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Discounted optimal stopping for maxima of some jump-diffusion processes

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Abstract

We present solutions to some discounted optimal stopping problems for the maximum process in a model driven by a Brownian motion and a compound Poisson process with exponential jumps. The method of proof is based on reducing the initial problems to integro-differential free-boundary problems where the normal reflection and smooth fit may break down and the latter then be replaced by the continuous fit. The results can be interpreted as pricing perpetual American lookback options with fixed and floating strikes in a jump-diffusion model.

1 Introduction

The main aim of this paper is to present solutions to the discounted optimal stopping problems (2.4) and (5.1) for the maximum associated with the process X defined in (2.1) that solves the stochastic differential equation (2.2) driven by a Brownian motion and a compound Poisson process with exponentially distributed jumps. These problems are related to the option pricing theory in mathematical finance, where the process X can describe the price of a risky asset (e.g., a stock) on a financial market. In that case the values (2.4) and (5.1) can be formally interpreted as *fair prices* of *perpetual lookback options* of American type with *fixed* and *floating strikes* in a jump-diffusion market model, respectively. For a continuous model the problems (2.4) and (5.1) were solved by Pedersen [21], Guo and Shepp [13], and Beibel and Lerche [4].

Observe that when $K = 0$ the problems (2.4) and (5.1) turn into the classical Russian option problem introduced and explicitly solved by Shepp and Shiryaev [30] by means of reducing the initial problem to an optimal stopping problem for a (continuous) two-dimensional Markov process and solving the latter problem using the smooth-fit and normal-reflection conditions. It was further observed in [31] that the change-of-measure theorem allows to reduce the Russian option problem to a one-dimensional optimal stopping problem that explained the simplicity of the solution in [30]. Building on the optimal stopping analysis of Shepp and Shiryaev [30]-[31], Duffie and Harrison [7] derived a rational economic value for the Russian option and then extended their arbitrage arguments to perpetual lookback options. More recently, Shepp, Shiryaev and Sulem [32] proposed a barrier version of the Russian option where the decision about stopping should be taken before the price process reaches a 'dangerous' positive level. Peskir [24] presented a solution to the Russian option problem in the finite horizon case (see also [8] for a numeric algorithm for solving the corresponding free-boundary problem and [10] for a study of asymptotic behavior of the optimal stopping boundary near expiration).

In the recent years, the Russian option problem in models with jumps was studied quite extensively. Gerber, Michaud and Shiu [12] and then Mordecki and Moreira [20] obtained closed form solutions to the perpetual Russian option problems for diffusions with negative exponential jumps. Asmussen, Avram and Pistorius [2] derived explicit expressions for the prices of perpetual Russian options in the dense class of Lévy processes with phase-type jumps in both directions by reducing the initial problem to the first passage time problem and solving the latter by martingale stopping and Wiener-Hopf factorization. Avram, Kyprianou and Pistorius [3] studied exit problems for spectrally negative Lévy processes and applied the results to solving optimal stopping problems associated with perpetual Russian and American put options.

In contrast to the Russian option problem, the problem (2.4) is necessarily *two-dimensional* in the sense that it cannot be reduced to an optimal stopping problem for a one-dimensional (time-homogeneous) Markov process. Some other two-dimensional optimal stopping problems for continuous processes were earlier considered in [6] and [22]. The main feature of the optimal stopping problems for the maximum process in continuous models is that the normal-reflection condition at the diagonal holds and the optimal boundary can be characterized as a unique solution of a (first-order) nonlinear ordinary differential equation (see, e.g., [6], [30]-[31], [22], [21] and [13]). The key point in solving optimal stopping problems for jump processes established in [25]-[26] is that the smooth fit at the optimal boundary may break down and then be replaced by the continuous fit (see also [1] for necessary and sufficient conditions for the occurrence of smooth-fit condition and references to the related literature and [27] for an extensive overview).

In the present paper we derive solutions to the problems (2.4) and (5.1) in a jump-diffusion model driven by a Brownian motion and a compound Poisson process with exponential jumps. Such model was considered in [18]-[19], [15]-[17] and [11] where the optimal stopping problems related to pricing American call and put options and convertible bonds were solved, respectively. We show that under some relationships on the parameters of the model the optimal stopping boundary can be uniquely determined as a component of a *two-dimensional system* of (first-order) nonlinear ordinary differential equations.

The paper is organized as follows. In Section 2, we formulate the optimal stopping problem for a two-dimensional Markov process related to the perpetual American *fixed-strike* lookback option problem and reduce it to an equivalent integro-differential free-boundary problem. In Section 3, we present a solution to the free-boundary problem and derive (first-order) nonlinear ordinary differential equations for the optimal stopping boundary under different relationships on the parameters of the model as well as specify the asymptotic behavior of the boundary. In Section 4, we verify that the solution of the free-boundary problem turns out to be a solution of the initial optimal stopping problem. In Section 5, we give some concluding remarks as well as present an *explicit solution* to the optimal stopping problem related to the perpetual American *floating-strike* lookback option problem. The main results of the paper are stated in Theorems 4.1 and 5.1.

2 Formulation of the problem

In this section we introduce the setting and notation of the two-dimensional optimal stopping problem which is related to the perpetual American fixed-strike lookback option problem and formulate an equivalent integro-differential free-boundary problem.

2.1. For a precise formulation of the problem let us consider a probability space $(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $B = (B_t)_{t \geq 0}$ and a jump process $J = (J_t)_{t \geq 0}$ defined by $J_t = \sum_{i=1}^{N_t} Y_i$, where $N = (N_t)_{t \geq 0}$ is a Poisson process of the intensity λ and $(Y_i)_{i \in \mathbb{N}}$ is a sequence of independent random variables exponentially distributed with parameter 1 (B, N and $(Y_i)_{i \in \mathbb{N}}$ are supposed to be independent). Assume that there exists a process $X = (X_t)_{t \geq 0}$ given by:

$$X_t = x \, \exp\Big(\big(r - \sigma^2/2 - \lambda\theta/(1-\theta) \big) \, t + \sigma \, B_t + \theta \, J_t \Big) \tag{2.1}$$

and hence solving the stochastic differential equation:

$$dX_t = rX_{t-} \, dt + \sigma X_{t-} \, dB_t + X_{t-} \int_0^\infty \Big(e^{\theta y} - 1 \Big) \left(\mu(dt, dy) - \nu(dt, dy) \right) \quad (X_0 = x) \tag{2.2}$$

where $\mu(dt, dy)$ is the measure of jumps of the process J with the compensator $\nu(dt, dy) = \lambda dt I(y > 0) e^{-y} dy$, and $x > 0$ is given and fixed. It can be assumed that the process X describes a stock price on a financial market, where $r > 0$ is the interest rate, and $\sigma \geq 0$ and $\theta < 1$, $\theta \neq 0$, are the volatilities of continuous and jump part, respectively. Note that the assumption $\theta < 1$ guarantees that the jumps of X are integrable and that is not a restriction. With the process X let us associate the *maximum* process $S = (S_t)_{t \geq 0}$ defined by:

$$S_t = \Big(\max_{0 \leq u \leq t} X_u \Big) \vee s \tag{2.3}$$

for an arbitrary $s \geq x > 0$. The main purpose of the present paper is to derive a solution to the optimal stopping problem for the time-homogeneous (strong) Markov process $(X, S) = (X_t, S_t)_{t \geq 0}$ given by:

$$V_*(x, s) = \sup_\tau E_{x,s} \big[e^{-(r+\delta)\tau} (S_\tau - K)^+ \big] \tag{2.4}$$

where the supremum is taken over all stopping times τ of the process X (i.e., stopping times with respect to the natural filtration of X), and $P_{x,s}$ is a probability measure under which the (two-dimensional) process (X, S) defined in (2.1)-(2.3) starts at

$(x,s) \in E$. Here by $E = \{(x,s) \mid 0 < x \le s\}$ we denote the state space of the process (X,S). The value (2.4) coincides with an *arbitrage-free price* of a fixed-strike lookback American option with the strike price $K > 0$ and the discounting rate $\delta > 0$ (see, e.g., [34]). Note that in the continuous case $\sigma > 0$ and $\theta = 0$ the problem (2.4) was solved in [21] and [13]. It is also seen that if $\sigma = 0$ and $0 < \theta < 1$ with $r - \lambda\theta/(1-\theta) \ge 0$, then the optimal stopping time in (2.4) is infinite.

