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Simulation of stochastic auto–oscillating systems through variable stepsize algorithms with small noise

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ABSTRACT.
The paper considers some questions of the numerical analysis of stochastic auto–oscillating systems and their simulation on computers. A low computer costs, variable stepsize algorithm based on local error estimation of stochastic Runge–Kutta–Fehlberg methods is stated for solving nonlinear stochastic differential equations. In particular, it turns out to be very efficient for dynamical systems with small noise intensity. Results of numerical experiments for a plenty of well–known examples from Physics, Chemistry, Biology and Ecology are illustrated with the help of the dialogue system 'Dynamics and Control'.

CONTENTS

1. INTRODUCTION

In recent years chaotic oscillations in dynamical systems of different nature have aroused considerable interest among physicists and mathematicians [1, 33]. Since deterministic chaos is observed in nonlinear systems of Ordinary Differential Equations (ODEs) when their dimension is $N \geq 3$, obviously, the role of numerical simulation in analyzing ODEs is rising. Using classical Runge–Kutta methods of 4-th order for the numerical solution of auto–oscillating ODEs with a constant integration stepsize may lead to quite untrue conclusions about properties of the solution oscillations of ODEs. The complicated nonregular behaviour of solution trajectories of such ODEs requires the compulsory presence of the estimation of the error of numerical solutions and the procedure of automatic choice of the integration stepsize in numerical algorithms. A number of very effective variable stepsize algorithms for solving ODEs has been constructed up to now [12, 17].

Random fluctuations affecting auto–oscillating systems may have principal significance, because they can determine the type of newly established oscillations [1]. The numerical simulation of oscillating ODEs under the influence of random fluctuations merges to statistical simulation of solution trajectories of systems of nonlinear Stochastic Differential Equations (SDEs). As in the deterministic analysis, variable stepsize algorithms for numerical solution of oscillating SDEs are also urgently required. An already existing algorithm with variable stepsizes is based on embedded 5- and 6-stage Runge–Kutta methods [5]. It is intended for solving those problems of optimal control, where usually there is no need in the simulation of a large number of trajectories. However, for obtaining different probabilistic characteristics of stochastic oscillations the simulation of a large number of trajectories is needed. Then it is a very time–consuming task. Therefore variable stepsize algorithms for solving stochastic oscillation systems have to cause relatively low computer costs. In this paper we will use generalized 3-stage Runge–Kutta methods as a basis for low computer cost variable stepsize algorithms for solving SDEs. The main requirement for such algorithms is to provide the possibility of simulating oscillating trajectories with sufficiently high accuracy and efficiency, with stable integration stepsizes and with lowest possible information on the σ–algebra generated by the underlying noise sources (e.g. using only local noise increments). By a result of Clark & Cameron [8] the global error of mean square convergence of such numerical solutions can not exceed the level one while using only noise increments for approximating diffusion parts of SDEs.

The paper is organized as follows. Section 2 briefly describes the dynamical sytems to be considered here, as well as it recalls some basic notions. In the next section we introduce a variable stepsize algorithm for solving SDEs. This algorithm is based on well–known deterministic Runge–Kutta–Fehlberg methods with lower order of convergence, but under low smoothness conditions on drift and diffusion of SDEs. Section 4 exhibits the results of a large variety of numerical experiments for such systems as stochastically perturbed Lorenz, Rössler and Brusselator equations. These experiments are carried out by using the system 'Dynamics and Control' which includes the constructed variable stepsize algorithm, and they indicate the efficiency of the proposed algorithm under small noise to some extent. The paper ends with some conclusions and an appendix on the interactive system 'Dynamics

and Control'.

2. Strange Attractors, Bifurcation, Phase Transitions

Consider a system of ODEs in the form:

$$\begin{aligned} \frac{dy(t)}{dt} &= f(t, y(t), \mu), \quad t_0 \le t \le t_{end}, \\ y(t_0) &= y_0, \end{aligned} \tag{2.1}$$

where f is d-dimensional vector–valued function, $\mu = (\mu_1, ..., \mu_k)$ is a vector of real parameters. Throughout the paper we assume that system 2.1 for a certain μ is auto–oscillating, i.e. it has a *limit cycle.* The limit cycle is a particular case of the *attractor* – a bounded, attractive limit set. Attractors which have a nonperiodic auto–oscillating mode are called *strange.* Only auto–oscillating systems with dimension $d \ge 3$ can have strange attractors. The classical example of a system of ODEs with a strange attractor is the Lorenz system

$$\begin{aligned} \frac{dy_1(t)}{dt} &= -\mu_1(y_1 - y_2), \\ \frac{dy_2(t)}{dt} &= \mu_2 y_1 - y_2 - y_1 y_3, \\ \frac{dy_3(t)}{dt} &= -\mu_3 y_3 + y_1 y_2. \end{aligned} \tag{2.2}$$

With μ continuously changing along some curve γ in the space of parameters it can occur that a qualitative rebuilding of the phase portrait takes place in passing some points on this curve. Such values of parameters are called *bifurcation points* of the phase portrait, and this phenomenon is said to be *bifurcation.* Then μ^* is the point of bifurcation if in an arbitrary small vicinity of this point there are points with qualitatively different phase portraits. The transition from one phase portrait to another while changing the vector of parameters μ is called *phase transition.* Bifurcation of strange attractors can occur as phase transition of the types 'chaos – chaos' or 'chaos – order'.

Any movement of real dynamical objects takes place under the influence of random fluctuations. The role of fluctuations gains special importance near bifurcation points, when even small fluctuations of parameters or external noise can initiate various phase transitions. The investigation of influence of random disturbances on a dynamical system usually reduces to the analysis of SDEs

$$dy(t) = f(t, y(t), \mu)dt + \sigma(t, y(t))dw(t), \tag{2.3}$$

where $\sigma(t, y)$ is a matrix–valued function with dimension $d \times m$, $w(\cdot)$ – is a m - dimensional standard Wiener process. Solution of SDEs have such probabilistic characteristics as the mean, matrix of covariance, function of correlation and probability distribution density. For stationary, ergodic random processes their spectral density is also a probabilistic characteristic.

