Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Error estimates for the finite-element approximation of an elliptic control problem with pointwise state and control constraints

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submitted: August 22, 2006

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No. 1159
Berlin 2006

2000 *Mathematics Subject Classification.* 49K20, 49M25, 65N30.

Key words and phrases. Linear-quadratic optimal control problems, elliptic equations, state constraints, numerical approximation.

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Abstract. We consider a linear-quadratic elliptic optimal control problem with pointwise state constraints. The problem is fully discretized using linear ansatz functions for state and control. Based on a Slater-type argument, we investigate the approximation behavior for mesh size tending to zero. The obtained convergence order for the L^2-error of the control and for H^1-error of the state amounts $1-\varepsilon$ in the two-dimensional case and $1/2-\varepsilon$ in three dimension. In a second step, a state-constrained problem with additional control constraints is considered. Here, the control is discretized by constant ansatz functions. It is shown that the convergence theory can be adapted to this case yielding the same order of convergence. The theoretical findings are confirmed by numerical examples.

1. Introduction. In this paper, we focus on the error analysis for a finite element discretization of linear elliptic optimal control problems with pointwise state constraints. It is well known that, in contrast to the control-constrained case, these problems provide some particular difficulties. This especially concerns the regularity of the Lagrange multipliers associated to the state constraints that are generally regular Borel measures (see for instance Casas [6] or Alibert and Raymond [1]). As a consequence, the optimal controls are in general only elements of $W^{1,\sigma}(\Omega)$ with some $\sigma < 2$ (cf. [6]). This lack of regularity naturally affects the behavior of finite element discretization and numerical optimization algorithms. Consequently, several articles addressed the numerical treatment of state-constrained problems in the recent past. We only mention Bergounioux and Kunisch [4] and the regularization approaches proposed by Meyer, Rösch and Tröltzsch [19] and Hintermüller and Kunisch [17]. In contrast to the control-constrained case, where the finite element discretization is well investigated (see for instance [14, 2, 9] and the references therein), finite element convergence analysis for state-constrained problems still provides several open questions. Here, we refer to Casas [7], Casas and Mateos [8], and, in particular, to Deckelnick and Hinze [12] and [13]. The first two articles deal with finitely many state constraints, whereas in the latter, Deckelnick and Hinze established error estimates for a semi-discrete approach in the spirit of [18]. In [12], they considered the following purely state-constrained problem

$$
\text{(P)}\quad
\begin{cases}
\text{minimize} & J(y,u) := \dfrac{1}{2}\displaystyle\int_\Omega |y-y_d|^2\,dx + \dfrac{\alpha}{2}\displaystyle\int_\Omega u^2\,dx\\[2ex]
\text{subject to} & -\Delta\, y(x) + y(x) = u(x) \quad \text{in } \Omega\\
& \qquad\qquad \partial_n y(x) = 0 \qquad \text{on } \Gamma\\[2ex]
\text{and} & y(x) \le y_b(x) \quad \text{a.e. in } \Omega
\end{cases}
$$

and derived a convergence order of $h^{1-\varepsilon}$, $\varepsilon > 0$, in the two-dimensional case and $h^{1/2-\varepsilon}$ in three dimensions. Furthermore, it turns out that, in the purely state-constrained case, the semi-discrete solution coincides with the solution of the fully discretized problem using linear ansatz functions for the control. In other words, the results of [12] also apply to a full discretization of (P) (see [12, Remark 2.2]). Here, we will confirm their results for the fully discretized case by using a completely different technique. Based on a Slater-point assumption, we establish the existence of a function which is, in some sense, close the solution of (P) and, on the other hand, feasible for the discrete version of (P). By similar arguments, one shows the existence of another function which is feasible for (P) and close to the discrete solution. Together with the variational inequalities for (P) and its discretization, this two-way feasibility is the basis for the overall error analysis. In the second part of the paper, we use this technique to verify a similar result for the case with additional control constraints, i.e.

$$
(\mathrm{Q})\quad \begin{cases} \text{minimize} & J(y,u) := \dfrac{1}{2}\displaystyle\int_\Omega |y-y_d|^2\,dx + \dfrac{\alpha}{2}\int_\Omega u^2\,dx \\ \text{subject to} & -\Delta\, y(x) + y(x) = u(x) \quad \text{in } \Omega \\ & \partial_n y(x) = 0 \quad \text{on } \Gamma \\ \text{and} & y_a(x) \le y(x) \le y_b(x) \quad \text{a.e. in } \Omega \\ & u_a \le u(x) \le u_b \quad \text{a.e. in } \Omega. \end{cases}
$$

In contrast to (P), the controls are now discretized with piecewise constant functions. The error analysis for (Q) represents the genuine result of this article since, in case of (Q), the discrete solution differs from the semi-discrete one. Hence, the theory developed in [13] for the semi-discretization of (Q) cannot be applied to the full discretization.

The paper is organized as follows: In Section 2, we specify the assumptions for the analysis of problem (P) and describe the discretization of (P). After stating some basic properties of (P) and its state equation in Section 3, we derive some auxiliary results in Section 4. These are needed for the proof of the main convergence result Section 5 is devoted to. In Section 6, we turn to problem (Q) and derive an analogous convergence result for this problem by using the same technique. The obtained error estimates are discussed in Section 7, whereas Section 8 finally presents some numerical examples.

2. Notation and Assumptions. In the following, we state the assumptions required for discussion of the finite element discretization of (P). The additional assumptions for the analysis of problem (Q) are mentioned in Section 6.

ASSUMPTION 2.1. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^N$, $N = 2, 3$. Moreover, we assume that y_d is a given function in $L^2(\Omega)$, while the bound y_b is defined in $C(\bar\Omega)$. The Tikhonov parameter α is a real positive number.*

For an interpolation of y_d and y_b, higher regularity is required. This is discussed in detail in Section 7. It is well known that, under Assumption 2.1, to every $u \in L^2(\Omega)$ there exists a unique solution of the state equation in $H^2(\Omega) \subset C(\bar\Omega)$ (cf. for instance [15]). Thus, we introduce the control-to-state mapping $S : L^2(\Omega) \to H^2(\Omega)$ that maps u to y. In the subsequent sections, the control-to-state mapping is considered with different ranges. For simplicity, the associated operators are also denoted by S. In view of the definition of S, we are in the position to introduce the reduced optimal control problem by

$$
(\mathrm{P})\quad \begin{cases} \text{minimize} & f(u) := \dfrac{1}{2}\,\|S\,u - y_d\|^2_{L^2(\Omega)} + \dfrac{\alpha}{2}\,\|u\|^2_{L^2(\Omega)} \\ \text{subject to} & u \in L^2(\Omega) \text{ and } (S\,u)(x) \le y_b(x) \text{ a.e. in } \Omega. \end{cases}
$$

Now, we turn to the discretization of (P). To that end, let us introduce a family of triangulation of $\bar\Omega$, denoted by $\{\mathcal{T}_h\}_{h>0}$. Each triangulation is assumed to exactly fit the boundary of Ω such that

$$\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T.$$

Hence, the elements of $\mathcal{T}_h$ lying on the boundary of Ω are curved. With each element $T \in \mathcal{T}_h$, we associate two parameters $\rho(T)$ and $R(T)$, where $\rho(T)$ denotes the diameter of the set T and $R(T)$ is the diameter of the largest ball contained in T. The mesh size of $\mathcal{T}_h$ is defined by $h = \max_{T \in \mathcal{T}_h} \rho(T)$. We suppose the following regularity assumptions for $\mathcal{T}_h$:

Assumption 2.2. *There exist two positive constants ρ and R such that*

$$\frac{\rho(T)}{R(T)} \leq R, \quad \frac{h}{\rho(T)} \leq \rho$$

hold for all $T \in \mathcal{T}_h$ and all $h > 0$.

With this setting at hand, we are in the position to introduce the discretized control space:

Definition 1. *The space of discrete controls is given by*

$$V_h = \{u_h \in C(\bar{\Omega}) \mid u|_T \in \mathcal{P}_1 \; \forall \, T \in \mathcal{T}_h\}.$$

Notice that $V_h \in H^1(\Omega) \cap C(\bar{\Omega})$.

Furthermore, we define by $\{x_i\}_{i=1}^n$ the set of all nodes of $\mathcal{T}_h$ and denote the standard continuous and piecewise linear finite element ansatz function associated to x_i, $1 \leq i \leq n$, by ϕ_i. In other words, ϕ_i satisfies $\phi_i \in V_h$ with $\phi_i(x_i) = 1$ and $\phi_i(x_j) = 0$ for all $1 \leq j \leq n$ with $j \neq i$. In the same way as the control, the state is also discretized by linear ansatz functions such that the discrete state is equivalent to

$$\int_\Omega \nabla y_h \cdot \nabla v_h \, dx + \int_\Omega y_h \, v_h \, dx = \int_\Omega u \, v_h \, dx \qquad \forall \, v_h \in V_h$$

with an arbitrary $u \in L^2(\Omega)$. The associated discrete solution operator is denoted by $S_h : L^2(\Omega) \to V_h$ and hence, the discrete counterpart of (P) is given by

$$(\mathrm{P}_h) \quad \begin{cases} \text{minimize} & f_h(u) := \dfrac{1}{2} \, \|S_h \, u - y_d\|^2_{L^2(\Omega)} + \dfrac{\alpha}{2} \, \|u\|^2_{L^2(\Omega)} \\ \text{subject to} & u \in V_h \text{ and } (S_h \, u)(x) \leq y_b(x) \text{ a.e. in } \Omega \end{cases}$$

For the derivation of first-order necessary optimality conditions to (P), one needs the following *Slater condition*:

Assumption 2.3. *A function $\hat{u} \in H^2(\Omega)$ exists such that $(S \, \hat{u})(x) \geq \tau$ with some $\tau > 0$ for all $x \in \bar{\Omega}$.*

This condition is also essential for the overall convergence analysis (see Lemma 4.4).