2.2. Let us first determine the structure of the optimal stopping time in the problem (2.4).

Applying the arguments from [6; Subsection 3.2] and [22; Proposition 2.1] to the optimal stopping problem (2.4) we see that it is never optimal to stop when $X_t = S_t$ for $t \ge 0$ (this fact will be also proved independently below). It follows directly from the structure of (2.4) that it is never optimal to stop when $S_t \le K$ for $t \ge 0$. In other words, this shows that all points (x,s) from the set:

$$C' = \{(x,s) \in E \mid 0 < x \le s \le K\} \tag{2.5}$$

and from the diagonal $\{(x,s) \in E \mid x = s\}$ belong to the continuation region:

$$C_* = \{(x,s) \in E \mid V_*(x,s) > (s-K)^+\}. \tag{2.6}$$

Let us fix $(x,s) \in C_*$ and let $\tau_* = \tau_*(x,s)$ denote the optimal stopping time in (2.4). Then, taking some point (y,s) such that $0 < y \le s$, by virtue of the structure of optimal stopping problem (2.4) and (2.3) with (2.1) we get:

$$V_*(y,s) \ge E_{y,s}\big[e^{-\lambda\tau_*}(S_{\tau_*} - K)^+\big] \ge E_{x,s}\big[e^{-\lambda\tau_*}(S_{\tau_*} - K)^+\big] = V_*(x,s) > (s-K)^+. \tag{2.7}$$

These arguments together with the comments in [6; Subsection 3.3] and [22; Subsection 3.3] as well as the assumption that $V_*(x,s)$ is continuous show that there exists a function $g_*(s)$ for $s > K$ such that the continuation region (2.6) is an open set consisting of (2.5) and of the set:

$$C''_* = \{(x,s) \in E \mid g_*(s) < x \le s, s > K\} \tag{2.8}$$

while the stopping region is the closure of the set:

$$D_* = \{(x,s) \in E \mid 0 < x < g_*(s), s > K\}. \tag{2.9}$$

Let us now show that in (2.8)-(2.9) the function $g_*(s)$ is increasing on (K,∞) (this fact will be also proved independently below). Since in (2.4) the function $s-K$ is linear in s on (K,∞), by means of standard arguments it is shown that $V_*(x,s) - (s-K)$ is decreasing in s on (K,∞). Hence, if for given $(x,s) \in C''_*$ we take s' such that $K < s' < s$, then $V_*(x,s') - (s'-K) \ge V_*(x,s) - (s-K) > 0$ so that $(x,s') \in C''_*$, and thus the desired assertion follows.

Let us denote by $W_*(x,s)$ and a_*s the value function and the boundary of the optimal stopping problem related to the Russian option problem. It is easily seen that in case

$K = 0$ the function $W_*(x, s)$ coincides with (2.4) and (5.1), while under different relationships on the parameters of the model $a_* < 1$ can be uniquely determined by (5.11), (5.13), (5.15) and (5.17), respectively. Suppose that $g_*(s) > a_* s$ for some $s > K$. Then for any $x \in (a_* s, g_*(s))$ given and fixed we have $W_*(x, s) - K > s - K = V_*(x, s)$ contradicting the obvious fact that $W_*(x, s) - K \le V_*(x, s)$ for all $(x, s) \in E$ with $s > K$ as it is clearly seen from (2.4). Thus, we may conclude that $g_*(s) \le a_* s < s$ for all $s > K$.

2.3. Standard arguments imply that in this case the infinitesimal operator $\mathbb{L}$ of the process (X, S) acts on a function $F \in C^{2,1}(E)$ (or $F \in C^{1,1}(E)$ when $\sigma = 0$) according to the rule:

$$(\mathbb{L}F)(x, s) = (r+\zeta)x\, F_x(x, s) + \frac{\sigma^2}{2}x^2\, F_{xx}(x, s) + \int_0^\infty \Big(F\big(xe^{\theta y}, xe^{\theta y} \vee s\big) - F(x, s) \Big)\, \lambda e^{-y}\, dy \tag{2.10}$$

for all $0 < x < s$ with $\zeta = -\lambda\theta/(1-\theta)$. Using standard arguments based on the strong Markov property it follows that $V_* \in C^{2,1}(C_* \equiv C' \cup C''_*)$ (or $V_* \in C^{1,1}(C_* \equiv C' \cup C''_*)$ when $\sigma = 0$). In order to find analytic expressions for the unknown value function $V_*(x, s)$ from (2.4) and the unknown boundary $g_*(s)$ from (2.8)-(2.9) using the results of general theory of optimal stopping problems for Markov processes (see, e.g., [33; Chapter III, Section 8] or [27]) we can formulate the following *integro-differential free-boundary problem*:

$$(\mathbb{L}V)(x, s) = (r+\delta)V(x, s) \quad \text{for} \quad (x, s) \in C \equiv C' \cup C'' \tag{2.11}$$

$$V(x, s)\big|_{x=g(s)+} = s - K \quad (\textit{continuous fit}) \tag{2.12}$$

$$V(x, s) = (s-K)^+ \quad \text{for} \quad (x, s) \in D \tag{2.13}$$

$$V(x, s) > (s-K)^+ \quad \text{for} \quad (x, s) \in C \tag{2.14}$$

where C'' and D are defined as C''_* and D_* in (2.8) and (2.9) with $g(s)$ instead of $g_*(s)$, respectively, and (2.12) playing the role of instantaneous-stopping condition is satisfied for all $s > K$. Observe that the superharmonic characterization of the value function (see [9] and [33]) implies that $V_*(x, s)$ is the smallest function satisfying (2.11)-(2.13) with the boundary $g_*(s)$. Moreover, under some relationships on the parameters of the model which are specified below, the following conditions can be satisfied or break down:

$$V_x(x, s)\big|_{x=g(s)+} = 0 \quad (\textit{smooth fit}) \tag{2.15}$$

$$V_s(x, s)\big|_{x=s-} = 0 \quad (\textit{normal reflection}) \tag{2.16}$$

for all $s > K$. Note that in the case $\sigma > 0$ and $\theta = 0$ the free-boundary problem (2.11)-(2.16) was solved in [21] and [13].

2.4. In order to specify the boundary $g_*(s)$ as a solution of the free-boundary problem (2.11)-(2.14) and (2.15)-(2.16), for further considerations we need to observe that from (2.4) it follows that the inequalities:

$$0 \le \sup_\tau E_{x,s}\big[e^{-(r+\delta)\tau}\, S_\tau\big] - K \le \sup_\tau E_{x,s}\big[e^{-(r+\delta)\tau}\, (S_\tau - K)^+\big] \le \sup_\tau E_{x,s}\big[e^{-(r+\delta)\tau}\, S_\tau\big] \tag{2.17}$$

which are equivalent to:

$$0 \le W_*(x,s) - K \le V_*(x,s) \le W_*(x,s) \tag{2.18}$$

hold for all $(x,s) \in E$ with $s > K$. Thus, setting $x = s$ in (2.18) we get:

$$0 \le \frac{W_*(s,s)}{s} - \frac{K}{s} \le \frac{V_*(s,s)}{s} \le \frac{W_*(s,s)}{s} \tag{2.19}$$

for all $s > K$ so that letting s go to infinity in (2.19) we obtain:

$$\liminf_{s\to\infty} \frac{V_*(s,s)}{s} = \limsup_{s\to\infty} \frac{V_*(s,s)}{s} = \lim_{s\to\infty} \frac{W_*(s,s)}{s}. \tag{2.20}$$

3 Solution of the free-boundary problem

In this section we obtain solutions to the free-boundary problem (2.11)-(2.16) and derive ordinary differential equations for the optimal boundary under different relationships on the parameters of the model (2.1)-(2.2).

3.1. By means of straightforward calculations we reduce equation (2.11) to the form:

$$(r+\zeta)x\, V_x(x,s) + \frac{\sigma^2}{2} x^2\, V_{xx}(x,s) - \alpha\lambda x^\alpha\, G(x,s) = (r+\delta+\lambda)V(x,s) \tag{3.1}$$

with $\alpha = 1/\theta$ and $\zeta = -\lambda\theta/(1-\theta)$, where taking into account conditions (2.12)-(2.13) we set:

$$G(x,s) = -\int_x^s V(z,s)\, \frac{dz}{z^{\alpha+1}} - \int_s^\infty V(z,z)\, \frac{dz}{z^{\alpha+1}} \;\text{ if }\; \alpha = 1/\theta > 1 \tag{3.2}$$

$$G(x,s) = \int_{g(s)}^x V(z,s)\, \frac{dz}{z^{\alpha+1}} - \frac{s-K}{\alpha g(s)^\alpha} \;\text{ if }\; \alpha = 1/\theta < 0 \tag{3.3}$$

for all $0 < x < g(s)$ and $s > K$. Then from (3.1) and (3.2)-(3.3) it follows that the function $G(x,s)$ solves the following (third-order) ordinary differential equation:

$$\begin{aligned} &\frac{\sigma^2}{2} x^3\, G_{xxx}(x,s) + \big[\sigma^2(\alpha+1) + r + \zeta\big] x^2\, G_{xx}(x,s) \\ &+ \left[(\alpha+1)\left(\frac{\sigma^2\alpha}{2} + r + \zeta\right) - (r+\delta+\lambda)\right] x\, G_x(x,s) - \alpha\lambda\, G(x,s) = 0 \end{aligned} \tag{3.4}$$

for $0 < x < g(s)$ and $s > K$, which has the following general solution:

$$G(x,s) = C_1(s)\, \frac{x^{\beta_1}}{\beta_1} + C_2(s)\, \frac{x^{\beta_2}}{\beta_2} + C_3(s)\, \frac{x^{\beta_3}}{\beta_3} \tag{3.5}$$

where $C_1(s)$, $C_2(s)$ and $C_3(s)$ are some arbitrary functions and $\beta_3 < \beta_2 < \beta_1$ are the real roots of the corresponding (characteristic) equation:

$$\frac{\sigma^2}{2}\,\beta^3 + \left[\sigma^2\left(\alpha - \frac{1}{2}\right) + r + \zeta\right]\beta^2 + \left[\alpha\left(\frac{\sigma^2(\alpha-1)}{2} + r + \zeta\right) - (r+\delta+\lambda)\right]\beta - \alpha\lambda = 0. \tag{3.6}$$