3. Variable Stepsize Algorithms Based on Drift Comparison

A s-stage Runge–Kutta method for solving ODEs 2.1 has the form

$$y_{n+1} = y_n + \sum_{i=1}^{s} p_i k_i, \tag{3.1}$$

$$k_i = hf(t_n + c_i h, y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j, \mu), \quad i = 1, ..., s,$$

$$\beta_{10} = 0, \quad c_i = \sum_{j=1}^{i-1} \beta_{ij}, \quad i = 2, ..., s,$$

where p_i, $\beta_{i,j}$ are the coefficients of this method, y_n is a numerical solution at the mesh node t_n and h the integration stepsize at the mesh node t_n. It can be generalized to the solution of SDEs in the sense of Itô 2.3 by methods

$$y_{n+1} = y_n + \sum_{i=1}^{s} p_i k_i + \sqrt{h}\sigma(t_n, y_n)\zeta_n \tag{3.2}$$

where ζ_n are m-dimensional random vectors of independent, standard Gaussian distributed components. More detailed description of various families of numerical methods for solving SDEs can be found, e.g. in [3, 4, 27, 28, 31, 47]. An alternative approach to that numerical analysis is given by the construction of Markov chain approximations in stochastic control problems, cf. Kushner and Dupuis [29], whereas we will follow the suggestions made by references above.
A deterministic Taylor expansion of numerical solution 3.2 in a neighbourhood of point t_n on a uniform mesh takes the form

$$y_{n+1} = y_n + hf(t_n, y_n, \mu) + \sqrt{h}\sigma(t_n, y_n)\zeta_n + C(h) \tag{3.3}$$

under sufficient smoothness of function f. The moments of the remainder term $C(h)$ possess the following asymptotic behaviour

$$< C(h) >= O(h^2), \quad < C^2(h) >= O(h^3)$$

as $h \to 0$. Here $< \cdot >$ denotes the operation of mean expectation. Comparing the Taylor expansion 3.3 with the Itô – Taylor expansion of the exact solution of SDEs (see [27]) we see at once that numerical methods 3.2 have first order of convergence in mean square sense for arbitrary SDEs and have second order for SDEs with constant matrices σ. The stochastic notion of convergence mentioned here is also refered to *global mean square convergence*, i.e. it exists a constant $K > 0$ such that

$$\sup_{t_n \le T} < \|y(t_n) - y_n\|^2 > \le K(T) \cdot \left(\max_{t_n \le T} h_n\right)^{\gamma}$$

for sufficiently smooth systems 2.3, where $h_n = |t_n - t_{n-1}|$ $(n = 1, 2, ...)$. $\gamma > 0$ is called the *convergence order* for a fixed terminal time $T > 0$. The order of global mean square convergence can be generally verified by the help of a theorem from [31].

The well–known embedded Runge–Kutta–Fehlberg methods of second and third order (see [17]) for solving ODEs 2.1 have the form

$$
\begin{aligned}
& y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + 4k_3), \quad y^*_{n+1} = y_n + \frac{1}{2}(k_1 + k_2), \qquad (3.4)\\
& k_1 = hf(t_n, y_n, \mu),\\
& k_2 = hf(t_n + h, y_n + k_1, \mu),\\
& k_3 = hf(t_n + \frac{h}{2}, y_n + \frac{1}{4}(k_1 + k_2), \mu)
\end{aligned}
$$

where y_{n+1} is the numerical solution at point t_{n+1} which is obtained by the method of third order, y^*_{n+1} the numerical solution at point t_{n+1} which is obtained by the method of second order. According to 3.2 the following methods are a generalization of 3.4 for solving SDEs in the sense of Itô

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}(k_1 + k_2 + 4k_3) + \sqrt{h}\sigma(t_n, y_n)\zeta_n, \qquad (3.5)\\
y^*_{n+1} &= y_n + \frac{1}{2}(k_1 + k_2) + \sqrt{h}\sigma(t_n, y_n)\zeta_n \, .
\end{aligned}
$$

Variable stepsize algorithms automatically choose a stepsize at any integration step such that the local error of the methods should not exceed a given quantity ϵ. The following procedure for choosing the current integration stepsize is usually used in variable stepsize algorithms for ODEs 2.1 based on methods 3.4, cf. [17] and [45]. After choosing an initial stepsize h at point t_n, y_{n+1} and y^*_{n+1} are calculated with this stepsize. Then the local, weighted error of the numerical solution at node t_{n+1} is estimated by the formula

$$
\delta_{n+1} = \sqrt{\frac{1}{d}\sum_{i=1}^{d}\left(\frac{y^*_{n+1,i} - y_{n+1,i}}{d_i}\right)^2}, \qquad (3.6)
$$

where the scaling factors are equal to $d_i = max(1, |y_{n+1,i}|, |y_{n,i}|)$. The obtained estimation δ_{n+1} is compared with a given error tolerance ϵ. Based on extensive numerical experimentation, this allows us to predict the new stepsize

$$
h_{new} = \frac{h}{max(fac1, min(fac2, (\frac{\delta_{n+1}}{\epsilon})^{1/3}/fac))}, \qquad (3.7)
$$

where $fac = 0.9$ represents a suitable adjustment factor which is used for the estimation of the next mesh node being admissible with high probability. The coefficient for the maximum increasing stepsize is set as $fac1 = 0.1$, the coefficient for minimum decreasing stepsize is taken as $fac2 = 5$. The step from node t_n to t_{n+1} is considered as successful, if

$$
\delta_{n+1} \leq \epsilon \qquad (3.8)
$$

and for further calculation from the mesh node t_{n+1} to t_{n+2} the initial stepsize is chosen according to 3.7. If inequality 3.8 is untrue then one repeats the whole procedure with h_{new} instead of h as initial stepsize according to 3.7 for integration from mesh node t_n to t_{n+1}.
This procedure for choosing adaptively stepsizes can be transferred to solving SDEs 2.3 in the sense of Itô without considerable change. Inequality 3.8 is replaced by inequality

$$\delta_{n+1} \leq 5\epsilon \,. \tag{3.9}$$

Fluctuations of numerical solutions at the expense of diffusion terms are not taken into account in the error estimation 3.6, since

$$y_{n+1} - y^*_{n+1} = -\frac{1}{3}(k_1 + k_2 - 2k_3) \,.$$

Thus, we obtained an adaptive algorithm for pathwise stepsize control without time–consuming, statistical estimation procedures.
Method 3.4 can be generalized to the solution of SDEs in the sense of Stratonovich in the following way. Consider

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}(k_1 + k_2 + 4k_3) + \sqrt{h}(2G_3 - G_1)\zeta_n, \\ G_1 &= \sigma(t_n, y_n), \\ G_2 &= \sigma(t_n, y_n + \frac{6}{5}k_1 + \frac{1}{2}G_1\sqrt{h}\zeta_n), \\ G_3 &= \sigma(t_n + \frac{h}{2}, y_n + \frac{1}{4}(k_1 + k_2) + \frac{1}{24}(G_1 + 5G_2)\sqrt{h}\zeta_n) \end{aligned} \tag{3.10}$$

where k_i are the same as in 3.4. These numerical methods 3.10 have first order of convergence in mean square sense for arbitrary SDEs and second order for SDEs with constant matrices σ. The procedure of estimation of the local error and the choice of integration stepsizes for solving SDEs in the sense of Itô is analogously transferred to solving SDEs in the sense of Stratonovich. Another alternative is given by the transformation of Stratonovich systems to corresponding Itô systems. For the transformation formula, e.g. see [27]. This happens without changing the dynamical system for SDEs with constant matrices.