Notations. Due to the strict convexity of $f(u)$ and $f_h(u)$, (P) and (P$_h$) admit unique optimal solutions that are denoted by $\bar{u} \in L^2(\Omega)$ and $\bar{u}_h \in V_h$ in all what follows. The admissible set of (P) is defined by $U_{ad} := \{u \in L^2(\Omega) \mid (S \, u)(x) \leq y_b(x) \text{ a.e. in } \Omega\}$, and a function v is called feasible for (P) if $v \in U_{ad}$. Analogously, we set $U_{ad}^h := \{u_h \in V_h \mid (S_h\, u_h)(x) \le y_b(x) \text{ a.e. in } \Omega\}$ and say $v_h \in V_h$ is feasible for (P$_h$) if $v \in U_{ad}^h$. Given a real number σ with $1 \le \sigma < N/(N-1)$, $N = 2, 3$, we introduce the abbreviation $W = W^{1,\sigma}(\Omega)$ and denote the dual space of W with respect to the L^2-inner product by W^*. Furthermore, for a given $1 \le p \le \infty$, we define $\|\,.\,\|_p := \|\,.\,\|_{L^p(\Omega)}$, except $p = 2$, i.e. the $L^2(\Omega)$-norm, which is denoted by $\|\,.\,\|$. Moreover, $(\,.\,,\,.\,)$ is natural inner product in $L^2(\Omega)$. The set $C(\bar\Omega)^+$ is defined by $C(\bar\Omega)^+ := \{v \in C(\bar\Omega) \,|\, v(x) \ge 0 \;\forall\, x \in \bar\Omega\}$. Finally, throughout the paper, c is a generic constant.

3. Known results. The subsequent section states some basic results needed for the error analysis of (P). We start with the well known L^2-projection that is defined in a standard way as follows:

DEFINITION 2. *Let V_h be an arbitrary subspace of $L^2(\Omega)$. Then, for an arbitrary $u \in L^2(\Omega)$, the L^2-projection on V_h, denoted by $\Pi_h u$, is defined by*

$$\Pi_h u := \arg\min_{v_h \in V_h} \|u - v_h\|^2. \tag{3.1}$$

The first-order optimality conditions for (3.1) immediately imply

$$(u - \Pi_h u\,,\, v_h) = 0 \quad \forall\, v_h \in V_h, \tag{3.2}$$

which will be used several times in the subsequent. Now, let us consider the control-to-state mapping S that was introduced in Section 2.

THEOREM 3.1. *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded Lipschitz domain. Then, there is a $p > N$ such that, for all $N \le q \le p$, the control-to-state operator is continuous from $W^{-1,q}(\Omega)$ to $W^{1,q}(\Omega)$. In other words, if the right-hand side in the state equation is an element of $W^{-1,q}(\Omega)$, then the solution belongs to $W^{1,q}(\Omega)$ and satisfies*

$$\|y\|_{W^{1,q}(\Omega)} \le c\,\|u\|_{W^{-1,q}(\Omega)}$$

with a constant c independent of u. Moreover, if Ω is of class $C^{1,1}$, then, for every right-hand side in $L^p(\Omega)$, $2 \le p < \infty$, there exists a unique solution of the state equation in $W^{2,p}(\Omega)$ that depends continuously on the inhomogeneity.

For the first part of Theorem 3.1, we refer to Gröger [16] if $N = 2$. In the three dimensional case, a corresponding result can be found in Zanger [23]. The second part of the Theorem 3.1 is a standard result that is for instance proven in Grisvard [15].

REMARK 3.2. *Using well known imbedding theorems, we find $W^{1,q}(\Omega) \hookrightarrow C(\bar\Omega)$ for $q > N$ such that $S : W^{-1,q}(\Omega) \to C(\bar\Omega)$ continuously.*

As indicated in the introduction, under Assumption 2.3, the generalized Karush-Kuhn-Tucker theory implies the existence of the Lagrange multipliers associated to the pointwise state constraints in the space $C(\bar\Omega)^*$, whose elements can be identified as regular Borel measures. This is also covered by the following theorem that states the first-order necessary optimality conditions for (P). The corresponding proof can be found in Casas [6] or Alibert and Raymond [1].

THEOREM 3.3. *There exists a unique optimal solution to* (P) *in* $H^2(\Omega) \times L^2(\Omega)$, *denoted by* $(\bar{y}, \bar{u})$, *that satisfies the following optimality system*

$$\left.\begin{array}{c} \begin{array}{rlrl} -\Delta\,\bar{y} + \bar{y} = \bar{u} & \textit{in } \Omega & -\Delta\,p + p = \bar{y} - y_d + \mu_\Omega & \textit{in } \Omega \\ \partial_n\,\bar{y} = 0 & \textit{on } \Gamma & \partial_n\,p = \mu_\Gamma & \textit{on } \Gamma \end{array} \\ \alpha\,\bar{u}(x) + p(x) = 0 \quad \textit{a.e. in } \Omega \\ \displaystyle\int_{\bar{\Omega}} (\bar{y} - y_b)\,d\mu = 0\,, \quad \bar{y}(x) \leq y_b(x) \quad \forall\, x \in \bar{\Omega} \\ \displaystyle\int_{\bar{\Omega}} y\,d\mu \geq 0 \quad \forall\, y \in C(\bar{\Omega})^+ \end{array}\right\} \tag{3.3}$$

with a Lagrange multiplier $\mu \in \mathcal{M}(\Omega)$ *and an adjoint state* $p \in W$.

REMARK 3.4. *The gradient equation in* (3.3) *immediately implies* $\bar{u} \in W$ *and hence* $u \notin C(\bar{\Omega})$.

This remark illustrates an essential difference to the control-constrained case, where the optimal control is even Lipschitz continuous. Due to this lack of regularity, we need a generalized interpolation operator for functions in $H^t(\Omega)$, $t \leq 1$, that employs local L^2-projections. In case of polyhedral domains, this operator is given by the well known Clément interpolation operator (cf. [10]) that is defined by

$$(I_h u)(x) := \sum_{i=1}^{n} (\Pi_i u)(x_i)\phi_i(x),$$

where Π_i denotes the L^2-projection on $\operatorname{supp}\{\phi_i\}$, i.e. the solution of

$$(\Pi_i u\,,\, u) = (u_h\,,\, u) \quad \forall\, u_h \in V_h \cap H^t(\operatorname{supp}\{\phi_i\}).$$

In [5], Bernardi generalized this concept for domains with curved boundary and proved the following result:

LEMMA 3.5. *Let* $t \in [0,1]$ *be given. Then there exists an interpolation operator* $I_h : H^t(\Omega) \to V_h$ *such that, for all* $u \in H^t(\Omega)$,

$$\|u - I_h u\| \leq c\,h^t\,\|u\|_{H^t(\Omega)}$$

is satisfied with a constant c *independent of* t, h, *and* u.

For the particular form of I_h in case of curved domains, we refer to [5]. The operator I_h will be called quasi-interpolation in all what follows. Next, we turn to the finite element approximation of the state equation in (P). Using again Bernardi's results for interpolation error estimates on curved domains (cf. [5]), the standard theory for linear finite elements yields that, for all $u \in L^2(\Omega)$, the discrete solution operator S_h satisfies the following error estimates

$$\|(S - S_h)u\| \leq c\,h^2\,\|u\| \tag{3.4}$$

$$\|(S - S_h)u\|_\infty \leq c\,h^{2-N/2}\,\|u\|. \tag{3.5}$$

However, if u is more regular, then this result can be improved as shown by Deckelnick and Hinze in [12].

LEMMA 3.6. *For every $u \in W$ and all $\varepsilon > 0$,*

$$\|(S - S_h)u\|_\infty \leq c\, h^{4-N-\varepsilon}\, |\log h|\; \|u\|_W$$

holds true with a constant c only depending on Ω.

To improve the readability, we use the notation

$$\delta(h) := h^{4-N-\varepsilon}\, |\log h| \tag{3.6}$$

in all what follows. The Tikhonov regularization term within the objective function immediately implies that the discrete controls are uniformly bounded in $L^2(\Omega)$. Moreover, because of $\bar{u}_h \in V_h \subset H^1(\Omega)$, we have $\bar{u}_h \in W$. In addition to that, Deckelnick and Hinze proved that, for the semi-discrete approach, the discrete solutions are uniformly bounded in W (cf. [12, Lemma 3.5]). It is easy to see that the same arguments can also be applied in case of the full discretization such that the following result is obtained:

LEMMA 3.7. *The sequence of discrete optimal solutions, denoted by $\{\bar{u}_h\}_{h>0}$, is uniformly bounded in W.*

4. Auxiliary results. Before we are in the position to prove the main convergence theorem, we have to derive some auxiliary results. In particular, Lemma 4.4 is essential for the overall theory. Nevertheless, let us start with a result on the Slater point $\hat{u}$ that follows immediately from the required regularity of $\hat{u}$ (cf. Assumption 2.3).

LEMMA 4.1. *There is an $h_0 > 0$ such that, for all $h \leq h_0$,*

$$(S_h\, \Pi_h \hat{u})(x) \geq \tau_0 > 0 \quad a.e.\ in\ \Omega$$

is valid with a constant $\tau_0 > 0$ independent of h.