Therefore, differentiating both sides of the formulas (3.2)-(3.3) we get that the integro-differential equation (3.1) has the general solution:

$$V(x,s) = C_1(s)\,x^{\gamma_1} + C_2(s)\,x^{\gamma_2} + C_3(s)\,x^{\gamma_3} \tag{3.7}$$

where we set $\gamma_i = \beta_i + \alpha$ for $i = 1, 2, 3$. Further we assume that the functions $C_1(s)$, $C_2(s)$ and $C_3(s)$ as well as the boundary $g(s)$ are continuously differentiable for $s > K$. Observe that if $\sigma = 0$ and $r + \zeta < 0$ then it is seen that (3.4) degenerates into a second-order ordinary differential equation, and in that case we can set $C_3(s) \equiv 0$ in (3.5) as well as in (3.7), while the roots of equation (3.6) are explicitly given by:

$$\beta_i = \frac{r+\delta+\lambda}{2(r+\zeta)} - \frac{\alpha}{2} - (-1)^i\sqrt{\left(\frac{r+\delta+\lambda}{2(r+\zeta)} - \frac{\alpha}{2}\right)^2 + \frac{\alpha\lambda}{r+\zeta}} \tag{3.8}$$

for $i = 1, 2$.

3.2. Let us first determine the boundary $g_*(s)$ for the case $\sigma > 0$ and $\alpha = 1/\theta < 0$. Then we have $\beta_3 < 0 < \beta_2 < -\alpha < 1 - \alpha < \beta_1$ so that $\gamma_3 < \alpha < \gamma_2 < 0 < 1 < \gamma_1$ with $\gamma_i = \beta_i + \alpha$, where β_i for $i = 1, 2, 3$ are the roots of equation (3.6). Since in this case the process X can leave the part of continuation region $g_*(s) < x \le s$ and hits the diagonal $\{(x,s) \in E \,|\, x = s\}$ only continuously, we may assume that both the smooth-fit and normal-reflection conditions (2.15) and (2.16) are satisfied. Hence, applying conditions (3.3), (2.12) and (2.15) to the functions (3.5) and (3.7), we get that the following equalities hold:

$$C_1(s)\,\frac{g(s)^{\gamma_1}}{\beta_1} + C_2(s)\,\frac{g(s)^{\gamma_2}}{\beta_2} + C_3(s)\,\frac{g(s)^{\gamma_3}}{\beta_3} = -\frac{s-K}{\alpha} \tag{3.9}$$

$$C_1(s)\,g(s)^{\gamma_1} + C_2(s)\,g(s)^{\gamma_2} + C_3(s)\,g(s)^{\gamma_3} = s - K \tag{3.10}$$

$$\gamma_1 C_1(s)\,g(s)^{\gamma_1} + \gamma_2 C_2(s)\,g(s)^{\gamma_2} + \gamma_3 C_3(s)\,g(s)^{\gamma_3} = 0 \tag{3.11}$$

for $s > K$. Thus, by means of straightforward calculations, from (3.9)-(3.11) we obtain that the solution of system (2.11)-(2.13)+(2.15) takes the form:

$$\begin{aligned} V(x,s;g(s)) = {} & \frac{\beta_1\gamma_2\gamma_3(s-K)/\alpha}{(\gamma_2-\gamma_1)(\gamma_1-\gamma_3)}\left(\frac{x}{g(s)}\right)^{\gamma_1} \\ & + \frac{\beta_2\gamma_1\gamma_3(s-K)/\alpha}{(\gamma_2-\gamma_1)(\gamma_3-\gamma_2)}\left(\frac{x}{g(s)}\right)^{\gamma_2} \\ & + \frac{\beta_3\gamma_1\gamma_2(s-K)/\alpha}{(\gamma_1-\gamma_3)(\gamma_3-\gamma_2)}\left(\frac{x}{g(s)}\right)^{\gamma_3} \end{aligned} \tag{3.12}$$

for $0<x<g(s)$ and $s>K$. Then applying condition (2.16) to the function (3.7) we get:

$$C_1'(s)\, s^{\gamma_1} + C_2'(s)\, s^{\gamma_2} + C_3'(s)\, s^{\gamma_3} = 0 \tag{3.13}$$

from where using the solution of system (3.9)-(3.11) it follows that the function $g(s)$ solves the following (first-order) ordinary differential equation:

$$\begin{aligned} g'(s) = &\ \frac{g(s)}{\gamma_1\gamma_2\gamma_3(s-K)} \\ \times &\ \frac{\beta_1\gamma_2\gamma_3(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1} - \beta_2\gamma_1\gamma_3(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2} + \beta_3\gamma_1\gamma_2(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3}}{\beta_1(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1} - \beta_2(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2} + \beta_3(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3}} \end{aligned} \tag{3.14}$$

for $s>K$ with $\gamma_i=\beta_i+\alpha$, where β_i for $i=1,2,3$ are the roots of equation (3.6). By means of standard arguments it can be shown that the right-hand side of equation (3.14) is positive so that the function $g(s)$ is strictly increasing on (K,∞).

Let us denote $h_*(s)=g_*(s)/s$ for all $s>K$ and set $\overline{h}=\limsup_{s\to\infty} h_*(s)$ and $\underline{h}=\liminf_{s\to\infty} h_*(s)$. In order to specify the solution of equation (3.14) which coincides with the optimal stopping boundary $g_*(s)$, we observe that from the expression (3.12) it follows that (2.20) directly implies:

$$\begin{aligned} &\beta_1\gamma_2\gamma_3(\gamma_3-\gamma_2)\underline{h}^{-\gamma_1} + \beta_2\gamma_1\gamma_3(\gamma_1-\gamma_3)\overline{h}^{-\gamma_2} + \beta_3\gamma_1\gamma_2(\gamma_2-\gamma_1)\overline{h}^{-\gamma_3} \\ &= \beta_1\gamma_2\gamma_3(\gamma_3-\gamma_2)\overline{h}^{-\gamma_1} + \beta_2\gamma_1\gamma_3(\gamma_1-\gamma_3)\underline{h}^{-\gamma_2} + \beta_3\gamma_1\gamma_2(\gamma_2-\gamma_1)\underline{h}^{-\gamma_3} \\ &= \beta_1\gamma_2\gamma_3(\gamma_3-\gamma_2)a_*^{-\gamma_1} + \beta_2\gamma_1\gamma_3(\gamma_1-\gamma_3)a_*^{-\gamma_2} + \beta_3\gamma_1\gamma_2(\gamma_2-\gamma_1)a_*^{-\gamma_3} \end{aligned} \tag{3.15}$$

where a_* is uniquely determined by (5.11) under $K=0$. Then, using the fact that $h_*(s)=g_*(s)/s\le a_*$ for $s>K$ and thus $\underline{h}\le\overline{h}\le a_*<1$, from (3.15) we get that $\underline{h}=\overline{h}=a_*$. Hence, we obtain that the optimal boundary $g_*(s)$ should satisfy the property:

$$\lim_{s\to\infty}\frac{g_*(s)}{s}=a_* \tag{3.16}$$

which gives a condition on the infinity for the equation (3.14). By virtue of the results on the existence and uniqueness of solutions for first-order ordinary differential equations, we may therefore conclude that condition (3.16) uniquely specifies the solution of equation (3.14) that corresponds to the problem (2.4). Taking into account the expression (3.12), we also note that from inequalities (2.18) it follows that the optimal boundary $g_*(s)$ satisfies the properties:

$$g_*(K+)=0 \quad \text{and} \quad g_*(s)\sim A_*(s-K)^{1/\gamma_1} \quad \text{under} \quad s\downarrow K \tag{3.17}$$

for some constant $A_*>0$ which can be also determined by means of condition (3.16) above.

3.3. Let us now determine the boundary $g_*(s)$ for the case $\sigma=0$ and $\alpha=1/\theta<0$. Then we have $0<\beta_2<-\alpha<1-\alpha<\beta_1$ so that $\alpha<\gamma_2<0<1<\gamma_1$ with $\gamma_i=\beta_i+\alpha$, where β_i for $i=1,2$ are given by (3.6). In this case, applying conditions

