounded by a constant multiple of $\log n$, uniformly in $x \in \mathbb{Z}^4$. On the one hand, if $|x| \le 2n$ this follows easily from the fact that the sum can now be taken over all $z \in \mathbb{Z}^4$ with $|z| \le 3n$. On the other hand, if $|x| > 2n$ the sum can be taken over the annulus $|x| - n \le |z| \le |x| + n$ and is thus easily seen to be bounded by a constant. ☐

The proof of Proposition 11 requires the following 'folklore' lemma about the intersection of two independent random walks $\{S_n\colon n \ge 0\}$ and $\{S'_n\colon n \ge 0\}$ with $S_0 = S'_0$. Denote

$$A_n := \sum_{i=1}^{n} \sum_{j=0}^{n-1} 1\{S_i = S'_j\} \quad \text{for } n \ge 1.$$

Lemma 14. *There exists a constant $\vartheta > 0$ such that,*

(a) *if $d > 4$, then* $\sup_{n\ge 2} \mathbb{E}\exp\big\{\vartheta A_n^{1/2}\big\} < \infty$;

(b) *if $d = 4$, then* $\sup_{n\ge 2} \mathbb{E}\exp\big\{\vartheta \frac{1}{\sqrt{\log n}} A_n^{1/2}\big\} < \infty$;

(c) *if $d = 3$, then* $\sup_{n\ge 2} \mathbb{E}\exp\big\{\vartheta \big(\frac{A_n}{\sqrt{n}}\big)^{2/3}\big\} < \infty$.

Proof. From the definition of A_n we obtain, for moments of order $m \geq 1$,

$$\begin{aligned}\mathbb{E}A_n^m &\leq m! \sum_{1\leq j_1\leq\cdots\leq j_m\leq n} \sum_{0\leq k_1,\dots,k_m<n} \mathbb{E}\prod_{l=1}^m 1\{S_{j_l}=S'_{k_l}\}\\ &\leq m! \sum_{\sigma\in\mathfrak{S}_m} \sum_{1\leq j_1\leq\cdots\leq j_m\leq n} \sum_{0\leq k_1\leq\cdots\leq k_m<n} \sum_{x_1,\dots,x_m} \mathbb{E}\prod_{l=1}^m 1\{S_{j_l}=x_l\}\,\mathbb{E}\prod_{l=1}^m 1\{S'_{k_l}=x_{\sigma(l)}\}\\ &\leq m! \sum_{\sigma\in\mathfrak{S}_m} \sum_{x_1,\dots,x_m} \prod_{l=1}^m G_n(x_l-x_{l-1})G_n(x_{\sigma(l)}-x_{\sigma(l-1)}),\end{aligned}$$

where $\mathfrak{S}_m$ denotes the group of all permutations of $\{1,\dots,m\}$, and we set $x_0 := 0 =: x_{\sigma(0)}$ for convenience. Applying Hölder's inequality,

$$\mathbb{E}A_n^m \leq (m!)^2 \sum_{x_1,\dots,x_m} \prod_{l=1}^m G_n^2(x_l-x_{l-1}) = (m!)^2 \Big(\sum_{x\in\mathbb{Z}^d} G_n^2(x)\Big)^m,$$

and from Lemma 13 we obtain, for all $n \geq 2$,

$$\mathbb{E}A_n^m \leq \begin{cases} (m!)^2\, C^m\, n^{m/2} & \text{if } d=3,\\ (m!)^2\, C^m\, (\log n)^m & \text{if } d=4,\\ (m!)^2\, C^m & \text{if } d>4. \end{cases}$$

If $d > 4$ this implies $\mathbb{E}\big(\sqrt{A_n}\big)^m \leq \sqrt{\mathbb{E}A_n^m} \leq m!\,C^m$, and (a) follows by considering the exponential series. The analogous argument for $d = 4$ gives (b). In $d = 3$ we need an extra argument to complete the proof: We write $\ell(m,n) := \lceil n/m \rceil + 1$. Using an inequality of Chen, [Ch04, Theorem 5.1] (with $p = 2$ and $a = m$), we get, for $n \geq m$,

$$\begin{aligned}\sqrt{\mathbb{E}A_n^m} &\leq \sum_{\substack{k_1+\cdots+k_m=m\\ k_1,\dots,k_m\geq 0}} \frac{m!}{k_1!\cdots k_m!}\sqrt{\mathbb{E}A_{\ell(m,n)}^{k_1}}\cdots\sqrt{\mathbb{E}A_{\ell(m,n)}^{k_m}}\\ &\leq \sum_{\substack{k_1+\cdots+k_m=m\\ k_1,\dots,k_m\geq 0}} \frac{m!}{k_1!\cdots k_m!}\sqrt{(k_1!)^2\,C^{k_1}\ell(m,n)^{k_1/2}}\cdots\sqrt{(k_m!)^2\,C^{k_m}\ell(m,n)^{k_m/2}}\\ &\leq \binom{2m-1}{m}\, m!\,C^m\, \big(\tfrac{n}{m}\big)^{m/4} \leq (m!)^{3/4}\, C^m\, n^{m/4},\end{aligned}$$

and therefore $\mathbb{E}A_n^m \leq (m!)^{3/2}\,C^m\,n^{m/2}$. For $n \leq m$ we get the same estimate immediately from the trivial inequality $A_n^m \leq n^{2m} \leq (m!)^{3/2}\,C^m\,n^{m/2}$. We thus obtain, for all n, m, that

$$\mathbb{E}\big(n^{-1/3}\,A_n^{2/3}\big)^m = n^{-m/3}\,\mathbb{E}\big(A_n^m\big)^{2/3} \leq m!\,C^m,$$

and (c) follows by taking the exponential series. □

Introduce, for $n \geq 1$,

$$\Lambda_n := \sum_{i=1}^n \sum_{j,k=0}^{n-1} 1\{S_i = S'_j = S'_k\} \qquad \text{and} \qquad \Lambda_n^* := \sum_{i=0}^{n-1} \sum_{j,k=1}^{n} 1\{S_i = S'_j = S'_k\}.$$

Lemma 15. *There exists a constant $\vartheta > 0$ such that,*

(a) *if $d > 4$, then* $\sup_{n\geq 2} \mathbb{E}\exp\left\{\vartheta\,\Lambda_n^{1/3}\right\} < \infty$;

(b) *if $d = 4$, then* $\sup_{n\geq 2} \mathbb{E}\exp\left\{\vartheta\,\frac{\Lambda_n^{1/3}}{(\log n)^{1/2}}\right\} < \infty$.

The same statements hold when Λ_n is replaced by Λ_n^*.

Proof. We only consider Λ_n, as Λ_n^* can be treated analogously. From the definition of Λ_n we obtain, for moments of order $m \geq 1$,

$$\begin{aligned}\mathbb{E}\Lambda_n^m &\leq m! \sum_{1\leq j_1\leq\cdots\leq j_m\leq n} \sum_{\substack{0\leq k_1,\dots,k_m<n\\ 0\leq l_1,\dots,l_m<n}} \mathbb{E}\prod_{i=1}^m 1\{S_{j_i} = S'_{k_i} = S'_{l_i}\}\\ &\leq m! \sum_{x_1,\dots,x_m} \sum_{\substack{0\leq k_1,\dots,k_m<n\\ 0\leq l_1,\dots,l_m<n}} \prod_{i=1}^m G_n(x_i - x_{i-1})\,\mathbb{E}\prod_{i=1}^m 1\{S'_{k_i} = S'_{l_i} = x_i\},\end{aligned}$$

where we set $x_0 := 0$ for convenience. Continuing with Cauchy-Schwarz, we get

$$\leq m!\Big(\sum_{x_1,\dots,x_m}\prod_{i=1}^m G_n^2(x_i - x_{i-1})\Big)^{1/2}\Big(\sum_{x_1,\dots,x_m}\Big(\sum_{\substack{0\leq k_1,\dots,k_m<n\\ 0\leq l_1,\dots,l_m<n}}\mathbb{E}\prod_{i=1}^m 1\{S'_{k_i} = S'_{l_i} = x_i\}\Big)^2\Big)^{1/2}.$$

By Lemma 13 the first bracket is bounded by C^m if $d > 4$, and by $C^m(\log n)^m$ if $d = 4$. To analyse the second bracket we denote by $\mathcal{T}_m$ the set of all mappings $\tau\colon \{1,\dots,2m\} \to \{1,\dots,m\}$ such that $\#\tau^{-1}\{j\} = 2$ for all $j \in \{1,\dots,m\}$. For the cardinality of $\mathcal{T}_m$ we get

$$\#\mathcal{T}_m \leq \binom{2m}{m}(m!)^2 \leq C^m\,(m!)^2. \tag{8}$$

Given $(k_1,\dots,k_m)$ and $(l_1,\dots,l_m)$ there exists at least one ordered tuple $(k'_1,\dots,k'_{2m})$ with $k'_1 \leq \cdots \leq k'_{2m}$ with $\{k_1,\dots,k_m,l_1,\dots,l_m\} = \{k'_1,\dots,k'_{2m}\}$ and $\tau \in \mathcal{T}_m$ such that $\tau(i) = j$ if $k'_i = l_j$ or $k'_i = k_j$. Hence we obtain,

$$\begin{aligned}\sum_{\substack{0\leq k_1,\dots,k_m<n\\ 0\leq l_1,\dots,l_m<n}} \mathbb{E}\prod_{i=1}^m 1\{S'_{k_i} = S'_{l_i} = x_i\} &\leq \sum_{\tau\in\mathcal{T}_m}\sum_{0\leq k'_1\leq\cdots\leq k'_{2m}<n}\prod_{i=1}^{2m}\mathbb{P}\{S'_{k'_i} - S'_{k'_{i-1}} = x_{\tau(i)} - x_{\tau(i-1)}\}\\ &\leq \sum_{\tau\in\mathcal{T}_m}\prod_{i=1}^{2m} G_n(x_{\tau(i)} - x_{\tau(i-1)}),\end{aligned}$$

and, using the triangle inequality,

$$\begin{aligned}\Big(\sum_{x_1,\dots,x_m}\Big(\sum_{\tau\in\mathcal{T}_m}\prod_{i=1}^{2m} G_n(x_{\tau(i)} - x_{\tau(i-1)})\Big)^2\Big)^{1/2} &\leq \sum_{\tau\in\mathcal{T}_m}\Big(\sum_{x_1,\dots,x_m}\prod_{i=1}^{2m} G_n^2(x_{\tau(i)} - x_{\tau(i-1)})\Big)^{1/2}\\ &\leq \#\mathcal{T}_m\Big(\sum_{x_1,\dots,x_{2m}}\prod_{i=1}^{2m} G_n^2(x_i - x_{i-1})\Big)^{1/2}.\end{aligned}$$

By Lemma 13 the bracket is bounded by C^m if $d > 4$, and by $C^m(\log n)^{2m}$ if $d = 4$. Thus, together with (8), we obtain the estimates

$$\mathbb{E}\Lambda_n^m \le \begin{cases} \big(m!\,C^m\big)^3 & \text{if } d > 4, \\ \big(m!\,C^m\,(\log n)^{m/2}\big)^3 & \text{if } d = 4. \end{cases}$$

But $\mathbb{E}(\Lambda_n^{1/3})^m \le \big(\mathbb{E}\Lambda_n^m\big)^{1/3}$, and both statements follow by taking exponential series. □

For any $N \ge 0$ we use the classical decomposition

$$\ell_{2^N}^{(2)} - \mathbb{E}\ell_{2^N}^{(2)} = 2\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}} \overline{A}_{j,k},$$

where

$$\overline{A}_{j,k} := \overline{A}_{j,k}(N) := \sum_{\substack{(2k-2)2^{N-j} < l \le (2k-1)2^{N-j} \\ (2k-1)2^{N-j} < m \le (2k)2^{N-j}}} \Big(1\{S_l = S_m\} - \mathbb{P}\{S_l = S_m\}\Big).$$

For fixed $1 \le j \le N$ the random variables $\overline{A}_{j,k}$, for $k = 1, \ldots, 2^{j-1}$, are independent, identically distributed with the law of $A_{2^{N-j}} - \mathbb{E}A_{2^{N-j}}$. The next proposition exploits this independence, and the moment results of Lemma 14 to give large deviation upper bounds.

Proposition 16 (Large deviation upper bounds). *For every $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$ such that, for all $1 \le j \le N$,*

(a) *if $d > 4$, then* $\mathbb{P}\Big\{\Big|\sum_{k=1}^{2^{j-1}} \overline{A}_{j,k}(N)\Big| \ge \varepsilon\, x\Big\} \le \exp\big\{-c\sqrt{x}\big\}$ *for all $x \ge (2^N)^{2/3}$;*

(b) *if $d = 4$, then* $\mathbb{P}\Big\{\Big|\sum_{k=1}^{2^{j-1}} \overline{A}_{j,k}(N)\Big| \ge \varepsilon\, x\Big\} \le \exp\Big\{-c\sqrt{\frac{x}{N}}\Big\}$ *for all $x \ge N(2^N)^{2/3}$;*

(c) *if $d = 3$, then* $\mathbb{P}\Big\{\Big|\sum_{k=1}^{2^{j-1}} \overline{A}_{j,k}(N)\Big| \ge \varepsilon\, x\Big\} \le \exp\Big\{-c\,\frac{x^2}{2^N}\Big\} + \exp\Big\{-c\,\frac{x^{2/3}2^{j/3}}{2^{N/3}}\Big\}$

for all $x \ge N^{9/2}(2^N)^{1/2}$.