4. Models and Numerical Experiments

Numerical experiments were made on a PC/AT 486DX-2 using the dialogue system 'Dynamics and Control' (DS) written at Novosibirsk Computing Center (Russia). Numerical tests for the constructed algorithm with variable stepsize are carried out for stochastic dynamical systems with strange attractors. A preliminary attempt was made to calculate such systems with the help of several numerical methods with constant stepsize. They failed because of computer overflows, even while using very small stepsizes.
All examples given below have been taken from books [1, 33]. The occuring ODEs were solved by the algorithm RKF45 with variable stepsizes, constructed on the

basis of embedded 5- and 6-stage Runge–Kutta–Fehlberg methods [12]. Parameters of ODEs are chosen such that the solutions have complex nonregular oscillations. Then the solution of SDEs in the sense of Itô obtained from ODEs by 'noising' of parameters are simulated with the help of the constructed algorithm with variable stepsizes. In all examples the initial values of SDEs solution consist of Gaussian random vectors with independent components having the same variance $D = 0.01$. The desired accuracy of calculations is equal to $\epsilon = 10^{-3}$ (error tolerance). Main objective of the following numerical experiments is to demonstrate the possibility of the constructed variable stepsize algorithm for simulating solutions of SDEs with complex oscillating character and to show what happens with SDEs solution if parameters of the system start to be 'noisy'. Note that the proposed algorithm is not intended for integration under very large noise influence.

Example 1. Auto–oscillations are found in the generator with inertial nonlinearity governed by

$$\begin{aligned} \frac{dy_1(t)}{dt} &= y_2 + \mu_1 y_1 - y_1 y_3, \\ \frac{dy_2(t)}{dt} &= -y_1, \\ \frac{dy_3(t)}{dt} &= \mu_2(-y_3 + \chi(y_1) y_1^2), \end{aligned} \tag{4.1}$$

where $\chi(y)$ is the Heaviside function, i.e.

$$\chi(y) = \begin{cases} 1, & \text{if} \quad y > 0, \\ 0, & \text{if} \quad y \leq 0. \end{cases}$$

The change in the phase diagram by the presence of noise in parameter μ is recognizable by Figures 1 and 2. We observe a more and more destructuring process under increasing noise in comparison with the deterministic movement in the phase plane.

One trajectory of the SDEs solution in the interval [0,200] on the grid having 2000 nodes was simulated. The number of algorithm steps is equal to 2346 at 'inessential' noise and 2412 at 'more intensive' noise. There we could not establish any wrongly predicted integration stepsizes (i.e. no repititions/rejections in the algorithm during local stepsize selection). Thus the recommended choice 3.7 seems to be optimal in this sense. Figure 3 presents the graph of the estimated spectral density of component y_1 of the SDEs solution 4.1 when

$$\mu_1 = 1.45 + 0.01\frac{dw_1}{dt}, \quad \mu_2 = 0.3 + 0.01\frac{dw_2}{dt}.$$

With such parameters, the SDEs have a strange attractor and continuous spectrum of their solutions.

Example 2. An example for Lorenz equations 2.2 is the simple three–mode model of convective turbulence. Assume that parameters $\mu_1 = 10$ and $\mu_3 = \frac{8}{3}$ in system 2.2 are deterministic, whereas parameter μ_2 is 'noising'.

In this example larger noise intensity of parameter μ_2 brings about a qualitative change of the phase portrait. Figure 4 shows a phase trajectory (y_1, y_3) of the SDEs

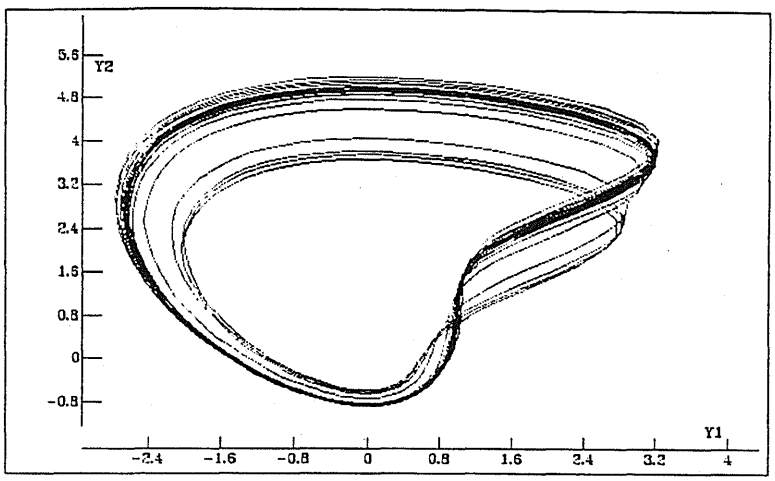

FIGURE 1. Phase trajectory (y_1, y_2) with $\mu_1 = 1.09$, $\mu_2 = 0.9$ (noise is absent)

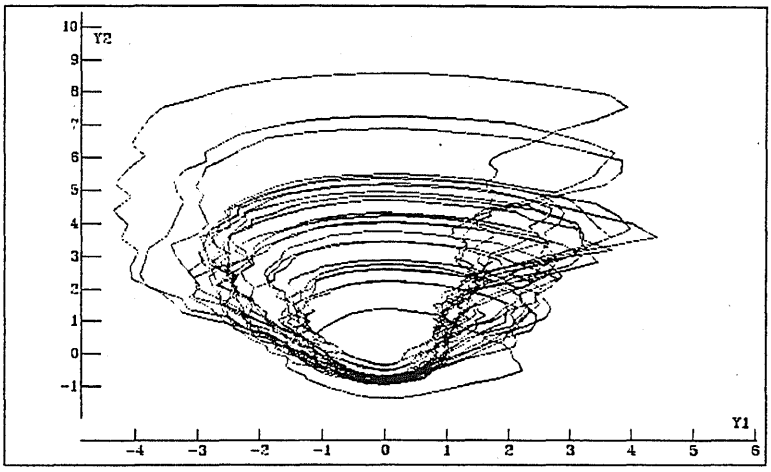

FIGURE 2. Phase trajectory (y_1, y_2) with $\mu_1 = 1.09 + 0.2\frac{dw_1}{dt}$, $\mu_2 = 0.9 + 0.2\frac{dw_2}{dt}$