(3.3) and (2.12) to the functions (3.5) and (3.7) with $C_3(s) \equiv 0$, we get that the following equalities hold:

$$C_1(s)\,\frac{g(s)^{\gamma_1}}{\beta_1} + C_2(s)\,\frac{g(s)^{\gamma_2}}{\beta_2} = -\frac{s-K}{\alpha} \tag{3.18}$$

$$C_1(s)\,g(s)^{\gamma_1} + C_2(s)\,g(s)^{\gamma_2} = s-K \tag{3.19}$$

for $s > K$. Thus, by means of straightforward calculations, from (3.18)-(3.19) we obtain that the solution of system (2.11)-(2.13) takes the form:

$$V(x,s;g(s)) = \frac{\beta_1\gamma_2(s-K)}{\alpha(\gamma_1-\gamma_2)}\Big(\frac{x}{g(s)}\Big)^{\gamma_1} - \frac{\beta_2\gamma_1(s-K)}{\alpha(\gamma_1-\gamma_2)}\Big(\frac{x}{g(s)}\Big)^{\gamma_2} \tag{3.20}$$

for $0 < x < g(s)$ and $s > K$. Since in this case $r + \zeta > 0$ so that the process X hits the diagonal $\{(x,s) \in E \,|\, x = s\}$ only continuously, we may assume that the normal-reflection condition (2.16) holds. Hence, applying condition (2.16) to the function (3.7) with $C_3(s) \equiv 0$, we get:

$$C_1'(s)\,s^{\gamma_1} + C_2'(s)\,s^{\gamma_2} = 0 \tag{3.21}$$

from where using the solution of system (3.18)-(3.19) it follows that the function $g(s)$ solves the differential equation:

$$g'(s) = \frac{g(s)}{\gamma_1\gamma_2(s-K)}\,\frac{\beta_1\gamma_2(s/g(s))^{\gamma_1} - \beta_2\gamma_1(s/g(s))^{\gamma_2}}{\beta_1(s/g(s))^{\gamma_1} - \beta_2(s/g(s))^{\gamma_2}} \tag{3.22}$$

for $s > K$ with $\gamma_i = \beta_i + \alpha$, where β_i for $i = 1, 2$ are given by (3.8). By means of standard arguments it can be shown that the right-hand side of equation (3.22) is positive so that the function $g(s)$ is strictly increasing on (K, ∞). Note that in this case the smooth-fit condition (2.15) fails to hold, that can be explained by the fact that leaving the part of continuation region $g_*(s) < x \le s$ the process X can pass through the boundary $g_*(s)$ only by jumping. Such an effect was earlier observed in [25]-[26] by solving some other optimal stopping problems for jump processes. According to the results in [1] we may conclude that this property appears because of finite intensity of jumps and exponential distribution of jump sizes of the compound Poisson process J.

Let us recall that $\overline{h} = \limsup_{s\to\infty} h_*(s)$ and $\underline{h} = \liminf_{s\to\infty} h_*(s)$ with $h_*(s) = g_*(s)/s$ for all $s > K$. In order to specify the solution of equation (3.22) which coincides with the optimal stopping boundary $g_*(s)$, we observe that from the expression (3.20) it follows that (2.20) directly implies:

$$\beta_1\gamma_2\overline{h}^{-\gamma_1} - \beta_2\gamma_1\underline{h}^{-\gamma_2} = \beta_1\gamma_2\underline{h}^{-\gamma_1} - \beta_2\gamma_1\overline{h}^{-\gamma_2} = \beta_1\gamma_2 a_*^{-\gamma_1} - \beta_2\gamma_1 a_*^{-\gamma_2} \tag{3.23}$$

where a_* is uniquely determined by (5.13) under $K = 0$. Then, using the fact that $h_*(s) = g_*(s)/s \le a_*$ for $s > K$ and thus $\underline{h} \le \overline{h} \le a_* < 1$, from (3.23) we get that $\underline{h} = \overline{h} = a_*$. Hence, we obtain that the optimal boundary $g_*(s)$ should satisfy the property (3.16) which gives a condition on the infinity for the equation (3.22).

By virtue of the results on the existence and uniqueness of solutions for first-order ordinary differential equations, we may therefore conclude that condition (3.16) uniquely specifies the solution of equation (3.22) that corresponds to the problem (2.4). Taking into account the expression (3.20), we also note that from inequalities (2.18) it follows that the optimal boundary $g_*(s)$ satisfies the properties (3.17) for some constant $A_* > 0$ which can be also determined by means of condition (3.16) above.

3.4. Let us now determine the optimal boundary $g_*(s)$ for the case $\sigma > 0$ and $\alpha = 1/\theta > 1$. Then we have $\beta_3 < -\alpha < 1 - \alpha < \beta_2 < 0 < \beta_1$ so that $\gamma_3 < 0 < 1 < \gamma_2 < \alpha < \gamma_1$ with $\gamma_i = \beta_i + \alpha$, where β_i for $i = 1, 2, 3$ are the roots of equation (3.6). By virtue of the same arguments as mentioned above, in this case we may also assume that both the smooth-fit and normal-reflection conditions (2.15) and (2.16) hold. Hence, applying conditions (3.3), (2.12) and (2.15) to the functions (3.5) and (3.7), respectively, we get that the following equalities hold:

$$C_1(s)\,\frac{s^{\gamma_1}}{\beta_1} + C_2(s)\,\frac{s^{\gamma_2}}{\beta_2} + C_3(s)\,\frac{s^{\gamma_3}}{\beta_3} = f(s) s^{\alpha} (s-K) \tag{3.24}$$

$$C_1(s)\, g(s)^{\gamma_1} + C_2(s)\, g(s)^{\gamma_2} + C_3(s)\, g(s)^{\gamma_3} = s - K \tag{3.25}$$

$$\gamma_1 C_1(s)\, g(s)^{\gamma_1} + \gamma_2 C_2(s)\, g(s)^{\gamma_2} + \gamma_3 C_3(s)\, g(s)^{\gamma_3} = 0 \tag{3.26}$$

where we set:

$$f(s) = -\frac{1}{s-K} \int_s^{\infty} V(z,z)\, \frac{dz}{z^{\alpha+1}} \tag{3.27}$$

for $s > K$. Thus, by means of straightforward calculations, from (3.24)-(3.26) we obtain that the solution of system (2.11)-(2.13)+(2.15) takes the form:

$$\begin{aligned} &V(x,s;g(s)) \qquad\qquad (3.28) \\ &= \frac{\beta_1 (s-K)[\beta_2\beta_3(\gamma_2-\gamma_3) s^{\alpha} f(s) + \beta_3\gamma_3 (s/g(s))^{\gamma_2} - \beta_2\gamma_2 (s/g(s))^{\gamma_3}]}{\beta_2\beta_3(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3}} \Big(\frac{x}{g(s)}\Big)^{\gamma_1} \\ &+ \frac{\beta_2 (s-K)[\beta_1\beta_3(\gamma_3-\gamma_1) s^{\alpha} f(s) - \beta_3\gamma_3 (s/g(s))^{\gamma_1} + \beta_1\gamma_1 (s/g(s))^{\gamma_3}]}{\beta_2\beta_3(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3}} \Big(\frac{x}{g(s)}\Big)^{\gamma_2} \\ &+ \frac{\beta_3 (s-K)[\beta_1\beta_2(\gamma_1-\gamma_2) s^{\alpha} f(s) + \beta_2\gamma_2 (s/g(s))^{\gamma_1} - \beta_1\gamma_1 (s/g(s))^{\gamma_2}]}{\beta_2\beta_3(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3}} \Big(\frac{x}{g(s)}\Big)^{\gamma_3} \end{aligned}$$

for $0 < x < g(s)$ and $s > K$. Inserting the expressions (3.5) and (3.7) into the formula (3.2), letting $x = s$ and differentiating the both sides of the obtained equality, we get:

$$C_1'(s)\,\frac{s^{\gamma_1}}{\beta_1} + C_2'(s)\,\frac{s^{\gamma_2}}{\beta_2} + C_3'(s)\,\frac{s^{\gamma_3}}{\beta_3} = 0 \tag{3.29}$$

from where using the solution of system (3.24)-(3.26) it follows that the function

$f(s)$ solves the differential equation:

$$f'(s) = -\frac{f(s)}{s-K} \tag{3.30}$$
$$+ \frac{\beta_1\beta_2\beta_3 f(s)[(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1} - (\gamma_1-\gamma_3)(s/g(s))^{\gamma_2} + (\gamma_1-\gamma_2)(s/g(s))^{\gamma_3}]}{s[\beta_2\beta_3(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3}]}$$
$$+ \frac{\beta_3\gamma_3(\gamma_1-\gamma_2)(s/g(s))^{\gamma_1+\gamma_2} - \beta_2\gamma_2(\gamma_1-\gamma_3)(s/g(s))^{\gamma_1+\gamma_3} + \beta_1\gamma_1(\gamma_2-\gamma_3)(s/g(s))^{\gamma_2+\gamma_3}}{s^{\alpha+1}[\beta_2\beta_3(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3}]}$$

for $s > K$. Applying the condition (2.16) to the function (3.7), we get that the equality (3.13) holds, from where it follows that the function $g(s)$ solves the differential equation:

$$g'(s) = \frac{g(s)}{s-K} \tag{3.31}$$
$$\times \frac{\beta_3\gamma_3(\gamma_1-\gamma_2)(s/g(s))^{\gamma_1+\gamma_2} - \beta_2\gamma_2(\gamma_1-\gamma_3)(s/g(s))^{\gamma_1+\gamma_3} + \beta_1\gamma_1(\gamma_2-\gamma_3)(s/g(s))^{\gamma_2+\gamma_3}}{\beta_3(\gamma_1-\gamma_2)(s/g(s))^{\gamma_1+\gamma_2} - \beta_2(\gamma_1-\gamma_3)(s/g(s))^{\gamma_1+\gamma_3} + \beta_1(\gamma_2-\gamma_3)(s/g(s))^{\gamma_2+\gamma_3}}$$
$$\times \frac{\beta_2\beta_3(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3}}{\eta_2\eta_3(\gamma_2-\gamma_3)(s/g(s))^{\gamma_1} - \eta_1\eta_3(\gamma_1-\gamma_3)(s/g(s))^{\gamma_2} + \eta_1\eta_2(\gamma_1-\gamma_2)(s/g(s))^{\gamma_3} - \rho f(s)s^{\alpha}}$$

for $s > K$ with $\eta_i = \beta_i\gamma_i$ for $i = 1, 2, 3$, and $\rho = \beta_1\beta_2\beta_3(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)(\gamma_2-\gamma_3)$.