The proof of this result will be postponed to the next section.

Completion of the proof of Proposition 11(a) – (c). We use two simple ingredients, stated below as (9) and (10). *First*, note that, for any $N \ge 0$ and any choice of nonnegative weights p_j, $1 \le j \le N$, with $\sum p_j \le 1$, we have

$$\mathbb{P}\big\{|\ell_{2^N}^{(2)} - \mathbb{E}\ell_{2^N}^{(2)}| \ge \varepsilon y\big\} = \mathbb{P}\Big\{2\Big|\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}} \overline{A}_{j,k}\Big| \ge \varepsilon y\Big\} \le \sum_{j=1}^{N} \mathbb{P}\Big\{\Big|\sum_{k=1}^{2^{j-1}} \overline{A}_{j,k}\Big| \ge \frac{\varepsilon y p_j}{2}\Big\}. \tag{9}$$

Second, for any $n \ge 2$ there exists the representation

$$n = 2^{N_1} + \cdots + 2^{N_l},$$

where $l \ge 1$ and $N_1 > \cdots > N_l \ge 0$ are integers. Note that $l \le c\log n$. Write $n_0 := 0$ and $n_i := 2^{N_1} + \cdots + 2^{N_i}$ for $1 \le i \le l$, and denote

$$B_i := \sum_{n_{i-1} < j < k \le n_i} 1\{S_j = S_k\}, \quad \text{and} \quad D_i := \sum_{\substack{n_{i-1} < j \le n_i \\ n_i < k \le n}} 1\{S_j = S_k\}.$$

Then $\sum_{1\le j<k\le n} 1\{S_j = S_k\} = \sum_{i=1}^l B_i + \sum_{i=1}^{l-1} D_i$. We thus have, for any choice of nonnegative weights q_i, $1 \le i \le l$, with $\sum q_i \le 1$, for x large enough to satisfy $xq_i > 4\mathbb{E}D_i$,

$$\mathbb{P}\big\{|\ell_n^{(2)} - \mathbb{E}\ell_n^{(2)}| \ge x\big\} \le \sum_{i=1}^{l} \mathbb{P}\big\{|B_i - \mathbb{E}B_i| \ge \tfrac{xq_i}{4}\big\} + \sum_{i=1}^{l-1} \mathbb{P}\big\{D_i \ge \tfrac{xq_i}{4}\big\}. \tag{10}$$

Depending on the dimension, we use the ingredients (9) and (10) with different choice of weights. If $d = 3$ we define $q_i = b2^{(N_i-N_1)/2}$ with $b = (\sum_{j=1}^\infty 2^{-j/2})^{-1}$, and apply (9) for

$$N = N_i,\;\; y = \frac{xq_i}{4\varepsilon} \text{ and weights } p_j = aj^{-2} \text{ with } a = \Big(\sum_{j=1}^{\infty} j^{-2}\Big)^{-1},$$

where $\varepsilon > 0$ may be chosen independently of i, j such that $yp_j/2 \ge N_i^{9/2}(2^{N_i})^{1/2}$. Using (9), Proposition 16 (c) and that $l \le c\log n$, this gives

$$\begin{aligned}\sum_{i=1}^{l} \mathbb{P}\big\{|B_i - \mathbb{E}B_i| \ge \tfrac{xq_i}{4}\big\} &\le \sum_{i=1}^{l}\sum_{j=1}^{N_i} \exp\Big\{-c\,\tfrac{(yp_j)^2}{2^{N_i}}\Big\} + \exp\Big\{-c\,\tfrac{(yp_j)^{2/3}2^{j/3}}{2^{N_i/3}}\Big\} \\ &\le \exp\Big\{-c\,\tfrac{x^{2/3}}{n^{1/3}}\Big\}.\end{aligned} \tag{11}$$

As (with $\stackrel{d}{=}$ denoting equality of distributions)

$$D_i \stackrel{d}{=} \sum_{j=1}^{2^{N_i}} \sum_{k=1}^{n-n_i} 1\{S_j = S'_k\} \le \sum_{j=1}^{2^{N_i}} \sum_{k=0}^{2^{N_i}-1} 1\{S_j = S'_k\} = A_{2^{N_i}},$$

the second sum in (10) can be estimated using Chebyshev's inequality and Lemma 14,

$$\begin{aligned}\sum_{i=1}^{l-1} \mathbb{P}\big\{D_i \ge \tfrac{xq_i}{4}\big\} &\le \sum_{i=1}^{l-1} \mathbb{P}\big\{\tfrac{A_{2^{N_i}}}{2^{N_i/2}} \ge \tfrac{xq_i}{4 2^{N_i/2}}\big\} \\ &\le \sum_{i=1}^{l-1} \exp\big\{-c\,\big(\tfrac{xq_i}{2^{N_i/2}}\big)^{2/3}\big\} \le \exp\big\{-c\,\tfrac{x^{2/3}}{n^{1/3}}\big\},\end{aligned} \tag{12}$$

and the proof of (c) follows by plugging (11) and (12) into (10). The proof of (a), (b) is analogous, but now the weights are chosen to be equal, i.e. $p_j = 1/N$ and $q_i = 1/l$. We leave the obvious details to the reader. □

An analogous argument can be carried out for triple self-intersections. Indeed, for any $N \ge 0$ we have

$$\ell_{2^N}^{(3)} - \mathbb{E}\ell_{2^N}^{(3)} = \sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}} \overline{\Lambda}_{j,k} + \sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}} \overline{\Lambda}^*_{j,k} \tag{13}$$

where

$$\overline{\Lambda}_{j,k} := \sum_{\substack{(2k-2)2^{N-j} < l \le (2k-1)2^{N-j} \\ (2k-1)2^{N-j} < m,n \le (2k)2^{N-j}}} \Big(1\{S_l = S_m = S_n\} - \mathbb{P}\{S_l = S_m = S_n\}\Big)$$

and

$$\overline{\Lambda}^*_{j,k} := \sum_{\substack{(2k-2)2^{N-j} < l,m \le (2k-1)2^{N-j} \\ (2k-1)2^{N-j} < n \le (2k)2^{N-j}}} \Big(1\{S_l = S_m = S_n\} - \mathbb{P}\{S_l = S_m = S_n\}\Big).$$

Again, for fixed $1 \le j \le N$ the random variables $\overline{\Lambda}_{j,k}$, for $k = 1, \dots, 2^{j-1}$, are independent, identically distributed with the law of $\Lambda_{2^{N-j}} - \mathbb{E}\Lambda_{2^{N-j}}$, and the random variables $\overline{\Lambda}^*_{j,k}$, for $k = 1, \dots, 2^{j-1}$, are independent, identically distributed with the law of $\Lambda^*_{2^{N-j}} - \mathbb{E}\Lambda^*_{2^{N-j}}$.

Proposition 17 (Large deviation upper bounds). *For any $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$ such that, for all $1 \le j \le N$,*

(a) *if $d > 4$, then* $\mathbb{P}\Big\{\Big|\sum_{k=1}^{2^{j-1}} \overline{\Lambda}_{j,k}\Big| \ge \varepsilon\, x\Big\} \le \exp\big\{ - c\, x^{1/3}\big\}$, *for all $x \ge (2^N)^{3/5}$;*

(b) *if $d = 4$, then* $\mathbb{P}\Big\{\Big|\sum_{k=1}^{2^{j-1}} \overline{\Lambda}_{j,k}\Big| \ge \varepsilon\, x\Big\} \le \exp\big\{ - c\,\big(\frac{x}{N^{3/2}}\big)^{1/3}\big\}$, *for all $x \ge N^{3/2}(2^N)^{3/5}$.*

The same estimates hold for $\overline{\Lambda}_{j,k}$ replaced by $\overline{\Lambda}^*_{j,k}$.

Again we postpone the proof of Proposition 17 to the next section and first complete the details of the remaining parts of Proposition 11.

Proof of Proposition 11(d),(e). For any $N \ge 0$, we have by (13),

$$\begin{aligned}\mathbb{P}\big\{|\ell^{(3)}_{2^N} - \mathbb{E}\ell^{(3)}_{2^N}| \ge \varepsilon y\big\} &= \mathbb{P}\Big\{\Big|\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}} \overline{\Lambda}_{j,k}\Big| \ge \frac{\varepsilon y}{2}\Big\} + \mathbb{P}\Big\{\Big|\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}} \overline{\Lambda}^*_{j,k}\Big| \ge \frac{\varepsilon y}{2}\Big\} \\ &\le \sum_{j=1}^{N}\mathbb{P}\Big\{\Big|\sum_{k=1}^{2^{j-1}} \overline{\Lambda}_{j,k}\Big| \ge \frac{\varepsilon y}{2N}\Big\} + \sum_{j=1}^{N}\mathbb{P}\Big\{\Big|\sum_{k=1}^{2^{j-1}} \overline{\Lambda}^*_{j,k}\Big| \ge \frac{\varepsilon y}{2N}\Big\}.\end{aligned} \tag{14}$$

For any $n \ge 2$ there exists the representation $n = 2^{N_1} + \cdots + 2^{N_l}$, where $N_1 > \cdots > N_l \ge 0$ are integers. Note that $l \le c \log n$. Write $n_0 := 0$ and $n_i := 2^{N_1} + \cdots + 2^{N_i}$ for $1 \le i \le l$, and denote

$$B_i := \sum_{n_{i-1} < j,k,l \le n_i} 1\{S_j = S_k = S_l\},$$

$$D_i := \sum_{\substack{n_{i-1} < j,k \le n_i \\ n_i < l \le n}} 1\{S_j = S_k = S_l\} \qquad \text{and} \qquad E_i := \sum_{\substack{n_{i-1} < j \le n_i \\ n_i < k,l \le n}} 1\{S_j = S_k = S_l\}.$$

Then $\ell^{(3)}_n = \sum_{i=1}^{l} B_i + \sum_{i=1}^{l-1} D_i + \sum_{i=1}^{l-1} E_i$. As $\mathbb{E}D_i$ and $\mathbb{E}E_i$ are bounded by a constant multiple of $\log n$, we get for all sufficiently large x,

$$\mathbb{P}\big\{|\ell^{(3)}_n - \mathbb{E}\ell^{(3)}_n| \ge x\big\} \le \sum_{i=1}^{l}\mathbb{P}\big\{|B_i - \mathbb{E}B_i| \ge \tfrac{x}{3l}\big\} + \sum_{i=1}^{l-1}\mathbb{P}\big\{D_i \ge \tfrac{x}{3l}\big\} + \sum_{i=1}^{l-1}\mathbb{P}\big\{E_i \ge \tfrac{x}{3l}\big\}. \tag{15}$$

We now look at the case $d = 4$. Using (14) with $y = x/(3l\varepsilon)$, Proposition 17(b) and that $l \le c \log n$, this gives

$$\sum_{i=1}^{l}\mathbb{P}\big\{|B_i - \mathbb{E}B_i| \ge \tfrac{x}{3l}\big\} \le 2\sum_{i=1}^{l}\sum_{j=1}^{N_i}\exp\Big\{ - c\,\Big(\frac{x}{l N_i^{5/2}}\Big)^{1/3}\Big\} \le \exp\Big\{ - c\,\frac{x^{1/3}}{\log^{7/6} n}\Big\}. \tag{16}$$

As we have

$$D_i \stackrel{d}{=} \sum_{j,k=0}^{2^{N_i}-1}\sum_{m=1}^{n-n_i} 1\{S_j = S_k = S'_m\} \le \sum_{j,k=0}^{2^{N_i}-1}\sum_{m=1}^{2^{N_i}} 1\{S_j = S_k = S'_m\} = \Lambda^*_{2^{N_i}},$$

the second sum in (15) can be estimated using Chebyshev's inequality and Lemma 15(b),

$$\sum_{i=1}^{l-1} \mathbb{P}\{D_i \geq \tfrac{x}{3l}\} \leq \sum_{i=1}^{l-1} \mathbb{P}\Big\{\frac{\Lambda^*_{2^{N_i}}}{N_i^{3/2}} \geq \frac{x}{3lN_i^{3/2}}\Big\} \leq l\,\exp\Big\{-c\Big(\frac{x}{lN_1^{3/2}}\Big)^{1/3}\Big\} \leq \exp\Big\{-c\,\frac{x^{1/3}}{\log^{5/6} n}\Big\}. \tag{17}$$

The same estimate holds for E_i in place of D_i, using the estimate for $\Lambda_{2^{N_i}}$ instead of $\Lambda^*_{2^{N_i}}$. The proof of (c) follows by plugging this, (17) and (16) into (15). The case $d \geq 5$ is analogous. □

3.2 Proof of Propositions 16 and 17

Proof of Proposition 16. We first give the argument in the case $d \geq 5$. Take a continuously differentiable function $g\colon (0,\infty) \to \mathbb{R}$ with non-increasing derivative, such that