Proof. Using Assumption 2.3, the proof follows from standard interpolation error estimates for curved domains (cf. Bernardi [5]) and approximation arguments:

$$\begin{aligned}(S_h\, \Pi_h \hat{u})(x) &= (S\, \hat{u})(x) + \big(S(\Pi_h \hat{u} - \hat{u})\big)(x) + \big((S_h - S)\Pi_h \hat{u}\big)(x)\\ &\geq \tau - \|S\|_{\mathcal{L}(L^2(\Omega), L^\infty(\Omega))}\, \|\Pi_h \hat{u} - \hat{u}\| - c\, h^{2-N/2}\, \|\Pi_h \hat{u}\|\\ &\geq \tau - c\, h^{2-N/2}\, \|\hat{u}\| =: \tau_0.\end{aligned}$$

Hence, if h_0 is chosen sufficiently small, we obtain $\tau_0 > 0$ for all $h \leq h_0$. ∎

Now, we turn to the approximation error for the optimal control $\bar{u}$. As stated above, one has to apply quasi-interpolation to approximate $\bar{u}$. Based on Lemma 3.5, we find the following estimates:

LEMMA 4.2. *For every function $u \in W$, there exists a constant c , independent of u and h, such that*

$$\|u - \Pi_h u\| \leq c\, h^{2-N/2-\varepsilon}\, \|u\|_W \tag{4.1}$$

$$\|u - \Pi_h u\|_{W^*} \leq c\, h^{4-N-\varepsilon}\, \|u\|_W \tag{4.2}$$

hold true for all $\varepsilon > 0$.

Proof. Imbedding theorems imply that $W \hookrightarrow H^t(\Omega)$ with $t = 2 - N/2 - \varepsilon$. Hence, Lemma 3.5 yields

$$\|u - \Pi_h u\| \le c\, h^t\, \|u\|_{H^t(\Omega)} \le c\, h^{2-N/2-\varepsilon}\, \|u\|_W. \tag{4.3}$$

For the second statement, we argue in a standard way: due to (3.2), for every $v_h \in V_h$, it follows

$$\begin{aligned} \|u - \Pi_h u\|_{W^*} &= \sup_{\varphi\in W, \varphi\neq 0} \frac{(u - \Pi_h u\,,\, \varphi)}{\|\varphi\|_W} \\ &= \sup_{\varphi\in W, \varphi\neq 0} \frac{(u - \Pi_h u\,,\, \varphi - v_h)}{\|\varphi\|_W} \\ &= \|u - \Pi_h u\| \sup_{\varphi\in W, \varphi\neq 0} \frac{\|\varphi - v_h\|}{\|\varphi\|_W}. \end{aligned} \tag{4.4}$$

Now, we choose the quasi-interpolant for v_h, i.e. $v_h = I_h\varphi$, such that, analogously to above, Lemma 3.5 implies

$$\|\varphi - I_h\varphi\| \le c\, h^{2-N/2-\varepsilon}\, \|\varphi\|_W.$$

Inserting this together with (4.3) in (4.4) finally yields the assertion. □

LEMMA 4.3. *Let u be an arbitrary function in W. Then, the following estimate holds with a constant c, independent of h and u,*

$$\|S_h(\Pi_h u - u)\|_\infty \le c\, h^{4-N-\varepsilon}\, \|u\|_W$$

for all $\varepsilon > 0$.

Proof. We start with the triangle inequality that implies

$$\|S_h(\Pi_h u - u)\|_\infty \le \|(S_h - S)(\Pi_h u - u)\|_\infty + \|S(\Pi_h u - u)\|_\infty. \tag{4.5}$$

For the first addend, (3.5) and (4.1) yield

$$\|(S_h - S)(\Pi_h u - u)\|_\infty \le c\, h^{4-N-\varepsilon}\, \|u\|_W.$$

It remains to estimate the second addend in (4.5). According to Remark 3.2, we have $S\,u \in L^\infty(\Omega)$ if $u \in W^{-1,N+\varepsilon}(\Omega)$, $\varepsilon > 0$. Moreover, due to $\sigma < N/(N-1)$, the associated conjugate exponent σ', defined by $1/\sigma + 1/\sigma' = 1$, satisfies $\sigma' > N$ and hence $W^* = W^{1,\sigma}(\Omega)^* \hookrightarrow W^{-1,N+\varepsilon}(\Omega)$. Consequently, we obtain

$$\|S(\Pi_h u - u)\|_\infty \le c\, \|\Pi_h u - u\|_{W^*} \le c\, h^{4-N-\varepsilon}\, \|u\|_W,$$

where we used (4.2) for the last estimate. □

With these results at hand, we are now able to show the key point of our convergence theory. Here, by using the Slater condition, we prove the feasibility of $\bar{u}_h - c\,\delta(h)\hat{u}$ for the infinite dimensional problem (P), where $\delta(h)$ is as defined in (3.6). On the other hand, $\Pi_h\bar{u} - c\,\delta(h)\,\Pi_h\hat{u}$ is feasible for the discrete problem (P_h). This two-way feasibility represents the basis for the convergence theory in Section 5.

LEMMA 4.4. *Let δ be defined by* (3.6). *Then there exist positive constants γ_1 and γ_2, each independent of h, such that, the function v_1, defined by*

$$v_1 := \bar{u}_h - \gamma_1\, \delta(h)\, \hat{u},$$

is feasible for (P), *whereas, for all $h < h_0$,*

$$v_2 := \Pi_h \bar{u} - \gamma_2\, \delta(h)\, \Pi_h \hat{u}$$

is feasible for (P_h).

Proof. First, we show $(S\, v_1)(x) \leq y_b(x)$ a.e. in Ω. Together with Assumption 2.3 and Lemma 3.6, the feasibility of $\bar{u}_h$ for (P_h) implies

$$\begin{aligned}(S\, v_1)(x) &= (S_h\, \bar{u}_h)(x) + \big((S - S_h)\bar{u}_h\big)(x) - \gamma_1\, \delta(h)\, (S\hat{u})(x)\\ &\leq y_b(x) + \|(S - S_h)\bar{u}_h\|_\infty - \gamma_1\, \delta(h)\, \tau\\ &\leq y_b(x) - \big(\gamma_1\, \tau - c\, \|\bar{u}_h\|_W\big)\delta(h)\end{aligned} \tag{4.6}$$

for almost all $x \in \Omega$. Because of Lemma 3.7, $\|\bar{u}_h\|_W$ is bounded by a constant independent of h and hence, (4.6) yields the feasibility of v_1 for sufficiently large γ_1. Next, let us turn to the feasibility of v_2 for (P_h). First, we have $v_2 \in V_h$ by construction. To verify the inequality constraints in (P_h), we deduce from Lemma 4.3, Lemma 3.6, and Lemma 4.1 that

$$\begin{aligned}(S_h\, v_2)(x) &= (S\, \bar{u})(x) + \big(S_h(\Pi_h\bar{u} - \bar{u})\big)(x) + \big((S_h - S)\bar{u}\big)(x) - \gamma_2\, \delta(h)\, (S_h\, \Pi_h\hat{u})(x)\\ &\leq y_b(x) + \|S_h(\Pi_h\bar{u} - \bar{u})\|_\infty + \|(S - S_h)\bar{u}\|_\infty - \gamma_2\, \delta(h)\, \tau_0\\ &\leq y_b(x) + c\, h^{4-N-\varepsilon}\, \|u\|_W + c\, \delta(h)\, \|\bar{u}\|_W - \gamma_2\, \delta(h)\, \tau_0,\end{aligned} \tag{4.7}$$

and hence

$$(S_h\, v_2)(x) \leq y_b(x) - \big(\gamma_2\, \tau_0 - c\, \|u\|_W\big)\, \delta(h).$$

Due to $\bar{u} \in W$, the expression in the brackets is non-negative, if γ_2 is chosen sufficiently large, giving in turn the assertion. ∎

The following lemma is an immediate consequence of the variational inequalities for (P) and (P_h).

LEMMA 4.5. *For every $v \in U_{ad}$ and every $v_h \in U_{ad}^h$, we find*

$$\begin{aligned}&\alpha\, \|\bar{u} + \bar{u}_h\|^2 + \|S\, \bar{u} - S_h\, \bar{u}_h\|^2\\ &\quad\leq \alpha\, (\bar{u}\,,\, v - \bar{u}_h) + \alpha\, (\bar{u}\,,\, v_h - \bar{u}) + \alpha\, (\bar{u}_h - \bar{u}\,,\, v_h - \bar{u})\\ &\qquad + \Big(S_h\, \bar{u}_h - S\, \bar{u}\,,\, (S_h - S)v_h + S(v_h - \bar{u})\Big)\\ &\qquad + \Big(S\, \bar{u} - y_d\,,\, S(v - \bar{u}_h) + S(v_h - \bar{u}) + (S - S_h)\bar{u}_h + (S_h - S)v_h\Big).\end{aligned} \tag{4.8}$$

Proof. The proof is completely analogous to the control-constrained case presented by Falk in [14] and follows from straight forward computation. We start with the variational inequalities for (P) and (P_h), respectively, given by

$$(S\, \bar{u} - y_d\,,\, S\, v - S\, \bar{u}) + \alpha\, (\bar{u}\,,\, v - \bar{u}) \geq 0 \quad \forall\, v \in U_{ad} \tag{4.9}$$

$$(S_h\, \bar{u}_h - y_d\,,\, S_h\, v_h - S_h\, \bar{u}_h) + \alpha\, (\bar{u}_h\,,\, v_h - \bar{u}_h) \geq 0 \quad \forall\, v_h \in U_{ad}^h. \tag{4.10}$$