In order to specify the solution of equation (3.30) let us define the function:

$$f_*(s) = -\frac{1}{s-K}\int_s^\infty V_*(z,z)\,\frac{dz}{z^{\alpha+1}} \tag{3.32}$$

for all $s > K$. Then by virtue of the inequalities (2.18), using the expression (5.14) we obtain the function (3.32) is well-defined and should satisfy the property:

$$\begin{aligned}\lim_{s\to\infty} f_*(s)\,s^{\alpha} &= \gamma_2(\gamma_3-1)/[(\gamma_2-\gamma_1)(\beta_1(\gamma_3-1)a_*^{\gamma_1} - \beta_3(\gamma_1-1)a_*^{\gamma_3})] \\ &+ \gamma_3(\gamma_1-1)/[(\gamma_3-\gamma_2)(\beta_2(\gamma_1-1)a_*^{\gamma_2} - \beta_1(\gamma_2-1)a_*^{\gamma_1})] \\ &+ \gamma_1(\gamma_2-1)/[(\gamma_1-\gamma_3)(\beta_3(\gamma_2-1)a_*^{\gamma_3} - \beta_2(\gamma_3-1)a_*^{\gamma_2})]\end{aligned} \tag{3.33}$$

where a_* is uniquely determined by (5.15) under $K = 0$. From (3.27) and (3.32) it therefore follows that (3.33) gives a condition on the infinity for the equation (3.30).

Let us recall that $\overline{h} = \limsup_{s\to\infty} h_*(s)$ and $\underline{h} = \liminf_{s\to\infty} h_*(s)$ with $h_*(s) = g_*(s)/s$ for all $s > K$. In order to specify the solution of equation (3.31) which coincides with the optimal stopping boundary $g_*(s)$, we observe that from the expressions (3.28) and (3.33) it follows that (2.20) directly implies:

$$\begin{aligned}&\frac{(\gamma_2-\gamma_3)\overline{h}^{-\gamma_1} + (\gamma_3-\gamma_1)\underline{h}^{-\gamma_2} + (\gamma_1-\gamma_2)\underline{h}^{-\gamma_3}}{\beta_2\beta_3(\gamma_2-\gamma_3)\underline{h}^{-\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)\underline{h}^{-\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)\underline{h}^{-\gamma_3}} \\ &= \frac{(\gamma_2-\gamma_3)\underline{h}^{-\gamma_1} + (\gamma_3-\gamma_1)\overline{h}^{-\gamma_2} + (\gamma_1-\gamma_2)\overline{h}^{-\gamma_3}}{\beta_2\beta_3(\gamma_2-\gamma_3)\overline{h}^{-\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)\overline{h}^{-\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)\overline{h}^{-\gamma_3}} \\ &= \frac{(\gamma_2-\gamma_3)a_*^{-\gamma_1} + (\gamma_3-\gamma_1)a_*^{-\gamma_2} + (\gamma_1-\gamma_2)a_*^{-\gamma_3}}{\beta_2\beta_3(\gamma_2-\gamma_3)a_*^{-\gamma_1} - \beta_1\beta_3(\gamma_1-\gamma_3)a_*^{-\gamma_2} + \beta_1\beta_2(\gamma_1-\gamma_2)a_*^{-\gamma_3}}.\end{aligned} \tag{3.34}$$

Then, using the fact that $h_*(s) = g_*(s)/s \le a_*$ for $s > K$ and thus $\underline{h} \le \overline{h} \le a_* < 1$, from (3.34) we get that $\underline{h} = \overline{h} = a_*$. Hence, we obtain that the optimal boundary $g_*(s)$ should satisfy the property (3.16) which gives a condition on the infinity for the equation (3.31). By virtue of the results on the existence and uniqueness of solutions for systems of first-order ordinary differential equations, we may therefore conclude that conditions (3.33) and (3.16) uniquely specifies the solution of system (3.30)+(3.31) that corresponds to the problem (2.4). Taking into account the expression (3.28), we also note that from inequalities (2.18) it follows that the optimal boundary $g_*(s)$ satisfies the properties (3.17) for some constant $A_* > 0$ which can be also determined by means of the condition (3.16) above.

3.5. Let us finally determine the boundary $g_*(s)$ for the case $\sigma = 0$ and $\alpha = 1/\theta > 1$ with $r + \zeta = r - \lambda\theta/(1-\theta) < 0$. Then we have $\beta_2 < -\alpha < 1 - \alpha < \beta_1 < 0$ so that $\gamma_2 < 0 < 1 < \gamma_1$ with $\gamma_i = \beta_i + \alpha$, where β_i for $i = 1, 2$ are given by (3.6). Since in this case the process X can leave the continuation region $g_*(s) < x \le s$ only continuously, we may assume that the smooth-fit condition (2.15) holds. Hence, applying conditions (2.12) and (2.15) to the function (3.7), we get that the following equalities hold:

$$C_1(s)\, g(s)^{\gamma_1} + C_2(s)\, g(s)^{\gamma_2} = s - K \tag{3.35}$$

$$\gamma_1 C_1(s)\, g(s)^{\gamma_1} + \gamma_2 C_2(s)\, g(s)^{\gamma_2} = 0 \tag{3.36}$$

for $s > K$. Thus, by means of straightforward calculations, from (3.35)-(3.36) we obtain that the solution of system (2.11)-(2.13)+(2.15) takes the form:

$$V(x, s; g(s)) = \frac{\gamma_2(s-K)}{\gamma_2 - \gamma_1} \left(\frac{x}{g(s)}\right)^{\gamma_1} - \frac{\gamma_1(s-K)}{\gamma_2 - \gamma_1} \left(\frac{x}{g(s)}\right)^{\gamma_2} \tag{3.37}$$

for $0 < x < g(s)$ and $s > K$. Inserting the expressions (3.5) and (3.7) with $C_3(s) \equiv 0$ into the formula (3.2), letting $x = s$ and differentiating the both sides of the obtained equality, we get:

$$C_1'(s)\, \frac{s^{\gamma_1}}{\beta_1} + C_2'(s)\, \frac{s^{\gamma_2}}{\beta_2} = 0 \tag{3.38}$$

from where using the solution of system (3.35)-(3.36) it follows that the function $g(s)$ satisfies the differential equation:

$$g'(s) = \frac{g(s)}{\gamma_1\gamma_2(s-K)}\, \frac{\beta_2\gamma_2(s/g(s))^{\gamma_1} - \beta_1\gamma_1(s/g(s))^{\gamma_2}}{\beta_2(s/g(s))^{\gamma_1} - \beta_1(s/g(s))^{\gamma_2}} \tag{3.39}$$

for $s > K$ with $\gamma_i = \beta_i + \alpha$, where β_i for $i = 1, 2$ are given by (3.8). By means of standard arguments it can be shown that the right-hand side of equation (3.39) is positive so that the function $g(s)$ is strictly increasing on (K, ∞). Note that in this case the normal-reflection condition (2.16) fails to hold, that can be explained by the fact that the process X can hit the diagonal $\{(x, s) \in E \,|\, x = s\}$ only by jumping.

Let us recall that $\overline{h} = \limsup_{s\to\infty} h_*(s)$ and $\underline{h} = \liminf_{s\to\infty} h_*(s)$ with $h_*(s) = g_*(s)/s$ for all $s > K$. In order to specify the solution of equation (3.39) which coincides with the optimal stopping boundary $g_*(s)$, we observe that from the expression (3.37) it follows that (2.20) directly implies:

$$\gamma_2 \overline{h}^{-\gamma_1} - \gamma_1 \underline{h}^{-\gamma_2} = \gamma_2 \underline{h}^{-\gamma_1} - \gamma_1 \overline{h}^{-\gamma_2} = \gamma_2 a_*^{-\gamma_1} - \gamma_1 a_*^{-\gamma_2} \tag{3.40}$$