(a) $g'(x) > 2/x$ for all $x > 0$,
(b) $g(x) = \vartheta\sqrt{x}$ for all $x \geq x_0$,

where ϑ is chosen as in Lemma 14. For $1 \leq j \leq N$ denote

$$b_j(N) := \mathbb{E}\Big[\exp\big\{g\big(\overline{A}_{j,1}(N)\big)\big\}\,1\{\overline{A}_{j,1}(N) > 0\}\Big],$$

and recall from Lemma 14(a) that $b_j(N)$ is uniformly bounded in j and N. By Theorem 2.3 of [Na79] (with $\gamma_1 = \gamma_2 = \gamma_3 = 1/3$, $\gamma = 2/3$ and $\delta = 2$) we obtain the bound

$$\mathbb{P}\Big\{\sum_{k=1}^{2^{j-1}} \overline{A}_{j,k} \geq \varepsilon\, x\Big\} \leq e^{1/2}\exp\Big\{-\tfrac{a^2\,\varepsilon^2\,x^2}{2(a+1)2^{j-1}V_j(N)}\Big\} \tag{18}$$

$$+\, e^{1/2}\exp\Big\{-\frac{2a\,\varepsilon\,x}{3S^{-1}\big(\frac{a\varepsilon x}{3e^a 2^{j-1}b_j(N)}\big)}\Big\} \tag{19}$$

$$+\, 2^j\,b_j(N)\,e^{1/2}\exp\Big\{-g\big(\tfrac{2}{3}\,\varepsilon\,x\big)\Big\} + 2^{j-1}\,\mathbb{P}\Big\{\overline{A}_{j,1} \geq \tfrac{2}{3}\,\varepsilon\,x\Big\}, \tag{20}$$

where $V_j(N)$ is the variance of $\overline{A}_{j,1}$, the constant a is the unique solution of the equation $(u+1) = e^{u-1}$, and S^{-1} is the inverse of the strictly decreasing function $u \mapsto S(u) := e^{-g(u)}g'(u)u^2$, see [Na79, p.765]. By Chebyshev's inequality,

$$\mathbb{P}\{\overline{A}_{j,1} \geq x\} \leq \big(\sup_N \sup_{j\leq N} b_j(N)\big)\,e^{-g(x)},$$

and therefore the two terms in (20) are bounded by a constant multiple of

$$2^N\exp\Big\{-g\big(\tfrac{2}{3}\,\varepsilon\,x\big)\Big\} \qquad \text{for all } j \leq N.$$

Recalling the definition of g we arrive at an upper bound of

$$C\exp\big\{-c\,\sqrt{x}\big\} \qquad \text{for all } N \geq 1. \tag{21}$$

If $x \geq (2^N)^{2/3}$, then $x^2/2^{j-1} = x^{1/2}\,x^{3/2}/2^{j-1} > \sqrt{x}$ for all $j \leq N$. Further, using this inequality and the boundedness of $V_j(N)$, the term in (18) is also bounded by a constant multiple of $\exp\{-c\sqrt{x}\}$.

To show that also the term in (19) is negligible, recall that the function S is strictly decreasing. Hence, the term in (19) is bounded by

$$C\exp\Big\{-c\frac{x}{S^{-1}\big(\frac{c}{2^{N/3}}\big)}\Big\}.$$

From the definition of the functions g and S it is easy to see that

$$S^{-1}\big(\tfrac{c}{2^{N/3}}\big) \leq CN^2.$$

This implies that the term in (19) is bounded by a constant multiple of $\exp\{-c\,x/N^2\}$, and is therefore also negligible compared to (21). This completes the bound for $\sum \overline{A}_{j,k}$. The same reasoning can be applied with $-\overline{A}_{j,k}$ in place of $\overline{A}_{j,k}$, using only the trivial fact that $-\overline{A}_{j,1}$ is bounded from above, uniformly in j. Hence we get the same bound for $-\sum \overline{A}_{j,k}$. This completes the proof in dimensions $d \geq 5$. The result in $d = 4$ is a modification of this argument, using the random variable $(N-j)^{-1}\overline{A}_{j,k}$ instead of $\overline{A}_{j,k}$, and details are left to the reader.

Turning to dimension $d = 3$, we use that

$$\mathbb{P}\Big\{\sum_{k=1}^{2^{j-1}} \overline{A}_{j,k} \geq \varepsilon x\Big\} = \mathbb{P}\Big\{\sum_{k=1}^{2^{j-1}} \frac{\overline{A}_{j,k}}{2^{(N-j)/2}} \geq \varepsilon \frac{x}{2^{(N-j)/2}}\Big\},$$

and choose a function $g\colon (0,\infty) \to \mathbb{R}$ which satisfies the same conditions as above, except that we now replace condition (b) by $g(x) = \vartheta\, x^{2/3}$ for all $x \geq x_0$, and ϑ as in Lemma 14. We define

$$b_j(N) := \mathbb{E}\big[\exp\big\{g\big(\overline{A}_{j,1}/2^{(N-j)/2}\big)\, 1\{\overline{A}_{j,1} > 0\}\big],$$

and by Theorem [Na79, Theorem 2.3] we obtain

$$\mathbb{P}\Big\{\sum_{k=1}^{2^{j-1}} \frac{\overline{A}_{j,k}}{2^{(N-j)/2}} \geq \varepsilon\, \frac{x}{2^{(N-j)/2}}\Big\} \leq \exp\Big\{-c\,\frac{x^2}{2^N}\Big\} + \exp\Big\{-c\,\frac{x}{2^{(N-j)/2}\, S^{-1}\big(\frac{cx}{2^{(N+j)/2}}\big)}\Big\} \tag{22}$$

$$+\, C\, 2^j\, b_j(N)\, \exp\Big\{-g\big(c\,\frac{x}{2^{(N-j)/2}}\big)\Big\} + 2^{j-1}\,\mathbb{P}\Big\{\frac{\overline{A}_{j,1}}{2^{(N-j)/2}} \geq c\,\frac{x}{2^{(N-j)/2}}\Big\}. \tag{23}$$

The two terms in (23) are bounded by $2^N\, \exp\{-c\,x^{2/3}/2^{(N-j)/3}\}$. To bound the last term in (22) we use that, for $x \geq 2^{N/2}/N^2$,

$$S^{-1}\Big(\frac{cx}{2^{(N+j)/2}}\Big) \leq S^{-1}\Big(\frac{cx}{2^N}\Big) \leq S^{-1}\Big(\frac{c}{N^2 2^{N/2}}\Big) \leq C N^{3/2},$$

to get

$$\exp\Big\{-c\frac{x}{2^{(N-j)/2}\, S^{-1}\big(\frac{cx}{2^{(N+j)/2}}\big)}\Big\} \leq \exp\Big\{-c\,\frac{x 2^{j/2}}{2^{N/2} N^{3/2}}\Big\}.$$

As $x \geq 2^{N/2} N^{9/2}$ this term is also bounded by $\exp\{-c\,x^{2/3}/2^{(N-j)/3}\}$, completing the proof. □

Proof of Proposition 17. We use the same arguments as in Proposition 16, but now for a function $g\colon (0,\infty) \to \mathbb{R}$ with condition (b) replaced by $g(x) = \vartheta x^{1/3}$ for $x \geq x_0$. Then both terms in (20) give contributions bounded by $\exp\{-c\,x^{1/3}\}$. If $x \geq (2^N)^{3/5}$, then $x^2/2^{j-1} \geq x^{1/3}$, and hence we obtain the same bound for (18). Under the same condition $x \geq (2^N)^{3/5}$, we have

$$S^{-1}\big(cx/2^{j-1}\big) \leq S^{-1}\big(c/(2^N)^{2/5}\big) \leq C\, N^3,$$

hence the term in (19) is of smaller order. □

3.3 A large deviation bound for the maximum of the local times

We complete this section with an easy lemma, which provides bounds for the large deviation probabilities of the maximum $\ell_n^{(\infty)}$ of the local times. Ideas for this proof are taken from Gantert and Zeitouni [GZ98].

Lemma 18 (Large deviation bounds for the maximal local time). *There exists $c > 0$ such that*

(a) *if $d \geq 3$, then for each sequence $a_n \to \infty$ and all $n \geq 2$,*

$$\mathbb{P}\big\{\ell_n^{(\infty)} > a_n\big\} \leq n \exp\big\{ - c\, a_n\big\} ;$$

(b) *if $d = 2$, then for each sequence $a_n / \log n \to \infty$ and all $n \geq 2$,*

$$\mathbb{P}\big\{\ell_n^{(\infty)} > a_n\big\} \leq n \exp\Big\{ - c\, \frac{a_n}{\log n}\Big\} .$$

Proof. Without loss of generality we may assume that all a_n are positive integers. We first reduce the problem to a large deviation bound for $\ell_n(0)$. Defining the stopping times $T_z := \min\{k \geq 1 \colon S_k = z\}$ we have, for all nonnegative integers x,

$$\begin{aligned}\mathbb{P}\big\{\ell_n^{(\infty)} > x\big\} &\leq \sum_{z\in\mathbb{Z}^d} \mathbb{P}\big\{\ell_n(z) > x\big\} = \sum_{z\in\mathbb{Z}^d}\sum_{k=1}^{n} \mathbb{P}\{T_z = k\}\, \mathbb{P}\big\{\ell_{n-k}(0) \geq x\big\} \\ &\leq \mathbb{P}\big\{\ell_n(0) \geq x\big\} \sum_{z\in\mathbb{Z}^d} \mathbb{P}\{T_z \leq n\} .\end{aligned}$$

Now $\sum_z \mathbb{P}\{T_z \leq n\} \leq \sum_z \sum_{k=1}^n \mathbb{P}\{S_k = z\} = n$, so that it suffices to bound the large deviation probabilities of $\ell_n(0)$. By the strong Markov property applied at the successive hitting times of the origin, we get

$$\mathbb{P}\big\{\ell_n(0) \geq a_n\big\} \leq \mathbb{P}\{T_0 \leq n\}^{a_n} . \tag{24}$$

In the transient case, $d \geq 3$, this gives (a) with $c := -\log \mathbb{P}\{T_0 < \infty\} > 0$. In the recurrent case $d = 2$, we use the last exit decomposition, for all $2 \leq k \leq n$,

$$1 \leq \sum_{j=0}^{k} \mathbb{P}\{S_j = 0\}\, \mathbb{P}\{\ell_{n-k}(0) = 0\} + \sum_{j=k+1}^{n} \mathbb{P}\{S_j = 0\} .$$

By [Sp76, Proposition P7.6, p.72] we have $\mathbb{P}\{S_j = 0\} \leq \frac{c}{j}$ for $j \geq 1$. This implies that

$$(\log k)\, \mathbb{P}\{\ell_{n-k}(0) = 0\} \geq C\Big[1 - c\Big(\sum_{j=k+1}^{n} \frac{1}{j}\Big)\Big] .$$

Now let $k = \lceil \eta n \rceil$ and choose $\eta \in (0,1)$ sufficiently close to one, so that the right hand side is bounded from zero by a positive constant. Hence,

$$\mathbb{P}\{T_0 > n(1-\eta)\} = \mathbb{P}\{\ell_{\lfloor n(1-\eta)\rfloor}(0) = 0\} \geq \frac{c}{\log n} ,$$

and thus $\log \mathbb{P}\{T_0 \leq n\} = \log(1 - \mathbb{P}\{T_0 > n\}) \leq -c/\log n$. Plugging this into (24) completes the proof of (b). □

4. Precise asymptotics in dimensions $d \geq 4$: Proof of Theorem 1

The main ingredient of the proof is the following proposition. Recall that the probability P refers exclusively to the scenery variables with fixed random walk samples, and the Landau symbols are uniform in these samples.

Proposition 19. *Assume that, for some $A>0$ and all sufficiently large n,*

$$\Gamma_n := \sum_{z\in\mathbb{Z}^d} \ell_n^3(z) \le n\log^2 n \qquad \text{and} \qquad V_n^2 := \sigma^2 \sum_{z\in\mathbb{Z}^d} \ell_n^2(z) \le An\,.$$

Then, for $\sqrt{n} \ll b_n \ll n^{2/3}/\log^{3/2} n$, we have

$$\mathsf{P}\Big\{ \sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \ge b_n \Big\} = \frac{V_n}{\sqrt{2\pi} b_n} \exp\Big\{ -\frac{b_n^2}{2V_n^2}\Big\} (1+o(1))\,. \tag{25}$$

Proof of Theorem 1. On the event

$$\big\{ |\ell_n^{(2)} - \mathbb{E}\ell_n^{(2)}| \le n^{2/3}\log^3 n,\ \ell_n^{(3)} \le n\log^2 n \big\}$$

we have

$$V_n^2 = \sigma^2\, \mathbb{E}\ell_n^{(2)} + O(n^{2/3}\log^3 n).$$

Since for $d \ge 4$,

$$\mathbb{E}\ell_n^{(2)} - n\,(2G(0)-1) = O(\log n),$$

we obtain

$$V_n^2 = n\,\sigma^2\,(2G(0)-1) + O(n^{2/3}\log^3 n)\,.$$

Thus, if we assume $\sqrt{n} \ll b_n \ll n^{2/3}/\log^{3/2} n =: a_n$, we have

$$-\frac{b_n^2}{2V_n^2} = -\frac{b_n^2}{2n\sigma^2(2G(0)-1)} + o(1)\,.$$

Using that

$$1-\Phi(x) = \frac{1}{\sqrt{2\pi}x}\, e^{-\frac{x^2}{2}} \big(1+O(x^{-2})\big), \qquad \text{as } x\to\infty\,, \tag{26}$$

and abbreviating $\rho_n^2 := 2n\sigma^2(2G(0)-1)$ we obtain, on the same event,

$$\frac{\frac{V_n}{\sqrt{2\pi}b_n}\,\exp\Big\{-\frac{b_n^2}{2V_n^2}\Big\}}{1-\Phi(b_n/\rho_n)} = 1+o(1).$$