Adding both inequalities yields

$$\overbrace{(S\,\bar u - y_d\,,\,S\,v - S\,\bar u) + (S_h\,\bar u_h - y_d\,,\,S_h\,v_h - S_h\,\bar u_h)}^{=:\,A} \\ + \alpha\,\underbrace{\big[(\bar u\,,\,v - \bar u) + (\bar u_h\,,\,v_h - \bar u_h)\big]}_{=:\,B} \geq 0 \tag{4.11}$$

for all $v \in U_{ad}$ and all $v_h \in U_{ad}^h$. Straight forward computations show for A and B

$$\begin{aligned} B &= (\bar u\,,\,v - \bar u_h) + (\bar u\,,\,\bar u_h - \bar u) + (\bar u_h\,,\,v_h - \bar u) + (\bar u_h\,,\,\bar u - \bar u_h) \\ &\leq -\|\bar u - \bar u_h\|^2 + (\bar u\,,\,v - \bar u_h) + (\bar u\,,\,v_h - \bar u) + (\bar u_h - \bar u\,,\,v_h - \bar u) \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} A &= \Big(S\,\bar u - y_d\,,\,S(v - \bar u_h) + (S - S_h)\bar u_h + S_h\,\bar u_h - S\,\bar u\Big) \\ &\quad + \Big(S_h\,\bar u_h - y_d\,,\,(S_h - S)v_h + S(v_h - \bar u) + S\,\bar u - S_h\bar u_h\Big) \\ &= \Big(S\,\bar u - y_d\,,\,S(v - \bar u_h) + S(v_h - \bar u) + (S - S_h)\bar u_h + (S_h - S)v_h\Big) \\ &\quad + \Big(S_h\,\bar u_h - S\,\bar u\,,\,(S_h - S)v_h + S(v_h - \bar u)\Big) - \|S_h\,\bar u_h - S\,\bar u\|^2. \end{aligned} \tag{4.13}$$

Inserting (4.12) and (4.13) in (4.11) finally implies the assertion. ∎

5. Convergence analysis. With the results of the previous section at hand, in particular Lemma 4.4, we are now able to prove our main result, which is the following convergence theorem:

THEOREM 5.1. *Let $\bar u$ denote the optimal solution of* (P), *while $\bar u_h$ is the optimal solution of* (P_h). *Then the following estimate holds true*

$$\|\bar u - \bar u_h\| + \|S\,\bar u - S_h\,\bar u_h\| \leq C\,h^{2-N/2-\varepsilon}$$

for all $\varepsilon > 0$ with a constant C depending on ε, Ω, α, $\bar u$, and $\hat u$.

Proof. We start by estimating the right hand side of (4.8). For the first two expressions, we obtain

$$(\bar u\,,\,v - \bar u_h) + (\bar u\,,\,v_h - \bar u) \leq \|\bar u\|_W\big(\|v - \bar u_h\|_{W^*} + \|v_h - \bar u\|_{W^*}\big).$$

The next two addends are estimated by using Young's inequality such that

$$(\bar u_h - \bar u\,,\,v_h - \bar u) \leq \frac{1}{2}\,\|\bar u_h - \bar u\|^2 + \frac{1}{2}\,\|v_h - \bar u\|^2$$

and

$$\begin{aligned} &\Big(S_h\,\bar u_h - S\,\bar u\,,\,(S_h - S)v_h + S(v_h - \bar u)\Big) \\ &\qquad\qquad \leq \frac{1}{2}\,\|S_h\,\bar u_h - S\,\bar u\|^2 + \|(S_h - S)v_h\|^2 + \|S(v_h - \bar u)\|^2 \\ &\qquad\qquad \leq \frac{1}{2}\,\|S_h\,\bar u_h - S\,\bar u\|^2 + \|(S_h - S)v_h\|^2 + c\,\|v_h - \bar u\|_{W^*}^2, \end{aligned}$$

are obtained. Here, we used the continuity of S from W^* to $H^1(\Omega)$ that follows from $S : H^1(\Omega)^* \to H^1(\Omega)$ continuously and $W^* \subset H^1(\Omega)^*$ because of $H^1(\Omega) \subset W$. The last term on the right hand side of (4.8) is estimated by the Cauchy-Schwarz inequality, i.e.

$$\begin{aligned}
&\Big(S\,\bar u - y_d\,,\, S(v - \bar u_h) + S(v_h - \bar u) + (S - S_h)\bar u_h + (S_h - S)v_h\Big)\\
&\quad \le c\,\|S\,\bar u - y_d\|\Big(\|v - \bar u_h\|_{W^*} + \|v_h - \bar u\|_{W^*} + \|(S - S_h)\bar u_h\| + \|(S_h - S)v_h\|\Big),
\end{aligned}$$

where we again used $S : W^* \to H^1(\Omega)$ continuously. Inserting these estimates in (4.8) yields

$$\begin{aligned}
&\frac{\alpha}{2}\,\|\bar u + \bar u_h\|^2 + \frac{1}{2}\,\|S\,\bar u - S_h\,\bar u_h\|^2\\
&\quad \le \frac{\alpha}{2}\,\|v_h - \bar u\|^2\\
&\qquad + \Big(\alpha\,\|\bar u\|_W + c\,\|S\,\bar u - y_d\|\Big)\Big(\|v - \bar u_h\|_{W^*} + \|v_h - \bar u\|_{W^*}\Big)\\
&\qquad + c^2\,\|v_h - \bar u\|_{W^*}^2 + \|(S - S_h)v_h\|^2\\
&\qquad + \|S\,\bar u - y_d\|\Big(\|(S - S_h)\bar u_h\| + \|(S - S_h)v_h\|\Big) \qquad \forall\, v \in U_{ad},\ v_h \in U_{ad}^h.
\end{aligned} \tag{5.1}$$

Thanks to Lemma 4.4, we are now allowed to insert $v = v_1$ and $v_h = v_2$. Let t again be defined by $t = 2 - N/2 - \varepsilon$. Then, by means of Lemma 4.2, we obtain

$$\begin{aligned}
\|v_h - \bar u\| &\le \|\Pi_h\bar u - \bar u\| + \gamma_2\,\delta(h)\,\|\Pi_h\hat u\|\\
&\le \big(c\,\|\bar u\|_W + \gamma_2\,\|\hat u\|\big)\max\{h^t, \delta(h)\} = c_1\,h^t,
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
\|v_h - \bar u\|_{W^*} &\le \|\Pi_h\bar u - \bar u\|_{W^*} + \gamma_2\,\delta(h)\,\|\Pi_h\hat u\|_{W^*}\\
&\le \big(c\,\|\bar u\|_W + c\,\gamma_2\,\|\hat u\|\big)\max\{h^{2t}, \delta(h)\} = c_2\,\delta(h),
\end{aligned} \tag{5.3}$$

and in case of $v = v_1$

$$\|v - \bar u_h\|_{W^*} \le c\,\gamma_1\,\delta(h)\,\|\hat u\| = c_3\,\delta(h). \tag{5.4}$$

For the remaining expressions in (5.1), one can apply (3.4), i.e.

$$\begin{aligned}
\|(S_h - S)v_h\| &\le c\,h^2\,\|\Pi_h\bar u - \gamma_2\,\delta(h)\,\Pi_h\hat u\|\\
&\le c\,h^2\,\big(\|\bar u\| + \gamma_2\,\|\hat u\|\big) = c_4\,h^2
\end{aligned} \tag{5.5}$$

and

$$\|(S_h - S)\bar u_h\| \le c\,h^2\,\|\bar u_h\| = c_5\,h^2, \tag{5.6}$$

where the optimality of $\bar u_h$ guarantees its uniform boundedness in $L^2(\Omega)$ such that c_5 is independent of h. If, we now insert (5.2)–(5.6) in (5.1), we obtain

$$\begin{aligned}
\frac{\alpha}{2}\,\|\bar u + \bar u_h\|^2 &+ \frac{1}{2}\,\|S\,\bar u - S_h\,\bar u_h\|^2\\
&\le \frac{\alpha}{2}\,c_1^2\,h^{2t} + \Big(\alpha\,\|\bar u\|_W + c\,\|S\,\bar u - y_d\|\Big)\,(c_2 + c_3)\,\delta(h)\\
&\quad + c^2\,c_2^2\,\delta(h)^2 + c_4\,h^2 + \|S\,\bar u - y_d\|\,(c_4 + c_5)\,h^2\\
&\le C\,\delta(h).
\end{aligned}$$

With the definition of $\delta(h)$ and t, we therefore end up with

$$\|\bar u - \bar u_h\|^2 + \|S\,\bar u - S_h\,\bar u_h\|^2 \le C\,h^{4-N-\varepsilon}\,|\log h|, \tag{5.7}$$

where C is independent of ε. Now, for a fixed $\varepsilon > 0$, there exists a constant $c(\varepsilon)$ such that

$$h^{4-N-\varepsilon}\,|\log h| = h^{4-N-2\varepsilon}\,h^{\varepsilon}\,|\log h| \le c(\varepsilon)\,h^{4-N-2\varepsilon}.$$

Notice however that $c(\varepsilon) \to \infty$ if $\varepsilon \downarrow 0$. Hence, the right hand side in (5.7) can be estimated by $C\,h^{4-N-2\varepsilon}$ with C depending on ε, which gives in turn the assertion. ∎

Using standard finite element error estimates, we deduce

$$\begin{aligned}\|S\,u - S_h\,u_h\|_{H^1(\Omega)} &\le \|S(u-u_h)\|_{H^1(\Omega)} + \|(S-S_h)u_h\|_{H^1(\Omega)}\\ &\le c\,\|u-u_h\| + c\,h\,\|u_h\|.\end{aligned}$$

Hence, Theorem 5.1 implies the following result:

COROLLARY 5.2. *For the optimal states of* (P) *and* (P_h)*, we have*

$$\|\bar y - \bar y_h\|_{H^1(\Omega)} \le c\,h^{2-N/2-\varepsilon}.$$

6. A problem with pointwise state and control constraints. As already mentioned in the introduction, the previous theory for (P) can be adapted to problem (Q) with additional box-constraints on the control. Analogously to (P), we introduce the reduced optimal control problem by

$$(\mathrm{Q})\quad\left\{\begin{array}{ll} \min\limits_{u\in L^2(\Omega)} & f(u) := \dfrac{1}{2}\,\|S\,u - y_d\|^2 + \dfrac{\alpha}{2}\,\|u\|^2\\ \text{subject to} & y_a(x) \le (S\,u)(x) \le y_b(x) \quad \text{a.e. in } \Omega\\ & \quad u_a \le \quad u(x) \quad \le u_b \qquad \text{a.e. in } \Omega.\end{array}\right.$$

Beside Assumption 2.1, we need the following assumptions on the additional quantities in (Q):

ASSUMPTION 6.1. *The bounds y_a and y_b are given in $C(\bar\Omega)$. Moreover, u_a and u_b are real numbers satisfying $u_a \le u_b$.*

In contrast to the discretization of problem (P), the control is now discretized by *piecewise constant ansatz functions*, while the discrete state is still an element of V_h as defined in Definition 1.