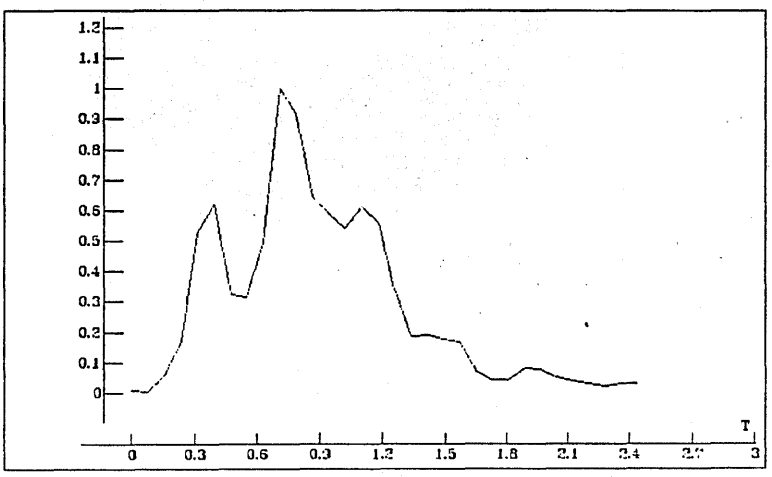

FIGURE 3. Spectral density of the first component of solution 4.1

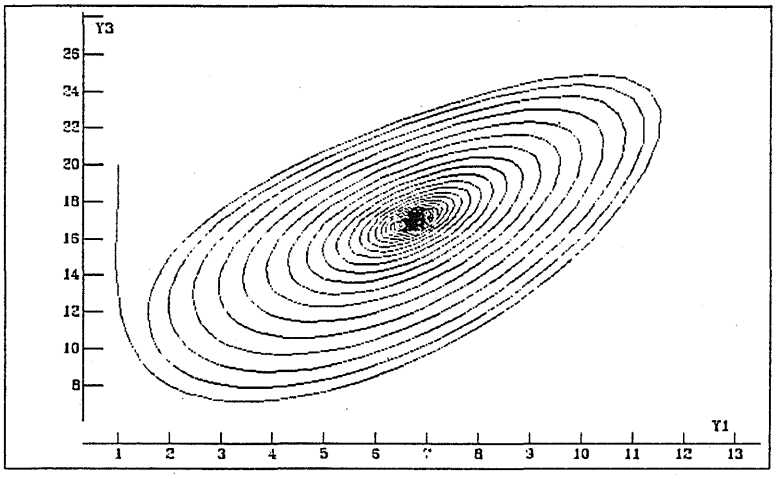

FIGURE 4. Phase trajectory (y_1, y_3) at inessential noise

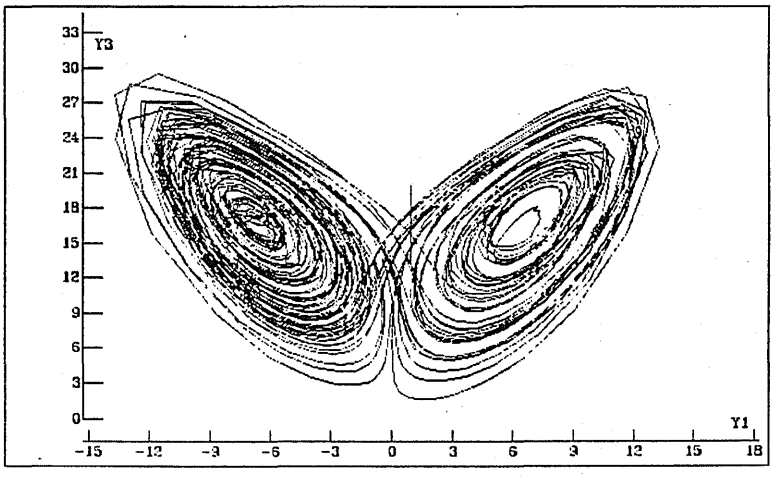

FIGURE 5. Phase trajectory (y_1, y_3) at intensive noise

solution 2.2 with $\mu_2 = 18+0.04\frac{dw_1}{dt}$, and Figure 5 with $\mu_2 = 18+0.4\frac{dw_1}{dt}$. As there is seen, SDEs 2.2 have a regular attractor at inessential noise and a strange attractor at more intensive noise. One trajectory of the SDEs solution in the interval [0,100] on the grid having 2000 nodes was simulated. The number of algorithm steps is equal to 2330 at inessential noise and 4603 at intensive noise. There we could not establish any wrongly predicted integration stepsizes (no rejections).

Example 3. The system of Rössler equations

$$\begin{aligned} \frac{dy_1(t)}{dt} &= -y_2 - y_3, \\ \frac{dy_2(t)}{dt} &= y_1 + \mu_1 y_2, \\ \frac{dy_3(t)}{dt} &= \mu_2 + y_1 y_3 - \mu_3 y_3 \end{aligned} \tag{4.2}$$

describes a hypothetic chemical reaction. Figure 6 visualizes a phase trajectory (y_1, y_3) of the ODEs solution 4.2 with $\mu_1 = 0.2$, $\mu_2 = 0.2$, $\mu_3 = 2.83$.
Figure 7 presents the graph of joint density of the first and second components of the SDEs solution 4.2 with the noisy parameters

$$\begin{aligned} \mu_1 &= 0.2 + 0.01\frac{dw_1}{dt}, \\ \mu_2 &= 0.2 + 0.01\frac{dw_2}{dt}, \\ \mu_3 &= 2.83 + 0.1\frac{dw_3}{dt}. \end{aligned}$$

The estimation of the joint density is obtained along one trajectory of the SDEs solution in the interval [0,20000] on the grid having $2 \cdot 10^5$ nodes. The number of algorithm steps is equal to 266750. There were not any wrongly predicted integration stepsizes. With increasing noise intensity of parameters in the algorithm, the integration stepsize can decrease up to the computer zero, and simulation of the problem terminates with the message 'The demanded accuracy of computation is not attained' while using the computer package 'Dynamics and Control'. Figure 8 presents a graph of the estimation of the spectral density of the component y_1 of SDEs solution with

$$\begin{aligned} \mu_1 &= 0.3 + 0.001\frac{dw_1}{dt}, \\ \mu_2 &= 0.4 + 0.001\frac{dw_2}{dt}, \\ \mu_3 &= 8.5 + 0.001\frac{dw_3}{dt}. \end{aligned}$$

With these parameters SDEs 4.2 have a strange attractor.