Therefore, for a constant $c>0$ and all large n,

$$\begin{aligned}
&\left|\frac{\mathbb{P}\{X_n \ge b_n\}}{1-\Phi(b_n/\rho_n)} - 1\right| \\
&\le \mathbb{E}\left[\left|\frac{\mathsf{P}\{\sum \ell_n(z)\xi(z)\ge b_n\}}{\frac{V_n}{\sqrt{2\pi}b_n}\,\exp\{-\frac{b_n^2}{2V_n^2}\}} - 1\right| 1\big\{|\ell_n^{(2)} - \mathbb{E}\ell_n^{(2)}| \le n^{2/3}\log^3 n,\ \ell_n^{(3)} \le n\log^2 n\big\}\right] + o(1) \\
&\qquad + \mathbb{P}\big\{|\ell_n^{(2)} - \mathbb{E}\ell_n^{(2)}| > n^{2/3}\log^3 n\big\}\, e^{c\frac{b_n^2}{n}} + \mathbb{P}\big\{\ell_n^{(3)} > n\log^2 n\big\}\, e^{c\frac{b_n^2}{n}}\,.
\end{aligned}$$

By Proposition 11 both probabilities in the last line are bounded by $\exp\{-cn^{1/3}\}$ if $d\ge 5$, and by $\exp\{-cn^{1/3}/\log^{1/2} n\}$ if $d=4$. As $b_n \ll a_n$ we have $b_n^2/n \ll n^{1/3}$ if $d\ge 5$, and $b_n^2/n \ll (n/\log^2 n)^{1/3}$ if $d=4$, hence the summands in the last line go to zero, and together with Proposition 19 this implies Theorem 1. □

Proof of Proposition 19. Recall Cramér's condition (1) and denote $f(h) := \mathbb{E}e^{h\xi(0)}$ for all $h\in[0,\theta)$. For fixed $n\ge 1$ and $h>0$ satisfying the condition

$$h\,\ell_n^{(\infty)} \le \tfrac{\theta}{2} \tag{27}$$

we introduce a family $\{Y_z : z \in \mathbb{Z}^d\}$ of independent auxiliary random variables with distributions

$$P\{Y_z < x\} = \big(f(h\ell_n(z))\big)^{-1} \int_{-\infty}^{x} e^{hy}\, d\mathsf{P}\{\ell_n(z)\xi(z) < y\}\,.$$

We define

$$m_z := EY_z, \qquad \sigma_z^2 := E[(Y_z - m_z)^2], \quad \gamma_z := E|Y_z - m_z|^3,$$
$$M_n(h) := \textstyle\sum_{z\in\mathbb{Z}^d} m_z, \quad V_n^2(h) := \sum_{z\in\mathbb{Z}^d} \sigma_z^2, \quad \Gamma_n(h) := \sum_{z\in\mathbb{Z}^d} \gamma_z\,.$$

From the definition of Y_z we infer that

$$\mathsf{P}\big\{\ell_n(z)\xi(z) < x\big\} = f(h\ell_n(z)) \int_{-\infty}^{x} e^{-hy}\, dP\{Y_z < y\}\,,$$

and therefore

$$\mathsf{P}\Big\{\sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \geq b_n\Big\} = \prod_{z\in\mathbb{Z}^d} f(h\ell_n(z)) \int_{b_n}^{\infty} e^{-hy}\, dP\Big\{\sum_{z\in\mathbb{Z}^d} Y_z < y\Big\}\,.$$

Substituting $y = M_n(h) + xV_n(h)$ and denoting $T := (\sum Y_z - M_n(h))/V_n(h)$, we get

$$\mathsf{P}\Big\{\sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \geq b_n\Big\} = \exp\Big\{-hM_n(h) + \sum_{z\in\mathbb{Z}^d} \log f(\ell_n(z)h)\Big\} \times \int_{\frac{b_n - M_n(h)}{V_n(h)}}^{\infty} \exp\{-hxV_n(h)\}\, dP(T<x). \tag{28}$$

Now we show that (27) implies that, for some constant $c > 0$, we have

$$h\,V_n^2 - c\,h^3\,\Gamma_n \leq M_n(h) \leq h\,V_n^2 + c\,h^2\,\Gamma_n\,. \tag{29}$$

Obviously,

$$m_z = \frac{\ell_n(z)\,f'(\ell_n(z)h)}{f(\ell_n(z)h)} \qquad \text{and thus} \qquad M_n(h) = \sum_{z\in\mathbb{Z}^d} \frac{\ell_n(z)\,f'(\ell_n(z)h)}{f(\ell_n(z)h)}\,.$$

On the one hand, using that all derivatives of f are increasing, we get

$$f'(\ell_n(z)h) \leq f''(0)\,\ell_n(z)\,h + \tfrac{1}{2}\,f'''(\ell_n(z)h)\,\ell_n^2(z)\,h^2 \leq \sigma^2\,\ell_n(z)\,h + \tfrac{1}{2}\,f'''(\theta/2)\,\ell_n^2(z)\,h^2,$$

and the second inequality in (29) readily follows from this together with the fact that $f(\ell_n(z)h) \geq 1$. On the other hand, noting that $f'(\ell_n(z)h) \geq \sigma^2\,\ell_n(z)\,h$ and

$$f(\ell_n(z)h) \leq 1 + f'(\ell_n(z)h)\,\ell_n(z)\,h \leq 1 + f'(\theta/2)\,\ell_n(z)\,h\,,$$

we obtain the bound

$$M_n(h) \geq \sum_{z\in\mathbb{Z}^d} \frac{\sigma^2\,\ell_n(z)\,h}{1 + f'(\theta/2)\,\ell_n(z)\,h} = h\,\sigma^2 \sum_{z\in\mathbb{Z}^d} \ell_n^2(z) - f'(\theta/2)\,\sigma^2\,h^2 \sum_{z\in\mathbb{Z}^d} \ell_n^3(z)\,.$$

Summarizing, we see that (29) holds with $c := \max\{\sigma^2 f'(\theta/2), \frac{1}{2}\,f'''(\theta/2)\}$.

Let $h_n^{\pm}$ denote the positive solutions of the quadratic equations

$$V_n^2\,h \pm c\Gamma_n\,h^2 = b_n\,.$$

It is easy to see that

$$h_n^{\pm} = \frac{b_n}{V_n^2} + O\Big(\frac{\Gamma_n b_n^2}{V_n^6}\Big) \qquad \text{as } n \to \infty\,, \tag{30}$$

provided that $\Gamma_n b_n = O(V_n^4)$.

From our assumption $\Gamma_n \leq n\log^2 n$ we get $\ell_n^{(\infty)} \leq n^{1/3}\log^{2/3} n$ and thus (27) holds for all $h \leq \theta/(2n^{1/3}\log^{2/3} n)$. Since $b_n \leq n^{2/3}/\log n$ and $\Gamma_n b_n^2 \leq n^{7/3}$ but $V_n^2 \geq n$ we obtain that $h_n^- \leq n^{-1/3}/\log n + O(n^{-2/3})$ and thus h_n^- is in the domain given by (27), for all large n. Hence the inequalities (29) hold for all $0 < h \leq h_n^-$ and so, on the one hand, we have $M(h_n^-) \geq b_n$, and on the other hand, as $h_n^+ < h_n^-$, we have $M(h_n^+) \leq b_n$. Therefore there exists $h_n \in [h_n^+, h_n^-]$ such that $M(h_n) = b_n$. Applying (30) gives

$$h_n = \frac{b_n}{V_n^2} + O\Big(\frac{\Gamma_n b_n^2}{V_n^6}\Big) \qquad \text{as } n \to \infty\,. \tag{31}$$

Clearly,

$$\log f\big(\ell_n(z)h_n\big) = \log\big(1 + \tfrac{\sigma^2}{2}\,\ell_n^2(z)\,h_n^2 + O(\ell_n^3(x)h_n^3)\big) = \tfrac{\sigma^2}{2}\,\ell_n^2(z)\,h_n^2 + O(\ell_n^3(x)h_n^3)\,.$$

Thus, in view of (31),

$$-h_n M_n(h_n) + \sum_{z\in\mathbb{Z}^d} \log f\big(\ell_n(z)h_n\big) = -h_n\,b_n + \tfrac12\,V_n^2\,h_n^2 + O\big(\Gamma_n h_n^3\big) = -\frac{b_n^2}{2V_n^2} + O\Big(\frac{\Gamma_n b_n^3}{V_n^6}\Big)\,. \tag{32}$$

Putting $h = h_n$ in (28) and using (32), we obtain

$$\mathsf{P}\big\{\ell_n(z)\xi(z) \geq b_n\big\} = \exp\Big\{-\frac{b_n^2}{2V_n} + O\Big(\frac{\Gamma_n b_n^3}{V_n^6}\Big)\Big\}\int_0^\infty e^{-xh_nV_n(h_n)}\,dP(T<x). \tag{33}$$

Integrating by parts gives, for a standard normal random variable N,

$$\begin{aligned}\int_0^\infty e^{-xh_nV_n(h_n)}\,dP\{T<x\} &= \int_0^\infty P\{T<x\}\,h_n\,V_n(h_n)\,e^{-h_nV_n(h_n)\,x}\,dx\\ &= \int_0^\infty P\{N<x\}\,h_n\,V_n(h_n)\,e^{-h_n\,V_n(h_n)\,x}\,dx + \int_0^\infty \Delta(x)\,h_n\,V_n(h_n)e^{-h_n\,V_n(h_n)\,x}\,dx\\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\big\{-h_n\,V_n(h_n)\,x - \tfrac{x^2}{2}\big\}\,dx + \int_0^\infty \Delta(x)\,h_n\,V_n(h_n)\,e^{-h\,V_n(h_n)\,x}\,dx,\end{aligned}$$

where $\Delta(x) := P\{T<x\} - P\{N<x\}$. By Esseen's inequality, see for example [Pe75, Theorem V.3], there exists an abolute constant $C>0$, such that

$$\sup_x |\Delta(x)| \leq C\,\frac{\Gamma_n(h_n)}{V_n^3(h_n)}\,.$$

Therefore

$$\Big|\int_0^\infty e^{-xh_nV_n(h_n)}\,dP\{T<x\} - \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\big\{-h_n\,V_n(h_n)\,x - \tfrac{x^2}{2}\big\}\,dx\Big| \leq C\,\frac{\Gamma_n(h_n)}{V_n^3(h_n)}\,.$$

Evidently,

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\big\{-h_n\,V_n(h_n)\,x - \tfrac{x^2}{2}\big\}\,dx\\ &\qquad = \frac{1}{\sqrt{2\pi}}\exp\Big\{\frac{h_n^2V_n^2(h_n)}{2}\Big\}\int_0^\infty \exp\Big\{-\frac{(x+h_nV_n(h_n))^2}{2}\Big\}\,dx\\ &\qquad = \exp\Big\{\frac{h_n^2V_n^2(h_n)}{2}\Big\}\Big(1-\Phi\big(h_nV_n(h_n)\big)\Big)\,.\end{aligned} \tag{34}$$

We now show that, for a suitable constant $C>0$,

$$V_n^2(h_n) = V_n^2 + O(\Gamma_n h_n) \qquad \text{and} \qquad \Gamma_n(h_n) \leq C\,\Gamma_n. \tag{35}$$

First, we obtain that

$$V_n^2(h_n) = \sum_{z\in\mathbb{Z}^d} \sigma_z^2 = \sum_{z\in\mathbb{Z}^d} \ell_n^2(z)\, \frac{f''(\ell_n(z)h_n) - (f'(\ell_n(z)h_n))^2}{f(\ell_n(z)h_n)}$$
$$= \sum_{z\in\mathbb{Z}^d} \ell_n^2(z)\, \big(\sigma^2 + O(\ell_n(z)\, h_n)\big) = V_n^2 + O(\Gamma_n h_n)\,.$$

Second, for an upper estimate of $\Gamma_n(h_n)$, we note that

$$E|Y_z|^3 = 2\int_{-\infty}^0 |y|^3\, dP\{Y_z < y\} + EY_z^3\,.$$

From the definition of Y_z we get, on the one hand,

$$\int_{-\infty}^0 |y|^3\, dP\{Y_z < y\} = \tfrac{1}{f(\ell_n(z)h)} \int_{-\infty}^0 |y|^3\, e^{hy}\, d\mathsf{P}\{\xi(z) < \tfrac{y}{\ell_n(z)}\}$$
$$\le \ell_n^3(z) \int_{-\infty}^0 |x|^3\, d\mathsf{P}\{\xi(z) < x\} \le \ell_n^3(z)\, \mathsf{E}|\xi(0)|^3\,,$$

and, on the other hand,

$$EY_z^3 = \frac{f'''(\ell_n(z)h)\, \ell_n^3(z)}{f(\ell_n(z)h)} \le \ell_n^3(z)\, f'''(\theta/2)\,.$$

The two bounds imply that $E|Y_z|^3 \le \big(f'''(\theta/2)+2\gamma\big)\ell_n^3(z)$, and combining this with $m_z \le f'(\theta/2)\, \ell_n(z)$ gives $\gamma_z \le E|Y_z|^3 + m_z^3 \le C\ell_n^3(z)$ and therefore we have proved (35).
From (31) and (35) we thus get

$$h_n V_n(h_n) = \Big(\frac{b_n}{V_n^2} + O\big(\tfrac{\Gamma_n b_n^2}{V_n^6}\big)\Big) \Big(V_n^2 + O\big(\tfrac{\Gamma_n b_n}{V_n^2}\big)\Big)^{1/2} = \frac{b_n}{V_n}\Big(1 + O\big(\tfrac{\Gamma_n b_n}{V_n^4}\big)\Big)\,.$$