DEFINITION 3. *The space of discrete controls is given by*

$$U_h = \{u_h \in L^2(\Omega) \mid u|_T = const.\ \forall\, T \in \mathcal{T}_h\}.$$

Notice that $U_h \not\subseteq W$. With the discrete control-to-state mapping, again denoted by S_h, the discrete optimal control problem now reads

$$(\mathrm{Q}_h)\quad\left\{\begin{array}{ll} \min\limits_{u\in U_h} & f_h(u) := \dfrac{1}{2}\,\|S_h\,u - y_d\|^2 + \dfrac{\alpha}{2}\,\|u\|^2\\ \text{subject to} & y_a(x) \le (S_h\,u)(x) \le y_b(x) \quad \text{a.e. in } \Omega\\ & \quad u_a \le \quad u(x) \quad \le u_b \qquad \text{a.e. in } \Omega.\end{array}\right.$$

Again, a *Slater condition* is needed to derive first-order necessary conditions Similarly to Assumption 2.3, it is now given by:

ASSUMPTION 6.2. *A function $\hat{u} \in W$ exists such that*

$$\begin{aligned} y_a(x) + \tau \leq (S\,\hat{u})(x) &\leq y_b(x) - \tau \\ u_a \leq \quad \hat{u}(x) \quad &\leq u_b \end{aligned}$$

holds for all $x \in \bar{\Omega}$ with some $\tau > 0$.

As in case of (P), the Karush-Kuhn Tucker theory implies the existence of Lagrange multipliers $\mu_a, \mu_b \in \mathcal{M}(\Omega)$ associated to the state constraints in (Q). Again, the lack of regularity of the multipliers impairs the regularity of p such that $p \in W$. Moreover, the pointwise control constraints in (Q) can be discussed in a standard way such that the overall optimality system reads as follows:

$$\left.\begin{aligned} -\Delta\,\bar{y} + \bar{y} = \bar{u} \quad \text{in } \Omega \qquad & -\Delta\,p + p = \bar{y} - y_d + \mu_{b,\Omega} - \mu_{a,\Omega} \quad \text{in } \Omega \\ \partial_n\,\bar{y} = 0 \quad \text{on } \Gamma \qquad & \partial_n\,p = \mu_{b,\Gamma} - \mu_{a,\Gamma} \quad \text{on } \Gamma \\ \bar{u}(x) = \Pi_{ad}\Big\{ -\frac{1}{\alpha}\,p(x)\Big\} & \\ y_a(x) \leq \bar{y}(x) \leq y_b(x) \quad \forall\, x \in \bar{\Omega} & \\ \int_{\bar{\Omega}} (y_a - \bar{y})\,d\mu_a = 0\,, \quad \int_{\bar{\Omega}} (\bar{y} - y_b)\,d\mu_b = 0 & \\ \int_{\bar{\Omega}} y\,d\mu_a \geq 0\,, \quad \int_{\bar{\Omega}} y\,d\mu_b \geq 0 \quad \forall\, y \in C(\bar{\Omega})^+, & \end{aligned}\right\} \tag{6.1}$$

where Π_{ad} denotes the pointwise projection operator on $[u_a, u_b]$. Hence, we have $u \in W \cap L^\infty(\Omega)$.

Our aim is now to derive results analogous to the ones in Section 4 for the new discrete control space U_h. Therefore, let us define the projection of a function $u \in L^2(\Omega)$ on U_h. Based on (3.2), it is straight forward to see that $\Pi_h : L^2(\Omega) \to U_h$ is given by

$$\Pi_h u|_T = \frac{1}{|T|} \int_T u\,dx \quad \forall\, T \in \mathcal{T}_h.$$

LEMMA 6.3. *For every $u \in W$, it holds*

$$\|u - \Pi_h u\| \leq c\,h^{2-N/2-\varepsilon}\,\|u\|_W,$$

for all $\varepsilon > 0$ with a constant c only depending on Ω.

Proof. Let T be an arbitrary element of $\mathcal{T}_h$. Then, according to Stampacchia [21, Theorem 6.6], one finds

$$\|u - \Pi_h u\|_{L^{\sigma^*}(T)} \leq c\,\frac{h^N}{|T|}\,\|u\|_{W^{1,\sigma}(T)},$$

where σ^* is defined by $\sigma^* = N\,\sigma/(N-\sigma)$. Together with the definition of σ, this yields $\sigma^* < N/(N-2)$. Applying Hölder's inequality then yields

$$\|u - \Pi_h u\|_{L^2(T)} \leq |T|^{(\sigma^*-2)/(2\sigma^*)}\,\|u - \Pi_h u\|_{L^{\sigma^*}(T)}$$

and hence

$$\|u - \Pi_h u\|_{L^2(T)} \le c\, h^N\, |T|^{(\sigma^*-2)/(2\,\sigma^*)-1}\, \|u\|_{W^{1,\sigma}(T)}. \tag{6.2}$$

Now, by definition of h, there is a constant c such that $|T| \le c\, h^N$. Thus, regarding $\sigma^* < N/(N-2)$, we obtain

$$h^N\, |T|^{(\sigma^*-2)/(2\,\sigma^*)-1} = h^{N(\sigma^*-2)/(2\,\sigma^*)} \le h^{2-N/2-\varepsilon}. \tag{6.3}$$

Now, given an arbitrary set of non-negative real numbers $\{a_i\}$, we have $\sum_i a_i^{2/\sigma} \le \big(\sum_i a_i\big)^{2/\sigma}$ since $2/\sigma > (2N-2)/N \ge 1$ for $N = 2, 3$. Hence, together with (6.3), (6.2) implies

$$\begin{aligned}\|u - \Pi_h u\|^2_{L^2(\Omega)} &= c\, h^{4-N-2\varepsilon} \sum_{T\in\mathcal{T}_h} \big(\|u\|^{\sigma}_{W^{1,\sigma}(T)}\big)^{2/\sigma} \\ &\le c\, h^{4-N-\varepsilon} \|u\|^2_W,\end{aligned} \tag{6.4}$$

giving in turn the assertion. ∎

Now, we can argue analogously to the proof of Lemma 4.2 and Lemma 4.3, respectively, (with Π_h instead of I_h) to obtain the following result:

Corollary 6.4. *Suppose that $u \in W$. Then, the following estimates hold true*

$$\|u - \Pi_h u\|_{W^*} \le c\, h^{4-N-\varepsilon}\, \|u\|_W \tag{6.5}$$

$$\|S(\Pi_h u - u)\|_\infty \le c\, h^{4-N-\varepsilon}\, \|u\|_W, \tag{6.6}$$

for all $\varepsilon > 0$.

Based on Lemma 6.3, a discussion analogous to the proof of Lemma 4.1 yields the following result:

Lemma 6.5. *There exists a $\tau_0 > 0$, independent of h such that,*

$$y_a(x) + \tau_0 \le (S_h\, \Pi_h \hat{u})(x) \le y_b(x) - \tau_0$$

holds for all $0 < h \le h_0$.

As mentioned above, we have $U_h \not\subseteq W$ such that one cannot use this additional smoothness for the estimation of $\|(S - S_h)\bar{u}_h\|_\infty$ as done in the proof of Lemma 3.6 (see [12]). However, here we benefit from the additional control constraints that guarantee $\bar{u}, \bar{u}_h \in L^\infty(\Omega)$. For a corresponding lemma, we argue analogously to Deckelnick and Hinze [12, Lemma 3.4].

Lemma 6.6. *Suppose that $u \in L^q(\Omega)$ is given with $N < q < \infty$. Then a constant c independent of h and u exists such that*

$$\|(S - S_h)u\|_\infty \le c\, h^{2-N/q}\, |\log h|\, \|u\|_q. \tag{6.7}$$

Proof. Let us introduce the notations $y = S\,u$ and $y_h = S_h\, u$. First, according to Grisvard [15], $u \in L^q(\Omega)$ implies $y = S\,u \in W^{2,q}(\Omega) \subset W^{1,\infty}(\Omega)$, where the embedding is guaranteed by the assumption $q > N$. For $y \in W^{1,\infty}(\Omega)$, Schatz proved in [20, Theorem 2.2] that

$$\|y - y_h\|_\infty \le c\, |\log h|\, \|y - I_h y\|_\infty,$$

where I_h again denotes the interpolation operator. Now, together with interpolation error estimates for curved domains (cf. Bernardi [5]), the regularity of y grants

$$\|y - I_h y\|_{L^\infty(\Omega)} \leq c\, h^{2-N/q}\, \|y\|_{W^{2,q}(\Omega)} \leq c\, h^{2-N/q}\, \|u\|_q,$$

which concludes the proof. ∎

This immediately implies the following result:

REMARK 6.7. *For every $u \in L^\infty(\Omega)$, there exists a constant c, independent of u and h, such that for all $\varepsilon > 0$*

$$\|(S - S_h)u\|_\infty \leq c\, h^{2-\varepsilon}\, |\log\, h|\, \|u\|_\infty$$

is valid.

Notice that, thanks to the control constraints, we do not need the uniform boundedness of the discrete controls in W as stated by Lemma 3.7 for the analysis of (Q). Similarly to (3.6), we introduce the following abbreviation

$$\eta(h) := h^{2-\varepsilon}\, |\log\, h|.$$

Using the previous results, we are now ready to state the analogon to Lemma 4.4, which is again the crucial point in the overall convergence theory.