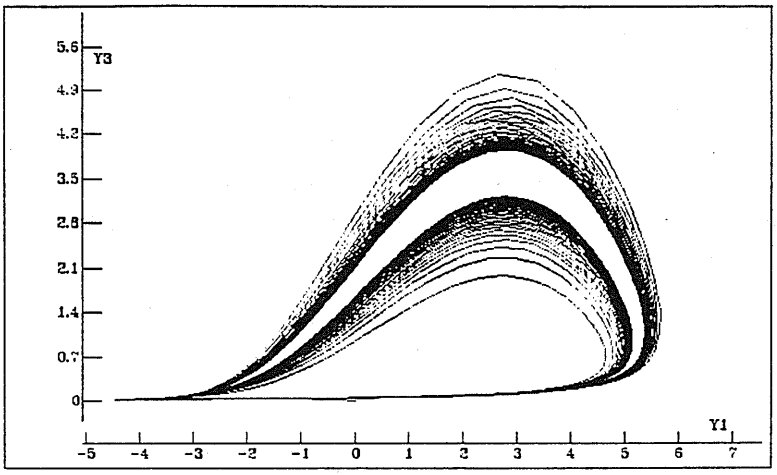

FIGURE 6. Phase trajectory (y_1, y_3)

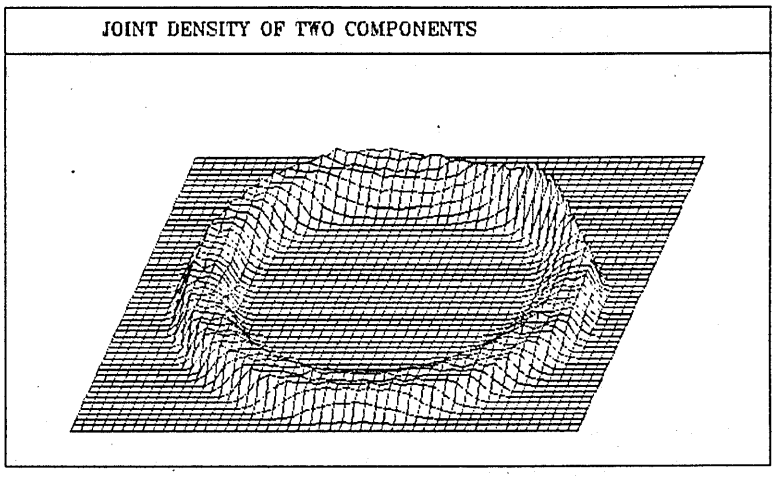

FIGURE 7. Graph of the joint density

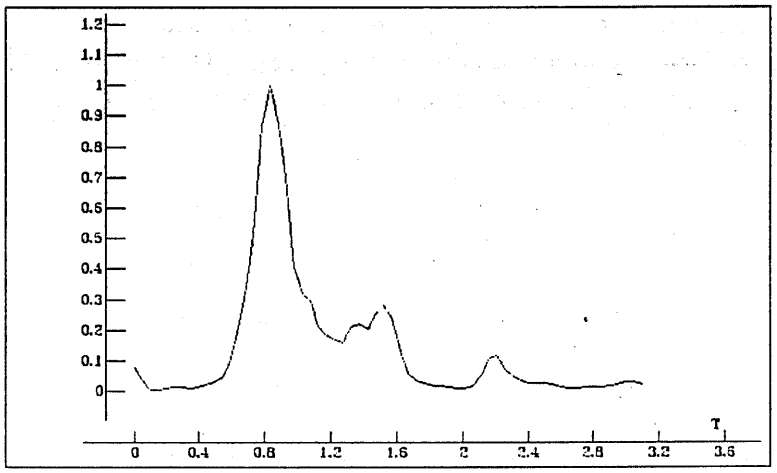

FIGURE 8. Spectral density

Example 4. Two connected brusselators

$$\begin{aligned}
\frac{dy_1(t)}{dt} &= \mu_1 - 5.56y_1 + y_1^2y_3 + \mu_2(y_2 - y_1),\\
\frac{dy_2(t)}{dt} &= \mu_3 - 3.308y_2 + y_2^2y_4 + \mu_2(y_1 - y_2),\\
\frac{dy_3(t)}{dt} &= 4.56y_1 - y_1^2y_3 + \mu_2(y_4 - y_3),\\
\frac{dy_4(t)}{dt} &= 2.308y_2 - y_2^2y_4 + \mu_2(y_3 - y_4)
\end{aligned} \tag{4.3}$$

describe the temporal evolution of the concentration of corresponding substances in a chemical reaction. For constant parameters $\mu_1 = 1.6$, $\mu_2 = 0.125$, $\mu_3 = 0.555$ the phase trajectory (y_2, y_4) of the SDEs solution 4.3 is of the form presented in Figure 9. For component y_3 of the SDEs solution 4.3, Figure 10 shows the estimated correlation function $R(t, t+\tau)$ at point $t = 10$, computed with the noisy parameters

$$\begin{aligned}
\mu_1 &= 1.6 + 0.1\frac{dw_1}{dt},\\
\mu_2 &= 0.125 + 0.1\frac{dw_2}{dt},\\
\mu_3 &= 0.555 + 0.1\frac{dw_3}{dt}.
\end{aligned}$$

The correlation function estimation is obtained by the simulation of an ensemble of 100 trajectories of the SDEs solution 4.3 in the interval [0,20] on the grid having 200 nodes. The total number of steps of the algorithm is equal to 33563. 21 wrongly predicted integration stepsizes were fixed in the course of computation.

Example 5. A seven-dimensional discrete model of Navier-Stokes equation, as a system of Lorenz equations, describes a convective turbulence

$$\begin{aligned}
\frac{dy_1(t)}{dt} &= -2y_1 + \mu_1(y_2y_3 + y_4y_5),\\
\frac{dy_2(t)}{dt} &= -9y_2 + \mu_2(y_1y_3 + y_6y_7),\\
\frac{dy_3(t)}{dt} &= -5y_3 + \mu_3y_1y_7 - 7\sqrt{5}y_1y_2 + \mu_6,\\
\frac{dy_4(t)}{dt} &= -5y_4 - \sqrt{5}y_1y_5,\\
\frac{dy_5(t)}{dt} &= -y_5 - \mu_4y_1y_4,\\
\frac{dy_6(t)}{dt} &= -8y_6 - 4\sqrt{5}y_2y_7,\\
\frac{dy_7(t)}{dt} &= -5y_7 + \sqrt{5}y_2y_6 - \mu_5y_1y_3,
\end{aligned} \tag{4.4}$$

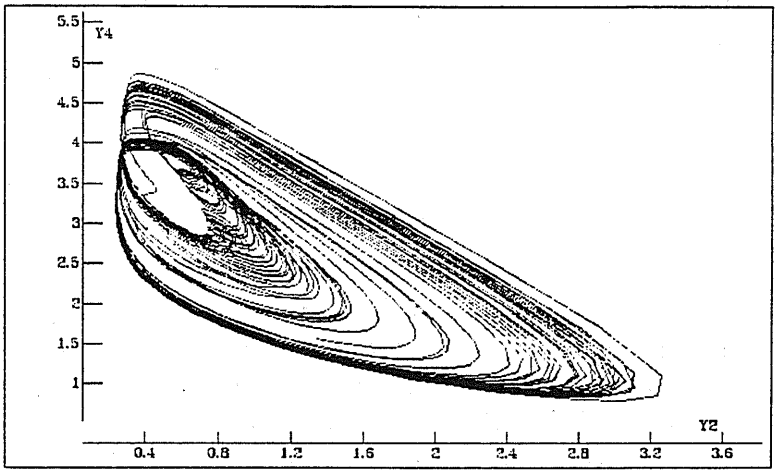

FIGURE 9. Phase trajectory (y_2, y_4)