Recalling that $b_n \gg \sqrt{n}$ and $V_n^2 \le An$ we conclude that $h_n V_n(h_n) \to \infty$. Then, using (26),

$$e^{h_n^2 V_n^2(h_n)/2}\big(1 - \Phi(h_n V_n(h_n)\big) = \frac{1}{\sqrt{2\pi} h_n V_n(h_n)}\Big(1 + O\big(\tfrac{1}{h_n^2 V_n^2(h_n)}\big)\Big)$$
$$= \frac{V_n}{\sqrt{2\pi} b_n}\Big(1 + O\big(\tfrac{\Gamma_n b_n}{V_n^4}\big) + O\big(\tfrac{V_n^2}{b_n^2}\big)\Big)\,.$$

Substituting this into (34) gives

$$\int_0^\infty e^{-h_n V_n(h_n)x}\, dP\{T < x\} = \frac{V_n}{\sqrt{2\pi} b_n}\Big(1 + O\big(\tfrac{\Gamma_n b_n}{V_n^4}\big) + O\big(\tfrac{V_n^2}{b_n^2}\big)\Big)\,,$$

and the result follows by plugging this into (33). □

5. Moderate deviations in dimensions $d \ge 3$: Proof of Theorem 3

5.1 Proof of the upper bound in Theorem 3

We fix $\epsilon > 0$ and let $A := 2G(0) - 1 + 3\epsilon$. Our aim is to show that

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\Big\{\sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \ge b_n\Big\} \le -\frac{1}{2\sigma^2 A}. \tag{36}$$

We note that, for any fixed $\eta > 0$,

$$\mathbb{P}\Big\{ \sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \geq b_n \Big\} \leq \mathbb{P}\Big\{ \sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \geq b_n,\ \ell_n^{(\infty)} \leq \tfrac{\eta n}{b_n},\ \ell_n^{(2)} \leq A\,n \Big\} + \mathbb{P}\Big\{ \ell_n^{(\infty)} \geq \tfrac{\eta n}{b_n} \Big\} + \mathbb{P}\big\{ \ell_n^{(2)} \geq A\,n \big\}. \tag{37}$$

To see that the second summand is negligible apply Lemma 18 with $a_n = \eta n/b_n$, which gives

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\big\{ \ell_n^{(\infty)} > \tfrac{\eta n}{b_n} \big\} \leq \limsup_{n\to\infty} \frac{n \log n}{b_n^2} - c\frac{\eta n^2}{b_n^3} = -\infty. \tag{38}$$

To see that the third term in (37) is negligible, recall from (6) that $\mathbb{E}\ell_n^{(2)} \sim n(2G(0)-1)$ and therefore, for all large n,

$$\mathbb{P}\big\{ \ell_n^{(2)} \geq A\,n \big\} \leq \mathbb{P}\Big\{ \ell_n^{(2)} - \mathbb{E}\ell_n^{(2)} \geq \big(\tfrac{A-1-\epsilon}{2} - G(0)\big)n \Big\} = \mathbb{P}\big\{ \ell_n^{(2)} - \mathbb{E}\ell_n^{(2)} \geq \epsilon n \big\}.$$

From Proposition 11 we know that for $b_n \ll n^{2/3}$, if $d \geq 4$,

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\big\{ \ell_n^{(2)} - \mathbb{E}\ell_n^{(2)} \geq \epsilon n \big\} \leq \limsup_{n\to\infty} -c\frac{n^{3/2}}{b_n^2 \log n} = -\infty,$$

and, if $d = 3$,

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\big\{ \ell_n^{(2)} - \mathbb{E}\ell_n^{(2)} \geq \epsilon n \big\} \leq \limsup_{n\to\infty} -c\frac{n^{4/3}}{b_n^2} = -\infty.$$

Combining this, we get

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log \mathbb{P}\big\{ \ell_n^{(2)} \geq A\,n \big\} = -\infty. \tag{39}$$

It remains to investigate the first term on the right hand side of (37). For this purpose, for the moment fix $\{\ell_n(z) \,:\, z \in \mathbb{Z}^d\}$ such that

$$\ell_n^{(\infty)} \leq \frac{\eta n}{b_n} \qquad \text{and} \qquad \ell_n^{(2)} \leq An,$$

and just look at probabilities for the i.i.d. variables $\{\xi(z) \,:\, z \in \mathbb{Z}^d\}$. Denote $f(h) := \mathsf{E}e^{h\xi(0)}$ for all $h < \theta$, which is well-defined by Cramér's condition. Recall that

$$f(h) = \exp\big\{ \tfrac{1}{2}\, h^2\, \sigma^2 (1 + o(h)) \big\} \qquad \text{as } h \downarrow 0.$$

In particular, given any $\delta > 0$, we may choose a small $\eta > 0$ such that

$$f\Big(\frac{b_n\, \ell_n(x)}{\sigma^2\, \ell_n^{(2)}} \Big) \leq \exp\Big\{ (1+\delta)\, \frac{b_n^2 \ell_n^2(x)}{2\sigma^2\, (\ell_n^{(2)})^2} \Big\}, \tag{40}$$

where we use that $b_n \ell_n(x)/\ell_n^{(2)} \leq \eta$. From Chebyshev's inequality and independence we get that

$$\begin{aligned} \mathsf{P}\Big\{ \sum_{x\in\mathbb{Z}^d} \ell_n(x)\xi(x) \geq b_n \Big\} &\leq \prod_{x\in\mathbb{Z}^d} f\Big(\frac{b_n \ell_n(x)}{\sigma^2\, \ell_n^{(2)}} \Big)\ \exp\Big\{ -\frac{b_n^2}{\sigma^2\, \ell_n^{(2)}} \Big\} \\ &\leq \exp\Big\{ (1+\delta)\, \frac{b_n^2}{2\sigma^2\, \ell_n^{(2)}} \Big\}\ \exp\Big\{ -\frac{b_n^2}{\sigma^2\, \ell_n^{(2)}} \Big\} \leq \exp\Big\{ -\big(1-\delta\big)\, \frac{b_n^2}{2\sigma^2\, A\, n} \Big\}. \end{aligned}$$

We can now average over the random walk again, and get (36) from (37) together with (38) and (39), recalling that $\delta > 0$ was arbitrary. This completes the proof. ☐

5.2 Proof of the lower bound in Theorem 3

We impose 'typical behaviour' on $\ell_n^{(2)}$ and $\ell_n^{(\infty)}$. More precisely, fix an arbitrary $\epsilon \in (0,1)$, and also fix $\eta > 0$ which we specify later. We have

$$\begin{aligned}\mathbb{P}\Big\{ \sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \geq b_n \Big\} &\geq \mathbb{P}\Big\{ \sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \geq b_n,\ \ell_n^{(\infty)} \leq \tfrac{\eta n}{b_n},\ \ell_n^{(2)} \leq A\,n \Big\} \\ &= \mathbb{E}\Big\{ \mathsf{P}\Big\{ \sum_{z\in\mathbb{Z}^d} \ell_n(z)\,\xi(z) \geq b_n \Big\}\, 1\{\ell_n^{(\infty)} \leq \tfrac{\eta n}{b_n},\ \ell_n^{(2)} \leq A\,n\} \Big\},\end{aligned} \tag{41}$$

where $A := 2G(0) - 1 + 3\epsilon$ and P refers to the probability with respect to the scenery only. To study the inner probability we now suppose that, for the moment, a random walk sample is fixed, such that

$$\ell_n^{(\infty)} \leq \tfrac{\eta n}{b_n} \qquad \text{and} \qquad \ell_n^{(2)} \leq An.$$

Denote $\gamma := \mathbb{E}|\xi(0)|^3 < \infty$. Hence the variance of the random variable $\sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z)$ with respect to P is given by $V_n^2 := \sigma^2 \sum_{z\in\mathbb{Z}^d} \ell_n^2(z)$ and the Lyapunov ratio by $L_n := \gamma\, V_n^{-3} \sum_{z\in\mathbb{Z}^d} \ell_n^3(z)$. By [Na02, Theorem 2] there exist constants $c_1, c_2 > 0$ such that, for all $\frac{3}{2}V_n \leq x \leq \frac{V_n}{196 L_n}$,

$$\mathsf{P}\Big\{ \sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \geq x \Big\} \geq \big(1 - \Phi(\tfrac{x}{V_n})\big) \exp\big\{ - c_1 x^3\, L_n V_n^{-3} \big\} \big(1 - c_2 x L_n V_n^{-1}\big). \tag{42}$$

Now suppose that $\eta > 0$ is chosen to satisfy the three inequalities

$$\eta < \sigma^4/(196\gamma),\ c_1\eta\gamma\sigma^{-6} < \epsilon, \text{ and } c_2\eta\gamma\sigma^{-4} < \epsilon.$$

Using the upper bound on $\ell_n^{(\infty)}$, we get that $L_n \leq \frac{\gamma\eta n}{\sigma^2 b_n} V_n^{-1}$. Therefore,

$$\mathsf{P}\Big\{ \sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \geq x \Big\} \geq \big(1 - \Phi(\tfrac{x}{V_n})\big) \exp\big\{ - c_1\, \eta\, \tfrac{x^3}{b_n\,\sigma^2}\, n\, V_n^{-4} \big\} \big(1 - c_2\,\gamma\,\eta\, \tfrac{x}{b_n\,\sigma^2}\, n\, V_n^{-2}\big),$$

for all $(3/2)V_n \leq x \leq (b_n V_n)/(196\eta n)$. We can use this inequality for $x = b_n$. Indeed, as $V_n^2 \leq A\sigma^2\, n$ we get $b_n \geq (3/2)V_n$, if n exceeds some constant depending only on σ^2. Also $V_n^2 \geq \sigma^2 n$ and $\eta < \sigma^4/(196\gamma)$, therefore

$$b_n \leq b_n \sigma^2 V_n^2/(196\gamma\eta n) \leq V_n/(196 L_n).$$

Hence,

$$\mathsf{P}\Big\{ \sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \geq b_n \Big\} \geq \big(1 - \Phi(\tfrac{b_n}{V_n})\big) \exp\big\{ - c_1\, \eta\, \gamma\, \sigma^{-6}\, \tfrac{b_n^2}{n} \big\} \big(1 - c_2 \gamma\, \sigma^{-4}\, \eta\big). \tag{43}$$

Substituting (43) into (41) gives

$$\begin{aligned}&\mathbb{P}\Big\{ \sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \geq b_n \Big\} \\ &\geq \big(1 - c_2\gamma\sigma^{-4}\,\eta\big) \exp\big\{ - c_1\, \gamma\sigma^{-6}\, \eta\, \tfrac{b_n^2}{n} \big\}\, \mathbb{E}\Big[\big(1 - \Phi(\tfrac{b_n}{V_n})\big)\, 1\big\{ V_n^2 \leq A\sigma^2\, n,\ \ell_n^{(\infty)} \leq \tfrac{\eta n}{b_n} \big\} \Big] \\ &\geq \big(1 - \epsilon\big) \exp\big\{ - \epsilon\, \tfrac{b_n^2}{n} \big\}\, \mathbb{E}\Big[\big(1 - \Phi(\tfrac{b_n}{V_n})\big)\, 1\big\{ V_n^2 \leq A\sigma^2\, n \big\} \Big] - \mathbb{P}\Big\{ \ell_n^{(\infty)} \geq \tfrac{\eta n}{b_n} \Big\}.\end{aligned} \tag{44}$$

Since, by a standard estimate, $(1 - \Phi(z)) \geq \exp\{-(1+\eta)\, z^2/2\}$ for all sufficiently large z, we get

$$\mathbb{E}\Big[\big(1 - \Phi(\tfrac{b_n}{V_n})\big)\, 1\big\{ V_n^2 \leq A\sigma^2\, n \big\} \Big] \geq \mathbb{E} \exp\Big\{ - \frac{(1+\eta) b_n^2}{2V_n^2} \Big\} - \mathbb{P}\big\{ V_n^2 \geq A\sigma^2\, n \big\}. \tag{45}$$

By Jensen's inequality, we obtain

$$\mathbb{E} \exp\Big\{ - \frac{(1+\eta) b_n^2}{2V_n^2} \Big\} \geq \exp\Big\{ - \frac{(1+\eta) b_n^2}{2\sigma^2 n}\, \mathbb{E} \frac{n}{\ell_n^{(2)}} \Big\}.$$

Using Proposition 11 and the Borel-Cantelli lemma,

$$\lim_{n\to\infty}\frac{\ell_n^{(2)}}{n}=\lim_{n\to\infty}\frac{\mathbb{E}\ell_n^{(2)}}{n}=2G(0)-1\quad\text{almost surely,}$$

and using further that $n/\ell_n^{(2)}\le 1$, we obtain that

$$\lim_{n\to\infty}\mathbb{E}\frac{n}{\ell_n^{(2)}}=\frac{1}{2G(0)-1}.$$

Then, for all n sufficiently large,

$$\mathbb{E}\Big[\big(1-\Phi(\tfrac{b_n}{V_n})\big)\Big]\ge\exp\Big\{-\frac{(1+2\eta)b_n^2}{2\sigma^2 n(2G(0)-1-\epsilon)}\Big\}. \tag{46}$$

Combining (44), (45) and (46) gives

$$\mathbb{P}\Big\{\sum_{z\in\mathbb{Z}^d}\ell_n(z)\xi(z)\ge b_n\Big\}\ge(1-\epsilon)\;\exp\Big\{-\Big(\epsilon+\frac{1+2\eta}{2\sigma^2(2G(0)-1-\epsilon)}\Big)\frac{b_n^2}{n}\Big\}$$
$$-\mathbb{P}\Big\{\ell_n^{(\infty)}\ge\tfrac{\eta n}{b_n}\Big\}-\mathbb{P}\big\{\ell_n^{(2)}\ge A\,n\big\}.$$

The required lower bound follows from the estimates (38) and (39) for the subtracted probabilities, and the fact that $\epsilon>0$ can be chosen arbitrarily small, whence η also becomes arbitrarily small. □

6. Moderate deviations in dimension $d=2$: Proof of Theorem 6

We use the following moderate deviation principle for the self-intersection local time in the planar case, which is due to Bass, Chen and Rosen [BCR06, Theorem 1.1 and (3.2)]: If $x_n\to\infty$ and $x_n=o(n)$, then for every $\lambda>0$,

$$\lim_{n\to\infty}\frac{1}{x_n}\log\mathbb{P}\big\{\ell_n^{(2)}-\mathbb{E}\ell_n^{(2)}\ge\lambda\,nx_n\big\}=\lim_{n\to\infty}\frac{1}{x_n}\log\mathbb{P}\big\{|\ell_n^{(2)}-\mathbb{E}\ell_n^{(2)}|\ge\lambda\,nx_n\big\}=-\frac{\lambda\sqrt{\det\Gamma}}{2\varkappa^4}, \tag{47}$$

where again $\varkappa$ is the optimal constant in the Gagliardo-Nirenberg inequality.