LEMMA 6.8. *There exists a positive constant γ, independent of h, such that, the function v_1, defined by*

$$v_1 := \bar{u}_h + \gamma\, \eta(h)\, (\hat{u} - \bar{u}_h),$$

is feasible for (Q)*. On the other hand, there is an h_0 such that*

$$v_2 := \Pi_h \bar{u} + \gamma\, \eta(h)\, (\Pi_h \hat{u} - \Pi_h \bar{u})$$

is feasible for (Q_h) *for all $h < h_0$.*

Proof. With the previous results at hand, the proof is similar to the one of Lemma 4.4. We exemplarily show the feasibility of v_2. In case of v_1, the arguments are analogous. First, we have $v_2 \in U_h$ by construction. Hence, it remains to show that v_2 satisfies the inequality constraints in (Q_h). Clearly, if $u(x) \in [u_a, u_b]$ for almost all $x \in \Omega$, then $(\Pi_h u)(x) \in [u_a, u_b]$ follows a.e. in Ω. Hence, we have $(\Pi_h \bar{u})(x), (\Pi_h \hat{u})(x) \in [u_a, u_b]$ a.e. in Ω due to Assumption 6.2. Moreover, for h sufficiently small, we have $\gamma\, \eta(h) \leq 1$ such that v_2 is a convex linear combination of two functions in $[u_a, u_b]$ and consequently $u_a \leq v_2(x) \leq u_b$ a.e. in Ω. For the upper state constraint in (Q_h), Lemma 6.5, Corollary 6.4, and Remark 6.7 imply

$$\begin{aligned}
(S_h\, v_2)(x) = {} & [1 - \gamma\, \eta(h)](S\, \bar{u})(x) + [1 - \gamma\, \eta(h)]\big(S(\Pi_h \bar{u} - \bar{u})\big)(x) \\
& + [1 - \gamma\, \eta(h)]\big((S_h - S)\Pi_h \bar{u}\big)(x) + \gamma\, \eta(h)\, (S_h\, \Pi_h \hat{u})(x) \\
\leq {} & [1 - \gamma\, \eta(h)]\, y_b(x) + \gamma\, \eta(h)\, (y_b(x) - \tau_0) \\
& + [1 - \gamma\, \eta(h)]\, \Big(\|S(\Pi_h \bar{u} - \bar{u})\|_\infty + \|(S - S_h)\Pi_h \bar{u}\|_\infty\Big) \\
\leq {} & y_b(x) - \gamma\, \eta(h)\, \tau_0 + c\, [1 - \gamma\, \eta(h)]\, \big(h^{4-N-\varepsilon}\, \|\bar{u}\|_W + \eta(h)\, \|\Pi_h \bar{u}\|_\infty\big) \\
\leq {} & y_b(x) - \Big(\gamma\, \tau_0 - c\big(\|\bar{u}\|_W + \|\bar{u}\|_\infty\big)\Big)\, \max\{h^{4-N-\varepsilon}, \eta(h)\}.
\end{aligned}$$

Here, we used that $\|\Pi_h \bar{u}\|_\infty \leq \|\bar{u}\|_\infty$. Since $\bar{u}$ is bounded in W and $L^\infty(\Omega)$ because of the control constraints, the expression in the brackets is non-negative if γ is chosen sufficiently large. Notice that γ depends on $\bar{u}$, u_a, and u_b, but not on h. The lower state constraint, i.e. $(S_h\, v_2)(x) \geq y_a(x)$ a.e. in Ω, can be discussed analogously giving the assertion on v_2. Using again Remark 6.7 and Assumption 6.2, it is straight forward to show the feasibility of v_1 for (Q). Here, one again benefits from the control constraints in (Q_h) that imply $\|\bar{u}_h\|_\infty \leq \max\{|u_a|, |u_b|\}$ for all h. ∎

The remaining analysis follows the lines of the previous sections. First, Lemma 4.5 clearly also holds in case of (Q), with

$$
\begin{aligned}
U_{ad} &:= \{u \in L^2(\Omega) \mid u_a \leq u(x) \leq u_b \;\text{ and }\; y_a(x) \leq (S\,u)(x) \leq y_b(x) \text{ a.e. in } \Omega\}\\
U_{ad}^h &:= \{u_h \in U_h \mid u_a \leq u_h(x) \leq u_b \;\text{ and }\; y_a(x) \leq (S_h\,u_h)(x) \leq y_b(x) \text{ a.e. in } \Omega\}.
\end{aligned}
$$

Furthermore, with Lemma 6.3, Corollary 6.4, and Lemma 6.8, we obtain the following estimates instead of (5.2)–(5.4):

$$
\begin{aligned}
\|v_2 - \bar{u}\| &\leq \|\Pi_h \bar{u} - \bar{u}\| + \gamma\, \eta(h)\, \|\Pi_h \hat{u} - \Pi_h \bar{u}\|\\
&\leq \Big(c\, \|\bar{u}\|_W + \gamma\big(\|\hat{u}\| + \|\bar{u}\|\big)\Big) \max\{h^t, \eta(h)\} = c_1\, h^t,\\
\|v_2 - \bar{u}\|_{W^*} &\leq \|\Pi_h \bar{u} - \bar{u}\|_{W^*} + \gamma\, \eta(h)\, \|\Pi_h \hat{u} - \Pi_h \bar{u}\|_{W^*}\\
&\leq \Big(c\, \|\bar{u}\|_W + c\,\gamma\big(\|\hat{u}\| + \|\bar{u}\|\big)\Big) \max\{h^{2t}, \eta(h)\}\\
&\leq c_2 \max\{h^{2t}, \eta(h)\},\\
\|v_1 - \bar{u}_h\|_{W^*} &= c\,\gamma\, \eta(h)\, \|\hat{u} - \bar{u}_h\| = c_3\, \eta(h),
\end{aligned}
$$

where t is as above defined by $t = 2 - N/2 - \varepsilon$. Moreover, using (3.4) for the L^2-approximation error, one finds analogously to (5.5) and (5.6)

$$
\begin{aligned}
\|(S_h - S)v_2\| &\leq c\, h^2\, \|\Pi_h \bar{u} - \gamma\, \eta(h)\, (\Pi_h \hat{u} - \Pi_h \bar{u})\|\\
&\leq c\, h^2 \left((1+\gamma)\|\bar{u}\| + \gamma\, \|\hat{u}\|\right) = c_4\, h^2,\\
\|(S_h - S)\bar{u}_h\| &\leq c\, h^2\, \|\bar{u}_h\| = c_5\, h^2.
\end{aligned}
$$

Therefore, with these estimates at hand, we can proceed analogously to the proof of Theorem 5.1 and in this way, one obtains the following result:

THEOREM 6.9. *Suppose that $\bar{u}$ and $\bar{u}_h$ are the optimal solutions of* (Q) *and* (Q_h)*, respectively. Then the following estimate holds true*

$$
\|\bar{u} - \bar{u}_h\| + \|S\,\bar{u} - S_h\,\bar{u}_h\| \leq C\, h^{2-N/2-\varepsilon}
$$

for all $\varepsilon > 0$ with a constant C depending on ε, Ω, α, $\bar{u}$, and $\hat{u}$.

REMARK 6.10. *Notice that, in case of $N = 3$, the overall error is not longer dominated by the L^∞-error of the finite element approximation (cf. Lemma 6.7), but by the interpolation error of $\bar{u}$ (see Lemma 6.3). Consequently, the constant C in Theorem 6.9 does not depend on ε in the three-dimensional case.*

Analogously to Corollary 5.2, one shows the following estimate:

COROLLARY 6.11. *For the optimal states of* (Q) *and* (Q_h)*, it follows*

$$
\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)} \leq c\, h^{2-N/2-\varepsilon}.
$$

7. Discussion of the error estimates. In the following section, we highlight several aspects of the error analysis presented before. First, one observes that, for simplicity, we have not considered the discretization of the desired state y_d as well as the bounds y_a and y_b. However, it is easy to see that, if y_d, y_a, and y_b are sufficiently smooth, then the arguments can be modified such that the presented theory still holds in case of a discretizion of y_d and the bounds. For convenience of the reader, we shortly present the corresponding arguments. In case of a discretization of y_d, the variational inequality (4.10) for the discrete problem has to be replaced by

$$\begin{aligned}(S_h\,\bar u_h - y_d\,,\,S_h\,v_h - S_h\,\bar u_h) + \alpha\,(\bar u_h\,,\,v_h - \bar u_h)\\ + (y_d - I_h\,y_d\,,\,S_h\,v_h - S_h\,\bar u_h) \geq 0 \quad \forall\, v_h \in U_{ad}^h.\end{aligned}$$

If we assume $y_d \in H^2(\Omega)$, the additional term is estimated by

$$(y_d - I_h\,y_d\,,\,S_h\,v_h - S_h\,\bar u_h) \leq \|y_d - I_h\,y_d\|\,\|S_h\,(v_h - \bar u_h)\| \leq c\,h^2\,\|v_h - \bar u_h\|$$

with $v_h = \Pi_h\bar u - \gamma_2\,\delta(h)\,\Pi_h\hat u$ in case of problem (P) and $v_h = \Pi_h\bar u + \gamma\,\eta(h)\,(\Pi_h\hat u - \Pi_h\bar u)$ for problem (Q). Clearly, in both cases, $\|v_h - \bar u_h\|$ is uniformly bounded by a constant because of the optimality of $\bar u$ and $\bar u_h$ such that the additional term does not influence the theory. If y_a and y_b are discretized, the proofs of Lemma 4.4 and Lemma 6.8, respectively, have to be modified. We exemplarily study the first part of Lemma 4.4. The other cases can be discussed analogously. To derive the feasibility of $v_1 := \bar u_h - \gamma_1\,\delta(h)\,\hat u$ for (P), we argue similarly to the original proof of Lemma 4.4:

$$\begin{aligned}(S\,v_1)(x) &= (S_h\,\bar u_h)(x) + \big((S - S_h)\bar u_h\big)(x) - \gamma_1\,\delta(h)\,(S\hat u)(x)\\ &\leq I_h\,y_b(x) + \|(S - S_h)\bar u_h\|_\infty - \gamma_1\,\delta(h)\,\tau\\ &\leq y_b(x) + \|I_h\,y_b - y_b\|_\infty - \big(\gamma_1\,\tau - c\,\|\bar u_h\|_W\big)\delta(h).\end{aligned}$$

If y_b is sufficiently smooth, i.e. $y_b \in W^{2,\infty}(\Omega)$, then interpolation error estimates for curved domains yield

$$\|I_h\,y_b - y_b\|_\infty \leq c\,h^2\,\|y_b\|_{W^{2,\infty}(\Omega)}, \tag{7.1}$$

giving in turn the feasibility of v_1 for (P) provided that γ_1 is chosen sufficiently large.