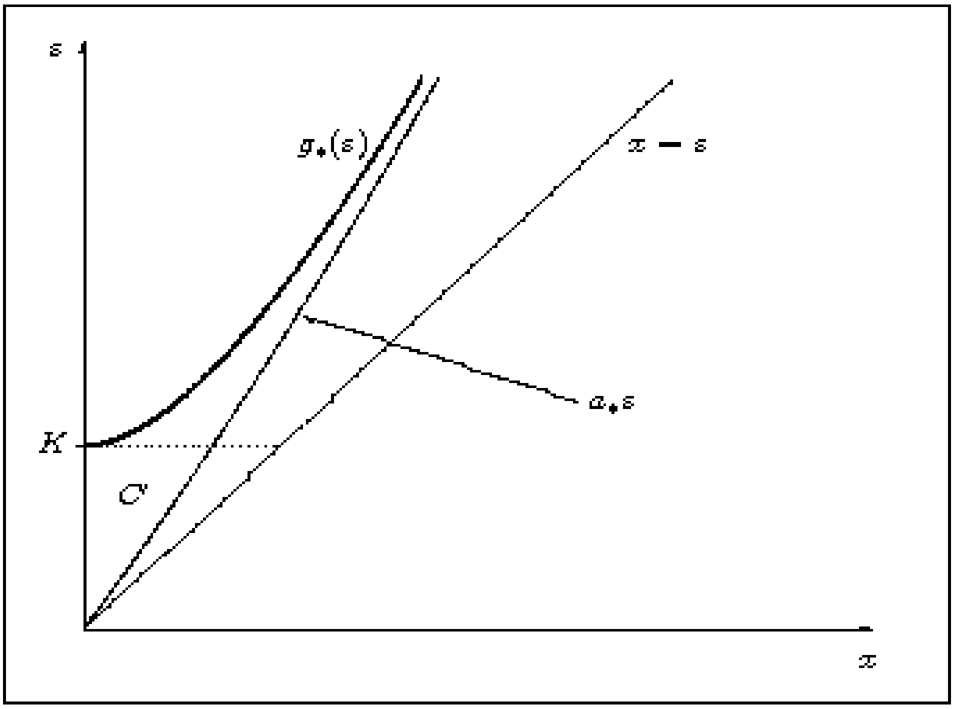

Figure 1. A computer drawing of the optimal stopping boundary $g_*(s)$.

where a_* is uniquely determined by (5.17) under $K = 0$. Then, using the fact that $h_*(s) = g_*(s)/s \le a_*$ for $s > K$ and thus $\underline{h} \le \overline{h} \le a_* < 1$, from (3.40) we get that $\underline{h} = \overline{h} = a_*$. Hence, we obtain that the optimal boundary $g_*(s)$ should satisfy the property (3.16) which gives a condition on the infinity for the equation (3.39). By virtue of the results on the existence and uniqueness of solutions for first-order ordinary differential equations, we may therefore conclude that condition (3.16) uniquely specifies the solution of equation (3.39) that corresponds to the problem (2.4). Taking into account the expression (3.37), we also note that from inequalities (2.18) it follows that the optimal boundary $g_*(s)$ satisfies the properties (3.17) for some constant $A_* > 0$ which can be also determined by means of the condition (3.16) above.

3.6. Observe that the arguments above show that if we start at the point $(x, s) \in C'$ then it is easily seen that the process (X, S) can be stopped optimally after it passes through the point (K, K). Thus, using standard arguments based on the strong Markov property it follows that:

$$V_*(x, s) = U(x; K)\, V_*(K, K) \tag{3.41}$$

for all $(x, s) \in C'$ with $V_*(K, K) = \lim_{s \downarrow K} V_*(K, s)$, where we set:

$$U(x; K) = E_x\big[e^{-(r+\delta)\sigma_*}\big] \tag{3.42}$$

and

$$\sigma_* = \inf\{t \geq 0 \mid X_t \geq K\}. \tag{3.43}$$

Here E_x denotes the expectation under the assumption that $X_0 = x$ for some $0 < x \leq K$.

By means of straightforward calculations based on solving the corresponding boundary value problem (see also [2]-[3] and [17]) it follows that when $\alpha = 1/\theta < 0$ holds, we have:

$$U(x; K) = \Big(\frac{x}{K}\Big)^{\gamma_1} \tag{3.44}$$

with $\gamma_1 = \beta_1 + \alpha$, where if $\sigma > 0$ then β_1 is the largest root of equation (3.6), while if $\sigma = 0$ then β_1 is given by (3.8). It also follows that when $\alpha = 1/\theta > 1$ holds, then we have:

$$U(x; K) = \frac{\beta_1 \gamma_2}{\alpha(\gamma_1 - \gamma_2)} \Big(\frac{x}{K}\Big)^{\gamma_1} - \frac{\beta_2 \gamma_1}{\alpha(\gamma_1 - \gamma_2)} \Big(\frac{x}{K}\Big)^{\gamma_2} \tag{3.45}$$

with $\gamma_i = \beta_i + \alpha$, where if $\sigma > 0$ then β_i for $i = 1, 2$ are the two largest roots of equation (3.6), while if $\sigma = 0$ and $r + \zeta = r - \lambda\theta/(1 - \theta) < 0$ then β_i for $i = 1, 2$ are given by (3.8).

4 Main result and proof

In this section using the facts proved above we formulate and prove the main result of the paper.

Theorem 4.1. *Let the process (X, S) be defined in (2.1)-(2.3). Then the value function of the problem (2.4) takes the expression:*

$$V_*(x, s) = \begin{cases} V(x, s; g_*(s)), & \textit{if} \quad g_*(s) < x < s \quad \textit{and} \quad s > K \\ U(x; K) V_*(K, K), & \textit{if} \quad 0 < x \leq s \leq K \\ s - K, & \textit{if} \quad 0 < x \leq g_*(s) \quad \textit{and} \quad s > K \end{cases} \tag{4.1}$$

[with $V_*(K, K) = \lim_{s \downarrow K} V_*(K, s)$] and the optimal stopping time is explicitly given by:

$$\tau_* = \inf\{t \geq 0 \,|\, X_t \leq g_*(S_t)\} \tag{4.2}$$

where the functions $V(x, s; g(s))$ and $U(x; K)$ as well as the increasing boundary $g_*(s) \leq a_* s < s$ for $s > K$ satisfying $g_*(K+) = 0$ and $g_*(s) \sim A_*(s - K)^{1/\gamma}$ under $s \downarrow K$ [see Figure 1 above] are specified as follows:

(i): if $\sigma > 0$ and $\theta < 0$ then $V(x,s;g(s))$ is given by (3.12), $U(x;K)$ is given by (3.44), and $g_*(s)$ is uniquely determined from the differential equation (3.14) and the condition (3.16), where $\gamma_i = \beta_i + 1/\theta$ and β_i for $i = 1,2,3$ are the roots of equation (3.6), while a_* is found from equation (5.11) under $K = 0$;

(ii): if $\sigma = 0$ and $\theta < 0$ then $V(x,s;g(s))$ is given by (3.20), $U(x;K)$ is given by (3.44), and $g_*(s)$ is uniquely determined from the differential equation (3.22) and the condition (3.16), where $\gamma_i = \beta_i + 1/\theta$ and β_i for $i = 1,2$ are given by (3.8), while a_* is found from equation (5.13) under $K = 0$;

(iii): if $\sigma > 0$ and $0 < \theta < 1$ then $V(x,s;g(s))$ is given by (3.28), $U(x;K)$ is given by (3.45), and $g_*(s)$ is uniquely determined from the system of differential equations (3.30)+(3.31) and the conditions (3.33)+(3.16), where $\gamma_i = \beta_i + 1/\theta$ and β_i for $i = 1,2,3$ are the roots of equation (3.6), while a_* is found from equation (5.15) under $K = 0$;

(iv): if $\sigma = 0$ and $0 < \theta < 1$ with $r - \lambda\theta/(1-\theta) < 0$ then $V(x,s;g(s))$ is given by (3.37), $U(x;K)$ is given by (3.45), and $g_*(s)$ is uniquely determined from the differential equation (3.39) and the condition (3.16), where $\gamma_i = \beta_i + 1/\theta$ and β_i for $i = 1,2$ are given by (3.8), while a_* is found from equation (5.17) under $K = 0$.

Proof. In order to verify the assertions stated above, it remains us to show that the function (4.1) coincides with the value function (2.4) and the stopping time τ_* from (4.2) with the boundary $g_*(s)$ specified above is optimal. For this, let us denote by $V(x,s)$ the right-hand side of the expression (4.1). In this case, by means of straightforward calculations and the assumptions above it follows that the function $V(x,s)$ solves the system (2.11)-(2.13), and condition (2.15) is satisfied when either $\sigma > 0$ or $r - \lambda\theta/(1-\theta) < 0$ holds, while condition (2.16) is satisfied when either $\sigma > 0$ or $\theta < 0$ holds. Then taking into account the fact that the boundary $g_*(s)$ is assumed to be continuously differentiable for $s > K$ and applying the change-of-variable formula from [23; Theorem 3.1] to $e^{-(r+\delta)t}V(X_t,S_t)$, we obtain:

$$\begin{aligned} e^{-(r+\delta)t}\,V(X_t,S_t) &= V(x,s) \\ &+ \int_0^t e^{-(r+\delta)u}\,(\mathbb{L}V-(r+\delta)V)(X_u,S_u)I(X_u\neq g_*(S_u))\,du \\ &+ \int_0^t e^{-(r+\delta)u}\,V_s(X_{u-},S_{u-})\,dS_u - \sum_{0<u\le t} e^{-(r+\delta)u}\,V_s(X_{u-},S_{u-})\,\Delta S_u + M_t \end{aligned} \tag{4.3}$$

where the process $(M_t)_{t\ge0}$ defined by:

$$\begin{aligned} M_t = &\int_0^t e^{-(r+\delta)u}\,V_x(X_{u-},S_{u-})\,\sigma X_{u-}\,dB_u \\ + &\int_0^t\int_0^\infty e^{-(r+\delta)u}\Big(V\big(X_{u-}e^{\theta y},X_{u-}e^{\theta y}\vee S_{u-}\big)-V(X_{u-},S_{u-})\Big)\,(\mu(du,dy)-\nu(du,dy)) \end{aligned} \tag{4.4}$$

is a local martingale under $P_{x,s}$. Observe that when either $\sigma > 0$ or $0 < \theta < 1$, the time spent by the process X at the diagonal $\{(x,s)\in E \mid 0 < x \le s\}$ is of Lebesgue measure zero that allows to extend $(\mathbb{L}V - (r+\delta)V)(x, s)$ arbitrarily to $x = s$. When either $\sigma > 0$ or $\theta < 0$, the time spent by the process X at the boundary $g_*(S)$ is of Lebesgue measure zero that allows to extend $(\mathbb{L}V - (r+\delta)V)(x, s)$ to $x = g_*(s)$ and set the indicator in the formula (4.3) to one. Note that when either $\sigma > 0$ or $\theta < 0$, the process S increases only continuously, and hence in (4.3) the sum with respect to ΔS_u is zero and the same is the integral with respect to dS_u, since at the diagonal $\{(x, s) \in E \mid x = s\}$ we assume (2.16). When $\sigma = 0$ and $0 < \theta < 1$, the process S increases only by jumping, and thus in (4.3) the integral with respect to dS_u is compensated by the sum with respect to ΔS_u.