6.1 Proof of Theorem 6(a)

The proof is largely analogous to that of Theorem 3 replacing Proposition 11 by (47). Starting with the *upper bound*, for any fixed $\epsilon>0$, we use the decomposition

$$\mathbb{P}\Big\{\sum_{z\in\mathbb{Z}^2}\ell_n(z)\xi(z)\ge b_n\Big\}\le\mathbb{P}\Big\{\sum_{z\in\mathbb{Z}^2}\ell_n(z)\xi(z)\ge b_n,\;\ell_n^{(\infty)}\le\tfrac{\sqrt{n}(\log n)^5}{b_n},\;\ell_n^{(2)}\le A\,n\log n\Big\}$$
$$+\mathbb{P}\Big\{\ell_n^{(\infty)}\ge\tfrac{\sqrt{n}(\log n)^5}{b_n}\Big\}+\mathbb{P}\big\{\ell_n^{(2)}\ge A\,n\log n\big\},$$

where $A:=(\pi\sqrt{\det\Gamma})^{-1}+4\epsilon$. The estimate for the last probability follows from (47). Indeed, by (6), for sufficiently large n,

$$\mathbb{P}\big\{\ell_n^{(2)}\ge A\,n\log n\big\}\le\mathbb{P}\big\{\ell_n^{(2)}-\mathbb{E}\ell_n^{(2)}\ge\big(A-(\pi\sqrt{\det\Gamma})^{-1}-\epsilon\big)\,n\log n\big\}\le n^{-\epsilon\sqrt{\det\Gamma}\,\varkappa^{-4}},$$

hence, as $b_n\ll n^{\frac12}\log n$,

$$\limsup_{n\to\infty}\frac{n\log n}{b_n^2}\log\mathbb{P}\big\{\ell_n^{(2)}\ge A\,n\log n\big\}=-\infty. \tag{48}$$

Moreover, applying Lemma 18, we get

$$\limsup_{n\to\infty} \frac{n\log n}{b_n^2} \log \mathbb{P}\big\{\ell_n^{(\infty)} > b_n^{-1}\sqrt{n}\,(\log n)^5\big\} \leq \limsup_{n\to\infty} \frac{n(\log n)^2}{b_n^2} - c\frac{n^{\frac32}(\log n)^4}{b_n^3} = -\infty. \tag{49}$$

We now look at fixed local times $\{\ell_n(z)\colon z\in\mathbb{Z}^2\}$ satisfying the conditions $\max \ell_n(z) \leq b_n^{-1}\sqrt{n}(\log n)^5$ and $\ell_n^{(2)} \leq A\, n\log n$. Note that, together with the trivial inequality $\ell_n^{(2)} \geq n$, this implies

$$\lim_{n\uparrow\infty} \frac{b_n \ell_n(z)}{\sigma^2\,\ell_n^{(2)}} = 0.$$

Hence, for arbitrary $\delta>0$, if n is sufficiently large, an application of Chebyshev's inequality and the estimate (40) for the Laplace transform f of $\xi(z)$, gives, for n larger than some absolute constant,

$$\mathsf{P}\Big\{\sum_{z\in\mathbb{Z}^2}\ell_n(z)\xi(z)\geq b_n\Big\} \leq \prod_{z\in\mathbb{Z}^d} f\big(\tfrac{b_n\ell_n(z)}{\sigma^2\,\ell_n^{(2)}}\big)\,\exp\big\{-\tfrac{b_n^2}{\sigma^2\,\ell_n^{(2)}}\big\} \leq \exp\big\{-(1-\delta)\,\tfrac{b_n^2}{2\sigma^2\,A\,n\log n}\big\}.$$

Averaging over the local times again, we obtain

$$\limsup_{n\uparrow\infty} \tfrac{n\log n}{b_n^2}\log\mathbb{P}\Big\{\sum_{z\in\mathbb{Z}^2}\ell_n(z)\xi(z)\geq b_n,\ \ell_n^{(\infty)}\leq \tfrac{\sqrt{n}(\log n)^5}{b_n},\ \ell_n^{(2)}\leq A\,n\log n\Big\} \leq \tfrac{-(1-\delta)}{2\sigma^2\,A},$$

so that the claimed upper bound follows, as $\epsilon,\delta>0$ were arbitrary.

Turning to the *lower bound*, we fix $\epsilon>0$ again, and use that

$$\mathbb{P}\Big\{\sum_{z\in\mathbb{Z}^2}\ell_n(z)\xi(z)\geq b_n\Big\} \geq \mathbb{E}\Big\{\mathsf{P}\big\{\sum_{z\in\mathbb{Z}^2}\ell_n(z)\,\xi(z)\geq b_n\big\} \times 1\big\{\ell_n^{(\infty)}\leq \tfrac{\sqrt{n}(\log n)^5}{b_n},\ \ell_n^{(2)}\leq A\,n\log n\big\}\Big\}, \tag{50}$$

where $A:=(\pi\sqrt{\det\Gamma})^{-1}+4\epsilon$. To obtain a lower bound for the inner probability we argue as in Theorem 3, relying on the estimates of [Na02, Theorem 2]. This gives

$$\mathsf{P}\Big\{\sum_{z\in\mathbb{Z}^d}\ell_n(z)\xi(z)\geq b_n\Big\} \geq \big(1-\Phi(\tfrac{b_n}{V_n})\big)\,\exp\big\{-c_1\,\gamma\sigma^{-6}\,b_n^2\,n^{-\frac32}\,(\log n)^3\big\}\,\big(1-c_2\,\gamma\sigma^{-4}\,n^{-\frac12}(\log n)^4\big).$$

We now show that

$$\lim_{n\uparrow\infty}\mathbb{E}\Big[\frac{n\log n}{\ell_n^{(2)}}\Big] = \pi\sqrt{\det\Gamma}\,. \tag{51}$$

For this purpose define the random variables $Y_n := \frac1n\,\ell_n^{(2)} - (\pi\sqrt{\det\Gamma})^{-1}\log n$ and note that

$$\frac{n\log n}{\ell_n^{(2)}} = \pi\sqrt{\det\Gamma} - \pi\sqrt{\det\Gamma}\,\frac{Y_n}{\frac1n\,\ell_n^{(2)}}.$$

It suffices to show that the expectation of the fraction on the right converges to zero. As $|Y_n|\leq\varepsilon\log n$ implies that $\frac1n\ell_n^{(2)}\geq((\pi\sqrt{\det\Gamma})^{-1}-\varepsilon)\log n$ we obtain, for any small $\varepsilon>0$, that

$$\mathbb{E}\Big[\frac{|Y_n|}{\frac1n\,\ell_n^{(2)}}\,1\{|Y_n|\leq\varepsilon\log n\}\Big] \leq \frac{\varepsilon}{(\pi\sqrt{\det\Gamma})^{-1}-\varepsilon}\,. \tag{52}$$

Also, as $\frac1n\ell_n^{(2)}\geq 1$ and using (47) with $\lambda=\varepsilon$ and $x_n=\log n$, for any $0<\varepsilon<\delta$,

$$\mathbb{E}\Big[\frac{|Y_n|}{\frac1n\,\ell_n^{(2)}}\,1\{\varepsilon\log n<|Y_n|\leq\delta\log n\}\Big] \leq \delta\,(\log n)\,\mathbb{P}\{|Y_n|>\varepsilon\,\log n\}\longrightarrow 0, \tag{53}$$

and, using (47) with $\lambda=\delta$ and $x_n=\log n$, if $\delta>0$ is sufficiently large,

$$\mathbb{E}\Big[\frac{|Y_n|}{\frac{1}{n}\,\ell_n^{(2)}}\,1\{|Y_n|>\delta\log n\}\Big]\le n\,\mathbb{P}\{|Y_n|>\delta\,\log n\}\longrightarrow 0\,. \tag{54}$$

We obtain that $\lim\mathbb{E}|Y_n|/\frac{1}{n}\ell_n^{(2)}=0$, and hence (51), by combining (52), (53), and (54). Repeating the arguments of the $d\ge 3$ case, given in Section 5.2, gives

$$\begin{aligned}&\mathbb{P}\Big\{\sum_{z\in\mathbb{Z}^d}\ell_n(z)\xi(z)\ge b_n\Big\}\\ &\ge\big(1-c_2\,\gamma\sigma^{-4}\,n^{-\frac{1}{2}}(\log n)^4\big)\,\exp\big\{-c_1\,\gamma\sigma^{-6}\,b_n^2\,n^{-\frac{3}{2}}\,(\log n)^3\big\}\,\exp\Big\{-\frac{(1+\varepsilon)^2\pi b_n^2}{2\sigma^2 n\log n}\Big\}\\ &\qquad-\mathbb{P}\big\{\ell_n^{(\infty)}\ge\tfrac{\sqrt{n}(\log n)^5}{b_n}\big\}-\mathbb{P}\big\{\ell_n^{(2)}\ge A\,n\log n\big\}.\end{aligned}$$

The result follows, by observing that the first two factors on the right converge to one, recalling (49), (48) and that $\epsilon>0$ was arbitrary. □

6.2 Proof of Theorem 6(b)

Again, we start with the *upper bound.* Since $\mathbb{E}\ell_n^{(2)}\sim(\pi\sqrt{\det\Gamma})^{-1}n\log n$, we can conclude from (47) that, for $\log n\ll x_n\ll n$,

$$\lim_{n\to\infty}\frac{1}{x_n}\log\mathbb{P}\{\ell_n^{(2)}\ge\lambda n x_n\}=-\frac{\lambda}{2\varkappa^4}\sqrt{\det\Gamma}\,. \tag{55}$$

For arbitrary $N\ge 1$ and $0<\delta<1$,

$$\mathbb{P}\{X_n\ge b_n\}\le\sum_{i=0}^{N-1}\mathbb{P}\big\{X_n\ge b_n,\,\ell_n^{(2)}\in(i\delta a_n,(i+1)\delta a_n]\big\}+\mathbb{P}\{\ell_n^{(2)}>N\delta a_n\}, \tag{56}$$

where $a_n:=b_n\sqrt{n}$. Note that $a_n\gg n\log n$. Hence, in view of (55),

$$\mathbb{P}\{\ell_n^{(2)}>N\delta a_n\}\le\exp\Big\{-\frac{N\delta a_n\sqrt{\det\Gamma}}{3\varkappa^4 n}\Big\} \tag{57}$$

for all sufficiently large n. Fix $i\ge 1$ and $\eta\in(0,\theta\sigma^2)$. Then,

$$\begin{aligned}&\mathbb{P}\big\{X_n\ge b_n,\,\ell_n^{(2)}\in(i\delta a_n,(i+1)\delta a_n]\big\}\\ &\le\mathbb{P}\Big\{X_n\ge b_n,\,\ell_n^{(2)}\in(i\delta a_n,(i+1)\delta a_n],\,\ell_n^{(\infty)}\le\eta i\delta\sqrt{n}\Big\}+\mathbb{P}\big\{\ell_n^{(\infty)}>\eta i\delta\sqrt{n}\big\}\,.\end{aligned}$$

Using Lemma 18, we get

$$\mathbb{P}\big\{\ell_n^{(\infty)}>\eta i\delta\sqrt{n}\big\}\le\exp\Big\{-c\frac{\eta i\delta\sqrt{n}}{\log n}\Big\}. \tag{58}$$

On the event $\big\{\ell_n^{(\infty)}\le\eta i\delta\sqrt{n},\ell_n^{(2)}\in(i\delta a_n,(i+1)\delta a_n]\big\}$, we obtain,

$$\frac{b_n\ell_n(z)}{\sigma^2\ell_n^{(2)}}\le\frac{b_n\eta i\delta\sqrt{n}}{\sigma^2 i\delta a_n}=\frac{\eta}{\sigma^2}<\theta.$$