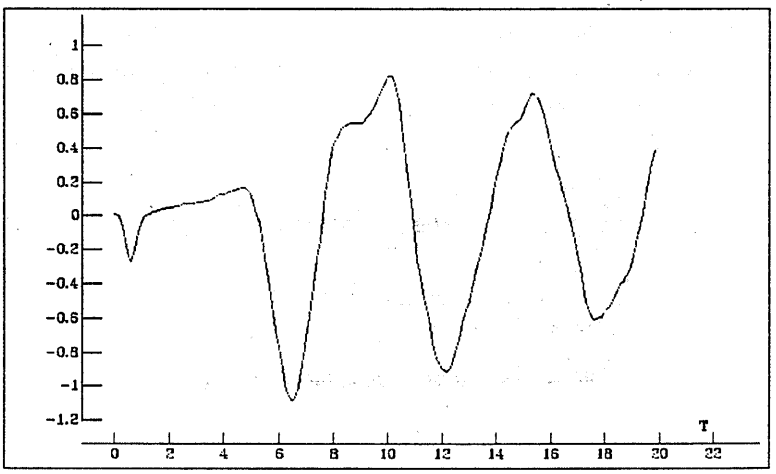

FIGURE 10. Graph of the correlation function

where μ_6 is an anology to Reinold's number. Figure 11 views a phase trajectory (y_3, y_5) of the ODEs solution 4.4 with parameters $\mu_1 = 4\sqrt{5}$, $\mu_2 = 3\sqrt{5}$, $\mu_3 = 9$, $\mu_4 = 3\sqrt{5}$, $\mu_5 = 9$, $\mu_6 = 360$.
Six noisy parameters will be set in the following manner

$$\mu_1 = 4\sqrt{5} + 0.001\frac{dw_1}{dt},$$
$$\mu_2 = 3\sqrt{5} + 0.001\frac{dw_2}{dt},$$
$$\mu_3 = 9 + 0.001\frac{dw_3}{dt},$$
$$\mu_4 = 3\sqrt{5} + 0.001\frac{dw_4}{dt},$$
$$\mu_5 = 9 + 0.001\frac{dw_5}{dt},$$
$$\mu_6 = 360 + 0.1\frac{dw_6}{dt}.$$

Figure 12 presents the graph of the estimated joint density of the third and fifth components of the SDEs solution 4.4, and Figure 13 the one–dimensional density of the third component.
Density estimations are obtained along one trajectory of the SDEs solution in the interval [0,500] on the grid having $2 \cdot 10^5$ nodes. There were no difficulties in the numerical simulation. The number of steps of the algorithm is equal to 379766, no wrongly predicted integration stepsizes were established. For testing the procedure of varying integration stepsizes, similar calculations for larger noise intensity of parameters

$$\mu_1 = 4\sqrt{5} + 0.1\frac{dw_1}{dt},$$
$$\mu_2 = 3\sqrt{5} + 0.1\frac{dw_2}{dt},$$
$$\mu_3 = 9 + 0.1\frac{dw_3}{dt},$$
$$\mu_4 = 3\sqrt{5} + 0.1\frac{dw_4}{dt},$$
$$\mu_5 = 9 + 0.1\frac{dw_5}{dt},$$
$$\mu_6 = 360 + \frac{dw_6}{dt}$$

were carried out. The number of steps has increased up to 427068, and 22 wrongly predicted integration stepsizes were found.
Example 6. A model of three–wave resonance interaction serves for the description of combinative light scattering in a dielectric with

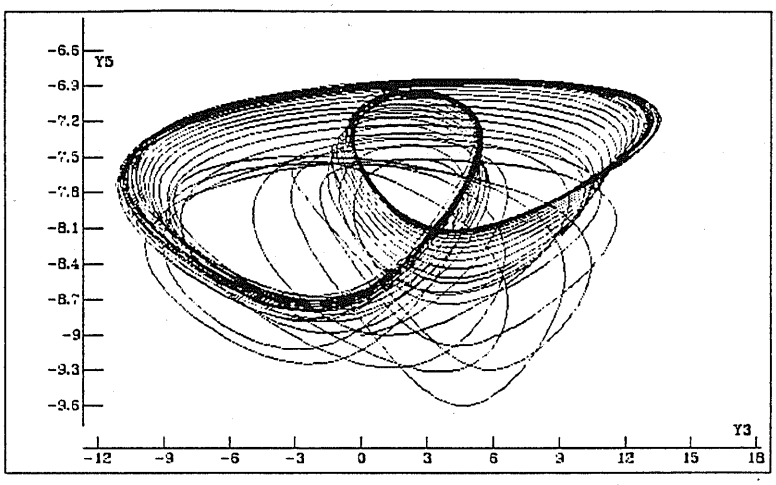

FIGURE 11. Phase trajectory (y_3, y_5)