By virtue of the arguments from the previous section we may conclude that $(\mathbb{L}V - (r+\delta)V)(x, s) \le 0$ for all $(x, s) \in E$. Moreover, by means of straightforward calculations it can be shown that the property (2.14) also holds that together with (2.12)-(2.13) yields $V(x, s) \ge (s - K)^+$ for all $(x, s) \in E$. From the expression (4.3) it therefore follows that the inequalities:

$$e^{-(r+\delta)\tau} (S_\tau - K)^+ \le e^{-(r+\delta)\tau} V(X_\tau, S_\tau) \le V(x, s) + M_\tau \tag{4.5}$$

hold for any finite stopping time τ of the process X.

Let $(\sigma_n)_{n \in \mathbb{N}}$ be an arbitrary localizing sequence of stopping times for the process $(M_t)_{t \ge 0}$. Then taking in (4.5) expectation with respect to $P_{x,s}$, by means of the optional sampling theorem we get:

$$\begin{aligned} &E_{x,s}\big[e^{-(r+\delta)(\tau\wedge\sigma_n)} (S_{\tau\wedge\sigma_n} - K)^+\big] \le E_{x,s}\big[e^{-(r+\delta)(\tau\wedge\sigma_n)} V(X_{\tau\wedge\sigma_n}, S_{\tau\wedge\sigma_n})\big] \\ &\le V(x, s) + E_{x,s}\big[M_{\tau\wedge\sigma_n}\big] = V(x, s) \end{aligned} \tag{4.6}$$

for all $(x, s) \in E$. Hence, letting n go to infinity and using Fatou's lemma, we obtain that for any finite stopping time τ the inequalities:

$$E_{x,s}\big[e^{-(r+\delta)\tau} (S_\tau - K)^+\big] \le E_{x,s}\big[e^{-(r+\delta)\tau} V(X_\tau, S_\tau)\big] \le V(x, s) \tag{4.7}$$

are satisfied for all $(x, s) \in E$.

By virtue of the fact that the function $V(x, s)$ together with the boundary $g_*(s)$ satisfy the system (2.11)-(2.14), by the structure of stopping time τ_* in (4.2) and the expression (4.3) it follows that the equality:

$$e^{-(r+\delta)(\tau_*\wedge\sigma_n)} V(X_{\tau_*\wedge\sigma_n}, S_{\tau_*\wedge\sigma_n}) = V(x, s) + M_{\tau_*\wedge\sigma_n} \tag{4.8}$$

holds. Then, using the expression (4.5), by virtue of the fact that the function $V(x, s)$ is increasing, we may conclude that the inequalities:

$$-V(x, s) \le M_{\tau_*\wedge\sigma_n} \le V(g_*(S_{\tau_*\wedge\sigma_n}), S_{\tau_*\wedge\sigma_n}) - V(x, s) \tag{4.9}$$

are satisfied for all $(x, s) \in E$, where $(\sigma_n)_{n \in \mathbb{N}}$ is a localizing sequence for $(M_t)_{t \ge 0}$. Taking into account conditions (3.16) and (3.33), from the structure of the functions (3.12), (3.20), (3.28) and (3.37) it follows that:

$$V(g_*(S_t), S_t) \le K' \, S_t \tag{4.10}$$

for some $K' > 0$. Hence, letting n go to infinity in the expression (4.8) and using the conditions (2.12)-(2.13) as well as the property:

$$E_{x,s}\Big[\sup_{t\geq 0} e^{-(r+\delta)t}\, S_t\Big] = E_{x,s}\Big[\sup_{t\geq 0} e^{-(r+\delta)t}\, X_t\Big] < \infty \tag{4.11}$$

(the latter can be proved by means of the same arguments as in [31] and using the fact that the processes B and J are independent and the jumps of J are integrable), by means of the Lebesgue dominated convergence theorem we obtain the equality:

$$E_{x,s}\big[e^{-(r+\delta)\tau_*}\,(S_{\tau_*} - K)^+\big] = V(x,s) \tag{4.12}$$

for all $(x,s) \in E$, from where the desired assertion follows directly. □

5 Conclusions

In this section we give some concluding remarks and present an explicit solution to the optimal stopping problem which is related to the perpetual American fixed-strike lookback option problem.

5.1. We have considered the perpetual fixed-strike lookback American option optimal stopping problem in a jump-diffusion model. In order to be able to derive (first-order) nonlinear differential equations for the optimal boundary that separates the continuation and stopping regions, we have let the jumps of the driving compound Poisson process be exponentially distributed. It was shown that not only the smooth-fit condition at the optimal boundary, but also the normal-reflection condition at the diagonal may break down because of the occurrence of jumps in the model. We have seen that under some relationships on the parameters of the model the optimal boundary can be found as a component of the solution of a two-dimensional system of ordinary differential equations that shows the difference of the jump-diffusion case from the continuous case. We have also derived special conditions that specify in the family of solutions of the system of nonlinear differential equations the unique solution that corresponds to the initial optimal stopping problem. The existence and uniqueness of such a solution can be obtained by standard methods of first-order ordinary differential equations.

In the rest of the paper we derive a solution to the floating-strike lookback American option problem in the jumps-diffusion model (2.1)-(2.3). In contrast to the fixed-strike case, by means of the change-of-measure theorem, the related two-dimensional optimal stopping problem can be reduced to an optimal stopping problem for a one-dimensional strong Markov process $(S_t/X_t)_{t\geq 0}$ that explains the simplisity of the structure of the solution in (5.18)-(5.19) (see [31] and [4]).

5.2. Let us now consider the following optimal stopping problem:

$$\widetilde{V}_*(x,s) = \sup_{\tau} E_{x,s}\big[e^{-(r+\delta)\tau}(S_\tau - KX_\tau)^+\big] \tag{5.1}$$

where the supremum is taken over all stopping times τ of the process X. The value (2.4) coincides with an *arbitrage-free price* of a floating-strike lookback American option (or 'partial lookback' as it is called in [5]) with $K > 0$ and the discounting rate $\delta > 0$. Note that in the continuous case $\sigma > 0$ and $\theta = 0$ the problem (5.1) was solved in [4]. It is also seen that if $\sigma = 0$ and $0 < \theta < 1$ with $r - \lambda\theta/(1-\theta) \geq 0$, then the optimal stopping time in (5.1) is infinite in case $K < 1$ and equals zero in case $K \geq 1$.

Using the same arguments as in [4] it can be shown that the continuation region for the problem (5.1) is an open set of the form:

$$\widetilde{C}_* = \{(x,s) \in E \mid b_* s < x \leq s\} \tag{5.2}$$

while the stopping region is the closure of the set:

$$\widetilde{D}_* = \{(x,s) \in E \mid 0 < x < b_* s\}. \tag{5.3}$$

From (5.1) it is easily seen that $b_* \leq 1/K$ in (5.2)-(5.3).

In order to find analytic expressions for the unknown value function $\widetilde{V}_*(x,s)$ from (5.1) and the unknown boundary $b_* s$ from (5.2)-(5.3), we can formulate the following *integro-differential free-boundary problem*:

$$(\mathbb{L}\widetilde{V})(x,s) = (r+\delta)\widetilde{V}(x,s) \quad \text{for} \quad (x,s) \in \widetilde{C} \tag{5.4}$$

$$\widetilde{V}(x,s)\big|_{x=bs+} = s(1-Kb) \quad (\textit{continuous fit}) \tag{5.5}$$

$$\widetilde{V}(x,s) = (s-Kx)^+ \quad \text{for} \quad (x,s) \in \widetilde{D} \tag{5.6}$$

$$\widetilde{V}(x,s) > (s-Kx)^+ \quad \text{for} \quad (x,s) \in \widetilde{C} \tag{5.7}$$

where $\widetilde{C}$ and $\widetilde{D}$ are defined as $\widetilde{C}_*$ and $\widetilde{D}_*$ in (5.2) and (5.3) with b instead of b_*, respectively, and (5.5) playing the role of instantaneous-stopping condition is satisfied for all $s > 0$. Moreover, under some relations on the parameters of the model which are specified below, the following conditions can be satisfied or break down:

$$\widetilde{V}_x(x,s)\big|_{x=bs+} = -K \quad (\textit{smooth fit}) \tag{5.8}$$

$$\widetilde{V}_s(x,s)\big|_{x=s-} = 0 \quad (\textit{normal reflection}) \tag{5.9}$$

for all $s > 0$. Note that in the case $\sigma > 0$ and $\theta = 0$ the free-boundary problem (5.4)-(5.9) was solved in [4].