Therefore, we can use Chebyshev's inequality as before, which gives

$$\mathsf{P}\Big\{\sum_{z\in\mathbb{Z}^2}\ell_n(z)\xi(z)\ge b_n\Big\}\le\exp\Big\{-\frac{(1-\epsilon/2)b_n^2}{2\sigma^2\ell_n^{(2)}}\Big\}\le\exp\Big\{-\frac{(1-\epsilon)b_n^2}{2\sigma^2(i+1)\delta a_n}\Big\},$$

and thus, applying (55) again and recalling the definition of a_n, for sufficiently large n,

$$\begin{aligned}&\mathbb{P}\Big\{X_n \ge b_n,\, \ell_n^{(\infty)} \le \eta i\delta\sqrt{n},\, \ell_n^{(2)} \in (i\delta a_n, (i+1)\delta a_n]\Big\}\\ &\le \exp\Big\{-\frac{(1-\epsilon)b_n^2}{2\sigma^2(i+1)\delta a_n}\Big\}\,\mathbb{P}\{\ell_n^{(2)} > i\delta a_n\} \le \exp\Big\{-\frac{(1-\epsilon)b_n}{2\sigma^2(i+1)\delta\sqrt{n}} - \frac{(1-\epsilon)\sqrt{\det\Gamma}\,i\delta b_n}{2\varkappa^4\sqrt{n}}\Big\}.\end{aligned} \tag{59}$$

It remains to consider the summand corresponding to $i=0$ in (56), which for any $\eta > 0$ is bounded by

$$\mathbb{P}\Big\{X_n \ge b_n,\, \ell_n^{(2)} \le \delta a_n,\, \ell_n^{(\infty)} \le \eta\sqrt{n}\Big\} + \mathbb{P}\Big\{\ell_n^{(\infty)} > \eta\sqrt{n}\Big\}\,. \tag{60}$$

Applying Chebyshev's inequality on the event $\{\ell_n^{(2)} \le \delta a_n,\, \ell_n^{(\infty)} \le \eta\sqrt{n}\}$ we get, for any $a > 0$ and $\eta < \theta/a$,

$$\mathsf{P}\Big\{\sum_{z\in\mathbb{Z}^2} \ell_n(z)\xi(z) \ge b_n\Big\} \le \exp\Big\{-a\frac{b_n}{\sqrt{n}} + C\sum_{z\in\mathbb{Z}^2}\frac{a^2}{n}\,\ell_n^2(z)\Big\},$$

for a constant $C > 0$ depending only on the distribution of the scenery and the random walk. Using this estimate for $a = 1/(4C\delta)$ and $\eta < 4C\delta\theta$ we get

$$\mathbb{P}\{X_n \ge b_n,\, \ell_n^{(2)} \le \delta a_n,\, \ell_n^{(\infty)} \le \eta\sqrt{n}\} \le \exp\Big\{-\frac{b_n}{\sqrt{n}}\frac{1}{8\delta C}\Big\}\,. \tag{61}$$

Combining (56) – (61) gives us

$$\begin{aligned}\mathbb{P}\{X_n \ge b_n\} \le &\sum_{i=1}^{N-1}\exp\Big\{-\frac{(1-\epsilon)b_n}{2\sigma^2(i+1)\delta\sqrt{n}} - \frac{(1-\epsilon)i\delta b_n\sqrt{\det\Gamma}}{2\varkappa^4\sqrt{n}}\Big\}\\ &+\exp\Big\{-\frac{b_n}{\sqrt{n}}\frac{1}{8\delta C}\Big\} + N\,\exp\Big\{-c\frac{\eta\delta\sqrt{n}}{\log n}\Big\} + \exp\Big\{-\frac{N\delta b_n\sqrt{\det\Gamma}}{2\varkappa^4\sqrt{n}}\Big\}.\end{aligned} \tag{62}$$

It is easily seen, that

$$\begin{aligned}\lim_{n\to\infty}\frac{\sqrt{n}}{b_n}\log\sum_{i=1}^{N-1}&\exp\Big\{-\frac{(1-\epsilon)b_n}{2\sigma^2(i+1)\delta\sqrt{n}} - \frac{(1-\epsilon)i\delta b_n\sqrt{\det\Gamma}}{2\varkappa^4\sqrt{n}}\Big\}\\ &= -(1-\epsilon)\min_{1\le i\le N-1}\Big(\frac{1}{2\sigma^2(i+1)\delta} + \frac{i\delta\sqrt{\det\Gamma}}{2\varkappa^4}\Big).\end{aligned}$$

Furthermore, if we choose $\delta > 0$ small and N large, we get

$$\min_{1\le i\le N-1}\Big(\frac{1}{2\sigma^2(i+1)\delta} + \frac{i\delta\sqrt{\det\Gamma}}{2\varkappa^4}\Big) \ge (1-\epsilon)\,\min_{x>0}\Big(\frac{1}{2\sigma^2 x} + \frac{x\sqrt{\det\Gamma}}{2\varkappa^4}\Big) = (1-\epsilon)\frac{(\det\Gamma)^{1/4}}{\sigma\varkappa^2}.$$

Therefore, for all n large enough,

$$\sum_{i=1}^{N-1}\exp\Big\{-\frac{(1-\epsilon)b_n}{4\sigma^2(i+1)\delta n^{1/2}} - \frac{(1-\epsilon)i\delta b_n\sqrt{\det\Gamma}}{\varkappa^4\sqrt{n}}\Big\} \le \exp\Big\{-(1-\epsilon)^3\frac{b_n\,(\det\Gamma)^{1/4}}{\sigma\varkappa^2\sqrt{n}}\Big\}. \tag{63}$$

Making first δ smaller, and then N larger, if necessary, we see that all other terms in (62) are of smaller order than (63). Taking into account that $\epsilon > 0$ was arbitrary, we have

$$\limsup_{n\to\infty}\frac{\sqrt{n}}{b_n}\log\mathbb{P}\{X_n \ge b_n\} \le -\frac{(\det\Gamma)^{1/4}}{\sigma\varkappa^2}.$$

To obtain a *lower bound*, note that for all $0 < \mu < \lambda$ and $\eta > 0$,

$$\mathbb{P}\{X_n \ge b_n\} \ge \mathbb{P}\{X_n \ge b_n,\, \ell_n^{(2)} \in [\mu a_n, \lambda a_n],\, \ell_n^{(\infty)} \le \eta\sqrt{n}\}\,, \tag{64}$$

where we still use $a_n = b_n\sqrt{n}$. Recall (42) and the definition of L_n and V_n. Note that on the set $\{\ell_n^{(2)} \in [\mu a_n, \lambda a_n],\, \ell_n^{(\infty)} \le \eta\sqrt{n}\}$ and for sufficiently large n, we have $\frac{3}{2}V_n \le \frac{3}{2}\sigma(\lambda b_n)^{1/2}n^{1/4} \le b_n \le \eta a_n/\ell_n^{(\infty)} \le (\eta/\mu\sigma^2)V_n^2/\ell_n^{(\infty)} \le V_n/(196 L_n)$ if $\eta > 0$ is sufficiently small. Hence,

$$\mathsf{P}\Big\{\sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \ge b_n\Big\} \ge \big(1-\Phi(\tfrac{b_n}{V_n})\big)\,\exp\big\{-c_1 b_n^3\, L_n V_n^{-3}\big\}\,\big(1 - c_2 b_n L_n V_n^{-1}\big).$$

We observe that $L_n \le \frac{\gamma\eta}{\sigma^2}\frac{\sqrt{n}}{V_n}$ and hence

$$b_n^3 L_n V_n^{-3} \le \frac{\gamma\eta}{\sigma^2}\, b_n^3\, \frac{\sqrt{n}}{V_n^4} \le \frac{\gamma\eta}{\sigma^6\mu^2}\,\frac{b_n}{\sqrt{n}} \qquad \text{and} \qquad b_n L_n V_n^{-1} \le \frac{\gamma\eta}{\sigma^2}\, b_n\, \frac{\sqrt{n}}{V_n^2} \le \frac{\gamma\eta}{\mu\sigma^4}\,.$$

Therefore, for all large n,

$$\begin{aligned}\mathbb{P}\{X_n \ge b_n\} \ge \exp\Big\{-(1+\epsilon)\frac{b_n}{2\mu\sigma^2\sqrt{n}} - \frac{c_1\gamma\eta b_n}{\mu^2\sigma^6\sqrt{n}}\Big\}\Big(1 - c_2\,\frac{\gamma\eta}{\mu\sigma^4}\Big)&\\ \times\Big[\mathbb{P}\big\{\ell_n^{(2)} \in [\mu a_n, \lambda a_n])\big\} - \mathbb{P}\big\{\ell_n^{(\infty)} > \eta\sqrt{n}\big\}\Big].&\end{aligned} \tag{65}$$

From (55) we conclude that for all $\mu < \lambda$,

$$\log \mathbb{P}\big\{\ell_n^{(2)} \in [\mu a_n, \lambda a_n]\big\} \sim -\frac{\mu b_n\sqrt{\det\Gamma}}{2\varkappa^4 n^{1/2}}. \tag{66}$$

Applying (66) and (58) to the right hand side of (65), we get for $n^{1/2}\log n \ll b_n \ll n/\log n$,

$$\liminf_{n\to\infty} \frac{\sqrt{n}}{b_n}\log \mathbb{P}\{X_n \ge b_n\} \ge -\frac{1+\epsilon}{2\mu\sigma^2} - \frac{c_1\gamma\eta}{\mu^2\sigma^6} - \frac{\mu\sqrt{\det\Gamma}}{2\varkappa^4}.$$

Since $\epsilon, \eta > 0$ can be chosen arbitrarily small, and μ is arbitrary,

$$\liminf_{n\to\infty} \frac{\sqrt{n}}{b_n}\log \mathbb{P}\{X_n \ge b_n\} \ge -\min_{\mu>0}\Big(\frac{1}{2\mu\sigma^2} + \frac{\mu\sqrt{\det\Gamma}}{2\varkappa^4}\Big) = -\frac{(\det\Gamma)^{1/4}}{\sigma\varkappa^2}.$$

This completes the proof of Theorem 6(b). □

6.3 Proof of Theorem 6(c)

We now assume that $b_n := a\sqrt{n}\log n$. In this case we use the following decomposition,

$$\begin{aligned}\mathbb{P}\{X_n \ge b_n\} \;\le\; & \mathbb{P}\big\{X_n \ge b_n,\, \ell_n^{(\infty)} \le \gamma_n,\, \ell_n^{(2)} - \mathbb{E}\ell_n^{(2)} \le \delta a_n\big\}\\ &+ \sum_{i=1}^{N} \mathbb{P}\big\{X_n \ge b_n,\, \ell_n^{(\infty)} \le \gamma_n,\, \ell_n^{(2)} - \mathbb{E}\ell_n^{(2)} \in (i\delta a_n, (i+1)\delta a_n]\big\}\\ &+ \mathbb{P}\big\{\ell_n^{(\infty)} > \gamma_n\big\} + \mathbb{P}\big\{\ell_n^{(2)} - \mathbb{E}\ell_n^{(2)} > N\delta a_n\big\},\end{aligned}$$

here $a_n := n\log n$, $\gamma_n := \eta n \log n/b_n$. Estimating every term as in the proof of the upper bound in (b) and using the relation $\mathbb{E}\ell_n^{(2)} \sim (\pi\sqrt{\det\Gamma})^{-1} n\log n$, one can get

$$\limsup_{n\to\infty} \frac{1}{\log n}\mathbb{P}\{X_n \ge b_n\} \le -\min_{x\ge 0}\Big(\frac{a^2}{2\sigma^2((\pi\sqrt{\det\Gamma})^{-1} + x)} - \frac{x\sqrt{\det\Gamma}}{2\varkappa^4}\Big) = I(a).$$

In order to get a lower bound we consider the cases $a \le \sigma/(\pi\varkappa^2(\det\Gamma)^{1/4})$ and $a > \sigma/(\pi\varkappa^2(\det\Gamma)^{1/4})$ separately. In the first case we use

$$\mathbb{P}\{X_n \ge b_n\} \ge \mathbb{P}\Big\{X_n \ge b_n,\, \ell_n^{(\infty)} \le \gamma_n,\, |\ell_n^{(2)} - \mathbb{E}\ell_n^{(2)}| \le \delta a_n\Big\},$$

and in the second case

$$\mathbb{P}\{X_n \ge b_n\} \ge \mathbb{P}\Big\{X_n \ge b_n,\, \ell_n^{(\infty)} \le \gamma_n,\, \ell_n^{(2)} - \mathbb{E}\ell_n^{(2)} \in (\mu a_n, \lambda a_n]\Big\}$$

for some $0 < \mu < \lambda$. The further proof is similar to that of the lower bound in Theorem 6(b) and details are left to the reader. □

7. Large deviations in dimension $d = 2$: Proof of Proposition 9

We first derive an *upper bound* for $\mathbb{P}\{X_n \ge b_n\}$. For arbitrary $N \ge 1$ and $0 < \delta < 1$,

$$\mathbb{P}\{X_n \ge b_n\} \le \sum_{i=0}^{N-1} \mathbb{P}\{X_n \ge b_n,\ \ell_n^{(\infty)} \in (i\delta a_n, (i+1)\delta a_n]\} + \mathbb{P}\{\ell_n^{(\infty)} \ge \delta N a_n\}, \tag{67}$$

where $a_n := (b_n \log n)^{1/2}$. By assumption (4), there exists C_δ such that

$$\mathbb{E}e^{h\xi(0)} \le \exp\{C_\delta h^2\} \quad \text{for} \quad h \le (1-\delta)D.$$