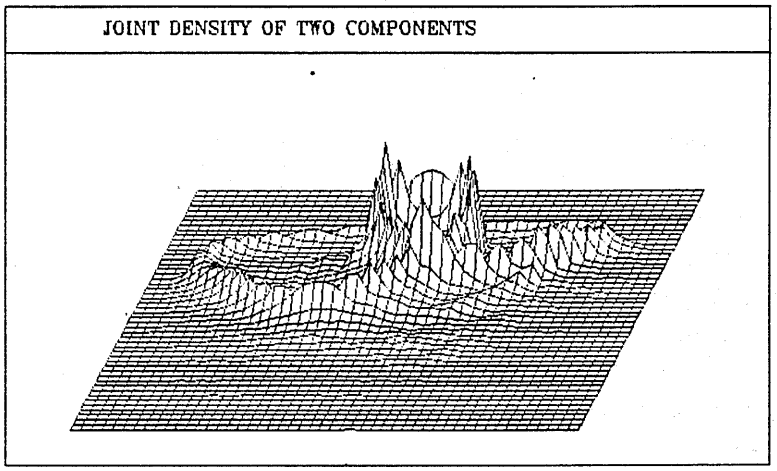

FIGURE 12. Graph of the joint density

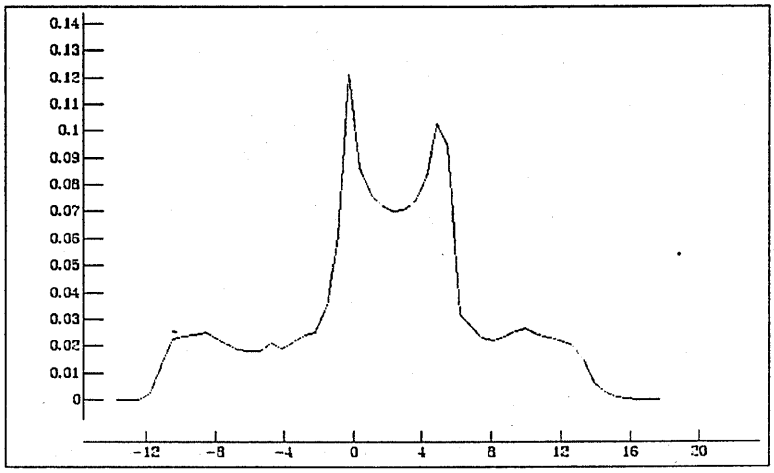

FIGURE 13. Graph of the one–dimensional density

$$\begin{aligned}
\frac{dy_1(t)}{dt} &= y_1 - \mu_1 y_2 + y_2(y_3 + y_1^2), \\
\frac{dy_2(t)}{dt} &= y_2 + \mu_1 y_1 + y_1(3y_3 - y_1^2), \\
\frac{dy_3(t)}{dt} &= -\mu_2 y_3 - 2y_1 y_2 y_3 \,.
\end{aligned} \tag{4.5}$$

Figure 14 displays a phase trajectory (y_2, y_3) of the ODEs solution 4.5 for the parameters $\mu_1 = 1.15$, $\mu_2 = 2.52$.
Figure 15 shows a graph of the integral curve $y_1(t)$ of the SDEs solution 4.5 for the case of 'weakly noising' parameters

$$\begin{aligned}
\mu_1 &= 1.15 + 0.0001\frac{dw_1}{dt}, \\
\mu_2 &= 2.52 + 0.001\frac{dw_2}{dt},
\end{aligned}$$

and Figure 16 for the case of 'intensively noising' parameters

$$\begin{aligned}
\mu_1 &= 1.15 + 0.1\frac{dw_1}{dt}, \\
\mu_2 &= 2.52 + 0.1\frac{dw_2}{dt}.
\end{aligned}$$

Integral curves are obtained by simulation of trajectories of the SDEs solution 4.5 in the interval [0,200] on the grid having 2000 nodes. The number of steps of the algorithm at inessential noise is equal to 3785. At intensive noise we failed to do simulations due to decreasing integration stepsize up to the computer zero.
Example 7. In a model of the ecological system 'plunderer – victim'

$$\begin{aligned}
\frac{dy_1(t)}{dt} &= -\mu_1 y_1 - 2\frac{y_1 y_2}{1 + 0.08y_1} + \frac{y_1 M_0}{1 + 0.08M_0}, \\
\frac{dy_2(t)}{dt} &= -\mu_1 y_2 + \frac{y_1 y_2}{1 + 0.08y_1}, \\
\frac{dy_3(t)}{dt} &= -\mu_2 y_3 - 2\frac{y_3 y_4}{1 + 0.08y_3} + \frac{y_3 M_0}{1 + 0.08M_0}, \\
\frac{dy_4(t)}{dt} &= -\mu_2 y_4 + \frac{y_3 y_4}{1 + 0.08y_3},
\end{aligned} \tag{4.6}$$

where M_0 is the quantity of the biogenic element with $M_0 = 20 - y_1 - y_2 - y_3 - y_4$, y_1 and y_3 are the biogenic contents in victims, y_2 and y_4 are the biogenic contents in plunderers. Figure 17 shows a phase trajectory (y_3, y_4) of the ODEs solution 4.6 with parameters $\mu_1 = 1$, $\mu_2 = 2$.
Two noisy parameters of the system are given as follows

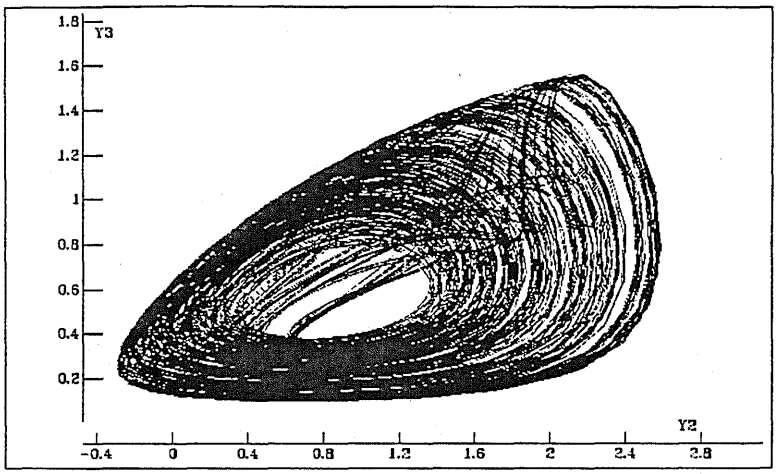

FIGURE 14. Phase trajectory (y_2, y_3)