Following the schema of arguments from the previous section, by means of straight-forward calculations it can be shown that in case $\sigma > 0$ and $\alpha = 1/\theta < 0$ the solution of system (5.4)-(5.7)+(5.8) takes the form:

$$\widetilde{V}(x,s;bs) = \frac{\beta_1[(1-\alpha)\gamma_2\gamma_3 + \alpha(\gamma_2-1)(\gamma_3-1)Kb]s}{\alpha(1-\alpha)(\gamma_2-\gamma_1)(\gamma_1-\gamma_3)}\Big(\frac{x}{bs}\Big)^{\gamma_1} + \frac{\beta_2[(1-\alpha)\gamma_1\gamma_3 + \alpha(\gamma_1-1)(\gamma_3-1)Kb]s}{\alpha(1-\alpha)(\gamma_2-\gamma_1)(\gamma_3-\gamma_2)}\Big(\frac{x}{bs}\Big)^{\gamma_2} + \frac{\beta_3[(1-\alpha)\gamma_1\gamma_2 + \alpha(\gamma_1-1)(\gamma_2-1)Kb]s}{\alpha(1-\alpha)(\gamma_1-\gamma_3)(\gamma_3-\gamma_2)}\Big(\frac{x}{bs}\Big)^{\gamma_3} \tag{5.10}$$

and from condition (5.9) it follows that b solves the equation:

$$\frac{\beta_1(\gamma_1-1)[(1-\alpha)\gamma_2\gamma_3 + \alpha(\gamma_2-1)(\gamma_3-1)Kb]}{(\gamma_2-\gamma_1)(\gamma_1-\gamma_3)b^{\gamma_1}} + \frac{\beta_2(\gamma_2-1)[(1-\alpha)\gamma_1\gamma_3 + \alpha(\gamma_1-1)(\gamma_3-1)Kb]}{(\gamma_2-\gamma_1)(\gamma_3-\gamma_2)b^{\gamma_2}} = \frac{\beta_3(\gamma_3-1)[(1-\alpha)\gamma_1\gamma_2 + \alpha(\gamma_1-1)(\gamma_2-1)Kb]}{(\gamma_3-\gamma_1)(\gamma_3-\gamma_2)b^{\gamma_3}}; \tag{5.11}$$

in case $\sigma = 0$ and $\alpha = 1/\theta < 0$ the solution of system (5.4)-(5.7) takes the form:

$$\widetilde{V}(x,s;bs) = \frac{\beta_1[(1-\alpha)\gamma_2 + \alpha(\gamma_2-1)Kb]s}{\alpha(1-\alpha)(\gamma_1-\gamma_2)}\Big(\frac{x}{bs}\Big)^{\gamma_1} - \frac{\beta_2[(1-\alpha)\gamma_1 + \alpha(\gamma_1-1)Kb]s}{\alpha(1-\alpha)(\gamma_1-\gamma_2)}\Big(\frac{x}{bs}\Big)^{\gamma_2} \tag{5.12}$$

and from condition (5.9) it follows that b solves the equation:

$$b^{\gamma_1-\gamma_2} = \frac{\beta_2(\gamma_2-1)}{\beta_1(\gamma_1-1)}\,\frac{(1-\alpha)\gamma_1 + \alpha(\gamma_1-1)Kb}{(1-\alpha)\gamma_2 + \alpha(\gamma_2-1)Kb}; \tag{5.13}$$

in case $\sigma > 0$ and $\alpha = 1/\theta > 1$ the solution of system (5.4)-(5.7)+(5.9) takes the form:

$$\widetilde{V}(x,s;bs) = \frac{\beta_1(\gamma_3-1)[\gamma_2-(\gamma_2-1)Kb]b^{\gamma_1}s}{(\gamma_2-\gamma_1)[\beta_1(\gamma_3-1)b^{\gamma_1}-\beta_3(\gamma_1-1)b^{\gamma_3}]}\Big(\frac{x}{bs}\Big)^{\gamma_1} + \frac{\beta_2(\gamma_1-1)[\gamma_3-(\gamma_3-1)Kb]b^{\gamma_2}s}{(\gamma_3-\gamma_2)[\beta_2(\gamma_1-1)b^{\gamma_2}-\beta_1(\gamma_2-1)b^{\gamma_1}]}\Big(\frac{x}{bs}\Big)^{\gamma_2} + \frac{\beta_3(\gamma_2-1)[\gamma_1-(\gamma_1-1)Kb]b^{\gamma_3}s}{(\gamma_1-\gamma_3)[\beta_3(\gamma_2-1)b^{\gamma_3}-\beta_2(\gamma_3-1)b^{\gamma_2}]}\Big(\frac{x}{bs}\Big)^{\gamma_3} \tag{5.14}$$

and from condition (5.8) it follows that b solves the equation:

$$\frac{\beta_1(\gamma_1-1)(\gamma_3-1)[\gamma_2-(\gamma_2-1)Kb]}{(\gamma_2-\gamma_1)[\beta_1(\gamma_3-1)b^{\gamma_1}-\beta_3(\gamma_1-1)b^{\gamma_3}]} + \frac{\beta_2(\gamma_1-1)(\gamma_2-1)[\gamma_3-(\gamma_3-1)Kb]}{(\gamma_3-\gamma_2)[\beta_2(\gamma_1-1)b^{\gamma_2}-\beta_1(\gamma_2-1)b^{\gamma_1}]} = \frac{\beta_3(\gamma_2-1)(\gamma_3-1)[\gamma_1-(\gamma_1-1)Kb]}{(\gamma_3-\gamma_1)[\beta_3(\gamma_2-1)b^{\gamma_3}-\beta_2(\gamma_3-1)b^{\gamma_2}]}; \tag{5.15}$$

while in case $\sigma = 0$ and $\alpha = 1/\theta > 1$ with $r + \zeta = r - \lambda\theta/(1-\theta) < 0$ the solution of system (5.4)-(5.7) takes the form:

$$\widetilde{V}(x, s; bs) = \frac{[\gamma_2 - (\gamma_2 - 1)Kb]s}{\gamma_2 - \gamma_1}\Big(\frac{x}{bs}\Big)^{\gamma_1} - \frac{[\gamma_1 - (\gamma_1 - 1)Kb]s}{\gamma_2 - \gamma_1}\Big(\frac{x}{bs}\Big)^{\gamma_2} \tag{5.16}$$

and from condition (5.8) it follows that b solves the equation:

$$b^{\gamma_1 - \gamma_2} = \frac{\beta_2}{\beta_1}\,\frac{\gamma_2(\gamma_1 - 1) + [\gamma_1 - \gamma_2(\gamma_1 - 1)]Kb}{\gamma_1(\gamma_2 - 1) + [\gamma_2 - \gamma_1(\gamma_2 - 1)]Kb}. \tag{5.17}$$

Summarizing the facts proved above we formulate the following assertion.

Theorem 5.1. *Let the process (X, S) be defined in (2.1)-(2.3). Then the value function of the problem (5.1) takes the expression:*

$$\widetilde{V}_*(x, s) = \begin{cases} \widetilde{V}(x, s; b_* s), & \text{if} \quad b_* s < x < s \\ s - Kx, & \text{if} \quad 0 < x \le b_* s \end{cases} \tag{5.18}$$

and the optimal stopping time is explicitly given by:

$$\widetilde{\tau}_* = \inf\{t \ge 0 \mid X_t \le b_* S_t\} \tag{5.19}$$

where the function $\widetilde{V}(x, s; bs)$ and the boundary $b_* s \le s/K$ for $s > 0$ are specified as follows:

(i): if $\sigma > 0$ and $\theta < 0$ then $\widetilde{V}(x, s; bs)$ is given by (5.10) and b_* is uniquely determined from equation (5.11), where $\gamma_i = \beta_i + 1/\theta$ and β_i for $i = 1, 2, 3$ are the roots of equation (3.6);

(ii): if $\sigma = 0$ and $\theta < 0$ then $\widetilde{V}(x, s; bs)$ is given by (5.12) and b_* is uniquely determined from equation (5.13), where $\gamma_i = \beta_i + 1/\theta$ and β_i for $i = 1, 2$ are given by (3.8);

(iii): if $\sigma > 0$ and $0 < \theta < 1$ then $\widetilde{V}(x, s; bs)$ is given by (5.14) and b_* is uniquely determined from equation (5.15), where $\gamma_i = \beta_i + 1/\theta$ and β_i for $i = 1, 2, 3$ are the roots of equation (3.6);

(iv): if $\sigma = 0$ and $0 < \theta < 1$ with $r - \lambda\theta/(1-\theta) < 0$ then $\widetilde{V}(x, s; bs)$ is given by (5.16) and b_* is uniquely determined from equation (5.17), where $\gamma_i = \beta_i + 1/\theta$ and β_i for $i = 1, 2$ are given by (3.8).

These assertions can be proved by means of the same arguments as in Theorem 4.1 above.

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