From this bound and Chebyshev's inequality we get

$$\mathsf{P}\Big\{\sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \ge b_n\Big\} \le \exp\big\{-hb_n + C_\delta h^2 \ell_n^{(2)}\big\} \quad \text{for} \quad h \le (1-\delta)D/\ell_n^{(\infty)}. \tag{68}$$

Letting here $h = (1-\delta)D/\ell_n^{(\infty)}$, we obtain

$$\mathsf{P}\Big\{\sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \ge b_n\Big\} \le \exp\Big\{-\tfrac{(1-\delta)Db_n}{\ell_n^{(\infty)}}\big(1 - \tfrac{C_\delta(1-\delta)D}{\ell_n^{(\infty)}b_n}\,\ell_n^{(2)}\big)\Big\}.$$

Therefore, for any $i \ge 1$,

$$\begin{aligned}&\mathbb{P}\{X_n \ge b_n,\, \ell_n^{(\infty)} \in (i\delta a_n, (i+1)\delta a_n]\}\\ &\qquad \le \exp\big\{-\tfrac{(1-\delta)^2 Db_n}{(i+1)\delta a_n}\big\}\,\mathbb{P}\{\ell_n^{(\infty)} > i\delta a_n\} + \mathbb{P}\big\{\ell_n^{(2)} > \tfrac{i\delta^2}{(1-\delta)C_\delta D} b_n a_n\big\}.\end{aligned} \tag{69}$$

Using [GHK06, Lemma 1.3] and recalling the definition of a_n, we get

$$\log \mathbb{P}\{\ell_n(0) > xa_n\} \sim -K_2 x \frac{(b_n \log b_n)^{1/2}}{\log n - (1/2)\log b_n} \sim -\frac{2K_2 x}{2-\beta}\Big(\frac{b_n}{\log n}\Big)^{1/2}. \tag{70}$$

Hence, arguing as in Lemma 18, for all $x \ge \delta$ and n large enough n,

$$\mathbb{P}\{\ell_n^{(\infty)} > xa_n\} \le \exp\Big\{-(1-\delta)^2 \frac{2K_2 x}{2-\beta}\Big(\frac{b_n}{\log n}\Big)^{1/2}\Big\}. \tag{71}$$

Combining (69) and (71), and noting that $b_n/a_n = (b_n/\log n)^{1/2}$, we obtain

$$\begin{aligned}&\mathbb{P}\{X_n \ge b_n,\, \ell_n^{(\infty)} \in (i\delta a_n, (i+1)\delta a_n]\}\\ &\qquad \le \exp\Big\{-\big(\tfrac{(1-\delta)^2 D}{(i+1)\delta} + (1-\delta)^2 \tfrac{2K_2 i\delta}{2-\beta}\big)\big(\tfrac{b_n}{\log n}\big)^{1/2}\Big\} + \mathbb{P}\big\{\ell_n^{(2)} > \tfrac{i\delta^2}{(1-\delta)C_\delta D} b_n a_n\big\}.\end{aligned} \tag{72}$$

Now we consider the probability corresponding to $i = 0$. As $\ell_n^{(\infty)} \le \delta a_n$, we can use $h = (\delta^{-1} - 1)Da_n^{-1}$ in (68). This gives us the bound

$$\mathsf{P}\Big\{\sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \ge b_n\Big\} \le \exp\Big\{-\tfrac{(1-\delta)Db_n}{\delta a_n}\big(1 - \tfrac{C_\delta(1-\delta)D}{\delta a_n b_n}\,\ell_n^{(2)}\big)\Big\}.$$

Averaging over the random walk, we have

$$\mathbb{P}\{X_n \ge b_n,\ \ell_n^{(\infty)} \le \delta a_n\} \le \exp\big\{-\tfrac{(1-\delta)^2 Db_n}{\delta a_n}\big\} + \mathbb{P}\big\{\ell_n^{(2)} > \tfrac{\delta^2}{(1-\delta)C_\delta D}\, b_n a_n\big\}. \tag{73}$$

Applying (71) we obtain

$$\mathbb{P}\{\ell_n^{(\infty)} \geq \delta N a_n\} \leq \exp\Big\{-c\delta N\big(\tfrac{b_n}{\log n}\big)^{1/2}\Big\}. \tag{74}$$

Substituting (72) – (74) into (67) gives

$$\begin{aligned}\mathbb{P}\{X_n \geq b_n\} \leq \sum_{i=0}^{N-1} \exp\Big\{-(1-\delta)^2\Big(\frac{}{(i+1)\delta} + \frac{2K_2 i\delta}{2-\beta}\Big)\Big(\frac{b_n}{\log n}\Big)^{1/2}\Big\} \\ + N\,\mathbb{P}\{\ell_n^{(2)} > \tfrac{\delta^2}{(1-\delta)C_\delta D}\, b_n a_n\} + \exp\{-c\delta N\big(\tfrac{b_n}{\log n}\big)^{1/2}\}.\end{aligned} \tag{75}$$

It is easily seen that

$$\begin{aligned}\lim_{n\to\infty}\Big(\frac{\log n}{b_n}\Big)^{1/2} \log \sum_{i=0}^{N-1} \exp\Big\{-(1-\delta)^2\Big(\frac{D}{(i+1)\delta} + \frac{2K_2 i\delta}{2-\beta}\Big)\Big(\frac{b_n}{\log n}\Big)^{1/2}\Big\} \\ = -(1-\delta)^2 \min_{0\leq i<N}\Big(\frac{D}{(i+1)\delta} + \frac{2K_2 i\delta}{2-\beta}\Big).\end{aligned}$$

Further, for small δ and large N we have the inequality

$$\min_{0\leq i<N}\Big(\frac{D}{(i+1)\delta} + \frac{2K_2 i\delta}{2-\beta}\Big) \geq (1-\delta)\min_{x>0}\Big(\frac{D}{x} + \frac{2K_2 x}{2-\beta}\Big) = (1-\delta)\Big(\frac{8\,K_2 D}{2-\beta}\Big)^{1/2}.$$

Consequently, for all n large enough,

$$\sum_{i=0}^{N-1} \exp\Big\{-(1-\delta)^2\Big(\frac{}{(i+1)\delta} + \frac{2K_2 i\delta}{2-\beta}\Big)\Big(\frac{b_n}{\log n}\Big)^{1/2}\Big\} \leq \exp\Big\{-(1-\delta)^4\Big(\frac{8\,K_2 D}{2-\beta}\Big)^{1/2}\Big(\frac{b_n}{\log n}\Big)^{1/2}\Big\}. \tag{76}$$

Making N larger, we see that the last term in (75) is of smaller order than (76). By (47) we obtain, for some constant $c > 0$,

$$\log \mathbb{P}\{\ell_n^{(2)} > t b_n a_n\} \sim -ct\Big(\frac{a_n b_n}{n}\Big).$$

By our assumption, $b_n \log n \gg n$. Therefore, $n^{-1}a_n b_n = n^{-1} b_n^{3/2} \log^{1/2} n \gg (b_n/\log n)^{1/2}$. This means that the probability term in (75) is negligible compared to (76). As a result we have

$$\limsup_{n\to\infty}\Big(\frac{\log n}{b_n}\Big)^{1/2} \log \mathbb{P}\{X_n \geq b_n\} \leq -(1-\delta)^4\Big(\frac{8\,K_2 D}{2-\beta}\Big)^{1/2}. \tag{77}$$

To derive a *lower bound* we note that

$$\mathsf{P}\Big\{\sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \geq b_n\Big\} \geq \mathsf{P}\{\ell_n(0)\xi(0) \geq (1+\delta)b_n\}\,\mathsf{P}\Big\{\sum_{z\neq 0} \ell_n(z)\xi(z) \geq -\delta b_n\Big\}.$$

Applying Chebyshev's inequality with second moments gives us

$$\mathsf{P}\Big\{\sum_{z\in\mathbb{Z}^d} \ell_n(z)\xi(z) \geq b_n\Big\} \geq \mathsf{P}\Big\{\ell_n(0)\xi(0) \geq (1+\delta)b_n\Big\}\Big(1 - \frac{\sigma^2 \ell_n^2}{\delta^2 b_n^2}\Big).$$

Consequently,

$$\mathbb{P}\{X_n \geq b_n\} \geq (1-\delta)\,\mathbb{P}\{\ell_n(0)\xi(0) \geq (1+\delta)b_n\} - \mathbb{P}\{\ell_n^{(2)} > \delta^2\, b_n^2/\sigma^2\}. \tag{78}$$

From (4) and (70) we get, for every $x > 0$,

$$\mathbb{P}\{\ell_n(0)\xi(0) \geq (1+\delta)b_n\} \geq \mathbb{P}\{\ell_n(0) > x a_n\} \geq \exp\Big\{-(1+\delta)^2\Big(\frac{D}{x} + \frac{2K_2 x}{2-\beta}\Big)\Big(\frac{b_n}{\log n}\Big)^{1/2}\Big\}.$$

Minimizing over x, we see that

$$\mathbb{P}\{\ell_n(0)\xi(0) \geq (1+\delta)b_n\} \geq \exp\Big\{-(1+\delta)^2\Big(\frac{8\,K_2 D}{2-\beta}\Big)^{1/2}\Big(\frac{b_n}{\log n}\Big)^{1/2}\Big\}. \tag{79}$$

As in the proof of the upper bound one can show that the last term in (78) is of smaller order than the right hand side in (79). Therefore,

$$\liminf_{n\to\infty}\Big(\frac{\log n}{b_n}\Big)^{1/2}\log\mathbb{P}\{X_n \geq b_n\} \geq -(1+\delta)^2\Big(\frac{8\,K_2 D}{2-\beta}\Big)^{1/2}. \tag{80}$$

Combining (77) and (80), and taking into account that δ is arbitrary, we get (5). □

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Bibliography

[As06] A. Asselah. Large deviations estimates for self-intersection local times for simple random walk in $\mathbb{Z}^3$. Preprint, arXiv:math.PR/0602074v1 (2006).

[AC03] A. Asselah and F. Castell. Large deviations for Brownian motion in a random scenery. *Probab. Theory Related Fields* 126, 497–527 (2003).

[AC05] A. Asselah and F. Castell. A note on random walk in random scenery. Preprint, arXiv:math.PR/0501068 (2005).

[AC06] A. Asselah and F. Castell. Self-intersection times for random walk and random walk in random scenery in dimensions $d \geq 5$. Preprint, arXiv:math.PR/0509721 (2006).

[BCR05] R.F. Bass, X. Chen and J. Rosen. Large deviations for renormalized self-intersection local times of stable processes. *Ann. Probab.* 33, 984–1013 (2005).

[BCR06] R.F. Bass, X. Chen and J. Rosen. Moderate deviations and laws of the iterated logarithm for the renormalized self-intersection local times of planar random walks. Preprint, arXiv:math.PR/0602001 (2006).

[Bo89] E. Bolthausen. A central limit theorem for two-dimensional random walk in random sceneries. *Ann. Probab.* 17, 108–115 (1989).

[Bo79a] A.N. Borodin. Limit theorems for sums of independent random variables defined on a transient random walk. (In Russian). *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI).* 85, 17-29 (1979).

[Bo79b] A.N. Borodin. A limit theorem for sums of independent random variables defined on a recurrent random walk. *Soviet Math. Dokl.* 20, 528–530 (1979).

[Ca04] F. Castell. Moderate deviations for diffusions in a random Gaussian shear flow drift. *Ann. Inst. H. Poincaré: Probab. Stat.* 40, 337–366 (2004).

[CP01] F. Castell and F. Pradeilles. Annealed large deviations for diffusions in a random Gaussian shear flow drift. *Stoch. Proc. Appl.* 94, 171–197 (2001).

[Ch04] X. Chen. Exponential asymptotics and law of the iterated logarithm for intersection local times of random walks. *Ann. Probab.* 32, 3248–3300 (2004).

[GHK06] N. Gantert, R. van der Hofstad and W. König. Deviations of a random walk in a random scenery with stretched exponential tails. *Stoch. Proc. Appl.* 116, 480–492 (2006).

[GKS05] N. GANTERT, W. KÖNIG and Z. SHI. Annealed deviations of random walk in random scenery. *Ann. Inst. H. Poincaré: Probab. Stat.* (to appear) (2005).

[GZ98] N. GANTERT and O. ZEITOUNI. Large and moderate deviations for the local time of a recurrent Markov chain. *Ann. Inst. H. Poincaré: Probab. Stat.* 34, 687–704 (1998).

[GP02] N. GUILLOTIN-PLANTARD. Large deviations for a Markov chain in a random landscape. *Adv. Appl. Prob.* 34, 375–393 (2002).

[KS79] H. KESTEN and F. SPITZER. A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. verw. Gebiete* 50, 5–25 (1979).

[Na79] S.V. NAGAEV. Large deviations of sums of independent random variables. *Ann. Probab.* 7, 745–789 (1979).

[Na02] S.V. NAGAEV. Lower bounds for the probabilities of large deviations of sums of independent random variables. *Theory Probab. Appl.* 46, 79–102 (2002).

[Pe75] V.V. PETROV. *Sums of independent random variables.* Springer, Berlin 1975.

[Sp76] F. SPITZER. *Principles of random walk*, 2nd edition. Springer, New York 1976.

[Uc98] K. UCHIYAMA. Wiener's test for random walks with mean zero and finite variance. *Ann. Probab.* 26, 368–376 (1998).

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