The analysis, presented in the sections before, is developed for triangulations that exactly fit a $C^{1,1}$-domain. Naturally, this assumption is fairly artificial. However, the regularity of Ω is required for the second part of Theorem 3.1, i.e. $S : L^p(\Omega) \to W^{2,p}(\Omega)$ for all $p < \infty$. This property of S is needed within the proof of Lemma 3.6 and Lemma 6.6, respectively. In case of polyhedral domains, where exact triangulations are evident, this additional regularity can in general not be expected. Nevertheless, if Ω is a convex and polyhedral domain in two dimensions, one has $S : L^p(\Omega) \to W^{2,p}(\Omega)$ for all $p \leq q$ with some $q > N$ depending on the maximum angle of all corners of Γ (cf. Dauge [11]). Moreover, if the maximum angle is less or equal than $\pi/2$, then the assertion again holds for all $p < \infty$ and thus, in this case, the presented error analysis applies in this case.

Next, let us turn to the semi-discrete approach according to Deckelnick and Hinze [12]. As already mentioned in the introduction, this approach coincides with the full discretization in the absence of additional control constraints, i.e. in case of problem (P). In contrast to that, the corresponding solutions differ from each other in case of problem (Q). However, one can easily verify that the theory, presented in Section 6, also applies to the semi-discretization of (Q), which reads

$$
(\mathrm{Q}_{sh}) \quad \begin{cases} \min\limits_{u \in L^2(\Omega)} & f_h(u) := \dfrac{1}{2}\,\|S_h\, u - y_d\|^2 + \dfrac{\alpha}{2}\,\|u\|^2 \\ \text{subject to} & y_a(x) \le (S_h\, u)(x) \le y_b(x) \quad \text{a.e. in } \Omega \\ & \quad u_a \le \quad u(x) \quad \le u_b \qquad \text{a.e. in } \Omega. \end{cases}
$$

In this case, the arguments are even simpler since we do not have to account for the approximation error of the control (see Lemma 6.3), as it is not discretized here. Therefore, the error is dominated by the FEM-discretization error (cf. Lemma 6.6), and one obtains the following result:

THEOREM 7.1. *Let $\bar{u}$ and $\bar{u}_{sh}$ denote the optimal solutions of* (Q) *and* (Q_{sh})*, respectively. Then the following estimate holds true*

$$
\|\bar{u} - \bar{u}_{sh}\| + \|S\,\bar{u} - S_h\,\bar{u}_{sh}\| \le C\, h^{1-\varepsilon}\, \sqrt{|\log h|}
$$

for all $\varepsilon > 0$ with a constant C independent of h.

Notice that, in the three dimensional case, the semi-discrete approach achieves a higher order of convergence than the full discretization (cf. Theorem 6.9). Moreover, similarly to purely control-constrained problems, $\bar{u}_{sh}$ is not an element of the discrete space spanned by the linear ansatz functions (see also Hinze [18]).

Now assume that problem (Q) is discretized by using linear ansatz functions for the control as done in case of (P). Then one cannot proceed as carried out in Section 6 since $u \in U_c := \{u \in L^2(\Omega)\,|\, u_a(x) \le u(x) \le u_b(x) \;\; \text{a.e. in } \Omega\}$ does in general not imply $\Pi_h\, u \in U_c$. It might be possible to work with a convex projection P_h defined by

$$
\|u - P_h u\| = \min_{v_h \in V_h \cap U_c} \|u - v_h\|.
$$

However, up to the author's knowledge, there is no convergence result of the form of Lemma 6.3 for this type of projection.

8. Numerical examples. In the following, we test the presented error analysis with two different examples. The first one refers to the purely state-constrained case, i.e. problem (P), see Section 8.1. Here, we use a primal-dual active set strategy to solve the discretized optimal control problem (see for instance [3] or [4]). The latter test case corresponds to problems with control and state constraints as discussed in Section 6. Here, we apply two different methods for the different inequality constraints in (Q). More precisely, the state constraints are penalized by a logarithmic barrier function (cf. for example [22]), while the box-constrains on the control are treated by a primal-dual active set method (see for instance [3]). The corresponding results are shown in Section 8.2. Both examples are performed on the unit square such that the remarks on polyhedral domains in Section 7 apply.

8.1. Example 1: pure state constraints. The test case for purely state-constrained problems is given by the following optimal control problem

$$
\text{(PT)} \quad \begin{cases} \text{minimize} & J(y,u) := \dfrac{1}{2}\displaystyle\int_\Omega |y - y_d|^2\,dx + \dfrac{\alpha}{2}\int_\Omega u^2\,dx \\ \text{subject to} & -\Delta\, y(x) + y(x) = u(x) + f(x) \quad \text{in } \Omega \\ & \qquad\qquad \partial_n y(x) = 0 \qquad\qquad\quad \text{on } \Gamma \\ \text{and} & y_a(x) \le y(x) \quad \text{a.e. in } \Omega, \end{cases}
$$

which can be reformulated by introducing the control-to-state operator into

$$
\text{(PT)} \quad \begin{cases} \text{minimize} & f(u) := \dfrac{1}{2}\,\|S\,u - (y_d - S\,f)\|^2 + \dfrac{\alpha}{2}\,\|u\|^2 \\ \text{subject to} & u \in L^2(\Omega) \text{ and } (S\,u)(x) \le (y_b - S\,f)(x) \text{ a.e. in } \Omega. \end{cases}
$$

Let us assume that $f \in W^{2,\infty}(\Omega)$ such that

$$
\begin{aligned} \|S\,f - S_h\,I_h\,f\|_\infty &\le \|(S - S_h)f\|_\infty + \|S(f - I_h\,f)\|_\infty \\ &\le c\,h^2\,|\log\,h|\,\|f\|_\infty + c\,h^2\,\|f\|_{W^{2,\infty}(\Omega)}. \end{aligned}
$$

Then, by using similar arguments as in Section 7, it is easy to see that the error analysis for (P) can be adapted to problem (PT). For the optimal state, control, and adjoint state, we choose

$$
\bar{y}(x) = -16\,x_1^4 + 32\,x_1^3 - 16\,x_1^2 + 1\;, \quad p(x) = 2\,x_1^3 - 3\,x_1^2\;, \quad \bar{u}(x) = -(1/\alpha)\,p(x),
$$

such that the gradient equation in the optimality system (3.3) is satisfied. Moreover, the lower bound is given by

$$
y_a(x) = \begin{cases} \bar{y}(x_1 = 0.2)\;, & x_1 \le 0.2 \\ \quad \bar{y}(x) & , \; 0.2 < x_1 < 0.8 \\ \bar{y}(x_1 = 0.8)\;, & 0.8 \le x_1. \end{cases}
$$

Notice that $y_a \notin W^{2,\infty}(\Omega)$, which was required in Section 7. However, the used meshes are constructed such that the lines $\{(x_1,x_2) \in \Omega \mid x_1 = 0.2\}$ and $\{(x_1,x_2) \in \Omega \mid x_1 = 0.8\}$ coincide with edges of the triangulation. Therefore, the kinks of y_a at $x_1 = 0.2$ and $x_2 = 0.8$ are captured by the mesh and thus, estimate (7.1) also holds in this case. The definition of y_a implies that the state constraint is active in $\Omega_a := \{(x_1,x_2) \in \Omega \mid 0.2 \le x_1 \le 0.8\}$. Hence, to fulfill the complementary slackness conditions the associated Lagrange multiplier must vanish on $\Omega\backslash\Omega_a$. Here, we choose a continuous multiplier given by

$$
\mu(x) = \begin{cases} 0 & , \; x_1 \le 0.2 \\ -\bar{y}(x) + y(x_1 = 0.2)\;, & 0.2 < x_1 < 0.8 \\ 0 & , \; 0.8 \le x_1. \end{cases}
$$

Finally, the state equation and the adjoint equation imply

$$
\begin{aligned} f &= -\Delta\bar{y} + \bar{y} - \bar{u}, \\ y_d &= \Delta p - p + \bar{y} - \mu. \end{aligned}
$$

Figures 8.1–8.4 show the numerical solution for $h = 0.02$ and $\alpha = 10^{-4}$. If the bound y_a is discretized like the state y, then the inequality constraint in the discretization of (PT) is equivalent to $y_{a,h}(x_i) \leq y_h(x_i)$, $i = 1, ..., n$, where x_i denote the nodes of the triangulation. The discrete Lagrange multiplier associated to this constraint is an element of $\mathbb{R}^n$, whose components can be interpreted as coefficients in the following discretization of the infinite dimensional multiplier

$$\mu_h = \sum_{i=1}^{N} \mu_i \, \delta_{x_i},$$

where δ_{x_i} denotes the Dirac measure at x_i (see also [12]). However, in this example, the multiplier is a continuous function that can be interpolated by linear ansatz functions, i.e. $\mu(x) \approx \tilde{\mu}_h(x) = \sum_{i=1}^{n} \tilde{\mu}_i \, \phi_i(x)$. In view of $\int_{\bar{\Omega}} v_h \, d\mu_h = \int_{\Omega} \tilde{\mu}_h \, v_h \, dx$ for all $v_h \in V_h$, we have $\tilde{\mu}_i = \sum_j M_{ij}^{-1} \, \mu_j$, where M denotes the mass matrix corresponding to linear ansatz functions. As approximation of the multiplier, the function $\tilde{\mu}_h$ is shown in Figure 8.4.