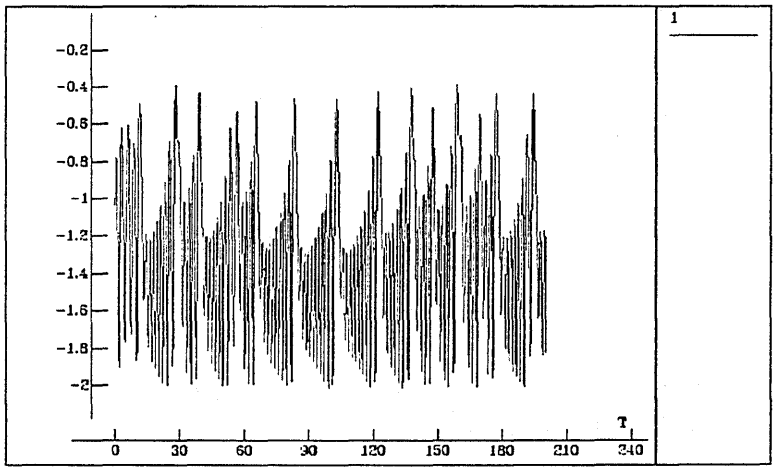

FIGURE 15. Graph of the integral curve $y_1(t)$ at inessential noise

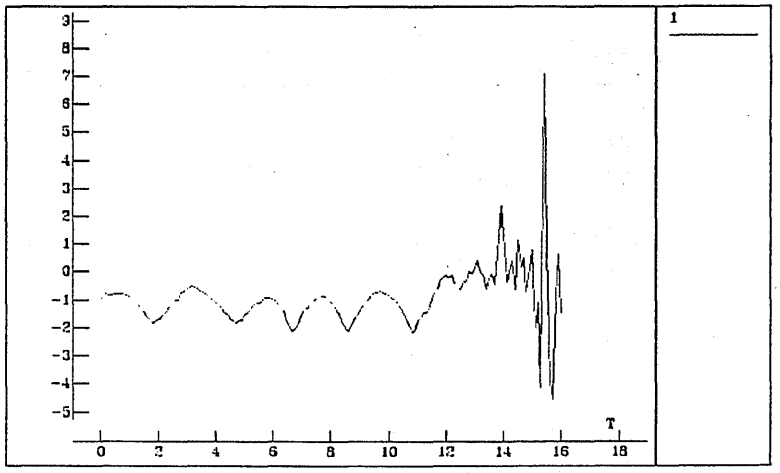

FIGURE 16. Graph of the integral curve $y_1(t)$ at intensive noise

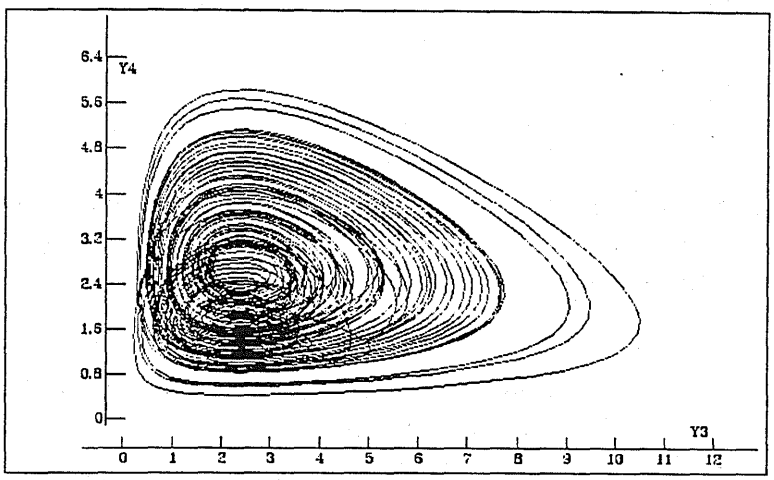

FIGURE 17. Phase trajectory (y_3, y_4)

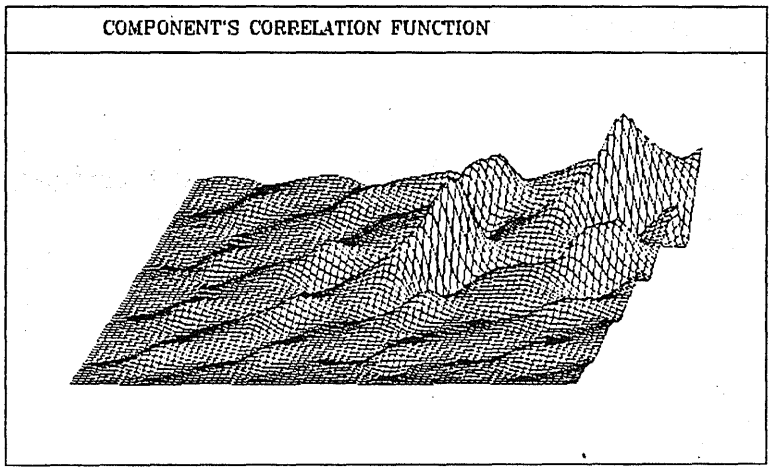

FIGURE 18. Correlation function

$$\mu_1 = 1 + 0.1\frac{dw_1}{dt},$$
$$\mu_2 = 2 + 0.1\frac{dw_2}{dt}.$$

Figure 18 presents the graph of the estimated correlation function $R(t, t+\tau)$ of the third component of the SDEs solution 4.6. Estimation of the correlation function is obtained by the simulation of 200 trajectories of the SDEs solution 4.6 in the interval [0,10] on the grid with 80 nodes. The total number of steps of the algorithm is equal to 38245. Three wrongly predicted integration stepsizes were found in the course of computation.

5. Conclusion and Remarks

Based on the results of numerical experiments we can conclude relatively high efficiency of the variable stepsize algorithm applied to statistical simulation of auto–oscillating stochastic systems. In particular, it is very appropriate for simulation studies under small noise perturbations. For those dynamical systems where noise decisively influences their behaviour it still exists the task of constructing very efficient, adaptive algorithms. Then the proposed algorithm does not work. This fact can be easily seen while simulating solutions of SDEs without drift terms. Thus, it would be important to incorporate the diffusion terms as well in the local error estimation and adaptive stepsize selection. The advantage of the suggested algorithm consists of its low computer costs, relatively low smoothness conditions on drift and diffusion terms and avoiding of the generation of fairly complicated multiple integrals. For the estimation of sufficiently smooth functionals of the form $< g(y(t)) >$ one can also make use of well–known deterministic methods. A corresponding exposition in this respect is worked out in [21], relying on extrapolation methods and Talay's global error expansion [48].

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Appendix A. The Interactive System 'Dynamics and Control'

The dialogue system (DS) has been worked out at the Computing Center of Sibirian Division of the Russian Academy of Sciences in Novosibirsk, Russia. This interactive system (DS) is intended for numerical experiments for solving problems of analysis and synthesis of the automatic control of dynamical objects. The system works on IBM–compatible computers under MS–DOS with some minimum requirements.

DS has the following algorithms for statistical simulation of solutions of SDEs:

- Euler–Maruyama method for SDEs in the sense of Itô;
- Generalized two-stage Runge–Kutta method for SDEs in the sense of Stratonovich;
- Generalized one-stage Rosenbrock type method for SDEs in the sense of Itô;
- Generalized two-stage Rosenbrock type method for SDEs in the sense of Itô and Stratonovich;
- Mil'shtein method for SDEs in the sense of Itô with single noise;
- Platen method for SDEs in the sense of Itô with single noise;
- Newton method for SDEs in the sense of Itô and Stratonovich with single noise;
- Two variable stepsize algorithms for SDEs in the sense of Itô and Stratonovich.

The DS allows to evaluate the following functionals of SDEs solutions:

- mean;
- covariance matrix;
- correlation function of a desired component of the solution;
- distribution density of a desired component of the solution;
- spectral density of a desired component of the solution;
- joint distribution density of two desired components of the solution;
- two-dimensional distribution density of a desired component of the solution at two required grid points.

New algorithms can be implemented, as well as special problems and computing tasks. For further details, please contact the second author. The publisher of Computing Center SD RAS plans to issue the monograph S.S. Artemiev, M.A. Yakunin, Y.G. Michaylichenko, I.O. Shkurko 'Dynamics and Control' (pp. 270) by the end of 1994.

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