We observe that the discrete Lagrange multiplier is fairly unregular, in particular at

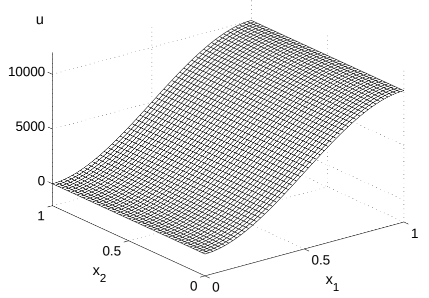

FIG. 8.1. *Example 1: optimal control for* $h = 0.02$.

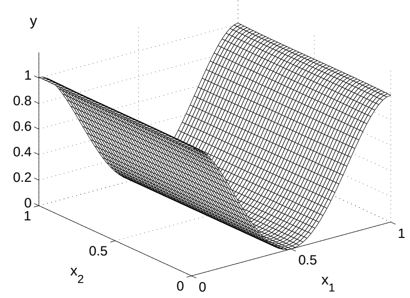

FIG. 8.2. *Example 1: optimal state for* $h = 0.02$.

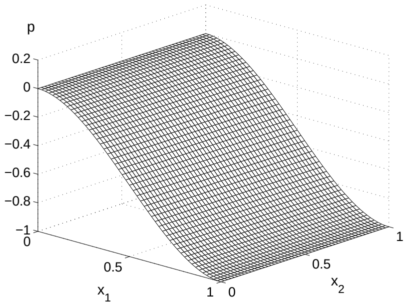

FIG. 8.3. *Example 1: adjoint state for* $h = 0.02$.

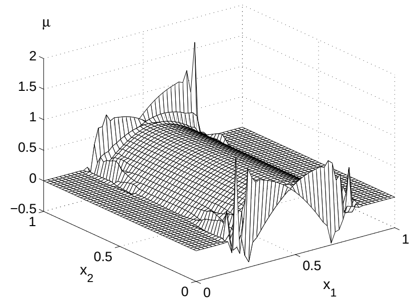

FIG. 8.4. *Example 1: multiplier for* $h = 0.02$.

the boundaries of Ω and the active set. However, since the multiplier is in general only an element of $C(\bar{\Omega})^*$, one cannot expect convergence of the discrete multipliers in $L^2(\Omega)$. Table 8.1 displays the relative errors of control and state for this example and $\alpha = 10^{-1}$. Here, e_2 refers to the relative error in the L^2-norm, whereas $e_{1,2}$ denotes the relative error in the H^1-norm. The experimental order of convergence is shown in Table 8.2. In case of u, it is computed by

$$EOC_2(u) := \frac{\log\big(e_2(u, h_1)\big) - \log\big(e_2(u, h_2)\big)}{\log(h_1) - \log(h_2)},$$

TABLE 8.1
Relative errors in the first example.

$h/\sqrt{2}$	$e_2(u)$	$e_2(y)$	$e_{1,2}(y)$
0.04	9.4654e-03	3.9105e-03	7.2099e-02
0.02	2.5233e-03	9.8588e-04	3.6141e-02
0.01	6.9207e-04	2.4815e-04	1.8084e-02
0.005	1.9484e-04	6.2375e-05	9.0437e-03
0.0025	5.6086e-05	1.5663e-05	4.5221e-03

where h_1 and h_2 denote two consecutive mesh sizes. Similarly, $EOC_{1,2}(y)$ is computed with $e_{1,2}(y)$ instead of $e_2(u)$. As one can see, the order of convergence in case of $e_2(u)$ is better

TABLE 8.2
Experimental order of convergence in the first example.

$h_2/\sqrt{2}$	0.02	0.01	0.005	0.0025
$EOC_2(u)$	1.907	1.866	1.829	1.797
$EOC_{1,2}(y)$	0.996	0.998	0.999	0.999

than expected. This agrees with the numerical findings in [12], where a similar purely state-constrained problem is solved with the semi-discrete approach. Notice that, as mentioned above, the semi-discrete and the fully discretized problem coincide in this case. A possible explanation for this superconvergence observation could be that the optimal control in this example is much smoother than in the general state-constrained case. However, if linear finite elements are used, then the convergence order with respect to the L^∞-error of $S-S_h$ is at best equal to $h^2\,|\log h|$. Thus, also in case of higher regularity, the presented theory only yields $h^{1-\varepsilon}$ for $e_2(u)$ and is therefore not appropriate to explain this superconvergence effect. In contrast to that, $EOC_{1,2}(y)$ fits to the theoretical predictions.

8.2. Example 2: state and control constraints. Now, let us turn to an example with pointwise state and control constraints. The test case is identical to (PT), except that we have additional box constraints on the control given by

$$u_a \le u(x) \le u_b \quad \text{a.e. in } \Omega.$$

The exact solution nearly coincides with the one of Section 8.1. The only difference is the optimal control, which is now given by

$$\bar u(x) = \max(u_a, \min(u_b, -(1/\alpha)p))$$

Hence, we also have to modify the correction term in the state equation, i.e. $f = -\Delta\bar y + \bar y - \bar u$. Consequently, $f \notin W^{2,\infty}(\Omega)$. However, u_a and u_b are chosen such that the kinks of f coincide with edges of the triangulation, and hence the same arguments as in case of y_a can be applied. Figures 8.5–8.8 show the numerical solution for $\alpha = 10^{-1}$ and $h = 0.02$. Notice that the active sets associated to the control constraints and the active set corresponding to the state constraint are not disjoint. Since u is discretized by

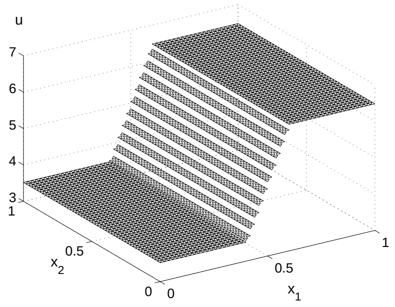

Fig. 8.5. *Example 2: optimal control for* $h = 0.02$.

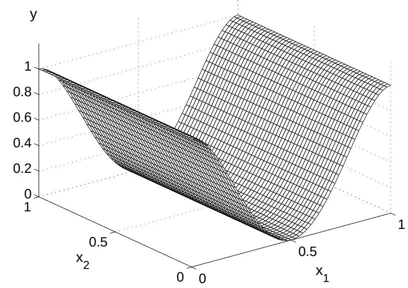

Fig. 8.6. *Example 2: optimal state for* $h = 0.02$.

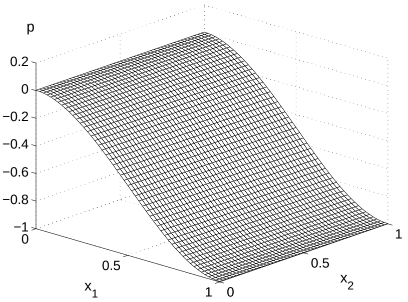

Fig. 8.7. *Example 2: adjoint state for* $h = 0.02$.

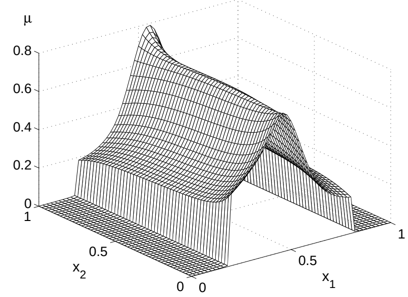

Fig. 8.8. *Example 2: multiplier for* $h = 0.02$.

constant ansatz functions, Figure 8.5 shows the values of u_h at each triangle. Furthermore, the multiplier is approximated by $\varepsilon/(y_h - y_a)$, where ε denotes the homotopy parameter of the associated interior point method (see [22] for details). As before, Tables 8.3 and 8.4 present the relative errors and orders of convergence, respectively. We observe that,

Table 8.3
Relative errors in the first example.

$h/\sqrt{2}$	$e_2(u)$	$e_2(y)$	$e_{1,2}(y)$
0.04	2.3751e-02	5.6749e-03	7.2242e-02
0.02	8.1245e-03	1.3651e-03	3.6158e-02
0.01	3.3656e-03	3.3680e-04	1.8086e-02
0.005	1.5419e-03	1.9980e-04	9.0441e-03
0.0025	7.8607e-04	1.6666e-04	4.5224e-03

Table 8.4
Experimental order of convergence in the first example.

$h_2/\sqrt{2}$	0.02	0.01	0.005	0.0025
$EOC_2(u)$	1.548	1.271	1.127	0.972
$EOC_{1,2}(y)$	0.998	0.995	0.998	0.999

for larger values of h, the experimental order of convergence in case of u is again higher than the expected one. However, the difference is smaller than in the first example and decreases if the mesh size is reduced such that, in the last step, it agrees with theoretical predictions. Moreover, as above, $EOC_{1,2}(y)$ coincides with the theoretical findings.

Acknowledgment. The author is very grateful to Dr. Joachim Rehberg for his helpful advice with respect to Theorem 3.1. Moreover, I would like to thank Prof. Michael Hinze for some helpful discussions concerning interpolation on curved domains.

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